Matching relations for decoupling in the Standard Model at two loops and beyond

Stephen P. Martin

Department of Physics, Northern Illinois University, DeKalb IL 60115

I discuss the matching relations for the running renormalizable parameters when the heavy particles (top quark, Higgs scalar, Z and W vector bosons) are simultaneously decoupled from the Standard Model. The complete two-loop order matching for the electromagnetic coupling and all light fermion masses are obtained, augmenting existing results at 4-loop order in pure QCD and complete two-loop order for the strong coupling. I also review the further sequential decouplings of the lighter fermions (bottom quark, tau lepton, and charm quark) from the low-energy effective theory.

Contents

I. INTRODUCTION

The discovery of the Higgs scalar boson at the Large Hadron Collider has put the Standard Model of particle physics on a firm footing. At the same time, searches for physics beyond the Standard Model have not produced confirmed hints of any more fundamental structure. It therefore seems worthwhile to consider the Standard Model as quite possibly valid and complete up to well above the TeV energy scale, and to study its precise parameters and predictions, assuming that the next layer of fundamental new physics particles is heavy enough to be irrelevant at energy scales now within direct reach at colliders.

The Standard Model has within it an interesting hierarchy, with four fundamental particles (the top quark, the Higgs scalar, and the Z and W vector bosons) having masses within a factor of 2.2 each other, and heavier than all others by well over an order of magnitude. This makes it sensible to consider a low-energy effective theory consisting of the b, c, s, u, d quarks, the τ, μ, e leptons, and their neutrinos, with renormalizable interactions coming from the unbroken $SU(3)_c \times U(1)_{\rm EM}$ gauge group, and non-renormalizable four-fermion couplings to describe the weak interactions. This low-energy effective field theory can be matched onto the full $SU(3)_c \times SU(2)_L \times U(1)_Y$ high-energy theory with no particles decoupled, by considering common physical observables calculated in each theory in terms of parameters defined in the $\overline{\rm MS}$ renormalization scheme [1, 2] based on dimensional regularization [3–7].

In this paper, I will consider the decoupling relations that govern the matching at an arbitrary $\overline{\rm MS}$ renormalization scale, denoted Q. Specifically, the pertinent running $\overline{\rm MS}$ parameters of the full Standard Model will be called

$$g_3,\ g,\ g',\ y_t,\ y_b,\ y_c,\ y_s,\ y_u,\ y_d,\ y_\tau,\ y_\mu,\ y_e,\ \lambda,\ v. \tag{1.1}$$

Here, g_3, g, and g' are the gauge couplings, the y_f are the Yukawa couplings, λ is the Higgs self-interaction coupling, and v is the Higgs vacuum expectation value (VEV), defined in this paper as the minimum of the effective potential in Landau gauge. This definition implies that scalar tadpole sub-graphs vanish identically when summed to all orders in perturbation theory (including the tree-level tadpole), and so can be omitted from all Feynman diagrams.† The very small effects of Cabibbo-Kobayashi-Maskawa mixing and neutrino masses are ne-

† The Landau gauge Standard Model effective potential and its minimization condition are presently known to full 2-loop [8, 9] and 3-loop [10, 11] orders, and the 4-loop part only at leading order in QCD [12]. These results make use of Goldstone boson resummation [13, 14], and employ 3-loop vacuum integral basis functions defined and evaluated by [15]; for an alternative evaluation method see [16]. In particular, refs. [11, 12] provide the formulas relating the VEV v used here to the tree-level VEV $v_{\rm tree} = \sqrt{-m_H^2/\lambda}$ used in many other works, which therefore must [17] include tadpole graphs. Outside of Landau gauge, the effective potential is much more complicated at 2-loop order [18], and not known at 3-loop order.

glected. The running $\overline{\rm MS}$ squared masses of the Standard Model states are then denoted:

$$Z = (g^2 + g'^2)v^2/4, \tag{1.2}$$
$$W = g^2v^2/4, \tag{1.3}$$
$$h = 2\lambda v^2, \tag{1.4}$$
$$t = y_t^2 v^2/2, \tag{1.5}$$
$$b = y_b^2 v^2/2, \qquad \text{etc.} \tag{1.6}$$

Due to the choice of the definition of the VEV v, these quantities are specific to Landau gauge. As a matter of preference, I find the convenience (and increased accuracy) of not having tadpole graphs (with their associated $1/\lambda$ factors in perturbation theory, coming from zero-momentum Higgs propagators) to be well worth the price of a Landau-gauge-specific VEV and running masses, especially since these are not renormalization group scale-invariant observables anyway. The high-energy non-decoupled electromagnetic coupling is defined by

$$e \equiv gg'/\sqrt{g^2 + g'^2}. \tag{1.7}$$

In the low-energy $SU(3)_c \times U(1)_{\rm EM}$ effective field theory, the renormalizable $\overline{\rm MS}$ parameters will be denoted in this paper as

$$\alpha_S,\ \alpha,\ m_b,\ m_c,\ m_s,\ m_u,\ m_d,\ m_\tau,\ m_\mu,\ m_e. \tag{1.8}$$

To avoid confusion, α_S and α are only used for the low-energy effective theory, and never for the gauge couplings of the non-decoupled full Standard Model theory. Conversely, the symbols g_3, g, g', and e are used exclusively to refer to quantities in the full non-decoupled theory. Note also that α is used in this paper to refer to the $\overline{\rm MS}$ quantity, not the so-called "on-shell" electromagnetic coupling. All of the parameters in eqs. (1.1)-(1.8) depend on the $\overline{\rm MS}$ renormalization scale Q.

There are several complementary paths that one can take to relating these parameters to experimental results. In one approach, one makes direct use of low-energy experimental observables as the basic inputs, which then determine the parameters in eq. (1.8), and then infer the full Standard Model parameters in eq. (1.1) from them. In this paper, I will instead take the basic input parameters to be those of eq. (1.1); then the low-energy observable data can be derived and used as the subjects of global fits. The purpose of this paper is limited to finding the matching relations that give the parameters of eq. (1.8) as functions of those in eq. (1.1). This will be done treating the matching scale Q as arbitrary, with the assumption that, typically, it should be chosen not much smaller than the W-boson mass and not much larger than the top-quark mass, in order to avoid unnecessary large logarithms. Note that $\ln(M_t/M_W) = 0.77$, so that any choice of $M_W \lesssim Q \lesssim M_t$ for the matching scale should be fine. (It is not necessary that each particle is automatically decoupled at the scale Q equal to its mass, which is ambiguous in any case.)

Some observables, notably the pole masses of the top, Higgs, Z, and W, are only accessible in the high energy theory. The Higgs boson mass has been connected to the self-coupling λ including 2-loop QCD corrections [19] and at full 2-loop order in terms of interpolating formulas [20, 21]. Analytical results and computer code for the Higgs mass at complete 2-loop order have been presented in the tadpole-free scheme consistent with the present paper in ref. [22], which also includes leading 3-loop corrections, and in the scheme with a tree-level VEV and tadpoles in refs. [23, 24]. Multi-loop corrections to the W and Z boson masses, their ratio (the ρ parameter), and their relationships with other observables have been discussed in [25–58],[23, 24]. In particular, refs. [57, 58] provide the complete 2-loop analytic results for the W and Z pole masses, respectively, in the tadpole-free $\overline{\rm MS}$ scheme consistent with the conventions and notations of the present paper. For the top-quark pole mass, the pure QCD contributions are known at 1-loop [59], 2-loop [60], 3-loop [61–63], and 4-loop [64, 65] orders; these results also apply to the light quark pole masses in the decoupled theory. Contributions and uncertainty estimates from higher orders in QCD are discussed in [66–70]. The non-QCD 1-loop corrections to fermion pole masses were given in [71, 72]. Mixed 2-loop QCD corrections to the top-quark pole mass were obtained in refs. [73–77], and the 2-loop electroweak corrections in the "gaugeless" limit (where W, Z masses are neglected compared to the top-quark mass) are given in refs. [78, 79]. The full 2-loop top-quark pole mass corrections have been given in the tree-level-VEV scheme in [23], and in the tadpole-free scheme used in the present paper in ref. [80].

For computations at characteristic energies much lower or much higher than the matching scale, one should use the renormalization group equations to run the $\overline{\rm MS}$ parameters to an appropriate comparable Q, thus resumming the potentially large logarithms that would otherwise occur. For the full Standard Model, the beta functions are presently known at full 2-loop [81–85] and 3-loop [86–93] orders. The beta function for the Higgs self-coupling is also known at 4 loops in the leading order in QCD [12, 94]. For the strong gauge coupling, the pure QCD contributions to the beta function are known at 4-loop [95, 96] and 5-loop [97, 98] orders, and the QCD contributions to the beta functions of the quark Yukawa couplings (or equivalently, the running quark masses) are likewise known at 3-loop [99], 4-loop [100, 101], and 5-loop [102] orders. These QCD results also apply to the α_S and quark masses of the low-energy effective theory, by changing the variable number of active quarks.

There are also already extensive multi-loop results on the decoupling matching relations involving the strong interactions. The 1-loop and 2-loop decoupling of the QCD coupling at quark thresholds were discussed long ago in refs. [103], and [104, 105], respectively. The pure QCD 3-loop and 4-loop threshold corrections for α_S were obtained in refs. [106, 107] and [108, 109], respectively. The complete 2-loop threshold corrections for α_S including electroweak and top-quark Yukawa effects were given in ref. [110], and have been checked as part of the present work. For the pure QCD contributions to quark mass threshold relations, the 3-loop results were obtained in ref. [106, 107], and the 4-loop results in ref. [111]. All of the pure QCD results for running and decoupling of α_S and quark masses have been incorporated into the RunDec [112] computer software packages.

The electromagnetic coupling is usually related to the very precisely known low-energy Thomson scattering value $\alpha_{\rm Thomson} = 1/137.0359991\ldots$ as the basic input parameter, through radiative corrections to the photon self-energy function [25, 32, 113–126], [23, 56]. The bottleneck to accuracy in running α to very high energies (where it can be matched to g, g') comes from the non-perturbative hadronic contributions, often parameterized as $\Delta\alpha^{(5)}_{\rm had}(m_Z)$. For recent evaluations of this important quantity, see refs. [127–129] and references therein. In this paper, I will instead concentrate on the connection to the far-ultraviolet, fundamental definition of the Standard Model, by obtaining the complete 2-loop relationship between the $\overline{\rm MS}$ parameters $g, g', \ldots$ of the Standard Model and the $\overline{\rm MS}$ running coupling $\alpha(Q)$ in the low-energy theory when t, h, Z, W are simultaneously decoupled.‡ The relationship between $\alpha(Q)$ and the very-low energy input $\alpha_{\rm Thomson}$ is in this paper left as a separate issue, as addressed in [25, 32, 113–126], [23, 56].

The other new result to be obtained below is the complete 2-loop matching for all of the light fermion masses listed in eq. (1.8). The relation between the Yukawa couplings and the pole masses of the lightest 5 quarks were obtained to order $\alpha_S\alpha$ in [130]. In ref. [79], the relationship between the bottom quark on-shell mass and its Yukawa coupling and running mass were obtained at 2-loop order in the gaugeless limit, for both a tree-level VEV scheme and for an "on-shell" definition of the VEV, $v^2_{\rm on-shell} \equiv 1/\sqrt{2}G_F$. This has been extended to full 2-loop order in ref.[23], with results given in terms of numerical linear interpolation formulas. In ref. [131], the matching formulas for decoupling were given for the bottom quark mass, again using numerical interpolation formulas. In this paper, I will give the analytic results for the matching relations for the bottom quark as well as all other light quark masses, using the tadpole-free scheme to define the Standard Model VEV (and thus the running masses) in the non-decoupled theory.

