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Bounds for integration matrices that arise in Gauss and Radau collocation

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Abstract
Bounds are established for integration matrices that arise in the convergence analysis of discrete approximations to optimal control problems based on orthogonal collocation. Weighted Euclidean norm bounds are derived for both Gauss and Radau integration matrices; these weighted norm bounds yield sup-norm bounds in the error analysis.

Keywords Integration matrix · Differentiation matrix · Gauss quadrature · Radau quadrature · Collocation methods

Mathematics Subject Classification 33C45 · 65L60 · 49M25 · 47A30

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1 Introduction

In a companion paper [8], a convergence result is established for an *hp*-Radau collocation method applied to an optimal control problem. The convergence analysis for this direct transcription method employs bounds for integration matrices associated with the collocated system dynamics. This paper establishes the required bounds, not only for the Radau scheme, but also for the Gauss scheme analyzed in [9,11].

Let P_N denote the Legendre polynomial of degree N, and let $\tau_i \in (-1, 1]$, $1 \le i \le N$, be either the zeros of P_N (the Gauss points) or the zeros of $P_N - P_{N-1}$ (the Radau points), arranged in increasing order. Note that $\tau_N = 1$ for the version of the Radau points that we use, sometimes called the flipped Radau points, while the points $-\tau_i$ are the ordinary Radau points [5,6], which can be analyzed in exactly the same way.

Let $\mathcal{P}_N$ denote the space of polynomials of degree at most N. In a collocation method applied to an initial value problem, we search for a polynomial in $\mathcal{P}_N$ which satisfies the differential equation at N collocation points. For ease of exposition, let us consider the trivial differential equation $\dot{x}(\tau) = f(\tau)$, $-1 \le \tau \le 1$, where $x(-1) = x_0$ is given. Let $\tau_0 = -1$ denote an additional noncollocated point corresponding to the initial condition. The Lagrange basis associated with the τ_i is

$$L_j(\tau) := \prod_{\substack{i=0\\ i\ne j}}^{N} \frac{\tau - \tau_i}{\tau_j - \tau_i}, \quad 0 \le j \le N.$$

Any $x \in \mathcal{P}_N$ has the expansion

$$x(\tau) = \sum_{j=0}^{N} x_j L_j(\tau), \quad x_j = x(\tau_j).$$

We enforce the differential equation at the collocation points to obtain a system of N equations in N unknowns

$$\dot{x}(\tau_i) = \sum_{j=0}^{N} x_j D_{ij} = f(\tau_i), \quad 1 \le i \le N, \quad D_{ij} = \dot{L}_j(\tau_i),$$

where x_0, the initial condition, is assumed to be given. In matrix notation, the system is $\mathbf{Dx} = \mathbf{f}$ where $\mathbf{f} \in \mathbb{R}^N$ is the right side $f(\cdot)$ evaluated at the collocation points.

Suppose that $f(\cdot)$ is continuous, let $x^*(\cdot)$ denote the solution of the differential equation, and let vectors $\mathbf{x}^* \in \mathbb{R}^{N+1}$ and $\dot{\mathbf{x}}^* \in \mathbb{R}^N$ be defined by

$$x_0^* = x_0, \quad x_i^* = x^*(\tau_i) \quad \text{and} \quad \dot{x}_i^* = \dot{x}^*(\tau_i), \quad 1 \le i \le N.$$

Subtracting $\mathbf{Dx}^*$ from each side of $\mathbf{Dx} = \mathbf{f}$ yields an equation for the error:

$$\mathbf{D}(\mathbf{x} - \mathbf{x}^*) = \mathbf{f} - \mathbf{Dx}^* = \dot{\mathbf{x}}^* - \mathbf{Dx}^* := \mathbf{r}. \tag{1.1}$$

The residual $\mathbf{r} = \dot{\mathbf{x}}^* - \mathbf{D}\mathbf{x}^*$, the difference between the derivative of $x^*(\cdot)$ and the derivative of the interpolant of $x^*(\cdot)$ evaluated at the collocation points, measures how accurately the continuous $x^*(\cdot)$ satisfies the discrete collocation equations. Bounds for $\mathbf{r}$ are given in [8–11].

Since the 0-th components of $\mathbf{x}$ and $\mathbf{x}^*$ are equal, the 0-th component of $\mathbf{x} - \mathbf{x}^*$ vanishes. If $\mathbf{e}$ is the vector obtained by deleting the 0-th component of $\mathbf{x} - \mathbf{x}^*$, then (1.1) reduces to $\bar{\mathbf{D}}\mathbf{e} = \mathbf{r}$, where $\bar{\mathbf{D}} \in \mathbb{R}^{N\times N}$ is obtained from $\mathbf{D}$ by deleting the 0-th column. As shown in [7], $\bar{\mathbf{D}}$ is invertible.

If our goal was to only analyze the error in the solution of the discretized differential equation, then the sup-norm could be used to obtain an error bound: $\|\mathbf{e}\|_\infty \le \beta_\infty \|\mathbf{r}\|_\infty$, where $\beta_\infty = \|\bar{\mathbf{D}}^{-1}\|_\infty$, the maximum absolute row sum in $\bar{\mathbf{D}}^{-1}$. However, when a collocation scheme is used to solve a constrained optimization problem, another norm arises since the convergence analysis requires showing that the optimization problem is stable under perturbations, and the stability analysis is performed in a 2-norm. In particular, it is necessary to obtain an estimate for $\mathbf{e}$ in the sup-norm relative to a bound for $\mathbf{r}$ in a 2-norm. We now explain how such a bound is obtained.

