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Modified Radau Collocation Method for Solving Optimal Control Problems with Nonsmooth Solutions Part I: Lavrentiev Phenomenon and the Search Space

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Abstract— **A new method is developed for solving optimal control problems whose solutions contain a nonsmooth optimal control. The method developed in this paper employs a modified form of the Legendre-Gauss-Radau (LGR) orthogonal direct collocation method in which an additional variable and two additional constraints are included at the end of a mesh interval. The additional variable is the switch time where a discontinuity occurs. The two additional constraints are a collocation condition on each differential equation that is a function of control along with a control constraint at the endpoint of the mesh interval that defines the location of the nonsmoothness. These additional constraints modify the search space of the NLP in a manner such that an accurate approximation to the location of the nonsmoothness is obtained. An example with a nonsmooth solution is used throughout the paper to illustrate the improvement of the method over the standard Legendre-Gauss-Radau collocation method.**

I. Introduction

Over the past few decades, direct collocation methods have become a popular tool to computationally solve nonlinear optimal control problems. In a direct collocation method, the state and control of a continuous time optimal control problem are discretized at a set of points along a given time interval. The infinite-dimensional optimal control problem is then transcribed to a finite-dimensional nonlinear programming problem (NLP) which can be solved using established NLP solvers [1], [2]. In recent years, a significant amount of research has focused on direct *Gaussian quadrature orthogonal collocation methods* [3]–[6] where the dynamics are collocated at points associated with a Gaussian quadrature. Most commonly used Gaussian quadrature methods employ Legendre-Gauss (LG) points [3], Legendre Gauss Radau (LGR) points [4]–[7], and Legendre-Gauss-Lobatto (LGL) points [8]. In addition, in recent years a convergence theory has been developed using Gaussian quadrature collocation. This theory has led to recent work where it has been shown that, under certain assumptions of the smoothness of solution and coercivity, an *hp* Gaussian quadrature method that employs either LG or LGR collocation points converges to a local minimizer of the optimal control problem [9]–[13].

While Gaussian quadrature orthogonal collocation methods are well suited to solving optimal control problems whose solutions are smooth, it is often the case that the solution of an optimal control problem has a nonsmooth optimal control [14]. The difficulty in solving problems with discontinuous control lies in determining when the discontinuity occurs. For example, dynamical systems where the control appears linearly or problems that have state inequality path constraints often have solutions where the control and state may be nonsmooth. One approach to handling nonsmoothness is to employ a mesh refinement method where the optimal control problem is partitioned into a mesh and a mesh that meets a specified solution accuracy tolerance is obtained iteratively. In the context of Gaussian quadrature collocation, so called *hp*-adaptive mesh refinement methods [15]–[19] have been developed more recently in order to improve accuracy in a wide variety of optimal control problems including those whose solutions are nonsmooth. It is noted, however, that mesh refinement methods often place an unnecessarily large number of collocation points and mesh intervals near points of nonsmoothness in the solution. Thus, it is beneficial to develop techniques that take advantage of the rapid convergence of a Gaussian quadrature collocation methods in segments where the solution is smooth and only increase the size of the mesh when necessary (thus, maintaining a smaller mesh than might be possible with a standard mesh refinement approach).

Now, as it turns out, for optimal control problems where the solution is nonsmooth the convergence theory developed in Refs. [9]–[13] is not applicable. Consequently, when the solution of an optimal control problem is nonsmooth, an *hp* method may not converge to a local minimizer of the optimal control problem. A well studied class of problems where the smoothness and coercivity conditions found in Ref. [10] are not met are those where the control appears linearly in the problem formulation [14], [20]–[22]. One approach for estimating the location of discontinuities is to introduce a variable called a breakpoint [23] that defines the location of a discontinuity and to include this variable in the NLP. The key issue that arises by introducing a breakpoint is that the NLP has an extra degree of freedom. As a result, the NLP may converge to a solution where this additional variable does not correspond to the location of the nonsmoothness. Next, Ref. [24] introduced a variable that defines the switch time and collocate the dynamics at both the end of a mesh interval and the start of the subsequent mesh interval using Legendre-Gauss-Lobatto collocation. Note, however, that the LGL method used in Ref. [24] employs a square and singular differentiation matrix. Therefore, unlike the approach of Ref. [23], which used Legendre-Gauss collocation, the scheme used in Ref. [24] is not a Gauss quadrature integrator.

This goal of this research is to develop a method that employs Gaussian quadrature collocation for accurately solving optimal control problems whose solution are nonsmooth. In this paper, which is Part I of a two-part sequence, an ap-

