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$\mathbf{L}(\mathbb{R})$ with determinacy satisfies the Suslin hypothesis

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ABSTRACT

The Suslin hypothesis states that there are no nonseparable complete dense linear orderings without endpoints which have the countable chain condition. $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ proves the Suslin hypothesis. In particular, if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R})$ satisfies the Suslin hypothesis, which answers a question of Foreman.

1. Introduction

Cantor had shown that $\mathbb{R}$ with its usual ordering is the unique complete dense separable linear ordering without endpoints up to isomorphism. A linear ordering has the

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countable chain condition if there are no uncountable sets of disjoint nonempty open intervals. Every separable linear ordering has the countable chain condition. Suslin asked if $\mathbb{R}$ is the unique (up to order isomorphism) complete dense linear ordering without endpoints that satisfies the countable chain condition. This question has come to be known as the Suslin problem. The study of the Suslin problem under the axiom of choice, AC, has led to a number of developments in set theory such as in constructibility and iterated forcing.

A Suslin line is a complete dense linear ordering without endpoints which has the countable chain condition but is not separable. The existence of a Suslin line gives a negative answer to the Suslin problem. The Suslin hypothesis, SH, is the statement that there are no Suslin lines.

The Suslin problem can not be resolved under ZFC. However, [4] showed that no linear ordering which comes from a $\mathbf{\Delta}^1_1$ prelinear ordering on $\mathbb{R}$ can be a counterexample to the Suslin problem. This suggests that no linear ordering which comes from a definable prelinear ordering on $\mathbb{R}$ should be a Suslin line. Since the determinacy axiom AD^+ implies that every set of reals is definable in a very absolute sense, $\mathsf{ZF} + \mathsf{AD}^+$ is a natural setting to ask the question of whether any definable prelinear ordering on $\mathbb{R}$ can induce a Suslin line. This paper will show under AD^+ that no linear ordering which comes from any prelinear ordering on $\mathbb{R}$ is a Suslin line. Assuming the universe satisfies $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ (which are known as natural models of AD^+), the paper will show that there are no Suslin lines at all. In particular, the most natural model of determinacy $L(\mathbb{R}) \models \mathsf{AD}$ will always satisfy SH, which answers a question of Foreman [2]. (This question was brought to the authors' attention by Hamkins [3].)

The following gives a brief introduction to the Suslin problem and a summary of the main results of the paper:

A tree is a partially ordered set $(T, \prec)$ so that for any $t \in T$, $\{s \in T : s \prec t\}$ is a wellordering under $\prec$. An ω_1-tree is an uncountable tree so that each level is countable. An Aronszajn tree is an ω_1-tree with no uncountable branch. A Suslin tree is an ω_1-tree with no uncountable branch or uncountable antichain. Under AC, the existence of a Suslin line is equivalent to the existence of a Suslin tree.

Tennenbaum [15] and Jech [6] independently showed that if ZF is consistent, then $\mathsf{ZFC} + \neg\mathsf{SH}$ is consistent. They used a forcing construction to produce a model of ZFC with a Suslin tree or Suslin line. With the development of iterated forcing, Solovay and Tennenbaum [14] showed that if ZF is consistent, then $\mathsf{ZF} + \mathsf{SH}$ is consistent. In fact, they showed Martin's axiom, MA, and the failure of the continuum hypothesis, CH, imply there are no Suslin lines. Thus SH is independent of ZFC.

One can also ask if SH holds in certain natural models of ZFC. Gödel's constructible universe L is the smallest inner model of ZFC. Jensen showed the axiom $\mathrm{V} = \mathrm{L}$ implies there is a Suslin line or tree. In fact, the Jensen's diamond principle $\diamondsuit$, which holds in L, implies there is a Suslin tree. Jensen also showed that there is a model of $\mathsf{ZFC} + \mathsf{CH} + \mathsf{SH}$. Thus CH is independent of SH.

Results from descriptive set theory have shown that Borel objects are well-behaved and have nice regularity properties. This suggests that no Borel linear ordering on $\mathbb{R}$ should be a Suslin line. Friedman and Shelah showed that there are no Borel Suslin lines. Harrington, Marker, and Shelah strengthened this result using effective methods and the Gandy–Harrington forcing: A prelinear order is a binary relation that is total and transitive (but may not be antisymmetric). [4] showed that every Δ^1_1 prelinear ordering has a perfect set of disjoint closed intervals or there is a Δ^1_1 order preserving function which maps the prelinear ordering into $\mathbb{R}$ with its usual ordering.

The intuition would be that every definable linear ordering which is the surjective image of $\mathbb{R}$ (that is, a collapse of a prelinear ordering on $\mathbb{R}$) is not a Suslin line assuming that certain descriptive set theoretic arguments are valid for this definable context. One approach to formalize this idea of extending descriptive set theoretic methods to the largest possible context is to assume determinacy axioms.

Let ${}^{\omega}\omega$ denote the Baire space which consists of all function from ω into ω. Let $A \subseteq {}^{\omega}\omega$. The game G_A consists of player 1 and 2 alternatingly picking integers a_i. Player 1 wins if $\bar{a} = \langle a_i : i \in \omega \rangle$ belongs to A. Player 2 wins if $\bar{a} \notin A$. The axiom of determinacy, AD, asserts that for all $A \subseteq {}^{\omega}\omega$, one of the players has a winning strategy in G_A.

AD implies that every set of reals has the perfect set property, has the Baire property, and is Lebesgue measurable. It is reasonable to expect that under AD every linear ordering on a set which is the surjective image of $\mathbb{R}$ is not a Suslin line.

AD implies the failure of AC. As noted above, under AC, the study of the Suslin problem can be reduced to the study of Suslin trees. The proof that the existence of a Suslin line implies the existence of a Suslin tree seems to require AC. Under determinacy assumptions, the existence of Suslin trees and Suslin lines are considered separately.

AD by its very nature is in general restricted to providing information about set which are surjective images of $\mathbb{R}$. However, SH is a statement about all linear orderings. $L(\mathbb{R})$ is the smallest transitive inner model of ZF containing all the reals. Woodin has shown that if $V \models \mathsf{ZFC}$ and has a measurable cardinal with infinitely many Woodin cardinals below it, then $L(\mathbb{R})^V \models \mathsf{AD}$. (See [10].) Sometimes results about all sets can be proved in this minimal model of AD containing the reals. Kechris [9] showed that if $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R}) \models \mathsf{DC}$. Caicedo and Ketchersid [1] extended the Silver's dichotomy [12] to show that in $L(\mathbb{R})$, every set is either wellorderable or $\mathbb{R}$ inject into it. Moreover, if $V \models \mathsf{ZFC}$ and has a proper class of Woodin cardinals, then for any forcing $\mathbb{P} \in V$ and $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over V, $L(\mathbb{R})^V$ and $L(\mathbb{R})^{V[G]}$ are elementarily equivalent. An external forcing cannot change the theory of $L(\mathbb{R})$ and in particular the status of SH in $L(\mathbb{R})^V$ and $L(\mathbb{R})^{V[G]}$. At the end of [2], Foreman asked whether $L(\mathbb{R}) \models \mathsf{SH}$ if $L(\mathbb{R}) \models \mathsf{AD}$.

First, the paper will consider the existence of Suslin trees.

Theorem 3.4. *If $L(\mathbb{R}) \models \mathsf{AD}$, then $L(\mathbb{R})$ has no Aronszajn tree and hence no Suslin trees.*

To study linear orderings on surjective images of $\mathbb{R}$, one will work in a strengthening of AD isolated by Woodin known as AD^+. It includes $\mathsf{DC}_{\mathbb{R}}$ and the statement that all sets

of reals have an absolute definition provided by an ∞-Borel code. AD^+ holds in every model of AD that has been produced. It is open whether AD and AD^+ are equivalent.

Theorem 4.2. ($\mathsf{ZF} + \mathsf{AD}^+$) *Let $\preceq$ be a prelinear order on $\mathbb{R}$. Exactly one of the following holds.*

(i) *There is a perfect set of disjoint closed intervals in $\preceq$. (That is, this set of intervals is in bijection with $\mathbb{R}$.)*
(ii) *There is a wellordered separating family for $\preceq$.*

Here, a separating family is a collection $\mathcal{S}$ of $\preceq$-downward closed sets so that for any $a \prec b$, there is some $A \in \mathcal{S}$ so that $a \in A$ and $b \notin A$. Note that (ii) cannot be replaced with the statement that $\preceq$ order embeds into $\mathbb{R}$ as in the case for Borel linear orderings. For example, there is a Σ^1_1 prelinear ordering whose quotient has ordertype ω_1.

The argument associated with (i) is a modification of the Gandy–Harrington forcing argument from [4] using the Vopěnka algebra. The argument associated with (ii) follows an idea of Hjorth from [5].

The proof has a clear descriptive set theoretic flavor: Instead of considering a set as an static object, one uses a sufficiently absolute definition of a set provided by the ∞-Borel code. This allows the definition to be interpreted in various inner models containing the necessary parameters to derive information about the true object in the real world.

The theorem implies the following:

Theorem 4.4. ($\mathsf{ZF} + \mathsf{AD}^+$) *There are no Suslin lines on a set which is the surjective image of $\mathbb{R}$.*

The previous results will be used to establish the full SH in models satisfying $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Woodin showed that such model take one of two forms: If $\mathsf{AD}_{\mathbb{R}}$ (the determinacy axiom for games with moves from $\mathbb{R}$) fails, then $V = L(J, \mathbb{R})$ for some set of ordinals J. Model of the form $L(J, \mathbb{R})$ cannot satisfy $\mathsf{AD}_{\mathbb{R}}$.

In such models, an arbitrary linear ordering is uniformly a union of sublinear orderings which are surjective images of $\mathbb{R}$. The dichotomy result is applied uniformly to each sublinear ordering to produce wellordered separating family for each sublinear ordering. Then these wellordered separating family need to be coherently patched together to form a wellordered separating family for the original linear ordering. In models of the form $L(J, \mathbb{R})$, this is relatively straightforward. In $L(\mathscr{P}(\mathbb{R})) \models \mathsf{AD}_{\mathbb{R}}$, one will need to use the unique supercompactness measure on $\mathscr{P}_{\omega_1}(\lambda)$ for each $\lambda < \Theta$. These patching arguments are similar to those used in [1].

Theorem 4.5. *Let J be a set of ordinals. If $L(J, \mathbb{R}) \models \mathsf{AD}$, then $L(J, \mathbb{R}) \models \mathsf{SH}$. In particular, one has that $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R})$ proves SH.*

Theorem 4.8. $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R})) \vdash \mathsf{SH}$.

Theorem 4.9. $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R})) \vdash \mathsf{SH}$.

The Solovay model is a choiceless model of ZF which possesses many of the descriptive set theory regularity properties which are consequences of AD. Woodin observed that the methods used above in the determinacy setting can be adapted to establish SH in some Solovay models. The final section provides some details on the modification of the earlier arguments to analyze when SH holds in Solovay models.

Theorem 5.7. *(Woodin) Let $V \models \mathsf{ZFC}$ and κ be a weakly compact cardinal of V. Let G be $\mathrm{Coll}(\omega, < \kappa)$-generic over V. $V(\mathbb{R}^{V[G]}) \models \mathsf{SH}$.*

Theorem 5.12. *Suppose $V \models \mathsf{ZFC}$ and κ is an inaccessible cardinal of V. Let $G \subseteq \mathrm{Coll}(\omega, < \kappa)$ be $\mathrm{Coll}(\omega, < \kappa)$-generic over V. The Solovay model $V(\mathbb{R}^{V[G]})$ has a Suslin line if and only if $V(\mathbb{R}^{V[G]})$ has a Suslin tree on ω_1.*

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2. Basics

Definition 2.1. Let $(L, \prec)$ be a (strict) linear ordering. L is dense if for all $a, c \in L$ with $a \prec c$, there is some $b \in L$ with $a \prec b \prec c$. L has the countable chain condition if there are no uncountable collection of disjoint nonempty open intervals in L. L is separable if there is a countable dense subset of L. L is complete if every nonempty subset which is bounded above has a supremum.

A Suslin line is a complete dense linear ordering without endpoints which has the countable chain condition and is not separable.

It should be noted that the existence of a nonseparable linear ordering with the countable chain condition is sufficient to imply the existence of a Susline line.