Let us define the ω-norm by

$$\|\mathbf{r}\|_\omega = \left(\sum_{i=1}^{N} \omega_i r_i^2\right)^{1/2} = \|\mathbf{W}^{1/2}\mathbf{r}\|_2, \tag{1.2}$$

where the ω_i are the quadrature weights associated with the collocation points, $\mathbf{W} \in \mathbb{R}^{N\times N}$ is the diagonal matrix with the ω_i on the diagonal, and $\|\cdot\|_2$ is the Euclidean norm. The ω-norm is a discrete version of the standard L^2-norm. Observe that

$$\mathbf{e} = \bar{\mathbf{D}}^{-1}\mathbf{r} = (\mathbf{W}^{1/2}\bar{\mathbf{D}})^{-1}\mathbf{W}^{1/2}\mathbf{r}. \tag{1.3}$$

Hence, e_i is the dot product between the i-th row of $(\mathbf{W}^{1/2}\bar{\mathbf{D}})^{-1}$ and the vector $\mathbf{W}^{1/2}\mathbf{r}$. From the Schwarz inequality and Eqs. (1.2) and (1.3), it follows that

$$\|\mathbf{e}\|_\infty \le \beta_2 \|\mathbf{r}\|_\omega,$$

where β_2 denotes the maximum Euclidean norm of a row from $(\mathbf{W}^{1/2}\bar{\mathbf{D}})^{-1}$. In [8,11] it is shown that $\beta_\infty \le \sqrt{2}\beta_2$.

In the convergence analysis of Gauss and Radau collocation schemes for constrained optimization problems, one needs to consider not only the original discrete dynamics $\mathbf{D}\mathbf{x} = \mathbf{f}$, but also the adjoint dynamics. The adjoint dynamics are obtained by multiplying the original dynamics by a Lagrange multiplier and differentiating the Lagrangian with respect to the state [2,14]. After some manipulations [8,11], the matrix $\mathbf{D}^\ddagger$ in the discrete dynamics for the adjoint that is the analogue of the matrix $\bar{\mathbf{D}}$ in the original discrete dynamics is given by $\mathbf{D}^\ddagger = -\mathbf{W}^{-1}\bar{\mathbf{D}}^T\mathbf{W}$. As a result, for constrained optimization, the convergence analysis of the Gauss and Radau collocation schemes also requires a bound for $\beta_2^\ddagger$, the maximum Euclidean norm for a row from $(\mathbf{W}^{1/2}\mathbf{D}^\ddagger)^{-1}$. In this paper, it is shown that both β_2 and $\beta_2^\ddagger$ are bound by $\sqrt{2}$, which provides the foundation for the convergence analysis of the Gauss and Radau collocation schemes in [8,9,11]. Note that the bound $\beta_2 \leq \sqrt{2}$ is also established in the appendix of [8], however, it is not clear how to extend the analysis to obtain a bound for $\beta_2^{\ddagger}$ associated with the Radau collocation points. In contrast, the approach developed in this paper for the analysis of β_2 can be extended to prove the corresponding bound for $\beta_2^{\ddagger}$; moreover, it is shown that the Euclidean norm of the rows of $(\mathbf{W}^{1/2}\bar{\mathbf{D}})^{-1}$ is a monotone increasing function of the row number, which implies that the last row has the largest Euclidean norm.

The paper is organized as follows: In Sect. 2, properties of Gauss and Radau quadrature are summarized. Sections 3 and 4 derive bounds for β_2 and $\beta_2^{\ddagger}$ respectively. Section 5 examines how the Euclidean norm of rows from $(\mathbf{W}^{1/2}\bar{\mathbf{D}})^{-1}$ and $(\mathbf{W}^{1/2}\mathbf{D}^{\ddagger})^{-1}$ depend on the row number.

Historical Note The four bounds $\beta_\infty \leq 2$, $\beta_2 \leq \sqrt{2}$, $\beta_\infty^{\ddagger} \leq 2$, and $\beta_2^{\ddagger} \leq \sqrt{2}$ have been referred to in the literature [8,9,11] as properties (P1)–(P4) respectively. (P1) was announced on William Hager's web page in 2015 as the 10,000 Yen Prize Problem, and it was first solved in 2017 [11]. A proof for (P2) was first given in 2018 [8]. (P3)–(P4) have remained open until this paper, which provides a unified approach to all four properties. Note that (P1) is equivalent to the following: If $p \in \mathcal{P}_N$ with $p(-1) = 0$ and $|\dot{p}(\tau_i)| \leq 1$ for all $1 \leq i \leq N$, then $|p(\tau_i)| \leq 2$ for all $1 \leq i \leq N$. In fact, the proof in [11] shows that $|p(\tau)| \leq 2$ for all $\tau \in [-1, 1]$; the extreme case is $p(\tau) = 1 + \tau$.