The method used to find each decoupling matching relation is to compute a gauge-invariant physical quantity two ways, in terms of the parameters of the decoupled and the non-decoupled theories, and then require that the results agree. For the gauge couplings, the physical quantity is the residue of the pole in a scattering amplitude at $p^2 = 0$, where p^μ is the 4-momentum of the gauge boson mediating the interaction. In the case of the fermion masses m_f, the physical quantity is the pole mass. The methods used here for the necessary calculations are very similar to those in refs. [11, 22, 57, 58, 80], and all notations are chosen

‡ Note, however, that the $\widehat{\alpha}^{(5)}(M_Z)$ quoted as the $\overline{\rm MS}$ coupling in the Review of Particle Properties (RPP) [126] decouples the top quark but not the W boson, and so is not the same as the $\overline{\rm MS}$-scheme $\alpha(Q)$ as defined here within the 5-quark, 3-lepton, $SU(3)_c \times U(1)_{\rm EM}$ gauge theory. In fact, strictly speaking $\widehat{\alpha}^{(5)}$ as defined in the RPP (following refs. [113, 115]) is not really an $\overline{\rm MS}$ coupling in the usual sense, because once the top quark has been decoupled, $SU(2)_L$ gauge invariance is explicitly and irretrievably broken, so that the W, Z bosons should also be decoupled in order to have a renormalizable effective theory.

to be consistent with those papers. In particular, logarithms involving the renormalization scale will be denoted by

$$\overline{\ln}(x) \equiv \ln(x/Q^2), \tag{1.9}$$

where $x = t, h, Z, W, \ldots$ are $\overline{\rm MS}$ squared masses. In the following, only vacuum graphs occur in the final results, so they can be written in terms of $\overline{\ln}(x)$ and the 2-loop renormalized vacuum basis integral function [8, 132]. The notation used here is, in terms of dilogarithms, for $x, y \leq z$:

$$\begin{aligned} I(x,y,z) \;=\;& \frac{1}{2}(x-y-z)\,\overline{\ln}(y)\,\overline{\ln}(z) + \frac{1}{2}(y-x-z)\,\overline{\ln}(x)\,\overline{\ln}(z) + \frac{1}{2}(z-x-y)\,\overline{\ln}(x)\,\overline{\ln}(y) \\ &+2x\,\overline{\ln}(x) + 2y\,\overline{\ln}(y) + 2z\,\overline{\ln}(z) - \frac{5}{2}(x+y+z) \\ &+r\left[\mathrm{Li}_2(k_1) + \mathrm{Li}_2(k_2) - \ln(k_1)\,\ln(k_2) + \frac{1}{2}\ln(x/z)\,\ln(y/z) - \zeta_2\right], \end{aligned} \tag{1.10}$$

with $r = \sqrt{\lambda(x,y,z)}$ and $k_1 = (z+x-y-r)/2z$ and $k_2 = (z+y-x-r)/2z$, where

$$\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{1.11}$$

The function $I(x,y,z)$ implicitly depends on Q through the $\overline{\ln}$ functions, and it is symmetric under interchange of any of its arguments x, y, z. Some useful special cases are:

$$I(0,0,x) = -\frac{1}{2}x\,\overline{\ln}^2(x) + 2x\,\overline{\ln}(x) - \frac{5}{2}x - \zeta_2 x, \tag{1.12}$$

$$I(0,x,x) = -x\,\overline{\ln}^2(x) + 4x\,\overline{\ln}(x) - 5x, \tag{1.13}$$

$$\begin{aligned} I(0,x,y) \;=\;& (y-x)\left[\mathrm{Li}_2(1-y/x) + \frac{1}{2}\overline{\ln}^2(x)\right] + y\,\overline{\ln}(y)[2-\overline{\ln}(x)] + 2x\,\overline{\ln}(x) \\ &-\frac{5}{2}(x+y). \end{aligned} \tag{1.14}$$

For use below in the matching relations for gauge couplings, it is convenient to define a Q-independent combination function:

$$\begin{aligned} F(x,y) \equiv\;& I(x,x,y) + (x-y/2)\,\overline{\ln}^2(x) + y\,\overline{\ln}(x)\,\overline{\ln}(y) + (4x-2y)\,\overline{\ln}(x) \\ &-8x\,\overline{\ln}(y) + [(4x-y)^2/6x]\ln(y/x) - x/3 + 31y/6 - y^2/3x, \end{aligned} \tag{1.15}$$

which has the nice property that the following limit is finite:

$$\lim_{y\to 4x} F(x,y)/(y-4x)^3 = [2\ln(2)-1]/60x^2. \tag{1.16}$$

Although the definitions in terms of ordinary dilogarithms are convenient for computer numerical evaluation, it is perhaps worth nothing that for $y \leq 4x$, one can also write

$$F(x,y) \;=\; (4x-y)\left[\frac{1}{2}\Phi(y/4x) + \left(\frac{4}{3}+\frac{y}{6x}\right)\ln(x/y) - \frac{4}{3} + \frac{y}{3x}\right], \tag{1.17}$$

where

$$\Phi(z) \;=\; 4\sqrt{\frac{z}{1-z}}\,\mathrm{Cl}_2(2\arcsin(\sqrt{z})), \tag{1.18}$$

with the Clausen integral function defined by

$$\mathrm{Cl}_2(x) = -\int_0^x dt\, \ln(2\sin(t/2)). \tag{1.19}$$

II. DECOUPLING RELATIONS IN THE STANDARD MODEL

Consider simultaneous decoupling of t, h, Z, and W from the Standard Model at a scale Q. The matching relations for the low-energy effective theory renormalizable parameters in the $\overline{\mathrm{MS}}$ scheme can be written as:

$$\alpha \;=\; \frac{e^2}{4\pi}\left[1+\sum_{\ell=1}^{\infty}\frac{1}{(16\pi^2)^\ell}\,\theta_\alpha^{(\ell)}\right], \tag{2.1}$$

$$\alpha_S \;=\; \frac{g_3^2}{4\pi}\left[1+\sum_{\ell=1}^{\infty}\frac{1}{(16\pi^2)^\ell}\,\theta_{\alpha_S}^{(\ell)}\right], \tag{2.2}$$

$$m_f \;=\; \frac{y_f v}{\sqrt{2}}\left[1+\sum_{\ell=1}^{\infty}\frac{1}{(16\pi^2)^\ell}\,\theta_{m_f}^{(\ell)}\right], \qquad (f = b, c, s, u, d, \tau, \mu, e), \tag{2.3}$$

where the ℓ-loop contributions $\theta_X^{(\ell)}$ on the right sides are functions of the parameters g_3, g, g', y_t, v, and the matching renormalization scale Q. The effects of y_f for $f \neq t$ are negligible and therefore neglected, except of course for the leading factor of y_f in eq. (2.3). Results below are expressed† in terms of the running $\overline{\mathrm{MS}}$ squared masses defined in eq. (1.2)-(1.5).

† It is also easy to re-express these results in terms of pole squared masses M_h^2, M_W^2, M_Z^2, M_t^2, by using the 1-loop expressions relating them to h, W, Z, t (found e.g. in refs. [22, 57, 58, 80], in the notations and VEV convention of the present paper), but that will not be done explicitly here.

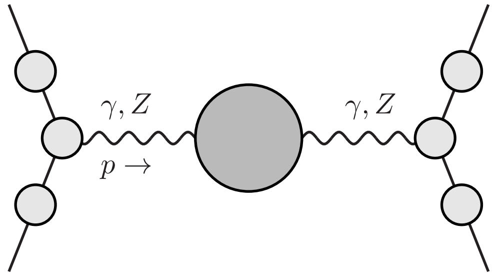

FIG. 2.1: The decoupling matching relation for α in the Standard Model is obtained from the residue of the pole at $p^2 = 0$ in the amplitude for neutral current scattering of charged particles, represented by the straight lines. By choosing the scattering charged particles to be vector-like singlets under $SU(2)_L$ and to have infinitesimal $U(1)_Y$ charges, the one-particle irreducible vertex corrections and external state propagator corrections (depicted as the smaller light gray blobs) are parametrically suppressed by an arbitrary amount, so that only the mixed γ, Z propagator corrections (larger, darker gray blob) contribute.

A. Matching of α

To determine the matching condition for α in the low-energy theory, consider the residue of the pole at $p^2 = 0$ in the neutral current channel amplitude for scattering of charged particles, as depicted in Figure 2.1. This is done first in the full Standard Model including both γ and Z contributions to the neutral current, and then in the low-energy effective theory where only γ exchange contributes. Requiring that the results of the two calculations agree gives the matching condition.

In order to avoid complications involving charged particle propagator and vertex corrections, it is convenient to use a trick, by taking the charged particles to be vector-like singlets under weak isospin $SU(2)_L$ and to carry infinitesimal electric charges, which are therefore also equal to their $U(1)_Y$ charges. This ensures that the one-particle-irreducible vertex corrections and the charged particle propagator corrections to the amplitude are parametrically suppressed by an arbitrary amount relative to the bosonic propagator corrections, due to higher powers of the infinitesimal charges, and so can be neglected. The result for the matching of the electromagnetic coupling then follows only from consideration of the corrections to the γ, Z system propagator. The idea behind this trick is that $U(1)_{\rm EM}$ gauge invariance guarantees that the matching condition for the electromagnetic coupling cannot depend on the quantum numbers of the charged states to which the neutral current couples, so the same result must obtain for other scattering processes involving chiral fermions including $SU(2)_L$ doublets such as those in the Standard Model, where vertex and fermion propagator corrections are non-trivial.

The propagator matrix for the γ, Z system can be written in terms of the components of the (transverse) one-particle-irreducible self-energy functions $\Pi_{ab}(s)$ for $a, b = \gamma, Z$ and

$s = -p^2$, as $iG(\eta^{\mu\nu} - p^\mu p^\nu/p^2)$, where

$$G^{-1} \;=\; \begin{pmatrix} s - \Pi_{\gamma\gamma}(s) & -\Pi_{\gamma Z}(s) \\ -\Pi_{\gamma Z}(s) & s - m_Z^2 - \Pi_{ZZ}(s) \end{pmatrix}. \tag{2.4}$$

It follows that

$$G_{\gamma\gamma} \;=\; \frac{1}{s - \widetilde{\Pi}_{\gamma\gamma}}\,, \tag{2.5}$$

$$G_{\gamma Z} \;=\; \frac{\Pi_{\gamma Z}}{\left(s - m_Z^2 - \Pi_{ZZ}\right)\left(s - \widetilde{\Pi}_{\gamma\gamma}\right)}\,, \tag{2.6}$$

$$G_{ZZ} \;=\; \frac{s - \Pi_{\gamma\gamma}}{\left(s - m_Z^2 - \Pi_{ZZ}\right)\left(s - \widetilde{\Pi}_{\gamma\gamma}\right)}\,, \tag{2.7}$$

where

$$\widetilde{\Pi}_{\gamma\gamma} \;\equiv\; \Pi_{\gamma\gamma} + \left(\Pi_{\gamma Z}\right)^2 / (s - m_Z^2 - \Pi_{ZZ}). \tag{2.8}$$

Now the neutral current interaction amplitude between two $SU(2)_L$ singlet states with infinitesimal $U(1)_Y$ charges is just proportional to

$$\mathcal{A} \;=\; g'^2 G_{YY} \;=\; g'^2 \left[c_W^2 G_{\gamma\gamma} - 2 c_W s_W G_{\gamma Z} + s_W^2 G_{ZZ}\right], \tag{2.9}$$

where $c_W = g/\sqrt{g^2 + g'^2}$ and $s_W = g'/\sqrt{g^2 + g'^2}$. It follows that

$$\mathcal{A} \;=\; e^2 \left[1 + \frac{(g'/g)^2 (s - \Pi_{\gamma\gamma}) - (2g'/g)\Pi_{\gamma Z}}{s - m_Z^2 - \Pi_{ZZ}}\right] \Big/ \left[s - \widetilde{\Pi}_{\gamma\gamma}\right]. \tag{2.10}$$

The existence of a massless photon pole in the amplitude at $s = 0$ therefore implies

$$\widetilde{\Pi}_{\gamma\gamma}(0) = 0, \tag{2.11}$$

and the residue of the pole in $\mathcal{A}$ is a gauge-invariant physical observable:

$$\mathrm{res}(\mathcal{A}) \;=\; e^2 \left[1 + \frac{(g'/g)\,\Pi_{\gamma Z}(0)}{m_Z^2 + \Pi_{ZZ}(0)}\right]^2 \Big/ \left[1 - \widetilde{\Pi}'_{\gamma\gamma}(0)\right]. \tag{2.12}$$

Here eq. (2.11) has been used to eliminate $\Pi_{\gamma\gamma}(0)$ from the numerator. So far no perturbative expansions or approximations or assumptions about the particular choice of gauge-fixing scheme have been used.

The calculation of the $\Pi_{ab}(s)$ is then performed in Landau gauge in a loop expansion in terms of bare parameters (with no counterterm diagrams) in $d = 4 - 2\epsilon$ dimensions, and at the end the result for the residue of the pole, res($\mathcal{A}$), is translated to the $\overline{\rm MS}$ scheme by the standard parameter redefinitions that give bare parameters (including the VEV) in terms of $\overline{\rm MS}$ ones.† This procedure is simpler and easier than using separate counterterm Feynman rules from the start, and the cancellation of poles in ϵ provides a check. The verification of eq. (2.11) through 2-loop order gives another check.

