Bergman Kernel Asymptotics for Singular Metrics on Punctured Riemann Surfaces

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ABSTRACT. We consider singular metrics on a punctured Riemann surface and on a line bundle, and study the behavior of the Bergman kernel in the neighborhood of the punctures. The results have an interpretation in terms of the asymptotic profile of the density-of-states function of the lowest Landau level in quantum Hall effect.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to study the behavior of the Bergman kernel function of a Hermitian holomorphic line bundle over a punctured Riemann surface. A quite general result about asymptotics of Bergman kernel on non-compact manifolds was given in [25, 34–36]. Let (Y, ω) be a complete Kähler manifold of dimension n, and $(L, h) \to X$ be a holomorphic Hermitian line bundle such that

$$(1.1) \qquad c_1(L, h) \geq \varepsilon \omega, \quad \mathrm{Ric}_\omega \geq -C\omega,$$

for some constants $\varepsilon, C > 0$. If $(L, h) = (K_Y, h^{K_Y})$, where $K_Y = \det(T^{*(1,0)}Y)$ is the canonical bundle and h^{K_Y} is induced by ω, condition (1.1) is to be replaced by

$$(1.2) \qquad\qquad\qquad \mathrm{Ric}_\omega \leq -\varepsilon\omega.$$

Under these assumptions, it is known that the Bergman kernel function $P_p(x)$ of the space of L^2-holomorphic sections $H^0_{(2)}(Y, L^p)$ has the following expansion:

$$(1.3) \qquad P_p(x) = b_0(x)p^n + b_1(x)p^{n-1} + \cdots = \sum_{j=0}^{\infty} b_j(x)p^{n-j},$$

uniformly on compact sets relative to any C^ℓ-topology.

More precisely, there exist coefficients $b_j \in C^\infty(Y)$, $j \in \mathbb{N}$, such that for any compact set $K \subset Y$, any $k, \ell \in \mathbb{N}$, there exists $C_{k,\ell,K} > 0$ such that for $p \in \mathbb{N}$

$$\left| \frac{1}{p^n} P_p(x) - \sum_{j=0}^{k} b_j(x)p^{-j} \right|_{C^\ell(K)} \leq C_{k,\ell,K} p^{-k-1}.$$

Moreover, we have

$$b_0 = \frac{c_1(L,h)^n}{\omega^n}, \quad b_1 = \frac{1}{8\pi} \frac{c_1(L,h)^n}{\omega^n} \left(r^Y - 2\Delta \log \left(\frac{c_1(L,h)^n}{\omega^n} \right) \right),$$

where r^Y and Δ are the scalar curvature and the Bochner Laplacian of the metric associated with the Kähler form $c_1(L, h)$.

Assume, now, that X is compact and $c_1(L, h) = \omega$; hence, $b_0 = 1$. The expansion $P_p(x) = p^n + O(p^{n-1/2})$ was proved by Tian [41, Section 3] in the C^4-topology and generalized by Ruan [37] to $P_p(x) = p^n + O(p^{n-1})$ in any C^ℓ-topology. Berndtsson [4] gave a simple proof of the uniform convergence $P_p(x) = b_0(x)p^n + o(p^n)$. The asymptotics (1.3) were proved by Catlin [10] and Zelditch [43].

In the quantum Hall effect (QHE), the density of states for the lowest Landau level on a Riemann surface, or more generally on a Kähler manifold, is given by the Bergman kernel on the diagonal (see [20], where (1.3) was derived using perturbation theory for the quantum mechanical path integral). The metric dependence and gravitational anomaly in the quantum Hall states has recently been studied using the asymptotic expansion of the Bergman kernel [26, 30], as well as other methods [1, 5–7, 22, 24, 29] (see [27] for a review). The quantum Hall states and the density function have been studied recently for surfaces with conical singularities [8, 28], singular surfaces with Z_n-symmetry [23] and cusps [9] (see [2] for the results for the Bergman kernel). Remarkably, the quantum Hall effect on

a cone can also be realized experimentally (see [38], where synthetic Landau levels on a cone were constructed in a photon resonator). In this paper, we study the asymptotic profile of the Bergman kernel for more general singular geometries.

If X is non-compact, (1.3) was deduced in [34, Theorem 6.1.1] under hypothesis (1.1), in [35, Theorem 3.11] under hypothesis (1.2) (see also Theorem 1.6 in [25]). We refer the reader to the book [34] for a comprehensive study of the Bergman kernel and its applications, and also to the survey [33].

By the above mentioned works, the asymptotic expansion of the Bergman kernel is well understood on fixed compact sets. In this paper, we consider very general metrics on a punctured Riemann surface and on a line bundle, and study the behavior of the Bergman kernel in the neighborhood of the punctures. The asymptotics depend on the singularities of the metrics on the base manifold and on the bundle.

In the case of metrics with Poincaré singularities, [2] provides a weighted estimate in the C^m-norm near the punctures for the global Bergman kernel compared to the Bergman kernel of the punctured disc, uniformly in the tensor powers of the given bundle. Our estimates complement the results of [2].

More precisely, we consider in this paper the following setting:

(A) X is a Riemann surface and $\Sigma = \{x_j : j \geq 1\} \subset X$ is a discrete closed subset. We fix a smooth Hermitian metric Ω on X and denote by dist the induced distance.

(B) ω is a Hermitian metric of class C^2 on $X \setminus \Sigma$ such that $\omega \geq c\,\Omega$, where $c : X \to (0, \infty)$ is a continuous function and $\mathrm{Ric}_\omega \geq -2\pi B\omega$, for some constant $B > 0$.

(C) L is a holomorphic line bundle on X and h is a Hermitian metric of class C^3 on $L|_{X \setminus \Sigma}$ such that $c_1(L, h) \geq 2\varepsilon\omega$ on $X \setminus \Sigma$, for some constant $\varepsilon > 0$.

Let $h_p := h^{\otimes p}$ be the metric induced by h on $L^p|_{X \setminus \Sigma}$, where $L^p := L^{\otimes p}$. We denote by $H^0_{(2)}(X \setminus \Sigma, L^p)$ the Bergman space of L^2-holomorphic sections of L^p relative to the metric h_p and ω,

$$(1.4) \quad H^0_{(2)}(X \setminus \Sigma, L^p) = \left\{ S \in H^0(X \setminus \Sigma, L^p) : \|S\|_p^2 := \int_{X \setminus \Sigma} |S|^2_{h_p} \omega < \infty \right\},$$

endowed with the obvious inner product.

Let $d_p \in \mathbb{N} \cup \{\infty\}$ be the dimension of $H^0_{(2)}(X \setminus \Sigma, L^p)$. We denote by P_p the Bergman kernel function of the space $H^0_{(2)}(X \setminus \Sigma, L^p)$, which is defined as follows. For $p \geq 1$, if $\{S^p_\ell\}_{\ell \geq 1}$ is an orthonormal basis of $H^0_{(2)}(X \setminus \Sigma, L^p)$, then

$$(1.5) \qquad\qquad P_p(x) = \sum_{\ell=1}^{d_p} |S^p_\ell(x)|^2_{h_p}.$$

Note that P_p is independent of the choice of basis (see [11, Lemma 3.1]).

Fix $x_j \in \Sigma$ and a constant $R_j > 0$ with the property that x_j has a coordinate neighborhood U_{x_j} centered at x_j such that the coordinate disc $B(x_j, 2R_j) \Subset U_{x_j}$ and

$$(1.6) \qquad \mathrm{dist}(B(x_j, 2R_j), \Sigma \setminus \{x_j\}) \geq R_j.$$

Let e_j be a local holomorphic frame of L on $B(x_j, 2R_j)$, and let φ_j be the sub-harmonic weight of h on $B(x_j, 2R_j) \setminus \{x_j\}$ corresponding to e_j, so $|e_j|_h = e^{-\varphi_j}$. We assume that, in local coordinate z on U_{x_j}, φ_j has the form

$$(1.7) \qquad \varphi_j(z) = v_j \log|z| + \psi_j(z),$$
$$\text{where } v_j \in \mathbb{R}, \ \psi_j \in C^3(B(x_j, 2R_j) \setminus \{x_j\}).$$

Moreover, we assume that there exist the constants $A_j > 0$, $\alpha_j \geq 0$ such that the third-order derivatives of ψ_j verify

$$(1.8) \qquad |D^\mu \psi_j(z)| \leq A_j |z|^{-\alpha_j}, \quad \text{for all } z \in B(x_j, 2R_j) \setminus \{x_j\}, \ |\mu| = 3.$$

In particular, equation (1.7) lets us consider the special cases when ψ_j is bounded or smooth near x_j; that is, the metric h has logarithmic singularities at Σ.

Next, we can write on $B(x_j, 2R_j) \setminus \{x_j\}$,

$$(1.9) \qquad \omega(z) = \frac{i}{2}\rho_j(z)\, dz \wedge d\bar{z} = \rho_j(z)\, dm(z),$$

where $dm(z)$ is the Lebesgue measure in the coordinate z. We assume that there exist the constants $A_j' > 0$, $\beta_j \geq 0$ such that the first-order derivatives of ρ_j verify

$$(1.10) \qquad |D\rho_j(z)| \leq A_j' |z|^{-\beta_j}, \quad \text{for all } z \in B(x_j, 2R_j) \setminus \{x_j\}.$$

Finally, we let

$$(1.11) \qquad \delta_j = \max\left\{\frac{8}{3}, \frac{8\beta_j}{3}, 8\alpha_j\right\}.$$

In [4, Section 2], Berndtsson gave a simple proof for the first-order asymptotics of the Bergman kernel function $P_p(x) = b_0(x)p^n + o(p^n)$ in the case of powers of an ample line bundle on a projective manifold. By adapting his methods to our situation, we prove the following asymptotics near the singular points. They show explicitly how the estimates depend on the distance to the singular points, and on the parameters α_j, β_j which encode the singularities of the metrics.

Theorem 1.1. *Let $(X, \Sigma, \omega, L, h)$ verify assumptions* (A)–(C), *and let $x_j \in \Sigma$. Let R_j be defined by* (1.6), *δ_j by* (1.11), *and assume that h and ω satisfy inequalities*

(1.8) *and* (1.10), *respectively. Then, there exists a constant* $C_j > 1$ *such that if* $x \in B(x_j, R_j) \setminus \{x_j\}$ *and*

$$p > C_j \operatorname{dist}(x, x_j)^{-\delta_j},$$

then

(1.12)
$$\left| \frac{P_p(x)}{p} \frac{\omega_x}{c_1(L, h)_x} - 1 \right|$$
$$\leq C_j (p^{-1/8} \operatorname{dist}(x, x_j)^{-\alpha_j} + p^{-3/8} \operatorname{dist}(x, x_j)^{-\beta_j}).$$

Next, we consider Bergman kernels for powers of the canonical bundle of a punctured Riemann surface. In addition to the setting in (A), let us consider the following condition:

(B') ω is a smooth Hermitian metric on $X \setminus \Sigma$ such that $\omega \geq c \, \Omega$, where $c : X \to (0, \infty)$ is a continuous function and $\operatorname{Ric}_\omega \leq -\lambda \omega$ on $X \setminus \Sigma$, for some constant $\lambda > 0$.

The Hermitian metric ω induces a Hermitian metric h^{K_X} on $K_X|_{X \setminus \Sigma}$. We denote by h_p the metric induced by h^{K_X} on K_X^p and by $H_{(2)}^0(X \setminus \Sigma, K_X^p)$ the space of holomorphic sections of $K_X^p|_{X \setminus \Sigma}$ which are L^2, with respect to the metrics h_p and volume form ω (cf. (1.4)).

Let $z : U_{x_j} \to \mathbb{C}$ be a local holomorphic coordinate, with respect to which the metric ω has the form (1.9). We define the weight φ_j on $B(x_j, 2R_j) \setminus \{x_j\}$ of h^{K_X} by

(1.13)
$$e^{-2\varphi_j} = |dz|_\omega^2, \quad \varphi_j = \frac{1}{2}(\log \rho_j - \log 2),$$

and we further write φ_j as in (1.7).

Theorem 1.2. *We let* (X, Σ, ω) *verify the assumptions* (A) *and* (B'). *We assume that* ω *satisfies* (1.8) *and* (1.10). *We let* P_p *be the Bergman kernel function of the space* $H_{(2)}^0(X \setminus \Sigma, K_X^p)$. *Then, there exists a constant* $C_j > 1$ *such that if* $x \in B(x_j, R_j) \setminus \{x_j\}$ *and*

(1.14)
$$p > C_j \operatorname{dist}(x, x_j)^{-\delta_j},$$

then

(1.15)
$$\left| \frac{P_p(x)}{p} \left(-\frac{2\pi}{R_\omega(x)} \right) - 1 \right|$$
$$\leq C_j (p^{-1/8} \operatorname{dist}(x, x_j)^{-\alpha_j} + p^{-3/8} \operatorname{dist}(x, x_j)^{-\beta_j}),$$

where R_ω *is the Gauss curvature of* ω.