Definition 2.2. The Suslin hypothesis, denoted SH, is the statement that there are no Suslin lines.

Definition 2.3. A (nonreflexive) partially ordered set $(T, \prec)$ is a tree if and only if for all $t \in T$, $\{s \in T : s \prec t\}$ is a wellordered by $\prec$. For $t \in T$, let $|t|_\prec$ denote the ordertype of $\{s \in T : s \prec t\}$. If α is an ordinal, then let $L^T_\alpha = \{t \in T : |t|_\prec = \alpha\}$. Let $|T|_\prec = \sup\{|t|_\prec + 1 : t \in T\}$ be the height of the tree T. A branch through T is a

maximal $\prec$-linearly ordered subset of T. $A \subseteq T$ is an antichain if every pair of elments from A is $\prec$-incomparable.

Let κ be a cardinal. $(T, \prec)$ is a κ-tree if and only if $(T, \prec)$ is a tree with $|T|_\prec = \kappa$ and for each ordinal α, L^T_α injects into κ but does not biject onto κ. (In particular, the levels are wellorderable.) A κ-Aronszajn tree is a κ-tree so that each chain has cardinality less than κ. A κ-Suslin tree is a κ-tree so that all chains have cardinality less than κ and κ does not inject into any antichain. (Every κ-Suslin tree is a κ-Aronszajn tree.)

An Aronszajn or Suslin tree is an ω_1-Aronszajn or ω_1-Suslin tree, respectively.

ZFC shows that there is a Suslin line if and only if there is a Suslin tree. However, the usual proof does seem to use AC. Suslin trees and Suslin lines will be studied in the choiceless context of ZF augmented with determinacy axioms or in specific natural models of these determinacy axioms.

Definition 2.4. Let $X \subseteq \mathbb{R}$. An ∞-Borel code for X is a pair (S, φ) where S is a set of ordinals and φ is a formula in the language of set theory such that for all $x \in \mathbb{R}$, $x \in X \Leftrightarrow L[S, x] \models \varphi(S, x)$.

Definition 2.5. ([17] Section 9.1) AD^+ consists of the following statements:

(i) $\mathsf{DC}_\mathbb{R}$.
(ii) Every $X \subseteq \mathbb{R}$ has an ∞-Borel code.
(iii) For all $\lambda < \Theta$, $X \subseteq \mathbb{R}$, and continuous function $\pi : {}^\omega\lambda \to \mathbb{R}$, $\pi^{-1}(X)$ is determined.

If J is a set of ordinals and $L(J, \mathbb{R}) \models \mathsf{AD}$, then $L(J, \mathbb{R}) \models \mathsf{AD}^+$. Also $L(J, \mathbb{R}) \models \mathsf{DC}$ by [9]. Models of $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$ are considered natural models of AD. No models of the form $L(J, \mathbb{R})$ can satisfy $\mathsf{AD}_\mathbb{R}$. Woodin ([1] Corollary 3.2) showed that if $V \models \mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R})) + \neg\mathsf{AD}_\mathbb{R}$, then V is of the form $L(J, \mathbb{R})$ for some set of ordinals J. Of particular importance to this paper is the existence of ∞-Borel codes for sets of reals. Although it is open whether $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ implies $\mathsf{ZF} + \mathsf{AD}^+$, Woodin has shown that $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$ can prove that every set of reals has an ∞-Borel code. However, it is known that $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_\mathbb{R}$ implies $\mathsf{ZF} + \mathsf{AD}^+$. (See [1] for more information about AD^+.)

Definition 2.6. (Vopěnka) Let S be a set of ordinals. Let $\mathbb{O}_S$ be the forcing of nonempty OD_S subsets of reals ordered by $\subseteq$. By using the canonical bijection of OD_S with ON, one will assume that this forcing has been transfered onto ON and is hence an element of HOD_S. $\mathbb{O}_S$ adds a generic real. Let τ denote the canonical $\mathbb{O}_S$-name for the canonical real.

More specifically, if G is $\mathbb{O}_S$-generic, then $n \in \tau[G] \Leftrightarrow \{x \in \mathbb{R} : n \in x\} \in G$.

Fact 2.7. *(Vopěnka's Theorem) Let S be a set of ordinals.*

Let $M \models \mathsf{ZF}$ be a transitive inner model containing S. For all $x \in \mathbb{R}^M$, there is a filter $G_x \in M$ which is $\mathbb{O}_S^M$-generic over HOD_S^M so that $\tau[G_x] = x$.

Suppose φ is a formula and $\bar{\alpha}$ is a tuple of ordinals. Let K be a set of ordinals in OD_S^M. Suppose $N \models \mathsf{ZF}$ is an inner model with $N \supseteq \mathrm{HOD}_S^M$. Suppose $p = \{x \in \mathbb{R}^M : L[K, x] \models \varphi(K, \bar{\alpha}, x)\}$ is a condition of $\mathbb{O}_S^M$, that is, it is a nonempty OD_S set. Then $N \models p \Vdash_{\mathbb{O}_S^M} L[\check{K}, \tau] \models \varphi(\check{K}, \bar{\alpha}, \tau)$.

Proof. The first statement is a classical result which can be found in [7] Theorem 15.46 or [5] Theorem 2.4. The set G_x is $\{p \in \mathbb{O}_S^M : x \in p\}$, where $\mathbb{O}_S^M$ is considered as the collection of OD_S sets of reals in M.

The second statement appears in [5] Theorem 2.4. A brief sketch will be given:

Suppose not. Then there is some $q' \leq_{\mathbb{O}_S^M} p$ such that $N \models q' \Vdash_{\mathbb{O}_S^M} L[\check{K}, \tau] \models \neg\varphi(\check{K}, \bar{\alpha}, \tau)$. Since every $\mathbb{O}_S^M$-generic filter over N is generic over HOD_S^M, there is some $q \leq_{\mathbb{O}_S^M} q'$ so that $\mathrm{HOD}_S^M \models q \Vdash_{\mathbb{O}_S^M} L[\check{K}, \tau] \models \neg\varphi(\check{K}, \bar{\alpha}, \tau)$. Let $y \in q$. Let G_y be the $\mathbb{O}_S^M$-generic filter over HOD_S^M derived from y. $q \in G_y$. By the forcing theorem, $\mathrm{HOD}_S^M[G_y] \models L[K, y] \models \neg\varphi(K, \bar{\alpha}, y)$. Hence $L[K, y] \models \neg\varphi(K, \bar{\alpha}, y)$. This contradicts $q \subseteq p$. □

Fact 2.8. *Let S be a set of ordinals. Let M be an inner model of ZF containing S. Suppose N is an inner model of ZF containing S and $\mathrm{HOD}_S^M \subseteq N$. If $n \geq 1$ is a natural number, let ${}_n\mathbb{O}_S^M$ be the Vopěnka forcing on $\mathbb{R}^n$. Suppose $(g_0, ..., g_{n-1})$ is a ${}_n\mathbb{O}_S^M$-generic over N n-tuple of reals. Then each of g_0,..., g_{n-1} is a $\mathbb{O}_S^M$ generic real over N.*

Proof. Here a real g is $\mathbb{O}_S^M$-generic over N if and only if there is a filter G which is $\mathbb{O}_S^M$-generic over N such that g is the canonical real that is added by G.

Consider the case when $n = 2$.

For each $p \in {}_2\mathbb{O}_S^M$, let $\Psi(p) = \{x \in \mathbb{R} : (\exists y)(x, y) \in p\}$. Note that $\Psi(p) \in \mathbb{O}_S^M$.

Let (g_0, g_1) be ${}_2\mathbb{O}_S^M$-generic over N. Let $G_{(g_0,g_1)}$ be the ${}_2\mathbb{O}_S^M$-generic filter over N which adds (g_0, g_1). Let $G = \{\Psi(p) : p \in G_{(g_0,g_1)}\}$. G is a filter on $\mathbb{O}_S^M$.

Suppose $D \subseteq \mathbb{O}_S^M$ is dense open and belongs to N. Let $D' = \{p : \Psi(p) \in D\}$. Suppose $r \in {}_2\mathbb{O}_S^M$. Since D is dense, there is some $r' \leq_{\mathbb{O}_S^M} \Psi(r)$ with $r' \in D$. Let $s = (r' \times \mathbb{R}) \cap r$. Note that $s \in {}_2\mathbb{O}_S^M$, $\Psi(s) = r' \in D$, and $s \leq_{{}_2\mathbb{O}_S^M} r$. Hence $s \in D'$. This shows that D' is dense in ${}_2\mathbb{O}_S^M$.

By genericity, there is some $r \in D'$ such $r \in G_{(g_0,g_1)}$. Then $\Psi(r) \in D \cap G$. G is $\mathbb{O}_S^M$-generic over N. g_0 is the real added by G. □

For this paper, one will need a uniform procedure for taking an OD definition for a set of reals to an OD ∞-Borel code for that set of reals.

Fact 2.9. *(Woodin, [1] Theorem 3.4) Assume $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. Let J be a set of ordinals. Let $X \subseteq \mathbb{R}$ be an OD_J set. Then X has an ∞-Borel code in HOD_J.*

Definition 2.10. Let $\leq_T$ denote the relation of Turing reducibility. If $x, y \in \mathbb{R}$, then $x \equiv_T y$ if and only if $x \leq_T y$ and $y \leq_T x$. A Turing degree is an $\equiv_T$-equivalence class. Let $\mathcal{D}$ denote the set of Turing degrees. If $X, Y \in \mathcal{D}$, then let $X \leq_T Y$ if and only if for all $x \in X$ and $y \in Y$, $x \leq_T y$. The Turing cone above X is the set $\{Y \in \mathcal{D} : X \leq_T Y\}$. The Martin measure μ is a measure on $\mathcal{D}$ defined by $A \in \mu$ if and only if A contains a Turing cone. Under AD, μ is a countably complete ultrafilter on $\mathcal{D}$.

Fact 2.11. *(Woodin, [1] Section 2.2) Assume* $\mathsf{ZF} + \mathsf{AD}^+$. *Let* $\prod_{X \in \mathcal{D}} \mathrm{ON}$ *be the collection of function* $f : \mathcal{D} \to \mathrm{ON}$. *If* $f, g \in \prod_{X \in \mathcal{D}} \mathrm{ON}$, *then define* $f \sim g$ *if and only if* $\{X \in \mathcal{D} : f(X) = g(X)\} \in \mu$. *Let* $[f]_\sim < [g]_\sim$ *if and only if* $\{X \in \mathcal{D} : f(X) < g(X)\} \in \mu$. *Then* $\prod_{X \in \mathcal{D}} \mathrm{ON}/\mu$, *the set of* $\sim$*-equivalence classes, is wellordered under* $<$.

3. No Aronszajn trees

Fact 3.1. $(\mathsf{ZF} + \mathsf{AD})$ *There are no Aronszajn trees on a wellorderable set.*

Proof. This is a well known result using standard techniques involving measures on ω_1. The following provides some details under AD.

Using AD, let U be a countably complete ultrafilter on ω_1. (For example, if μ is Martin's Turing cone measure on $\mathcal{D}$, let $A \in U \Leftrightarrow \{X \in \mathcal{D} : \omega_1^X \in A\} \in \mu$, where for $X \in \mathcal{D}$, $\omega_1^X = \omega_1^x$, the least x-admissible ordinal for any $x \in X$.)

Let $(T, \prec)$ be an ω_1-tree. Since T is wellorderable, one may assume that $T = \omega_1$. For each $s \in T$, let $A_s = \{t \in T : s \prec t\}$.

Note $\omega_1 = T = \bigcup_{s \in L_0^T} A_s$. Since $(T, \prec)$ is an ω_1-tree, L_0^T is countable. Since U is countably complete, there is some $s \in L_0^T$ so that $A_s \in U$. Since $T = \omega_1$, let s_0 be the least such object according to the wellordering of ω_1.

Suppose $s_\alpha \in L_\alpha^T$ has been defined so that $A_{s_\alpha} \in U$. Note that $A_{s_\alpha} = \bigcup_{s \in L_{\alpha+1}^T \cap A_{s_\alpha}} A_s$. By countably completeness, let $s_{\alpha+1}$ be the least $s \in L_{\alpha+1}^T \cap A_{s_\alpha}$ so that $A_s \in U$.