The calculation of res($\mathcal{A}$) is then repeated in the low-energy theory with t, h, Z, W absent, and therefore no $\Pi_{\gamma Z}$ or Π_{ZZ}, so that $\Pi_{\gamma\gamma}(0) = 0$, and res$(\mathcal{A}) = 4\pi\alpha/[1-\Pi'_{\gamma\gamma}(0)]$. Requiring that the two results for the observable res($\mathcal{A}$) are equal gives the matching condition for the electromagnetic coupling, after taking into account the 1-loop matching for the light fermion masses between the two theories, from subsection II C below. Note that non-perturbative effects from confined light quarks are common to the two versions of res($\mathcal{A}$), and so cancel out.

At 1-loop order, one obtains the well-known result:

$$\theta_\alpha^{(1)} \;=\; e^2\left[\frac{2}{3} - 7\,\overline{\ln}(W) + \frac{16}{9}\,\overline{\ln}(t)\right]. \tag{2.13}$$

For the 2-loop contribution to the matching, I obtain:

$$\begin{aligned}
\theta_\alpha^{(2)} \;=\;& e^2 g_3^2\left[-\frac{64}{9}\,\overline{\ln}(t) - \frac{208}{27}\right] + e^2 y_t^2\bigg\{\frac{16t(h-t)}{3h(4t-h)^2}F(t,h) - \frac{16t}{9h}[1+\ln(h/t)] \\
&+\frac{16t}{9(4t-Z)^3}\left[t\left(80W/Z - 7 - 64W^2/Z^2\right) + 8Z - 40W + 32W^2/Z\right]F(t,Z) \\
&+4\Big[\left(80W/Z - 64W^2/Z^2\right)[1+\ln(Z/t)] + 2 + 3\,\overline{\ln}(h) - 7\,\overline{\ln}(Z) - 14\,\overline{\ln}(t)\Big]/27 \\
&+22\,\overline{\ln}(t) - \frac{43}{4}\bigg\} + e^2 g^2\bigg\{\frac{3W(3h^2 - 12hW + 4W^2)}{h(4W-h)^3}F(W,h) + \left(\frac{W}{h} - 2\right)\overline{\ln}(h) \\
&+\frac{9W(4W^2 - 4WZ + 3Z^2)}{Z^2(4W-Z)^2}F(W,Z) + \left(\frac{661Z}{108W} - \frac{491}{27} + \frac{319W}{27Z} + \frac{12W^2}{Z^2}\right)\overline{\ln}(Z) \\
&+\left(\frac{20}{3} + \frac{37W}{3Z} - \frac{12W^2}{Z^2} - \frac{W}{h}\right)\overline{\ln}(W) + \frac{5t}{3(t-W)}\ln(t/W) \\
&+\frac{31}{81} - \frac{3h}{4W} + \frac{W}{h} + \frac{12W^2}{Z^2} - \frac{799W}{27Z} - \frac{1057Z}{324W}\bigg\} \\
&+e^4\bigg\{49\,\overline{\ln}^2(W) - \frac{224}{9}\,\overline{\ln}(t)\,\overline{\ln}(W) + \frac{256}{81}\,\overline{\ln}^2(t)\bigg\}.
\end{aligned} \tag{2.14}$$

The g_3^2 part of eq. (2.14) can be checked to be consistent with previously known results for

† In the same notations and conventions as the present paper, they can be found in eqs. (2.5)-(2.24) of ref. [22], eqs. (2.3)-(2.12) of ref. [57], and eqs. (2.5)-(2.8) of ref. [58].

the relation between the Thomson scattering value of α and its $\overline{\rm MS}$ version, e.g. in ref. [23].

The presentation of eq. (2.14) is made simpler by the use of the function $F(x, y)$ defined in eq. (1.15) above. Equation (1.16) shows immediately that $\theta_\alpha^{(2)}$ is finite and well-defined for $h = 4t$ and $Z = 4t$ and $h = 4W$ and $Z = 4W$ as well as $t = W$, despite the presence of denominators proportional to $(4t-h)^2$ and $(4t-Z)^3$ and $(4W-h)^3$ and $(4W-Z)^2$ and $t-W$ in eq. (2.14). This is a useful check, since there is no physical reason why anything untoward should happen at these special cases, even though of course none of them are close to being realized in our world. Additional checks are provided by the absence of poles $1/\epsilon$ and $1/\epsilon^2$, and by the cancellation[‡] of dependence on the Landau gauge Goldstone boson squared mass, after Goldstone boson resummation [13, 14]. I have further checked that renormalization group invariance is satisfied by eqs. (2.1), (2.13), and (2.14), by computing the Q derivative of each side using the known beta functions of the low-energy and high-energy theories and the direct Q dependence of the function $\overline{\ln}(x)$. [Note that $F(x, y)$ has no Q dependence.] In principle, this check should be merely equivalent to requiring the absence of poles in ϵ, but in practice it also checks intermediate steps in the calculations.

B. Matching of α_S

For the decoupling relation of α_S, the result has already been obtained in pure QCD to 1-loop [103], 2-loop [104, 105], 3-loop [106], and 4-loop order [108, 109], and at complete 2-loop order by Bednyakov [110]. I have re-calculated the latter result, finding complete agreement:

$$\theta_{\alpha_S}^{(1)} = \frac{2}{3} g_3^2 \,\overline{\ln}(t), \tag{2.15}$$

$$\begin{aligned}
\theta_{\alpha_S}^{(2)} = {}& g_3^4 \left[\frac{22}{9} + \frac{22}{3}\overline{\ln}(t) + \frac{4}{9}\overline{\ln}^2(t)\right] + g_3^2 y_t^2 \Bigg\{ \frac{2t(h-t)}{h(4t-h)^2} F(t,h) - \frac{2t}{3h}\left[1 + \ln(h/t)\right] \\
& + \frac{2t}{3(4t-Z)^3}\left[t\left(80W/Z - 7 - 64W^2/Z^2\right) + 8Z - 40W + 32W^2/Z\right] F(t,Z) \\
& + \left[\left(80W/Z - 64W^2/Z^2\right)\left[1 + \ln(Z/t)\right] + 2 + 3\,\overline{\ln}(h) - 7\,\overline{\ln}(Z) - 14\,\overline{\ln}(t)\right]/18 \Bigg\} \\
& + g_3^2 g^2 \Bigg\{ \frac{8(W-Z)}{9Z}\overline{\ln}(t) + 3\,\overline{\ln}(W) + \left(\frac{25Z}{18W} - \frac{13}{9} + \frac{14W}{9Z}\right)\overline{\ln}(Z) \\
& + \frac{t}{t-W}\ln(t/W) - \frac{49}{27} - \frac{W}{18Z} - \frac{163Z}{216W} \Bigg\}.
\end{aligned} \tag{2.16}$$

‡ More generally, in the Landau-gauge tadpole-free scheme this check is a non-trivial counterpart to the gauge-invariance check that one would obtain by instead working in a general gauge fixing with a tree-level VEV and non-vanishing tadpoles.

Although equivalent, the presentation in eq. (2.16) is somewhat more compact than the expression given in ref. [110]. This is due in part to the use of the function $F(x, y)$ defined in eq. (1.15) above, and also because the results are given in terms of running $\overline{\rm MS}$ squared masses here, rather than pole masses as in ref. [110]; converting to the top-quark pole mass in eq. (2.15) just contributes some additional 2-loop terms involving the 1-loop top-quark on-shell self-energy.

The pure QCD contributions to decoupling the top quark at 3-loop [106] and 4-loop [108, 109] order are also reproduced here for the sake of completeness:

$$\theta_{\alpha_S}^{(3)} = g_3^6 \left[\frac{8}{27}\overline{\ln}^3(t) - 3\,\overline{\ln}^2(t) + \frac{620}{9}\overline{\ln}(t) + 35.123151\right], \tag{2.17}$$

$$\theta_{\alpha_S}^{(4)} = g_3^8 \left[\frac{16}{81}\overline{\ln}^4(t) + \frac{4706}{81}\overline{\ln}^3(t) - \frac{1231}{27}\overline{\ln}^2(t) + 245.856958\,\overline{\ln}(t) - 109.765121\right]. \tag{2.18}$$

The coefficients involving irrational numbers (available in their full glory in refs. [106, 108, 109, 133]) have been reduced to decimal approximations here and in similar expressions below, for the sake of brevity.

C. Matching of running fermion masses

Now consider the decoupling relations for the masses of the fermions lighter than the top quark. The matching functions can be given generically for fermions other than the bottom quark, which is different because it has a direct coupling to the top quark and W boson. For a generic fermion,

$$(Q_f, I_3^f, C_f) = \begin{cases} (2/3,\ \ 1/2,\ \ 4/3) & (f = t, c, u), \\ (-1/3,\ -1/2,\ \ 4/3) & (f = b, s, d), \\ (-1,\ -1/2,\ \ 0) & (f = \tau, \mu, e), \end{cases} \tag{2.19}$$

are the notations for electric charge Q_f, I_3^f for the third component of weak isospin of the left-handed fermion, and C_f for the $SU(3)_c$ Casimir invariant.

The method used is to require equality between two computations of the pole mass for each light fermion, first in the full Standard Model theory and then again in the low-energy effective theory without t, h, Z, W. The strategy and details of the calculation of the light fermion pole masses that I have used are very similar to those described already in ref. [80] for the top quark, and so will not be reviewed here.

The resulting 1-loop order threshold corrections to the light fermion masses are:

$$\theta_{m_f}^{(1)} = \frac{9g^2 + 3g'^2}{16} + Q_f g'^2 \left[I_3^f + Q_f(W/Z - 1)\right]\left[3\,\overline{\ln}(Z) - 5/2\right], \tag{2.20}$$

$$\theta^{(1)}_{m_b} = \theta^{(1)}_{m_d} + \frac{3}{4}y_t^2\left[\frac{5}{6} - \overline{\ln}(t) + \left(\frac{W}{t-W}\right)^2 \ln(t/W) - \frac{W}{t-W}\right], \tag{2.21}$$

for a generic fermion other than the bottom quark, and for the bottom quark, respectively. In the case of the bottom quark, only the leading order in an expansion in y_b^2 has been kept. The next term in the expansion is

$$\begin{aligned}\Delta\theta^{(1)}_{m_b} = {}& y_b^2\Bigg[\frac{3}{4}\overline{\ln}(h) + \frac{1}{4}\overline{\ln}(t) + \frac{W^2(3t^2+4tW-W^2)}{4(t-W)^4}\ln(t/W) - \frac{W^2(7t-W)}{4(t-W)^3} \\ & + \frac{7Z^2+16WZ-32W^2}{36Z^2}\overline{\ln}(Z) - \frac{4(Z-W)(2W+Z)}{9Z^2}\overline{\ln}(b) \\ & - \frac{91}{216} - \frac{8W}{27Z} + \frac{16W^2}{27Z^2}\Bigg].\end{aligned} \tag{2.22}$$

However, since $y_b^2/16\pi^2 < 2\times 10^{-6}$, this contribution is negligible.