2 Properties of Gauss and Radau points

Let ω_i, $1 \leq i \leq N$, denote the quadrature weights associated with the collocation points. The quadrature formula

$$\int_{-1}^{1} p(\tau)\, d\tau = \sum_{i=1}^{N} \omega_i\, p(\tau_i) \tag{2.1}$$

is exact both for the Gauss points and polynomials of degree at most $2N - 1$ and for the Radau points and polynomials of degree at most $2N - 2$. The Gauss points are symmetric on the open interval $(-1, 1)$, while the Radau points lie on $(-1, 1]$. Axelsson [1, p. 74] provides the following formula for the matrix $\mathbf{A} = \bar{\mathbf{D}}^{-1}$:

$$A_{ij} = \frac{\omega_j}{2}\left\{1 + \tau_i + \sum_{k=1}^{N-1} P_k\left(\tau_j\right)\left[P_{k+1}\left(\tau_i\right) - P_{k-1}\left(\tau_i\right)\right]\right\}, \tag{2.2}$$

The matrix $\mathbf{A}$ is referred to as the integration matrix in [4].

The orthogonality properties of the Legendre polynomials are

$$\int_{-1}^{1} P_n(\tau)\, P_m(\tau)\, d\tau = \begin{cases} 0 & \text{if} \quad n \neq m, \\ \frac{2}{2n+1} & \text{if} \quad n = m. \end{cases}$$

By (2.1), it follows that if $m+n \le 2N-1$ for Gauss quadrature and $m+n \le 2N-2$ for Radau quadrature, then

$$\sum_{i=1}^{N} \omega_i P_n(\tau_i) P_m(\tau_i) = \begin{cases} 0 & \text{if} \quad n \ne m, \\ \frac{2}{2n+1} & \text{if} \quad n = m. \end{cases} \tag{2.3}$$

In particular, when $m = n = 0$, this shows that the sum of the quadrature weights is 2 since $P_0(x) = 1$. Moreover, for $m = 0$ and $n > 0$, we have

$$\sum_{i=1}^{N} \omega_i P_n(\tau_i) = 0, \tag{2.4}$$

provided $n \le 2N-1$ for Gauss quadrature and $n \le 2N-2$ for Radau quadrature. For convenience in the analysis, we define

$$a_n = \sum_{i=1}^{N} \omega_i P_n^2(\tau_i). \tag{2.5}$$

If $n < N$, then $a_n = 2/(2n+1)$ by (2.3) since the degree of P_n^2 is $\le 2N-2$. For the Radau points, $P_N(\tau_i) = P_{N-1}(\tau_i)$, so we have

$$\begin{aligned} a_N &= \sum_{i=1}^{N} \omega_i P_N^2(\tau_i) = \sum_{i=1}^{N} \omega_i P_N(\tau_i) P_{N-1}(\tau_i) \\ &= \sum_{i=1}^{N} \omega_i P_{N-1}^2(\tau_i) = 2/(2N-1) = a_{N-1}. \end{aligned} \tag{2.6}$$

Thus $a_N = a_{N-1}$ for the Radau points.

3 Analysis of β_2

Let E_i denote the square of the Euclidean norm of the i-th row of $(\mathbf{W}^{1/2}\bar{\mathbf{D}})^{-1}$:

$$E_i = \sum_{j=1}^{N} \frac{1}{\omega_j} \left(D_{ij}^{-1} \right)^2, \quad 1 \le i \le N, \tag{3.1}$$

where D_{ij}^{-1} denotes the (i, j) element of $\bar{\mathbf{D}}^{-1}$. The bound $\beta_2 \le \sqrt{2}$ is equivalent to $E_i \le 2$ for each i. Replace D_{ij}^{-1} in (3.1) by A_{ij} and utilize Axelsson's formula (2.2) to obtain

$$E_i = \sum_{j=1}^{N} \frac{\omega_j}{4} \left\{ 1 + \tau_i + \sum_{k=1}^{N-1} P_k\left(\tau_j\right) \left[P_{k+1}\left(\tau_i\right) - P_{k-1}\left(\tau_i\right) \right] \right\}^2, \quad 1 \le i \le N. \tag{3.2}$$

We write $E_i = E_{i1} + E_{i2} + E_{i3}$, where

$$E_{i1} = \sum_{j=1}^{N} \frac{\omega_j}{4} (1 + \tau_i)^2,$$

$$E_{i2} = \sum_{j=1}^{N} \sum_{k=1}^{N-1} \frac{\omega_j}{2} \left(1 + \tau_i\right) P_k\left(\tau_j\right) \left[P_{k+1}\left(\tau_i\right) - P_{k-1}\left(\tau_i\right) \right], \tag{3.3}$$

$$E_{i3} = \sum_{j=1}^{N} \frac{\omega_j}{4} \left\{ \sum_{k=1}^{N-1} P_k\left(\tau_j\right) \left[P_{k+1}\left(\tau_i\right) - P_{k-1}\left(\tau_i\right) \right] \right\}^2. \tag{3.4}$$

Since the sum of the quadrature weights is 2, $E_{i1} = (1 + \tau_i)^2/2$. Interchange the j and k sums in (3.3) and utilize (2.4) to obtain $E_{i2} = 0$. Due to the square in E_{i3}, it is equivalent to a triple sum:

$$E_{i3} = \frac{1}{4} \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} \left([P_{m+1}(\tau_i) - P_{m-1}(\tau_i)][P_{n+1}(\tau_i) - P_{n-1}(\tau_i)] \sum_{j=1}^{N} \omega_j P_m(\tau_j) P_n(\tau_j) \right).$$

If $n \neq m$, then the sum over j is zero due to the orthogonality of the Legendre polynomials and (2.3). The only nonzero terms correspond to $n = m$. Adding the nonzero terms in E_{i3} to E_{i1} yields

$$E_i = \frac{1}{2}(1 + \tau_i)^2 + \frac{1}{4} \sum_{n=1}^{N-1} a_n \left[P_{n+1}\left(\tau_i\right) - P_{n-1}\left(\tau_i\right) \right]^2, \quad 1 \le i \le N, \tag{3.5}$$

where a_n defined in (2.5) is $2/(2n + 1)$ since $n \le N - 1$ and the degree of P_n^2 is $\le 2N - 2$. The following lemma evaluates the summation in (3.5).

