

# ASYMPTOTICS OF CHEBYSHEV POLYNOMIALS, II: DCT SUBSETS OF $\mathbb{R}$

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## Abstract

*We prove Szegő–Widom asymptotics for the Chebyshev polynomials of a compact subset of  $\mathbb{R}$  which is regular for potential theory and obeys the Parreau–Widom and DCT conditions.*

## 1. Introduction

Let  $\mathfrak{e} \subset \mathbb{C}$  be a compact subset with logarithmic capacity  $C(\mathfrak{e}) > 0$ . Define

$$\|f\|_{\mathfrak{e}} = \sup_{x \in \mathfrak{e}} |f(x)|. \quad (1.1)$$

The Chebyshev polynomial,  $T_n(z)$ , is the monic polynomial with

$$t_n \equiv \|T_n\|_{\mathfrak{e}} = \inf\{\|P\|_{\mathfrak{e}} \mid \deg P = n, P \text{ monic}\}. \quad (1.2)$$

It is a consequence of the theory of best approximation that  $T_n(z)$  exists and is unique (see, e.g., [16, Chapter 2.5]).

When  $\mathfrak{e} \subset \mathbb{R}$ , the alternation theorem (a result of Borel [4] and Markov [17] using ideas that go back to Chebyshev; see [6] for a statement and proof or [5, Chapter 3.4]) implies that  $T_n$  is unique and that

$$\mathfrak{e}_n \equiv T_n^{-1}([-t_n, t_n]) = \{z \in \mathbb{C} \mid -t_n \leq T_n(z) \leq t_n\} \quad (1.3)$$

is a subset of  $\mathbb{R}$ . Clearly, by definition of  $t_n$ ,

$$\mathfrak{e} \subset \mathfrak{e}_n. \quad (1.4)$$

Recall that the Green’s function,  $G_{\mathfrak{e}}(z)$ , is the unique function on  $\mathbb{C}$  which is positive and harmonic on  $\mathbb{C} \setminus \mathfrak{e}$ , upper semicontinuous on  $\mathbb{C}$ , so that  $G_{\mathfrak{e}}(z) = \log(|z|) +$

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$O(1)$  near  $z = \infty$  and so that  $G_\epsilon(x) = 0$  for quasi-every  $x \in \epsilon$ . A set,  $\epsilon$ , is called regular (for potential theory) if  $G_\epsilon(x) = 0$  for all  $x \in \epsilon$  (which implies that  $G_\epsilon$  is continuous on  $\mathbb{C}$ ). We will assume that  $\epsilon$  is regular. One has that near infinity

$$G_\epsilon(z) = \log(|z|) - \log(C(\epsilon)) + O(1/|z|). \quad (1.5)$$

Moreover, if  $d\rho_\epsilon$  is the potential theoretic equilibrium measure for  $\epsilon$ , then

$$G_\epsilon(z) = -\log(C(\epsilon)) + \int \log(|z - x|) d\rho_\epsilon(x). \quad (1.6)$$

For more on potential theory, see [20] or [24, Section 3.6].

It is not hard to see ([6, Theorem 2.2]) that the Green's function,  $G_n$ , for  $\epsilon_n$  is

$$G_n(z) = \frac{1}{n} \log \left( \left| \frac{T_n(z)}{t_n} + i \sqrt{1 - \left( \frac{T_n(z)}{t_n} \right)^2} \right| \right), \quad (1.7)$$

which implies that

$$t_n = 2(C(\epsilon_n))^n. \quad (1.8)$$

In particular, since  $C(\epsilon) \leq C(\epsilon_n)$ , we get Schiefermayr's bound (see [21])

$$t_n \geq 2(C(\epsilon))^n. \quad (1.9)$$

In [6], we introduced the term *Totik–Widom bound* (after [27], [29]) if for some constant  $D$ , one has that

$$t_n \leq D(C(\epsilon))^n. \quad (1.10)$$

A compact set  $\epsilon \subset \mathbb{C}$  is said to obey a Parreau–Widom (PW) condition (after [19], [30]) if and only if

$$PW(\epsilon) \equiv \sum_{z_j \in \mathcal{C}} G_\epsilon(z_j) < \infty, \quad (1.11)$$

where  $\mathcal{C}$  is the set of points,  $z_j$ , where  $\nabla G_\epsilon(z_j) = 0$ . For regular subsets of  $\mathbb{R}$ , all these critical points are real and there is exactly one such point in each bounded open component,  $K_j$ , of  $\mathbb{R} \setminus \epsilon$  and  $G_\epsilon(z_j) = \max_{x \in K_j} G_\epsilon(x)$ .

In [6], we proved that if  $\epsilon \subset \mathbb{R}$  is a regular PW set, then one has an explicit Totik–Widom bound

$$t_n \leq 2 \exp(PW(\epsilon)) (C(\epsilon))^n. \quad (1.12)$$

Our methods there say nothing about the complex case. In this regard, we mention the recent interesting paper of Andrievskii [2], who has proven Totik–Widom bounds for a class of sets that, for example, includes the Koch snowflake (see also [3]).

One of our results in this paper (see Theorem 1.4 and Section 2) will be a kind of weak converse—that is, under an additional condition on  $\epsilon$  which should hold generically, if  $\epsilon \subset \mathbb{C}$  is compact, regular, and obeys a Totik–Widom bound, then  $\epsilon$  is a PW set.

For a general positive capacity, regular, compact set  $\epsilon \subset \mathbb{C}$ , we define  $\Omega$  to be its complement in the Riemann sphere, that is,

$$\Omega = (\mathbb{C} \cup \{\infty\}) \setminus \epsilon, \quad (1.13)$$

which we suppose is connected (this always holds if  $\epsilon \subset \mathbb{R}$ ). We let  $\widetilde{\Omega}$  be its universal cover and  $\pi : \widetilde{\Omega} \rightarrow \Omega$  the covering map. It is a consequence of the uniformization theorem (see [23, Section 8.7]) that  $\widetilde{\Omega}$  is conformally equivalent to the disk,  $\mathbb{D}$ , a fact we will use. We denote by  $\mathbf{x} : \mathbb{D} \rightarrow \Omega$  the unique covering map normalized by  $\mathbf{x}(0) = \infty$  and near  $z = 0$ ,  $\mathbf{x}(z) = Dz^{-1} + O(1)$  with  $D > 0$ .

There is an important multivalued analytic function,  $B_\epsilon(z)$ , on  $\Omega$  determined by

$$|B_\epsilon(z)| = e^{-G_\epsilon(z)} \quad (1.14)$$

and, near  $\infty$ ,

$$B_\epsilon(z) = C(\epsilon)z^{-1} + O(z^{-2}). \quad (1.15)$$

One way of constructing it is to use the fact that  $-G_\epsilon$  has a harmonic conjugate locally so that locally on  $\mathbb{C} \setminus \epsilon$ , it is the real part of an analytic function whose exponential is  $B_\epsilon(z)$ . It follows that  $B_\epsilon$  can be continued along any curve in  $\widetilde{\Omega}$ , and so by the monodromy theorem (see [23, Section 11.2]),  $B_\epsilon(z)$  has an analytic continuation to  $\widetilde{\Omega}$  which defines a multivalued analytic function on  $\Omega$ .

By analyticity, (1.14) holds for all branches of  $B_\epsilon(z)$ . In particular, going around a closed curve,  $\gamma$ , can only change  $B_\epsilon$  by a phase factor which implies there is a character,  $\chi_\epsilon$ , of the fundamental group,  $\pi_1(\Omega)$ , so that going around  $\gamma$  changes  $B_\epsilon$  by  $\chi_\epsilon([\gamma])$ . One can show (see [6, Theorem 2.7]) that

$$\chi_\epsilon(\gamma) = \exp\left(-2\pi i \int_\epsilon N(\gamma, x) d\rho_\epsilon(x)\right), \quad (1.16)$$

where  $N(\gamma, x)$  is the winding number for the curve  $\gamma$  about  $x$ . Thus  $B_\epsilon$  is a character-automorphic function.

An alternate construction is to consider elementary Blaschke factors  $b(z, w)$  ( $= (\bar{w}/|w|)[(w - z)/(1 - \bar{w}z)]$  if  $w \neq 0$ ) for  $z, w \in \mathbb{D}$ . Then, lifted to  $\mathbb{D}$ ,

$$B_\epsilon(z) = \prod_{\{w_j | \mathbf{x}(w_j) = \infty\}} b(z, w_j). \quad (1.17)$$

We will call  $B_\epsilon$  the *canonical Blaschke product* for  $\epsilon$  and  $\chi_\epsilon$ , the *canonical character*.