The 2-loop order threshold function for the bottom quark mass takes the form:

$$\begin{aligned}\theta^{(2)}_{m_b} = {}& \frac{4}{3}g_3^4\left[\overline{\ln}^2(t) + \frac{5}{3}\overline{\ln}(t) + \frac{89}{36}\right] + g_3^2 y_t^2\Big[(8t^2 - 8tW + 6W^2)I(0,t,W) \\ & + t(7t^2 - 17tW + 22W^2)\,\overline{\ln}^2(t) + 2tW(4t-7W)\,\overline{\ln}(t)\,\overline{\ln}(W) \\ & + (35t^2W - 23t^3 - 56tW^2 + 16W^3)\,\overline{\ln}(t) - (2t-3W)(7t-3W)W\overline{\ln}(W) \\ & + 92t^3/3 - 19t^2W + 17tW^2 + 4W^3/3\Big]/(t-W)^3 \\ & + \frac{4}{3}g_3^2\bigg\{\frac{g'^2}{6}(1+2W/Z)\left[\overline{\ln}(Z) - 17/12\right] + \frac{9}{4}g^2\overline{\ln}(W) \\ & + \frac{9}{8}(g^2+g'^2)\overline{\ln}(Z) - \frac{15}{32}(3g^2+g'^2)\bigg\} \\ & + \sum_{j=1}^{12} b_j^{(2)}\mathcal{I}_j^{(2)} + \sum_{j=1}^{4}\sum_{k=1}^{j} b_{j,k}^{(1,1)}\mathcal{I}_j^{(1)}\mathcal{I}_k^{(1)} + \sum_{j=1}^{4} b_j^{(1)}\mathcal{I}_j^{(1)} + b^{(0)}.\end{aligned} \tag{2.23}$$

The part that does not contain the strong coupling g_3 involves coefficients of 2-loop integral functions and logarithms from the lists

$$\begin{aligned}\mathcal{I}^{(2)} = {}& \big\{\zeta_2,\ I(0,h,W),\ I(0,h,Z),\ I(0,W,Z),\ I(h,W,W),\ I(h,Z,Z), \\ & I(t,t,Z),\ I(W,W,Z),\ I(0,t,W),\ I(h,t,t),\ I(h,t,W),\ I(t,W,Z)\big\},\end{aligned} \tag{2.24}$$

$$\mathcal{I}^{(1)} = \{\overline{\ln}(t),\ \overline{\ln}(h),\ \overline{\ln}(Z),\ \overline{\ln}(W)\}, \tag{2.25}$$

respectively. It cannot be simplified to a length reasonable for printing, and so is not given explicitly above in its full form, but instead in an ancillary electronic file distributed with this paper, called `theta2mb`. The individual coefficients $b_j^{(2)}$, $b_{j,k}^{(1,1)}$, $b_j^{(1)}$, and $b^{(0)}$ are rational functions of the input parameters t, h, Z, W, and v. Many of them have poles in one or more of the quantities $t-W$ and $4W-h$ and $4Z-h$ and $4t-Z$ and $\lambda(t,W,Z)$ and $\lambda(t,W,h)$, but I have checked that the total $\theta^{(2)}_{m_b}$ is nevertheless finite when each of these quantities vanishes. The format used in the ancillary file `theta2mb` is compatible with inclusion in computer code for easy numerical evaluation using eqs. (1.9)-(1.15). Additional checks follow, as usual, from the absence of poles $1/\epsilon^2$ and $1/\epsilon$ upon translating to the $\overline{\rm MS}$ scheme, and by the cancellation of contributions involving the Landau gauge Goldstone boson mass.

For generic fermions $f = (c, s, u, d, \tau, \mu, e)$, the 2-loop threshold functions are similarly found to be:

$$\begin{aligned}
\theta^{(2)}_{m_f} \;=\;& C_f g_3^4 \left[\overline{\ln}^2(t) + \frac{5}{3}\overline{\ln}(t) + \frac{89}{36}\right] + C_f g_3^2 \Biggl\{ 3g'^2 Q_f \left[I_3^f + Q_f(W/Z-1)\right]\left[\overline{\ln}(Z) - 17/12\right] \\
&+\frac{9}{4}g^2\overline{\ln}(W) + \frac{9}{8}(g^2+g'^2)\overline{\ln}(Z) - \frac{15}{32}(3g^2+g'^2)\Biggr\} \\
&+\sum_{j=1}^{8} c^{(2)}_j \mathcal{I}^{(2)}_j + \sum_{j=1}^{4}\sum_{k=1}^{j} c^{(1,1)}_{j,k}\mathcal{I}^{(1)}_j\mathcal{I}^{(1)}_k + \sum_{j=1}^{4} c^{(1)}_j \mathcal{I}^{(1)}_j + c^{(0)},
\end{aligned} \tag{2.26}$$

where the contributions independent of g_3 involve coefficients that are again too complicated to show in print, and so are relegated to an electronic file called `theta2mf` distributed as an ancillary to this paper. Note that the last four functions in the list eq. (2.24) do not appear in eq. (2.26). The individual coefficients $c^{(2)}_j$, $c^{(1,1)}_{j,k}$, $c^{(1)}_j$, and $c^{(0)}$ are again rational functions of t, h, Z, W, and v, with pole singularities at $4Z-h$ and $4t-Z$, but the total is free of these singularities.

The pure QCD threshold corrections for light quark masses were already known up to 3-loop order from Chetyrkin, Kniehl, and Steinhauser in ref. [106] and Liu and Steinhauser at 4-loop order in ref. [111]. They are listed here for the sake of completeness. For each quark $q = (b, c, s, u, d)$:

$$\theta^{(3)}_{m_q} \;=\; g_3^6\left[-\frac{152}{27}\,\overline{\ln}^3(t) + \frac{700}{27}\,\overline{\ln}^2(t) + 111.047973\,\overline{\ln}(t) + 126.160947\right], \tag{2.27}$$

$$\begin{aligned}
\theta^{(4)}_{m_q} \;=\;& g_3^8\Biggl[\frac{830}{27}\,\overline{\ln}^4(t) - \frac{10984}{81}\,\overline{\ln}^3(t) - 543.379386\,\overline{\ln}^2(t) + 452.388432\,\overline{\ln}(t) \\
&+236.908052\Biggr].
\end{aligned} \tag{2.28}$$

Note that the preceding equations apply specifically to the decoupling of the top quark from the theory. Again the known irrational parts have been replaced by decimal approximations.

III. DECOUPLING OF LIGHTER FERMIONS IN THE QCD+QED EFFECTIVE THEORY

In this section, I provide the decoupling relations appropriate for further sequential decoupling of fermions within the QCD+QED theory. None of the results in this section are new, as the QCD parts of these are now well-known, and the QED contributions at up to 2-loop order and certain light mass expansions can be easily inferred from those found in the existing literature. They are collected here for the sake of completeness.

The notation adopted here assumes that a generic fermion, denoted F, is to be decoupled.† The charge and QCD Casimir quantum numbers of F are to be denoted Q_F and C_F respectively, just as in eq. (2.19), and the index T_F equals 1/2 when the decoupled fermion F is a quark, and is 0 if it is a lepton, while the number of colors N_F is 3 when F is a quark and 1 when F is a lepton. The decoupling scale Q associated with the matching of parameters is again arbitrary, but typically should be chosen to be comparable to the mass of F, in order to avoid large logarithms in observables calculated after using the renormalization group equations to run the surviving parameters to lower energies. The running $\overline{\mathrm{MS}}$ parameters of the high-energy $SU(3)_c \times U(1)_{\mathrm{EM}}$ theory will be denoted α, α_S, $F \equiv m_F^2$, and m_f, where f runs over the list of the lighter fermions which are not being decoupled. For the low-energy theory with F decoupled, the parameters are distinguished by an underline, so they are $\underline{\alpha}$, $\underline{\alpha}_S$, and $\underline{m}_f$. The number of light quark flavors among the fermions f in the decoupled theory (which will also include leptons) will be denoted n_q.

The decoupling relations can then be written in the form:

$$\underline{\alpha}(Q) = \alpha(Q)\left[1+\sum_{\ell=1}^{\infty}\frac{1}{(4\pi)^\ell}\vartheta_\alpha^{(\ell)}\right], \tag{3.1}$$

$$\underline{\alpha}_S(Q) = \alpha_S(Q)\left[1+\sum_{\ell=1}^{\infty}\frac{1}{(4\pi)^\ell}\vartheta_{\alpha_S}^{(\ell)}\right], \tag{3.2}$$

$$\underline{m}_f(Q) = m_f(Q)\left[1+\sum_{\ell=1}^{\infty}\frac{1}{(4\pi)^\ell}\vartheta_{m_f}^{(\ell)}\right] \qquad (f \neq F). \tag{3.3}$$

(Note that the symbol ϑ is used to denote the threshold corrections within the QCD+QED theory in this section, in distinction with the symbol θ used in the previous section for decoupling t, h, Z, W.) Then for the matching coefficients for the electromagnetic coupling, one has at the scale Q where F is decoupled:

$$\vartheta_\alpha^{(1)} = \frac{4}{3}N_F Q_F^2 \alpha\, \overline{\ln}(F), \tag{3.4}$$

† In the Standard Model, the formulas below are not practically applicable with $F = u, d, s$, because QCD perturbation theory is not under control. Instead, the RPP [126] quotes the $\overline{\mathrm{MS}}$ masses at $Q = 2$ GeV.

$$\vartheta^{(2)}_{\alpha} = \left[\frac{4}{3}N_F Q_F^2 \alpha\, \overline{\ln}(F)\right]^2 - N_F Q_F^2 \alpha \left(C_F \alpha_S + Q_F^2 \alpha\right) \left[4\,\overline{\ln}(F) + \frac{13}{3}\right]. \tag{3.5}$$

For the QCD coupling, the results through 2-loop order including QED effects are:

$$\vartheta^{(1)}_{\alpha_S} = \frac{4}{3} T_F \alpha_S\, \overline{\ln}(F), \tag{3.6}$$

$$\begin{aligned}\vartheta^{(2)}_{\alpha_S} &= \left[\frac{4}{3} T_F \alpha_S\, \overline{\ln}(F)\right]^2 - T_F \alpha_S \left(C_F \alpha_S + Q_F^2 \alpha\right) \left[4\,\overline{\ln}(F) + \frac{13}{3}\right] \\ &\quad + T_F C_A \alpha_S^2 \left[\frac{20}{3}\overline{\ln}(F) + \frac{32}{9}\right],\end{aligned} \tag{3.7}$$

where $T_F = 1/2$ when F is a quark, and $T_F = 0$ when F is a lepton, and $C_A = 3$. The pure QCD contributions at 3-loop and 4-loop order, which apply only if F is a quark, are found from refs. [106] and [108, 109]:

$$\begin{aligned}\vartheta^{(3)}_{\alpha_S} &= \alpha_S^3 \left[\frac{8}{27}\,\overline{\ln}^3(F) + \left(\frac{53}{9} - \frac{16}{9} n_q\right) \overline{\ln}^2(F) + \left(\frac{955}{9} - \frac{67}{9} n_q\right) \overline{\ln}(F)\right. \\ &\quad \left. + 62.211628 - \frac{2633}{486} n_q\right],\end{aligned} \tag{3.8}$$

$$\begin{aligned}\vartheta^{(4)}_{\alpha_S} &= \alpha_S^4 \left[\frac{16}{81}\,\overline{\ln}^4(F) + \left(\frac{3766}{81} + \frac{508}{81} n_q - \frac{64}{81} n_q^2\right) \overline{\ln}^3(F)\right. \\ &\quad + \left(\frac{4354}{27} - \frac{2966}{81} n_q - \frac{77}{81} n_q^2\right) \overline{\ln}^2(F) \\ &\quad + \left(2157.863053 - 335.316171 n_q - \frac{6865}{729} n_q^2\right) \overline{\ln}(F) \\ &\quad \left. + 1323.608830 - 258.542470 n_q - 5.626464 n_q^2\right].\end{aligned} \tag{3.9}$$

These can be used with $n_q = 4$ when F is the bottom quark, and $n_q = 3$ when F is the charm quark. The formulas with $n_q = 5$ of course coincide with that for decoupling the top quark, as in eqs. (2.27)-(2.28) above.

The 1-loop and 2-loop threshold corrections for each light fermion mass m_f when decoupling the fermion F in the $SU(3)_c \times U(1)_{\rm EM}$ theory are:

$$\vartheta^{(1)}_{m_f} = 0, \tag{3.10}$$

$$\vartheta^{(2)}_{m_f} = 2\left(T_F C_f \alpha_S^2 + N_F Q_F^2 Q_f^2 \alpha^2\right) \left[\overline{\ln}^2(F) + \frac{5}{3}\overline{\ln}(F) + \frac{89}{36} + \Delta_2(f/F)\right], \tag{3.11}$$

where the last term is the power-suppressed mass correction, with f, F being the $\overline{\rm MS}$ squared masses and

$$\Delta_2(r) \;=\; r\left(\frac{8}{15}\ln(r) - \frac{76}{75}\right) + r^2\left(\frac{9}{70}\ln(r) - \frac{1389}{9800}\right) + \mathcal{O}(r^3). \tag{3.12}$$

This effect is mentioned because the squared mass ratios occurring in the decoupling of the light fermions (notably, $c/b \sim 0.1$) are not quite as suppressed as b/t in the decoupling of the top quark in the previous section, but its numerical impact is still quite small. It can be obtained from the 2-loop result for a quark pole mass in the presence of other massive and massless quarks, in ref. [60]. The pure QCD corrections are also known at 3-loop and 4-loop orders from refs. [106] and [111] respectively:

$$\begin{aligned}
\vartheta^{(3)}_{m_f} \;=\;& \alpha_S^3\Bigg[\left(\frac{16}{27}n_q - \frac{232}{27}\right)\overline{\ln}^3(F) + \frac{700}{27}\overline{\ln}^2(F) + \left(\frac{212}{27}n_q + 71.788714\right)\overline{\ln}(F) \\
&+118.248112 + 1.582567n_q + \Delta_3(f/F)\Bigg], \qquad (3.13)\\
\vartheta^{(4)}_{m_f} \;=\;& \alpha_S^4\Bigg[\left(\frac{8}{27}n_q^2 - \frac{80}{9}n_q + \frac{610}{9}\right)\overline{\ln}^4(F) + \left(\frac{184}{9}n_q - \frac{19264}{81}\right)\overline{\ln}^3(F) \\
&+\left(\frac{496}{81}n_q^2 - \frac{15650}{81}n_q + 269.583577\right)\overline{\ln}^2(F) \\
&+\big(286.364218 + 39.625147n_q - 1.284061n_q^2\big)\,\overline{\ln}(F) \\
&+14.375890n_q^2 - 375.221169n_q + 1753.616640\Bigg]. \qquad (3.14)
\end{aligned}$$

In the 3-loop part, the small mass correction is

$$\begin{aligned}
\Delta_3(r) \;=\;& \frac{8}{9}(2n_q - 31)\,\overline{\ln}(F)\Delta_2(r) \\
&+r\left\{\left(\frac{64}{135}n_q - \frac{451}{81}\right)\ln^2(r) + \left(\frac{84887}{7290} - \frac{128}{135}n_q\right)\ln(r) + 2.77670 - 0.22452n_q\right\} \\
&+r^2\left\{\left(\frac{4}{35}n_q - \frac{239}{270}\right)\ln^2(r) + \left(\frac{580157}{396900} - \frac{6}{35}n_q\right)\ln(r) + 0.52092 + 0.03556n_q\right\} \\
&+\mathcal{O}(r^3), \qquad (3.15)
\end{aligned}$$

which can be gleaned from the expansion of the pole mass given in ref. [131] based on the results in [134, 135]. In the 4-loop part, the expansion is not known beyond the lowest order in $r = f/F$.