In [2], we consider the particular situation of Theorem 1.2 when the metric ω equals the Poincaré metric near the punctures (hence, $\alpha_j = \beta_j = 3$, cf. Lemma 4.3) and obtain estimates which are different in nature from those in Theorem 1.2.

See Corollaries 4.4, 4.6, 4.7 for applications of Theorem 1.2.

Theorem 1.1 has the following interesting corollary which gives a uniform estimate on the Bergman kernel $P_p(x)$ in the regime where the distance from x to Σ decreases slower than some power of $1/p$.

Corollary 1.3. *In the setting of Theorems 1.1 or 1.2, there exists a constant $C_j > 1$ such that if $\eta \in [0,1]$, $p > C_j$, and $x \in B(x_j, R_j)$ satisfies $\mathrm{dist}(x, x_j) > (C_j/p)^{\eta/\delta_j}$, then*

$$\left| \frac{P_p(x)}{p} \frac{\omega_x}{c_1(L,h)_x} - 1 \right| \le 2C_j p^{-(1-\eta)/8}.$$

The paper is organized as follows. In Section 2, we recall some facts about singular Hermitian metrics on holomorphic line bundles and the solution of the $\bar{\partial}$ equation. Section 3 is devoted to the proof of the main results announced in the Introduction. In Section 4, we apply these results to interesting metrics for which the parameters α_j and β_j can be explicitly given (metrics with logarithmic, cuspidal, and conical singularities). In Section 5, we calculate the Bergman kernel of the Riemann sphere with two conical singularities and study its asymptotics near the singularities.

2. PRELIMINARIES

2.1. Singular Hermitian holomorphic line bundles.   Let L be a holomorphic line bundle on a complex manifold Y. The notion of singular Hermitian metric h on L is defined as follows (see [18], [34, p. 97]): if e_ℓ is a holomorphic frame of L over an open set $U_\ell \subset Y$, then $|e_\ell|_h^2 = e^{-2\varphi_\ell}$, where $\varphi_\ell \in L^1_{\mathrm{loc}}(U_\ell)$. If $g_{\ell k} = e_k/e_\ell \in \mathcal{O}_Y^*(U_\ell \cap U_k)$ are the transition functions of L, then $\varphi_\ell = \varphi_k + \log|g_{\ell k}|$. The curvature current $c_1(L,h)$ of h is the current of bidegree $(1,1)$ on Y defined by $c_1(L,h) = dd^c \varphi_\ell$ on U_ℓ, where $d^c = 1/(2\pi i)(\partial - \bar{\partial})$. If $c_1(L,h) \ge 0$, then the weight φ_ℓ is plurisubharmonic on U_ℓ. When Y is compact, the space $H^0(Y, L)$ of global holomorphic sections of L is finite dimensional.

Now, let $(X, \Sigma, \omega, L, h)$ be as in Theorem 1.1, and P_p be the Bergman kernel functions of the spaces $H^0_{(2)}(X \setminus \Sigma, L^p)$ defined in (1.5). Then, for all $x \in X \setminus \Sigma$,

$$(2.1) \qquad P_p(x) = \max\left\{ |S(x)|^2_{h_p} : S \in H^0_{(2)}(X \setminus \Sigma, L^p), \ \|S\|_p = 1 \right\}.$$

Let (Y, J, ω) be a Kähler manifold, where J is the complex structure of Y, and let g^{TY} be the Riemannian metric associated with ω by $g^{TY}(u,v) = \omega(u, Jv)$ for

all $u, v \in T_x Y$, $x \in Y$. Let Ric be the Ricci curvature of g^{TY}. The Ricci form Ric_ω is defined as the $(1, 1)$-form associated with Ric by

(2.2) $\qquad \text{Ric}_\omega(u, v) = \text{Ric}(Ju, v), \quad$ for any $u, v \in T_x Y$, $x \in Y$.

The volume form ω^n induces a metric $h_\omega^{K_Y^*}$ on K_Y^*, whose dual metric on K_Y is denoted by $h_\omega^{K_Y}$. For simplicity, we denote by $h_p := (h_\omega^{K_Y})^{\otimes p}$ the induced metric on K_Y^p. Since the metric g^{TY} is Kähler, we have (see, e.g., [34, Problem 1.7])

(2.3) $\qquad \text{Ric}_\omega = iR^{K_Y^*} = -iR^{K_Y} = -2\pi c_1(K_Y, h^{K_Y}).$

Let us consider now the case of dimension $n = 1$. The canonical bundle of Y is just $K_Y = T^{(1,0)*}Y$ and $K_Y^* = T^{(1,0)}Y$; moreover, the metric $h_\omega^{K_Y^*}$ on K_Y^* is directly given by ω. In local holomorphic coordinates $z : U \to \mathbb{C}$, we write $\omega(z) = (i/2)\rho(z)\, dz \wedge d\bar{z}$, so $g^{TY}(z) = \rho(z)|dz|^2$. The Gauss curvature of g^{TY} (and, by a slight abuse, of ω) is defined by

(2.4) $$R_\omega = -\frac{2}{\rho}\frac{\partial^2}{\partial z\, \partial \bar{z}} \log \rho.$$

Since the metric $h_\omega^{K_Y^*}$ on K_Y^* is directly given by ω, we have that $\partial/\partial z$ is a frame of K_Y^*, dz is the dual frame of K_Y on U, and

(2.5) $$\left|\frac{\partial}{\partial z}\right|_\omega^2 = \frac{\rho(z)}{2}, \quad |dz|_\omega^2 = \frac{2}{\rho(z)}.$$

The weight φ of $h_\omega^{K_Y}$ on U is given by

$$e^{-2\varphi} = |dz|_\omega^2, \quad \varphi = \frac{1}{2}(\log \rho - \log 2);$$

hence,

(2.6) $\qquad \text{Ric}_\omega = -2\pi c_1(K_Y, h^{K_Y}) = -2\pi dd^c \varphi = -\pi dd^c \log \rho = R_\omega \omega.$

In local normal coordinates associated with ω near an arbitrary point $x_0 \in Y$, we have $\omega|_{x_0} = (i/2)\, dz \wedge d\bar{z}$, and the scalar curvature r_ω of (Y, ω) is given at x_0 by

$$r_\omega = 4R^{T^{(1,0)}Y}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right).$$

Thus,

(2.7) $\qquad -\frac{i}{2}r_\omega \omega = R^{T^{(1,0)}Y} = -R^{K_Y} = \bar{\partial}\, \partial \log |\sigma|^2,$

where σ is a local holomorphic frame of $T^{(1,0)}Y$. From (2.6) and (2.7), we deduce the relation between scalar and Gaussian curvature,

$$(2.8) \qquad\qquad r_\omega = 2R_\omega.$$

2.2. L^2-estimates for $\bar{\partial}$. The following version of Demailly's estimates for the $\bar{\partial}$ operator [17, Theorem 4.1] will be needed in our proofs (see Theorem 2.5 of [14]).

Theorem 2.1 ([17]). *Let Y be a complete Kähler manifold, $\dim Y = n$, and let ω be a Kähler form of class C^2 on Y (not necessarily complete) such that its Ricci form $\mathrm{Ric}_\omega \geq -2\pi B\omega$ on Y, for some constant $B > 0$. Let (L,h) be a Hermitian holomorphic line bundle on Y such that h is of class C^2 and $c_1(L,h) \geq 2\varepsilon\omega$. If $p \geq B/\varepsilon$ and $g \in L^2_{0,1}(Y,L^p,\mathrm{loc})$ verifies*

$$\bar{\partial}g = 0 \quad \text{and} \quad \int_Y |g|^2_{h_p}\omega^n < \infty,$$

then there exists $u \in L^2_{0,0}(Y,L^p,\mathrm{loc})$ such that

$$\bar{\partial}u = g \quad \text{and} \quad \int_Y |u|^2_{h_p}\omega^n \leq \frac{1}{p\varepsilon}\int_Y |g|^2_{h_p}\omega^n.$$

We also need the following version for powers of the canonical bundle.

Theorem 2.2. *Let Y be a complete Kähler manifold, $\dim Y = n$, and let ω be a Kähler form on Y (not necessarily complete) such that its Ricci form satisfies $\mathrm{Ric}_\omega \leq -\lambda\omega$ on Y, for some constant $\lambda > 0$. If $p \geq 2$ and $g \in L^2_{0,1}(Y,K_Y^p,\mathrm{loc})$ verifies*

$$\bar{\partial}g = 0 \quad \text{and} \quad \int_Y |g|^2_{h_p}\omega^n < \infty,$$

then there exists $u \in L^2_{0,0}(Y,K_Y^p,\mathrm{loc})$ such that

$$\bar{\partial}u = g \quad \text{and} \quad \int_Y |u|^2_{h_p}\omega^n \leq \frac{1}{(p-1)\lambda}\int_Y |g|^2_{h_p}\omega^n.$$

3. Proof of Main Results

In this section, we prove Theorems 1.1 and 1.2 together with Corollary 1.3. We then give a semi-global version of these results (Theorem 3.1).

Proof of Theorem 1.1. We use methods from [4, Section 2] (see Theorem 1.3 of [14]), and divide the proof into three steps. Recall the definition (1.6) of R_j.

Step 1. Given $x \in B(x_j, R_j) \setminus \{x_j\}$ we estimate the growth of the functions ρ_j and ψ_j defined in (1.9) and (1.7), respectively.

Note that, since $\omega \geq c\,\Omega$, we have

$$(3.1) \qquad \rho_j(z) \geq c_0, \quad \forall\, z \in B(x_j, 2R_j) \setminus \{x_j\},$$

for some constant $c_0 > 0$. Let $x \in B(x_j, R_j)$, $r < |x|/2$, and set

$$M_j(x, r) = \max\{\rho_j(z) : |z - x| \leq r\},$$
$$m_j(x, r) = \min\{\rho_j(z) : |z - x| \leq r\}.$$

Hence,

$$(3.2) \qquad m_j(x, r)\, dm(z) \leq \omega(z) \leq M_j(x, r)\, dm(z) \quad \text{on } B(x, r).$$

Since $r < |x|/2$, we obtain by (1.10)

$$|\rho_j(z) - \rho_j(x)| \leq \frac{A'_j r}{(|x| - r)^{\beta_j}} \leq \frac{2^{\beta_j} A'_j r}{|x|^{\beta_j}}, \quad z \in B(x, r).$$

Therefore, by using (3.1), we get

$$M_j(x, r) \leq \rho_j(x) + \frac{2^{\beta_j} A'_j r}{|x|^{\beta_j}} \leq \rho_j(x) \left(1 + \frac{2^{\beta_j} A'_j r}{c_0 |x|^{\beta_j}} \right),$$

$$m_j(x, r) \geq \rho_j(x) - \frac{2^{\beta_j} A'_j r}{|x|^{\beta_j}} \geq \rho_j(x) \left(1 - \frac{2^{\beta_j} A'_j r}{c_0 |x|^{\beta_j}} \right).$$

If $r < |x|/4$ and $r < c_0 |x|^{\beta_j}/(2^{\beta_j + 2} A'_j)$, these estimates yield

$$(3.3) \qquad \frac{M_j(x, 2r)}{m_j(x, r)} \leq \frac{1 + \dfrac{2^{\beta_j + 1} A'_j r}{c_0 |x|^{\beta_j}}}{1 - \dfrac{2^{\beta}_j A'_j r}{c_0 |x|^{\beta_j}}} \leq 1 + \frac{C'_1 r}{|x|^{\beta_j}},$$

with some constant $C'_1 > 0$. Note that (3.3) holds also with $\rho_j(x)/m_j(x, r)$ and $M_j(x, 2r)/\rho_j(x)$ in place of $M_j(x, 2r)/m_j(x, r)$, since the first two quantities are bounded above by the third.