Suppose α is a limit ordinal and s_γ has been defined for all $\gamma < \alpha$. Since α is countable and each $A_{s_\gamma} \in U$, $\bigcap_{\gamma<\alpha} A_{s_\gamma} \in U$. Note that for all $s \in \bigcap_{\gamma<\alpha} A_{s_\gamma}$, one has $s_\gamma \prec s$ for all $\gamma < \alpha$. Observe $\bigcap_{\gamma<\alpha} A_{s_\gamma} = \bigcup_{s \in L_\alpha^T \cap \bigcap_{\gamma<\alpha} A_{s_\gamma}} A_s$. By countable completeness, let s_α be the least element $s \in L_\alpha^T \cap \bigcap_{\gamma<\alpha} A_{s_\gamma}$ so that $A_s \in U$.

$\langle s_\alpha : \alpha < \omega_1 \rangle$ is an uncountable branch through $(T, \prec)$. □

Fact 3.2. *([1] Theorem 1.4) Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = L(\mathscr{P}(\mathbb{R}))$. *For any set* X, *either* X *is wellorderable or* $\mathbb{R}$ *injects into* X.

Many of the ideas used in [1] to prove Fact 3.2 will be used in this paper to investigate Suslin lines. Fact 3.2 gives the following result about κ-trees.

Fact 3.3. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Let* κ *be a cardinal. There are no* κ*-trees on a nonwellorderable set.*

Proof. Let $(T, \prec)$ be a κ-tree where T cannot be wellordered. By Fact 3.2, there is an injection $\Phi : \mathbb{R} \to T$. Define a prewellordering $\sqsubseteq$ on $\mathbb{R}$ as follows: $x \sqsubseteq y$ if and only if $|\Phi(x)|_\prec \leq |\Phi(y)|_\prec$. Since each level of T is wellorderable and AD implies there are no uncountable wellordered sequences of distinct reals, each $\sqsubseteq$-prewellordering class is countable.

However, there are no prewellorderings of $\mathbb{R}$ with every prewellordering class countable (or more generally meager) under AD. To see this: Suppose there was such a prewellordering. Under AD, all sets of reals have the Baire property. By taking an appropriate homeomorphic image, one may assume one has a prewellordering $\preceq$ on all of $\mathbb{R}$ so that each initial segment of the prewellordering is meager. By AD, $\preceq$ itself has the Baire property as a subset of $\mathbb{R}^2$. Then the Kuratowski–Ulam theorem would yield a contradiction after considering the various horizontal and vertical sections of $\preceq$. □

Theorem 3.4. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = L(\mathscr{P}(\mathbb{R}))$. *There are no Aronszajn trees and hence no Suslin trees.*

In particular, if $L(\mathbb{R}) \models \mathsf{AD}$, *then* $L(\mathbb{R})$ *has no Aronszajn trees and hence no Suslin trees.*

4. No Suslin lines

Definition 4.1. A prelinear ordering $\preceq$ on a set P is a total transitive binary relation on P (which may not be antisymmetric).

Let $(P, \preceq)$ be a prelinear ordering. For each $x, y \in P$, let $x \prec y$ if and only if $x \preceq y$ and $\neg(y \preceq x)$. $\mathcal{S} \subseteq \mathscr{P}(P)$ is a separating family for P if and only if every $A \in \mathcal{S}$ is $\preceq$-downward closed and for all $x, y \in P$ with $x \prec y$, there is some $A \in \mathcal{S}$ with $x \in A$ and $y \notin A$.

Theorem 4.2. $(\mathsf{ZF} + \mathsf{AD}^+)$ *Let* $\preceq$ *be a prelinear order on* $\mathbb{R}$. *Exactly one of the following holds.*

(i) *There is a perfect set of disjoint closed intervals in* $\preceq$. *(That is, this set of intervals is in bijection with* $\mathbb{R}$.*)*
(ii) *There is a wellordered separating family for* $\preceq$.

Proof. Let (S, φ) be an ∞-Borel code for $\preceq$. Let (S, ψ) be an ∞-Borel code for $\prec$ which is uniformly derived from (S, φ). Although $\mathbb{O}_S$ is a forcing on the ordinals which belong to HOD_S, it will also be considered as the forcing of nonempty OD_S sets of reals. Let $\mathbb{U}_S$ denote the sets in $\mathbb{O}_S$ which are $\preceq$-downward closed.

Throughout this proof, within any transitive inner model M of ZF containing S, $\preceq$ and $\prec$ will always be defined using the ∞-Borel code (S, φ) and (S, ψ), respectively. Therefore, if $a, b \in \mathbb{R}^M$, $M \models L[S, a, b] \models \varphi(S, a, b)$ if and only if $L[S, a, b] \models \varphi(S, a, b)$ if only if $V \models L[S, a, b] \models \varphi(S, a, b)$ if and only if $V \models a \preceq b$.

(Case I) For all $x \in \mathbb{R}$, for all $a, b \in \mathbb{R}^{L[S,x]}$ with $a \prec b$, there exists some $A \in \mathbb{U}_S^{L[S,x]}$ with $a \in A$ and $b \notin A$.

For each $[f]_\sim \in \prod_{X \in \mathcal{D}} \omega_1/\mu$, let $A_{[f]_\sim}$ be the set of $y \in \mathbb{R}$ so that on a cone of $X \in \mathcal{D}$, y belongs to the $f(X)^{\text{th}}$ element of $\mathbb{U}_S^{L[S,x]}$ according to the canonical global wellordering of $\text{HOD}_S^{L[S,x]}$, where x is any real in X. The $f(X)^{\text{th}}$ element of $\mathbb{U}_S^{L[S,x]}$ is formally defined to be $\emptyset$ if there is no $f(X)^{\text{th}}$ element of $\mathbb{U}_S^{L[S,x]}$. This is well-defined. (Note that if $x \equiv_T y$, then $L[S,x] = L[S,y]$ and their canonical wellorderings of their HOD_S's are the same.)

For each $[f]_\sim$, $A_{[f]_\sim}$ is $\preceq$-downward closed. To see this, suppose $b \in A_{[f]_\sim}$. Let $a \preceq b$. There is some $z \in \mathbb{R}$ so that for all $X \geq_T [z]_{\equiv_T}$, b belongs to the $f(X)^{\text{th}}$ set in $\mathbb{U}_S^{L[S,x]}$, where $x \in X$. Then for any $X \geq_T [a \oplus z]_{\equiv_T}$, b belongs to the $f(X)^{\text{th}}$ set in $\mathbb{U}_S^{L[S,x]}$, where $x \in X$. Since $x \geq_T a$, $a \in L[S,x]$. By $\preceq$-downward closure, a also belongs to the $f(X)^{\text{th}}$ set in $\mathbb{U}_S^{L[S,x]}$. This verifies that $a \in A_{[f]_\sim}$.

By Fact 2.11, $\prod_{X \in \mathcal{D}} \omega_1/\mu$ is wellordered. Hence $\{A_{[f]_\sim} : [f]_\sim \in \prod_{X \in \mathcal{D}} \omega_1/\mu\}$ is a wellordered set. The next claim is that it is a separating family for $\preceq$.

To see this: Let $a \prec b$. Suppose $X \geq_T [a \oplus b]_{\equiv_T}$. Define $f(X)$ to be the rank under the canonical wellordering of $\text{HOD}_S^{L[S,x]}$ of the least element of $\mathbb{U}_S^{L[S,x]}$ according to the canonical wellordering of $\text{HOD}_S^{L[S,x]}$ (for any $x \in X$) containing a but not b. Note that this set exists by the Case I assumption. By definition, $a \in A_{[f]_\sim}$ and $b \notin A_{[f]_\sim}$.

This shows that Case I implies that (ii) holds.

(Case II) There exists $x \in \mathbb{R}$, there exists $a, b \in \mathbb{R}^{L[S,x]}$ with $a \prec b$ so that for all $A \in \mathbb{U}_S^{L[S,x]}$, $a \notin A$ or $b \in A$.

Let

$$u = \{(c_0, c_1) \in (\mathbb{R}^2)^{L[S,x]} : c_0 \prec c_1 \wedge (\forall A)(A \in \mathbb{U}_S^{L[S,x]} \Rightarrow (c_0 \notin A \vee c_1 \in A))\}.$$

Observe that $u \in {}_2\mathbb{O}_S^{L[S,x]}$ and in particular is nonempty due to the Case II assumption.

Claim 1: Let $M \models \mathsf{ZF}$ be an inner model of V such that $M \supseteq \text{HOD}_S^{L[S,x]}$. Let $r \in \mathbb{R}^M$. Then

$$M \models u \Vdash_{{}_2\mathbb{O}_S^{L[S,x]}} \neg(\tau_0 < \check{r} < \tau_1)$$

where τ_0 and τ_1 are the canonical names for the first and second real in the generic pair.

To prove Claim 1: Suppose it was not true. Then there is some $v \leq_{{}_2\mathbb{O}_S^{L[S,x]}} u$ so that $M \models v \Vdash_{{}_2\mathbb{O}_S^{L[S,x]}} \tau_0 < \check{r} < \tau_1$.

(Subcase 1.1.) There is some $(c_0, c_1) \in v$ and $(d_0, d_1) \in v$ so that $c_1 \preceq d_0$.

Let $w = \{(e_0, e_1, e_2, e_3) \in (\mathbb{R}^4)^{L[S,x]} : (e_0, e_1) \in v \wedge (e_2, e_3) \in v \wedge e_1 \preceq e_2\}$. $w \neq \emptyset$ by the Subcase 1.1. assumption and $w \in {}_4\mathbb{O}_S^{L[S,x]}$. Now let (g_0, g_1, g_2, g_3) be ${}_4\mathbb{O}_S^{L[S,x]}$-generic over M containing w. By Fact 2.8 (or essentially the proof), (g_0, g_1) and (g_2, g_3) are ${}_2\mathbb{O}_S^{L[S,x]}$-generic over M.

Let $u' = \{(c_0, c_1) \in (\mathbb{R}^2)^{L[S,x]} : c_0 \prec c_1\}$. As $u \leq_{{}_2\mathbb{O}_S^{L[S,x]}} u'$, u' belongs to the ${}_2\mathbb{O}_S^{L[S,x]}$-generic filter over M derived from (g_0, g_1) and (g_1, g_2). Using the ∞-Borel code for $\prec$, u' is a condition of the form for which Fact 2.7 applies. Hence Fact 2.7 implies

that $g_0 \prec g_1$ and $g_2 \prec g_3$. Let $w' = \{(e_0, e_1, e_2, e_3) \in (\mathbb{R}^4)^{L[S,x]} : e_1 \preceq e_2\}$. The condition w' is also a condition of the form for which Fact 2.7 applies. Since $w \leq_{{}_4\mathbb{O}_S^{L[S,x]}} w'$, this gives that $g_1 \preceq g_2$. By the forcing theorems and the above, $g_0 \prec r \prec g_1 \preceq g_2 \prec r \prec g_4$. Hence $r \prec r$ gives a contradiction.

(Subcase 1.2) For all $(c_0, c_1) \in v$ and $(d_0, d_1) \in v$, $d_0 \prec c_1$.

Let $A = \{x : (\exists (e_0, e_1) \in v)(x \preceq e_0)\}$. A is $\mathrm{OD}_S^{L[S,x]}$ and $A \neq \emptyset$ since for any element $(e_0, e_1) \in v$, $e_0 \in A$. A is $\preceq$-downward closed so $A \in \mathbb{U}_S^{L[S,x]}$. Now fix any $(e_0, e_1) \in v$. As observed just above, $e_0 \in A$. Note that $e_1 \notin A$. This is because if $e_1 \in A$, then there has to be some $(f_0, f_1) \in v$ so that $e_1 \preceq f_0$. This contradicts the Subcase 1.2 assumption. Thus A witnesses that $(e_0, e_1) \notin u$. This shows that v and u are disjoint which contradicts the fact that $v \leq_{{}_2\mathbb{O}_S^{L[S,x]}} u$.

Claim 1 has been established. (Claim 1 is enough to produce a perfect set of disjoint open intervals. However, a perfect set of disjoint closed intervals can actually be constructed using the next claim.)