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proach is developed to improve upon the approach originally developed in Ref. [23] by gaining a better understanding as to why an incorrect location of the nonsmoothness is obtained in the optimal control when solving an optimal control problem using Legendre-Gauss-Radau collocation. Specifically, it is shown in this paper that the incorrect discontinuity location is obtained due to so called Lavrentiev phenomenon [25]. Lavrentiev phenomenon occurs in a practical situation when it is desired to minimize a numerical approximation of a continuous (functional) optimization problem. Whenever a numerical approximation of a functional leads to a minimizer that is strictly greater than or less than the true minimizer of the functional, the continuous optimization problem may be subject to Lavrentiev phenomenon [26], [27]. Simple examples of problems that possess Lavrentiev phenomenon are given in Ref. [28], and the concept of Lavrentiev phenomenon has been extended to optimal control through the Lavrentiev gap [20]. The reason that the approximation of the continuous optimization problem has a higher or lower optimal cost arises from the possibility that the space over which the numerical optimization is performed may be different from the space over which the optimization needs to be performed in order to converge to the optimal solution. Therefore, the existence and the behavior of Lavrentiev phenomenon depends upon the choice of the approximation method. Moreover, any numerical scheme that gives rise to Lavrentiev phenomenon must somehow be augmented to compensate for any errors caused by the Lavrentiev phenomenon itself. An initial exploration of Lavrentiev phenomenon using Gaussian quadrature collocation methods was given in Ref. [27]. In order to properly account for Lavrentiev phenomenon it is first necessary to understand the circumstances in which it occurs for any given numerical scheme.

The approach developed in this paper is fundamentally different from the approaches developed in Refs. [23] and [24]. The key difference between the approach of this paper and that of Ref. [23] is that the search space is modified to include collocation constraints on the differential equations that are a function of control whereas the approach of Ref. [23] introduces no such additional collocation constraints. Moreover, the key difference between the approach of this paper and the work of Ref. [24] is that the work of Ref. [24] collocates all of the differential equations at the end of a mesh interval where a discontinuity may lie whereas in this work collocation constraints are included at the end of a mesh interval on only those differential equations that are a function of control. Second, the method of Ref. [24] uses Legendre-Gauss-Lobatto which employs a square and singular differentiation matrix whereas the approach developed in this paper employs Legendre-Gauss-Radau collocation where the differentiation matrix is rectangular and has been shown previously to be a Gaussian quadrature integrator [4].

This paper is organized as follows. Section II provides a brief introduction to solving optimal control problems using an LGR collocation method. Section III examines a motivating example to demonstrate the difficulties of solving optimal control problems with discontinuities using LGR methods. Section IV provides a brief introduction to Lavrentiev phenomenon and examines the polynomial search space of the LGR discretization scheme for an example problem. Section VI introduces the modified LGR method. Finally, the motivating example studied in Section III is revisited using the newly developed method to show the improvement in locating the nonsmoothness in the numerical approximation using the modified LGR method. Finally, Section VIII provides conclusions on this work.

II. Legendre-Gauss-Radau Collocation

In this paper we focus on second-order controlled dynamical systems of the form $\ddot{\mathbf{x}}(\tau) = \mathbf{f}(\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau), \mathbf{u}(\tau))$. Such a form is quite broad in applicability in that it arises frequently in mechanical systems (Newton-Euler or Lagrangian mechanics). With such a class of dynamical systems as the focus, consider the following optimal control problem defined on $\tau \in [-1, +1]$. Minimize the cost functional

$$\begin{aligned}\mathcal{J} = {} & \mathcal{M}(\mathbf{x}(-1), \mathbf{v}(-1), \mathbf{x}(+1), \mathbf{v}(+1)) \\ & + \int_{-1}^{+1} \mathcal{L}(\mathbf{x}(\tau), \mathbf{v}(\tau), \mathbf{u}(\tau)) d\tau,\end{aligned} \tag{1}$$

subject to the dynamic constraints

$$\begin{aligned}\dot{\mathbf{x}}(\tau) &= \mathbf{v}(\tau), \\ \dot{\mathbf{v}}(\tau) &= \mathbf{f}(\mathbf{x}, (\tau), \mathbf{v}(\tau), \mathbf{u}(\tau)),\end{aligned} \tag{2}$$

inequality path constraints

$$\mathbf{c}(\mathbf{x}(\tau), \mathbf{v}(\tau), \mathbf{u}(\tau)) \leq \mathbf{0}, \tag{3}$$

and boundary conditions

$$\mathbf{b}(\mathbf{x}(-1), \mathbf{v}(-1), \mathbf{x}(+1), \mathbf{v}(+1)) = \mathbf{0}, \tag{4}$$

where $(\mathbf{x}(\tau), \mathbf{v}(\tau)) \in \mathbb{R}^{2n}$ is the state (such that $\mathbf{x}(\tau) \in \mathbb{R}^n$ and $\mathbf{v}(\tau) \in \mathbb{R}^n$), $\mathbf{u}(\tau) \in \mathbb{R}^m$ is the control, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{c} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^c$, $\mathbf{b} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^b$,$\mathcal{M} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. For convenience with the mathematical development that follows, all vector quantities are treated as row vectors. For example, $\mathbf{x}(\tau)$ and $\mathbf{u}(\tau)$ are defined as row vectors, respectively, as

$$\begin{aligned}\mathbf{x}(\tau) &= \left[\begin{array}{c} x_1(\tau) \cdots x_n(\tau) \end{array}\right] \in \mathbb{R}^n, \\ \mathbf{u}(\tau) &= \left[\begin{array}{c} u_1(\tau) \cdots u_m(\tau) \end{array}\right] \in \mathbb{R}^m.\end{aligned} \tag{5}$$

All other vector quantities are defined in a similar manner to that shown for $\mathbf{x}(\tau)$ and $\mathbf{u}(\tau)$ given in Eq. (5).