Similarly, we can define, for each  $w \in \Omega$ ,  $B_\epsilon(z, w)$  either by using (1.17) with  $\{w_j \mid \mathbf{x}(w_j) = \infty\}$  replaced by  $\{w_j \mid \mathbf{x}(w_j) = w\}$  or by using the Green's function  $G_\epsilon(z, w)$  with pole at  $w$  and demanding that  $|B_\epsilon(z, w)| = \exp(-G_\epsilon(z, w))$  and fixing the phase by demanding that  $B_\epsilon(\infty, w) > 0$ .

One can consider character-automorphic functions for general characters,  $\chi \in \pi_1(\Omega)^*$ , the full character group. In this regard the following theorem of Widom [30] (see also Hasumi [14, Theorem 5.2B]) is important.

**THEOREM 1.1 (Widom)**

*Suppose that  $\epsilon \subset \mathbb{C}$  is a compact set regular for potential theory. Then  $\epsilon$  is a PW set if and only if, for every character,  $\chi \in \pi_1(\Omega)^*$ , there is a nonzero analytic  $\chi$ -automorphic function on  $\tilde{\Omega}$  which is bounded.*

Single-valued analytic functions on  $\tilde{\Omega}$  correspond to multivalued functions on  $\Omega$ , and we will often refer to them as if they are ordinary functions. In essence, when  $\epsilon \subset \mathbb{R}$  we view  $\Omega$  with the convex hull of  $\epsilon$  removed as a subset of  $\tilde{\Omega}$ .

For a PW set,  $\epsilon$ , and any character,  $\chi$ , we let  $H^\infty(\Omega, \chi)$  be the set of bounded analytic  $\chi$ -automorphic functions on  $\tilde{\Omega}$  and denote by  $\|\cdot\|_\infty$  the corresponding norm. We use  $H^2(\Omega, \chi)$  or  $\mathcal{H}_\chi$  for the set of analytic  $\chi$ -automorphic functions,  $f$ , for which  $|f|^2$  has a harmonic majorant in  $\Omega$ . Evidently,  $H^\infty(\Omega, \chi) \subset H^2(\Omega, \chi)$ . One can show (using the same ideas as in [12, Proposition 4.1]) that  $H^2(\Omega, \chi)$  is precisely those  $\chi$ -automorphic functions,  $f$ , on  $\Omega$  whose lifts to  $\mathbb{D}$  under  $\mathbf{x}$  are in  $H^2(\mathbb{D})$ .

When  $\epsilon$  is a PW set, there exist  $h \in H^\infty(\Omega, \chi)$  with  $h(\infty) \neq 0$ , for if  $f \in H^\infty(\Omega, \chi)$  with  $f(z) = Cz^{-n} + O(z^{-n-1})$ ,  $C \neq 0$ , then  $h(z) = z^n f(z)$  is also in  $H^\infty(\Omega, \chi)$  and  $h(\infty) = C$ .

For any  $\chi$ , the Widom trial functions for  $\chi$  is the set,  $\{h \in H^\infty(\Omega, \chi) \mid h(\infty) = 1\}$ . The *Widom minimizer*,  $F_\chi(z)$ , is a bounded  $\chi$ -character-automorphic function with  $F_\chi(\infty) = 1$  so that

$$\|F_\chi\|_\infty = \inf\{\|h\|_\infty \mid h \in H^\infty(\Omega, \chi); h(\infty) = 1\}. \quad (1.18)$$

Knowing that there are Widom trial functions, it follows from Montel's theorem (see [23, Section 6.2]) that minimizers exist. In Section 2, we will prove that minimizers are unique. (This is not a new result, although our proof is simpler than previous ones.)

We will also consider a dual problem. The dual Widom trial functions are  $\{g \in H^\infty(\Omega, \chi) \mid \|g\|_\infty = 1\}$ . The *dual Widom maximizer* is that function  $Q_\chi$  in the dual Widom trial functions with

$$Q_\chi(\infty) = \sup\{g(\infty) \mid g \in H^\infty(\Omega, \chi), \|g\|_\infty = 1, g(\infty) > 0\}. \quad (1.19)$$

If  $g$  is a dual Widom trial function with  $g(\infty) \neq 0$ , then  $g/g(\infty)$  is a Widom trial function. Conversely, if  $h$  is a Widom trial function, then  $h/\|h\|_\infty$  is a dual Widom trial function. This shows that for the two problems, either both or neither have unique solutions and

$$Q_\chi = F_\chi / \|F_\chi\|_\infty, \quad F_\chi = Q_\chi / Q_\chi(\infty), \quad Q_\chi(\infty) = 1 / \|F_\chi\|_\infty. \quad (1.20)$$

Suppose now that  $\epsilon \subset \mathbb{C}$  is compact connected and simply connected. Then  $\Omega$  is simply connected and  $B_\epsilon$  is analytic (rather than multivalued analytic) and is, in fact, the Riemann map of  $\Omega$  to  $\mathbb{D}$  (uniquely specified by  $B_\epsilon(\infty) = 0$  and that near  $\infty$ ,  $B_\epsilon(z) = Cz^{-1} + O(z^{-2})$  with  $C > 0$ ). In 1919, assuming that  $\partial\Omega$  is an analytic Jordan curve, Faber [10] proved that in this case

$$\frac{T_n(z)B_\epsilon(z)^n}{C(\epsilon)^n} \rightarrow 1 \quad (1.21)$$

uniformly on  $\overline{\Omega}$ .

In 1969, Widom [29] considered  $\epsilon \subset \mathbb{C}$ , which is a finite union of  $C^{2+}$  Jordan curves and arcs. He noted that (1.21) could not hold when there was more than one arc or curve since, in that case,  $B_\epsilon(z)^n$  is now a character-automorphic function with character  $\chi_\epsilon^n$ . If  $F_n \equiv F_{\chi_\epsilon^n}$ , Widom suggested what we call the *Widom surmise*, that

$$\frac{T_n(z)B_\epsilon(z)^n}{C(\epsilon)^n} - F_n(z) \rightarrow 0 \quad (1.22)$$

uniformly on compact subsets of  $\widetilde{\Omega}$ . He proved this when  $\epsilon$  consisted only of (closed) Jordan curves, and, in [6], we proved it for  $\epsilon$ , a finite gap set in  $\mathbb{R}$ .

A bounded function,  $f$ , on  $\mathbb{Z}$  is called *almost periodic* (in the Bochner sense) if the family  $\{f(\cdot - m)\}_{m \in \mathbb{Z}}$  has a compact closure with respect to the topology of uniform convergence. For a discussion of almost periodic functions, we refer to [7, Chapter I.6] or [25, Section 6.6].

We say that  $T_n$  has *strong Szegő–Widom asymptotics* if

- (a) (1.22) holds uniformly on compact subsets of  $\widetilde{\Omega}$ ;
  - (b)  $n \mapsto \|F_n\|_\infty$  is an almost periodic function;
  - (c)  $n \mapsto F_n(z)$  is an almost periodic function uniformly on compact subsets of  $\widetilde{\Omega}$ .
- We note that the above results of Widom [29] and [6] prove (b) and (c) also.

A final element we need before stating our main theorem is the notion of the Direct Cauchy Theorem (DCT) property. There are many equivalent definitions of DCT (see Hasumi [14] or Volberg and Yuditskii [28]). Rather than stating a formal definition, we first of all quote a theorem that could be used as one definition of DCT.

THEOREM 1.2 (see Hayashi [15], Hasumi [14])

A PW set  $\mathfrak{e} \subset \mathbb{C}$  obeys a DCT if and only if the function  $\chi \mapsto Q_\chi(\infty)$  of the dual Widom maximizer problem is a continuous function on  $\pi_1(\Omega)^*$ .

We will also quote as needed some other results that rely on the DCT condition. We note that any homogeneous subset of  $\mathbb{R}$  (in the sense of Carleson [26]) obeys DCT (see [26]). On the other hand, Hasumi [14] found rather simple explicit examples (with thin components) of subsets of  $\mathbb{R}$  which obey PW but not DCT. Volberg and Yuditskii [28] have even found examples, all of whose reflectionless measures are absolutely continuous. In this regard, see also [31].

We can now state the main result of this paper.

