ARTICLE IN PRESS

Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The maximum number of colorings of graphs of given order and size: A survey

Felix Lazebnik

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

ARTICLE INFO

Article history:
Received 13 May 2018
Accepted 9 August 2018
Available online xxxx

Dedicated to the memory of Herbert S. Wilf (1931–2012)

Keywords:
Graph coloring
Chromatic polynomial
Maximum number of colorings
Extremal graphs
Turán graph

ABSTRACT

Let m, n, λ be positive integers. What is the maximum number of proper vertex colorings in (at most) λ colors a graph with n vertices and m edges can have? On which graphs is this maximum attained? The question can be rephrased as the one of maximizing $\chi(G, \lambda)$, the value of the chromatic polynomial of G at λ, over all graphs G with n vertices and m edges.

This problem was stated independently by Wilf and Linial, and is still unsolved. In this article we survey the current state of the research directed at solving the problem.

Contents

E-mail address: fellaz@udel.edu.

https://doi.org/10.1016/j.disc.2018.08.010
0012-365X/

1. Introduction

All graphs in this paper are finite, undirected, and have neither loops nor multiple edges. For all missing definitions, we refer the reader to Bollobás [5].

For a graph G, let $V = V(G)$ and $E = E(G)$ denote the vertex set of G and the edge set of G, respectively. Let $|A|$ denote the cardinality of a set A. Let $n = |V(G)|$ and $m = |E(G)|$ denote the number of vertices (the *order*) of G, and number of edges (the *size*) of G, respectively. By $\mathcal{F}_{n,m}$ we denote the family of all graphs of order n and size m. We refer to graphs from this family as (n, m)-*graphs*.

The *Turán* graph $T_r(n)$, $r \geq 1$, is the complete r-partite graph of order n with all parts of size either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. It is easy to show that such a graph is unique. For example, an n-partite graph of order n is the complete graph K_n. If $r = 2$, $T_2(n)$ is $K_{s,s}$ for $n = 2s$, and $K_{s+1,s}$ for $n = 2s + 1$. Let $t_r(n) = |E(T_r(n))|$ denote the number of edges of $T_r(n)$.

The *edit distance* of two graphs of the same order is the minimum number of edges that need to be added and deleted from one graph to make it isomorphic to the other. A graph G is said to be *d-close* to a graph H, if the edit distance between the graphs is at most d.

An edge $\{x, y\}$ of a graph will also be denoted by xy, or yx. For a positive integer λ, let $[\lambda] = \{1, 2, \ldots, \lambda\}$. A function $c : V(G) \to [\lambda]$ such that $c(x) \neq c(y)$ for every edge xy of G is called a *proper vertex coloring of G in at most λ colors*, or simply a *λ-coloring* of G. The smallest value λ for which a λ-coloring of G exists is called the *chromatic number* of G, and is denoted $\chi(G)$. Let $\chi(G, \lambda)$ denote the number of λ-colorings of G. For a fixed G, $\chi(G, \lambda)$ is known to be a polynomial of λ, called the *chromatic polynomial of G*.

For readers familiar with Tutte polynomial, which will not be used in this paper, we wish to mention that the chromatic polynomial $\chi(G, \lambda)$ of a graph G can be viewed as a specialization of the Tutte polynomial $T_G(x, y)$ of G:

$$\chi(G, \lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)}T_G(1 - \lambda, 0),$$

where $k(G)$ is the number of components of G. See, e.g., [5] for more details.

We consider the following optimization question concerning chromatic polynomials: For given positive integers m, n, λ, what is the maximum number of λ-colorings of a graph with n vertices and m edges? For which graphs is this maximum attained? This problem was stated independently by Wilf [42] and Linial [22], and is still unsolved. The question can be rephrased as the question on maximizing $\chi(G, \lambda)$ over all graphs with n vertices and m edges. Let $f(n, m, \lambda)$ denote this maximum, i.e., $f(n, m, \lambda) = \max\{\chi(G, \lambda) : G \in \mathcal{F}_{m,n}\}$.

In this article we survey the current state of the research directed at solving the problem.

By log we will always denote the natural logarithm. For functions f and g from the set of positive integers to $(0, \infty)$, we write $f = o(g)$ for $n \to \infty$, if $\lim_{n\to\infty} f(n)/g(n) = 0$. We write $f = O(g)$ as $n \to \infty$, if there exist $c, n_0 > 0$ such that $f(n) \leq cg(n)$ for all $n \geq n_0$. And we write $f = \Omega(g)$ for $n \to \infty$, if $g = O(f)$.

The article is organized as follows. In Section 2 we provide a background for the problem; in Section 3 we present results on $f(n, m, \lambda)$; in Section 4 we discuss the ideas and techniques used in obtaining these results; and Section 5 contains several open problems.

2. Background

Investigation of proper colorings of planar graphs was originally motivated by a map coloring problem, that was generalized later to a graph coloring problem. The latter lead to the notions of the chromatic number and the chromatic polynomial of a graph, see Birkhoff [2,3], Whitney [40,41], and Birkhoff and Lewis [4].

A source of related problems, that is closer to the subject of this survey, is the sequence of papers by Read and by Wright. Generalizing the result of Gilbert [14], Read [27] found a method of computing the total number of λ-colorings of all graphs of order n, i.e.,

$$\sum_{G,|V(G)|=n} \chi(G, \lambda).$$

Using this Wright [44,45] determined the asymptotic behavior of the sum for fixed λ as $n \to \infty$. Continuing the investigation Wright [46] found an asymptotic approximation to the number $\sum_{G\in\mathcal{F}_{n,m}} \chi(G, \lambda)$ for large n and all $m = m(n)$. The latter leads easily to an asymptotic approximation of the average number of λ-colorings of graphs from $\mathcal{F}_{n,m}$ for large n and all $m = m(n)$, and this gives a lower bound for $f(n, m, \lambda)$.

