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On the edit distance of powers of cycles

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ABSTRACT

The edit distance between two graphs on the same labeled vertex set is defined to be the size of the symmetric difference of their edge sets. The edit distance function of a hereditary property $\mathcal{H}$ is a function of $p \in [0, 1]$ that measures, in the limit, the maximum normalized edit distance between a graph of density p and $\mathcal{H}$. The expression $\mathcal{H} = \mathrm{Forb}(H)$ denotes the property of having no induced subgraph isomorphic to H.

In this paper, we address the edit distance function for the hereditary property $\mathrm{Forb}\left(C_h^t\right)$, where C_h^t denotes the t^{th} power of the cycle of length h. For $h \geq 2t(t+1)+1$ and h not divisible by $t+1$, we determine the function for all values of p. For $h \geq 2t(t+1)+1$ and h divisible by $t+1$, the function is obtained for all but small values of p. We also obtain edit distance functions for some smaller values of h.

1. Introduction

The edit distance in graphs was introduced independently by Axenovich, Kézdy, and Martin [2] and by Alon and Stav [1]. The question considered is "Given a class of graphs $\mathcal{H}$ what is the minimum number $m = m(n)$ such that for every graph on n vertices, there is a set of m edge-additions plus edge-deletions that ensure the resultant graph is a member of $\mathcal{H}$?" A *hereditary property* is a family of graphs that is closed under isomorphism and the taking of induced subgraphs. For every hereditary property $\mathcal{H}$, there is a $p = p(\mathcal{H})$ such that the Erdős–Rényi random graph $G(n, p)$ is asymptotically extremal [1].

The edit distance function of a hereditary property $\mathcal{H}$ is a function of $p \in [0, 1]$ that measures, in the limit, the maximum normalized edit distance between a graph of density p and $\mathcal{H}$. A *principal hereditary property*, denoted $\mathrm{Forb}(H)$, is a hereditary property that consists of the graphs with no induced copy of a single graph H. Most of the known edit distance functions concern principal hereditary properties. These include the cases where H is a split graph [7] (including cliques and independent sets), complete bipartite graphs $K_{2,t}$ [8] and $K_{3,3}$ [5] and cycles C_h where $h \leq 10$ [6]. In this paper, we compute the edit distance function for powers of cycles.

For positive integers t and h, the tth power of a cycle of length h is denoted C_h^t and has vertex set $\{1, \ldots, h\}$, where two vertices are adjacent in C_h^t if and only if their distance is at most t in C_h.

The notation in this paper primarily comes from Martin [8]. The *edit distance* between graphs G and G' on the same labeled vertex set is denoted $\mathrm{dist}(G, G') = |E(G) \triangle E(G')|$. The edit distance between a graph G and a hereditary property $\mathcal{H}$ is

$$\mathrm{dist}(G, \mathcal{H}) = \min\{\mathrm{dist}(G, G') : V(G) = V(G'), G' \in \mathcal{H}\}.$$

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The edit distance function of a hereditary property $\mathcal{H}$ measures the maximum distance of a density p graph from $\mathcal{H}$, i.e.

$$\mathrm{ed}_{\mathcal{H}}(p) = \lim_{n\to\infty} \max\{\mathrm{dist}(G,\mathcal{H}) : |V(G)| = n, |E(G)| = \lfloor p\tbinom{n}{2}\rfloor\}/\tbinom{n}{2}.$$

Balogh and Martin [3] showed that this limit exists and is equal to $\lim_{n\to\infty}\mathbb{E}[\mathrm{dist}(G(n,p),\mathcal{H})]$, with an argument similar to one by Alon and Stav in [1]. The function has a number of interesting properties:

Proposition 1 (*Balogh–Martin [3]*). *If $\mathcal{H}$ is a hereditary property, then $\mathrm{ed}_{\mathcal{H}}(p)$ is continuous and concave down over $p \in [0,1]$.*

By the proposition above, the function $\mathrm{ed}_{\mathcal{H}}$ achieves its maximum in $[0,1]$. We denote this maximum value by $d^*_{\mathcal{H}}$, and the set of all values of p for which the maximum is achieved by $p^*_{\mathcal{H}}$.