In applications of the above formulas, the renormalization group running between scales requires the beta functions for the two gauge couplings and the fermion masses, which are known in the $SU(3)_c \times U(1)_{\rm EM}$ theory at full 3-loop order including electromagnetic effects; see for example ref. [131] (and the Appendix of ref. [136] for a general product gauge group with an arbitrary reducible fermion representation). The higher-order QCD corrections to the beta function for α are given in ref. [123] at order $\alpha^2\alpha_S^3$ and in ref. [137] at order $\alpha^2\alpha_S^4$. The 4-loop and 5-loop pure QCD contributions to the α_S beta function are found in refs. [95, 96] and [97, 98], respectively. The 3-loop, 4-loop and 5-loop pure QCD contributions to the quark mass betas functions are in [99], [100, 101], and [102]. Also useful in this context are the fermion pole masses, which are given for a general product gauge group with an arbitrary reducible fermion representation (but assuming just one non-zero fermion mass) in the Appendix of ref. [136], with 4-loop pure QCD contributions in refs. [64, 65]. In the case of more than one non-zero quark mass, expansions for small and large mass ratios in the 3-loop pole masses have been given in refs. [134] and [131].

IV. NUMERICAL RESULTS

In this section, I will illustrate the numerical impact of the matching conditions, concentrating on the new results of this paper, i.e. the shifts in the electromagnetic coupling and the light fermion masses from decoupling t, h, Z, W in the Standard Model, as a function of the matching scale Q. For a benchmark model, I consider the following numerical values for Standard Model parameters at a reference scale $Q_0 = 173.34$ GeV:

$$g_3 = 1.1666, \tag{4.1}$$
$$g = 0.647550, \tag{4.2}$$
$$g' = 0.358521, \tag{4.3}$$
$$y_t = 0.93690, \tag{4.4}$$
$$\lambda = 0.12597, \tag{4.5}$$
$$v = 246.647\,\text{GeV}. \tag{4.6}$$

These are then run to a matching scale 80 GeV $< Q <$ 180 GeV, and the figures below show the resulting matching corrections obtained in subsections II A and II C.

First, Figure 4.1 shows results for the various contributions to the fractional shift in α,

$$\delta\alpha/\alpha \equiv \frac{1}{16\pi^2}\theta_\alpha^{(1)} + \frac{1}{(16\pi^2)^2}\theta_\alpha^{(2)} + \ldots. \tag{4.7}$$

The left panel of Figure 4.1 shows the dominant 1-loop contribution from eq. (2.13), as well as the total from eqs. (2.13) and (2.14). The right panel shows the breakdown of the 2-loop contribution in eq. (2.14) into the part proportional to g_3^2, the part proportional to y_t^2, the remaining pure electroweak part, and the total of these 2-loop corrections. As might be expected, the pure electroweak 2-loop contributions are quite small over the entire range of Q, never exceeding 1 part in 10^5. The 2-loop g_3^2 and y_t^2 parts are larger, but for

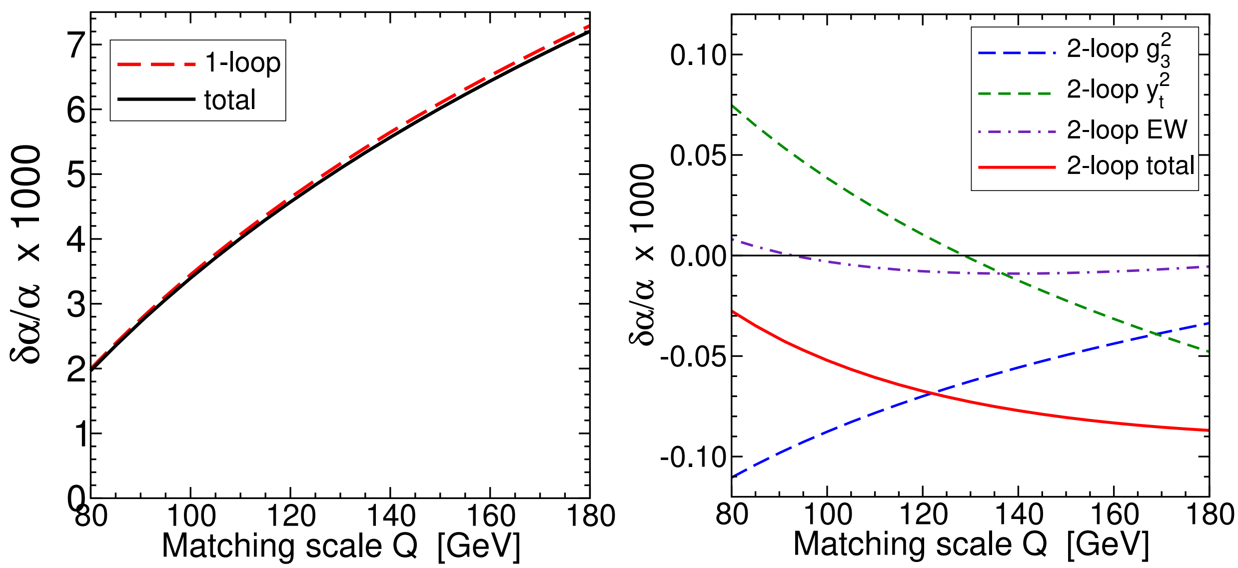

FIG. 4.1: Contributions to the matching relation fractional shift in α from decoupling t, h, Z, W in the Standard Model, as a function of the matching renormalization scale Q. The left panel shows the dominant 1-loop contribution (dashed line) from eq. (2.13), as well as the total (solid line). The right panel shows the breakdown of the total 2-loop contribution from eq. (2.14) (solid line) into the part proportional to g_3^2 (long-dashed line), the part proportional to y_t^2 (short-dashed line), and the remaining electroweak part (dot-dashed line).

lower Q there is significant cancellation between them. The total 2-loop contribution ranges from about -3×10^{-5} to -9×10^{-5}, depending on the choice of Q. This is comparable to the present uncertainty on $\Delta\alpha^{(5)}_{\rm had}(m_Z)$ estimated in the RPP [126], which is 7×10^{-5}. Therefore the total 2-loop correction is just barely numerically relevant at the present time. If improvements in the hadronic uncertainty are forthcoming, then the 2-loop corrections will become correspondingly more significant. However, it seems unlikely that further 3-loop corrections to the matching of α from decoupling t, h, Z, W will be needed in the foreseeable future.

The fractional shifts

$$\delta m_f/m_f \;\equiv\; \frac{1}{16\pi^2}\theta^{(1)}_{m_f} + \frac{1}{(16\pi^2)^2}\theta^{(2)}_{m_f} + \ldots \tag{4.8}$$

are shown in Figures 4.2-4.4. For each of the quark masses, the solid line is the total matching fractional shift, and the separate contributions from 1-loop (to which QCD does not contribute) and the combined 2, 3, and 4-loop QCD contributions are shown as the long-dashed and short-dashed lines, respectively. In the case of the bottom quark as shown in Figure 4.2, the remaining 2-loop mixed QCD and non-QCD contributions are each comparable in magnitude to the 3-loop pure QCD part and much larger than the 4-loop pure QCD part (not shown separately), but they have opposite signs from each other and have a significant

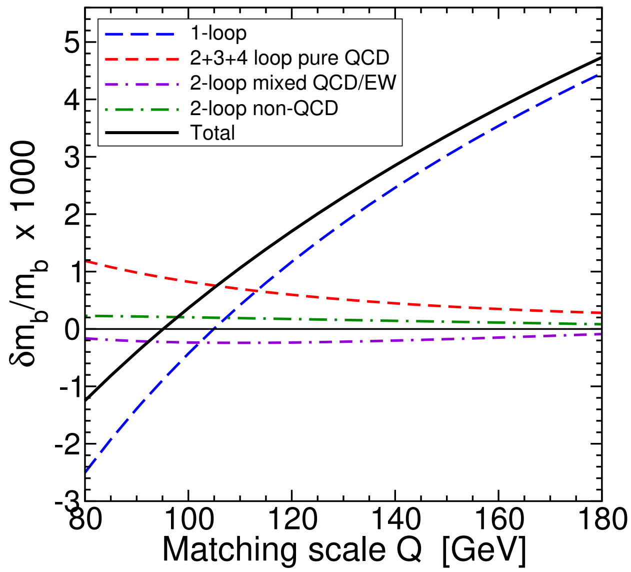

FIG. 4.2: Contributions to the matching relation fractional shift in the $\overline{\rm MS}$ bottom quark mass from decoupling t, h, Z, W in the Standard Model, as a function of the matching renormalization scale Q. The long-dashed line is the 1-loop contribution from eq. (2.21). The short-dashed line is the total QCD (2, 3, and 4-loop) contribution, from the g_3^4 part of eq. (2.23) and eqs. (2.27) and (2.28). The lower and upper dash-dotted lines are from the g_3^2 (mixed QCD) and g_3^0 (non-QCD) parts of eq. (2.23), respectively. The solid line is the total.

cancellation. The total fractional shift in m_b from decoupling t, h, Z, W is always less than 5×10^{-3}, and happens to be very small for Q near M_Z due to accidental cancellation between the different contributions. (A similar numerical study of the threshold correction for m_b was conducted in ref. [131], but with different details because that reference uses a different definition of the high-energy running bottom-quark mass, based on the VEV definition $v^2_{\rm on-shell} = 1/\sqrt{2}G_F$.)

In Figure 4.3, the results for the down and strange quark masses are shown in the left panel and for the charm and up quark masses in the right panel. In both cases, the 2-loop non-QCD corrections are quite tiny, in part because there is no y_t enhancement as there was for the bottom quark. The 2-loop mixed QCD corrections are larger in magnitude than the 4-loop and comparable to the 3-loop QCD corrections, but still less than 2×10^{-4} over most of the range of choices of Q. For each of the c, s, u, d quark masses, the total fractional shifts are slightly larger than 2×10^{-3} for Q near M_Z, and decrease with increasing Q. So, they are considerably smaller than the present experimental uncertainties in the masses. This situation is likely to persist for some time, pending dramatic improvements in the low-energy $\overline{\rm MS}$ quark mass determinations from e.g. lattice QCD.