Claim 2: Let $M \models \mathsf{ZF}$ be an inner model of V such that $M \supseteq \mathrm{HOD}_S^{L[S,x]}$. Let $r \in \mathbb{R}^M$. Then

$$M \models u \Vdash_{{}_2\mathbb{O}_S^{L[S,x]}} \tau_0 \neq \check{r} \wedge \tau_1 \neq \check{r}.$$

To prove this: Suppose there is some $v \leq_{{}_2\mathbb{O}_S^{L[S,x]}} u$ so that (without loss of generality) $M \models v \Vdash_{{}_2\mathbb{O}_S^{L[S,x]}} \tau_0 = \check{r}$.

Suppose there is some $(c_0, c_1) \in v$ so that there is some $n \in \omega$ with $c_0(n) \neq r(n)$. Let $v' = \{(c_0, c_1) \in (\mathbb{R}^2)^{L[S,x]} : c_0(n) \neq r(n)\}$. Note $v' \in \mathbb{O}_S^{L[S,x]}$. (Observe that v' is a condition which can be expressed in the form for which Fact 2.7 applies.) By the assumption, $v' \cap v \in \mathbb{O}_S^{L[S,x]}$. Let (g_0, g_1) be a ${}_2\mathbb{O}_S^{L[S,x]}$-generic over M whose associated generic-filter contains the condition $v \cap v'$. Since v' belong to this generic filter, Fact 2.7 implies that $g_0(n) \neq r(n)$. However since v belongs to this generic filter, the forcing theorem implies that $g_0(n) = r(n)$. Contradiction.

It has been shown that for all $(c_0, c_1) \in v$, $c_0 = r$. Hence $\{r\} = \{x : (\exists (c_0, c_1) \in v) : x = c_0\}$. Thus r is $\mathrm{OD}_S^{L[S,x]}$. Now let (r, c_1) be some element of v. The set $\{x \in \mathbb{R}^{L[S,x]} : x \leq r\}$ is in $\mathbb{U}_S^{L[S,x]}$. Clearly r is in this set but c_1 is not. Thus $(r, c_1) \notin u$. This contradicts $(r, c_1) \in v \subseteq u$. This completes the proof of Claim 2.

Since $V \models \mathsf{AD}$, there are only countably many dense open subsets of ${}_2\mathbb{O}_S^{L[S,x]}$ in $\mathrm{HOD}_S^{L[S,x]}$. In V, let $(D_n : n \in \omega)$ enumerate all of these dense open sets. By intersecting, one may assume that for all n, $D_{n+1} \subseteq D_n$. Similarly, let $(E_n : n \in \omega)$ enumerate all the dense open subsets of ${}_2\mathbb{O}_S^{L[S,x]} \times {}_2\mathbb{O}_S^{L[S,x]}$ in $\mathrm{HOD}_S^{L[S,x]}$. Assume again that the sequence is decreasing.

Let $p_\emptyset \leq_{{}_2\mathbb{O}_S^{L[S,x]}} u$ be any condition of ${}_2\mathbb{O}_S^{L[S,x]}$ that meets D_0.

Suppose for some $n \in \omega$, p_σ has been defined for all $\sigma \in {}^n2$. For each $\sigma\hat{}i$, let $p'_{\sigma\hat{}i}$ be some condition below p_σ that meets D_{n+1}.

Let $\tau_1, ..., \tau_{2^{n+1}}$ enumerate ${}^{n+1}2$, the set of binary strings of length $n+1$. For each $1 \leq m \leq 2^{n+1}$, let $q^0_m = p'_{\tau_m}$. For $0 \leq k < 2^{n+1}$, suppose q^k_m has been defined. For each $1 \leq m \leq 2^{n+1}$, find some $q^{k+1}_m \leq_{{}_2\mathbb{O}_S^{L[S,x]}} q^k_m$ so that for all $m \neq k+1$, $(q^{k+1}_m, q^{k+1}_{k+1})$ and $(q^{k+1}_{k+1}, q^{k+1}_m)$ meet E_{n+1}.

Let $p_{\tau_m} = q^{2^{n+1}}_m$. This completes the construction of $(p_\sigma : \sigma \in {}^{<\omega}2)$.

For each $x \in \mathbb{R}$, let G_x be the upward closure of $\{p_{x\restriction n} : n \in \omega\}$. By construction, G_x is ${}_2\mathbb{O}_S^{L[S,x]}$-generic over $\mathrm{HOD}_S^{L[S,x]}$. If $x \neq y$, then by construction $G_x \times G_y$ are ${}_2\mathbb{O}_S^{L[S,x]} \times {}_2\mathbb{O}_S^{L[S,x]}$-generic over $\mathrm{HOD}_S^{L[S,x]}$. For each $x \in \mathbb{R}$, let (c^x_0, c^x_1) denote the generic pair added by G_x. Using Claim 1 and 2, one has that if $x \neq y$, the intervals $[c^x_0, c^x_1]$ and $[c^y_0, c^y_1]$ are disjoint.

The proof is complete. □

Fact 4.3. (ZF) *Suppose $(P, \prec)$ is a complete dense nonseparable linear ordering with the countable chain condition and has a wellordered separating family, then there is a Suslin tree (on ω_1).*

Proof. In this case, the usual construction of a Suslin tree from a Suslin line works. The details follows:

Let $\langle A_\alpha : \alpha < \kappa\rangle$ for some κ be a sequence of $\preceq$-downward closed subsets of P which forms a separating family for $(P, \prec)$.

Let $B_0 = \emptyset$. Choose the least pair (α, β) so that $A_\alpha \subseteq A_\beta$ and $A_\beta \setminus A_\alpha$ is infinite. $A_\beta \setminus A_\alpha$ is a bounded infinite set. Using completeness, let $a_0 = \inf A_\beta \setminus A_\alpha$ and $b_0 = \sup A_\beta \setminus A_\alpha$. Let $B_1 = \{a_0, b_0\}$.

If δ is a limit ordinal, let $B_\delta = \bigcup_{\gamma<\delta} B_\gamma$.

Suppose for some $\delta < \omega_1$, (a_γ, b_γ) has been defined for all $\gamma < \delta$ and B_δ has been defined. Since B_δ is countable and P is not separable, there is some interval (a, b) so that $(a, b) \cap B_\delta = \emptyset$. By density, find $a', b' \in P$ so that $a \prec a' \prec b' \prec b$. There is some ν and ζ so that A_ν separates a from a' and A_ζ separates b' from b. Then $A_\zeta \setminus A_\nu \cap B_\delta = \emptyset$ and $A_\zeta \setminus A_\nu$ is an infinite set since $a', b' \in A_\zeta \setminus A_\nu$ and density.

Now let (α, β) be the least pair so that $A_\beta \setminus A_\alpha$ is infinite and $A_\beta \setminus A_\alpha \cap B_\delta = \emptyset$. $A_\beta \setminus A_\alpha$ is a bounded nonempty set so by completeness, let $a_\delta = \inf A_\beta \setminus A_\alpha$ and $b_\delta = \sup A_\beta \setminus A_\alpha$. Let $B_{\delta+1} = B_\delta \cup \{a_\delta, b_\delta\}$.

This construction produces points a_α and b_α for each $\alpha < \omega_1$. Let I_α be the open interval (a_α, b_α). Define a tree $(\omega_1, \sqsubset)$ by $\alpha \sqsubset \beta$ if and only if $I_\beta \subsetneq I_\alpha$. Note that $\alpha \sqsubset \beta$ implies $\alpha < \beta$. Hence $(\omega_1, \sqsubset)$ is a tree.

If α and β are $\sqsubset$-incomparable, then I_α and I_β are disjoint intervals of $(P, \prec)$. Since $(P, \prec)$ has the countable chain condition, $\sqsubset$ cannot have an uncountable antichain.

Suppose $(\epsilon_\alpha : \alpha < \omega_1)$ forms an uncountable chain in $\sqsubset$. Then $(a_{\epsilon_\alpha} : \alpha < \omega_1)$ is a $\prec$-increasing sequence. Then $((a_{\epsilon_\alpha}, a_{\epsilon_{\alpha+1}}) : \alpha < \omega_1)$ forms an uncountable collection of disjoint open intervals of $(P, \prec)$. This contradicts the countable chain condition.

$(\omega_1, \sqsubset)$ is a Suslin-tree. □

If J is a set of ordinals and $L(J,\mathbb{R}) \models \mathsf{AD}$, then every set in $L_\Theta(J,\mathbb{R})$ is the surjective image of $\mathbb{R}$. Hence every linear ordering in $L_\Theta(J,\mathbb{R})$ is the quotient of a prelinear ordering on $\mathbb{R}$. The following has been shown:

Theorem 4.4. ($\mathsf{ZF} + \mathsf{AD}^+$) *There is no Suslin line on a set which is the surjective image of* $\mathbb{R}$.

If J is a set of ordinals and $L(J,\mathbb{R}) \models \mathsf{AD}$, then there are no Suslin lines in $L_\Theta(J,\mathbb{R})$.

Theorem 4.5. *Let J be a set of ordinal. If $L(J,\mathbb{R}) \models \mathsf{AD}$, then $L(J,\mathbb{R}) \models \mathsf{SH}$.*

In particular, $\mathsf{ZF} + \mathsf{AD} + \mathsf{V} = \mathsf{L}(\mathbb{R}) \vdash \mathsf{SH}$.

Proof. Let $\Phi : \mathbb{R} \times \mathrm{ON} \to L(J,\mathbb{R})$ be a definable surjection using only J as a parameter. Using a definable surjection of On onto On $\times$ On so that the preimage of any element is a proper class, one can assume that for any $x, y \in L(J,\mathbb{R})$, there is a proper class of ordinals α such that there are $r, s \in \mathbb{R}$ with $\Phi(r,\alpha) = x$ and $\Phi(s,\alpha) = y$.

Let $(P, \prec)$ be a complete dense nonseparable linear ordering. $(P, \prec)$ is $\mathrm{OD}_{J,z}$ for some $z \in \mathbb{R}$.

Let $Q_\alpha = \{x \in \mathbb{R} : \Phi(x,\alpha) \in P\}$. Let $P_\alpha = \Phi[Q_\alpha \times \{\alpha\}]$. Let $(P_\alpha, \prec)$ be the restriction of $\prec$ to P_α. Let $(Q_\alpha, \sqsubseteq)$ be the induced prelinear ordering coming from Φ and $(P_\alpha, \prec)$. For all α, $(P_\alpha, \prec)$ and $(Q_\alpha, \sqsubseteq)$ are uniformly $\mathrm{OD}_{J,z}$ in the sense that using α and the formula and ordinal that gives the $\mathrm{OD}_{J,z}$ definition of $(P, \prec)$, one can explicitly produce the ordinal and formula giving the $\mathrm{OD}_{J,z}$-definition of $(P_\alpha, \prec)$ and $(Q_\alpha, \sqsubseteq)$.

If any $(Q_\alpha, \sqsubseteq)$ has a perfect set of disjoint closed intervals, then collapsing using Φ would give a perfect set of disjoint closed intervals in $(P_\alpha, \prec)$. This would imply that $(P, \prec)$ has a perfect set of disjoint closed intervals. In this case, $(P, \prec)$ does not have the countable chain condition.

Therefore, assume for all $\alpha \in \mathrm{ON}$, $(Q_\alpha, \sqsubseteq)$ does not have a perfect set of disjoint closed intervals. Theorem 4.2 implies that each $(Q_\alpha, \sqsubseteq)$ has a wellordered separating family. Using Fact 2.9, one can obtain a sequence of ∞-Borel codes of $(Q_\alpha, \sqsubset)$ by chosing the $\mathrm{HOD}_{J,z}$-least ∞-Borel code for $(Q_\alpha, \sqsubseteq)$. Since the argument in Case I of Theorem 4.2 gives an explicit procedure for obtaining the separating family from the ∞-Borel code of the prelinear ordering, one has a sequence $\langle \mathcal{E}'_\alpha : \alpha \in \mathrm{On} \rangle$ such that each $\mathcal{E}'_\alpha$ is a wellordered separating family for $(Q_\alpha, \sqsubseteq)$ along with the wellordering. Collapsing using Φ, let $\langle \mathcal{E}_\alpha : \alpha \in \mathrm{ON} \rangle$ be the derived sequence of wellordered separating family for $(P_\alpha, \prec)$ along with the wellordering.