Next, we turn our attention to the weight φ_j of the metric h corresponding to the local holomorphic frame e_j of L on $B(x_j, 2R_j)$ (see (1.7)). By using the Taylor expansion of order 2 of ψ_j at x on $B(x, |x|)$, we can write

$$\varphi_j(z) = \nu_j \log |z| + \operatorname{Re} f_j(z) + \lambda_x |z - x|^2 + \tilde{\psi}_j(z),$$

where f_j is a holomorphic polynomial, and $\bar{\psi}_j$ vanishes to order 3 at x. If, then, $r < |x|/2$, we have by (1.8) that

$$(3.4) \qquad \max\{|\bar{\psi}_j(z)| : z \in B(x,r)\} \leq \frac{A_j r^3}{(|x| - r)^{\alpha_j}} \leq \frac{2^{\alpha_j} A_j r^3}{|x|^{\alpha_j}}.$$

Since $c_1(L,h)_x \geq 2\varepsilon\omega_x$, it follows by (3.1) that

$$(3.5) \qquad \lambda_x \geq \pi\varepsilon\rho_j(x) \geq \pi\varepsilon c_0.$$

Note that the function $\log|z|$ is harmonic on the disc $B(x,|x|)$. Hence, there exists a holomorphic function $F_j(z)$ on $B(x,|x|)$ such that

$$\varphi_j(z) = \operatorname{Re} F_j(z) + \lambda_x |z - x|^2 + \bar{\psi}_j(z).$$

Consider the holomorphic frame $e_x = e^{F_j} e_j$ of L on $B(x,|x|)$, so

$$(3.6) \qquad \bar{\varphi}_j(z) = -\log|e_x(z)|_h = \varphi_j(z) - \operatorname{Re} F_j(z) = \lambda_x |z - x|^2 + \bar{\psi}_j(z)$$

is the corresponding weight of h. Note that $\bar{\varphi}_j(x) = 0$.

We conclude *Step 1* by introducing the following function, which is needed in the sequel:

$$E(r) := \int_{|\xi| \leq r} e^{-2|\xi|^2} \, dm(\xi) = \frac{\pi}{2}(1 - e^{-2r^2}),$$

where dm is the Lebesgue measure on $\mathbb{C}$. If $r \geq \delta > 0$, then

$$(3.7) \qquad \frac{\pi}{2E(r)} = 1 + \frac{e^{-2r^2}}{1 - e^{-2r^2}} \leq 1 + \frac{e^{-2r^2}}{1 - e^{-2\delta^2}}.$$

Step 2. We obtain here the upper estimate for $P_p(x)$ if $x \in B(x_j, R_j) \setminus \{x_j\}$. Let

$$S \in H^0_{(2)}(X \setminus \Sigma, L^p)$$

and write $S = s e_x^{\otimes p}$, where e_x is the local holomorphic frame of L on $B(x,|x|)$ from *Step 1* and $s \in \mathcal{O}_X(B(x,|x|))$. Let $r_p \in (0, |x|/4)$ be an arbitrary number which will be specified later. It follows from the sub-averaging inequality for subharmonic functions that

$$|S(x)|^2_{h_p} = |s(x)|^2 \leq \frac{\displaystyle\int_{B(x,r_p)} |s(z)|^2 e^{-2p\lambda_x |z-x|^2} \, dm(z)}{\displaystyle\int_{B(x,r_p)} e^{-2p\lambda_x |z-x|^2} \, dm(z)}.$$

By using (3.2), (3.6), and (3.4), we obtain

$$\int_{B(x,r_p)} |s(z)|^2 e^{-2p\lambda_x |z-x|^2} \, dm(z)$$

$$\leq \frac{\exp(2p \max_{B(x,r_p)} \widetilde{\psi_j})}{m_j(x,r_p)} \int_{B(x,r_p)} |s(z)|^2 e^{-2p\check{\varphi}_j(z)} \omega(z)$$

$$\leq \frac{\exp(2^{\alpha_j+1} A_j pr_p^3 |x|^{-\alpha_j})}{m_j(x,r_p)} \|S\|_p^2.$$

Moreover,

$$(3.8) \qquad \int_{B(x,r)} e^{-2p\lambda_x |z-x|^2} \, dm(z) = \frac{1}{p\lambda_x} E(r\sqrt{p\lambda_x}) \leq \frac{\pi}{2p\lambda_x}, \qquad \forall \, r > 0.$$

By combining these estimates, it follows that

$$(3.9) \qquad |S(x)|_{h_p}^2 \leq \|S\|_p^2 \frac{p\lambda_x}{E(r_p\sqrt{p\lambda_x})} \frac{\exp(2^{\alpha_j+1} A_j pr_p^3 |x|^{-\alpha_j})}{m_j(x,r_p)}.$$

Note that

$$\frac{c_1(L,h)_x}{\omega_x} = \frac{2\lambda_x}{\pi \rho_j(x)}.$$

By taking the supremum in (3.9) over $S \in H^0_{(2)}(X \setminus \Sigma, L^p)$ with $\|S\|_p = 1$, we get by (2.1)

$$P_p(x) \leq p \frac{c_1(L,h)_x}{\omega_x} \frac{\pi}{2E(r_p\sqrt{p\lambda_x})} \frac{\rho_j(x)}{m_j(x,r_p)} \exp(2^{\alpha_j+1} A_j pr_p^3 |x|^{-\alpha_j}).$$

By using (3.5), we obtain

$$P_p(x) \leq p \frac{c_1(L,h)_x}{\omega_x} \frac{\pi}{2E(r_p\sqrt{\pi\varepsilon c_0 p})} \frac{\rho_j(x)}{m_j(x,r_p)} \exp(2^{\alpha_j+1} A_j pr_p^3 |x|^{-\alpha_j}).$$

If $r_p\sqrt{p} \geq 1$, then by (3.7),

$$\frac{\pi}{2E(r_p\sqrt{\pi\varepsilon c_0 p})} \leq 1 + C_2' \exp(-2\pi\varepsilon c_0 pr_p^2)$$

with a constant $C_2' > 0$. Moreover, if $pr_p^3 < |x|^{\alpha_j}$, then

$$(3.10) \qquad \exp(2^{\alpha_j+1} A_j pr_p^3 |x|^{-\alpha_j}) \leq \exp(2^{\alpha_j+4} A_j pr_p^3 |x|^{-\alpha_j})$$
$$\leq 1 + C_3' pr_p^3 |x|^{-\alpha_j},$$

with a constant $C_3' > 0$. If, in addition, $r_p < c_0 |x|^{\beta_j}/(2^{\beta_j+2} A_j')$, then applying (3.3), together with these estimates, to the above upper bound on $P_p(x)$ yields

$$P_p(x) \leq p \frac{c_1(L,h)_x}{\omega_x} (1 + C_2' e^{-2\pi\varepsilon c_0 pr_p^2})(1 + C_1' r_p |x|^{-\beta_j})(1 + C_3' pr_p^3 |x|^{-\alpha_j}),$$

provided that

$$0 < r_p < \frac{|x|}{4}, \; r_p\sqrt{p} \ge 1, \; r_p < \frac{c_0|x|^{\beta_j}}{2^{\beta_j+2}A_j'}, \; pr_p^3 < |x|^{\alpha_j}.$$

Set $r_p = p^{-a}$, so $r_p\sqrt{p} = p^{1/2-a}$, $pr_p^3 = p^{1-3a}$. We have shown the following: if

$$\frac{1}{3} < a < \frac{1}{2}, \; p^{-a} < \frac{|x|}{4}, \; p^{-a} < \frac{c_0|x|^{\beta_j}}{2^{\beta_j+2}A_j'}, \; p^{1-3a} < |x|^{\alpha_j},$$

then

$$P_p(x) \le p\frac{c_1(L,h)x}{\omega_x}(1 + C_2'e^{-2\pi\varepsilon c_0 p^{1-2a}})$$
$$\times (1 + C_1'p^{-a}|x|^{-\beta_j})(1 + C_3'p^{1-3a}|x|^{-\alpha_j}).$$

With $a = \frac{3}{8}$, this implies that there exists a constant $C_4' > 0$ such that if

$$p > C_4'\max\{|x|^{-8/3}, |x|^{-8\beta_j/3}, |x|^{-8\alpha_j}\} = C_4'|x|^{-\delta_j},$$

where δ_j is defined in (1.11), then

$$\frac{P_p(x)}{p}\frac{\omega_x}{c_1(L,h)_x} \le 1 + C_4'(p^{-3/8}|x|^{-\beta_j} + p^{-1/8}|x|^{-\alpha_j}).$$

Step 3. We obtain now the lower estimate for $P_p(x)$ if $x \in B(x_j, R_j) \setminus \{x_j\}$. As before, let $r_p \in (0, |x|/4)$ be an arbitrary number which will be specified later. Let $\chi : \mathbb{C} \to [0, 1]$ be a smooth function such that $\chi = 1$ on the unit disc $B(0, 1)$ and $\operatorname{supp}\chi \subset B(0, 2)$. If e_x is the local holomorphic frame of L on $B(x, |x|)$ from *Step 1*, define

$$\chi_p(z) = \rho_j(x)^{-1/2}\chi\left(\frac{z-x}{r_p}\right), \quad F = \chi_p e_x^{\otimes p},$$

so

$$|F(x)|_{h_p}^2 = \rho_j(x)^{-1}e^{-2p\tilde{\varphi}_j(x)} = \rho_j(x)^{-1}.$$

By using (3.2), (3.6), (3.4), and (3.8), we obtain

(3.11) $$\int_{B(x,2r_p)} e^{-2p\tilde{\varphi}_j}\omega \le M_j(x, 2r_p)\exp(2p\max_{B(x,2r_p)}|\widetilde{\psi}_j|)$$
$$\times \int_{B(x,2r_p)} e^{-2p\lambda_x|z-x|^2}\,dm(z)$$
$$\le \frac{\pi M_j(x, 2r_p)}{2p\lambda_x}\exp(2^{\alpha_j+4}A_jpr_p^3|x|^{-\alpha_j}).$$

Since $\chi_p^2 \le \rho_j(x)^{-1}$, this implies

$$(3.12) \qquad \|F\|_p^2 = \int_{B(x,2r_p)} \chi_p^2 e^{-2p\check{\phi}_j} \omega$$

$$\le \frac{\pi}{2p\lambda_x} \frac{M_j(x,2r_p)}{\rho_j(x)} \exp(2^{\alpha_j+4} A_j p r_p^3 |x|^{-\alpha_j}).$$

Note that any non-compact Riemann surface admits a complete Kähler metric, since it is a Stein manifold by Behnke-Stein [3]. Hence, if X is a Riemann surface and Σ is a discrete closed set, then $X \setminus \Sigma$ admits a complete Kähler metric. By assumptions (B) and (C), $\mathrm{Ric}_\omega \ge -2\pi B\omega$, $c_1(L^p, h_p) \ge 2p\varepsilon\omega$ on $X \setminus \Sigma$. So, if $p \ge B/\varepsilon$, we can solve the $\bar{\partial}$-equation using [17] (see Theorem 2.1): if $\theta = \bar{\partial}F \in L_{0,1}^2(X \setminus \Sigma, L^p, \mathrm{loc})$, there exists $G \in L_{0,0}^2(X \setminus \Sigma, L^p, \mathrm{loc})$ such that $\bar{\partial}G = \theta = \bar{\partial}F$ and

$$\|G\|_p^2 = \int_{X \setminus \Sigma} |G|_{h_p}^2 \omega \le \frac{1}{p\varepsilon} \int_{X \setminus \Sigma} |\theta|_{h_p}^2 \omega.$$

Since

$$|\bar{\partial}\chi_p|^2 \le \|\bar{\partial}\chi\|^2 \rho_j(x)^{-1} r_p^{-2},$$

where $\|\bar{\partial}\chi\|$ denotes the maximum of $|\bar{\partial}\chi|$, we get by (3.11)

$$\int_{X \setminus \Sigma} |\theta|_{h_p}^2 \omega = \int_{B(x,2r_p)} |\bar{\partial}\chi_p|^2 e^{-2p\check{\phi}_j} \omega$$

$$\le \frac{\pi \|\bar{\partial}\chi\|^2}{2\lambda_x p r_p^2} \frac{M_j(x,2r_p)}{\rho_j(x)} \exp(2^{\alpha_j+4} A_j p r_p^3 |x|^{-\alpha_j}).$$

Thus,

$$(3.13) \qquad \|G\|_p^2 \le \frac{1}{p\varepsilon} \frac{\pi \|\bar{\partial}\chi\|^2}{2\lambda_x} \frac{1}{p r_p^2} \frac{M_j(x,2r_p)}{\rho_j(x)} \exp(2^{\alpha_j+4} A_j p r_p^3 |x|^{-\alpha_j}).$$

Since $\bar{\partial}G = \bar{\partial}F = 0$ on $B(x, r_p)$, G is holomorphic on $B(x, r_p)$. Hence, the estimate (3.9) applies to G on $B(x, r_p)$ and gives

$$|G(x)|_{h_p}^2 \le \|G\|_p^2 \frac{p\lambda_x}{E(r_p\sqrt{p\lambda_x})} \frac{\exp(2^{\alpha_j+1} A_j p r_p^3 |x|^{-\alpha_j})}{m_j(x,r_p)}.$$

By using (3.13), we obtain

$$|G(x)|_{h_p}^2 \le \rho_j(x)^{-1} \frac{\pi \|\bar{\partial}\chi\|^2}{2\varepsilon E(r_p\sqrt{p\lambda_x})} \frac{1}{p r_p^2} \frac{M_j(x,2r_p)}{m_j(x,r_p)} \exp(2^{\alpha_j+5} A_j p r_p^3 |x|^{-\alpha_j}).$$

If $r_p\sqrt{p} \ge 1$, then $E(r_p\sqrt{p\lambda_x}) \ge E(\sqrt{\pi\varepsilon c_0})$ by (3.5). So,

$$(3.14) \qquad |G(x)|_{h_p}^2 \le \rho_j(x)^{-1} \frac{C_5'}{p r_p^2} \frac{M_j(x,2r_p)}{m_j(x,r_p)} \exp(2^{\alpha_j+5} A_j p r_p^3 |x|^{-\alpha_j}),$$

with a constant $C_5' > 0$.