Alon and Stav [1] defined the following: A *colored regularity graph (CRG)*, K, is a complete graph with a partition of the vertices into white $\mathrm{VW}(K)$ and black $\mathrm{VB}(K)$, and a partition of the edges into white $\mathrm{EW}(K)$, gray $\mathrm{EG}(K)$, and black $\mathrm{EB}(K)$. We say that a graph H *embeds in* K, denoted $H \mapsto K$, if there is a function $\varphi : V(H) \to V(K)$ so that if $h_1h_2 \in E(H)$, then either $\varphi(h_1) = \varphi(h_2) \in \mathrm{VB}(K)$ or $\varphi(h_1)\varphi(h_2) \in \mathrm{EB}(K) \cup \mathrm{EG}(K)$, and if $h_1h_2 \notin E(H)$, then either $\varphi(h_1) = \varphi(h_2) \in \mathrm{VW}(K)$ or $\varphi(h_1)\varphi(h_2) \in \mathrm{EW}(K) \cup \mathrm{EG}(K)$.

Given a hereditary property $\mathcal{H}$, it is easy to see that it can be expressed as $\mathcal{H} = \bigcap\{\mathrm{Forb}(H) : H \in \mathcal{F}(\mathcal{H})\}$ for some family of graphs $\mathcal{F}(\mathcal{H})$. We denote $\mathcal{K}(\mathcal{H})$ to be the subset of CRGs such that no forbidden graph embeds into them, i.e. $\mathcal{K}(\mathcal{H}) = \{K : H \not\mapsto K, \forall H \in \mathcal{F}(\mathcal{H})\}$. In our case, $\mathcal{K}(\mathcal{H}) = \{K : H \not\mapsto K\}$ for $\mathcal{H} = \mathrm{Forb}(H)$. We define a CRG K to be a *sub-CRG* of $\tilde{K}$ if K can be obtained by deleting vertices of $\tilde{K}$.

For every CRG K we associate a function g on $[0,1]$ defined by

$$g_K(p) = \min\{\mathbf{x}^T\mathbf{M}_K(p)\mathbf{x} : \mathbf{x}^T\mathbf{1} = 1, \mathbf{x} \geq \mathbf{0}\}, \tag{1}$$

where

$$[\mathbf{M}_K(p)]_{ij} = \begin{cases} p, & \text{if } v_iv_j \in \mathrm{EW}(K) \text{ or } v_i = v_j \in \mathrm{VW}(K); \\ 1-p, & \text{if } v_iv_j \in \mathrm{EB}(K) \text{ or } v_i = v_j \in \mathrm{VB}(K); \\ 0, & \text{if } v_iv_j \in \mathrm{EG}(K). \end{cases} \tag{2}$$

The g function of CRGs can be used to compute the edit distance function. Balogh and Martin [3] proved that $\mathrm{ed}_{\mathcal{H}}(p) = \inf\{g_K(p) : K \in \mathcal{K}(\mathcal{H})\}$ and Marchant and Thomason [5] further proved that the infimum is achieved by some K, i.e. $\mathrm{ed}_{\mathcal{H}}(p) = \min\{g_K(p) : K \in \mathcal{K}(\mathcal{H})\}$. So, for every $p \in [0,1]$, there is a CRG $K \in \mathcal{K}(\mathcal{H})$ such that $\mathrm{ed}_{\mathcal{H}}(p) = g_K(p)$. It is also shown in that paper that in order to find such CRG we only need to look at so called p-core CRGs. A CRG $\tilde{K}$ is *p-core* if $g_{\tilde{K}}(p) < g_K(p)$ for every sub-CRG K of $\tilde{K}$.

The CRG with r white vertices, s black vertices and all edges gray is denoted $K(r,s)$. The *clique spectrum* of the hereditary property $\mathcal{H} = \mathrm{Forb}(H)$, denoted $\Gamma(\mathcal{H})$, is the set of all pairs (r,s) such that $H \not\mapsto K(r,s)$. It is easy to see that, for any hereditary property $\mathcal{H}$ its clique spectrum $\Gamma = \Gamma(\mathcal{H})$ can be expressed as a Ferrers diagram. That is, if $r \geq 1$ and $(r,s) \in \Gamma$, then $(r-1,s) \in \Gamma$ and if $s \geq 1$ and $(r,s) \in \Gamma$, then $(r,s-1) \in \Gamma$. An *extreme point* of a clique spectrum Γ is a pair $(r,s) \in \Gamma$ such that $(r+1,s)$ and $(r,s+1)$ do not belong to Γ. The set of all extreme points of Γ is denoted by Γ^*.