Figure 4.4 shows the results for the charged lepton (τ, μ, e) masses, for which there are of course no QCD-enhanced corrections through 2-loop order. As expected, the matching is dominated by the 1-loop part, which contributes of order 2×10^{-4} to 2×10^{-3} to $\delta m_e/m_e = \delta m_\mu/m_\mu = \delta m_\tau/m_\tau$, depending on the choice of matching scale Q. The 2-loop contribution to the fractional matching shift is seen to be always less than 6×10^{-6}. This can be compared to the fractional experimental uncertainty in the physical masses of the charged leptons from ref. [126]. For the tau lepton, this is presently about 7×10^{-5}, showing that the 2-loop contribution is already safely smaller than the accuracy needed under the most optimistic of circumstances. For the muon, the fractional uncertainty in the physical mass is about 2×10^{-8} and for the electron about 6×10^{-9}, so in those cases the 2-loop (and perhaps even

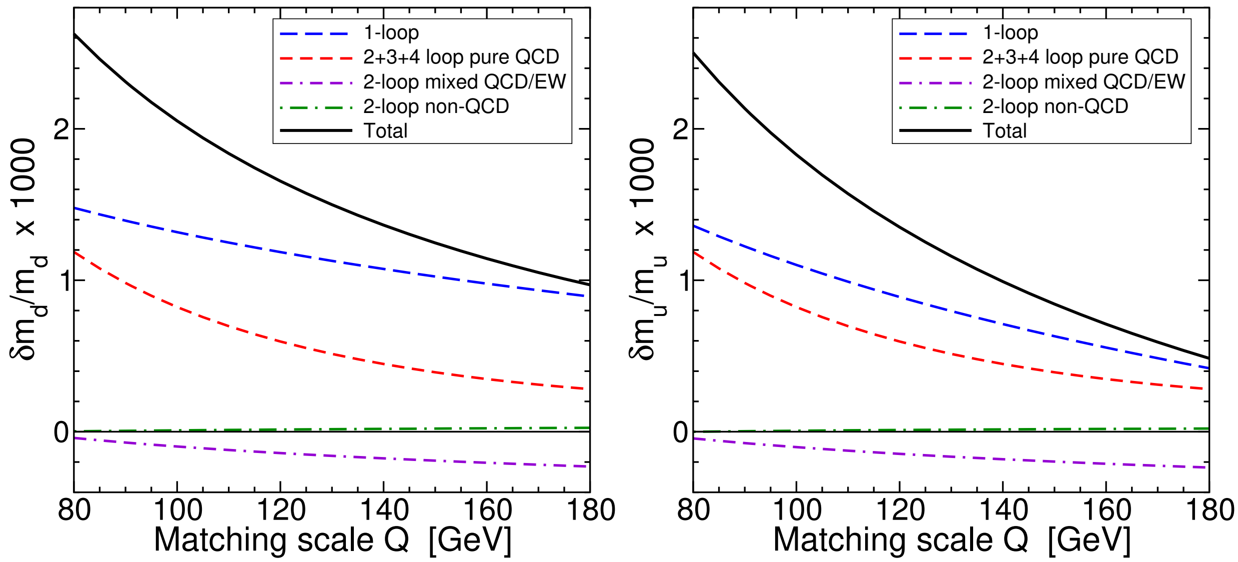

FIG. 4.3: Contributions to the matching relation fractional shift in the $\overline{\rm MS}$ quark masses from decoupling t, h, Z, W in the Standard Model, as a function of the matching renormalization scale Q. The left panel shows $\delta m_s/m_s = \delta m_d/m_d$, and the right panel shows $\delta m_c/m_c = \delta m_u/m_u$. In each case, the long-dashed line is the 1-loop contribution from eq. (2.20). The short-dashed line is the total QCD (2, 3, and 4-loop) contribution, from the g_3^4 part of eq. (2.26) and eqs. (2.27) and (2.28), and the lower and upper dash-dotted lines are from the g_3^2 (mixed QCD) and g_3^0 (non-QCD) parts of eq. (2.26), respectively. The solid line is the total.

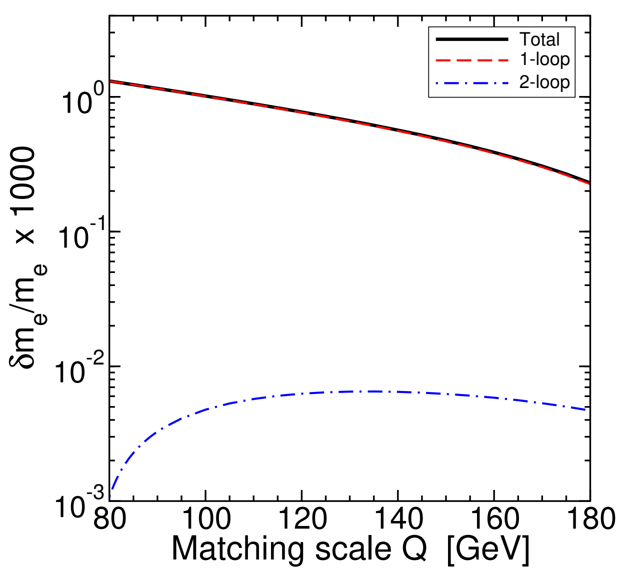

FIG. 4.4: Contributions to the matching relation fractional shift in the $\overline{\rm MS}$ charged lepton masses from decoupling t, h, Z, W in the Standard Model, as a function of the matching renormalization scale Q. The solid line is the total, and the long dashed line hiding just beneath it is the dominant 1-loop contribution from eq. (2.20). The small difference is the 2-loop contribution from eq. (2.26), shown as the dot-dashed line.

higher loop) threshold matching contributions are worthwhile, at least in principle. However, this does not yet take into account more subtle parametric uncertainties that are beyond the scope of this paper, for example the low-energy non-perturbative hadronic contribution to their pole masses induced through photon self-energy corrections, and even small loop effects from G_F-suppressed 4-fermion couplings in the low-energy effective field theory.

V. OUTLOOK

In this paper, I have discussed the matching relations for the renormalizable couplings in the low-energy effective $SU(3)_c \times U(1)_{\rm EM}$ gauge theory with 5 quarks and 3 leptons, when the top quark, Higgs scalar, and Z and W vector bosons are decoupled together at an $\overline{\rm MS}$ renormalization scale Q. This simultaneous decoupling ensures that the low-energy effective field theory has marginal and relevant couplings as part of a consistent renormalizable gauge theory. Also present in the low-energy theory are non-renormalizable couplings including 4-fermion terms for the effective weak interactions; the matching relations for those are not discussed in the present paper. The matching relations provide a connection to the far ultraviolet, fundamental, and complete definition of the Standard Model. The new results for the matching of the electromagnetic coupling α and the light quark and lepton masses augment previously known results for the strong coupling and the bottom quark mass, and the latter is given here in the tadpole-free scheme for the VEV, as part of a larger program [11–13, 22, 57, 58, 80] to relate Standard Model observables to the underlying Lagrangian parameters in that scheme. The matching corrections found here are reassuringly small, and in some cases much smaller than the present experimental uncertainties in the corresponding observables. They nevertheless are at least useful in providing informed bounds on the possible sources of theoretical error. They could become considerably more significant in the future when experimental uncertainties on input parameters, notably the low-energy quark masses and non-perturbative contributions to the fine-structure constant, are reduced.

Acknowledgments: I am grateful to Jens Erler and Ayres Freitas for helpful communications regarding ref. [126]. This work was supported in part by the National Science Foundation grant number PHY-1719273.

---

[1] W. A. Bardeen, A. J. Buras, D. W. Duke and T. Muta, "Deep Inelastic Scattering Beyond the Leading Order in Asymptotically Free Gauge Theories," Phys. Rev. D **18**, 3998 (1978).

[2] E. Braaten and J. P. Leveille, "Minimal Subtraction and Momentum Subtraction in QCD at Two Loop Order," Phys. Rev. D **24**, 1369 (1981).

[3] C. G. Bollini and J. J. Giambiagi, "Lowest order divergent graphs in nu-dimensional space," Phys. Lett. **40B**, 566 (1972).

[4] J. F. Ashmore, "A Method of Gauge Invariant Regularization," Lett. Nuovo Cim. **4**, 289 (1972).

[5] G. M. Cicuta and E. Montaldi, "Analytic renormalization via continuous space dimension," Lett. Nuovo Cim. **4**, 329 (1972).

[6] G. 't Hooft and M. J. G. Veltman, "Regularization and Renormalization of Gauge Fields," Nucl. Phys. B **44**, 189 (1972).

[7] G. 't Hooft, "Dimensional regularization and the renormalization group," Nucl. Phys. B **61**, 455 (1973).

[8] C. Ford, I. Jack and D. R. T. Jones, "The Standard model effective potential at two loops," Nucl. Phys. B **387**, 373 (1992) Erratum: [Nucl. Phys. B **504**, 551 (1997)] [hep-ph/0111190]. "The Effective potential and the differential equations method for Feynman integrals," Phys.

Lett. B **274**, 409 (1992) [Erratum-ibid. B **285**, 399 (1992)].

[9] S. P. Martin, "Two loop effective potential for a general renormalizable theory and softly broken supersymmetry," Phys. Rev. D **65**, 116003 (2002) [hep-ph/0111209].

[10] S. P. Martin, "Three-loop Standard Model effective potential at leading order in strong and top Yukawa couplings," Phys. Rev. D **89**, no. 1, 013003 (2014) [arXiv:1310.7553 [hep-ph]].

[11] S. P. Martin, "Effective potential at three loops," Phys. Rev. D **96**, no. 9, 096005 (2017) [arXiv:1709.02397 [hep-ph]].

[12] S. P. Martin, "Four-Loop Standard Model Effective Potential at Leading Order in QCD," Phys. Rev. D **92**, no. 5, 054029 (2015) [arXiv:1508.00912 [hep-ph]].

[13] S. P. Martin, "Taming the Goldstone contributions to the effective potential," Phys. Rev. D **90**, no. 1, 016013 (2014) [arXiv:1406.2355 [hep-ph]].

[14] J. Elias-Miro, J. R. Espinosa and T. Konstandin, "Taming Infrared Divergences in the Effective Potential," JHEP **1408**, 034 (2014) [arXiv:1406.2652 [hep-ph]].

[15] S. P. Martin and D. G. Robertson, "Evaluation of the general 3-loop vacuum Feynman integral," Phys. Rev. D **95**, no. 1, 016008 (2017) [arXiv:1610.07720 [hep-ph]]. The software package `3VIL` (3-loop Vacuum Integral Library) implementing these results is available from `http://www.niu.edu/spmartin/3VIL/`

[16] A. Freitas, "Three-loop vacuum integrals with arbitrary masses," JHEP **1611**, 145 (2016) [arXiv:1609.09159 [hep-ph]]. S. Bauberger and A. Freitas, "TVID: Three-loop Vacuum Integrals from Dispersion relations," arXiv:1702.02996 [hep-ph].

[17] See, for example, J. Fleischer and F. Jegerlehner, "Radiative Corrections to Higgs Decays in the Extended Weinberg-Salam Model," Phys. Rev. D **23**, 2001 (1981).

[18] S. P. Martin and H. H. Patel, "Two-loop effective potential for generalized gauge fixing," Phys. Rev. D **98**, no. 7, 076008 (2018) [arXiv:1808.07615 [hep-ph]].

[19] F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl and M. Shaposhnikov, "Higgs Boson Mass and New Physics," JHEP **1210**, 140 (2012) [arXiv:1205.2893 [hep-ph]].

[20] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori and A. Strumia, "Higgs mass and vacuum stability in the Standard Model at NNLO," JHEP **1208**, 098 (2012) [arXiv:1205.6497 [hep-ph]].

[21] D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, A. Salvio and A. Strumia, "Investigating the near-criticality of the Higgs boson," JHEP **1312**, 089 (2013) [arXiv:1307.3536 [hep-ph]].

[22] S. P. Martin and D. G. Robertson, "Higgs boson mass in the Standard Model at two-loop order and beyond," Phys. Rev. D **90**, no. 7, 073010 (2014) [arXiv:1407.4336 [hep-ph]].

[23] B. A. Kniehl, A. F. Pikelner and O. L. Veretin, "Two-loop electroweak threshold corrections in the Standard Model," Nucl. Phys. B **896**, 19 (2015) [arXiv:1503.02138 [hep-ph]].

[24] B. A. Kniehl, A. F. Pikelner and O. L. Veretin, "mr: a C++ library for the matching and running of the Standard Model parameters," Comput. Phys. Commun. **206**, 84 (2016) [arXiv:1601.08143 [hep-ph]].

[25] A. Sirlin, "Radiative Corrections in the $SU(2)_L \times U(1)$ Theory: A Simple Renormalization Framework," Phys. Rev. D **22**, 971 (1980).

[26] W. J. Marciano and A. Sirlin, "Radiative Corrections to Neutrino Induced Neutral Current Phenomena in the $SU(2)_L \times U(1)$ Theory," Phys. Rev. D **22**, 2695 (1980) Erratum: [Phys. Rev. D **31**, 213 (1985)].