THEOREM 1.3

Let  $\mathfrak{e} \subset \mathbb{R}$  be a compact set which is regular for potential theory and that obeys the PW and DCT conditions. Then its Chebyshev polynomials have strong Szegő–Widom asymptotics. Moreover,

$$\lim_{n \rightarrow \infty} \frac{t_n}{C(\mathfrak{e})^n \|F_n\|_\infty} = 2. \quad (1.23)$$

*Remarks*

1. Given the limit (1.22), the 2 in (1.23) may seem surprising. Widom noted the 2 in the easy special case  $\mathfrak{e} = [-1, 1]$  and proved (1.23) for general finite gap subsets of  $\mathbb{R}$ . This fact was used in our proof of (1.22) for the finite gap case in [6]. Here we will prove (1.22) first and then prove (1.23).
2. Our proof is based, in part, on a variant of the strategy employed in [6], and we believe it is simpler, even in the finite gap case (especially if you include the need there for some results of Widom that we do not need to prove a priori).

For our other main results, we need a new definition. We say a set  $\mathfrak{e} \subset \mathbb{R}$  has a *canonical generator* if  $\{\chi_\mathfrak{e}^n\}_{n=-\infty}^\infty$  is dense in the character group  $\pi_1(\Omega)^*$ . This holds if and only if for each decomposition  $\mathfrak{e} = \mathfrak{e}_1 \cup \cdots \cup \mathfrak{e}_\ell$  into closed disjoint sets and rational numbers  $\{q_j\}_{j=1}^{\ell-1}$ , we have that

$$\sum_{j=1}^{\ell-1} q_j \rho_\mathfrak{e}(\mathfrak{e}_j) \neq 0. \quad (1.24)$$

*Remarks*

1. The class of regular PW sets  $\mathfrak{e} \subset \mathbb{R}$  can be parameterized by comb domains of the form

$$\Pi = \{x + iy \mid 0 < x < 1, y > 0\} \setminus \bigcup_k \{\omega_k + iy \mid 0 < y \leq h_k\} \quad (1.25)$$

with  $\omega_k \in (0, 1)$ ,  $\omega_k \neq \omega_j$  for  $k \neq j$  and  $h_k > 0$ ,  $\sum_k h_k < \infty$ . Specifically, if  $\mathfrak{e}$  is scaled to the interval  $[0, 1]$ , then

$$\theta(z) = \frac{-\log B_{\mathfrak{e}}(z)}{\pi i} \quad (1.26)$$

is a conformal mapping of  $\mathbb{C}_+$  onto such a domain (see [9] for more details). In that parameterization, the property of a canonical generator is generic. For one can show that  $\omega_k = \rho_{\mathfrak{e}}(\{x \in \mathfrak{e} \mid x \leq a_k\})$  and the collection of comb domains with rationally independent  $\omega_k$ 's clearly form a dense  $G_\delta$  set.

2. It seems likely that the condition of a canonical generator holds in various other generic senses as well. For example, given a nowhere dense, infinite gap set, we can pick a positive integer labeling of the gaps and, for any  $\lambda \in \prod_1^\infty [1/2, 2]$ , consider the set obtained by scaling the  $j$ th gap by  $\lambda_j$ . We suspect that the set of  $\lambda$ 's for which this set has a canonical generator is a dense  $G_\delta$ . In the finite gap case, that this is true follows from results of Totik [27].

#### THEOREM 1.4

*Let  $\mathfrak{e} \subset \mathbb{C}$  be a compact set regular for potential theory with a canonical generator. If  $\mathfrak{e}$  has a Totik–Widom bound, then  $\mathfrak{e}$  is a PW set.*

#### Remarks

1. While we need to assume canonical generator, this result suggests that Totik–Widom fails if the set is not PW.
2. We emphasize that this result holds for  $\mathfrak{e} \subset \mathbb{C}$  and not just  $\mathfrak{e} \subset \mathbb{R}$ .

#### THEOREM 1.5

*Let  $\mathfrak{e} \subset \mathbb{C}$  be a compact set regular for potential theory with a canonical generator. Suppose that  $\mathfrak{e}$  is a PW set and that  $n \mapsto \|F_n\|_\infty$  is a bounded almost periodic function on  $\mathbb{Z}$ . Then  $\mathfrak{e}$  is a DCT set.*

#### Remarks

1. Again, we emphasize that this holds for all  $\mathfrak{e} \subset \mathbb{C}$ , not just  $\mathfrak{e} \subset \mathbb{R}$ .
2. So, one small part of Szegő–Widom asymptotics, namely, asymptotic almost periodicity of  $\|T_n\|_{\mathfrak{e}}/C(\mathfrak{e})^n$  and the limit result (1.23), implies that  $\mathfrak{e}$  is a DCT set (at least if  $\mathfrak{e}$  has a canonical generator).

We will note results from [6] as needed and mention now some that are needed to give an overview of the contents of the paper. Let  $B_n \equiv B_{\mathfrak{e}_n}$ . Then [6, Section 2]

proved that

$$\frac{2T_n(z)}{t_n} = B_n(z)^n + B_n(z)^{-n}. \quad (1.27)$$

Thus, instead of looking at

$$L_n(z) \equiv \frac{T_n(z)B_\epsilon(z)^n}{C(\epsilon)^n}, \quad (1.28)$$

we will look at

$$M_n(z) = B_\epsilon(z)^n / B_n(z)^n, \quad (1.29)$$

which obeys

$$|M_n(z)| = \exp(-nh_n(z)), \quad h_n(z) \equiv G_\epsilon(z) - G_n(z). \quad (1.30)$$

By (1.27),

$$L_n(z) = (1 + B_n(z)^{2n})H_n(z), \quad H_n(z) = \frac{C(\epsilon_n)^n}{C(\epsilon)^n} \frac{B_\epsilon(z)^n}{B_n(z)^n} = \frac{M_n(z)}{M_n(\infty)}. \quad (1.31)$$

The first equation in (1.31) explains the 2 in (1.23). By a simple argument,

$$\sup_{n, z \in K} |B_n(z)| < 1 \quad \text{for any compact set } K \subset \widetilde{\Omega} \quad (1.32)$$

so that  $B_n(z)^{2n}$  goes to zero, but for  $\sup_{z \in \Omega} |1 + B_n(z)^{2n}|$ , we get 2 since there are points  $x \in \epsilon_n$  with  $B_n(x + i0) = 1$ .

By the first equation in (1.31) and (1.32), (1.22) is equivalent to

$$H_n(z) - F_n(z) \rightarrow 0. \quad (1.33)$$

By the second equation in (1.31), it seems likely that it suffices to control limits of  $M_n$ , and that is what we will do. By the maximum principle for harmonic functions and (1.30),  $|M_n(z)| \leq 1$ . We will prove that  $\lim_{n \rightarrow \infty} \|M_n\|_\infty = 1$  and that limit points of  $M_n$  with  $n_j \rightarrow \infty$  so that  $\chi_\epsilon^{n_j} \rightarrow \chi_0$  for some  $\chi_0 \in \pi_1(\Omega)^*$  are dual Widom maximizers, which will let us prove (1.33).

Here is an overview of the rest of this paper. In Section 2, following ideas of Fisher [11], we prove uniqueness of solutions of the Widom minimization problem (this is not a new result, only a new proof; see the discussion there) and prove Theorem 1.4. In Section 3, we discuss continuity of  $F_\chi$  in  $\chi$  and prove Theorem 1.5. In Section 4, we prove that limit points of the  $M_n$  are Blaschke products of suitable  $B(z, x_j)$ , and in Section 5 we prove that these products are dual Widom maximizers. This result has been obtained by Volberg and Yuditskii [28], but we found an alternate proof using ideas of Eichinger and Yuditskii [8]. Finally, in Section 6, we put things together and prove Theorem 1.3.



## 2. Uniqueness of the dual Widom maximizer

In this section, we provide a proof of uniqueness of solutions of the dual Widom maximizer problem and so uniqueness of solutions of the Widom minimizer problem. If  $\epsilon$  obeys a PW condition, then  $H^\infty(\Omega, \chi)$  is nonempty (by Theorem 1.1) and so contains  $h$  with  $h(\infty) > 0$ . By Montel's theorem (see [23, Section 6.2]),  $\{h \in H^\infty(\Omega, \chi) \mid \|h\|_\infty \leq 1, h(\infty) \geq 0\}$  is compact in the topology of uniform convergence on compact subsets of  $\widetilde{\Omega}$ . Thus, there exists a maximizer. We need to prove that this is unique.

Recall that the Ahlfors problem for a compact set  $\epsilon \subset \mathbb{C}$  is to look for bounded analytic functions,  $f$ , on  $\Omega = (\mathbb{C} \cup \{\infty\}) \setminus \epsilon$  with  $\sup_{z \in \Omega} |f(z)| \leq 1$  and  $f(\infty) = 0$  that maximize  $f'(\infty)$  (defined by  $f(z) = f(\infty) + f'(\infty)z^{-1} + O(z^{-2})$  near  $z = \infty$ ). This maximum is called the *analytic capacity* (because if *analytic* is replaced by *harmonic*, then the maximum is the potential theoretic capacity). There is an enormous literature on the Ahlfors problem, in particular, two sets of lecture notes, [13] and [18], and a textbook presentation in [23, Section 8.8].