Lemma 3.1 *For $N \ge 2$, we have*

$$\frac{1}{2} \sum_{n=1}^{N-1} a_n \left[P_{n+1}\left(\tau\right) - P_{n-1}\left(\tau\right) \right]^2 = 2\tau P_N\left(\tau\right) P_{N-1}\left(\tau\right) - P_N^2\left(\tau\right) - P_{N-1}^2\left(\tau\right) + 1 - \tau^2. \tag{3.6}$$

Proof If g denotes the left side of (3.6), then

$$g' = \sum_{n=1}^{N-1} a_n \left[P_{n+1} - P_{n-1}\right]\left[P'_{n+1} - P'_{n-1}\right], \tag{3.7}$$

where the (τ) argument on the variables is dropped for simplicity. A well-known identity (see [12, p. 95] or [13, p. 83]) for the Legendre polynomials is

$$P'_{n+1} - P'_{n-1} = (2n+1)P_n = \left(\frac{2}{a_n}\right)P_n.$$

With this substitution in (3.7), the sum telescopes into

$$g' = 2\left[P_N P_{N-1} - P_1 P_0\right] = 2[P_N P_{N-1} - \tau]. \tag{3.8}$$

If h denotes the right side of (3.6), then

$$h' = 2[P_N P_{N-1} - \tau] + 2[\tau P'_N - P'_{N-1}]P_{N-1} + 2[\tau P'_{N-1} - P'_N]P_N. \tag{3.9}$$

The last two terms in (3.9) sum to zero due to the identities [13, p. 83]

$$N P_N = \tau P'_N - P'_{N-1} \quad \text{and} \quad -N P_{N-1} = \tau P'_{N-1} - P'_N.$$

Hence, $g' = h'$ and g and h differ by at most a constant. Since $P_n(1) = 1$ for any n, it follows that $g(1) = h(1) = 0$, which implies that g and h are identically equal. □

Let G_i and R_i be the values of E_i corresponding to the Gauss and Radau collocation points respectively.

Theorem 3.1 *For each* $1 \le i \le N$, *we have*

$$G_i = 1 + \tau_i - \frac{1}{2}P^2_{N-1}(\tau_i) \quad \textit{and} \quad R_i = 2 + (\tau_i - 1)\left\{P^2_N(\tau_i) + 1\right\}.$$

Proof Let us focus on the case $N \ge 2$ since the validity of the theorem for $N = 1$ can be checked by hand. Since the Gauss collocation points are the zeros of P_N, it follows that $P_N(\tau_i) = 0$. Hence, the P_N terms in Lemma 3.1 vanish at the τ_i and (3.5) yields the formula for G_i. Since the Radau collocation points are the zeros of $P_N - P_{N-1}$, it follows that $P_N(\tau_i) = P_{N-1}(\tau_i)$. Consequently, Lemma 3.1 and (3.5) yield the formula for R_i. □

Remark 3.1 The zeros of P_N lie on $(-1, 1)$, while the zeros of $P_N - P_{N-1}$ lie on $(-1, 1]$ with $\tau_N = 1$; in [13, (7.21.1)] it is shown that for each i,

$$|P_i(\tau)| \le 1 \text{ for all } \tau \in [-1, 1] \tag{3.10}$$

with equality achieved only for $\tau = \pm 1$. Hence, for each $i \in [1, N]$, $G_i < 2$ and $R_i \le 2$ with $R_N = 2$. Thus $\beta_2 < \sqrt{2}$ for Gauss collocation and $\beta_2 = \sqrt{2}$ for Radau collocation.

Next, let us examine how the E_i depend on i.

Proposition 3.1 *For both the Gauss and Radau collocation points, E_i is a monotone increasing function of i and $\beta_2 = \sqrt{E_N}$.*

Proof Let us define

$$E(\tau) = \frac{1}{2}\left\{(1+\tau)^2 + g(\tau)\right\}, \tag{3.11}$$

where g was introduced in the proof of Lemma 3.1. Based on the formula (3.8) for g', we have $E'(\tau) = P_N(\tau)P_{N-1}(\tau) + 1$. By (3.10), $E'(\tau) > 0$ for $\tau \in (-1, 1)$, which implies that $E(\cdot)$ is strictly increasing on $[-1, 1]$. Since the τ_i are arranged in increasing order, $E_i = E(\tau_i)$ is monotone increasing in i and E_N is the maximum of the E_i. □

The property that $\beta_2 = \sqrt{E_N}$ was observed numerically in [9], while Proposition 3.1 provides a rigorous proof.