For any α, if $A \in \mathcal{E}_\alpha$, let $\bar{A}$ be the $\preceq$-downward closure of A in $(P, \prec)$. Let $\mathcal{S} = \{\bar{A} : (\exists \alpha)(A \in \mathcal{E}_\alpha)\}$. Using the wellordering of ordinals and the wellordering of each $\mathcal{E}_\alpha$ which is given uniformly, $\mathcal{S}$ can be wellordered. Suppose $a, b \in P$ and $a \prec b$. By the assumption on Φ mentioned at the beginning of the proof, there is some α so that $a, b \in P_\alpha$. There is some $A \in \mathcal{E}_\alpha$ so that $a \in A$ and $b \notin A$. Then $\bar{A} \in \mathcal{S}$, $a \in \bar{A}$, and $b \notin \bar{A}$. It has been shown that $\mathcal{S}$ is a wellordered separating family for $(P, \prec)$. By Fact 3.1 and Fact 4.3, $(P, \prec)$ cannot have the countable chain condition. □

Woodin showed that assuming $\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, if $\mathsf{AD}_{\mathbb{R}}$ fails, then there is some set of ordinal J so that $V = L(J, \mathbb{R})$. To study the Suslin hypothesis in these natural models of AD^+, it remains to consider $L(\mathscr{P}(\mathbb{R})) \models \mathsf{AD}_{\mathbb{R}}$. The argument uses techniques from [1]. The following results of Woodin will be necessary.

Fact 4.6. *([1] Theorem 3.3) Assume* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$. *Then every set is ordinal definable from some element of* $\bigcup_{\lambda<\Theta} \mathscr{P}_{\omega_1}(\lambda)$.

Fact 4.7. *([16] and [1] Theorem 2.13)* $(\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}})$ *For each* $\lambda < \Theta$, *there is a unique fine normal measure on* $\mathscr{P}_{\omega_1}(\lambda)$ *which is also* OD. *(Such a measure is called a supercompactness measure on* $\mathscr{P}_{\omega_1}(\lambda)$.*)*

Theorem 4.8. $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R})) \vdash \mathsf{SH}$.

Proof. Suppose $(P, \prec)$ is a complete dense nonseparable linear ordering. By Fact 4.6, there is some $\alpha < \Theta$ and some $\sigma \in \mathscr{P}_{\omega_1}(\alpha)$ so that $(P, \prec)$ is OD_σ.

For each $\tau \in \bigcup_{\beta<\Theta} \mathscr{P}_{\omega_1}(\beta)$, $\zeta \in \mathrm{ON}$, and formula φ, define $P_{\tau,\zeta,\varphi}$ to be the set of $x \in P$ such that there is some $r \in \mathbb{R}$ so that x is the unique solution v to $\varphi(\sigma, \tau, \zeta, r, v)$. Let $(P_{\tau,\zeta,\varphi}, \prec)$ be the linear ordering resulting from restricting $\prec$ to $P_{\tau,\zeta,\varphi}$. Let $\star$ be some set not in P. There is a surjection of $\mathbb{R}$ onto $P_{\tau,\zeta,\varphi} \cup \{\star\}$ defined by letting r map to the unique solution v in $\varphi(\sigma, \tau, \zeta, r, v)$ if it exists and $\star$ otherwise. Let $(Q_{\tau,\zeta,\varphi}, \sqsubseteq)$ be the induced prelinear ordering on $\mathbb{R}$ coming from $(P_{\tau,\zeta,\varphi}, \prec)$ with $\star$ as the largest element via the surjection.

Observe that if $\rho \in \bigcup_{\beta<\Theta} \mathscr{P}_{\omega_1}(\beta)$ and $\rho \supseteq \tau$, then for all $\zeta \in \mathrm{ON}$ and formula φ, there is some other formula ψ so that $P_{\tau,\zeta,\varphi} \subseteq P_{\rho,\zeta,\psi}$.

Let Form be the collection of formulas. For each $\tau \in \bigcup_{\beta<\Theta} \mathscr{P}_{\omega_1}(\beta)$, let

$$P_\tau = \bigcup_{\zeta \in \mathrm{ON} \wedge \varphi \in \mathrm{Form}} P_{\tau,\zeta,\varphi}.$$

By the previous observation, if $\tau \subseteq \rho$, then $P_\tau \subseteq P_\rho$.

For each fixed τ and formula φ, $P_{\tau,\zeta,\varphi}$ and $Q_{\tau,\zeta,\varphi}$ are $\mathrm{OD}_{\sigma,\tau}$ with witnessing formulas obtained uniformly from the OD_σ witness to P. By Fact 2.9, choose the $\mathrm{HOD}_{\sigma,\tau}$-least ∞-Borel code for $(Q_{\tau,\zeta,\varphi}, \sqsubseteq)$ to be the canonical ∞-Borel code for this set. If any $(Q_{\tau,\zeta,\varphi}, \sqsubseteq)$ has a perfect set of disjoint closed intervals, then such a collection would yield a perfect set of disjoint closed intervals for $(P_{\tau,\zeta,\varphi}, \prec)$ and hence $(P, \prec)$. Therefore, one may assume that each $(Q_{\tau,\zeta,\varphi}, \sqsubseteq)$ satisfies Case I of Theorem 4.2. By the proof in Case I in Theorem 4.2, this gives a sequence $\mathcal{E}'_{\tau,\zeta,\varphi}$ of wellordered separating families (along with the wellordering) for $(Q_{\tau,\zeta,\varphi}, \sqsubset)$. Collapsing $\mathcal{E}'_{\tau,\zeta,\varphi}$, one obtains a separating family $\mathcal{E}_{\tau,\zeta,\varphi}$ for $(P_{\tau,\zeta,\varphi}, \prec)$. Using the wellordering of the ordinals and the wellordering of each $\mathcal{E}_{\tau,\zeta,\varphi}$, one obtains a wellordering of $\mathcal{E}'_\tau = \bigcup_{\zeta \in \mathrm{ON} \wedge \varphi \in \mathrm{Form}} \mathcal{E}'_{\tau,\zeta,\varphi}$. Downward $\preceq$-closing each set of $\mathcal{E}'_\tau$ in P_τ gives a wellordered separating family $\mathcal{E}_\tau$ for P_τ.

Now fix $\alpha \leq \beta < \Theta$. Define

$$P_\beta = \bigcup \{P_\tau : \tau \in \mathcal{P}_{\omega_1}(\beta)\}.$$

Note that the sequence $\langle \mathcal{E}_\tau : \tau \in \mathcal{P}_{\omega_1}(\beta)\rangle$ belongs to $\mathrm{OD}^V_{\mathscr{P}_{\omega_1}(\beta)}$. Let μ_β be the unique OD supercompactness measure for $\mathscr{P}_{\omega_1}(\beta)$. Let $U_\beta = (\prod_{\mathscr{P}_{\omega_1}(\beta)} \mathrm{ON}/\mu_\beta)^{\mathrm{HOD}^V_{\mathscr{P}_{\omega_1}(\beta)}}$. Now suppose $[f]_\sim \in U_\beta$ where $f \in \mathrm{HOD}^V_{\mathscr{P}_{\omega_1}(\beta)}$. Let $A_{[f]_\sim}$ be the set of $z \in P_\beta$ such that the set K of $\rho \in \mathscr{P}_{\omega_1}(\beta)$ such that z belongs to the $f(\rho)^{\text{th}}$ set in $\mathcal{E}_\rho$ (according to the wellordering of $\mathcal{E}_\rho$) belongs to μ_β.

Each $A_{[f]_\sim}$ is $\preceq$-downward closed in P_β. Suppose $z_1, z_1 \in P_\beta$, $z_1 \preceq z_2$, and $z_2 \in A_{[f]_\sim}$. There is a $K \in \mu_\beta$ so that for all $\rho \in K$, z_2 belong to the $f(\rho)^{\text{th}}$ set in $\mathcal{E}_\rho$. $z_1, z_2 \in P_\beta$ means that $z_1 \in P_{\xi_1}$ and $z_2 \in P_{\xi_2}$ for some $\xi_1, \xi_2 \in \mathscr{P}_{\omega_1}(\beta)$. For $\rho \in \mathscr{P}_{\omega_1}(\beta)$, let $R_\rho = \{\gamma \in \mathscr{P}_{\omega_1}(\beta) : \rho \subseteq \gamma\}$. By fineness and countable completeness, $R_\rho \in \mu_\beta$. For any $\rho \in R_{\xi_1} \cap R_{\xi_2} \cap K \in \mu$, $z_1, z_2 \in P_\rho$. This shows that $z_1 \in A_{[f]_\sim}$.

Now suppose $z_1, z_2 \in P_\beta$ and $z_1 \prec z_2$. There is some $\xi_1, \xi_2 \in \mathscr{P}_{\omega_1}(\beta)$ so that $z_1 \in P_{\xi_1}$ and $z_2 \in P_{\xi_2}$. Note that this implies that z_1 and z_2 are $\mathrm{OD}_{\mathbb{R}\cup\{\sigma,\xi_1,\xi_2\}}$. In particular, they belong to $\mathrm{OD}_{\mathscr{P}_{\omega_1}(\beta)}$. Hence if $\rho \supseteq \zeta_1 \cup \zeta_2$, then $z_1, z_2 \in P_\rho$. Define $f : \mathscr{P}_{\omega_1}(\beta) \to \mathrm{On}$ by $f(\rho)$ is the least ordinal α so that the α^{th} element of $\mathcal{E}_\rho$ contains z_1 but not z_2 whenever $\rho \in R_{\xi_1\cup\xi_2}$ and 0 otherwise. Note that $f \in \mathrm{HOD}^V_{\mathscr{P}_{\omega_1}(\beta)}$ so $[f]_\sim \in U_\beta$. Then $A_{[f]_\sim}$ separates z_1 from z_2.

Let $\mathcal{E}_\beta = \langle A_{[f]_\sim} : [f]_\sim \in U_\beta\rangle$. Note that $\mathrm{HOD}^V_{\mathscr{P}_{\omega_1}(\beta)} \models \mathsf{DC}$ since $V \models \mathsf{DC}_{\mathbb{R}}$. Hence U_β is wellfounded. $\mathcal{E}_\beta$ is a wellordered separating family for P_β.

One has produced a sequence of separating families $\langle \mathcal{E}_\beta : \beta < \Theta\rangle$ for $\langle P_\beta : \beta < \Theta\rangle$. Using the wellordering of the ordinals and the wellordering of each $\mathcal{E}_\beta$ for $\beta < \Theta$, one obtains, as before, a wellordered separating family $\mathcal{S}$ for $P = \bigcup_{\alpha\leq\beta<\Theta} P_\beta$.

Now Fact 3.1 and Fact 4.3 imply that there are no Suslin lines. □

Theorem 4.9. $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R})) \vdash \mathsf{SH}$.

Proof. This follows from Theorem 4.5, Theorem 4.8, and the comments after Definition 2.5. □

5. The Solovay model

Woodin has observed that the methods above can be applied to explore the Suslin hypothesis in the Solovay model. This section will give the details of Woodin's argument.

Let V denote the ground model satisfying ZFC. Let κ be an inaccessible cardinal. Let $\mathrm{Coll}(\omega, <\kappa)$ be the finite support product of $\langle \mathrm{Coll}(\omega, \xi) : \xi < \kappa\rangle$. Let $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ be $\mathrm{Coll}(\omega, <\kappa)$-generic over V. $V(\mathbb{R}^{V[G]}) = \mathrm{HOD}^{V[G]}_{V\cup\mathbb{R}^{V[G]}}$ is known as the Solovay model (of an inaccessible cardinal). If κ is weakly compact or measurable, one will refer to the resulting model as the Solovay model of a weakly compact cardinal or measurable

cardinal. For more about the Solovay model see [13], [7] Chapter 26, [11] Chapter 8, or [8] Section 11.

In the Solovay model, every set of reals has a code which is a generalization of the ∞-Borel code which allows a parameter from V:

Definition 5.1. Assume $V \models \mathsf{ZFC}$ and κ is an inaccessible cardinal. Let $G \subseteq \mathrm{Coll}(\omega, < \kappa)$ be $\mathrm{Coll}(\omega, < \kappa)$-generic over V. A Solovay code is a triple (v, r, φ) where $v \in V$, $r \in \mathbb{R}^{V[G]}$ and φ is formula. The set coded by (v, r, φ) is $\{x \in \mathbb{R}^{V[G]} : V[r][x] \models \varphi(v, r, x)\}$.