Suppose now that the state $(\mathbf{x}(\tau), \mathbf{v}(\tau))$ is approximated by a polynomial of degree at most N. Let ℓ_i $(i = 1, \ldots, N+1)$ be a basis of Lagrange polynomials given by

$$\ell_i(\tau) = \prod_{\substack{j=1 \\ j \neq i}}^{N+1} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \qquad i = 1, \ldots, N+1.$$

The j^{th} component of $\mathbf{x}(\tau)$ and $\mathbf{v}(\tau)$ are then approximated in terms of the Lagrange polynomial basis as

$$\begin{aligned}x_j(\tau) &\approx X_j(\tau) = \textstyle\sum_{i=1}^{N+1} X_{ij} \ell_i(\tau), \\ v_j(\tau) &\approx V_j(\tau) = \textstyle\sum_{i=1}^{N+1} V_{ij} \ell_i(\tau),\end{aligned} \tag{6}$$

Differentiating $x_j(\tau)$ and $v_j(\tau)$ in Eq. (6) and evaluating the result at $\tau = \tau_k$ gives

$$\begin{aligned}\dot{x}_j(\tau) &\approx \dot{X}_j(\tau) = \textstyle\sum_{i=1}^{N+1} X_{ij} \dot{\ell}_i(\tau_k) = \sum_{i=1}^{N+1} D_{ik} X_{ij}, \\ \dot{v}_j(\tau) &\approx \dot{V}_j(\tau) = \textstyle\sum_{i=1}^{N+1} V_{ij} \dot{\ell}_i(\tau_k) = \sum_{i=1}^{N+1} D_{ik} V_{ij}.\end{aligned} \tag{7}$$

The coefficients D_{ik}, $(i = 1, \ldots, N;\ k = 1, \ldots, N+1)$ form the $N \times (N+1)$ matrix $\mathbf{D}$ called the *LGR differentiation matrix*. For convenience $\mathbf{D}$ is partitioned as

$$\mathbf{D} = \left[\begin{array}{cccc} \mathbf{D}_1 & \mathbf{D}_2 & \cdots & \mathbf{D}_{N+1} \end{array}\right] = \left[\begin{array}{cc} \mathbf{D}_{1:N} & \mathbf{D}_{N+1} \end{array}\right], \tag{8}$$

where $\mathbf{D}_i$ denotes the i^{th} column of $\mathbf{D}$, $\mathbf{D}_{1:N} \in \mathbb{R}^{N\times N}$ is an $N \times N$ matrix formed from the first N columns of $\mathbf{D}$, and $\mathbf{D}_{N+1}$ is the last column of $\mathbf{D}$ [4]–[6]. Thus, unlike the state and control, which are treated as row vectors at an instant of time, in this exposition the differentiation matrix is dealt with column-wise. Using the row vector convention for the state and control, the matrices $\mathbf{X} \in \mathbb{R}^{(N+1)\times n}$ and $\mathbf{V} \in \mathbb{R}^{(N+1)\times n}$ correspond row-wise to the state approximations at times $(\tau_1, \ldots, \tau_{N+1})$, while the matrix $\mathbf{U} \in \mathbb{R}^{N\times m}$ corresponds row-wise to the approximations of the control at times $(\tau_1, \ldots, \tau_N)$. Therefore, the matrices $\mathbf{X}$, $\mathbf{V}$, and $\mathbf{U}$ are given respectively

$$\begin{aligned}
\mathbf{X} &= \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_{N+1} \end{bmatrix} \equiv \mathbf{X}_{1:N+1}, \\
\mathbf{V} &= \begin{bmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_{N+1} \end{bmatrix} \equiv \mathbf{V}_{1:N+1}, \\
\mathbf{U} &= \begin{bmatrix} \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_{N} \end{bmatrix} \equiv \mathbf{U}_{1:N},
\end{aligned} \tag{9}$$

where the notation $\mathbf{Y}_{i:j}$ denotes generically rows i through j of the matrix $\mathbf{Y}$. Also, the derivative approximations $\dot{\mathbf{X}}(\tau)$ and $\dot{\mathbf{V}}(\tau)$ at the k^{th} LGR point τ_k are then given as row vectors, respectively, as

$$\dot{\mathbf{X}}(\tau_k) = [\mathbf{D}\mathbf{X}]_k \quad , \quad \dot{\mathbf{V}}(\tau_k) = [\mathbf{D}\mathbf{V}]_k . \tag{10}$$