Define the function $\gamma_{\mathcal{H}}(p) = \min\{g_{K(r,s)}(p) : (r,s) \in \Gamma(\mathcal{H})\}$. Clearly, $\mathrm{ed}_{\mathcal{H}}(p) \leq \gamma_{\mathcal{H}}(p)$. Moreover, one only need consider the extreme points rather than all of Γ itself, that is, $\gamma_{\mathcal{H}}(p) = \min\{g_{K(r,s)}(p) : (r,s) \in \Gamma^*(\mathcal{H})\}$.

In this paper, the hereditary properties we consider are of the form $\mathcal{H} = \mathrm{Forb}(C_h^t)$. Since Martin [6] gives $\mathrm{ed}_{\mathrm{Forb}(K_h)}(p) = p/(h-1)$, we will assume that $h \geq 2t+2$. For convenience, we denote $\ell_r = \left\lceil \frac{h}{t+r+1} \right\rceil$, for $r \in \{0,\ldots,t\}$. We also denote $p_t = {\ell_t}^{-1}$. The motivation for these values will be discussed in Section 5.

The main results of this paper are Theorems 2 and 3.

Theorem 2. *Let $t \geq 1$ and $h \geq \max\{t(t+1),4\}$ be integers, and for $r \in \{0,1,\ldots,t\}$, let $\ell_r = \left\lceil \frac{h}{t+r+1} \right\rceil$ and let $\mathcal{H} = \mathrm{Forb}(C_h^t)$. If $p \in [0,1]$, then*

$$\gamma_{\mathcal{H}}(p) = \min_{r\in\{0,1,\ldots,t\}} \left\{ \frac{p(1-p)}{r(1-p)+(\ell_r-1)\,p} \right\}, \qquad \text{if } (t+1) \mid h;$$

$$\gamma_{\mathcal{H}}(p) = \min_{r\in\{0,1,\ldots,t\}} \left\{ \frac{p}{t+1}, \frac{p(1-p)}{r(1-p)+(\ell_r-1)\,p} \right\}, \qquad \text{if } (t+1) \nmid h.$$

Note: If $r = 0$, then $\frac{p(1-p)}{r(1-p)+(\ell_r-1)p} = \frac{p(1-p)}{(\ell_0-1)p}$, which we define to be $\frac{1-p}{\ell_0-1}$ at $p = 0$.

Theorem 3. *Let $t \geq 1$ and $h \geq 2t(t+1)+1$ be positive integers, let $p_t = \left\lceil \frac{h}{2t+1} \right\rceil^{-1}$, and let $\mathcal{H} = \mathrm{Forb}(C_h^t)$. If $(t+1) \nmid h$ and $0 \leq p \leq 1$ or $(t+1) \mid h$ and $p_t \leq p \leq 1$, then*

$$\mathrm{ed}_{\mathcal{H}}(p) = \gamma_{\mathcal{H}}(p). \tag{3}$$

Corollary 4. *Let $h \geq 5$ be a positive integer and let $\mathcal{H} = \mathrm{Forb}(C_h)$.*

- *If h is even, then for $\lceil h/3 \rceil^{-1} \leq p \leq 1$,*

$$\mathrm{ed}_{\mathcal{H}}(p) = \min\left\{\frac{p(1-p)}{1-p+(\lceil h/3\rceil - 1)p}, \frac{1-p}{\lceil h/2 \rceil - 1}\right\}.$$

- *If h is odd, then for $0 \leq p \leq 1$,*

$$\mathrm{ed}_{\mathcal{H}}(p) = \min\left\{\frac{p}{2}, \frac{p(1-p)}{1-p+(\lceil h/3\rceil - 1)p}, \frac{1-p}{\lceil h/2 \rceil - 1}\right\}.$$

It was shown by Martin [6] and by Marchant and Thomason [5], respectively, that

$$\mathrm{ed}_{\mathrm{Forb}(C_3)}(p) = p/2 \qquad \text{and} \qquad \mathrm{ed}_{\mathrm{Forb}(C_4)}(p) = p(1-p).$$