Fact 5.2. *Let $V \models \mathsf{ZFC}$ and let κ be an inaccessible cardinal of V. Let G be $\mathrm{Coll}(\omega, < \kappa)$-generic over V. Suppose $X \in V(\mathbb{R}^{V[G]})$ is a set of reals. Then there is some $v \in V$, $r \in \mathbb{R}^{V[G]}$, and formula φ so that $x \in X$ if and only if $V[G] \models \varphi(v, r, x)$. (The objects φ, v, and r are the witnesses to $X \in V(\mathbb{R}^{V[G]})$.) Then for all $x \in \mathbb{R}^{V[G]}$, $x \in X$ if and only if*

$$V[r][x] \models 1_{\mathrm{Coll}(\omega, < \kappa)} \Vdash_{\mathrm{Coll}(\omega, < \kappa)} \varphi(\check{v}, \check{r}, \check{x}).$$

Thus every set of reals in $V(\mathbb{R}^{V[G]})$ has a Solovay code. In particular, given the witnesses to $X \in V(\mathbb{R}^{V[G]})$, the above gives an explicit procedure to obtain a Solovay code for X.

Proof. This is a well-known result of Solovay. The following is a brief sketch.

Suppose $x \in X$. So $V[G] \models \varphi(v, r, x)$. Note that r and x are generic over V since they belong to some $V[G \restriction \xi]$ where $\xi < \kappa$ and $G \restriction \xi$ is the induced $\mathrm{Coll}(\omega, < \xi)$-generic over V coming from G. By a crucial property of the Lévy collapse, there is some $H \subseteq \mathrm{Coll}(\omega, < \kappa)$ which is $\mathrm{Coll}(\omega, < \kappa)$-generic over $V[r][x]$ so that $V[G] = V[r][x][H]$. Therefore, $V[r][x][H] \models \varphi(v, r, x)$. By the forcing theorem, there is some $p \in \mathrm{Coll}(\omega, < \kappa)$ so that

$$V[r][x] \models p \Vdash_{\mathrm{Coll}(\omega, < \kappa)} \varphi(\check{v}, \check{r}, \check{x}).$$

By the homogeneity of $\mathrm{Coll}(\omega, < \kappa)$,

$$V[r][x] \models 1_{\mathrm{Coll}(\omega, < \kappa)} \Vdash_{\mathrm{Coll}(\omega, < \kappa)} \varphi(\check{v}, \check{r}, \check{x}).$$

Now suppose

$$V[r][x] \models 1_{\mathrm{Coll}(\omega, < \kappa)} \Vdash_{\mathrm{Coll}(\omega, < \kappa)} \varphi(\check{v}, \check{r}, \check{x}).$$

As before, there is some $H \subseteq \mathrm{Coll}(\omega, < \kappa)$ which is $\mathrm{Coll}(\omega, < \kappa)$-generic over $V[r][x]$ so that $V[r][x][H] = V[G]$. Hence $V[G] \models \varphi(v, r, x)$. This shows that $x \in X$. □

If κ is a measurable cardinals and $G \subseteq \mathrm{Coll}(\omega, < \kappa)$ is generic over V, the prelinear order dichotomy result for the associated Solovay model $V(\mathbb{R}^{V[G]})$ follows by methods

similar to the arguments in the determinacy setting. This is done by replacing Martin's measure with a fine countably complete ultrafilter on $\mathscr{P}_{\omega_1}(\mathbb{R})$, ∞-Borel codes with Solovay codes, and the Vopěnka forcings with forcing of OD_V subsets of $\mathbb{R}$. The following is a brief sketch of the main modifications.

Subsequently, Woodin's argument for the prelinear ordering dichotomy theorem for Solovay models of inaccessible cardinals will be given. This will require more substantial modifications involving intervals generated by $\mathrm{Coll}(\omega, \xi)$-names for reals, where $\xi < \kappa$.

Fact 5.3. *Suppose $V \models \mathsf{ZFC}$ and κ is a measurable cardinal. Let $G \subseteq \mathrm{Coll}(\omega, < \kappa)$ be $\mathrm{Coll}(\omega, < \kappa)$-generic over V. In $V(\mathbb{R}^{V[G]})$, there is a fine countably complete ultrafilter on $\mathscr{P}_{\omega_1}(\mathbb{R})$.*

Theorem 5.4. *Let $V \models \mathsf{ZFC}$ and κ be a measurable cardinal of V. Let G be $\mathrm{Coll}(\omega, < \kappa)$-generic over V. The following holds in $V(\mathbb{R}^{V[G]})$, the Solovay model of a measurable cardinal:*

Let $\preceq$ be a prelinear order on $\mathbb{R}$. Exactly one of the following holds.

(i) *There is a perfect set of disjoint closed intervals in $\preceq$. (That is, this set of intervals is in bijection with $\mathbb{R}$.)*
(ii) *There is a wellordered separating family for $\preceq$.*

Proof. This can be proved by a modification of the argument in Theorem 4.2. Let μ denote the fine countably complete ultrafilter on $\mathscr{P}_{\omega_1}(\mathbb{R})$ in $V(\mathbb{R}^{V[G]})$ given by Fact 5.3.

For each $r \in \mathbb{R}^{V[G]}$ and $X \in \mathscr{P}_{\omega_1}(\mathbb{R})$, let M_r^X denote the model $\mathrm{HOD}^{V(\mathbb{R}^{V[G]})}_{V \cup X \cup \{r, X\}}$.

Using replacement in $V[G]$, choose $\delta' \in \mathrm{ON}$ so large so that for all $r \in \mathbb{R}^{V[G]}$ and $X \in \mathscr{P}_{\omega_1}(\mathbb{R})$, every $\mathrm{OD}^{M_r^X}_{V \cup \{r\}}$ set of reals has a definition whose parameters from V are actually from $V_{\delta'}$. Then choose $\delta \geq \delta'$ so that every $(\mathrm{OD}_{V \cup \mathbb{R}})^{V[G]}$ set of reals has a definition whose parameters from V are actually from V_δ.

Let $\preceq$ be a prelinear ordering on $\mathbb{R}$ in $V(\mathbb{R}^{V[G]})$. Let (v, r, φ) be the Solovay code for $\preceq$ given by Fact 5.2. (By the explicit procedure to produce the Solovay code in Fact 5.2 and the choice of δ, one may assume $v \in V_\delta$.) Let $\mathbb{O}_r$ denote the forcing of $\mathrm{OD}_{V \cup \{r\}}$ subsets of the reals. Let $\mathbb{U}_r$ denote the sets in $\mathbb{O}_r$ which are $\preceq$-downward closed. Using parameters in V_δ, for each $X \in \mathscr{P}_{\omega_1}(\mathbb{R})$, $\mathbb{O}_r^{M_r^X}$ can be coded as a set in V. Hence it may be considered a forcing in $\mathrm{HOD}^{M_r^X}_{V \cup \{r\}}$. This forcing still has the basic properties of the ordinary Vopěnka forcing. As $V \models \mathrm{AC}$, fix some wellordering of V_δ which belongs to V for the rest of the proof. This wellordering can be used to define a wellordering of $\mathbb{O}_r$.

Work in $V(\mathbb{R}^{V[G]})$. The proof splits into two cases. The following includes some details of how to handle each case.

(Case I) For all $X \in \mathscr{P}_{\omega_1}(\mathbb{R})$, for all $a, b \in \mathbb{R}^{M_r^X}$ with $a \prec b$, there exists some $A \in \mathbb{U}_r^{M_r^X}$ with $a \in A$ and $b \notin A$.

If $f \in \prod_{\mathscr{P}_{\omega_1}(\mathbb{R})} \omega_1$, then $[f]_\sim$ denotes the equivalence class of f modulo μ. For each $[f]_\sim \in \prod_{\mathscr{P}_{\omega_1}(\mathbb{R})} \omega_1/\mu$, let $A_{[f]_\sim}$ be the set of $y \in \mathbb{R}$ so that the set of $X \in \mathscr{P}_{\omega_1}(\mathbb{R})$ with the property that y belongs to the $f(X)^{\text{th}}$ element of $\mathbb{U}_r^{M_r^X}$ according to the wellordering of $\mathbb{O}_r^{M_r^X}$ (defined from the fixed wellordering of V_δ) belongs to μ.

For each $[f]_\sim$, $A_{[f]_\sim}$ is $\preceq$-downward closed. To see this: Suppose $a \preceq b$ and $b \in A_{[f]_\sim}$. Since $b \in A_{[f]_\sim}$, if K is the set of $X \in \mathscr{P}_{\omega_1}(\mathbb{R})$ so that b belongs to the $f(X)^{\text{th}}$ set in $\mathbb{U}_r^{M_r^X}$, then $K \in \mu$. For any $c \in \mathbb{R}$, let $O_c = \{X \in \mathscr{P}_{\omega_1}(\mathbb{R}) : c \in X\}$. By fineness, $O_a, O_b \in \mu$. Thus $K \cap O_a \cap O_b \in \mu$. For any $X \in K \cap O_a \cap O_b$, one has that $a, b \in M_r^X$. Hence for all $X \in K \cap O_a \cap O_b$, a belongs to the $f(X)^{\text{th}}$ set in $\mathbb{U}_r^{M_r^X}$.

Since $V(\mathbb{R}^{V[G]}) \models \mathsf{DC}$ and μ is countably complete, $\prod_{\mathscr{P}_{\omega_1}(\mathbb{R})} \omega_1/\mu$ is a wellordering. The claim is that $\{A_{[f]_\sim} : [f]_\sim \in \prod_{\mathscr{P}_{\omega_1}(\mathbb{R})} \omega_1/\mu\}$ is a wellordered separating family.

So see this: Suppose $a \prec b$. Define a function f by letting, for each $X \in \mathscr{P}_{\omega_1}(\mathbb{R})$, $f(X)$ be the least ordinal α so that the α^{th} set in $\mathbb{U}_r^{M_r^X}$ contains a but does not contain b. Such a set exists using the Case I assumption. Then $a \in A_{[f]_\sim}$ and $b \notin A_{[f]_\sim}$.

(Case II) There exists some $X \in \mathscr{P}_{\omega_1}(\mathbb{R})$, there exist $a, b \in \mathbb{R}^{M_r^X}$ with $a \prec b$ so that for all $A \in \mathbb{U}_r^{M_r^X}$, $a \notin A$ or $b \in A$.

The argument in this case is essentially the same as in Theorem 4.2. It should be noted that at the beginning of Case II in Theorem 4.2, one defines a condition which in the present situation would take the form

$$u = \{(c_0, c_1) \in (\mathbb{R}^2)^{M_r^X} : c_0 \prec c_1 \wedge (\forall A)(A \in \mathbb{U}_r^{M_r^X} \Rightarrow (c_0 \notin A \vee c_1 \in A))\}$$

This definition uses $\mathbb{U}_r^{M_r^X}$ as a parameter. By using V_δ as a parameter, $\mathbb{O}_r^{M_r^X}$ and also $\mathbb{U}_r^{M_r^X}$ can be identified as sets in $V[r]$. Hence u is indeed $\mathrm{OD}_{V \cup \{r\}}^{M_r^X}$.

With these modifications, the results follow. □

Theorem 5.5. *(Woodin) Let κ be an inaccessible cardinal. Let $G \subseteq \mathrm{Coll}(\omega, < \kappa)$ be generic over V. Then the following holds in $V(\mathbb{R}^{V[G]})$: Let $(\mathbb{R}, \preceq)$ be a dense prelinear ordering on $\mathbb{R}$. One of the following holds*

(i) *There is a wellordered separating family for $(\mathbb{R}, \preceq)$.*
(ii) *There is a perfect set of disjoint open intervals for $(\mathbb{R}, \preceq)$.*

Proof. Work in $V(\mathbb{R}^{V[G]})$. By Fact 5.2, $\preceq$ has a Solovay code (v, r, φ). Without loss of generality, assume that $r \in \mathbb{R}^V$. In the remainder of the proof, $\preceq$ will always refer to the set defined by this Solovay code.