It is noted that the state approximation is exact if the state is a polynomial of degree at most N. The LGR approximation of the state leads to the following nonlinear programming problem (NLP) that approximates the optimal control problem given in Eqs. (1)–(4):

$$\begin{aligned}
\text{minimize } J = \mathcal{M}(\mathbf{X}_1, \mathbf{V}_1, \mathbf{X}_{N+1}, \mathbf{V}_{N+1}) \\
+ \sum_{i=1}^{N} w_i \mathcal{L}(\mathbf{X}_i, \mathbf{V}_i, \mathbf{U}_i),
\end{aligned} \tag{11}$$

subject to

$$\begin{aligned}
\mathbf{D}\mathbf{X} - \mathbf{V}_{1:N} &= \mathbf{0}, \\
\mathbf{D}\mathbf{V} - \mathbf{f}(\mathbf{X}_{1:N}, \mathbf{V}_{1:N}, \mathbf{U}_{1:N}) &= \mathbf{0}, \\
\mathbf{c}(\mathbf{X}_{1:N}, \mathbf{V}_{1:N}, \mathbf{U}_{1:N}) &\leq \mathbf{0}, \\
\mathbf{b}(\mathbf{X}_1, \mathbf{V}_1, \mathbf{X}_{N+1}, \mathbf{V}_{N+1}) &\leq \mathbf{0},
\end{aligned} \tag{12}$$

where $w_i, \quad (i = 1, \ldots, N)$ are the LGR quadrature weights (and produce an exact integral if the integrand is a polynomial of degree at most $2N-2$). Equations (11) and (12) will be referred to as the *standard Legendre-Gauss-Radau collocation method*.

III. Illustrative Example

To motivate the study of optimal control problems with nonsmooth solutions and how Lavrentiev phenomenon manifests itself when such problems are solved using the standard LGR collocation method, consider the following optimal control problem [15]. Minimize the cost functional

$$\mathcal{J} = t_f \tag{13}$$

subject to the dynamic constraints

$$\dot{x}(\tau) = \frac{t_f}{2} v(\tau) \quad , \quad \dot{v}(\tau) = \frac{t_f}{2} u(\tau), \tag{14}$$

the inequality path constraints

$$(0, -10, -1) \leq (x(\tau), v(\tau), u(\tau)) \leq (\infty, 10, +1), \tag{15}$$

and the boundary conditions

$$(x(-1), x(+1), v(-1), v(+1)) = (10, 0, 0, 0). \tag{16}$$

The optimal solution to the optimal control problem given in Eqs. (13)–(16) is

$$\begin{aligned}
x^*(\tau) &= \tfrac{t_f}{2} \begin{cases} x_0 - \dfrac{x_0}{2}(\tau+1)^2 & , \quad -1 \leq \tau \leq \tau_s^*, \\ \dfrac{x_0}{2}(\tau-1)^2 & , \quad \tau_s^* \leq \tau \leq +1, \end{cases} \\
v^*(\tau) &= \tfrac{t_f}{2} \begin{cases} -\sqrt{x_0}(\tau+1) & , \quad -1 \leq \tau \leq \tau_s^*, \\ +\sqrt{x_0}(\tau-1) & , \quad \tau_s^* \leq \tau \leq +1, \end{cases} \\
u^*(\tau) &= \begin{cases} -1 & , \quad -1 \leq \tau \leq \tau_s^*, \\ +1 & , \quad \tau_s^* \leq \tau \leq +1, \end{cases}
\end{aligned} \tag{17}$$

where $\tau_s^* = 0$ and $t_f^* = 2\sqrt{x_0} \approx 6.32456$. It is seen that the trajectory given in Eq. (17) is piecewise quadratic with a single switch in the optimal control. Thus, it should be possible to obtain the exact solution to the problem given in Eqs. (13)–(16) can be obtained by dividing the time interval into two subintervals as follows. Minimize the objective functional

$$J = t_f \tag{18}$$

subject to the dynamic constraints in each interval $k \in [1, 2]$,

$$\dot{x}^{(k)}(\tau) = \alpha^{(k)} \tfrac{t_f}{2} v^{(k)}(\tau) \quad , \quad \dot{v}^{(k)}(\tau) = \alpha^{(k)} \tfrac{t_f}{2} u^{(k)}(\tau), \tag{19}$$

the inequality path constraints in each interval $k \in [1, 2]$,

$$(0, -10, -1) \leq (x^{(k)}(\tau), v^{(k)}(\tau), u^{(k)}(\tau)) \leq (\infty, 10, +1) \tag{20}$$

and the boundary conditions

$$(x^{(1)}(-1), x^{(2)}(+1), v^{(1)}(-1), v^{(2)}(+1)) = (10, 0, 0, 0), \tag{21}$$

where $\alpha^{(1)} = (t_s - t_0)/(t_f - t_0)$ and $\alpha^{(2)} = (t_f - t_s)/(t_f - t_0)$. Suppose now that the LGR collocation method is used to approximate the two-interval optimal control problem of Eqs. (18)–(21). Because the optimal trajectory is piecewise quadratic and the LGR quadrature is exact for polynomials of degree at most $2N-2$, it should be possible to obtain an exact solution using two collocation points in each subinterval (that is, $N^{(1)} = N^{(2)} = 2$) with t_s included as a variable in the optimization. Furthermore, the control function, known as the *approximate control*, can be obtained using $\dot{v}(\tau)$ as

$$u(\tau) = \dot{v}(\tau). \tag{22}$$

The NLP control and the approximate control obtained by solving the two-interval NLP are shown in Fig. 1. First, it is seen that the NLP solver returns a switch time in the control that is differs significantly from the optimal switch time. In addition, the optimal cost returned by the NLP solver is *less* than the known optimal cost. Finally, the approximate control given by Eq. (22) exceeds the upper limit, $u_{\max}$, given in Eq. (20) and, as a result, the NLP returns an approximate optimal control solution that is not a member of the admissible set of solutions for the original continuous optimal control problem described in Eqs. (13)–(16). Consequently, adding the switch time of the control as a variable results a solution with a lower cost and an incorrect switch time, thus making it the case that the allowable search space in the two-interval problem is larger than what should be permissible. The behavior of obtaining a lower than optimal cost simply by partitioning the time interval and adding the switch time as a variable in the optimization is an example of *Lavrentiev phenomenon*.