Linial [22] arrived at the problem of minimizing the chromatic polynomial over the family $\mathcal{F}_{n,m}$ by studying the worst-case computational complexity of a certain algorithm. At the end of this paper, Linial poses the problem of maximizing $\chi(G, \lambda)$ over $\mathcal{F}_{n,m}$.

Around the same time, Wilf [43] and Bender and Wilf [1] studied the backtrack algorithm for the decision problem on the existence of λ-coloring of a given graph G, in particular, for the family $\mathcal{F}_{n,m}$. This prompted Wilf [42] to ask the question of maximizing $\chi(G, \lambda)$ over $\mathcal{F}_{n,m}$.

Extremal problems on chromatic polynomials over the families of graphs other than $\mathcal{F}_{n,m}$ were also studied by Tomescu [31–38]. For other examples, see the monograph by Dong, Koh and Teo [10], and recent preprints by Erey [11,12], by Knox and Mohar [15,16], and by Fox, He and Manners [13].

Please cite this article in press as: F. Lazebnik, The maximum number of colorings of graphs of given order and size: A survey, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.08.010.

3. Results

Below we will survey results concerning $f(n, m, \lambda)$ and related extremal graphs. In general, we have grouped results by the number of colors λ. When a small value of λ is fixed, we present the results concerning $f(n, m, \lambda)$ where m is a function of n. Section 3.5 is an exception, as Turán graphs have played a special role in the subject.

3.1. $f(n, m, 2)$

The value of $f(n, m, 2)$ and a complete description of the extremal graphs achieving this value was obtained by Lazebnik in [17]. (See Chen [7] for a minor correction to [17].) To state the result we need the following definitions.

Suppose $p = \lfloor\sqrt{m}\rfloor$ and $p^2 < m \le (p+1+a)(p-a)$ for an integer $a \ge 0$. Let $H(p+1, p, a)$ be obtained by deleting $(p+1+a)(p-a) - m$ edges from some $K_{p+1+a,p-a}$. For example, for $m = 38$ we have $p = 6$; possible values for a are 0,1. Therefore $H(7, 6, a)$ can be obtained by deleting 4 edges from $K_{7,6}$ or 2 edges from $K_{8,5}$.

Similarly, if $p^2 + p < m \le (p+1)^2 - a^2$ for an integer $a \ge 0$, let $H(p+1, p+1, a)$ be obtained by deleting $(p+1)^2 - a^2 - m$ edges from some $K_{p+1+a,p+1-a}$. Let $G_1 + G_2$ denote the disjoint union of graphs G_1 and G_2.

Theorem 3.1 ([17]). *Let $0 \le m \le \lfloor n^2/4\rfloor$, $p = \lfloor\sqrt{m}\rfloor$, $G \in \mathcal{F}_{n,m}$ and $\chi(G, 2) = f(n, m, 2)$. Then*

$$f(n, m, 2) = \begin{cases} 2^n & \text{if } m = 0, \\ 2^{n-\lceil 2\sqrt{m}\rceil+1} & \text{if } 0 \le m \le \lfloor n^2/4\rfloor, \\ 0 & \text{if } m > \lfloor n^2/4\rfloor, \end{cases}$$

and

$$G = \begin{cases} K_{p,p} + \overline{K}_{n-2p} & \text{if } 4 \le m = p^2, \\ H(p+1, p, a) + \overline{K}_{v-2p-1} & \text{if } 4 \le p^2 < m \le p^2 + p, \\ H(p+1, p+1, a) + \overline{K}_{v-2p-2} & \text{if } 4 \le p^2 + p < m \le (p+1)^2. \end{cases}$$

3.2. $f(n, m, 3)$

The value of $f(n, m, 3)$ is not known in general. In [17], it was observed that for $m \le n^2/4$, extremal graphs appeared to be close to complete bipartite graphs with one partition much smaller than the other, plus isolated points. This observation motivated the following definition: let $0 \le p \le a \le b$ be positive integers. A *semi-complete bipartite graph* $K_{a,b,p}$ has $a+b+1$ vertices and $ab+p$ edges, and is obtained from $K_{a,b}$ by adding a vertex to the partition of size b and joining it to p vertices in the partition of size a. Notice that $K_{a,b,a} = K_{a,b+1,0} = K_{a,b+1}$.

Theorem 3.2 ([17]). *For $0 \le m \le n^2/4$,*

$$\chi(K_{a,b,p}, 3) = \begin{cases} 3 \cdot (2^a + 2^b - 2) & \text{if } p = 0, \\ 3 \cdot (2^{b+1} + 2^b + 2^{a-p+1}) & \text{if } 1 \le p \le a, \end{cases}$$

where $n = a + b + 1$ and $m = ab + p$.