This is clearly analogous to the dual Widom maximizer problem, so proofs of uniqueness for the Ahlfors problem should have analogs for our problem. In his original paper, Ahlfors [1] considered an  $n$ -connected domain  $\Omega$  (i.e.,  $\epsilon \subset \mathbb{C}$  has  $n$  connected components) and proved that any maximizer,  $g$ , has limiting values for almost every point in  $\partial\Omega$  (maybe only one sided if  $\epsilon$  has a one-dimensional component) with  $|g(w)| = 1$  for  $w \in \partial\Omega$ . This can be used to prove uniqueness. In [29], Widom proved uniqueness for the dual maximizer by proving that any maximizer had absolute value one on  $\partial\Omega$ . The same idea occurs for general Parreau–Widom sets in [28] by Volberg and Yuditskii, who had the first proof of the result in this section.

A simple, elegant approach to uniqueness of the Ahlfors problem is due to Fisher [11]. We will modify his approach to accommodate change of character and the fact that the vanishing at  $\infty$  is different.

### THEOREM 2.1

*Let  $\epsilon \subset \mathbb{C}$  be a PW set regular for potential theory. Then, for any character  $\chi \in \pi_1(\Omega)^*$ , the dual Widom maximizer (and so also the Widom minimizer) exists and is unique.*

### Remarks

1. As noted above this has already been proved by Volberg and Yuditskii [28], but starting from first principles, our proof is simpler.
2. Uniqueness implies that the maximizer in the dual problem is an extreme point in  $H^\infty(\Omega, \chi)_1$ , the closed unit ball in  $H^\infty(\Omega, \chi)$ . For if  $Q_\chi = \frac{1}{2}(q_1 + q_2)$

with  $q_j \in H^\infty(\Omega, \chi)_1$ , then by the maximum property,  $q_j(\infty) = Q_\chi(\infty)$ . So the  $q_j$  are also maximizers, and hence equal to  $Q_\chi$ .

*Proof*

Without loss, we can suppose  $\chi \not\equiv 1$  since if  $\chi \equiv 1$ , then the unique dual maximizer is  $f \equiv 1$ . In particular, since  $\chi \not\equiv 1$ , we have that  $f(\infty) < 1$  by the maximum principle. Let  $f_1$  and  $f_2$  be two maximizers, and define

$$f = \frac{1}{2}(f_1 + f_2), \quad k = \frac{1}{2}(f_1 - f_2). \quad (2.1)$$

Pick  $q \in H^\infty(\Omega, \overline{\chi})$  with  $q(\infty) \neq 0$  and  $\|q\|_\infty = 1$  which exists by the PW condition and Theorem 1.1.

Since  $\|f_j\|_\infty = 1$ , we have that  $\|f \pm k\|_\infty = 1$ , so

$$|f|^2 + |k|^2 = \frac{1}{2}(|f + k|^2 + |f - k|^2) \leq 1. \quad (2.2)$$

Define

$$g = qk^2/2, \quad (2.3)$$

so  $g \in H^\infty(\Omega, \chi)$ . By (2.2),

$$|g| \leq \frac{1 - |f|^2}{2} = (1 - |f|) \left( \frac{1 + |f|}{2} \right) \leq 1 - |f|,$$

so

$$|g| + |f| \leq 1. \quad (2.4)$$

Since  $f_1(\infty) = f_2(\infty)$  is the maximum value,  $g(\infty) = 0$ , and so if  $g \not\equiv 0$ , then, near  $\infty$ , we can write

$$g(z) = \sum_{k=\ell}^{\infty} a_k z^{-k}, \quad a_\ell \neq 0 \quad (2.5)$$

for some  $\ell \geq 1$ .

We will consider as a trial function

$$h_\epsilon(z) = f(z) + \epsilon \bar{a}_\ell z^\ell g(z), \quad (2.6)$$

where  $\epsilon$  will be picked below. Since  $f(\infty) \in (0, 1)$ , we can pick  $\epsilon_0 > 0$  so that

$$f(\infty) + \epsilon_0 |a_\ell|^2 < 1. \quad (2.7)$$

Therefore, we can find  $R > 0$  so that

$$|z| > R \Rightarrow |f(z)| + \epsilon_0 |a_\ell| |z|^\ell |g(z)| < 1. \quad (2.8)$$

Pick  $\epsilon_1 > 0$  so that

$$\epsilon_1 < \epsilon_0, \quad \epsilon_1 |a_\ell| R^\ell < 1. \quad (2.9)$$

We claim that  $\|h_{\epsilon_1}\| \leq 1$ , for by (2.8), if  $|z| > R$ , then  $|h_{\epsilon_1}(z)| \leq 1$ , and, if  $|z| \leq R$ , then by (2.9)

$$|h_{\epsilon_1}(z)| \leq |f(z)| + \epsilon_1 |a_\ell| R^\ell |g(z)| < |f(z)| + |g(z)| \leq 1$$

by (2.4). Thus  $h_{\epsilon_1}$  is a trial function for the dual Widom problem.

On the other hand,

$$h_{\epsilon_1}(\infty) = f(\infty) + \epsilon_1 |a_\ell|^2 > f(\infty), \quad (2.10)$$

violating maximality. We conclude that  $g \equiv 0$ , so  $k \equiv 0$ , and  $f_1 = f_2$ .  $\square$

#### *Proof of Theorem 1.4*

Suppose we have a Totik–Widom bound

$$t_n \leq D(C(\mathfrak{e}))^n. \quad (2.11)$$

Given  $\chi_\infty \in \pi_1(\Omega)^*$ , pick  $n_j \rightarrow \infty$  so that  $\chi_\epsilon^{n_j}$ , the character of  $B_\epsilon^{n_j}$ , converges to  $\chi_\infty$  (which we can do by the assumption of canonical generator). Let

$$f_j(z) = \frac{T_{n_j}(z) B_\epsilon(z)^{n_j}}{C(\mathfrak{e})^{n_j}}. \quad (2.12)$$

By the maximum principle,

$$\|f_j\|_\infty \leq \sup_{z \rightarrow \mathfrak{e}} |f_j(z)| \leq t_{n_j} C(\mathfrak{e})^{-n_j} \leq D,$$

and so by Montel's theorem, we can find  $j_k \rightarrow \infty$ , so that  $f_{j_k}$  converges to  $f_\infty$  uniformly on compacts. Since  $T_{n_j}$  is monic and  $B_\epsilon(z) = C(\mathfrak{e})/z + O(z^{-2})$ , we have  $f_j(\infty) = 1$  and, therefore,  $f_\infty$  is nonzero. Clearly,  $f_\infty \in H^\infty(\Omega, \chi_\infty)$ . By Theorem 1.1,  $\mathfrak{e}$  obeys a PW condition.  $\square$

### **3. Continuity of the Widom minimizer**

In this section, we study continuity properties (in  $\chi$ ) of  $Q_\chi(z)$ ,  $F_\chi(z)$ , and  $\|F_\chi\|_\infty$ . We will show that there is continuity if and only if the DCT holds. Applying this to  $n \rightarrow F_{\chi_\epsilon^n}$ , we will see that DCT implies almost periodicity.

## THEOREM 3.1

Let  $\mathfrak{e} \subset \mathbb{C}$  be a compact, PW, and DCT set that is regular for potential theory. Then  $\chi \mapsto Q_\chi$  and  $\chi \mapsto F_\chi$  are continuous in the topology of uniform convergence on compact subsets of  $\widetilde{\Omega}$ . Moreover,  $\chi \mapsto \|F_\chi\|_\infty$  is continuous. Conversely, if  $\chi \mapsto \|F_\chi\|_\infty$  is continuous for  $\mathfrak{e}$  a regular PW set, then  $\mathfrak{e}$  is a DCT set.

*Proof*

By Theorem 1.2, if  $\mathfrak{e}$  is a DCT set, then  $Q_\chi(\infty)$  is continuous. If  $\chi_n \rightarrow \chi$  for some sequence so that  $Q_{\chi_n}$  converges to a function  $g$  uniformly on compact subsets of  $\widetilde{\Omega}$ , then, by continuity,  $g(\infty) = Q_\chi(\infty)$  and  $\|g\|_\infty \leq 1$ . It follows by uniqueness of the minimizer that  $g = Q_\chi$ . By Montel's theorem,  $\chi \mapsto Q_\chi$  is continuous. Since  $F_\chi(z) = Q_\chi(z)/Q_\chi(\infty)$  and  $\|F_\chi\|_\infty = 1/Q_\chi(\infty)$ , we conclude continuity of  $F_\chi$  and  $\|F_\chi\|_\infty$ .