[27] W. J. Marciano and A. Sirlin, "Testing the Standard Model by Precise Determinations of $W^\pm$ and Z Masses," Phys. Rev. D **29**, 945 (1984) Erratum: [Phys. Rev. D **31**, 213 (1985)].

[28] A. Sirlin, "On the O(α^2) Corrections to τ_μ, m_W, m_Z in the $SU(2)_L \times U(1)$ Theory," Phys. Rev. D **29**, 89 (1984).

[29] A. Djouadi and C. Verzegnassi, "Virtual Very Heavy Top Effects in LEP / SLC Precision Measurements," Phys. Lett. B **195**, 265 (1987).

[30] A. Djouadi, "O($\alpha\alpha_S$) Vacuum Polarization Functions of the Standard Model Gauge Bosons,"

Nuovo Cim. A **100**, 357 (1988).
[31] M. Consoli, W. Hollik and F. Jegerlehner, "The Effect of the Top Quark on the M_W-M_Z Interdependence and Possible Decoupling of Heavy Fermions from Low-Energy Physics," Phys. Lett. B **227**, 167 (1989).
[32] B. A. Kniehl, "Two Loop Corrections to the Vacuum Polarizations in Perturbative QCD," Nucl. Phys. B **347**, 86 (1990).
[33] F. Halzen and B. A. Kniehl, "Δr beyond one loop," Nucl. Phys. B **353**, 567 (1991).
[34] A. Djouadi and P. Gambino, "Electroweak gauge bosons selfenergies: Complete QCD corrections," Phys. Rev. D **49**, 3499 (1994) Erratum: [Phys. Rev. D **53**, 4111 (1996)] [hep-ph/9309298].
[35] L. Avdeev, J. Fleischer, S. Mikhailov and O. Tarasov, "$\mathcal{O}(\alpha\alpha_S^2)$ correction to the electroweak ρ parameter," Phys. Lett. B **336**, 560 (1994) Erratum: [Phys. Lett. B **349**, 597 (1995)] [hep-ph/9406363].
[36] K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, "Corrections of order $\mathcal{O}(G_F M_t^2 \alpha_S^2)$ to the ρ parameter," Phys. Lett. B **351**, 331 (1995) [hep-ph/9502291].
[37] K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, "QCD corrections from top quark to relations between electroweak parameters to order α_S^2," Phys. Rev. Lett. **75**, 3394 (1995) [hep-ph/9504413].
[38] G. Degrassi, P. Gambino and A. Vicini, "Two loop heavy top effects on the $m_Z - m_W$ interdependence," Phys. Lett. B **383**, 219 (1996) [hep-ph/9603374].
[39] G. Degrassi, P. Gambino and A. Sirlin, "Precise calculation of M_W, $\sin^2\theta_W(M_Z)$, and $\sin^2\theta_{\rm eff}$(lept)," Phys. Lett. B **394**, 188 (1997) [hep-ph/9611363].
[40] G. Degrassi, P. Gambino, M. Passera and A. Sirlin, "The Role of M_W in precision studies of the standard model," Phys. Lett. B **418**, 209 (1998) [hep-ph/9708311].
[41] M. Passera and A. Sirlin, "Radiative corrections to W and quark propagators in the resonance region," Phys. Rev. D **58**, 113010 (1998) [hep-ph/9804309].
[42] A. Freitas, W. Hollik, W. Walter and G. Weiglein, "Complete fermionic two loop results for the M_W-M_Z interdependence," Phys. Lett. B **495**, 338 (2000) Erratum: [Phys. Lett. B **570**, no. 3-4, 265 (2003)] [hep-ph/0007091].
[43] A. Freitas, W. Hollik, W. Walter and G. Weiglein, "Electroweak two loop corrections to the M_W-M_Z mass correlation in the standard model," Nucl. Phys. B **632**, 189 (2002) Erratum: [Nucl. Phys. B **666**, 305 (2003)] [hep-ph/0202131].
[44] M. Awramik and M. Czakon, "Complete two loop bosonic contributions to the muon lifetime in the standard model," Phys. Rev. Lett. **89**, 241801 (2002) [hep-ph/0208113].
[45] A. Onishchenko and O. Veretin, "Two loop bosonic electroweak corrections to the muon lifetime and M(Z) - M(W) interdependence," Phys. Lett. B **551**, 111 (2003) [hep-ph/0209010].
[46] M. Awramik, M. Czakon, A. Onishchenko and O. Veretin, "Bosonic corrections to Delta r at the two loop level," Phys. Rev. D **68**, 053004 (2003) [hep-ph/0209084].
[47] M. Faisst, J. H. Kühn, T. Seidensticker and O. Veretin, "Three loop top quark contributions to the rho parameter," Nucl. Phys. B **665**, 649 (2003) [hep-ph/0302275].
[48] M. Awramik and M. Czakon, "Complete two loop electroweak contributions to the muon lifetime in the standard model," Phys. Lett. B **568**, 48 (2003) [hep-ph/0305248].
[49] M. Awramik, M. Czakon, A. Freitas and G. Weiglein, "Precise prediction for the W boson mass in the standard model," Phys. Rev. D **69**, 053006 (2004) [hep-ph/0311148].
[50] Y. Schroder and M. Steinhauser, "Four-loop singlet contribution to the rho parameter," Phys. Lett. B **622**, 124 (2005) [hep-ph/0504055].
[51] K. G. Chetyrkin, M. Faisst, J. H. Kühn, P. Maierhofer and C. Sturm, "Four-Loop QCD Corrections to the Rho Parameter," Phys. Rev. Lett. **97**, 102003 (2006) [hep-ph/0605201].
[52] R. Boughezal and M. Czakon, "Single scale tadpoles and O($G_F m_t^2 \alpha_S^3$) corrections to the rho parameter," Nucl. Phys. B **755**, 221 (2006) [hep-ph/0606232].
[53] F. Jegerlehner, M. Y. Kalmykov and O. Veretin, "MS versus pole masses of gauge bosons:

Electroweak bosonic two loop corrections," Nucl. Phys. B **641**, 285 (2002) [hep-ph/0105304].

[54] F. Jegerlehner, M. Y. Kalmykov and O. Veretin, "Full two loop electroweak corrections to the pole masses of gauge bosons," Nucl. Phys. Proc. Suppl. **116**, 382 (2003) [hep-ph/0212003].

[55] F. Jegerlehner, M. Y. Kalmykov and O. Veretin, "$\overline{\rm MS}$ versus pole masses of gauge bosons. 2. Two loop electroweak fermion corrections," Nucl. Phys. B **658**, 49 (2003) [hep-ph/0212319].

[56] G. Degrassi, P. Gambino and P. P. Giardino, "The $m_W - m_Z$ interdependence in the Standard Model: a new scrutiny," JHEP **1505**, 154 (2015) [arXiv:1411.7040 [hep-ph]].

[57] S. P. Martin, "Pole Mass of the W Boson at Two-Loop Order in the Pure $\overline{MS}$ Scheme," Phys. Rev. D **91**, no. 11, 114003 (2015) [arXiv:1503.03782 [hep-ph]].

[58] S. P. Martin, "Z-Boson Pole Mass at Two-Loop Order in the Pure $\overline{MS}$ Scheme," Phys. Rev. D **92**, no. 1, 014026 (2015) [arXiv:1505.04833 [hep-ph]].

[59] R. Tarrach, "The Pole Mass in Perturbative QCD," Nucl. Phys. B **183**, 384 (1981).

[60] N. Gray, D. J. Broadhurst, W. Grafe and K. Schilcher, "Three Loop Relation of Quark $\overline{\rm MS}$ and Pole Masses," Z. Phys. C **48**, 673 (1990).

[61] K. G. Chetyrkin and M. Steinhauser, "Short distance mass of a heavy quark at order α_S^3," Phys. Rev. Lett. **83**, 4001 (1999) [hep-ph/9907509].

[62] K. G. Chetyrkin and M. Steinhauser, "The Relation between the $\overline{\rm MS}$ and the on-shell quark mass at order α_S^3," Nucl. Phys. B **573**, 617 (2000) [hep-ph/9911434].

[63] K. Melnikov and T. v. Ritbergen, "The Three loop relation between the MS-bar and the pole quark masses," Phys. Lett. B **482**, 99 (2000) [hep-ph/9912391].

[64] P. Marquard, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, "Quark Mass Relations to Four-Loop Order in Perturbative QCD," Phys. Rev. Lett. **114**, no. 14, 142002 (2015) [arXiv:1502.01030 [hep-ph]].

[65] P. Marquard, A. V. Smirnov, V. A. Smirnov, M. Steinhauser and D. Wellmann, "$\overline{\rm MS}$-on-shell quark mass relation up to four loops in QCD and a general SU(N) gauge group," Phys. Rev. D **94**, no. 7, 074025 (2016) [arXiv:1606.06754 [hep-ph]].

[66] M. Beneke and V. M. Braun, "Naive nonAbelianization and resummation of fermion bubble chains," Phys. Lett. B **348**, 513 (1995) [hep-ph/9411229].

[67] P. Ball, M. Beneke and V. M. Braun, "Resummation of (beta0 alpha-s)**n corrections in QCD: Techniques and applications to the tau hadronic width and the heavy quark pole mass," Nucl. Phys. B **452**, 563 (1995) [hep-ph/9502300].

[68] A. L. Kataev and V. S. Molokoedov, "On the flavour dependence of the $\mathcal{O}(\alpha_s^4)$ correction to the relation between running and pole heavy quark masses," Eur. Phys. J. Plus **131** (2016) no.8, 271 [arXiv:1511.06898 [hep-ph]]. "Multiloop contributions to the $\overline{\rm MS}$-on-shell mass relation for heavy quarks in QCD and charged leptons in QED and the asymptotic structure of the perturbative QCD series," arXiv:1807.05406 [hep-ph]. "Dependence of five and six-loop estimated QCD corrections to the relation between pole and running masses of heavy quarks on the number of light flavours," arXiv:1811.02867 [hep-ph].

[69] M. Beneke, P. Marquard, P. Nason and M. Steinhauser, "On the ultimate uncertainty of the top quark pole mass," Phys. Lett. B **775**, 63 (2017) [arXiv:1605.03609 [hep-ph]].

[70] A. H. Hoang, C. Lepenik and M. Preisser, "On the Light Massive Flavor Dependence of the Large Order Asymptotic Behavior and the Ambiguity of the Pole Mass," JHEP **1709**, 099 (2017) [arXiv:1706.08526 [hep-ph]].

[71] M. Bohm, H. Spiesberger and W. Hollik, "On the One Loop Renormalization of the Electroweak Standard Model and Its Application to Leptonic Processes," Fortsch. Phys. **34**, 687 (1986).

[72] R. Hempfling and B. A. Kniehl, "On the relation between the fermion pole mass and MS Yukawa coupling in the standard model," Phys. Rev. D **51**, 1386 (1995) [hep-ph/9408313].

[73] F. Jegerlehner and M. Y. Kalmykov, "$\mathcal{O}(\alpha\alpha_S)$ correction to the pole mass of the t quark within the standard model," Nucl. Phys. B **676**, 365 (2004) [hep-ph/0308216].

[74] F. Jegerlehner and M. Y. Kalmykov, "$\mathcal{O}(\alpha\alpha_S)$ relation between pole- and MS-bar mass of

the t quark," Acta Phys. Polon. B **34**, 5335 (2003) [hep-ph/0310361].

[75] M. Faisst, J. H. Kühn and O. Veretin, "Pole versus MS mass definitions in the electroweak theory," Phys. Lett. B **589**, 35 (2004) [hep-ph/0403026].

[76] D. Eiras and M. Steinhauser, "Two-loop $\mathcal{O}(\alpha\alpha_S)$ corrections to the on-shell fermion propagator in the standard model," JHEP **0602**, 010 (2006) [hep-ph/0512099].

[77] F. Jegerlehner, M. Y. Kalmykov and B. A. Kniehl, "On the difference between the pole and the $\overline{\mathrm{MS}}$ masses of the top quark at the electroweak scale," Phys. Lett. B **722**, 123 (2013) [arXiv:1212.4319 [hep-ph]].

[78] S. P. Martin, "Fermion self-energies and pole masses at two-loop order in a general renormalizable theory with massless gauge bosons," Phys. Rev. D **72**, 096008 (2005) [hep-ph/0509115].

[79] B. A. Kniehl and O. L. Veretin, "Two-loop electroweak threshold corrections to the bottom and top Yukawa couplings," Nucl. Phys. B **885**, 459 (2014) Erratum: [Nucl. Phys. B **894**, 56 (2015)] [arXiv:1401.1844 [hep-ph]].