Theorem 3.2 can be used to obtain a lower for $f(n, m, 3)$. Several other nontrivial upper and lower bounds for $\chi(G, 3)$ have been established by Byer [6], Lazebnik [17,18], Liu [23], and Dohmen [8,9], but the bounds are widely separated. For $\lambda \ge 3$ and any $m \ge 4$, Simonelli [28] exhibited a subfamily of (n, m)-bipartite graphs $\mathcal{M}(n, m)$ (we omit its definition) which generalized the family of semi-complete bipartite graphs, and showed that the maximum number of λ-colorings of a bipartite (m, n)-graph is necessarily attained on a graph from $\mathcal{M}(n, m)$. The following conjecture was motivated by computations on small graphs, several bounds on $\chi(G, 3)$ mentioned above, and the result of Lazebnik [19] on the extremality of $K_{p,p}$ for large λ (see Theorem 3.12 ahead).

Conjecture 3.3 ([17]). *Let $0 \le m \le n^2/4$ and G be an (n, m)-graph. Then $f(n, m, 3) = \chi(G, 3)$ if and only if G is a semi-complete bipartite graph.*

Though Theorems 3.1 and 3.12 imply that for $(2p, p^2)$-graphs Conjecture 3.3 is true for $\lambda = 2$ and for large λ, a similar result for $\lambda = 3$ was obtained only fifteen years later by Lazebnik, Pikhurko and Woldar [20].

Theorem 3.4 ([20]). *For all $t \ge 1$,*

$$f(2t, t^2, 3) = \chi(K_{t,t}, 3) = 6(2^t - 1), \tag{1}$$

with $K_{t,t}$ being the only extremal graph.

Please cite this article in press as: F. Lazebnik, The maximum number of colorings of graphs of given order and size: A survey, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.08.010.

This result was extended in several ways. First, in a remarkable paper [24], Loh, Pikhurko and Sudakov proved Conjecture 3.3 for large m and in a strong way. To state their result we need the following definition. Given $\lambda \geq 2$, we define the constant

$$\kappa_\lambda = \left(\sqrt{\frac{\log(\lambda/(\lambda-1))}{\log\lambda}} + \sqrt{\frac{\log\lambda}{\log(\lambda/(\lambda-1))}}\right)^{-2}.$$

Clearly, $\kappa_\lambda = (1+o(1))\frac{1}{\lambda\log\lambda}$, when $\lambda \to \infty$.

Theorem 3.5 ([24]). *For every fixed $\lambda \geq 3$, and any $\kappa < \kappa_\lambda$, the following holds for sufficiently large m with $m \leq \kappa n^2$. Every (n, m)-graph that maximizes the number of λ-colorings is a semi-complete bipartite subgraph of $K_{a,b}$ plus isolated vertices, where $a = (1+o(1))\sqrt{m \cdot \log\frac{\lambda}{\lambda-1}/\log\lambda}$ and $b = (1+o(1))\sqrt{m \cdot \log\lambda/\log\frac{\lambda}{\lambda-1}}$, $m \to \infty$. The corresponding number of colorings is $\lambda^n e^{(c+o(1))\sqrt{m}}$, $m \to \infty$, where $c = 2\sqrt{\log\frac{\lambda}{\lambda-1}\cdot\log\lambda}$.*

The partition sizes of all extremal graphs above have the ratio roughly $\log\lambda/\log\frac{\lambda}{\lambda-1}$, that is similar to the case for 3-colorings from [17].

It was shown in [24] that for three colors more could be proven. The following notion is important in dealing with some rare exceptions. We say that we add a *pendant edge* to a graph, if we add a new vertex and connect it to any vertex of the graph.

Theorem 3.6 ([24]). *The following holds for all sufficiently large m with $m \leq n^2/4$. Every (n, m)-graph with the maximum number of 3-colorings is either*

(i) *a semi-complete bipartite graph $K_{a,b,p}$ plus isolated vertices if necessary, or*
(ii) *$K_{a,b}$ plus a pendant edge. Furthermore,*
(iii) *If $m \leq \kappa_3 n^2$, then*

$$a = (1+o(1))\sqrt{m \cdot \log\frac{3}{2}/\log 3} \quad \text{and} \quad b = (1+o(1))\sqrt{m \cdot \log 3/\log\frac{3}{2}},\ m \to \infty.$$

The corresponding number of 3- colorings is $3^n e^{(c+0(1))\sqrt{m}}$, $m \to \infty$, where $c = 2\sqrt{\log\frac{3}{2}\cdot\log 3}$.

(iv) *If $\kappa_3 n^2 \leq m \leq \frac{1}{4}n^2$, then*

$$a = \left(\frac{1}{2}+o(1)\right)(n+\sqrt{n^2-4m}) \quad \text{and} \quad b = \left(\frac{1}{2}+o(1)\right)\left(n-\sqrt{n^2-4m}\right),\ n \to \infty.$$

The corresponding number of 3-colorings is $2^{b+o(n)}$, $n \to \infty$.

Another way to generalize Theorem 3.4 is to obtain a similar result for all Turán $(n, t_r(n))$-graphs and $\lambda = r+1$ for all $n \geq r \geq 2$ (for $r = 2$, we get $\lambda = 3$). The following theorem from [24] establishes the extremality of $T_r(n)$ for a fixed r, $\lambda = r+1$ and large n.

Theorem 3.7 ([24]). *Fix an integer $r \geq 3$. For all sufficiently large n, the Turán graph $T_r(n)$ has more $(r+1)$-colorings than any other graph with the same number of vertices and edges.*

At about the same time, Lazebnik and Tofts [21] obtained a similar result but for all n, $2 \leq r \leq n$.

Theorem 3.8 ([21]). *Let $2 \leq r \leq n$. Then*

$$f(n, t_r(n), r+1) = \chi(T_r(n), r+1) = (r+1)!(s\,2^k + (r-s)2^{k-1} - (r-1)),$$

where $1 \leq r \leq n$, $k = \lfloor\frac{n}{r}\rfloor \geq 1$, and $0 \leq s = n - rk < r$. Moreover, $T_r(n)$ is the only extremal graph.