The converse follows from Theorem 1.2 and  $Q_\chi(\infty) = 1/\|F_\chi\|_\infty$ .  $\square$

## THEOREM 3.2

Let  $\mathfrak{e} \subset \mathbb{C}$  be a compact, PW, and DCT set that is regular for potential theory. Then  $n \mapsto F_{\chi_\epsilon^n}(z)$  and  $n \mapsto Q_{\chi_\epsilon^n}(z)$  are almost periodic uniformly for  $z$  in compact subsets of  $\widetilde{\Omega}$ . Moreover,  $n \mapsto \|F_{\chi_\epsilon^n}\|_\infty$  is a bounded almost periodic function.

*Proof*

Almost periodicity of a function,  $f$ , on  $\mathbb{Z}$  can be defined in terms of the family  $f_m \equiv f(\cdot - m)$  lying in a compact (with respect to the topology of uniform convergence) family of functions. Since  $\pi_1(\Omega)^*$  is compact and both  $\chi \mapsto F_\chi(z)$  and  $\chi \mapsto Q_\chi(z)$  are continuous,  $\{F_\chi\}_{\chi \in \pi_1(\Omega)^*}$  and  $\{Q_\chi\}_{\chi \in \pi_1(\Omega)^*}$  are the required compact families. Since  $Q_\chi(\infty)$  is a continuous function, it takes its minimum value which is always nonzero. Thus  $Q_\chi(\infty)$  is bounded away from zero and thus  $\|F_\chi\|_\infty = 1/Q_\chi(\infty)$  is bounded.  $\square$

We now turn to the proof of Theorem 1.5. The first two of four lemmas require neither almost periodicity nor canonical generator. We will focus on the dual maximizer,  $Q_\chi$ , given by (1.20).

## LEMMA 3.3

Let  $\mathfrak{e}$  be a regular PW set. Then  $\chi \mapsto Q_\chi(\infty)$ , the map from  $\pi_1(\Omega)^*$  to  $(0, 1]$ , is upper semicontinuous; that is,

$$\chi_j \rightarrow \chi \Rightarrow \limsup_{j \rightarrow \infty} Q_{\chi_j}(\infty) \leq Q_\chi(\infty). \quad (3.1)$$

*Proof*

By Montel's theorem, we can always pick a subsequence so that  $Q_{\chi_{j_n}}(\infty) \rightarrow \limsup_{j \rightarrow \infty} Q_{\chi_j}(\infty)$  and so that  $Q_{\chi_{j_n}}$  has a pointwise limit,  $g$ , on the universal cover which has  $\|g\|_\infty \leq 1$  and for which the convergence is uniform on compact subsets of the universal cover. Since  $\chi_{j_n} \rightarrow \chi$ ,  $g$  is a trial function for the dual Widom problem with character  $\chi$ . Since  $Q_\chi$  is a maximizer,  $g(\infty) \leq Q_\chi(\infty)$ ; that is, (3.1) holds.  $\square$

LEMMA 3.4

*Let  $\mathfrak{e}$  be a regular PW set. If  $\chi \mapsto Q_\chi(\infty)$  is continuous at  $\chi = \mathbf{1}$  (i.e., we know that  $\chi_j \rightarrow \mathbf{1} \Rightarrow Q_{\chi_j}(\infty) \rightarrow 1$ ), then  $\chi \mapsto Q_\chi(\infty)$  is continuous on  $\pi_1(\Omega)^*$ .*

*Proof*

Suppose  $\chi_j \rightarrow c$ . Then  $\chi_j/c \rightarrow \mathbf{1}$ . Since  $Q_c Q_{\chi_j/c}$  is a trial function for the  $\chi_j$  dual maximizer problem, we have that

$$Q_c(\infty) Q_{\chi_j/c}(\infty) \leq Q_{\chi_j}(\infty). \quad (3.2)$$

By hypothesis,  $Q_{\chi_j/c}(\infty) \rightarrow 1$ , so (3.2) implies that

$$Q_c(\infty) \leq \liminf_{j \rightarrow \infty} Q_{\chi_j}(\infty). \quad (3.3)$$

This and (3.1) imply that  $Q_{\chi_j}(\infty) \rightarrow Q_c(\infty)$ .  $\square$

LEMMA 3.5

*Let  $\mathfrak{e}$  be a regular PW set. Suppose that  $n \mapsto \|F_n\|_\infty$  is a bounded almost periodic function and that  $\chi_\mathfrak{e}^{n_j} \rightarrow \mathbf{1}$ . Then  $Q_{\chi_\mathfrak{e}^{n_j}} \rightarrow 1$ .*

*Proof*

By hypothesis, there exists a compact additive group  $\mathbb{K}$  and a bounded continuous function,  $B$ , on  $\mathbb{K}$  so that  $\mathbb{Z}$  is a dense subgroup in  $\mathbb{K}$  and  $B(n) = \|F_n\|_\infty$ . Let  $A(\alpha) = B(\alpha)^{-1}$  which is also continuous on  $\mathbb{K}$ , bounded away from 0 (and bounded above by 1) with

$$Q_{\chi_\mathfrak{e}^n}(\infty) = A(n). \quad (3.4)$$

By passing to a subsequence, we can suppose that  $n_j \rightarrow \alpha \in \mathbb{K}$  and that  $Q_{\chi_\mathfrak{e}^{n_j}}(\infty)$  has a limit  $q$ .

Fix  $n_s$ . By passing to a further subsequence, we can suppose that  $Q_{\chi_\mathfrak{e}^{n_s - n_j}}$  has a limit,  $g$ , on the universal cover. Since  $\chi_\mathfrak{e}^{n_j} \rightarrow \mathbf{1}$ ,  $g$  is a trial function for the  $\chi_\mathfrak{e}^{n_s}$  problem, so

$$Q_{\chi_\epsilon^{n_s}}(\infty) \geq g(\infty) = \lim_{n_j \rightarrow \infty} A(n_s - n_j) = A(n_s - \alpha) \quad (3.5)$$

by the continuity of  $A$ . Now take  $n_s \rightarrow \infty$ . By definition of  $q$ , we have

$$q = \lim_{n_s \rightarrow \infty} Q_{\chi_\epsilon^{n_s}}(\infty) \geq \limsup_{n_s \rightarrow \infty} A(n_s - \alpha) = A(0) = 1$$

since  $n_s \rightarrow \alpha$  and  $A(0) = 1$  by (3.4). Thus  $q \geq 1$ . Since  $Q_\chi(\infty) \in (0, 1]$ , we conclude that  $q = 1$ ; that is, 1 is the only limit point of  $Q_{\chi_\epsilon^{n_j}}(\infty)$ , proving the lemma.  $\square$

LEMMA 3.6

Let  $\epsilon$  be a regular PW set. Suppose that  $n \rightarrow \|F_n\|_\infty$  is a bounded almost periodic function and that  $\epsilon$  has a canonical generator. Then  $\chi \mapsto Q_\chi(\infty)$  is continuous at  $\chi = \mathbf{1}$ ; that is,

$$\chi_j \rightarrow \mathbf{1} \Rightarrow \lim_{j \rightarrow \infty} Q_{\chi_j}(\infty) = 1. \quad (3.6)$$

*Proof*

$\pi_1(\Omega)^*$  is a compact, separable group and thus metrizable. Let  $d$  be a metric on  $\pi_1(\Omega)^*$  yielding the usual topology. Since  $\{\chi_\epsilon^m\}$  is dense, we can pick integers  $m_j(\ell)$  for each  $j$  and  $\ell = 1, 2, \dots$  so that  $d(\chi_j, \chi_\epsilon^{m_j(\ell)}) \leq 2^{-\ell}$ .