Set $S = F - G \in H^0(X \setminus \Sigma, L^p)$, as $\bar{\partial} S = \bar{\partial} F - \bar{\partial} G = 0$. By (3.12) and (3.13), we have

$$\|S\|_p^2 \le (\|F\|_p + \|G\|_p)^2$$

$$\le \frac{\pi}{2p\lambda_x} \frac{M_j(x, 2r_p)}{\rho_j(x)} \exp(2^{\alpha_j + 4} A_j p r_p^3 |x|^{-\alpha_j}) \left(1 + \frac{\|\bar{\partial}\chi\|}{r_p \sqrt{p\varepsilon}}\right)^2.$$

Moreover, if

$$Q(x, r_p) := \frac{\sqrt{C_5'}}{r_p \sqrt{p}} \left(\frac{M_j(x, 2r_p)}{m_j(x, r_p)}\right)^{1/2} \exp(2^{\alpha_j + 4} A_j p r_p^3 |x|^{-\alpha_j}) < 1,$$

then, by using (3.14), it follows that

$$|S(x)|_{h_p}^2 \ge (|F(x)|_{h_p} - |G(x)|_{h_p})^2 \ge \rho_j(x)^{-1} (1 - Q(x, r_p))^2.$$

Therefore,

$$P_p(x) \ge \frac{|S(x)|_{h_p}^2}{\|S\|_p^2}$$

$$\ge p \frac{c_1(L, h)_x}{\omega_x} \frac{\rho_j(x)}{M_j(x, 2r_p)} \frac{(1 - Q(x, r_p))^2}{\left(1 + \frac{\|\bar{\partial}\chi\|}{r_p \sqrt{p\varepsilon}}\right)^2 \exp(2^{\alpha_j + 4} A_j p r_p^3 |x|^{-\alpha_j})}.$$

If $r_p < c_0 |x|^{\beta_j} / (2^{\beta_j + 2} A_j')$ and $p r_p^3 < |x|^{\alpha_j}$, then by (3.3) and (3.10),

$$Q(x, r_p) \le \frac{\sqrt{C_5'}}{r_p \sqrt{p}} (1 + C_1' r_p |x|^{-\beta_j})^{1/2} (1 + C_3' p r_p^3 |x|^{-\alpha_j}) \le \frac{C_6'}{r_p \sqrt{p}},$$

for some constant $C_6' > 1$. We may assume that $C_6' \ge \|\bar{\partial}\chi\| / \sqrt{\varepsilon}$. Applying this estimate on $Q(x, r_p)$, together with (3.3) and (3.10), to the above lower bound on $P_p(x)$ yields

$$(3.15) \quad P_p(x) \ge p \frac{c_1(L, h)_x}{\omega_x}$$

$$\times \frac{\left(1 - \frac{C_6'}{r_p \sqrt{p}}\right)^2}{(1 + C_1' r_p |x|^{-\beta_j})(1 + C_3' p r_p^3 |x|^{-\alpha_j}) \left(1 + \frac{C_6'}{r_p \sqrt{p}}\right)^2},$$

if

$$p > \frac{B}{\varepsilon}, \ 0 < r_p < \frac{|x|}{4}, \ r_p \sqrt{p} > C_6', \ r_p < \frac{c_0 |x|^{\beta_j}}{2^{\beta_j + 2} A_j'}, \ p r_p^3 < |x|^{\alpha_j}.$$

We again let $r_p = p^{-a}$, so $r_p\sqrt{p} = p^{1/2-a}$, $pr_p^3 = p^{1-3a}$. Then, (3.15) implies the following: if

$$\frac{1}{3} < a < \frac{1}{2}, \ p > \frac{B}{\varepsilon}, \ p^{1/2-a} > C_6', \ p^{-a} < \frac{|x|}{4}, \ p^{-a} < \frac{c_0|x|^{\beta_j}}{2^{\beta_j+2}A_j'}, \ p^{1-3a} < |x|^{\alpha_j},$$

then

$$P_p(x) \geq p\,\frac{c_1(L,h)_x}{\omega_x}\,\frac{(1 - C_6'p^{a-1/2})^2}{(1 + C_1'p^{-a}|x|^{-\beta_j})(1 + C_3'p^{1-3a}|x|^{-\alpha_j})(1 + C_6'p^{a-1/2})^2}.$$

By taking $a = \frac{3}{8}$, we conclude that there exists a constant $C_7' > 0$ such that if

$$p > C_7' \max\{|x|^{-8/3}, |x|^{-8\beta_j/3}, |x|^{-8\alpha_j}\} = C_7'|x|^{-\delta_j},$$

then

$$\frac{P_p(x)}{p}\,\frac{\omega_x}{c_1(L,h)_x} \geq 1 - C_7'(p^{-3/8}|x|^{-\beta_j} + p^{-1/8}|x|^{-\alpha_j}).$$

This concludes the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. Here, $L = K_X$. The proof is analogous to the proof of Theorem 1.1, with the only difference that we apply Theorem 2.2 instead of Theorem 2.1 in order to solve the $\bar{\partial}$-equation in *Step 3*. For the leading term of the expansion, observe that

$$c_1(K_Y, h^{K_Y}) = -\frac{R_\omega}{2\pi}\omega \quad \text{by (2.6)}. \qquad \square$$

Proof of Corollary 1.3. Let C_j be the constant from Theorem 1.1. Then,

$$\text{dist}(x, x_j) > (C_j/p)^{\eta/\delta_j} \geq (C_j/p)^{1/\delta_j}.$$

Hence, by Theorem 1.1, and by using that $\alpha_j/\delta_j \leq \frac{1}{8}$ and $\beta_j/\delta_j \leq \frac{3}{8}$, we obtain

$$\left|\frac{P_p(x)}{p}\,\frac{\omega_x}{c_1(L,h)_x} - 1\right| \leq C_j(p^{-1/8}\,\text{dist}(x, x_j)^{-\alpha_j} + p^{-3/8}\,\text{dist}(x, x_j)^{-\beta_j})$$

$$\leq C_j(p^{-1/8}(p/C_j)^{\eta\alpha_j/\delta_j} + p^{-3/8}(p/C_j)^{\eta\beta_j/\delta_j})$$

$$\leq C_j(p^{-(1-\eta)/8} + p^{-3(1-\eta)/8}) \leq 2C_jp^{-(1-\eta)/8}. \qquad \square$$

We give now a semi-global version of Theorem 1.1. Let $K \subset X$ be a compact set and let $\Sigma \cap K = \{x_1, \ldots, x_m\}$. Fix a constant $R_0 > 0$ with the property that

every point $x \in K$ has a coordinate neighborhood U_x centered at x such that the coordinate disc $B(x, 2R_0) \Subset U_x$ and

$$\operatorname{dist}(B(x_j, 2R_0), \Sigma \setminus \{x_j\}) \geq R_0, \quad 1 \leq j \leq m.$$

Define

$$\alpha := \max\{\alpha_j : 1 \leq j \leq m\}, \quad A := \max\{A_j : 1 \leq j \leq m\},$$
$$\beta := \max\{\beta_j : 1 \leq j \leq m\}, \quad A' := \max\{A'_j : 1 \leq j \leq m\},$$

where α_j, A_j and β_j, A'_j are as in (1.8) and (1.10), respectively. We have the following result.

Theorem 3.1. *Let $(X, \Sigma, \omega, L, h)$ verify assumptions* (A)–(C), *and let $K \subset X$ be a compact set with $\Sigma \cap K = \{x_1, \ldots, x_m\}$. Assume that h and ω satisfy* (1.8) *and* (1.10), *respectively.*

Then, there exists a constant $C = C(K, \Sigma, \omega, L, h) > 1$ such that if $x \in K \setminus \Sigma$ and $p > C \operatorname{dist}(x, \Sigma)^{-\delta}$, where $\delta = \max\{\frac{8}{3}, 8\beta/3, 8\alpha\}$, then

$$\left| \frac{P_p(x)}{p} \frac{\omega_x}{c_1(L, h)_x} - 1 \right| \leq C(p^{-1/8} \operatorname{dist}(x, \Sigma)^{-\alpha} + p^{-3/8} \operatorname{dist}(x, \Sigma)^{-\beta}).$$

Proof. Let

$$K' := K \setminus \bigcup_{j=1}^{m} B(x_j, R_j).$$

There exist a positive number $r_0 < R_0$ and points $y_j \in K'$, $1 \leq j \leq m'$ such that $K' \subset \bigcup_{j=1}^{m'} B(y_j, r_0)$ and

$$\operatorname{dist}(B(y_j, 2r_0), \Sigma) \geq r_0, \quad 1 \leq j \leq m'.$$

We have to estimate $P_p(x)$ for $x \in B(y_j, r_0)$. Note that h is of class C^3 and ω is of class C^2 in a neighborhood of $\bar{B}(y_j, 2r_0)$. As in (1.9), we write $\omega(z) = \rho_j(z) \, dm(z)$, where $\rho_j \geq c'_0$ on $B(y_j, 2r_0)$, $1 \leq j \leq m'$, with some constant $c'_0 > 0$. If $M_j(x, r)$ and $m_j(x, r)$ are defined as in *Step 1* of the proof of Theorem 1.1, we have that (3.3) holds for $r < r_0/2$ with $\beta_j = 0$ and some constant $C'_1 > 0$. Next, we can choose a holomorphic frame e_x of L on $B(x, r_0) \subset B(y_j, 2r_0)$ for which the corresponding weight $\tilde{\varphi}_j$ of h verifies (3.6) and

$$\max\{|\tilde{\psi}_j(z)| : z \in B(x, r)\} \leq C'_8 r^3, \quad r < r_0,$$

with some constant $C'_8 > 0$. Moreover, $\lambda_x \geq \pi \varepsilon c'_0$.

Proceeding as in *Step 2* and *Step 3* of the previous proof, we show that there exist $p_0 \in \mathbb{N}$ and a constant $C'_9 > 0$ such that if $\frac{1}{3} < a < \frac{1}{2}$, $p > p_0$, and $p^{1/2-a} > C'_9$, then

(3.16)
$$P_p(x) \le p\frac{c_1(L,h)_x}{\omega_x}(1 + C'_9(e^{-2\pi\varepsilon c'_0 p^{1-2a}} + p^{-a} + p^{1-3a})),$$

$$P_p(x) \ge p\frac{c_1(L,h)_x}{\omega_x}(1 - C'_9(p^{a-1/2} + p^{-a} + p^{1-3a})).$$

By choosing $a = \frac{3}{8}$ in (3.16), we see that there exists $p'_0 \in \mathbb{N}$ such that if $p > p'_0$, then

$$\left| \frac{P_p(x)}{p} \frac{\omega_x}{c_1(L,h)_x} - 1 \right| \le 3C'_9 p^{-1/8}.$$

Together with Theorem 1.1, this completes the proof of Theorem 3.1. □

4. APPLICATIONS

In this section we examine some situations when the parameters α and β can be explicitly calculated. We consider metrics with logarithmic singularities and hyperbolic metrics with parabolic singularities (cusps) or conical singularities.