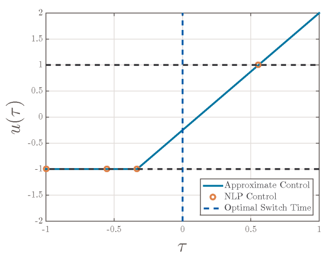


Fig. 1: Optimal control for example given in Eqs. 18–21 by including a variable corresponding to the location of the switch time, t_s, in the control.

IV. Search Space and Lavrentiev Gap

Given the incorrect result obtained in solving the two-interval optimal control problem of Eqs. (18)–(21), the goal of this section is to generate the correct search space for the modified optimal control problem. Identifying this new search space allows for the addition of the appropriate constraints to the optimal control problem of Eqs. (18)–(21). These new constraints reduce the allowable search space in the LGR discrete approximation of the continuous optimal control problem, thus improving the accuracy of both the switch time in the control and the minimum cost. In order to obtain the search space, let $\boldsymbol{\mathcal{P}}_N$ be the space of all polynomials of degree N on the interval $[-1,+1]$. Furthermore, let $\boldsymbol{\mathcal{A}} \subset \boldsymbol{\mathcal{P}}_N$ be the set of all polynomials of degree N that satisfies the collocation constraints of (12) at each LGR point $(\tau_1,\ldots,\tau_N)$. Next, let $\boldsymbol{\mathcal{U}}^p$ be the set of all control functions such that produce a state approximation that lies in $\boldsymbol{\mathcal{A}}$. Note that any state approximation that arises from a control in $\boldsymbol{\mathcal{U}}^p$ satisfies the collocation constraints of Eq. (12) at only the LGR points. Let $\boldsymbol{\mathcal{Y}} \subset \boldsymbol{\mathcal{P}}_N$ be the set of polynomials such that any polynomial $\boldsymbol{\mathcal{Y}}$ lies in a neighborhood of a solution to the continuous-time optimal control problem given in Eqs. (1)–(4). Finally, let $\boldsymbol{\mathcal{U}}$ be the set of controls such that any element in $\boldsymbol{\mathcal{U}}$ produces a state that lies in $\boldsymbol{\mathcal{Y}}$. Because $\boldsymbol{\mathcal{Y}}$ is a set of polynomials that lie in a neighborhood of an optimal solution, any state in $\boldsymbol{\mathcal{Y}}$ must also reside in $\boldsymbol{\mathcal{A}}$ (that is, $\boldsymbol{\mathcal{Y}} \subset \boldsymbol{\mathcal{A}}$) while any control that lies in $\boldsymbol{\mathcal{U}}$ must also lie in $\boldsymbol{\mathcal{U}}^p$ (that is, $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{U}}^p$). Suppose now that $\mathbf{u}^* \in \boldsymbol{\mathcal{U}}$ and $\mathbf{U}^* \in \boldsymbol{\mathcal{U}}^p$ are the optimal controls obtained when solving the LGR NLP with allowable search spaces $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{U}}^p$, respectively. Furthermore, let $\mathcal{J}_{\mathbf{u}^*}$ and $\mathcal{J}_{\mathbf{U}^*}$ be the costs obtained with $\mathbf{u}^*$ and $\mathbf{U}^*$, respectively. If $\mathcal{J}_{\mathbf{U}^*} < \mathcal{J}_{\mathbf{u}^*}$, then, because $\boldsymbol{\mathcal{U}} \subset \boldsymbol{\mathcal{U}}^p$, the solution obtained solving the LGR NLP with the allowable search space $\boldsymbol{\mathcal{U}}^p$ exhibits Lavrentiev phenomenon [25], [26] and the optimal control problem possess a Lavrentiev gap [20] defined by $\boldsymbol{\mathcal{U}}^p - \boldsymbol{\mathcal{U}}$. Figure 2 illustrates the Lavrentiev gap.

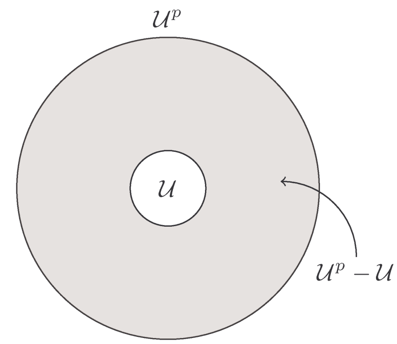


Fig. 2: Venn diagram of sets $\boldsymbol{\mathcal{U}}^p$ and $\boldsymbol{\mathcal{U}}$ where the Lavrentiev gap, $\boldsymbol{\mathcal{U}}^p - \boldsymbol{\mathcal{U}}$, is the shaded region.