By Lemma 3.3, we can pick  $\ell_j \geq j$  so that

$$Q_{\chi_\epsilon^{m_j(\ell_j)}}(\infty) \leq Q_{\chi_j}(\infty) + 2^{-j}. \quad (3.7)$$

Let  $k(j) = m_j(\ell_j)$ . Since  $d(\mathbf{1}, \chi_\epsilon^{k(j)}) \leq d(\mathbf{1}, \chi_j) + 2^{-j}$ , we see that  $\chi_\epsilon^{k(j)} \rightarrow \mathbf{1}$ , and so by Lemma 3.5,  $Q_{\chi_\epsilon^{k(j)}}(\infty) \rightarrow 1$ . By (3.7), we conclude that  $\liminf Q_{\chi_j}(\infty) \geq 1$ . Since  $Q_{\chi_j}(\infty) \in (0, 1]$ , we conclude that the limit is 1.  $\square$

*Proof of Theorem 1.5*

By the hypothesis, Lemma 3.6 applies, so we conclude that  $\chi \mapsto Q_\chi(\infty)$  is continuous at  $\mathbf{1}$ . By Lemma 3.4,  $\chi \mapsto Q_\chi(\infty)$  is continuous on all of  $\pi_1(\Omega)^*$ , so, by Theorem 1.2, the set  $\epsilon$  is DCT.  $\square$

#### 4. Limit points of $M_n$ are Blaschke products

In this section and the next, we consider the functions  $M_n(z) = [B_\epsilon(z)/B_n(z)]^n$  of (1.29). Since  $\epsilon \subset \epsilon_n$ , we have that  $G_n(z) \leq G_\epsilon(z)$ , so

$$|M_n(z)| \leq 1. \quad (4.1)$$

$M_n(z)$  is analytic on the universal cover of  $(\mathbb{C} \cup \{\infty\}) \setminus \epsilon_n$ . Since the harmonic measures of components of  $\epsilon_n$  are  $j/n$ ,  $B_n(z)^n$  is single-valued analytic on  $\mathbb{C} \setminus \epsilon_n$ , so  $M_n(z)$  has character  $\chi_n \equiv \chi_\epsilon^n$  for curves in  $\widetilde{\Omega}$  that avoid  $\epsilon_n$ .

In this section, we will prove that limit points of  $M_n$  (after removing some removable potential singular points) are Blaschke products analytic on  $\widetilde{\Omega}$  and, in the next, that these Blaschke products are dual Widom maximizers. This section will only require that  $\mathfrak{e} \subset \mathbb{R}$  is regular for potential theory and obeys a PW condition, while the next will also require the DCT condition.

$\mathbb{R} \setminus \mathfrak{e}$  is a disjoint union of bounded open components (plus two unbounded components),  $K \in \mathcal{G}$ . We will call these the gaps and  $\mathcal{G}$  the set of gaps. A *gap collection* is a subset  $\mathcal{G}_0 \subset \mathcal{G}$ . A *gap set* is a gap collection,  $\mathcal{G}_0$ , and for each  $K_k \in \mathcal{G}_0$  a point  $x_k \in K_k$ . For any gap  $K = (\beta - \alpha, \beta + \alpha)$ , we define

$$K^{(\epsilon)} = (\beta - (1 - \epsilon)\alpha, \beta + (1 - \epsilon)\alpha)$$

so that  $K^{(\epsilon)} \subset K$  and  $|K^{(\epsilon)}| = (1 - \epsilon)|K|$ .

For any gap set,  $S$ , we define the associated Blaschke product

$$B_S(z) = \prod_{K_k \in \mathcal{G}_0} B_{\mathfrak{e}}(z, x_k). \quad (4.2)$$

Lifted to  $\mathbb{D}$ , each  $B_{\mathfrak{e}}(z, x_k)$  is a product of elementary Blaschke factors and thus so is the product in (4.2). It is known (see [23, Theorem 9.9.4]) that such products either converge to 0 uniformly on compacts or else converge to an analytic function vanishing only at the individual zeros; in the latter case, the product has  $\lim_{r \uparrow 1} |B_S(\mathbf{x}(re^{i\theta}))| = 1$  for a.e.  $\theta$  (see [24, Theorem 5.3.1]). Since  $\sum_{K \in \mathcal{G}} \sup_{y \in K} G_{\mathfrak{e}}(\infty, y) < \infty$  by the PW condition, we see that the product in (4.2) converges to a nonzero value at  $z = \infty$ . Thus  $B_S(z)$  is an analytic function on  $\widetilde{\Omega}$  which vanishes exactly at points  $w$  with  $\pi(w) \in \{x_k\}_{K_k \in \mathcal{G}_0}$ . Moreover, for a.e. point  $y \in \mathfrak{e}$ ,

$$\lim_{\epsilon \downarrow 0} |B_S(y + i\epsilon)| = 1. \quad (4.3)$$

Recall (see [6, paragraph (b) following Theorem 1.1]) that any Chebyshev polynomial,  $T_n$ , has at most one zero in any gap  $K \in \mathcal{G}$ . Our main result in this section is the following.

#### THEOREM 4.1

Let  $n_j \rightarrow \infty$  so that for some gap set,  $S$ , we have that if  $K_k \in \mathcal{G}_0$ , then for large  $j$ ,  $T_{n_j}(z)$  has a zero  $z_j^{(k)}$  in  $K_k$  which converges to  $x_k$  as  $j \rightarrow \infty$  and so that for any  $K \in \mathcal{G} \setminus \mathcal{G}_0$ , and for all  $\epsilon > 0$ ,  $T_{n_j}(z)$  has no zero in  $K^{(\epsilon)}$  for all large  $j$ . Then, as  $j \rightarrow \infty$ ,  $M_{n_j}(z) \rightarrow B_S(z)$  uniformly on compact subsets of  $\widetilde{\Omega} \setminus \{w \mid \pi(w) \in \{x_k\}\}$ .

*Remarks*

1. The points  $w$  with  $\pi(w) = x_k$  for some  $k$  are removable singular points for  $B_S$ . In fact, it is easy to see that while  $M_{n_j}(x_k + i0)$  and  $M_{n_j}(x_k - i0)$  may be different, both values converge to 0, so, in a certain sense, one has convergence on all of  $\widetilde{\Omega}$ .
2. By Montel's theorem and (4.1), the functions  $M_n$  lie in a compact set in the Fréchet topology of uniform convergence on compact subsets. We can therefore make multiple demands and one might guess that, as in [6], we want to also demand that  $\chi_{n_j}$  has a limit, as does  $[C(\epsilon_{n_j})/C(\epsilon)]^{n_j}$  and the  $M_{n_j}$ . It turns out that the single condition on the limits of zeros will automatically imply that these other objects converge.

We will prove this result by controlling convergence for  $z$  near  $\infty$  using the following.

PROPOSITION 4.2

Let  $\Upsilon$  be a Riemann surface, and let  $U_n$  be open sets so that for any compact set  $K \subset \Upsilon$ , eventually,  $K \subset U_n$ . Let  $f_n$  be analytic functions on  $U_n$  so that

$$\sup_n \sup_{z \in U_n} |f_n(z)| < \infty. \quad (4.4)$$

Let  $f_\infty$  be analytic on  $\Upsilon$  so that for some  $z_0 \in \Upsilon$  and some neighborhood,  $V$ , of  $z_0$ , we have that

$$\lim_{n \rightarrow \infty} |f_n(z)| = |f_\infty(z)| \quad \text{for all } z \in V; \quad (4.5)$$

$$f_n(z_0) > 0, \quad f_\infty(z_0) > 0; \quad (4.6)$$

$$z \in V \Rightarrow \forall n : f_n(z) \neq 0 \text{ and } f_\infty(z) \neq 0. \quad (4.7)$$

Then  $f_n \rightarrow f$  uniformly on compact subsets of  $\Upsilon$ .

*Proof*

By shrinking  $V$ , we can suppose that it is simply connected and that  $\overline{V}$  is compact. By (4.6) and (4.7), we can define  $g_n(z) = \log f_n(z)$  uniquely if we demand that

$$\operatorname{Im} g_n(z_0) = 0. \quad (4.8)$$

By (4.5),  $\operatorname{Re} g_n \rightarrow \operatorname{Re} g_\infty$  on  $V$ , and so by the Cauchy–Riemann equations,  $\nabla(\operatorname{Im} g_n) \rightarrow \nabla(\operatorname{Im} g_\infty)$ . By (4.8),  $\operatorname{Im} g_n \rightarrow \operatorname{Im} g_\infty$ , so  $f_n \rightarrow f_\infty$  on  $V$ . By Vitali's theorem (see [23, Section 6.2]) and (4.4),  $f_n \rightarrow f_\infty$  uniformly on compacts.  $\square$

Thus instead of  $M_n(z)$ , we can look at



$$|M_n(z)| = \exp(-nh_n(z)), \quad h_n(z) = G_\epsilon(z) - G_n(z). \quad (4.9)$$

Let  $d\rho_n$  be the potential theoretic equilibrium measure of  $\epsilon_n$  (see [24, Sections 3.6–3.7] for background on potential theory). Then we have the following.