4.1. Metrics with logarithmic singularities. Let X be a Riemann surface and $\Sigma \subset X$ be a discrete closed subset. Let ω be a Hermitian metric of class C^2 on X. Let (L, h) be a holomorphic line bundle on X with singular metric h (see [18], [34, p. 97]). We assume that h is smooth on $X \setminus \Sigma$ and has weights with logarithmic singularities at Σ; that is, in (1.7) we have $\varphi_j(z) = v_j \log |z| + \psi_j(z)$, with $v_j \ge 0$ and $\psi_j \in C^3(B(x_j, 2R_j)$, where $\varphi_j(z) = -\log|e_j|_h$ is a local weight around x_j.

In this situation, we have $\alpha = \beta = 0$, so from Theorem 1.1 we obtain immediately the following result.

Corollary 4.1. Let X be a Riemann surface and $\Sigma \subset X$ be a discrete closed subset. Let ω be a Kähler metric of class C^2 on X such that $\mathrm{Ric}_\omega \ge -2\pi B\omega$, for some $B > 0$. Let (L, h) be a holomorphic line bundle on X, where h has weights with logarithmic singularities at Σ, it is smooth on $X \setminus \Sigma$, and $c_1(L, h) \ge 2\varepsilon\omega$ holds in the sense of currents on X, for some $\varepsilon > 0$. Let $P_p(x)$ be the Bergman kernel function of $H^0_{(2)}(X, L^p)$. Then, for any $x_j \in \Sigma$ and any compact set $K \subset X$ with $K \cap \Sigma = \{x_j\}$, there exists $C_j = C_j(K) > 0$ such that if $x \in K$, then

$$\left| \frac{P_p(x)}{p} \frac{\omega_x}{c_1(L,h)_x} - 1 \right| \le C_j p^{-1/8}, \quad \text{for } p > C_j \, \mathrm{dist}(x, x_j)^{-8/3}.$$

Note that on a compact Riemann surface, the Ricci curvature is automatically bounded below. Given a compact Riemann surface, a line bundle (L, h_0) with

smooth metric of positive curvature and a finite set $\Sigma \subset X$, we can always construct a singular Hermitian metric h with the properties of Corollary 4.1. We can take $h = h_0 \exp(-\varepsilon \psi)$, where $\varepsilon > 0$ is small enough and ψ a smooth function on $X \setminus \Sigma$ with $\psi(z) = \log |z - x_j|$ in a neighborhood of $x_j \in \Sigma$.

Consider a non-compact Riemann surface X endowed with a Kähler metric of class C^2 on X such that $\mathrm{Ric}_\omega \geq -2\pi B\omega$, for some $B > 0$. One can consider, for example, a hyperbolic domain in $\mathbb{P}^1$ endowed with the Poincaré metric, or a domain in $\mathbb{C}$ endowed with the Euclidean metric. Recall that a hyperbolic domain $X \subset \mathbb{P}^1$ is a domain such that $\mathbb{P}^1 \setminus X$ contains at least three points. Since X is Stein, X admits a strictly subharmonic exhaustion function φ. Let $\Sigma \subset X$ be a discrete closed subset. The metric $\exp(-\chi(\varphi) - \psi)$ on the trivial bundle $L = X \times \mathbb{C}$ satisfies the conditions of Corollary 4.1 for some convex increasing function $\chi : \mathbb{R} \to \mathbb{R}$ and for ψ as above.

Remark 4.2. In terms of the Nadel multiplier ideal sheaves (see, e.g., Definition 2.3.1 in [34]), we deal in Corollary 4.1 with a singular metric h on L with $c_1(L, h) \geq 2\varepsilon\omega$ such that the zero variety of the multiplier ideal sheaves $\mathcal{I}(h)$ of h equals Σ. We have $H^0_{(2)}(X, L^p) = H^0(X, \mathcal{O}(L^p) \otimes \mathcal{I}(h^p))$, the space of global sections of the sheaf $\mathcal{O}(L^p) \otimes \mathcal{I}(h^p)$. By [25, Theorem 1.8], if X is compact, the Bergman kernel function $P_p(x)$ of $H^0(X, \mathcal{O}(L^p) \otimes \mathcal{I}(h^p))$ has the full asymptotic expansion (1.3) on compact sets of $X \setminus \Sigma$.

4.2. Poincaré metric on the punctured disc. Let us consider $X = \mathbb{D}$ and $\Sigma = \{0\}$. We endow the punctured disc $Y := \mathbb{D}^* = \mathbb{D} \setminus \{0\}$ with the Poincaré metric $ds^2 = (|z| \log |z|^2)^{-2} |dz|^2$, that is,

$$(4.1) \qquad \omega = \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2 (\log |z|^2)^2}.$$

This is a complete Kähler metric with Gauss curvature $R_\omega = -4$, or equivalently, $\mathrm{Ric}_\omega = -4\omega$ (see (2.6)). The metric ω fulfills condition (B').

Lemma 4.3. We have $\alpha = \beta = 3$.

Proof. We use (1.13), (1.7), (1.8) and (1.9), (1.10). We have

$$\varphi(z) = -\log |z| - \log |\log |z|^2 | - \frac{1}{2} \log 2,$$

hence,

$$\psi(z) = -\log |\log(x^2 + y^2)| - \frac{1}{2} \log 2,$$

with $z = x + iy$. By explicitly calculating $\partial_x^3 \psi$ and $\partial_x^2 \partial_y \psi$, we obtain that for any $r \in (0, 1)$ there exists a constant $C = C_r$ such that $|\partial_x^3 \psi(z)| \leq C|z|^{-3}$, $|\partial_x^2 \partial_y \psi(z)| \leq C|z|^{-3}$ for $|z| \leq r$. By symmetry, we obtain the same estimates for $\partial_y^3 \psi$, $\partial_x \partial_y^2 \psi$. Thus, $\alpha = 3$.

We have $\rho(z) = (|z| \log |z|^2)^{-2}$. Note that, by direct calculation, we obtain $|\partial_z \rho(z)| \leq |z|^{-3}$, and by symmetry, we obtain the same estimate for $|\partial_{\bar{z}} \rho(z)|$. Thus, $\beta = 3$. $\qquad\blacksquare$

By Theorem 1.2 and Lemma 4.3, we obtain the following result.

Corollary 4.4. *Let* $P_p(x)$ *be the Bergman kernel function of* $H^0_{(2)}(\mathbb{D}^*, K^p_{\mathbb{D}^*})$. *Let* $r \in (0, 1)$. *Then, there exists* $C = C(r) > 1$ *such that if* $0 < |x| \leq r$ *and* $p > C|x|^{-24}$, *we have*

$$\left| \frac{P_p(x)}{p} \cdot \frac{\pi}{2} - 1 \right| \leq Cp^{-1/8}|x|^{-3}.$$

An explicit expression of the Bergman kernel on the punctured unit disc was derived in [2, (3.7)]. By modifying slightly [2, Proposition 3.3], we can show that

(4.2) $$P_p(x) = \frac{2}{\pi}p - \frac{4}{\pi} + O(e^{-cp}), \quad p \to \infty,$$

outside a fixed neighborhood of the origin (or more generally, outside a shrinking neighborhood [2, (3.9)]). Note that in [2], the metric (4.1) is normalized such that its Gauss curvature equals -2 and one works with a line bundle L satisfying $2\pi c_1(L, h) = \omega$; hence, the first two coefficients of the expansion differ from those in [2] (see also Remark 4.8). Corollary 4.4 is concerned with the behavior of the Bergman kernel inside a neighborhood of the origin.

4.3. Hyperbolic metrics with parabolic singularities. We consider the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \infty$ with $m \geq 3$ marked points $\Sigma = \{x_1, \ldots, x_m\}$. By using a Möbius map we can assume that $x_m = \infty$. Let $Y = \mathbb{P}^1 \setminus \Sigma$. By the uniformization theorem, Y is the quotient $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the upper half-plane, and $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a finitely generated torsion-free Fuchsian group acting on $\mathbb{H}$ by linear fractional transformations. The Poincaré metric on $\mathbb{H}$ descends to Y and gives a complete Kähler metric ds^2 of constant curvature -1. By [39, Lemma 2], the metric ds^2 has the form $ds^2 = e^\varphi |dz|^2$, where φ is a smooth function on Y verifying $\varphi_{z\bar{z}} = \frac{1}{2}e^\varphi$ and

$$\varphi(z) = \begin{cases} -2\log|z - x_j| - 2\log|\log|z - x_j|| + O(1), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{as } z \to x_j, \ 1 \leq j \leq m - 1, \\ -2\log|z| - 2\log|\log|z|| + O(1), \quad \text{as } z \to x_m = \infty. \end{cases}$$

Lemma 4.5. *For each* $j = 1, \ldots, m$, *we have* $\alpha_j = \beta_j = 3$.

Proof. Since the curvature is constant -1, we have $\varphi_{z\bar{z}} = \frac{1}{2}e^\varphi = \rho$. Without loss of generality, we may assume $j = 1$ and $x_1 = 0$. Near $z_1 = 0$, we have

$$\varphi(z) = -2\log|z| - 2\log|\log|z|| + O(1).$$

By [39, Lemma 2], there exist $C > 0$ such that near 0 we have

$$(4.3) \qquad\qquad |\varphi_z| \leq \frac{C}{|z|}.$$

In the rest of the proof, we denote by C a constant that may change from line to line. By [39, Lemma 2] (see also [39, (1.7), (1.8)]), there exists $c_1, \ldots, c_{m-1}$ such that

$$(4.4) \qquad\qquad \varphi_{zz} - \frac{1}{2}\varphi_z^2 = \sum_{i=1}^{m-1} \left(\frac{1}{2(z - x_i)^2} + \frac{c_i}{z - x_i} \right)$$

(the righthand side is actually the Schwarzian derivative of the inverse of the projection $\mathbb{H} \to \mathbb{H}/\Gamma = Y$, and $c_1, \ldots, c_{m-1}$ are called accessory parameters). By (4.3), (4.4),

$$|\varphi_{zz}| \leq \frac{C}{|z|^2} \quad \text{near } z_1 = 0,$$

and

$$\varphi_{zzz} = \varphi_z \varphi_{zz} - \sum_{i=1}^{m-1} \left(\frac{1}{(z - z_i)^3} + \frac{c_i}{(z - z_i)^2} \right).$$

Hence,

$$|\varphi_{zzz}| \leq \frac{C}{|z|^3} \quad \text{near } z_1 = 0.$$

Now, $\rho_z = \varphi_{zz\bar{z}} = \frac{1}{2}e^\varphi \varphi_z$; thus,

$$|\rho_z| = |\varphi_{zz\bar{z}}| \leq \frac{C}{|z|^3} \quad \text{near } z_1 = 0.$$

Further, $\psi = \varphi + 2\log|z|$; hence, $\psi_z = \varphi_z + z^{-1}$ and $\psi_{zzz} = \varphi_{zzz} + 2z^{-3}$, $\psi_{zz\bar{z}} = \varphi_{zz\bar{z}}$. We deduce that $\alpha_1 = \beta_1 = 3$. $\qquad\square$

By Theorem 1.2 and Lemma 4.5, we obtain the following result.

Corollary 4.6. *Let $Y := \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \{x_1, \ldots, x_m\}$, $m \geq 3$ be endowed with the induced Poincaré metric of constant curvature -1. Let $P_p(x)$ be the Bergman kernel function of $H^0_{(2)}(Y, K_Y^p)$. Then, for any $j = 1, \ldots, m$, and any compact set $K \subset \mathbb{P}^1$ with $K \cap \Sigma = \{x_j\}$, there exists $C_j = C_j(K) > 0$ such that if $x \in K$, then*

$$\left| \frac{2\pi P_p(x)}{p} - 1 \right| \leq C_j p^{-1/8} \operatorname{dist}(x, x_j)^{-3}, \quad \text{for } p > C_j \operatorname{dist}(x, x_j)^{-24}.$$

Note that the space $H^0_{(2)}(Y, K^p_Y)$ is the space of cusp forms of weight $2p$ on Y, so $P_p(x)$ is the Bergman kernel of the cusp forms.