It has been shown in Refs. [9]–[13] that under conditions of smoothness and coercivity, a Gaussian quadrature direct LGR collocation will converge to a local minimizer of the continuous-time optimal control problem. A locally minimizing solution may not, however, be obtained when the problem does not satisfy such coercivity conditions (for example, a problem with a nonsmooth optimal control). In such a situation, the solution of the LGR NLP may have a lower cost than the cost of the minimizing solution of the continuous-time optimal control problem. The continuous-time search space of control is found by considering a set of polynomial functions that approximate the state of the NLP and then solving for the control that produce the admissible state trajectories. Now the control search space of the LGR NLP arising from the two-interval optimal control problem given in Eqs. (18)–(21) is examined to demonstrate how Lavrentiev phenomenon can manifest in an LGR approximation of the continuous time problem.

V. Search Space Using Standard LGR Collocation

The differentiation matrix $\mathbf{D}$ of the optimal control of Section III using the chosen two-interval two-collocation-point LGR approximation is given as

$$\mathbf{D} = \begin{bmatrix} D_{11}^{(1)} & D_{12}^{(1)} & D_{13}^{(1)} & 0 & 0 \\ D_{21}^{(1)} & D_{22}^{(1)} & D_{23}^{(1)} & 0 & 0 \\ 0 & 0 & D_{11}^{(2)} & D_{12}^{(2)} & D_{13}^{(2)} \\ 0 & 0 & D_{21}^{(2)} & D_{22}^{(2)} & D_{23}^{(2)} \end{bmatrix}. \tag{23}$$

Furthermore, the collocation constraints for the two-interval approximation approximation of the dynamics given in Eq. (14) are given as

$$\begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{V} \end{bmatrix} - \frac{t_f}{2} \begin{bmatrix} \mathbf{V}_{1:4} \\ \mathbf{U} \end{bmatrix} = 0, \tag{24}$$

where $\mathbf{V}_{1:4}$ is the column vector formed using the first four rows of the column vector $\mathbf{V}$. Now, solving Eq. (14) gives $x(t_f) = x_0 + \int_{t_0}^{t_f} v(\tau) d\tau$. Suppose now that $v(\tau)$ is approximated as a Lagrange polynomial of degree two [as given in Eq. (6)] in each mesh interval. Given that the boundary conditions are fixed values, suppose now that V_2, V_3, and V_4 are defined to be the coefficients of the Lagrange polynomial approximation of $v(\tau)$ at the following points, respectively: (1) the second LGR point in the first mesh interval (that is, the first interior LGR point in the first mesh interval); (2)

the non-collocated point at the end of the first mesh interval (which is the same coefficient as that at the first point of the second interval); and (3) the second LGR point in the second mesh interval (that is, the first interior LGR point in the second mesh interval). Because $v(\tau)$ is approximated as a piecewise quadratic, varying these three coefficients results in polynomial approximations for the integral of $v(\tau)$ [that is, $x(\tau)$], and the derivative of $v(\tau)$ [that is $\dot{v}(\tau)$] along with a value for t_f. A feasible solution of the two-interval LGR NLP that approximates the optimal control problem defined in Eqs. (13)–(16) for any given values of V_2, V_3, V_4 is obtained by solving the following system of four linear equations for the unknowns X_2, X_3 X_4, and t_f:

$$\begin{bmatrix} D_{12}^{(1)} & D_{13}^{(1)} & 0 \\ D_{22}^{(1)} & D_{23}^{(1)} & 0 \\ 0 & D_{11}^{(2)} & D_{12}^{(2)} \\ 0 & D_{21}^{(2)} & D_{22}^{(2)} \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \\ X_4 \\ t_f \end{bmatrix} = \begin{bmatrix} -D_{11}^{(1)} x_1 - 0.5V_1 \\ -D_{12}^{(1)} x_1 - 0.5V_2 \\ -D_{11}^{(2)} x_5 - 0.5V_3 \\ -D_{12}^{(2)} x_5 - 0.5V_4 \end{bmatrix}. \tag{25}$$

To understand how the allowable values of the polynomial approximations of the state compare to feasible solutions of the corresponding continuous optimal control problem, it is necessary to analyze the associated polynomial approximation of the state. Let $(X^{(k)}(\tau), V^{(k)}(\tau))$ be Lagrange polynomial approximations of the state in mesh interval k that satisfy the constraints of Eqs. (14)–(16) at the points defined by the LGR approximation (that is, the state bounds in Eqs. (15)–(16) are satisfied at the collocation points and any non-collocated points while the control bounds in Eqs. (15)–(16) are satisfied at all collocation points). Figure 3 shows all possible control functions, that is, all functions (which in this case are polynomials because the state approximation is a polynomial and the dynamics of the example in Section III are linear in the control) of the form

$$U^{(k)}(\tau) = \dot{V}^{(k)}(\tau). \tag{26}$$

that arise from state approximations $(X^{(k)}(\tau), V^{(k)}(\tau))$ that satisfy the constraints of the LGR NLP. It is seen from Fig. 3 that the possible control function that satisfy the LGR NLP constraints violate the bounds on the control as given in Eq. (16). Consequently, the set of possible solutions of the LGR discrete approximation produce control approximations that are infeasible with respect to the continuous constraints given in Eqs. (15)–(16). The goal of the next section is to modify the search space for the standard LGR method by introducing constraints into the NLP that allow the admissible solutions of the NLP to more closely represent the feasible solutions of the continuous time problem.