For any $\xi < \kappa$, $p \in \mathrm{Coll}(\omega, \xi)$, and $\mathrm{Coll}(\omega, \xi)$-name τ such that $p \Vdash \tau \in \mathbb{R}$, let $\mathrm{Eval}(\xi, p, \tau)$ be the collection of $\tau[h]$ where $h \in V(\mathbb{R}^{V[G]})$, $h \subseteq \mathrm{Coll}(\omega, \xi)$ is generic over V, and $p \in h$. Let $I(\xi, p, \tau)$ be the $\preceq$-interval generated by $\mathrm{Eval}(\xi, p, \tau)$ in $V(\mathbb{R}^{V[G]})$. That is, $x \in I(\xi, p, \tau)$ if and only if there exists $a, b \in \mathrm{Eval}(\xi, p, \tau)$ so that $a \preceq x \preceq b$. Suppose

$g \subseteq \mathrm{Coll}(\omega, \xi)$ belongs to $V(\mathbb{R}^{V[G]})$ and is generic over V. Let $I(\xi, \tau, g) = \bigcap\{I(\xi, p, \tau) : p \in g \wedge p \Vdash \tau \in \mathbb{R}\}$. Note that $\tau[g] \in I(\xi, \tau, g)$.

(Case I) For all $\xi < \kappa$, $\mathrm{Coll}(\omega, \xi)$-name τ, and $g \subseteq \mathrm{Coll}(\omega, \xi)$ as above, $I(\xi, \tau, g)$ has only one $\preceq$-class (i.e. the $\preceq$-class of $\tau[g]$).

Choose λ large enough so that for every $\xi < \kappa$, $p \in \mathrm{Coll}(\omega, \xi)$, and $\mathrm{Coll}(\omega, \xi)$-name τ such that $p \Vdash \tau \in \mathbb{R}$, there is some $\mathrm{Coll}(\omega, \xi)$-name $\tau' \in V_\lambda$ so that $p \Vdash \tau = \tau'$. Since $V \models \mathsf{AC}$, use a fixed wellordering of V_λ to wellorder all (ξ, p, τ) such that $p \in \mathrm{Coll}(\omega, \xi)$, $\tau \in V_\lambda$ is a $\mathrm{Coll}(\omega, \xi)$-name such that $p \Vdash \tau \in \mathbb{R}$.

Suppose $a, c \in \mathbb{R}$ are such that $a \prec c$. By density, find some $b \in \mathbb{R}$ such that $a \prec b \prec c$. Find some $\xi < \kappa$, $\mathrm{Coll}(\omega, \xi)$-name $\tau \in V_\lambda$, and $g \subseteq \mathrm{Coll}(\omega, \xi)$ in $V(\mathbb{R}^{V[G]})$ so that $\tau[g] = b$. By the case assumption, there is some $p \in g$ so that $c \notin I(\xi, p, \tau)$. Let $A(\xi, p, \tau)$ be the $\preceq$-downward closure of $\mathrm{Eval}(\xi, p, \tau)$. Then $a \in A(\xi, p, \tau)$ but $c \notin A(\xi, p, \tau)$.

Using the wellordering of the collection of appropriate tuples (ξ, p, τ) from above, one can wellorder the collection of all appropriate $A(\xi, p, \tau)$. This gives a wellordered separating family for $\preceq$.

(Case II) For some $\xi < \kappa$, $\mathrm{Coll}(\omega, \xi)$-name τ, and $g \subseteq \mathrm{Coll}(\omega, \xi)$ as above, $I(\xi, \tau, g)$ has more than one $\preceq$-class.

Let $\dot{\mathbb{R}}$ be the canonical homogeneous $\mathrm{Coll}(\omega, < \kappa)$-name for the set of reals of the $\mathrm{Coll}(\omega, < \kappa)$-generic extension. By the basic properties of $\mathrm{Coll}(\omega, < \kappa)$, find some $H \subseteq \mathrm{Coll}(\omega, < \kappa)$ which is generic over $V[g]$ such that $V[G] = V[g][H]$. Note that $V[G] = V[g][H]$ models that $V(\mathbb{R}^{V[g][H]})$ thinks that $I(\xi, \tau, g)$ has more than one $\preceq$-class. Applying the forcing theorem and homogeneity of $\mathrm{Coll}(\omega, < \kappa)$ over $V[g]$, one has that $V[g]$ models that $1_{\mathrm{Coll}(\omega, <\kappa)}$ forces that $V(\dot{\mathbb{R}})$ thinks $I(\check{\xi}, \check{\tau}, \check{g})$ has more than one $\preceq$-class. Let $\dot{g}$ denote the canonical $\mathrm{Coll}(\omega, \xi)$-name for the generic filter. Then applying the forcing theorem in V, there is some $p^* \in \mathrm{Coll}(\omega, \xi)$ so that V models that p^* forces that $1_{\mathrm{Coll}(\omega, <\kappa)}$ forces that $V(\dot{\mathbb{R}})$ thinks $I(\check{\xi}, \check{\tau}, \dot{g})$ has more than one $\preceq$-class. The main observation is that for any generic $h \in V(\mathbb{R}^{V[G]})$ such that $p^* \in h$, $I(\xi, \tau, h)$ has more than one $\preceq$-class.

(Claim i) For any $p \leq p^*$, there exists $q_1, q_2 \leq p$ so that $(q_1, q_2) \Vdash \tau_{\text{left}} \prec \tau_{\text{right}}$, where τ_{left} and τ_{right} are the canonical $\mathrm{Coll}(\omega, \xi) \times \mathrm{Coll}(\omega, \xi)$-name for the evaluation of τ according to the left and right $\mathrm{Coll}(\omega, \xi)$-generic, respectively, added by a $\mathrm{Coll}(\omega, \xi) \times \mathrm{Coll}(\omega, \xi)$-generic filter.

To prove this: Since $p \leq p^*$, $\mathrm{Eval}(\xi, p, \tau)$ has representatives from more than one $\preceq$-class. Let h_1, h_2 be $\mathrm{Coll}(\omega, \xi)$-generics over V containing p and belongs to $V(\mathbb{R}^{V[G]})$ such that $\neg(\tau[h_1] \preceq \tau[h_2] \wedge \tau[h_2] \preceq \tau[h_1])$. Since $\mathrm{Coll}(\omega, \xi)$ and $\mathscr{P}(\mathrm{Coll}(\omega, \xi))^V$ is countable in $V(\mathbb{R}^{V[G]})$, find some $h \subseteq \mathrm{Coll}(\omega, \xi)$ generic over V, belonging to $V(\mathbb{R}^{V[G]})$, contains p, and such that $h_1 \times h$ and $h_2 \times h$ are $\mathrm{Coll}(\omega, \xi) \times \mathrm{Coll}(\omega, \xi)$-generic over V. Since $\neg(\tau[h_1] \preceq \tau[h_2] \wedge \tau[h_2] \preceq \tau[h_1])$ means that $\tau[h_1]$ and $\tau[h_2]$ are in two distinct $\preceq$-classes, $\tau[h]$ must not belong to the $\preceq$-class of $\tau[h_1]$ or $\tau[h_2]$. Without loss of generality, suppose $\tau[h]$ does not belong to the $\preceq$-class of $\tau[h_1]$ and in fact $\tau[h_1] \prec \tau[h]$. (The argument for the other possibilities are similar.) By the forcing theorem, there exists $(q_1, q_2) \leq (p, p)$ so that $(q_1, q_2) \Vdash \tau_{\text{left}} \prec \tau_{\text{right}}$. This proves Claim i.

(Claim ii) Suppose $(p_1, p_2) \leq (p^*, p^*)$ and $(p_1, p_2) \Vdash \tau_{\text{left}} \prec \tau_{\text{right}}$. Then for all $a \in \text{Eval}(\xi, \tau, p_1)$ and all $b \in \text{Eval}(\xi, \tau, p_2)$, one has that $a \prec b$.

To prove this: Let h_1 and h_2 be $\text{Coll}(\omega, \xi)$-generic filters over V so that $p_1 \in h_1$, $p_2 \in h_2$, $a = \tau[h_1]$, and $b = \tau[h_2]$. Let

$$D_1 = \{q \in \text{Coll}(\omega, \xi) : q \leq p_1 \wedge (\exists q')(q' \leq p_1 \wedge (q, q') \Vdash \tau_{\text{left}} \prec \tau_{\text{right}})\}$$
$$D_2 = \{q \in \text{Coll}(\omega, \xi) : q \leq p_2 \wedge (\exists q')(q' \leq p_2 \wedge (q', q) \Vdash \tau_{\text{left}} \prec \tau_{\text{right}})\}.$$

Claim i implies that D_1 and D_2 are dense below p_1 and p_2, respectively. Since $p_1 \in h_1$, $p_2 \in h_2$, genericity implies there are some $r_1 \in D_1 \cap h_1$ and $s_2 \in D_2 \cap h_2$. Hence there is some $r_2 < p_1$ and $s_1 < p_2$ so that $(r_1, r_2) \Vdash \tau_{\text{left}} \prec \tau_{\text{right}}$ and $(s_1, s_2) \Vdash \tau_{\text{left}} \prec \tau_{\text{right}}$. Let $k_1 \subseteq \text{Coll}(\omega, \xi)$ and $k_2 \subseteq \text{Coll}(\omega, \xi)$ belong to $V(\mathbb{R}^{V[G]})$ and be such that $k_1 \times k_2$, $h_1 \times k_1$, and $k_2 \times h_2$ are $\text{Coll}(\omega, \xi) \times \text{Coll}(\omega, \xi)$-generic over V with $(r_1, r_2) \in h_1 \times k_1$ and $(s_1, s_2) \in k_2 \times h_2$. Since $r_2 \leq p_1$ and $s_1 \leq p_2$, $(r_2, s_1) \in k_1 \times k_2$ implies that $(p_1, p_2) \in k_1 \times k_2$. Hence $\tau[k_1] \prec \tau[k_2]$. Since $(r_1, r_2) \in h_1 \times k_2$, $\tau[h_1] \prec \tau[k_1]$. Since $(s_1, s_2) \in k_2 \times h_2$, $\tau[k_2] \prec \tau[h_2]$. Hence $\tau[h_1] \prec \tau[k_1] \prec \tau[k_2] \prec \tau[h_2]$. This show $a = \tau[h_1] \prec \tau[h_2] = b$. This completes the proof of Claim ii.

(Claim iii) Suppose $g_1 \times g_2$ is $\text{Coll}(\omega, \xi) \times \text{Coll}(\omega, \xi)$-generic over V and contains (p^*, p^*). Then $\neg(\tau[g_1] \preceq \tau[g_2] \wedge \tau[g_2] \preceq \tau[g_1])$.

To prove this: If not, there is some $(p_1, p_2) \in g_1 \times g_1$ so that $(p_1, p_2) \Vdash \tau_{\text{left}} \preceq \tau_{\text{right}} \wedge \tau_{\text{right}} \preceq \tau_{\text{left}}$. By Claim ii, there is some $(q_1, q_2) \preceq p_1$ so that $(q_1, q_2) \Vdash \tau_{\text{left}} \prec \tau_{\text{right}}$. Let $k_1, k_2 \subseteq \text{Coll}(\omega, \xi)$ be such that $(q_1, q_2) \in k_1 \times k_2$ and $k_1 \times k_2$, $k_1 \times g_2$, and $k_2 \times g_2$ are $\text{Coll}(\omega, \xi) \times \text{Coll}(\omega, \xi)$-generic over V. Note that $(q_1, q_2) \in k_1 \times k_2$ implies that $\tau[k_1] \prec \tau[k_2]$. $(p_1, p_2) \in k_1 \times g_2$ and $(p_1, p_2) \in k_2 \times g_2$ implies $\tau[g_2] \preceq \tau[k_1]$ and $\tau[k_2] \preceq \tau[g_2]$. Hence $\tau[k_2] \preceq \tau[k_1]$. Contradiction. This proves Claim iii.

Now suppose $g_1 \times g_2$ is $\text{Coll}(\omega, \xi) \times \text{Coll}(\omega, \xi)$-generic over V such that $(p^*, p^*) \in g_1 \times g_2$. By Claim iii, without loss of generality, one may assume that $\tau[g_1] \prec \tau[g_2]$. Then there is some $(p_1, p_2) \in g_1 \times g_2$ such that $(p_1, p_2) \leq (p^*, p^*)$ and $(p_1, p_2) \Vdash \tau_{\text{left}} \prec \tau_{\text{right}}$. Claim ii implies that every element of $\text{Eval}(\xi, p_1, \tau)$ is less than any element of $\text{Eval}(\xi, p_2, \tau)$. Thus $I(\xi, \tau, g_1) \cap I(\xi, \tau, g_2) = \emptyset$.