[80] S. P. Martin, "Top-quark pole mass in the tadpole-free $\overline{MS}$ scheme," Phys. Rev. D **93**, no. 9, 094017 (2016) [arXiv:1604.01134 [hep-ph]].

[81] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization," Nucl. Phys. B **222**, 83 (1983).

[82] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations in a General Quantum Field Theory. 2. Yukawa Couplings," Nucl. Phys. B **236**, 221 (1984).

[83] I. Jack and H. Osborn, "General Background Field Calculations With Fermion Fields," Nucl. Phys. B **249**, 472 (1985).

[84] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations in a General Quantum Field Theory. 3. Scalar Quartic Couplings," Nucl. Phys. B **249**, 70 (1985).

[85] M. x. Luo and Y. Xiao, "Two loop renormalization group equations in the standard model," Phys. Rev. Lett. **90**, 011601 (2003) [hep-ph/0207271].

[86] L. N. Mihaila, J. Salomon and M. Steinhauser, "Gauge Coupling Beta Functions in the Standard Model to Three Loops," Phys. Rev. Lett. **108**, 151602 (2012) [arXiv:1201.5868 [hep-ph]].

[87] K. G. Chetyrkin and M. F. Zoller, "Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the Standard Model," JHEP **1206**, 033 (2012) [1205.2892].

[88] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin, "Anomalous dimensions of gauge fields and gauge coupling beta-functions in the Standard Model at three loops," JHEP **1301**, 017 (2013) [arXiv:1210.6873 [hep-ph]].

[89] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin, "Yukawa coupling beta-functions in the Standard Model at three loops," Phys. Lett. B **722**, 336 (2013) [arXiv:1212.6829 [hep-ph]].

[90] K. G. Chetyrkin and M. F. Zoller, "β-function for the Higgs self-interaction in the Standard Model at three-loop level," JHEP **1304**, 091 (2013) [1303.2890].

[91] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin, "Higgs self-coupling beta-function in the Standard Model at three loops," Nucl. Phys. B **875**, 552 (2013) [1303.4364].

[92] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin, "Three-loop Higgs self-coupling beta-function in the Standard Model with complex Yukawa matrices," Nucl. Phys. B **879**, 256 (2014) [arXiv:1310.3806 [hep-ph]].

[93] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin, "Three-loop SM beta-functions for matrix Yukawa couplings," Phys. Lett. B **737**, 129 (2014) [arXiv:1406.7171 [hep-ph]].

[94] K. G. Chetyrkin and M. F. Zoller, "Leading QCD-induced four-loop contributions to the β-function of the Higgs self-coupling in the SM and vacuum stability," JHEP **1606**, 175 (2016) [arXiv:1604.00853 [hep-ph]].

[95] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, "The Four loop beta function in quantum chromodynamics," Phys. Lett. B **400**, 379 (1997) [hep-ph/9701390].

[96] M. Czakon, "The Four-loop QCD beta-function and anomalous dimensions," Nucl. Phys. B **710**, 485 (2005) [hep-ph/0411261].
[97] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, "Five-Loop Running of the QCD coupling constant," Phys. Rev. Lett. **118**, no. 8, 082002 (2017) [arXiv:1606.08659 [hep-ph]].
[98] F. Herzog, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, "The five-loop beta function of Yang-Mills theory with fermions," JHEP **1702**, 090 (2017) [arXiv:1701.01404 [hep-ph]].
[99] O.V. Tarasov, "Anomalous Dimensions Of Quark Masses In Three Loop Approximation," preprint JINR-P2-82-900, (1982), unpublished.
[100] K. G. Chetyrkin, "Quark mass anomalous dimension to $\mathcal{O}(\alpha_S^4)$," Phys. Lett. B **404**, 161 (1997) [hep-ph/9703278].
[101] J. A. M. Vermaseren, S. A. Larin and T. van Ritbergen, "The four loop quark mass anomalous dimension and the invariant quark mass," Phys. Lett. B **405**, 327 (1997) [hep-ph/9703284].
[102] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, "Quark Mass and Field Anomalous Dimensions to $\mathcal{O}(\alpha_S^5)$," JHEP **1410**, 076 (2014) [arXiv:1402.6611 [hep-ph]].
[103] S. Weinberg, "Effective Gauge Theories," Phys. Lett. **91B**, 51 (1980). B. A. Ovrut and H. J. Schnitzer, "The Decoupling Theorem and Minimal Subtraction," Phys. Lett. **100B**, 403 (1981).
[104] W. Bernreuther and W. Wetzel, "Decoupling of Heavy Quarks in the Minimal Subtraction Scheme," Nucl. Phys. B **197**, 228 (1982) Erratum: [Nucl. Phys. B **513**, 758 (1998)].
[105] S. A. Larin, T. van Ritbergen and J. A. M. Vermaseren, "The Large quark mass expansion of $\Gamma(Z^0 \to \text{hadrons})$ and $\Gamma(\tau^- \to \nu_\tau + \text{hadrons})$ in the order α_S^3," Nucl. Phys. B **438**, 278 (1995) [hep-ph/9411260].
[106] K. G. Chetyrkin, B. A. Kniehl and M. Steinhauser, "Decoupling relations to $\mathrm{O}(\alpha_S^3)$ and their connection to low-energy theorems," Nucl. Phys. B **510**, 61 (1998) [hep-ph/9708255].
[107] A. G. Grozin, M. Hoeschele, J. Hoff, M. Steinhauser, "Simultaneous decoupling of bottom and charm quarks," JHEP **1109**, 066 (2011) [arXiv:1107.5970 [hep-ph]].
[108] Y. Schroder and M. Steinhauser, "Four-loop decoupling relations for the strong coupling," JHEP **0601**, 051 (2006) [hep-ph/0512058].
[109] K. G. Chetyrkin, J. H. Kühn and C. Sturm, "QCD decoupling at four loops," Nucl. Phys. B **744**, 121 (2006) [hep-ph/0512060].
[110] A. V. Bednyakov, "On the electroweak contribution to the matching of the strong coupling constant in the SM," Phys. Lett. B **741**, 262 (2015) [arXiv:1410.7603 [hep-ph]].
[111] T. Liu and M. Steinhauser, "Decoupling of heavy quarks at four loops and effective Higgs-fermion coupling," Phys. Lett. B **746**, 330 (2015) [arXiv:1502.04719 [hep-ph]].
[112] K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, "RunDec: A Mathematica package for running and decoupling of the strong coupling and quark masses," Comput. Phys. Commun. **133**, 43 (2000) [hep-ph/0004189]. B. Schmidt and M. Steinhauser, "CRunDec: a C++ package for running and decoupling of the strong coupling and quark masses," Comput. Phys. Commun. **183**, 1845 (2012) [arXiv:1201.6149 [hep-ph]]. F. Herren and M. Steinhauser, "Version 3 of RunDec and CRunDec," Comput. Phys. Commun. **224**, 333 (2018) [arXiv:1703.03751 [hep-ph]].
[113] W. J. Marciano and J. L. Rosner, "Atomic parity violation as a probe of new physics," Phys. Rev. Lett. **65**, 2963 (1990) Erratum: [Phys. Rev. Lett. **68**, 898 (1992)].
[114] G. Degrassi, S. Fanchiotti and A. Sirlin, "Relations Between the On-shell and $\overline{\mathrm{MS}}$ Frameworks and the M_W - M_Z Interdependence," Nucl. Phys. B **351**, 49 (1991).
[115] S. Fanchiotti, B. A. Kniehl and A. Sirlin, "Incorporation of QCD effects in basic corrections of the electroweak theory," Phys. Rev. D **48**, 307 (1993) [hep-ph/9212285].
[116] S. Eidelman and F. Jegerlehner, "Hadronic contributions to $g-2$ of the leptons and to the effective fine structure constant $\alpha(M_Z^2)$," Z. Phys. C **67**, 585 (1995) [hep-ph/9502298].
[117] H. Burkhardt and B. Pietrzyk, "Update of the hadronic contribution to the QED vacuum polarization," Phys. Lett. B **356**, 398 (1995).

[118] A. D. Martin and D. Zeppenfeld, "A Determination of the QED coupling at the Z pole," Phys. Lett. B **345**, 558 (1995) [hep-ph/9411377].

[119] R. Alemany, M. Davier and A. Hocker, "Improved determination of the hadronic contribution to the muon $g-2$ and to $\alpha(M_Z)$ using new data from hadronic tau decays," Eur. Phys. J. C **2**, 123 (1998) [hep-ph/9703220].

[120] M. Davier and A. Hocker, "Improved determination of $\alpha(M_Z^2)$ and the anomalous magnetic moment of the muon," Phys. Lett. B **419**, 419 (1998) [hep-ph/9801361].

[121] J. H. Kühn and M. Steinhauser, "A Theory driven analysis of the effective QED coupling at M_Z," Phys. Lett. B **437**, 425 (1998) [hep-ph/9802241].

[122] M. Steinhauser, "Leptonic contribution to the effective electromagnetic coupling constant up to three loops," Phys. Lett. B **429**, 158 (1998) [hep-ph/9803313].

[123] J. Erler, "Calculation of the QED coupling $\alpha(M_Z)$ in the modified minimal subtraction scheme," Phys. Rev. D **59**, 054008 (1999) [hep-ph/9803453].

[124] J. Erler, "Global fits to electroweak data using GAPP," hep-ph/0005084.

[125] G. Degrassi and A. Vicini, "Two loop renormalization of the electric charge in the standard model," Phys. Rev. D **69**, 073007 (2004) [hep-ph/0307122].

[126] M. Tanabashi *et al.* [Particle Data Group], "Review of Particle Physics," Phys. Rev. D **98**, no. 3, 030001 (2018).

[127] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, "Reevaluation of the hadronic vacuum polarisation contributions to the Standard Model predictions of the muon $g-2$ and $\alpha(m_Z^2)$ using newest hadronic cross-section data," Eur. Phys. J. C **77**, no. 12, 827 (2017) [arXiv:1706.09436 [hep-ph]].

[128] F. Jegerlehner, "Variations on Photon Vacuum Polarization," arXiv:1711.06089 [hep-ph].

[129] A. Keshavarzi, D. Nomura and T. Teubner, "Muon $g-2$ and $\alpha(M_Z^2)$: a new data-based analysis," Phys. Rev. D **97**, no. 11, 114025 (2018) [arXiv:1802.02995 [hep-ph]].

[130] B. A. Kniehl, J. H. Piclum and M. Steinhauser, "Relation between bottom-quark MS-bar Yukawa coupling and pole mass," Nucl. Phys. B **695**, 199 (2004) [hep-ph/0406254].

[131] A. V. Bednyakov, B. A. Kniehl, A. F. Pikelner and O. L. Veretin, "On the b-quark running mass in QCD and the SM," Nucl. Phys. B **916**, 463 (2017) [arXiv:1612.00660 [hep-ph]].

[132] A. I. Davydychev and J. B. Tausk, "Two loop selfenergy diagrams with different masses and the momentum expansion," Nucl. Phys. B **397**, 123 (1993). A. I. Davydychev, V. A. Smirnov and J. B. Tausk, "Large momentum expansion of two loop selfenergy diagrams with arbitrary masses," Nucl. Phys. B **410**, 325 (1993) [hep-ph/9307371]. F. A. Berends and J. B. Tausk, "On the numerical evaluation of scalar two loop selfenergy diagrams," Nucl. Phys. B **421**, 456 (1994).

[133] B. A. Kniehl, A. V. Kotikov, A. I. Onishchenko and O. L. Veretin, "Strong-coupling constant with flavor thresholds at five loops in the $\overline{\text{MS}}$ scheme," Phys. Rev. Lett. **97**, 042001 (2006) [hep-ph/0607202].

[134] S. Bekavac, A. Grozin, D. Seidel and M. Steinhauser, "Light quark mass effects in the on-shell renormalization constants," JHEP **0710**, 006 (2007) [arXiv:0708.1729 [hep-ph]].

[135] S. Bekavac, A. G. Grozin, D. Seidel and V. A. Smirnov, "Three-loop on-shell Feynman integrals with two masses," Nucl. Phys. B **819**, 183 (2009) [arXiv:0903.4760 [hep-ph]].

[136] S. P. Martin, "Refined gluino and squark pole masses beyond leading order," Phys. Rev. D **74**, 075009 (2006) [hep-ph/0608026].

[137] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn and J. Rittinger, "Vector Correlator in Massless QCD at Order O(α_S^4) and the QED beta-function at Five Loop," JHEP **1207**, 017 (2012) [arXiv:1206.1284 [hep-ph]].