3.3. $f(n, m, 4)$

The following theorem shows that the graph $T_2(2t)$ is asymptotically extremal for $\lambda = 4$.

Theorem 3.9 ([20]).

$$f(2t, t^2, 4) \sim \chi(T_2(2t), 4) \sim (6+o(1))4^t, \quad \text{as } t \to \infty. \tag{2}$$

In [26], Norine showed that results similar to the ones of Theorems 3.7, 3.8 (but for sufficiently large n), and 3.9, hold also for all other values of $\lambda \geq r+1 \geq 3$ as long as r is a divisor of λ. In particular, it holds for $(r, \lambda) = (2, 4)$ and sufficiently large n, that strengthens (2) to $f(n, t_r(n), \lambda) = \chi(T_r(n), \lambda)$, as $n \to \infty$.

Please cite this article in press as: F. Lazebnik, The maximum number of colorings of graphs of given order and size: A survey, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.08.010.

Theorem 3.10 ([26]). *For any positive integers r, λ, such that $2 \le r < \lambda$ and r divides λ, there exists $n_0 = n_0(r, \lambda)$, such that for all $n \ge n_0$,*

$$f(n, t_r(n), \lambda) = \chi(T_r(n), \lambda),$$

and that $T_r(n)$ is the only extremal graph.

This result was followed by a paper by Tofts [30], where the extremality of $T_2(n)$ with respect to the number of 4-colorings was proven for all $n \ge 4$.

Theorem 3.11 ([30]). *Let $n = 2k + s \ge 4$ and $s \in \{0, 1\}$. Then*

$$f(n, t_2(n), 4) = \chi(T_2(n), 4) = 6 \cdot 2^n + 4(1 + 3^s)3^k - 12(1 + 2^s)2^k + 12,$$

with $T_2(n)$ being the only extremal graph.

3.4. $f(n, m, \lambda)$ for large λ

Here we collect results that hold for some fixed (n, m) and λ large.

Theorem 3.12 ([19]). *(i) Let $p \ge 3$. Then for $\lambda \ge p^5$,*

$$f(2p, p^2, \lambda) = \chi(K_{p,p}, \lambda),$$

and $K_{p,p}$ is the unique extremal graph.
(ii) Let $p \ge 3$, $n \ge 2p$, $m = p^2$. Then for $\lambda \ge p^4/12$,

$$f(n, p^2, \lambda) = \chi(K_{p,p} + \overline{K}_{n-2p}, \lambda),$$

and $K_{p,p} + \overline{K}_{n-2p}$ is the unique extremal graph.

Part (i) of this theorem was generalized to all Turán graphs with a slightly greater lower bound on λ.

Theorem 3.13 ([19]). *Let p and $r \ge 2$ be positive integers, $n = pr$, $m = t_r(n)$, and $\lambda \ge 2\binom{m}{3}$. Then*

$$f(n, t_r(n), \lambda) = \chi(T_r(n), \lambda),$$

and $T_r(n)$ is the unique extremal graph.

If we allow λ to depend on n and m, then Theorems 3.5, 3.8, and 3.10 provide additional results.

3.5. $f(n, t_r(n), \lambda)$ and extremality of $\chi(T_r(n))$

The following conjecture by Lazebnik (unpublished, 1987) appeared in print in [20].

Conjecture 3.14 ([20]). *For all $n \ge r \ge 2$ and all $\lambda \ge r$,*

$$f(n, t_r(n), \lambda) = \chi(T_r(n), \lambda),$$

and $T_r(n)$ is the only extremal graph.

When $\lambda = r$, the statement follows from the celebrated Turán's theorem [39], since any $(n, t_r(n))$-graph different from $T_r(n)$ has chromatic number at least $r + 1$.
Conjecture 3.14 was widely believed to be true, and all related results from previous subsections of this survey supported it. It has been a surprise to many to learn that the conjecture was actually false. In [25], Ma and Naves presented counterexamples to the conjecture for some ranges of r and λ.

Theorem 3.15 ([25]). *(i) For all integers $r \ge 50000$ and λ_0 such that*

$$20r \le \lambda_0 \le \frac{r^2}{200 \log r},$$

there exists an integer λ within distance at most r from λ_0, such that Conjecture 3.14 is false for (r, λ).
(ii) If $r + 3 \le \lambda \le 2r - 7$, where r is an integer and $r \ge 10$, Conjecture 3.14 is false.

Nevertheless, in the same paper the authors confirmed that $T_r(n)$ is asymptotically extremal when $\lambda = \Omega(r^2/\log r)$, $r \to \infty$.

Please cite this article in press as: F. Lazebnik, The maximum number of colorings of graphs of given order and size: A survey, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.08.010.

Theorem 3.16 (*[25]*). *(i) For sufficiently large integers r and λ with* $\lambda \geq \frac{100r^2}{\log r}$, *the following holds for all sufficiently large n. Every extremal graph that maximizes the number of* λ*-colorings over all* $(n, \frac{r-1}{2r}n^2 + o(n^2))$*-graphs is* $o(n^2)$*-close to* $T_r(n)$.

(ii) If $r \leq \lambda \leq r+2$, *then for every integer* $r > 1$ *and sufficiently large integers n, every extremal graph that maximizes the number of* λ*-colorings over all* $(n, \frac{r-1}{2r}n^2 + o(n^2))$*-graphs is* $o(n^2)$*-close to* $T_r(n)$.