By the usual argument, one can construct, within $V(\mathbb{R}^{V[G]})$, a perfect set of mutual $\text{Coll}(\omega, \xi)$-generic filters over V containing p^*. This yields a perfect set of open intervals in $(\mathbb{R}, \preceq)$. This completes the proof. □

Fact 5.6. *Let $V \models \mathsf{ZF}$ and κ be a weakly compact cardinal of V. Let G be $\text{Coll}(\omega, < \kappa)$-generic over V. There are no Aronszajn trees on a wellorderable set in the Solovay model $V(\mathbb{R}^{V[G]})$.*

Proof. Suppose T is an ω_1-tree in $V(\mathbb{R}^{V[G]})$. Since T is an ω_1-tree on a wellorderable set, one may assume that the underlying domain of the tree T is ω_1. The tree T is $\text{OD}^{V[G]}_{V \cup \{r\}}$ for some $r \in \mathbb{R}^{V[G]}$. Using the homogeneity of $\text{Coll}(\omega, < \kappa)$ as in the proof of Fact 5.2, one can show that $T \in V[r]$. In $V[r]$, T is a κ-tree. However $r \in V[G \restriction \xi]$ for some $\xi < \kappa$.

Since $|\mathrm{Coll}(\omega, <\xi)|^V < \kappa$ and a forcing smaller than κ preserves the weak compactness of κ, $V[r]$ still thinks κ is weakly compact. (See [7] Theorem 21.2 and Lemma 9.26 for this basic forcing fact.) Using the tree property in $V[r]$, there is a branch through T. By absoluteness, it is still a branch in $V(\mathbb{R}^{V[G]})$. □

Theorem 5.7. *(Woodin) Let $V \models \mathsf{ZFC}$ and κ be a weakly compact cardinal of V. Let G be $\mathrm{Coll}(\omega, <\kappa)$-generic over V. $V(\mathbb{R}^{V[G]}) \models \mathsf{SH}$.*

Proof. Let $(P, \prec)$ be a complete dense nonseparable linear ordering without endpoints in $V(\mathbb{R}^{V[G]})$. There is some $s \in \mathbb{R}^{V[G]}$, $w \in V$, and formula φ witnessing $(P, \prec)$ is $\mathrm{OD}_{w,s}$. Choose V_λ so large that every element of P is $\mathrm{OD}_{v,r,s}$ for some $v \in V_\lambda$ and $r \in \mathbb{R}^{V[G]}$.

By fixing a wellordering B of V_λ and a wellordering of the formulas, one can define a surjection from $\Phi : \Lambda \times \mathbb{R} \to P$ where Λ is some ordinal. Modify Φ if necessary to ensure that for any two points of $x, y \in P$, there are cofinal in Λ many α's so that there are $a, b \in \mathbb{R}$ with $\Phi(\alpha, a) = x$ and $\Phi(\alpha, b) = y$. This map is ordinal-definable from s, w, V_λ, and B.

For $\alpha < \Lambda$, let $P_\alpha = \{\Phi(\alpha, r) : r \in \mathbb{R}^{V[G]}\}$. Let $(P_\alpha, \prec)$ be the linear ordering resulting from the restriction of $\prec$. Let $(Q_\alpha, \sqsubseteq)$ be the prelinear ordering on $\mathbb{R}$ induced by $\Psi_\alpha : \mathbb{R} \to P_\alpha$ defined by $\Psi_\alpha(r) = \Phi(\alpha, r)$. The witness to each $(Q_\alpha, \sqsubseteq)$ being ordinal-definable in V_λ, s, w, B is obtained uniformly. Hence Fact 5.2 gives uniformly the Solovay codes for each $(Q_\alpha, \sqsubseteq)$. The proof of Case I in Theorem 5.5 gives a uniform sequence of wellordered separating families for each $(Q_\alpha, \sqsubseteq)$. Collapsing, one obtain a uniform sequence $\mathcal{E}_\alpha$ of separating family for each $(P_\alpha, \prec)$. Using the wellordering of Λ and the wellordering of each $\mathcal{E}_\alpha$, one can define a wellordered separating family for $(P, \prec)$ just as in Theorem 4.5.

Fact 4.3 and Fact 5.6 imply that there are no Suslin line. □

Fact 5.8. (ZFC) *Let κ be a cardinal and T be a κ-Suslin tree. If $\mathbb{P}$ is a forcing with $|\mathbb{P}| < \kappa$, then in $\mathbb{P}$-generic extensions, T has no branches and no κ-sized antichains.*

Proof. This is a well known result that small forcing can not kill a κ-Suslin tree.

One may assume that $(T, \prec)$ is a tree on κ. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Suppose B is a branch of T in $V[G]$. Let $p \in G$ and $\dot{B}$ be a $\mathbb{P}$-name so that $\dot{B}[G] = B$ and p forces that $\dot{B}$ is a branch. Fix some $r \leq_{\mathbb{P}} p$. For each $\alpha < \kappa$, let $E_\alpha = \{q \in \mathbb{P} : q \leq_{\mathbb{P}} r \wedge q \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{B}\}$. Since $|\mathbb{P}| < \kappa$, there is some $q \leq_{\mathbb{P}} r$ so that $C^q = \{\alpha : q \in E_\alpha\}$ is size κ. Let $D = \{q \in \mathbb{P} : q \leq_{\mathbb{P}} p \wedge |C^q| = \kappa\}$. The above argument showed that D is dense below $p \in G$. By genericity, let $q \in G \cap D$. Then $B \in V$ since B is the $\prec$ downward closure of C^q. Contradiction.

$\mathbb{P}$ does not add any κ-sized antichains is proved similarly. □

Fact 5.9. (ZF) *If there is a Suslin tree on a wellorderable set, then there is a Suslin line.*

Proof. This is the usual argument with some details provided to see that no choice is needed. See [7] Chapter 9 for the definition of a normal Suslin tree.

If a Suslin tree is on a wellorderable set, one may assume the Suslin tree is on ω_1. Given a Suslin tree on ω_1, one can produce a normal Suslin tree on ω_1 in ZF. (See the construction in [7] Lemma 9.13.) Let $T = (\omega_1, \prec)$ be a normal Suslin tree on ω_1.

Let $\pi : \omega \to \mathbb{Q}$ be a fixed bijection of ω with $\mathbb{Q}$, the set of rationals numbers. One property required of a normal Suslin tree is that for each $x \in T$, x has countably infinite many immediate sucessors. Let S_x denote the set of immediate successors of x in T. Since $S_x \subseteq \omega_1$, S_x is in bijection (via the Mostowski collapse map) with its ordertype which is some countable ordinal $\omega \cdot \gamma + n$ where $\gamma < \omega_1$ and $n \in \omega$. By moving the n-many points to the front, S_x is uniformly in bijection with $\omega \cdot \gamma$. S_x is now uniformly in bijection with $\mathbb{Q} \times \gamma$ (which comes from apply π to each ω-block). $\mathbb{Q} \times \gamma$ can be given the reverse lexicographic ordering where $\mathbb{Q}$ has its usual dense linear order structure. Pulling this ordering back to S_x, one has uniformly defined a dense linear ordering $<_x$ on S_x. (Individually, $(S_x, <_x)$ is isomorphic to $\mathbb{Q}$ but this isomorphism can not be found uniformly in x.)

Let L be the collection of maximal branches in $T = (\omega_1, \prec)$. If $B \in L$ is a maximal branch, for α less than the length of B, let $B(\alpha)$ be the element of B that belongs to the α^{th} level of T. Now suppose $B, C \in L$ and $B \neq C$. Note that B can not be an initial segment of C or the other way around since both are maximal branches. Let $\epsilon_{C,D}$ denote the least level α so that $B(\alpha) \neq C(\alpha)$. Another requirement of being a normal Suslin tree is that for each limit level δ, if x, y belong to level δ and have the same $\prec$-initial segment in T, then $x = y$. This implies that if $B \neq C$, then $\epsilon_{C,D}$ must be a sucessor ordinal. Let $z_{B,C}$ denote the common immediate predecessor of $B(\epsilon_{B,C})$ and $C(\epsilon_{B,C})$. Finally, define $B \lhd C$ if and only if $B \neq C$ and $B(\epsilon_{B,C}) <_{z_{B,C}} C(\epsilon_{B,C})$, where recall that $<_{z_{B,C}}$ is the dense linear ordering of $S_{z_{B,C}}$, which is the collection of immediate sucessor of $z_{B,C}$. It can be checked that $(L, \lhd)$ is a linear ordering.

For each $x \in \omega_1$, let $I_x = \{B \in L : x \in B\}$. Note that I_x is an open interval. Suppose $B \lhd C$. Then $B(\epsilon_{B,C}) <_{z_{B,C}} C(\epsilon_{B,C})$. Since $(S_{z_{B,C}}, <_{z_{B,C}})$ is a dense linear ordering, find some $x \in S_{z_{B,C}}$ so that $B(\epsilon_{B,C}) <_{z_{B,C}} x <_{z_{B,C}} C(\epsilon_{B,C})$. One can check that $I_x \subseteq (B, C)$, where (B, C) refers to the open interval in $(L, \lhd)$ determined by B and C.

It has been shown that for any open interval (B, C), there is some $x \in \omega_1$ so that $I_x \subseteq (B, C)$. This implies that $(L, \lhd)$ is a dense linear ordering.

If I_x and I_y are disjoint, then one has that x and y are incomparable in $(T, \prec)$. Suppose $\mathcal{E}$ is a family of disjoint open intervals. For each open interval $K \in \mathcal{E}$, let x_K denote the least $x \in \omega_1$ so that $I_x \subseteq K$ which exists by the above observation. Thus $\{x_K : K \in \mathcal{E}\}$ is an antichain in T. Since T is a Suslin tree, this collection must be countable. Thus $\mathcal{E}$ must be countable. So $(L, \lhd)$ has the countable chain condition.

Let $\mathcal{D} \subseteq L$ be a countable set. Let $\delta < \omega_1$ be the supremum of the length of all the maximal branches in $\mathcal{D}$. Let $x \in T$ be some element of T on a level higher than δ. Then I_x is an open interval disjoint from $\mathcal{D}$. $(L, \lhd)$ is not separable. □

Fact 5.10. *(Jensen) In L, if κ is a regular uncountable non-weakly compact cardinal, then there is a κ-Suslin tree.*

Fact 5.11. *Suppose κ is an inaccessible cardinal which is not weakly compact in L. Let $G \subseteq \mathrm{Coll}(\omega, < \kappa)$ be generic over V. The Solovay model $L(\mathbb{R}^{L[G]})$ has a Suslin tree on ω_1 and hence a Suslin line.*

Proof. By Fact 5.10, let $(T, \prec) \in L$ be a κ-Suslin tree in L. In $L(\mathbb{R}^{L[G]})$, $(T, \prec)$ is an ω_1-tree. If $(T, \prec)$ fails to be a Suslin tree, then there an uncountable branch or uncountable chain. This object is OD_r for some $r \in \mathbb{R}^{L[G]}$. As this object is a set of ordinals, a homogeneity argument shows that it belongs to $L[r]$. This real r belongs to a $\mathrm{Coll}(\omega, \xi)$-generic extension for some $\xi < \kappa$. This object is then a κ-sized branch or antichain in $L[r]$. However, Fact 5.8 implies that $(T, \prec)$ is still a κ-Suslin tree in $L[r]$. Contradiction.

Thus $(T, \prec)$ is a Suslin tree in $L(\mathbb{R}^{L[G]})$. By Fact 5.9, there is a Suslin line in $L(\mathbb{R}^{L[G]})$. □

Theorem 5.12. *Suppose $V \models \mathsf{ZFC}$ and κ is an inaccessible cardinal of V. Let $G \subseteq \mathrm{Coll}(\omega, < \kappa)$ be $\mathrm{Coll}(\omega, < \kappa)$-generic over V. The Solovay model $V(\mathbb{R}^{V[G]})$ has a Suslin line if and only if $V(\mathbb{R}^{V[G]})$ has a Suslin tree on ω_1.*

Proof. This follows from Fact 4.3, Theorem 5.5, and Fact 5.9. □

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