4 Analysis of $\beta_2^{\ddagger}$

In this section, we exploit the results of Sect. 3 to show that $\beta_2^{\ddagger} \le \sqrt{2}$. Let $E_i^{\ddagger}$ denote the square of the Euclidean norm of the i-th row of the matrix $[\mathbf{W}^{1/2}\mathbf{D}^{\ddagger}]^{-1}$, we have

$$E_i^{\ddagger} = \sum_{j=1}^{N} \frac{1}{\omega_j}\left(D_{ij}^{\ddagger -1}\right)^2. \tag{4.1}$$

Hence, the inequality $\beta_2^{\ddagger} < \sqrt{2}$ is equivalent to $E_i^{\ddagger} < 2$. Let $G_i^{\ddagger}$ and $R_i^{\ddagger}$ denote the square of the Euclidean norm of the i-th row of the matrix $\mathbf{W}^{1/2}\mathbf{D}^{\ddagger -1}$ for the Gauss and Radau collocation points respectively.

Theorem 4.1 *For each $1 \le i \le N$, we have*

$$G_i^{\ddagger} = 1 - \tau_i - \frac{1}{2}P_{N-1}^2(\tau_i). \tag{4.2}$$

Proof Due to symmetry of the Gauss collocation points around $\tau = 0$, it follows that $\tau_i = -\tau_{N+1-i}$ and $\omega_i = \omega_{N+1-i}$. As a consequence, it is shown in [9, Proposition 10.1] that

$$D_{ij}^{\ddagger -1} = -D_{N+1-i,N+1-j}^{-1}, \quad 1 \le i \le N,\ 1 \le j \le N.$$

Inserting this in (4.1) and exploiting the symmetry of the collocation points and Theorem 3.1, we have

$$G_i^{\ddagger} = G_{N+1-i} = \tau_{N+1-i} + 1 - \frac{1}{2} P_{N-1}^2 \left(\tau_{N+1-i}\right) = 1 - \tau_i - \frac{1}{2} P_{N-1}^2 \left(\tau_i\right).$$

□

Remark 4.1 By Theorem 4.1, $G_i^{\ddagger} < 2$ for each i, which implies that $\beta_2^{\ddagger} < \sqrt{2}$.

Analogous to Proposition 3.1, we have the following:

Proposition 4.1 *$G_i^{\ddagger}$ is a monotone decreasing function of i and $\beta_2^{\ddagger} = \sqrt{G_1^{\ddagger}}$.*

Proof Let us define

$$G^{\ddagger}(\tau) = \frac{1}{2} \left\{ (1-\tau)^2 + g(\tau) \right\},$$

where g is defined in Lemma 3.1. The formula for g' given in (3.8) implies that $G^{\ddagger\prime}(\tau) = P_N(\tau) P_{N-1}(\tau) - 1$. By (3.10), $G^{\ddagger\prime}(\tau) < 0$ for all $\tau \in (-1, 1)$, which implies that $G^{\ddagger}(\cdot)$ is strictly decreasing on $[-1, 1]$. Since the τ_i are arranged in increasing order, $G^{\ddagger}(\tau_i)$ is monotone decreasing in i. By (3.6) and (4.2), $G_i^{\ddagger} = G^{\ddagger}(\tau_i)$. Hence, $G_1^{\ddagger}$ is the maximum of the $G_i^{\ddagger}$. □

Due to the symmetry of the Gauss collocation points, the formula for $G_i^{\ddagger}$ could be deduced from the previously derived formula for G_i. However, the Radau points are unsymmetric, and a new analysis is needed for $R_i^{\ddagger}$. Since $\mathbf{D}^{\ddagger} = -\mathbf{W}^{-1}\bar{\mathbf{D}}^T\mathbf{W}$, Axelsson's formula (2.2) gives

$$D_{ij}^{\ddagger-1} = -\frac{\omega_j}{2} \left\{ 1 + \tau_j + \sum_{k=1}^{N-1} P_k\left(\tau_i\right) \left[P_{k+1}\left(\tau_j\right) - P_{k-1}\left(\tau_j\right) \right] \right\}. \quad (4.3)$$

Therefore, $R_i^{\ddagger}$, the square of the Euclidean norm of the i-th row of $[\mathbf{W}^{1/2}\mathbf{D}^{\ddagger}]^{-1}$, can be expressed as

$$R_i^{\ddagger} = \sum_{j=1}^{N} \frac{\omega_j}{4} \left\{ 1 + \tau_j + \sum_{k=1}^{N-1} P_k\left(\tau_i\right) \left[P_{k+1}\left(\tau_j\right) - P_{k-1}\left(\tau_j\right) \right] \right\}^2. \quad (4.4)$$

Similar to the approach in Sect. 3, an explicit formula is derived for the sum in (4.4).