The king is dead. Long live the king! The refutation of the conjecture makes the original problem of determining $f(n, m, \lambda)$ even more challenging.

4. Ideas and techniques

In this section we make an attempt of covering some ideas and techniques used in the proofs of main results mentioned in the previous section. Here we arrange the material closer to the chronological order, trying to unite results which proofs share similar techniques, and trace the development of the ideas where it is possible.

4.1. Theorem 3.1

It is obvious that for a graph G to have at least one 2-coloring, it has to be bipartite. In this case, if G has $c(G)$ components, then $\chi(G, 2) = 2^{c(G)}$. In order to maximize the number of components of a bipartite graph for $m \leq n^2/4$, we pack all edges as tight as possible, so the number of isolated vertices becomes maximum. See [17] for details.

4.2. Theorems 3.12 and 3.13

Suppose we wish to understand which (n, m)-graph G has the largest number of λ-colorings for n and m fixed and large λ. We will use a characterization of the coefficients of $\chi(G, \lambda)$ in terms of so called 'broken circuits' of graph G [40,41]. Suppose we number the edges of G by integers from 1 to m in some manner. Next, from the edge set of each cycle of G we delete the edge with the highest index, obtaining, thereby, the set of edges called the *broken cycle*.

Theorem 4.1 (*[41], Whitney's 'Broken Circuits' Theorem*). *For a graph of order n,*

$$\chi(G, \lambda) = a_0\lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} - \cdots + (-1)^{n-1}a_{n-1}\lambda, \tag{3}$$

where the coefficient a_j *is equal to the number of j-subsets of edges of G which contain no broken cycles.*

For any (n, m)-graph G, the first two coefficients of (3) are fixed: $a_0 = 1$ and $a_1 = m$. Since we wish to maximize $\chi(G, \lambda)$ for large λ, we look for (n, m)-graphs with the largest positive coefficient a_2. Then among all such graphs, we will look for the ones with the least a_3. This will lead us to a unique extremal graph.

As an immediate corollary from Theorem 4.1 we get that $a_2 = \binom{m}{2} - c_3$, where $c_3 = c_3(G) =$ the number of triangles in G. This relation shows that a_2 is the greatest if and only if c_3 is the least. If $m \leq n^2/4$, then, by Turán's theorem, there exist (n, m)-graphs without triangles (e.g., bipartite graphs), and for them $c_3 = 0$. Hence, the coefficient $a_2 = \binom{m}{2}$ for each of them, i.e., at its maximum, and we focus our attention on the value of a_3 for this subfamily of graphs. It is easy to see that for triangle-free graphs, $a_3 = \binom{m}{3} - c_4$, where $c_4 = c_4(G) =$ the number of quadrilaterals in G. Thus the problem is reduced to the following: for $m \leq n^2/4$, among all triangle-free (n, m)-graphs find ones with the greatest number of quadrilaterals. It turns out that for $n \geq 2p$ and $m = p^2$, the only(!) triangle-free (n, m)-graph graph having the greatest number of quadrilaterals is $K_{p,p} + \overline{K}_{n-2p}$. Our search is finished. The lower bounds for λ are derived by using an upper bound on roots of polynomials in terms of their coefficients. For $(n, t_r(n))$-graphs, the approach is similar. See [19] for details.

4.3. Theorems 3.4, 3.8 and 3.11

The proof of Theorem 3.4 used induction on p, and the explicit formula $6(2^t - 1)$ for $\chi(K_{p,p}, 3)$. It begins with an observation that there exists an extremal graph H, i.e. $\chi(H, \lambda) = f(2p, p^2, \lambda)$, having at most one component with more than one vertex. If H is not isomorphic to $K_{p,p}$, then, by Turán's theorem it must contain a triangle T. Then either there is an edge uv of T, such that $d(u) + d(v) \leq 2p$, or for every edge uv of T, $d(u) + d(v) \geq 2p + 1$. In the first case, after deleting vertices u and v from H, the result easily follows. The second case does not use induction hypothesis, and is more involved. It was proven by partitioning of vertices of H not in T into several classes, and estimating the number of edges between each class and the vertices of T.

The proofs of Theorems 3.8 and 3.11 generalize the one of Theorem 3.4 that we just described. Though their logic is similar, the proofs are substantially harder.

Please cite this article in press as: F. Lazebnik, The maximum number of colorings of graphs of given order and size: A survey, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.08.010.

4.4. Theorem 3.9

In proving Theorem 3.9 the following approach was used.

First the formula $\chi(K_{p,p}, 4) = 6 \cdot 4^p + 8 \cdot 3^p - 24 \cdot 2^p + 12$ was established. It provided a lower bound on $f(2p, p^2, 4)$.

Then a weaker result was established, namely that if a graph is close to $K_{p,p}$ with respect to edit distance, then the number of its 4-colorings is at most $(6 + o(1))4^k$. In other words, Theorem 3.9 was first proven for these special graphs:

Theorem 4.2. *There exist constants $\epsilon > 0$ and p_0 such that for every $p \geq p_0$ and every $(2p, p^2)$-graph G which is ϵp^2-close to $K_{p,p}$ we have*

$$\chi(G, 4) \leq 6 \cdot 4^p + (4 - \epsilon)^p.$$

Thus, it suffices to consider graphs which are not close to $K_{p,p}$ with respect to edit distance. Let a *kite* be a graph F isomorphic to K_4 with one edge deleted, i.e., consisting of two triangles sharing an edge. Since $\chi(F) = 3$, the Stability Theorem of Simonovits [29] implies that, for sufficiently large p, any $(2p, p^2)$-graph G not close to $K_{p,p}$ contains a subgraph isomorphic to F.