Let us now point out the interpretation of Corollaries 4.4 and 4.6 in terms of classical Bergman kernels for function spaces. Let us consider a general hyperbolic domain $Y \subset \mathbb{C}$, that is, $\mathbb{C} \setminus Y$ contains at least two points. As above, Y admits an induced Poincaré metric $\mathrm{d}s^2 = \rho(z)|\mathrm{d}z|^2$ of constant curvature -1. For $p \in \mathbb{N}$, we define the Petersson scalar product

$$(4.5) \qquad \langle f, g \rangle_p := \int_D f(z)\overline{g(z)}\rho^{1-p}(z)\,\mathrm{d}m(z)$$

on the space $L^2_p(Y) := \{f : Y \to \mathbb{C} \text{ measurable} : \|f\|^2_p = \langle f, f \rangle_p < \infty\}$ (see, e.g., [31, p. 88]), where $\mathrm{d}m(z) = (i/2)\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$ is the Euclidean volume form. Set

$$\mathcal{A}^2_p(Y) := \{f \in L^2_p(Y) : f \text{ holomorphic}\},$$

which is a closed subspace of $L^2_p(Y)$. Denote by $\Pi_p : L^2_p(Y) \to \mathcal{A}^2_p(Y)$ the orthogonal projection and by $\Pi_p(\cdot, \cdot)$ its reproducing kernel. If $\{f_j\}$ is an orthonormal basis of $(\mathcal{A}^2_p(Y), \langle \cdot, \cdot \rangle)$, then $\Pi_p(z, w) = \sum_j f_j(z)\overline{f_j(w)}$. The restriction on the diagonal $z \mapsto \Pi_p(z) := \sum_j |f_j(z)|^2$ is the Bergman kernel function for the Petersson scalar product. Note that any element $S \in H^0_{(2)}(Y, K^p_Y)$ is of the form $S = f(\mathrm{d}z)^{\otimes p}$, where $f \in \mathcal{O}(Y)$. Let $S' = f'(\mathrm{d}z)^{\otimes p}$ be a further element of $H^0_{(2)}(Y, K^p_Y)$. By (2.5),

$$\langle S, S' \rangle_\omega \omega = f\overline{f'}|(\mathrm{d}z)^{\otimes p}|^2_\omega \omega = 2^p f\overline{f'}\rho^{1-p}\,\mathrm{d}m(z),$$

hence, $\langle S, S' \rangle = 2^p \langle f, f' \rangle_p$. The map

$$\mathcal{A}^2_p(Y) \to H^0_{(2)}(Y, K^p_Y),$$
$$f \mapsto 2^{-p/2} f(\mathrm{d}z)^{\otimes p}$$

is an isometry. Therefore,

$$(4.6) \qquad P_p(z) = \Pi_p(z)\rho(z)^{-p}, \quad z \in Y.$$

By Corollary 4.6 and (4.6), we obtain the following result.

Corollary 4.7. *Let $Y = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \{x_1, \ldots, x_m\}$, $m \geq 3$, and write $\mathrm{d}s^2 = \rho(z)|\mathrm{d}z|^2$ for the induced Poincaré metric of constant curvature -1 on Y. Let $\Pi_p(z)$ be the Bergman kernel function associated with the Petersson scalar product (4.5). Then, for any $j = 1, \ldots, m$, and any compact set $K \subset \mathbb{P}^1$ with $K \cap \Sigma = \{x_j\}$, there exists $C_j = C_j(K) > 0$ such that if $z \in K$, then*

$$\left| \frac{2\pi \Pi_p(z)\rho(z)^{-p}}{p} - 1 \right| \leq C_j p^{-1/8} \operatorname{dist}(z, x_j)^{-3}, \quad \text{for } p > C_j \operatorname{dist}(z, x_j)^{-24}.$$

A similar statement can be proved for the Bergman kernel $\Pi_p(z)$ of the Petersson scalar product on $\mathbb{D}^*$ by using Corollary 4.4 (see [21, Section 5] for the explicit formula of the Bergman kernel $\Pi_p(z)$ for $\mathbb{D}^*$).

Remark 4.8. Let $Y \subset \mathbb{P}^1$ be an arbitrary hyperbolic domain endowed with the induced Poincaré metric of constant Gauss curvature -1. Let $P_p(x)$ be the Bergman kernel function of $H^0_{(2)}(Y, K_Y^p)$. By Theorem 6.1.1 of [34], Theorem 3.11 of [35], $P_p(x)$ has a full asymptotic expansion (1.3) on compact sets of Y. Actually, by [2, Corollary 2.4] the expansion reads

$$P_p(x) = \frac{p}{2\pi} - \frac{1}{4\pi} + O(p^{-\infty}), \quad p \to \infty.$$

Indeed, $b_0 = 1/(2\pi)$ by (2.6). The Gauss curvature of $c_1(K_Y, h^{K_Y})$ being -2π, the scalar curvature of $c_1(K_Y, h^{K_Y})$ is $r^Y = -4\pi$ by (2.8). Hence, $b_1 = -1/(4\pi)$. All other coefficients vanish by [2, Corollary 2.4], so the remainder is $O(p^{-\infty})$.

If $Y = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \{x_1, \ldots, x_m\}$, $m \geq 3$, we refer to [2] for a weighted estimate near the punctures for the global Bergman kernel compared to the Bergman kernel of the punctured disc.

4.4. Hyperbolic metrics with conical singularities. We consider again the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \infty$ with m marked points $\Sigma = \{x_1, \ldots, x_m\}$, $m \geq 3$. By using a Möbius map we can assume that $x_m = \infty$. Suppose that a_j, $1 \leq j \leq m$ are real numbers such that

$$(4.7) \qquad\qquad a_j < 1, \quad \sum_{j=1}^{m} a_j > 2.$$

Then,
$$Y := \mathbb{P}^1 \setminus \{x_1, \ldots, x_m\} = \mathbb{C} \setminus \{x_1, \ldots, x_{m-1}\}$$

admits a unique Kähler metric of constant curvature -1, which on $\mathbb{P}^1$ is with conical singularities of order a_j (or angle $2\pi(1 - a_j)$) at x_j, $1 \leq j \leq m$ (see [40, Section 2] and references therein). This metric has the form $ds^2 = e^{\varphi}|dz|^2$, where φ is a smooth function on Y verifying

$$\varphi_{z\bar{z}} = \frac{1}{2} e^{\varphi},$$

$$(4.8) \qquad \varphi(z) = \begin{cases} -2a_j \log|z - x_j| + O(1), & \text{as } z \to x_j, \ 1 \leq j \leq m-1, \\ -2(2 - a_m) \log|z| + O(1), & \text{as } z \to x_m = \infty. \end{cases}$$

Lemma 4.9. *For each $j = 1, \ldots, m$, we have $\alpha_j = \beta_j = 1 + 2a_j$.*

Proof. Since the curvature is constant -1, we have $\varphi_{z\bar{z}} = \frac{1}{2}e^{\varphi} = \rho$. Without loss of generality, we may assume $j = 1$ and $x_1 = 0$. By [40, Lemma 2 and (9)], we have

$$(4.9) \qquad e^{\varphi} = \frac{4|w'|}{(1 - |w|^2)^2},$$

with $w(z) = z^{1-a_1}g(z)$, g holomorphic near $x_1 = 0$, $g(0) \neq 0$.

The function w is a multi-valued meromorphic function on $\mathbb{P}^1$ with ramification points at $\{x_1, \ldots, x_m\}$, and it becomes single-valued on the universal cover of Y.

Let us consider the function $F(z) = (1 - a_1)g(z) + zg'(z)$, holomorphic near 0, $F(0) = (1 - a_1)g(0) \neq 0$. By (4.9),

$$\varphi(z) = -2a_1 \log|z| + \psi(z) + \log 4, \quad \text{where } \psi(z) = \log|F(z)|^2 - 2\log(1 - |w|^2).$$

Let us denote $\bar{\psi}(z) = -\log(1 - |w|^2)$. We have $\psi_z = (F'/F) + 2\bar{\psi}_z$, with (F'/F) holomorphic near 0. Thus, we only have to estimate the derivatives of $\bar{\psi}$. By a direct computation, we obtain

$$\bar{\psi}_{zzz} = \frac{2(w'\bar{w})^3}{(1 - |w|^2)^3} + \frac{3w'w''\bar{w}^2}{(1 - |w|^2)^2} + \frac{w'''\bar{w}}{(1 - |w|^2)}.$$

Taking into account (4.9), there exists $C > 0$ such that near 0 we have

$$(4.10) \qquad \begin{aligned} |w| &\leq C|z|^{1-a_1}, & |w'| &\leq C|z|^{-a_1}, \\ |w''| &\leq C|z|^{-1-a_1}, & |w'''| &\leq C|z|^{-2-a_1}. \end{aligned}$$

By (4.9) and (4.10), we infer that near 0,

$$|\bar{\psi}_{zzz}| \leq C(|z|^{3-6a_1} + |z|^{1-4a_1} + |z|^{-1-2a_1}) \leq C|z|^{-1-2a_1}.$$

We have used here that $a_1 < 1$. The previous estimate implies

$$(4.11) \qquad |\psi_{zzz}| \leq C|z|^{-1-2a_1}.$$

We estimate now ρ_z and $\psi_{zz\bar{z}}$ simultaneously. We have

$$\varphi_{z\bar{z}} = \psi_{z\bar{z}} = 2\bar{\psi}_{z\bar{z}} = \frac{1}{2}e^{\varphi} = \rho$$

so $\rho_z = \psi_{zz\bar{z}} = \frac{1}{2}e^{\varphi}\varphi_z$. On the other hand, $\varphi_z = -a_1/z + F'/F + 2\bar{\psi}_z$, where $\bar{\psi}_z = w'\bar{w}/(1 - |w|^2)$, so $|\bar{\psi}_z| \leq C|z|^{1-2a_1}$. We deduce that $|\varphi_z| \leq C|z|^{-1}$. Since $e^{\varphi} \leq C|z|^{-2a_1}$, we deduce

$$(4.12) \qquad |\rho_z| = |\psi_{zz\bar{z}}| \leq C|z|^{-1-2a_1}.$$

By symmetry, we obtain estimate similar to those in (4.11) and (4.12) for $\psi_{\bar{z}\bar{z}\bar{z}}$ and $\rho_{\bar{z}}$, $\psi_{z\bar{z}\bar{z}}$, which show that $\alpha_1 = \beta_1 = 1 + 2a_1$. $\qquad\square$

The $(1,1)$-form associated with ds^2 is $\omega = (i/2)e^\varphi \, dz \wedge d\bar{z}$; thus,

$$\mathrm{Ric}_\omega = -i\,\partial\,\bar\partial \log \frac{e^\varphi}{2} = -i\varphi_{z\bar{z}}\, dz \wedge d\bar{z} = -\omega.$$

Note that ω induces a Hermitian metric h_ω on $K_Y = K_{\mathbb{P}^1}|_Y = \mathcal{O}_{\mathbb{P}^1}(-2)|_Y$ with curvature

$$c_1(K_Y, h_\omega) = -\frac{1}{2\pi}\,\mathrm{Ric}_\omega = \frac{1}{2\pi}\omega.$$

In the chart $\mathbb{C}$, the metric h_ω has weight

$$\varphi_0 = \frac{1}{2}\log\frac{e^\varphi}{2} = \frac{\varphi}{2} - \frac{\log 2}{2}.$$

By letting $z = 1/\zeta$, we obtain in coordinate ζ near $x_m = \infty$ that

$$ds^2 = e^{\varphi(1/\zeta)}\frac{|d\zeta|^2}{|\zeta|^4} = e^{\varphi'(\zeta)}|d\zeta|^2,$$

$$\varphi'(\zeta) = \varphi\left(\frac{1}{\zeta}\right) - 4\log|\zeta| = -2a_m \log|\zeta| + O(1) \quad \text{as } \zeta \to 0.$$

As before, in this chart the weight of h_ω is $\varphi_1 = \varphi'/2 - \log 2/2$. It follows that h_ω extends to a singular Hermitian metric on $K_{\mathbb{P}^1}$ which does not have positive curvature measure since a_j cannot be all less than or equal to 0.

Note that $\omega \geq c\,\Omega$ for some positive metric Ω on $\mathbb{P}^1$ if and only if $a_j \geq 0$ for all $1 \leq j \leq m$. We conclude that $(\mathbb{P}^1, \Sigma, \omega, K_{\mathbb{P}^1}, h_\omega)$ verify assumptions (A)–(C) if and only if $a_j \geq 0$, $1 \leq j \leq m$; hence, Theorem 1.1 applies in this case.