PROPOSITION 4.3

*One has that*

$$h_n(z) = \int_{\bigcup_{K_j \in \mathcal{G}} K_j} G_\epsilon(x, z) d\rho_n(x). \quad (4.10)$$

*Remark*

In [6], we proved the Totik–Widom bound (1.12) for PW sets,  $\epsilon \subset \mathbb{R}$ , by using this when  $z = \infty$ ; that is,

$$h_n(\infty) = \int_{\bigcup_{K_j \in \mathcal{G}} K_j} G_\epsilon(x) d\rho_n(x).$$

We proved this by thinking of  $d\rho_n$  as harmonic measure at  $\infty$ ; that is, if  $H$  is harmonic on  $(\mathbb{C} \cup \{\infty\}) \setminus \epsilon_n$  with boundary values  $H(x)$  on  $\epsilon_n$ , then

$$H(\infty) = \int_{\epsilon_n} H(x) d\rho_n(x).$$

If we wrote the analog of this for general  $z$ , then we would get

$$H(z) = \int_{\epsilon_n} H(x) d\rho_n(x, z)$$

varying the harmonic measure. Instead we think of (4.10), with  $G_\epsilon$  arising as the Green's function for solving Poisson's equation with zero boundary values on  $\epsilon$ , and  $d\rho_n$  occurs as the Laplacian of  $G_n$ .

*Proof*

Both sides of (4.10) are continuous functions of  $z \in \mathbb{C} \cup \{\infty\}$  (by regularity of  $\epsilon$  and  $\epsilon_n$ ), and both sides vanish on  $\epsilon$ . Off  $\epsilon$ , they have the same distributional Laplacian, namely,  $d\rho_n \upharpoonright (\epsilon_n \setminus \epsilon)$ . Thus the difference is harmonic on  $(\mathbb{C} \cup \{\infty\}) \setminus \epsilon$ , continuous on  $\mathbb{C} \cup \{\infty\}$ , vanishing on  $\epsilon$ , and bounded near  $\infty$ . The boundedness means the difference is also harmonic at  $\infty$  (see [24, Theorem 3.1.26]) and then the maximum principle implies that the difference is 0.  $\square$

The final step in the proof of Theorem 4.1 involves the form as  $n \rightarrow \infty$  of  $d\rho_n \upharpoonright K$  for  $K \in \mathcal{G}$ . Recall that  $\epsilon_n$  is a union of  $n$  bands which are closures of the connected

components of  $T_n^{-1}[(-t_n, t_n)]$ . On each of these, as  $x$  increases,  $T_n$  is either strictly monotone increasing or strictly decreasing from  $-t_n$  to  $t_n$  or vice versa. Recall also that each of the bands has  $\rho_n$  measure exactly  $1/n$  (see [6, Theorem 2.3]). In [6], it is proved that each gap,  $K$ , contains all or part of a single band so that

$$n\rho_n(K) \leq 1. \quad (4.11)$$

If there is  $x_\infty \in K$  which is a limit as  $j \rightarrow \infty$  of zeros,  $x_{n_j}$  of  $T_{n_j}$ , then for  $j$  large,  $\epsilon_{n_j} \cap K$  is a complete band of exponentially small width, and so, in that case,

$$n_j \rho_{n_j} \upharpoonright K \rightarrow \delta_{x_\infty} \quad (4.12)$$

weakly. If for each  $\epsilon$ , there is a large  $J_\epsilon$  so if  $j \geq J_\epsilon$  then  $T_{n_j}$  has no zero in  $K^{(\epsilon)}$ , then for all sufficiently large  $j$ ,  $\rho_{n_j}(K^{(\epsilon)}) = 0$ . Since  $G_\epsilon$  vanishes at the edges of  $K$  (and so  $\sup_{x \in K \setminus K^{(\epsilon)}} G_\epsilon(x, z) \rightarrow 0$  as  $\epsilon \downarrow 0$  uniformly as  $z$  runs through compact sets), we conclude that

$$n \int_K G_\epsilon(x, z) d\rho_n(x) \rightarrow \begin{cases} G_\epsilon(x_\infty, z) & \text{if } K \in \mathcal{G}_0; \\ 0 & \text{if } K \notin \mathcal{G}_0. \end{cases} \quad (4.13)$$

By the PW condition,  $\sum_{K \in \mathcal{G}} \sup_{y \in K} G_\epsilon(z, y) < \infty$  uniformly in  $z$  on compacts, and we can go from pointwise limits in (4.13) to limits on sums. We conclude the following.

**PROPOSITION 4.4**

*Under the hypotheses of Theorem 4.1, uniformly for  $z$  in compact subsets of  $\Omega \setminus \{x_k\}_{K_k \in \mathcal{G}_0}$ , we have that*

$$n \int_{\bigcup_{K_k \in \mathcal{G}} K_k} G_\epsilon(x, z) d\rho_n(x) \rightarrow \sum_{K_k \in \mathcal{G}_0} G_\epsilon(x_k, z). \quad (4.14)$$

*Proof of Theorem 4.1*

By (4.9), (4.10), and (4.14),

$$\lim_{n_j \rightarrow \infty} |M_{n_j}(z)| = \prod_{K_k \in \mathcal{G}_0} |B_\epsilon(z, x_k)| = |B_S(z)|. \quad (4.15)$$

That  $M_{n_j} \rightarrow B_S$  then follows from Proposition 4.2.  $\square$

## 5. Blaschke products are dual Widom maximizers

Given the setup of Theorem 4.1, the function  $B_S(z)$  is character-automorphic with some character  $\beta$ . In this section, we will prove that  $B_S$  is a dual Widom maximizer

for character  $\beta$ . One can deduce this from results of Volberg and Yuditskii (see [28, Lemma 6.4]). Instead, we will follow an approach of Eichinger and Yuditskii [8] (who study an Ahlfors problem rather than a dual Widom problem) that relies on results of Sodin and Yuditskii [26].

A basic technique of Sodin and Yuditskii is to consider the space,  $\mathcal{H}_\alpha$ , of all functions on  $\tilde{\Omega}$  which are in  $H^2(\mathbb{D})$  when moved to  $\mathbb{D}$  and which are character-automorphic with character  $\alpha \in \pi_1(\Omega)^*$ . Moreover,  $\mathcal{H}_\alpha$  is a family of functions on  $\tilde{\Omega}$  which is a reproducing kernel Hilbert space (see [22, Problems 4–11 of Section 3.3]) under the inner product of  $H^2$ . In particular, there is a function  $K^\alpha \in \mathcal{H}_\alpha$  so that for all  $f \in \mathcal{H}_\alpha$ ,

$$f(\infty) = \langle K^\alpha, f \rangle. \quad (5.1)$$

Note that our inner products are linear in the second factor and antilinear in the first, as in [22].

We will prove the following.

#### THEOREM 5.1

*For any gap set,  $S$ , if  $B_S$  is the associated Blaschke product and  $\beta$  is its character, then  $B_S$  is a dual Widom maximizer for  $\beta$ , that is,*

$$\|B_S\|_\infty = 1, \quad (5.2)$$

*and if  $f \in H^\infty(\Omega, \beta)$  with  $\|f\|_\infty \leq 1$ , then*

$$|f(\infty)| \leq B_S(\infty). \quad (5.3)$$

Equation (5.2) is, of course, true for any (convergent) Blaschke product. We prove (5.3) by proving two facts:

- (1) For any character,  $\gamma$ , and  $f \in H^\infty(\Omega, \beta)$  with  $\|f\|_\infty \leq 1$ , one has that

$$|f(\infty)|^2 \leq \frac{K^{\gamma\beta}(\infty)}{K^\gamma(\infty)}. \quad (5.4)$$

- (2) There exists at least one  $\alpha_0$  with

$$|B_S(\infty)|^2 = \frac{K^{\alpha_0\beta}(\infty)}{K^{\alpha_0}(\infty)}. \quad (5.5)$$

#### LEMMA 5.2

*Inequality (5.4) holds.*

*Proof*

Since  $f \in H^\infty(\Omega, \beta)$  and  $K^\gamma \in \mathcal{H}_\gamma$ , we have that  $fK^\gamma \in \mathcal{H}_{\gamma\beta}$ . Thus

$$\begin{aligned} |f(\infty)K^\gamma(\infty)|^2 &= |\langle K^{\gamma\beta}, fK^\gamma \rangle|^2 \\ &\leq \|fK^\gamma\|_2^2 \|K^{\gamma\beta}\|_2^2 \end{aligned} \quad (5.6)$$

$$\leq \|K^\gamma\|_2^2 \|K^{\gamma\beta}\|_2^2 \quad (5.7)$$

$$\begin{aligned} &= \langle K^\gamma, K^\gamma \rangle \langle K^{\gamma\beta}, K^{\gamma\beta} \rangle \\ &= K^\gamma(\infty)K^{\gamma\beta}(\infty), \end{aligned} \quad (5.8)$$

which is (5.4) since  $K^\gamma(\infty) > 0$ . In the above, (5.6) is the Schwarz inequality, (5.7) uses  $\|f\|_\infty \leq 1$ , and (5.8) is (5.1).  $\square$