Next the authors consider constants ϵ and p_0 which satisfy the statement of Theorem 4.2 and such that $\epsilon \gg 1/p_0$, and show that

$$f(2l, l^2, 4) \leq (6 + \epsilon)4^l$$

for all $l > p_0^2$. The proof of the theorem continues by induction on p. If it is possible to remove a pair of vertices occurring in a unique triangle of a kite so that at most $2p - 2 < p^2 - (p - 1)^2$ edges are deleted, then the number of colorings decreases by at least a factor of 4, and the proof is finished.

Suppose there is no such an edge. It follows that vertices of every kite is incident to many edges. Then all 4-colorings of the graph are split into two classes depending on whether or not there is a kite with all four vertices having different colors. Using an argument similar to the one of the proof of Theorem 3.4 (and much more), the sizes of both classes are bounded, and this implies an upper bound on the number of all 4-colorings. See [20] for details.

4.5. Theorems 3.5, 3.6, 3.7

In [24], Loh, Pikhurko and Sudakov developed the ideas of the proof of Theorem 3.9, and introduced new techniques which have been successfully used by others in further studies of $f(n, m, \lambda)$. It was certainly a breakthrough paper. It is long and is very well written. We cannot summarize the ideas of the paper better than its authors. Therefore, let us quote them (we just replace the authors q for the number of colors by λ).

> . . . Perhaps part of the difficulty for general m, n, λ stems from the fact that maximal graphs are substantially more complicated than the minimum graphs that Linial found. For number-theoretic reasons, it is essentially impossible to construct maximal graphs for general m, n. Furthermore, even their coarse structure depends on the density m/n^2. . . . Therefore, in order to tackle the general case of this problem, one must devise a unified approach that can handle all of the outcomes.
>
> In this paper, we propose such an approach, developing the machinery that one might be able to use to determine the maximal graphs in many nontrivial ranges of m, n. Our methodology can be roughly outlined as follows. We show, via Szemerédi's Regularity Lemma, that the asymptotic solution to the problem reduces to a certain quadratically-constrained linear program in $2^\lambda - 1$ variables. For any given λ, this task can in principle be automated by a computer code that symbolically solves the optimization problem, although a more sophisticated approach was required to solve this for all λ. Our solutions to the optimization problem then give us the approximate structure of the maximal graphs. Finally, we use various local arguments, such as the so-called "stability" approach introduced by Simonovits [29], to refine their structure into precise results. . . .

We conclude these comments with the statement of the optimization problem mentioned above. It reduces the asymptotic version of the original problem to the following linear optimization problem with quadratic constraints. We follow the exposition in [24] with some changes in notation.

Fix an integer $\lambda \geq 2$ and a real number γ. Consider the following objective and constraint functions:

$$\mathrm{OBJ}(\mathbf{x}) := \sum_{A \neq \emptyset} x_A \log|A|; \quad \mathrm{v}(\mathbf{x}) := \sum_{A \neq \emptyset} x_A, \quad \mathrm{E}(\mathbf{x}) := \sum_{A \cap B = \emptyset} x_A x_B.$$

The vector $\mathbf{x}$ has $2^\lambda - 1$ coordinates $x_A \in \mathbb{R}$ indexed by the nonempty subsets $A \subseteq [\lambda]$, and the sum in $\mathrm{E}(\mathbf{x})$ runs over unordered pairs of disjoint nonempty sets $\{A, B\}$. Let $\mathrm{FEAS}(\gamma)$ be the *feasible set* of vectors defined by the constraints $\mathbf{x} \geq \mathbf{0}$, $\mathrm{v}(\mathbf{x}) = 1$, and $\mathrm{E}(\mathbf{x}) \geq \gamma$. We seek to maximize $\mathrm{OBJ}(\mathbf{x})$ over the set $\mathrm{FEAS}(\gamma)$, and we define $\mathrm{OPT}(\gamma)$ to be this maximum value which exists by compactness. Vector $\mathbf{x}$ *solves* $\mathrm{OPT}(\gamma)$ when both $\mathbf{x} \in \mathrm{FEAS}(\gamma)$ and $\mathrm{OBJ}(\mathbf{x}) = \mathrm{OPT}(\gamma)$.

Given a vector $\mathbf{x} \in \mathrm{FEAS}(\gamma)$ for some γ, we construct a graph $G_{\mathbf{x}}(n)$ on n vertices as follows. Partition $V(G_{\mathbf{x}}(n))$ into clusters V_A such that $|V_A|$ differs from $x_A n$ by less than 1, and for every pair $A, B \subseteq [\lambda]$ with $A \cap B = \emptyset$ join every vertex in V_A to every

vertex of V_B by an edge. It is clear that any coloring that for each cluster V_A uses only colors from A is a proper coloring of $G = G_{\mathbf{x}}$ and so $\chi(G, \lambda) \geq \prod_A |A|^{x_A n - 1} = e^{\mathrm{OBJ}(\mathbf{x})n - O(1)}$.

The following theorem states that the graph $G_{\mathbf{x}}(n)$ represents an approximate structure of the extremal graphs.