It follows from this and the above corollary that when $t = 1$, the furthest graph from $\mathrm{Forb}(C_h)$ is a graph which has density $p^* = 1/(\lceil h/2\rceil - \lceil h/3 \rceil + 1)$ when $h \geq 3$ and $h \notin \{3, 4, 7, 8, 10, 16\}$, and has density $p^* = 1/(1 + \sqrt{\lceil h/3 \rceil - 1})$ when $h \in \{3, 4, 7, 8, 10, 16\}$. Observe that the maximum value of the edit distance function can be an irrational number.

Our proof techniques often require us to compare the g function of a CRG to one of the individual functions that are given in Theorem 2. However, when h is large enough at most 3 of these functions are necessary to define $\gamma_{\mathcal{H}}$.

Corollary 5. *Let $t \geq 2$ and $h \geq 4t^2+10t+24$ be positive integers. Let $\ell_0 = \left\lceil \frac{h}{t+1} \right\rceil$, $\ell_t = \left\lceil \frac{h}{2t+1} \right\rceil$, $p_t = \ell_t^{-1}$, and let $\mathcal{H} = \mathrm{Forb}(C_h^t)$.*

- *If $(t+1) \nmid h$, then for $0 \leq p \leq 1$,*

$$\mathrm{ed}_{\mathcal{H}}(p) = \min\left\{\frac{p}{t+1}, \frac{p(1-p)}{t(1-p)+(\ell_t - 1)p}, \frac{1-p}{\ell_0 - 1}\right\}.$$

- *If $(t+1) \mid h$, then for $p_t \leq p \leq 1$,*

$$\mathrm{ed}_{\mathcal{H}}(p) = \min\left\{\frac{p(1-p)}{t(1-p)+(\ell_t - 1)p}, \frac{1-p}{\ell_0 - 1}\right\}.$$

We prove this in Section 6.

The rest of the paper is organized as follows: Section 2 provides some definitions and basic results, Section 3 gives the proof of Theorem 2, Section 4 gives Lemma 13 which is the key lemma for the proof of Theorem 3, Section 5 gives the proof of Theorem 3, Section 6 gives the proofs of some helpful lemmas and facts, and Section 7 gives some concluding remarks.

2. Definitions and tools

All graphs considered in this paper are simple. For standard graph theory notation please see West [11], for the edit distance notation please see Martin [6]. A sub-CRG K' of a CRG K is a *component* if it is maximal with respect to the property that, for all distinct $v, w \in V(K')$, there exists a path consisting of white and black edges entirely within K'. It is easy to compute the g function of a CRG given the g function of its components:

Proposition 6 ([6]). *Let K be a CRG with components $K^{(1)}, \ldots, K^{(r)}$ and $p \in [0, 1]$. Then $(g_K(p))^{-1} = \sum_{i=1}^{r} \left(g_{K^{(i)}}(p)\right)^{-1}$.*

Note that by Proposition 6,

$$g_{K(r,s)}(p) = \left(\frac{r}{p} + \frac{s}{1-p}\right)^{-1}. \tag{4}$$

Let K be a CRG, $v \in V(K)$, and let $\mathbf{x}$ be an optimal solution to the quadratic program (1). The *weight* of v, denoted $\mathbf{x}(v)$, is the entry of the vector $\mathbf{x}$ that corresponds to v. We will often refer to $\mathbf{x}$ as the *optimal weight function* of K. We say that $w \in V(K)$ is a *gray neighbor* of $v \in V(K)$ if w is adjacent to v via a gray edge. White and black neighbors are defined analogously. The set of all gray neighbors of v is denoted by $N_G(v)$ and the number of vertices adjacent to v via gray edges is denoted by $\deg_G(v)$, i.e. $\deg_G(v) = |N_G(v)|$.

In contrast, the *gray degree* of v, denoted $d_G(v)$, is the sum of the weights of gray neighbors of v, i.e. $d_G(v) = \sum\{\mathbf{x}(w) : w \in N_G(v)\}$. Similarly, the *white degree* of v, denoted $d_W(v)$, is the sum of the weights of the white neighbors of v plus the weight of v if and only if it is a white vertex. The *black degree* of v, denoted $d_B(v)$, is the sum of the weights of the black neighbors of v plus the weight of v if and only if it is a black vertex. So, $d_G(v) + d_W(v) + d_B(v) = 1$ for all $v \in V(K)$.