Corollary 4.10. *Let $Y := \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \{x_1, \ldots, x_m\}$, $m \geq 3$, be endowed with the Kähler metric ω of constant curvature -1 and conical singularities of order $a_j \in [0,1)$ at x_j (cf. (4.7), (4.8)). Let P_p be the Bergman kernel function of $H^0_{(2)}(Y, K_Y^p)$ associated with ω and h_ω. Then, for any $j = 1,\ldots,m$ and any compact set $K \subset \mathbb{P}^1$ with $K \cap \Sigma = \{x_j\}$, there exists $C_j = C_j(K) > 0$ such that, for $x \in K$ and $p > C_j \, \mathrm{dist}(x, x_j)^{-8(1+2a_j)}$,*

$$\left| \frac{2\pi P_p(x)}{p} - 1 \right| \leq C_j p^{-1/8} \, \mathrm{dist}(x, x_j)^{-(1+2a_j)}.$$

5. RIEMANN SPHERE WITH TWO CONICAL SINGULARITIES

In this section, we calculate explicitly the Bergman kernel and study its scaling asymptotics near conical singularities by using rescaled coordinates involving the magnetic length, suggested by [8]. We also interpret our results in terms of the density of states on the lowest Landau level.

5.1. Metrics with conical singularities. We take the line bundle $L = \mathcal{O}(1)$ on the projective space $\mathbb{P}^1$, endowed with the Hermitian metric h_a given by the logarithmically homogeneous plurisubharmonic function on $\mathbb{C}^2$,

$$\varphi_a(t, z) = \frac{1}{2a} \log(|t|^{2a} + |z|^{2a}), \quad 0 < a \le 1.$$

Consider the standard embedding $z \in \mathbb{C} \hookrightarrow [1 : z] \in \mathbb{P}^1$. Then, $\omega_a := c_1(L, h_a)$ is given by

(5.1) $\qquad \omega_a|_{\mathbb{C}} = dd^c \varphi_a(1, z) = i \dfrac{a}{2\pi |z|^{2(1-a)}(1 + |z|^{2a})^2} \, dz \wedge d\bar{z},$

and is the Kähler form associated with the metric

$$ds^2 = \frac{a}{\pi |z|^{2(1-a)}(1 + |z|^{2a})^2} |dz|^2$$

on $\mathbb{P}^1$ with conical singularities of order a (or angle $2\pi(1 - a)$) at 0 and ∞. This surface is sometimes also called american football or spindle. Thus, ω_a is polarized by (L, h_a). Moreover, $\int_{\mathbb{P}^1} \omega_a = 1$ and

$$\begin{aligned} \mathrm{Ric}_{\omega_a} &= -i\partial\bar{\partial} \log \frac{a}{2\pi |z|^{2(1-a)}(1 + |z|^{2a})^2} \\ &= 4\pi a \omega_a + 2\pi(1 - a)(\delta(0) + \delta(\infty)) \end{aligned}$$

in the sense of currents on $\mathbb{P}^1$, where $\delta(0) = (i/2)\delta_0 \, dz \wedge d\bar{z}$ and δ_0 is the Dirac measure at 0. So, ω_a has constant Gauss curvature on $\mathbb{C}^\star$ (and also on $\mathbb{P}^1$, in the sense of distributions)

$$R_{\omega_a} = \frac{\mathrm{Ric}_{\omega_a}}{\omega_a} = 4\pi a.$$

Let us define the function ψ on $\mathbb{P}^1$,

(5.2) $\qquad \psi([t : z]) = \dfrac{\nu}{2a} \log \dfrac{|z|^{2a}}{|t|^{2a} + |z|^{2a}}, \quad$ where $\nu \in \mathbb{R},$

and consider on $L^p = \mathcal{O}(p)$ the singular Hermitian metric

(5.3) $\qquad\qquad\qquad h_p = e^{-2\psi} h_a^{\otimes p}.$

We have that

(5.4) $\qquad\qquad\qquad c_1(L^p, h_p) = (p - \nu)\omega_a + \nu\delta(0).$

The motivation for adding the weight ψ in the metric h_p is to create a δ-distribution independent of p in the curvature, which can be interpreted as the Aharonov-Bohm flux in the magnetic field (see Section 5.2).

Let $P_p^{a,v}$ be the Bergman kernel of the Hilbert space $H_{(2)}^0(\mathbb{P}^1 \setminus \{0, \infty\}, L^p)$ of L^2-integrable holomorphic sections of L^p relative to the metrics h_p and ω_a. Let $(\cdot, \cdot)_p$ denote the corresponding inner product.

Proposition 5.1. *In the above setting, we have that*

$$(5.5) \quad P_p(z) := P_p^{a,v}(z)$$

$$= \frac{|z|^{2(j_0 - v)}}{(1 + |z|^{2a})^{(p-v)/a}} \sum_{j=0}^{p-j_0} \frac{|z|^{2j}}{B\left(1 + \dfrac{j + j_0 - v}{a}, 1 + \dfrac{p - j - j_0}{a}\right)},$$

with $z \in \mathbb{C}$, where B is the Euler Beta function,

$$j_0 = \max\{\lfloor v - a \rfloor + 1, 0\} = \begin{cases} 0, & \text{if } v < a, \\ \lfloor v - a \rfloor + 1, & \text{if } v \geq a, \end{cases}$$

and $\lfloor x \rfloor$ is the largest integer $\leq x$. Moreover, $P_p(z) = O(|z|^{2(j_0 - v)})$ for z near 0, and

$$j_0 - v = \begin{cases} -v, & \text{if } v < 0, \\ -\{v\}, & \text{if } v \geq 0 \text{ and } 0 \leq \{v\} < a, \\ 1 - \{v\}, & \text{if } v \geq 0 \text{ and } a \leq \{v\} < 1, \end{cases}$$

where $\{x\} := x - \lfloor x \rfloor$. In particular, $0 < P_p(0) < +\infty$ if and only if $v \in \mathbb{N}$, in which case $P_p(0) = (p - v)/a + 1$.

Proof. For $0 \leq j \leq p$, we obtain that

$$\|z^j\|_p^2 = \int_{\mathbb{C}} \frac{|z|^{2(j-v)}}{(1 + |z|^{2a})^{(p-v)/a}} \frac{a}{2\pi |z|^{2(1-a)}(1 + |z|^{2a})^2} i\, dz \wedge d\bar{z}$$

$$= \int_0^{+\infty} \frac{2ar^{2(j-v)+2a-1}}{(1 + r^{2a})^{(p-v)/a+2}} \, dr.$$

By using the substitution $r^{2a} = x/(1 - x)$, it follows that

$$(5.6) \quad \|z^j\|_p^2 = B\left(1 + \frac{j - v}{a}, 1 + \frac{p - j}{a}\right) < \infty \quad \text{if and only if } j_0 \leq j \leq p.$$

Since $(z^j, z^k)_p = 0$ for $j \neq k$, we conclude that

$$P_p(z) = \sum_{j=j_0}^p \frac{|z^j|_{h_p}^2}{\|z^j\|_p^2},$$

which yields the desired formula for P_p. The remaining assertions are straightforward. $\qquad \square$

Remark 5.2. If $v > 0$, the function ψ defined in (5.2) is quasisubharmonic and has a pole at 0 with Lelong number v. When $a = 1$ and $0 < v < 1$, ω_1 is the Fubini-Study metric on $\mathbb{P}^1$; hence, it is smooth, and $P_p^{1,v}(z) \sim |z|^{-2v}$ blows up at 0. So, the presence of a logarithmic pole at 0 in the Hermitian metric h_p on L^p makes $P_p(0)$ become infinite. On the other hand, if $v = 0$ and $0 < a < 1$, then $P_p^{a,0}(0) \sim p/a$ while $P_p^{a,0}(z) \sim p$ for $z \neq 0$, by Theorem 1.1. So, the presence of a conical singularity in ω_a at 0 makes P_p peak at 0. Proposition 5.1 shows that the following "interference" can appear in the presence of both a logarithmic pole at 0 in h_p and a conical singularity at 0 in ω_a: if $a \leq v < 1$, then $P_p^{a,v}(0) = 0$.

Next, we let $|z|^{2a} = ay/p$ and, inspired by [8], we are interested in the limit as $p \to \infty$ of the scaled Bergman kernel function

$$(5.7) \qquad F_p(y) = F_p^{a,v}(y) := \frac{1}{p} P_p^{a,v}\left(\left(\frac{ay}{p}\right)^{1/(2a)}\right), \qquad y \geq 0.$$

Recall the definition of the Mittag-Leffler function

$$E_{r,s}(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(rj+s)},$$

where $r > 0$, $s \geq 0$, and Γ is the Euler Gamma function.

Theorem 5.3. *In the above setting, we have that*

$$F_p(y) \to \frac{1}{a} y^{(j_0-v)/a} e^{-y} E_{1/a, 1+(j_0-v)/a}(y^{1/a}) \qquad \text{as } p \to \infty,$$

locally uniformly for $y \in (0, +\infty)$ (or for $y \in [0, +\infty)$ when $j_0 - v \geq 0$).

Observe that the theorem gives a scaling asymptotics of the Bergman kernel for z approaching the singularity at 0 (at ∞, respectively) as $p \to \infty$. Specifically, if $|z|^{2a} = ay/p$, then

$$P_p(z) \simeq \frac{p}{a} y^{(j_0-v)/a} e^{-y} E_{1/a, 1+(j_0-v)/a}(y^{1/a}) \qquad \text{as } p \to \infty.$$

In particular, if $v = 0$, we obtain that

$$P_p(z) \simeq \frac{p}{a} e^{-y} E_{1/a, 1}(y^{1/a}) \qquad \text{as } p \to \infty,$$

and

$$P_p(0) = \frac{p}{a} + 1.$$

For the proof of Theorem 5.3, we need the following lemmas.

Lemma 5.4. *If $r \geq 0$ and $s \geq 1$, then*

$$\frac{\Gamma(r+s)}{\Gamma(s)} \leq e^{1/12}(r+s)^r.$$

Moreover,

$$\lim_{s \to \infty} \frac{\Gamma(r+s)}{\Gamma(s)s^r} = 1 \quad \text{locally uniformly in } r \geq 0.$$

Proof. By Stirling's formula, we have for all $x > 0$ that

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x e^{\mu(x)}, \quad \text{where } 0 < \mu(x) < \frac{1}{12x}.$$

Thus,

$$\frac{\Gamma(r+s)}{\Gamma(s)} = \sqrt{\frac{s}{r+s}} e^{-r} \left(1 + \frac{r}{s}\right)^s (r+s)^r e^{\mu(r+s)-\mu(s)}.$$

Since $s \geq 1$, $\mu(r+s) - \mu(s) < \frac{1}{12}$. Moreover, $(1 + r/s)^s \leq e^r$, so the inequality in the statement follows. Next,

$$\frac{\Gamma(r+s)}{\Gamma(s)s^r} = \sqrt{\frac{s}{r+s}} e^{-r} \left(1 + \frac{r}{s}\right)^s \left(1 + \frac{r}{s}\right)^r e^{\mu(r+s)-\mu(s)} \to 1$$

as $s \to \infty$, locally uniformly for $r \geq 0$. $\qquad\square$

The next lemma is very simple and we omit its proof.