Theorem 4.2 *For $N = 1$, we have $R_1^{\ddagger} = 2$, while for $N \ge 2$ and $1 \le i \le N$,*

$$R_i^{\ddagger} = 2 + (1+\tau_i)\left[P_{N-1}(\tau_i)\, P_{N-2}(\tau_i) - 1\right] - \frac{N-1}{2N-1}\left[P_{N-1}(\tau_i) + P_{N-2}(\tau_i)\right]^2. \quad (4.5)$$

Proof The case $N = 1$ can be checked by hand, so we focus on $N \geq 2$. Let us write $R_i^\ddagger = R_{i1}^\ddagger + R_{i2}^\ddagger + R_{i3}^\ddagger$, where

$$R_{i1}^\ddagger = \sum_{j=1}^{N} \frac{\omega_j}{4} \left(1 + \tau_j\right)^2,$$

$$R_{i2}^\ddagger = \sum_{j=1}^{N} \frac{\omega_j}{2} \left\{ \left(1 + \tau_j\right) \sum_{k=1}^{N-1} P_k\left(\tau_i\right) \left[P_{k+1}\left(\tau_j\right) - P_{k-1}\left(\tau_j\right)\right] \right\}, \tag{4.6}$$

$$R_{i3}^\ddagger = \sum_{j=1}^{N} \frac{\omega_j}{4} \left\{ \sum_{k=1}^{N-1} P_k\left(\tau_i\right) \left[P_{k+1}\left(\tau_j\right) - P_{k-1}\left(\tau_j\right)\right] \right\}^2. \tag{4.7}$$

Since $N \geq 2$, Radau quadrature is exact for polynomials of degree two, and

$$R_{i1}^\ddagger = \frac{1}{4} \int_{-1}^{1} (1 + \tau)^2 d\tau = \frac{2}{3}. \tag{4.8}$$

Substitute $1 = P_0(\tau_j)$ and $\tau_j = P_1(\tau_j)$ in (4.6) to obtain

$$R_{i2}^\ddagger = \sum_{k=1}^{N-1} P_k\left(\tau_i\right) \sum_{j=1}^{N} \frac{\omega_j}{2} \left[P_{k+1}\left(\tau_j\right) - P_{k-1}\left(\tau_j\right)\right] \left[P_1\left(\tau_j\right) + P_0\left(\tau_j\right)\right]. \tag{4.9}$$

For $N = 2$, the fact that $P_N(\tau_i) = P_{N-1}(\tau_i)$ is exploited to obtain

$$\begin{aligned} R_{i2}^\ddagger &= \frac{P_1(\tau_i)}{2} \sum_{j=1}^{2} \omega_j [P_2(\tau_j) - P_0(\tau_j)][P_1(\tau_j) + P_0(\tau_j)] \\ &= \frac{P_1(\tau_i)}{2} \sum_{j=1}^{2} \omega_j [P_1^2(\tau_j) - P_0^2(\tau_j)] = -\frac{2}{3} P_1(\tau_i), \quad N = 2. \end{aligned}$$

For $N \geq 3$, the product of the polynomials in the j-sum of (4.9) has degree at most $N + 1 \leq 2N - 2$. Hence, the quadrature corresponding to the j-sum is equivalent to the integral

$$\frac{1}{2} \int_{-1}^{1} [P_{k+1}(\tau) - P_{k-1}(\tau)][P_1(\tau) + P_0(\tau)]\, d\tau.$$

Due to the orthogonality of the Legendre polynomials, this integral is zero except when $k = 1$ or $k = 2$, and the values of the integral are -1 and $-1/3$ respectively. With these substitutions in (4.9), we obtain

$$R_{i2} = -P_1\left(\tau_i\right) - \frac{1}{3} P_2\left(\tau_i\right), \quad N \geq 3. \tag{4.10}$$

The expression (4.7) is equivalent to the triple sum

$$R_{i3}^{\ddagger} = \frac{1}{4}\sum_{n=1}^{N-1}\sum_{m=1}^{N-1}\left(P_m(\tau_i)P_n(\tau_i)\sum_{j=1}^{N}\omega_j[P_{m+1}(\tau_j)\right.$$
$$\left. -P_{m-1}(\tau_j)][P_{n+1}(\tau_j) - P_{n-1}(\tau_j)]\right).$$

In the case $N = 2$, we have

$$\begin{aligned} R_{i3}^{\ddagger} &= \frac{P_1^2(\tau_i)}{4}\sum_{j=1}^{N}\omega_j[P_2(\tau_j) - P_0(\tau_j)]^2 \\ &= \frac{P_1^2(\tau_i)}{4}\sum_{j=1}^{N}\omega_j[P_1(\tau_j) - P_0(\tau_j)]^2 = \frac{2}{3}P_1^2(\tau_i). \end{aligned}$$

Combining R_{i1}, R_{i2} and R_{i3} for $N = 2$ gives $R_i^{\ddagger} = (2/3)[1 + P_1^2(\tau_i) - P_1(\tau_i)]$, which is equivalent to (4.5) for $N = 2$. The remainder of the proof focuses on the case $N \geq 3$.

The j-sum in the expression for $R_{i3}^{\ddagger}$ expands into four sums of the form

$$\sum_{j=1}^{N}\omega_j P_\mu(\tau_j)P_\nu(\tau_j),$$

where $\mu = m+1$ or $m-1$ and $\nu = n+1$ or $n-1$. If $\mu \neq \nu$ and $\mu+\nu \leq 2N-2$, then the quadrature is exact, and the sum is zero due to the orthogonality of the Legendre polynomials. The only cases where $\nu \neq \mu$ and $\nu+\mu > 2N-2$ is the term $P_{m+1}P_{n+1}$ with $m = N-1$ and $n = N-2$ or $m = N-2$ and $n = N-1$. Exploiting (2.6), the contribution S_0 of these two terms to $R_{i3}^{\ddagger}$ is given by

$$S_0 = \frac{P_{N-1}(\tau_i)P_{N-2}(\tau_i)}{2}\sum_{j=1}^{N}\omega_j P_N(\tau_j)P_{N-1}(\tau_j) = \frac{a_N P_{N-1}(\tau_i)P_{N-2}(\tau_i)}{2}. \tag{4.11}$$