Theorem 4.3 ([24]). *For any $\epsilon, \kappa > 0$ the following holds for all sufficiently large n. Let G be an (n, m)-graph with $m \leq \kappa n^2$, which has at least as many λ-colorings as any other (n, m)-graph. Then G is ϵn^2-close to a graph $G_{\mathbf{x}}(n)$ for some $\mathbf{x}$ which solves $OPT(\gamma)$ for some γ such that $|\gamma - m/n^2| < \epsilon$ and $\gamma \leq \kappa$.*

4.6. Theorem 3.10

We wish to begin with a quote from Norine's paper [26] (again, in the text below the number of colors q is denoted by λ).

> ... In a recent breakthrough paper Loh, Pikhurko and Sudakov [24] ...remarked that "the remaining challenge is to find analytic arguments which solves the optimization problem for general λ". In this note we present one such argument. We relax the optimization problem to a certain fractional version and solve some natural instances of this relaxation.. . .

The condition r dividing λ corresponds to a "natural instance". Consider vector $\mathbf{x}$ from our discussion of the optimization problem in the previous section. For a vector $\mathbf{x} = (x_A)$, $\emptyset \neq A \subseteq [\lambda]$, define *support* of $\mathbf{x}$ as a collection of sets A such that $x_A \neq 0$. The vector $\mathbf{x}$ is a *balanced partition vector* if the support of $\mathbf{x}$ is a partition of $[\lambda]$ and all sets in the support have the same size. This is how the divisibility condition enters the scene. See [26] for the details.

Another interesting result of the same paper is a general upper bound on $\chi(G, \lambda)$.

Theorem 4.4 ([26]). *For any positive integer $\lambda \geq 2$ and a positive real ϵ, the following holds for any sufficiently large n. For any (n, m)-graph,*

$$\chi(G, \lambda) \leq \lambda^{(1+\epsilon)n} \left(1 - \frac{2m}{n^2}\right)^n.$$

4.7. Theorems 3.15 and 3.16

The comments below follow closely the ones in Ma and Naves [25].

Among the main contributions of this paper is a structural theorem that allows substantially to simplify the quadratically constrained linear problem for general instances. This structural theorem asserts that extremal graphs must be asymptotically close (with respect to edit distance) to the ones in some family of graphs $\mathcal{G}_k$, where $k > 1$ is an integer depending only on the edge density of graphs. To be precise, for a fixed λ, the family $\mathcal{G}_k$ consists of complete multipartite graphs with at least k and at most λ parts as well as graphs obtained from a complete k-partite graph by adding some additional vertices each of which adjacent to the vertices of all but two fixed parts. Then the following is true.

Theorem 4.5 ([25]). *For any real $r > 1$, the following holds for all sufficiently large n. Let G be a (n, m)-graph with $m = \frac{r-1}{2r}n^2 + o(n^2)$ such that $f(n, m, \lambda) = \chi(G, \lambda)$. Then there exists an n-vertex graph in $\mathcal{G}_{\lceil r \rceil}$ which is $o(n^2)$-close to G.*

The details of the proofs related to the counterexamples of Conjecture 3.14 can be found in [25].

5. Some open questions

As this survey demonstrates, the problem of finding $f(n, m, \lambda)$ and extremal graphs is still wide open. Going through conditions of many theorems from Section 3, one can easily identify the ranges of the parameters that require further work. We would like to finish this survey by posing just three questions.

Our first question is related to Theorems 3.4, 3.11, and 3.12.

Question 5.1. *Let n be a positive fixed integer, $n \geq 6$. Is it true that $f(n, t_2(n), \lambda) = \chi(T_2(n), \lambda)$ for all $\lambda \geq 2$, with $T_2(n)$ being the only extremal graph? The first unknown case is $\lambda = 5$.*

Our second question is related to Theorems 3.8, 3.11, 3.13, and 3.16(ii).

Question 5.2. *Let n, r be positive integers, $n \geq 9$, $r \geq 2$. Is it true that $f(n, t_r(n), r + 2) = \chi(T_r(n), r + 2)$ for all $r \geq 2$, with $T_r(n)$ being the only extremal graph?*

Question 5.3. *Does Theorem 3.10 hold if the condition that r divides λ is removed?*

Please cite this article in press as: F. Lazebnik, The maximum number of colorings of graphs of given order and size: A survey, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.08.010.

Acknowledgments

The author is thankful to anonymous reviewers for their careful reading of the manuscript and for providing many useful comments.

This work was partially supported by a grant from the Simons Foundation (#426092).