The number of common gray neighbors of vertices v and w is denoted by $\deg_G(v, w)$. The *gray codegree* of vertices v and w, denoted $d_G(v, w)$, is the sum of the weights of the common gray neighbors of v and w. For a set of vertices $\{v_1, v_2, \ldots, v_\ell\}$, we say $v_1v_2\cdots v_\ell$ is a *gray path* if $v_iv_{i+1} \in \mathrm{EG}(K)$ for $i = 1, \ldots, \ell - 1$. Analogously, we say $v_1v_2\cdots v_\ell v_1$ is a *gray cycle* if $v_1v_\ell \in \mathrm{EG}(K)$ and $v_iv_{i+1} \in \mathrm{EG}(K)$ for $i = 1, \ldots, \ell - 1$.

Proposition 7 gives a structural classification of p-core CRGs and this is an essential tool that is the basis of the proof of Theorem 3. We note that, in particular, the proposition ensures that all edges between white and black vertices of a p-core CRG must be gray.

Proposition 7 (*Marchant–Thomason [5]*). *Let K be a p-core CRG.*

- *If $p = 1/2$, then all of the edges of K are gray.*
- *If $p < 1/2$, then $\mathrm{EB}(K) = \emptyset$ and there are no white edges incident to white vertices.*
- *If $p > 1/2$, then $\mathrm{EW}(K) = \emptyset$ and there are no black edges incident to black vertices.*

Proposition 8 gives a formula for $d_G(v)$ for all $v \in V(K)$ and Proposition 9 uses this to give a bound on the weight of each v.

Proposition 8 (*[6]*). *Let $p \in (0, 1)$ and K be a p-core CRG with optimal weight function $\mathbf{x}$.*

- *If $p \leq 1/2$, then $\mathbf{x}(v) = g_K(p)/(1-p)$ for all $v \in \mathrm{VW}(K)$, and*
$$d_G(v) = \frac{p - g_K(p)}{p} + \frac{1-2p}{p}\mathbf{x}(v), \text{ for all } v \in \mathrm{VB}(K).$$
- *If $p \geq 1/2$, then $\mathbf{x}(v) = g_K(p)/p$ for all $v \in \mathrm{VB}(K)$, and*
$$d_G(v) = \frac{1 - p - g_K(p)}{1-p} + \frac{2p-1}{1-p}\mathbf{x}(v), \text{ for all } v \in \mathrm{VW}(K).$$

Proposition 9 (*[6]*). *Let $p \in (0, 1)$ and K be a p-core CRG with optimal weight function $\mathbf{x}$.*

- *If $p \leq 1/2$, then $\mathbf{x}(v) \leq g_K(p)/(1-p)$ for all $v \in \mathrm{VB}(K)$.*
- *If $p \geq 1/2$, then $\mathbf{x}(v) \leq g_K(p)/p$ for all $v \in \mathrm{VW}(K)$.*

3. Proof of Theorem 2 : Computation of the $\gamma_{\mathcal{H}}$ function

In this section we compute the $\gamma_{\mathcal{H}}$ function, which gives an upper bound for the edit distance function for the tth power of the h cycle, denoted C_h^t. Recall that for any $t \geq 1$, $h \geq 2t+2$ and $r \in \{0, \ldots, t\}$, we denote $\ell_r = \left\lceil \frac{h}{t+r+1} \right\rceil$.

Proof (*Theorem 2*). First, we state the value of the chromatic number of C_h^t, denoted $\chi(C_h^t)$.