The only remaining nonzero terms in the j-sum correspond to those values of m and n for which the associated subscript μ and ν are equal. Table 1 lists each of the four terms associated with the j-sum and the value of m that makes the subscripts on these terms equal. For example, in the term $-P_{m+1}P_{n-1}$, the two scripts are equal when $m = n-2$; since both m and n are constrained to lie between 1 and $N-1$, n should be further restricted to the range $3 : N-1$ (shown in column 3 of Table 1), so that m lies in $1 : N-3$. If n lies in $3 : N-1$ and $m = n-2$, then the corresponding term in the j-sum is

$$\sum_{j=1}^{N}\omega_j P_{n-1}^2(\tau_j) = a_{n-1},$$

Table 1 The value of m for which the subscripts in column 1 are equal

Term	Equality	n-range	j-sum
$P_{m+1}P_{n+1}$	$m = n$	$1 \le n \le N-1$	a_{n+1}
$-P_{m+1}P_{n-1}$	$m = n-2$	$3 \le n \le N-1$	a_{n-1}
$-P_{m-1}P_{n+1}$	$m = n+2$	$1 \le n \le N-3$	a_{n+1}
$P_{m-1}P_{n-1}$	$m = n$	$1 \le n \le N-1$	a_{n-1}

which appears in the fourth column of Table 1. Recall from (2.5)–(2.6) that $a_n = 2/(2n+1)$ for $n \le N-1$ and $a_N = a_{N-1}$.

Using Table 1, $R^{\ddagger}_{i3}$ is expressed as

$$R^{\ddagger}_{i3} = S_0 + \frac{1}{4}\left(\sum_{n=1}^{N-1} P_n^2(\tau_i)a_{n+1} - \sum_{n=3}^{N-1} P_{n-2}(\tau_i)P_n(\tau_i)a_{n-1} + \sum_{n=1}^{N-1} P_n^2(\tau_i)a_{n-1} - \sum_{n=1}^{N-3} P_{n+2}(\tau_i)P_n(\tau_i)a_{n+1}\right).$$

The τ_i terms are gotten by substituting into the original $R^{\ddagger}_{i3}$ expression, $m = n$ or $m = n \pm 2$ in accordance with Table 1, Next, the indexing on the sums is modified so that only a_n appears:

$$R^{\ddagger}_{i3} = S_0 + \frac{1}{4}\left(\sum_{n=2}^{N} P_{n-1}^2(\tau_i)a_n - \sum_{n=2}^{N-2} P_{n-1}(\tau_i)P_{n+1}(\tau_i)a_n + \sum_{n=0}^{N-2} P_{n+1}^2(\tau_i)a_n - \sum_{n=2}^{N-2} P_{n+1}(\tau_i)P_{n-1}(\tau_i)a_n\right).$$

Notice that the summations all share the common range $2 : N-2$. Taking into account the identity $a_N = a_{N-1}$ given in (2.6), the four terms outside this range (in the first and third summations) are

$$S_1 = \frac{1}{4}\left(a_N[P_{N-1}^2(\tau_i) + P_{N-2}^2(\tau_i)] + P_1^2(\tau_i)a_0 + P_2^2(\tau_i)a_1\right),$$

For the terms in $R^{\ddagger}_{i3}$ where $n \in [2, N-2]$, we complete the square to obtain

$$\begin{aligned} R^{\ddagger}_{i3} &= S_0 + S_1 + \frac{1}{4}\sum_{n=2}^{N-2} a_n[P_{n+1}(\tau_i) - P_{n-1}(\tau_i)]^2 \\ &= S_0 + S_1 - \frac{a_1}{4}[P_2(\tau_i) - P_0(\tau_i)]^2 + \frac{1}{4}\sum_{n=1}^{N-2} a_n[P_{n+1}(\tau_i) - P_{n-1}(\tau_i)]^2. \end{aligned}$$

Add together $R^{\ddagger}_{i1}$ in (4.8), $R^{\ddagger}_{i2}$ in (4.10), and $R^{\ddagger}_{i3}$ to obtain

$$R^{\ddagger}_i = \frac{1}{4}\left(a_N\left[P_{N-1}(\tau_i) + P_{N-2}(\tau_i)\right]^2 + 2(1-\tau_i)^2 + \sum_{n=1}^{N-2} a_n[P_{n+1}(\tau_i) - P_{n-1}(\tau_i)]^2\right).$$

Applying Lemma 3.1 and then completing the square, it follows that

$$\frac{1}{4}\sum_{n=1}^{N-2} a_n[P_{n+1}(\tau_i) - P_{n-1}(\tau_i)]^2$$
$$= (1+\tau_i)P_{N-1}(\tau_i)P_{N-2}(\tau_i) - \frac{1}{2}[P_{N-1}(\tau_i) + P_{N-2}(\tau_i)]^2 + \frac{1}{2}(1-\tau_i^2).$$

Inserting this in the formula for $R^{\ddagger}_i$ gives (4.5). □

Remark 4.2 For $N > 1$, it follows from (3.10) that $P_{N-1}(\tau_i)P_{N-2}(\tau_i) < 1$ for $\tau_i \neq 1$. Hence, $R^{\ddagger}_i < 2$ by (4.5). Consequently, $\beta^{\ddagger}_2 < \sqrt{2}$ for the Gauss collocation points, and for the Radau collocation points when $N > 1$.