Lemma 5.5. *Let $f_n(\zeta) = \sum_{j=0}^{\infty} c_{n,j}\zeta^j$ be entire functions such that $c_{n,j} \to d_j$ as $n \to \infty$, for all $j \geq 0$. Assume that there exists $\xi_j > 0$ such that $|c_{n,j}| < \xi_j$ for all n, j, and that the function $g(\zeta) = \sum_{j=0}^{\infty} \xi_j \zeta^j$ is entire. Then, the function $f(\zeta) = \sum_{j=0}^{\infty} d_j \zeta^j$ is entire and $f_n \to f$ as $n \to \infty$ locally uniformly on $\mathbb{C}$.*

Proof of Theorem 5.3. We have

$$F_p(y) = \left(1 + \frac{ay}{p}\right)^{-(p-v)/a} \frac{(ay)^{(j_0-v)/a}}{p^{1+(j_0-v)/a}}$$

$$\times \sum_{j=0}^{p-j_0} \frac{\Gamma\left(2 + \frac{p-v}{a}\right)(ay)^{j/a}}{\Gamma\left(1 + \frac{j+j_0-v}{a}\right)\Gamma\left(1 + \frac{p-j-j_0}{a}\right)p^{j/a}}$$

$$= \frac{1}{a} y^{(j_0-v)/a} \left(1 + \frac{ay}{p}\right)^{-(p-v)/a} G_p(y),$$

where

$$G_p(y) = \sum_{j=0}^{p-j_0} c_{p,j} y^{j/a},$$

$$c_{p,j} = \frac{\Gamma\left(2 + \dfrac{p-v}{a}\right) a^{1+(j+j_0-v)/a}}{\Gamma\left(1 + \dfrac{j+j_0-v}{a}\right) \Gamma\left(1 + \dfrac{p-j-j_0}{a}\right) p^{1+(j+j_0-v)/a}}.$$

By Lemma 5.4, we have that for all p, j,

$$\frac{\Gamma\left(2 + \dfrac{p-v}{a}\right) a^{1+(j+j_0-v)/a}}{\Gamma\left(1 + \dfrac{p-j-j_0}{a}\right) p^{1+(j+j_0-v)/a}}$$

$$\le e^{1/12} \left(\left(2 + \frac{p-v}{a}\right)\frac{a}{p}\right)^{1+(j+j_0-v)/a} < C^j,$$

for some constant $C > 1$. Moreover, for j fixed,

$$\lim_{p\to\infty} \frac{\Gamma\left(2 + \dfrac{p-v}{a}\right) a^{1+(j+j_0-v)/a}}{\Gamma\left(1 + \dfrac{p-j-j_0}{a}\right) p^{1+(j+j_0-v)/a}} = 1.$$

Hence,

$$0 < c_{p,j} < \frac{C^j}{\Gamma\left(1 + \dfrac{j+j_0-v}{a}\right)}, \quad \lim_{p\to\infty} c_{p,j} = \frac{1}{\Gamma\left(1 + \dfrac{j+j_0-v}{a}\right)}.$$

Now, by Lemma 5.5, $f_p(\zeta) := \sum_{j=0}^{p-j_0} c_{p,j} \zeta^j \to E_{1/a,1+(j_0-v)/a}(\zeta)$ locally uniformly on $\mathbb{C}$. So, $G_p(y) = f_p(y^{1/a}) \to E_{1/a,1+(j_0-v)/a}(y^{1/a})$ locally uniformly in $y \ge 0$, and the proof is complete. $\square$

We conclude this section with a closed formula for $P_p^{a,0}$ in the case when $a = 1/s$ for some positive integer s.

Proposition 5.6. *If $v = 0$ and $a = 1/s$, where $s > 0$ is an integer, then*

$$P_p(z) = P_p^{1/s,0}(z) = \left(p + \frac{1}{s}\right)\left(1 + \sum_{\ell=1}^{s-1} \left(\frac{1 + e^{2\pi\ell i/s}|z|^{2/s}}{1 + |z|^{2/s}}\right)^{ps}\right).$$

Notably, $P_p(0) = sp + 1$, while P_p has the following asymptotic expansion on $\mathbb{C}\setminus\{0\}$: for every $M > 1$, there exists $\theta = \theta(M) > 0$ such that if $1/M \le |z| \le M$, then

$$P_p(z) = p + \frac{1}{s} + O(e^{-\theta p}).$$

Proof. By Proposition 5.1,

$$P_p(z) = \frac{1}{(1+|z|^{2/s})^{ps}} \sum_{j=0}^{p} \frac{|z|^{2j}}{B(1+sj, 1+s(p-j))}$$

$$= \frac{sp+1}{(1+|z|^{2/s})^{ps}} \sum_{j=0}^{p} \binom{ps}{js} |z|^{2j}.$$

If $\zeta = e^{2\pi i/s}$, we have that

$$\sum_{\ell=0}^{s-1} (1+\zeta^\ell y)^{ps} = \sum_{k=0}^{ps} \binom{ps}{k} y^k \sum_{\ell=0}^{s-1} \zeta^{k\ell} = s \sum_{j=0}^{p} \binom{ps}{js} y^{sj},$$

since $\zeta^k = 1$ if s divides k, and $\sum_{\ell=0}^{s-1} \zeta^{k\ell} = 0$ if s does not divide k. The conclusion follows if we let $y = |z|^{2/s}$ in the above formula. $\qquad\square$

Thus, the Bergman kernel $P_p^{1/s,0}$ has the same structure (4.2) as in the case of the punctured disc endowed with the Poincaré metric; namely, the first two terms are non-vanishing and the remainder has exponential decay (cf. Remark 4.8).

5.2. Density of states on the lowest Landau level.   Here, we comment on the relation of the results in the previous subsection to the density-of-states function on the lowest Landau level (LLL) on singular surfaces. The surface with the metric (5.1) is pictured on Figure 5.1. In physics terms, (5.4) means that the constant magnetic field (not pictured on Figure 5.1) $B = p - v$ is turned on, with flux lines everywhere perpendicular to the surface.

FIGURE 5.1. Spindle with cone angle a and Aharonov-Bohm flux v.

In addition, there is a delta-function (Aharonov-Bohm) flux v localized exactly at the north pole (see Figure 5.1), so that the total flux of the magnetic field $p \in \mathbb{Z}$ through the compact surface is an integer. In terms of the singular metric (5.3), the AB-flux v is the Lelong number of the weight ψ from (5.2). This is the compact surface version of the setup of [8] where density of states was studied on a flat cone with a boundary.

As was already pointed out in Remark 5.2, the Bergman kernel (5.5) diverges as $P_p(z) \sim |z|^{2v}$ for $v < 0$ and for $0 \le \{v\} < a$. This is because the Hermitian norm $|z^0|_{h^p}^2$ of the identity section z^0 (5.6) corresponding to the LLL wave function with the smallest angular momentum is singular at $z = 0$ for these values of v, while the section is L^2-normalizable. The question may arise whether

this section shall be kept in the spectrum. Here, we answer this question in the affirmative. The singular value of $P_p(0)$ is an artifact of the delta-function form of the AB-flux, which should be smeared over some ε-neighborhood around zero. Hence, the density is also smoothed out in this neighborhood.

This leads to an interesting effect, when the AB-flux ν is allowed to vary over the real line, say, in the range $0 \le \nu < \infty$. Between $0 \le \nu < a$, the density $P_p(0)$ is peaked, while at $a \le \nu < 1$ it drops to zero: $P_p(0) = 0$. This pattern then repeats in the interval $[1, 2]$ and so on. This is a manifestation of the Laughlin's "shift register" on the LLL, first described in the original argument for the quantization of the Hall conductance [32]. As ν becomes greater than a, the identity section (wave function) z^0 becomes non-normalizable (5.6) and drops from the spectrum of physical states, that is, disappears into the conical singularity. At the same time, a new L^2-normalizable section localized at the equator emerges so that the total number of states is preserved. (In Laughlin's setting of annulus geometry, the wave functions travelled from the outer to the inner edge of the annulus, as AB-flux varied form 0 to 1.)

One consequence of Theorems 1.1 and 1.2 is that at a certain small distance (in units set by magnetic length $\ell_B^2 \sim 1/p$) away from the singular point the Bergman kernel tends to its constant value $2\pi P_p \sim p$. Thus the interesting behavior of the density profile happens around a small area near the singular point which shrinks as p tends to infinity. One way to study the density profile, suggested in [8], is to use the rescaled coordinate $y = p|z|^{2a}/a$ in order to zoom in on the point $z = 0$. Remarkably, this leads to the universal finite result for the density profile near the conical singularity in Theorem 5.3, in agreement with the results of [8] for the flat cone.

5.3. Metrics with a logarithmic pole.

We consider again $\mathbb{P}^1$ and the metric ω_a with conical singularities at 0 and ∞, defined in (5.1). But here, we endow the line bundle $L = \mathcal{O}(1)$ with the Hermitian metric h determined by the plurisubharmonic function

$$\varphi(t, z) = \nu \log |z| + \frac{1 - \nu}{2a} \log(|t|^{2a} + |z|^{2a}), \quad 0 < a \le 1, \ 0 < \nu \le 1.$$

We let $h_p = h^{\otimes p}$ be the induced metric on L^p. Note that $\varphi(1, z)$ has a logarithmic pole at 0 with Lelong number ν, and

$$c_1(L^p, h_p) = p(1 - \nu)\omega_a + p\nu\delta(0).$$

Let $P_p = P_p^{a, \nu}$ be the Bergman kernel of $H^0_{(2)}(\mathbb{P}^1 \setminus \{0, \infty\}, L^p)$, the Hilbert space of L^2-integrable holomorphic sections of L^p relative to the metrics h_p and ω_a. By a calculation similar to the proof of Proposition 5.1, we obtain the following result.

Proposition 5.7. *In the above setting,*

$$P_p^{a,v}(z) = \frac{|z|^{2(j_p - pv)}}{(1 + |z|^{2a})^{p(1-v)/a}}$$

$$\times \sum_{j=0}^{p-j_p} \frac{|z|^{2j}}{B\left(1 + \dfrac{j + j_p - pv}{a}, 1 + \dfrac{p - j - j_p}{a}\right)}, \quad z \in \mathbb{C},$$

where $j_p = \lfloor pv - a \rfloor + 1$. *We have* $j_p - pv \in (-a, 1-a]$, *and* $j_p - pv = -\{pv\}$ *when* $a = 1$. *Moreover, if* $v = 1$, *then* $j_p = p$ *and* $P^{a,1}(z) = 1$.

Next, we study the behavior of the scaled Bergman kernel function $F_p^{a,v}$ defined as in (5.7) by setting $|z|^{2a} = ay/p$. Specifically:

$$F_p(y) = F_p^{a,v}(y) := \frac{1}{p} P_p^{a,v}\left(\left(\frac{ay}{p}\right)^{1/(2a)}\right), \quad y \geq 0.$$

The difference with Section 5.1 is that now the sequence $\{F_p\}_{p \geq 1}$ no longer has a limit, but it is relatively compact and its limit points are determined by the limit points of the bounded sequence $\{j_p - pv\}_{p \geq 1}$. Note that when v is irrational, the latter sequence is dense in the interval $[-a, 1-a]$.

Theorem 5.8. *In the above setting, assume that* $p_k \to \infty$ *is a sequence of positive integers such that*

$$j_{p_k} - p_k v \to \theta \in [-a, 1-a]$$

as $k \to \infty$. *Then,*

$$F_{p_k}(y) \to \frac{1-v}{a}((1-v)y)^{\theta/a} e^{-(1-v)y} E_{1/a, 1+\theta/a}(((1-v)y)^{1/a}),$$

locally uniformly for $y \in (0, +\infty)$. *Hence, the Bergman kernel has the following scaling asymptotics near 0:*

$$P_{p_k}(z) \simeq \frac{p_k(1-v)}{a}((1-v)y)^{\theta/a} e^{-(1-v)y} E_{1/a, 1+\theta/a}(((1-v)y)^{1/a}),$$

where $|z|^{2a} = ay/p_k$.

Proof. Arguing as in the proof of Theorem 5.3, we write

$$F_p(y) = \frac{1}{a} y^{(j_p - pv)/a} \left(1 + \frac{ay}{p}\right)^{-p(1-v)/a} G_p(y),$$

where

$$G_p(y) = \sum_{j=0}^{p-j_p} c_{p,j} y^{j/a},$$

$$c_{p,j} = \frac{\Gamma\left(2 + \dfrac{p(1-\nu)}{a}\right) a^{1+(j+j_p-p\nu)/a}}{\Gamma\left(1 + \dfrac{j+j_p-p\nu}{a}\right)\Gamma\left(1 + \dfrac{p-j-j_p}{a}\right) p^{1+(j+j_p-p\nu)/a}}.$$

By Lemma 5.4, we have that for all p, j,

$$\frac{\Gamma\left(2 + \dfrac{p(1-\nu)}{a}\right) a^{1+(j+j_p-p\nu)/a}}{\Gamma\left(1 + \dfrac{p-j-j_p}{a}\right) p^{1+(j+j_p-p\nu)/a}}$$

$$\leq e^{1/12}\left(\left(2 + \dfrac{p(1-\nu)}{a}\right)\dfrac{a}{p}\right)^{1+(j+j_p-p\nu)/a} < C_1^j,$$

for some constant $C_1 > 1$. By Stirling's formula, $\Gamma(x) > x^{-1/2}(x/e)^x$. Then,

$$0 < c_{p,j} < C_2^j j^{-j/a}, \quad p \geq 1, \ 0 \leq j \leq p - j_p,$$

with some constant $C_2 > 1$ (here, $0^0 := 1$). By using Lemma 5.4 again, we conclude that for j fixed,

$$\lim_{k\to\infty} c_{p_k,j} = \frac{(1-\nu)^{1+(j+\theta)/a}}{\Gamma\left(1 + \dfrac{j+\theta}{a}\right)}.$$

Now, Lemma 5.5 implies that

$$G_{p_k}(y) \to (1-\nu)^{1+\theta/a} E_{1/a,1+\theta/a}(((1-\nu)y)^{1/a}),$$

as $k \to \infty$, locally uniformly in $y \geq 0$. This completes the proof. $\square$

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