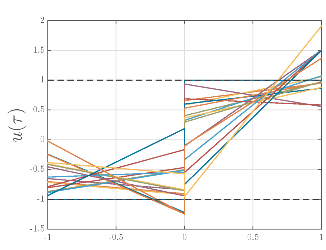


Fig. 3: Possible approximate control functions for the example given in Eqs. (18)–(21).

In order to modify the search space in the LGR collocation method, suppose now that the functions $(X^{(k)}(\tau), V^{(k)}(\tau))$ defined previously are restricted such that the only possible control functions $U^{(k)}(\tau)$ are those such that the state and control approximations, $(X^{(k)}(\tau), V^{(k)}(\tau)$, and $U^{(k)}(\tau))$, are feasible with respect to the control bounds $(u_{\min}, u_{\max})$given in Eq. (16). In other words, the only possible state approximations are those that are feasible with respect to the control bounds and simultaneously satisfy all other constraints in the LGR NLP. Using Eq. (26), a search space different from that of the standard LGR collocation method can now be constructed that provides those control function approximations that lead to state approximations $(X^{(k)}(\tau), V^{(k)}(\tau))$ that are feasible with respect to the constraints of the NLP arising from LGR collocation.

VI. Modification of Search Space

Using the results of Section V, additional constraints are now augmented to the standard collocation method presented in Section II in order to improve the approximation of the location of the nonsmoothness in the solution to the optimal control problem (thereby improving the accuracy of the solution itself). In particular collocation constraints are added at the end of a mesh interval where nonsmoothness occurs, but such constraints are added to only those differential equations *that are a function of control*. In this manner, and as stated in Section I, the approach developed in this research different fundamentally from the approaches developed in Refs. [23] and [24].

In order to better understand why collocation constraints are added to only those differential equations that are a function of control, consider first the first differential equation in the example of Section III, that is, consider the dynamics $\dot{x}(\tau) = v(\tau)$, where $x(\tau)$ and $v(\tau)$ are the two components of the state. Next, let $\mathcal{P}^N$ be the space of all polynomials of degree N on the domain $t \in [-1, +1]$. Let $x(\tau) \in \mathcal{P}^N$ and $v(\tau) \in \mathcal{P}^N$. Finally, let $(\tau_1, \ldots, \tau_N)$ be the LGR points and let $\tau_{N+1} = +1$ be what was originally the noncollocated point in the first of the two mesh intervals of the two-interval formulation that was used in Section III. Suppose now that both $X(\tau)$ and $V(\tau)$ are Lagrange polynomial approximations of degree N as given in Eq. (6). Then, the function $\dot{X}(\tau) - V(\tau)$ is a polynomial of degree N and, thus, has N roots. Consequently, the only possible way that the constraint $\dot{X}(\tau) - V(\tau)$ will be satisfied at the $N+1$ points $(\tau_1, \ldots, \tau_{N+1})$, is if this constraint is satisfied at every point in the domain $\tau \in [-1, +1]$. Now, because the state approximation is a polynomial of degree at most N, the approximation $\dot{X}(\tau) - V(\tau)$ is also a polynomial of degree at most N and can be zero at only N points. Thus, enforcing the constraint $\dot{X}(\tau) - V(\tau)$ at $N+1$ points will lead to an overdetermined system which is why collocation constraints are not added at $\tau = +1$ for a differential equation that is not a function of control.

Next, consider the second differential equation in the example of Section III, that is, consider the dynamics $\dot{v}(\tau) = \frac{t_f}{2} u(\tau)$, where $v(\tau)$ is the second component of the state and $u(\tau)$ is the control. Suppose again that the state approximation $V(\tau)$ of $v(\tau)$ is a polynomial of degree N in each of the two mesh intervals of a two-interval formulation of the optimal control problem given in Section III. Finally, suppose that the constraint $\dot{v}(\tau) = \frac{t_f}{2} u(\tau)$ is enforced at the N LGR points plus the final point of the first mesh interval. Because $V(\tau)$ is a polynomial of degree N and the differential equation is a function of control, it is possible to satisfy all $N+1$ conditions

$$\dot{V}(\tau_i) - U_i = 0, \quad (i = 1, \ldots, N+1) \tag{27}$$

in the first mesh interval because the control is a variable in Eq. (27). In other words, U_{N+1} can be varied in order to satisfy Eq. (27) at the endpoint of the first interval. Moreover, when adding this collocation condition, it is also necessary to add the constraint that $u_{\min} \leq U_{N+1} \leq u_{\max}$ in order to ensure that the control at the end of the first mesh interval satisfies the limits on the control.