For step 2, we need a deep result of Sodin and Yuditskii. For each gap  $K \in \mathcal{G}$ , we define  $C_K$  to be two copies glued together at the ends; that is, we take two copies  $\{(y, +), (y, -) \mid y \in \overline{K}\}$  and for  $y \in \partial K$  (two points), we set  $(y, +) = (y, -)$  so  $C_K$  is topologically a circle. According to Sodin and Yuditskii [26], there is a map,  $\mathfrak{A}$ , the Abel map, from  $\prod_{K \in \mathcal{G}} C_K$  to the character group, so that, in particular, the inner part of  $K^{\mathfrak{A}(y, \sigma)}$  is  $B_S$ , where  $S$  is the gap set with

$$\mathcal{G}_0 = \{K \mid (y_K, \sigma_K) \text{ has } \sigma_K = + \text{ and } y_K \in K\}$$

(i.e.,  $y_K \notin \partial K$ ) and for  $K \in \mathcal{G}_0$ , the point in  $K$  is  $y_K$ .

In particular, if  $S$  is given and  $(y, \sigma) = \{(y_K, \sigma_K)\}_{K \in \mathcal{G}}$  is picked so that, for  $K_k \in \mathcal{G}_0$ , we have that  $(y_{K_k}, \sigma_{K_k}) = (x_k, +)$  (and for  $K \notin \mathcal{G}_0$ ,  $(y_K, \sigma_K)$  is arbitrary in  $C_K$ ), then the inner factor of  $K^{\mathfrak{A}(y, \sigma)}$  is divisible by  $B_S$ ; that is, if  $\alpha_1 = \mathfrak{A}(y, \sigma)$ , then  $K^{\alpha_1}/B_S$  is in  $\mathcal{H}_{\alpha_0}$ , where  $\alpha_0 = \alpha_1\beta^{-1}$ . If  $g \in \mathcal{H}_{\alpha_0}$ , then because multiplication by  $B_S$  is an isometry on  $H^2$ , we have that

$$\begin{aligned} \langle K^{\alpha_0\beta} B_S^{-1}, g \rangle &= \langle K^{\alpha_0\beta}, B_S g \rangle \\ &= B_S(\infty)g(\infty) \end{aligned} \quad (5.9)$$

$$= B_S(\infty)\langle K^{\alpha_0}, g \rangle \quad (5.10)$$

$$= \langle \overline{B_S(\infty)} K^{\alpha_0}, g \rangle. \quad (5.11)$$

Since  $g$  is arbitrary in  $\mathcal{H}_{\alpha_0}$  and both  $K^{\alpha_0}$  and  $K^{\alpha_0\beta} B_S^{-1}$  lie in  $\mathcal{H}_{\alpha_0}$ , we conclude that

$$K^{\alpha_0\beta}(z)B_S(z)^{-1} = \overline{B_S(\infty)}K^{\alpha_0}(z). \quad (5.12)$$

Evaluating at  $z = \infty$ , we find the following.

## LEMMA 5.3

Equation (5.5) holds for  $\alpha_0 = \alpha_1 \beta^{-1}$ , where  $\alpha_1$  is the image under the Abel map of data  $\{(y_K, \sigma_K)\}_{K \in \mathcal{G}}$  which has  $(y_{K_k}, \sigma_{K_k}) = (x_k, +)$  if  $K_k \in \mathcal{G}_0$ .

*Proof of Theorem 5.1*

By Lemmas 5.2 and 5.3, if  $g \in H^\infty(\Omega, \beta)$  with  $\|g\|_\infty \leq 1$ , then

$$|g(\infty)|^2 \leq \frac{K^{\alpha_0 \beta}(\infty)}{K^{\alpha_0}(\infty)} = |B_S(\infty)|^2. \quad (5.13)$$

Thus, if  $g(\infty) > 0$ , we have that

$$0 < g(\infty) \leq B_S(\infty), \quad (5.14)$$

so  $B_S$  is a dual Widom maximizer.  $\square$

**6. Proof of the main theorem**

In this section, we will prove Theorem 1.3.

## PROPOSITION 6.1

Under the hypotheses of Theorem 4.1, we have that  $L_{n_j}(z)$  (given by (1.28)) converges uniformly on compact subsets of  $\widetilde{\Omega}$  to the Widom minimizer for the character,  $\beta$ , of  $B_S$ .

*Remark*

The  $M_n$ 's only converge away from the  $\{x_k\}_{K_k \in \mathcal{G}_0}$  because the  $M_n$ 's are not analytic on  $\widetilde{\Omega}$  and only analytic on those points whose images under  $\mathbf{x}$  are not in  $\epsilon_n$ . But  $L_n$  is analytic on all of  $\widetilde{\Omega}$ , so we can hope for convergence at the  $x_k$ 's, too. Indeed, the  $x_k$ 's are limit points of zeros and the Widom minimizers vanish at those points.

*Proof*

We have that  $M_{n_j}(\infty) = [C(\epsilon)/C(\epsilon_{n_j})]^{n_j}$ , so by Theorem 4.1,

$$B_S(\infty) = \lim_{j \rightarrow \infty} [C(\epsilon)/C(\epsilon_{n_j})]^{n_j}. \quad (6.1)$$

Thus, if  $H_n$  is given by (1.31), then

$$H_{n_j}(z) \rightarrow B_S(z)/B_S(\infty) \quad (6.2)$$

for  $z$  near  $\infty$  (in fact on compact subsets of  $\widetilde{\Omega} \setminus \{w \mid \pi(w) \in \{x_k\}\}$ ).

Since  $B_S$  is the dual Widom maximizer for  $\beta$ ,  $B_S(z)/B_S(\infty)$  is  $F_\beta$ , the Widom minimizer for  $\beta$ . By the first equation in (1.31), we get that  $L_{n_j}(z)$  converges to  $F_\beta(z)$  for  $z$  near  $\infty$ .

By the Totik–Widom bound,  $\|L_{n_j}\|_\infty$  are uniformly bounded, and so by Vitali’s theorem,  $L_{n_j}$  converges to  $F_\beta$  uniformly on compact subsets of  $\widetilde{\Omega}$ .  $\square$

PROPOSITION 6.2

*Under the hypotheses of Theorem 4.1, we have that*

$$\lim_{j \rightarrow \infty} \|L_{n_j}\|_\infty = 2\|F_\beta\|_\infty. \quad (6.3)$$

*Proof*

Since  $\log |L_{n_j}(z)|$  is harmonic on  $\Omega$  away from those zeros of  $T_{n_j}$  in the gaps where it goes to  $-\infty$ , its maximum occurs at limit points on  $\mathfrak{e}$ . Since  $|B_\mathfrak{e}(x)| = 1$  for  $x \in \mathfrak{e}$ , we conclude that

$$\|L_{n_j}\|_\infty = \frac{t_{n_j}}{C(\mathfrak{e})^{n_j}} = \frac{2C(\mathfrak{e}_{n_j})^{n_j}}{C(\mathfrak{e})^{n_j}} \quad (6.4)$$

by (1.8).

By (6.1), we conclude that

$$\lim_{j \rightarrow \infty} \|L_{n_j}\|_\infty = 2[B_S(\infty)]^{-1}, \quad (6.5)$$

and by (1.20), noting that  $Q_\beta = B_S$ ,

$$[B_S(\infty)]^{-1} = \|F_\beta\|_\infty, \quad (6.6)$$

proving (6.3).  $\square$

*Proof of Theorem 1.3*

By Theorem 3.2, we have the required almost periodicity of  $F_n(z)$  and  $\|F_n\|_\infty$ . By continuity of  $\|F_\chi\|_\infty$  and the Totik–Widom bound, the functions on the left of (1.22) lie in a compact set, so if the limit is not zero, by passing to suitable subsequences, we can find one whose limit is zero for which the hypotheses of Theorem 4.1 hold. But then the limit is zero by Proposition 6.1. We conclude that (1.22) holds.

Again, by continuity of  $\|F_\chi\|_\infty$  and the Totik–Widom bound, the numbers on the left-hand side of (1.23) are bounded above and away from zero, so if (1.23) fails we can find a subsequence for which the limit is not 2 and for which the hypotheses of Theorem 4.1 hold. This violates Proposition 6.2, so we conclude that (1.23) holds.  $\square$

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