Proposition 10 (*Prowse–Woodall [10]*). *Let $t \geq 1$ and $h \geq \max\{t+1, 3\}$ be positive integers, and for $\rho \in \{0, 1, \ldots, t\}$, let $h = q(t+1) + \rho$ where $\rho \in \{0, \ldots, t\}$. Then, $\chi(C_h^t) = t + \lceil \rho/q \rceil + 1$. In particular, if $h \geq \max\{t(t+1), 3\}$, then*
$$\chi(C_h^t) = \begin{cases} t+2, & \text{if } (t+1) \nmid h; \\ t+1, & \text{if } (t+1) \mid h. \end{cases}$$

Let $h \geq \max\{t(t+1), 2t+2\}$ and $\chi = \chi(C_h^t)$. Denote the vertices of C_h^t by $\{1, \ldots, h\}$ such that distinct i and j are adjacent if and only if $|i-j| \leq t \pmod h$. For each $r \in \{0, 1, \ldots, t\}$, we first show that $(r, \ell_r - 1) \in \Gamma$, where $\Gamma = \Gamma(\mathrm{Forb}(C_h^t))$ is the clique spectrum of $\mathrm{Forb}(C_h^t)$. We then show that $(r, \ell_r) \notin \Gamma$. We will also show that if $\chi > t+1$ then $\{(t+1, 0), \ldots, (\chi - 1, 0)\} \subset \Gamma$ but that $(t+1, 1) \notin \Gamma$.

This will imply that Γ^*, the extreme points of Γ, satisfy
$$\Gamma^* \subseteq \{(r, \ell_r - 1) : r = 0, 1, \ldots, t\} \cup \{(\chi - 1, 0)\},$$
which, together with (4), will complete the proof of Theorem 2.

Case 1: $r \in \{0, 1, \ldots, t\}$.

First, we show that $(r, \ell_r - 1) \in \Gamma$. By way of contradiction, assume there is a partition of $V(C_h^t)$ into r independent sets and $\ell_r - 1$ cliques. Let $k = \ell_r - 1$, and let $C_1, \ldots, C_k$ be the cliques. We may assume that the vertices in each C_i are consecutive. This is because if vertices j_1 and j_2 are in the same clique, then by the nature of adjacency in the power of a cycle, every vertex between j_1 and j_2 is adjacent to every member of the clique, and hence can be added to the clique. Thus, $|C_i| \leq t+1$ for $i = 1, \ldots, k$.

For $i = 1, \ldots, k-1$, let B_i be the set of vertices between C_i and C_{i+1}, and let B_k be the set of vertices between C_k and C_1. The sets B_i might or might not be empty. If some $|B_i| \geq r+1$, then the first $r+1 \leq t+1$ vertices form a clique and so must be in different independent sets, which is not possible since there are only r independent sets. Therefore, $|B_i| \leq r$ for $i = 1, \ldots, k$.

Consequently, we need $k(t+r+1) \geq h$ in order to cover C_h^t with r independent sets and k cliques. Hence, $k \geq \ell_r$, a contradiction to our choice of k. Thus $(r, \ell_r - 1) \in \Gamma$ for $r = 0, \ldots, t$.

Next, we show that $(r, \ell_r) \notin \Gamma$. Again, let $k = \ell_r - 1$. For $i = 1, \ldots, k$, let S_i be the vertex set $\{(i-1)(t+r+1)+1, \ldots, i(t+r+1)\}$ and let $S_{k+1} = \{1, \ldots, h\} - \cup_{i=1}^{k} S_i$. For $i = 1, \ldots, k$, let C_i be the first $t+1$ vertices of S_i and let C_{k+1}

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be the first $\min\{t+1, |S_{k+1}|\}$ vertices of S_{k+1}. For $j = 1, \dots, r$, let A_j consist of the $(t+1+j)$th vertex of $S_1, \dots, S_k$ and the $(t+1+j)$th vertex of S_{k+1} if $|S_{k+1}| \geq t+1+j$.

The sets $(A_1, \dots, A_r, C_1, \dots, C_{k+1})$ form a partition of $V(C_h^t)$. Clearly each C_i, $i = 1, \dots, k$, is a clique of size $t+1$ and since there is a clique of size $t+1$ between pairs of vertices in each A_j, each A_j is an independent set. Thus $(r, \ell_r) \notin \Gamma$ for $r = 0, \dots, t$.

Case 2: $r \geq t+1$.

If $(t+1) \mid h$, then Proposition 10 gives that C_h^t can be partitioned into $t+1$ independent sets and so $(t+1, 0) \notin \Gamma$.