The preceding argument leads to a modification of the LGR collocation method for the case where the solution may be nonsmooth. First, as stated, an additional variable that defines the location of the discontinuity is added. Second, a collocation condition similar to that given in Eq. (27) is included along with a constraint that enforces the control bounds at the end of the mesh interval. This modification leads to the use of the modified LGR differentiation matrix

$$\tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{D}_{1:N} & \mathbf{D}_{N+1} \\ \mathbf{E} & E_0 \end{bmatrix}, \tag{28}$$

in the mesh intervals where the additional collocation constraints are included for those differential equations that *are* a function of the control. It is noted in Eq. (28) that $[\mathbf{D}_{1:N} \;\; \mathbf{D}_{N+1}] \in \mathbb{R}^{N\times(N+1)}$ is the standard $N \times (N+1)$ LGR differentiation matrix [4]–[6]. Finally, the last row of the $\tilde{\mathbf{D}}$ matrix consists of $\mathbf{E} \in \mathbb{R}^{1\times N}$ and $E_0 \in \mathbb{R}$, where this last row corresponds to the fact that a collocation point has been added in the modified LGR method. With the inclusion of the new collocation constraint, the collocation equations given Eq. (12) are modified as

$$\begin{array}{rcl} \mathbf{D}\mathbf{X} - \mathbf{V}_{1:N} & = & \mathbf{0}, \\ \tilde{\mathbf{D}}\mathbf{V} - \mathbf{f}(\mathbf{X}, \mathbf{V}, \tilde{\mathbf{U}}) & = & \mathbf{0}. \end{array} \tag{29}$$

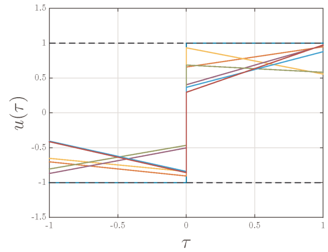
where

$$\tilde{\mathbf{U}} = \begin{bmatrix} \mathbf{U} \\ \mathbf{U}_{N+1} \end{bmatrix}$$

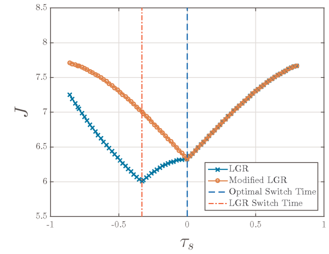
and $\mathbf{U}_{N+1}$ is the value of the control at $\tau = +1$. Observe that, consistent with the explanation provided earlier in this section, the first constraint in Eq. (29) is *not* a function of control and, as a result, is identical to the first constraint given in Eq. (12). The cost function given in Eq. (11), together with the constraints in Eq. (29), is referred to as the *modified Legendre-Gauss-Radau collocation method*.

VII. Search Space of Modified LGR Method

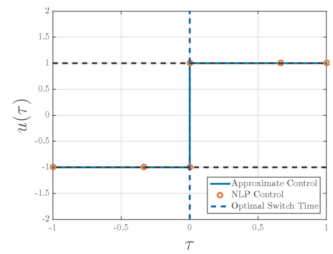
The example of Section III is now revisited using the modified LGR collocation method. Fig. 4a demonstrates that each admissible set for control now falls between the allowable control limits $(u_{\min}, u_{\max}) = (-1, +1)$. Next, to examine the effect that the modified LGR method has on the solution of the NLP for the example in Section III, Fig. 4b shows the cost of the modified LGR NLP as a function of the switch time, t_s, where it is assumed that the switch time is fixed. At the optimal switch time t_s^*, the cost of both the original and modified LGR methods is identical. Note, however, that when for $t_s < t_s^*$, the optimal cost of the standard LGR method is *smaller* than the modified LGR method. In fact, Fig. 4b shows that the optimal cost for the modified LGR method occurs when $t_s = t_s^*$. This last result indicates that the modified LGR method reduces the allowable search



(a) Admissible controls for example defined by Eqs. (18)–(21) using modified LGR collocation method (assuming a fixed switch time).



(b) Optimal cost, J^*, for example defined by Eqs. (18)–(21) as a function of switch time, t_s.



(c) Optimal control for the example defined by Eqs. (13)–(16) using the modified LGR method.

Fig. 4: Admissible controls for modified LGR collocation method, comparison of optimal cost for both standard and modified LGR collocation methods as a function of switch time, t_s, and optimal control obtained using modified LGR collocation method.

space such that the solution of the NLP leads to a state approximation that is closer to the solution of the continuous-time optimal control problem. Figure 4c shows the control solution obtained by solving for the control as a function of time using the Lagrange polynomial approximation of the state obtained using the modified LGR collocation method. It is seen that, not only does the control function lie within its allowable limits $(u_{\min}, u_{\max}) = (-1, +1)$, but the switch time obtained using the modified LGR collocation method matches the switch time of the solution of the continuous-time optimal control problem.

VIII. Conclusions

A modified Legendre-Gauss-Radau (LGR) collocation method has been developed for solving optimal control problems whose solutions are nonsmooth. The effect of Lavrentiev phenomenon on LGR collocation has been explored. It was shown that when a continuous-time optimal control problem is transcribed into an NLP using the original LGR collocation method that the possible solutions for the NLP may not correspond to possible solutions to the optimal control problem. The search space of the original LGR collocation method has been modified by including new constraints into the NLP. The new constraints include a collocation point for those differential equations that are functions of the control at the end of a mesh interval where the nonsmoothness in the solution occurs along with a control constraint at this same point. An example was considered throughout to describe the various phenomena and to justify the modifications to the original LGR collocation method. The results of this paper demonstrate that the modified LGR method developed in this paper significantly improves the numerical approximation for optimal control problems whose solutions are nonsmooth.

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