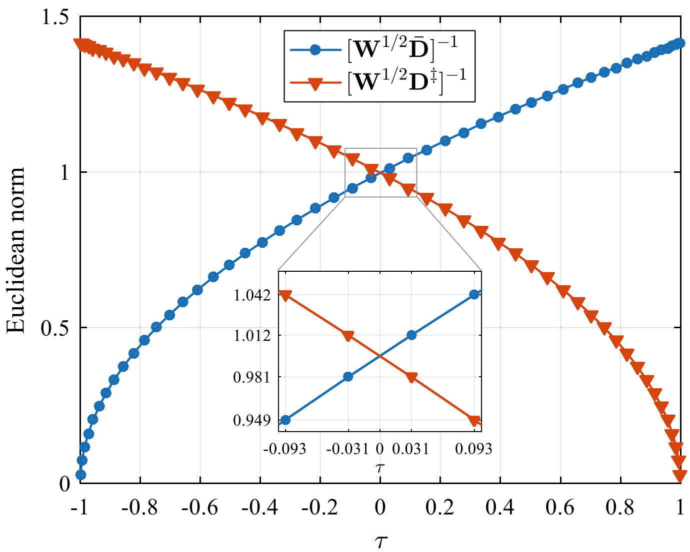

Fig. 1 The Euclidean norms of the rows of $[\mathbf{W}^{1/2}\bar{\mathbf{D}}]^{-1}$ and $[\mathbf{W}^{1/2}\mathbf{D}^{\ddagger}]^{-1}$ for the Gauss collocation points with $N = 50$, plotted as a function of τ_i, $1 \le i \le N$

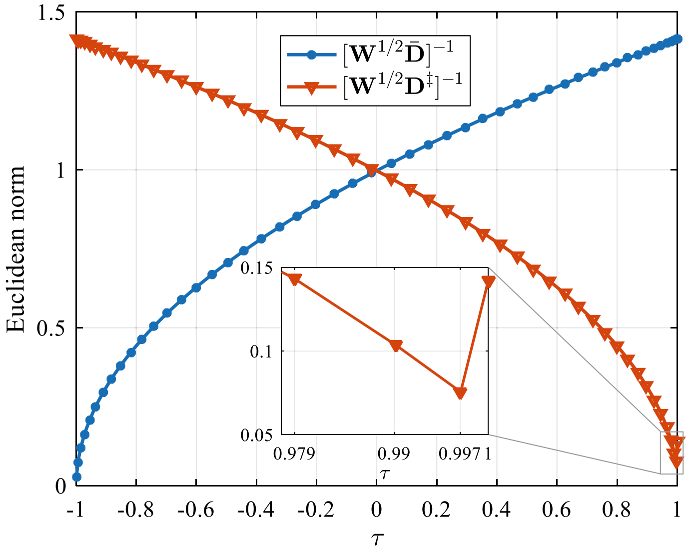

Fig. 2 The Euclidean norms of the rows of $[\mathbf{W}^{1/2}\bar{\mathbf{D}}]^{-1}$ and $[\mathbf{W}^{1/2}\mathbf{D}^{\ddagger}]^{-1}$ for the Radau collocation points with $N = 50$, plotted as a function of τ_i, $1 \leq i \leq N$

5 Numerical experiments

We plot the norm of the rows of $[\mathbf{W}^{1/2}\bar{\mathbf{D}}]^{-1}$ and $[\mathbf{W}^{1/2}\mathbf{D}^{\ddagger}]^{-1}$ as a function of τ_i for the Gauss and Radau collocation points. The plots are given for $N = 50$, however, the curves are similar for any choice of N. Figure 1 is based on the Gauss points. Both curves are monotone as shown in Propositions 3.1 and 4.1. As noted in Remark 4.1, the maximums are strictly less than $\sqrt{2}$. The fact that one curve is the flipped version of the other was established in the proof of Theorem 4.1. Figure 2 is the analogous plot for the Radau points. The monotonicity of the curve associated with the rows of $[\mathbf{W}^{1/2}\bar{\mathbf{D}}]^{-1}$ was established in Proposition 3.1, and the fact that the maximum is exactly $\sqrt{2}$ is explained in Remark 3.1. The curve for $[\mathbf{W}^{1/2}\mathbf{D}^{\ddagger}]^{-1}$ is not monotone, as shown in the magnified view of the lower right corner. Except for the glitch at $\tau = 1$, the numerically evaluated curve seems to be monotone, but currently, there is no proof of this property.

6 Conclusions

It is shown that integration matrices associated with both Gauss and Radau collocation schemes have the Euclidean norm of the rows bounded by $\sqrt{2}$. This property provides the foundation for the error analysis of the collocation schemes for optimal control problems developed in [2,3,5–9,11]. The analysis provides explicit expressions for the norm of each row of the matrix; the structure of the bounds and properties of Legendre polynomials lead to the upper bound $\sqrt{2}$. The analysis reveals that for three of the four integration matrices, the Euclidean norm of the row is a monotone function of the row number. As a result, the maximum norm corresponds to either the first or last row of the matrix. A key lemma in the analysis is a formula derived for a sum of squared differences of Legendre polynomials.

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