References

[1] E. Bender, H. Wilf, A theoretical analysis of backtracking in the graph coloring problem, J. Algorithms 6 (1985) 275–282.
[2] G. Birkhoff, A determinant formula for the number of ways of coloring a map, Ann. of Math. 14 (1912) 42–46.
[3] G. Birkhoff, On the number of ways of colouring a map, Proc. Edinburgh Math. Soc. (2) 2 (1930) 83–91.
[4] G. Birkhoff, D. Lewis, Chromatic polynomials, Trans. Amer. Math. Soc. 60 (1946) 355–451.
[5] B. Bollobás, Modern Graph Theory, Springer-Verlag, Berlin, 1998.
[6] O. Byer, Some new bounds for the maximum number of vertex colorings of a (v, e)-graph, J. Graph Theory 28 (1998) 115–128.
[7] X.B. Chen, The maximum number of proper 2-colorings of (v, e)-graphs and the extremal graphs, Zhangzhou Shiyuan Xuebao (Ziran Kexue Ban) 10 (1996) 25–27.
[8] K. Dohmen, Lower bounds and upper bounds for chromatic polynomials, J. Graph Theory 17 (1993) 75–80.
[9] K. Dohmen, Bounds to the chromatic polynomial of a graph, Results Math. 33 (1998) 87–88.
[10] F.M. Dong, K.M. Koh, K.L. Teo, Chromatic Polynomials and Chromaticity of Graphs, World Scientific, 2005.
[11] A. Erey, Maximizing the number of x-colorings of 4-chromatic graphs, 2016, arXiv:1610.07219 [math.CO].
[12] A. Erey, On the maximum number of colorings of a graph, 2017, arXiv:1610.07208 [math.CO].
[13] J. Fox, X. He, F. Manners, A proof of Tomescu's graph coloring conjecture, 2018, arXiv:1712.06067 [math.CO].
[14] E.N. Gilbert, Enumeration of labelled graphs, Can. J. Math. 8 (1956) 405–411.
[15] F. Knox, B. Mohar, Maximum number of colourings. I. 4-chromatic graphs, 2017, arXiv:1708.01781 [math.CO].
[16] F. Knox, B. Mohar, Maximum number of colourings. II. 5-chromatic graphs, 2017, arXiv:1710.06535 [math.CO].
[17] F. Lazebnik, On the greatest number of 2 and 3 colorings of a (v, e)-graph, J. Graph Theory 13 (1989) 203–214.
[18] F. Lazebnik, New upper bounds for the greatest number of proper colorings of a (V, E)-graph, J. Graph Theory 14 (1990) 25–29.
[19] F. Lazebnik, Some corollaries of a theorem of Whitney on the chromatic polynomial, Discrete Math. 87 (1991) 53–64.
[20] F. Lazebnik, O. Pikhurko, A. Woldar, Maximum number of colorings of $(2k, k^2)$-graphs, J. Graph Theory 56 (2007) 135–148.
[21] F. Lazebnik, S.N. Tofts, An extremal property of Turán's Graphs, Electron. J. Combin. 17 (1) (2010) 1–11 R170.
[22] N. Linial, Legal coloring of graphs, Combinatorica 6 (1986) 49–54.
[23] R. Liu, On the greatest number of proper 3-colorings of a graph, Math. Appl. 6 (1993) 88–91.
[24] P.-S. Loh, O. Pikhurko, B. Sudakov, Maximizing the number of q-colorings, Proc. London Math. Soc. 101 (2010) 655–696.
[25] J. Ma, H. Naves, Maximizing proper colorings on graphs, J. Combin. Theory Ser. B 115 (C) (2015) 236–275.
[26] S. Norine, Turán graphs and the number of colorings, SIAM J. Discrete Math. 25 (2011) 260–266.
[27] R. Read, The number of k-coloured graphs on labelled nodes, Canad. J. Math. 12 (1960) 409–413.
[28] I. Simonelli, Optimal graphs for chromatic polynomials, Discrete Math. 308 (11) (2008) 2228–2239.
[29] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: P. Erdős, G. Katona (Eds.), Theory of Graphs (Proceedings of the Colloquium, Tihany, 1966), Academic Press, New York, 1968, pp. 279–319.
[30] S.N. Tofts, An extremal property of Turán graphs, II, J. Graph Theory 75 (3) (2014) 275–283.
[31] I. Tomescu, Le nombre maximal de colorations d'un graphe, C. R. Math. Acad. Sci. Paris 272 (1971) 1301–1303.
[32] I. Tomescu, Le nombre maximal de 3-colorations d'un graphe connexe, Discrete Math. 1 (1972) 351–356.
[33] I. Tomescu, Le nombre minimal de colorations d'un graphe, C. R. Math. Acad. Sci. Paris 274 (1972) 539–542.
[34] I. Tomescu, Problèmes extremaux concernant le nombre des colorations des sommets d'un graphe fini, in: Combinatorial programming: Methods and applications (Proc. NATO Advanced Study Inst., Versailles, 1974), in: NATO Advanced Study Inst. Ser., Ser. C: Math. and Phys. Sci., vol. 19, Reidel, Dordrecht, 1975, pp. 327–336.
[35] I. Tomescu, Le nombre maximal de colorations d'un graphe hamiltonien, Discrete Math. 16 (1976) 353–359.
[36] I. Tomescu, Maximal chromatic polynomials of connected planar graphs, J. Graph Theory 14 (1990) 101–110.
[37] I. Tomescu, Maximum chromatic polynomials of 2-connected graphs, J. Graph Theory 18 (1994) 329–336.
[38] I. Tomescu, Maximum chromatic polynomial of 3-chromatic blocks, Discrete Math. 172 (1997) 131–139.
[39] P. Turán, On an extremal problem in graph theory (in Hungarian), Mat. Fiz. Lapok 48 (1941) 436–452.
[40] H. Whitney, The Coloring of Graphs, (Ph.D. thesis), Harvard University, 1932.
[41] H. Whitney, A logical expansion in mathematics, Bull. Amer. Math. Soc. 38 (1932) 572–579.
[42] H.S. Wilf, Personal communication. 1982.
[43] H.S. Wilf, Backtrack: an $O(1)$ expected time algorithm for the graph coloring problem, Inform. Process. Lett. 18 (1984) 119–121.
[44] E.M. Wright, Counting coloured graphs, Canad. J. Math. 13 (1961) 683–693.
[45] E.M. Wright, Counting coloured graphs II, Canad. J. Math. 16 (1964) 128–135.
[46] E. Wright, Counting coloured graphs III, Canad. J. Math. 1 (1972) 82–89.

Please cite this article in press as: F. Lazebnik, The maximum number of colorings of graphs of given order and size: A survey, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.08.010.