If $(t+1) \nmid h$, then Proposition 10 gives that $\chi \geq t+2$ and since C_h^t cannot be partitioned into fewer than χ independent sets, we have $(t+1, 0), \dots, (\chi-1, 0) \in \Gamma$. Since C_h^t can be partitioned into χ independent sets, $(\chi, 0) \notin \Gamma$. Finally, let $k = \lceil h/(t+1) \rceil - 1$. For $j = 1, \dots, t+1$, let $A_j = \{(i-1)(t+1)+j : i = 1, \dots, k\}$. Let $C_0 = \{k(t+1)+1, \dots, h\}$. The sets $(A_1, \dots, A_{t+1}, C_0)$ form a partition of $V(C_h^t)$. Clearly, C_0 is a clique of size at most $t+1$ and since there are at least t vertices between pairs of vertices in each A_j, each A_j is an independent set. Thus $(t+1, 1) \notin \Gamma$.

Summarizing, the extreme points of the clique spectrum, Γ, are a subset of $\{(r, \ell_r - 1) : r \in \{0, 1, \dots, t\}\}$ if $(t+1) \mid h$ and are a subset of $\{(r, \ell_r - 1) : r \in \{0, 1, \dots, t\}\} \cup \{\chi - 1, 0\}$ if $(t+1) \nmid h$.

Using Proposition 6, if $h = q(t+1) + \rho$ where $\rho \in \{0, \dots, t\}$, then

$$\gamma_{\mathcal{H}}(p) = \min_{r \in \{0,1,\dots,t\}} \left\{ \frac{p(1-p)}{r(1-p) + (\ell_r - 1)\, p} \right\}, \qquad \text{if } \rho = 0;$$

$$\gamma_{\mathcal{H}}(p) = \min_{r \in \{0,1,\dots,t\}} \left\{ \frac{p}{t + \lceil \rho/q \rceil}, \frac{p(1-p)}{r(1-p) + (\ell_r - 1)\, p} \right\}, \qquad \text{if } \rho \neq 0.$$

Restricting to the case $h \geq \max\{t(t+1), 4\}$, we have the result in the statement of the theorem.

4. Forbidden cycles

Recall that we may assume $h \geq 2t+2$. Before we can prove Theorem 3, we need to study the properties of the p-core CRGs into which C_h^t does not embed. According to Proposition 7, if we have a p-core CRG, then the part of the CRG induced on the black vertices has only gray or white edges (and only gray edges if $p \geq 1/2$). An important property of a p-core CRG, K such that $C_h^t \not\mapsto K$ is that the set of lengths of gray cycles on black vertices is restricted, as is shown in Lemma 13. Its proof needs the technical inequalities in Facts 11 and 12. For completeness, we give the proofs of these facts in Section 6.

Fact 11. *Let h, x, y be positive integers. Then*

(a) $\lfloor h/x \rfloor \geq y$ if and only if $\lfloor h/y \rfloor \geq x$.
(b) $\lceil h/x \rceil \leq y$ if and only if $\lceil h/y \rceil \leq x$.

Fact 12. *Let $t \geq 1$, $h \geq \max\{t(t-1), 2t+2\}$, and $r \in \{0, \dots, t-1\}$ be positive integers. Then $\left\lceil \frac{h}{t+r+1} \right\rceil \leq \left\lfloor \frac{h}{t} \right\rfloor$.*

Lemma 13 is a key lemma in proving our main result of Theorem 3.

Lemma 13. *Let $p \in (0, 1/2]$ and let $t \geq 1$ and $h \geq 2t+2$ be integers. Let $\tilde{K}$ be a p-core CRG with exactly r white vertices such that $C_h^t \not\mapsto \tilde{K}$. Let K be the sub-CRG of $\tilde{K}$ induced by the set of all black vertices of $\tilde{K}$. Then:*

(a) If $r \in \{0, \dots, t-1\}$ and $h \geq t^2 - t$, then K has no gray cycle that has length in $\left\{\left\lceil \frac{h}{t+r+1} \right\rceil, \dots, \left\lfloor \frac{h}{t} \right\rfloor\right\}$.
(b) If $r = t$, then $|V(K)| \leq \ell_t - 1$.
(c) If $r \geq t+1$, then $(t+1) \nmid h$ and $V(K) = \emptyset$.

Note: We interpret a gray cycle of length 2 to be a gray edge.

Proof (*Lemma* 13). By Proposition 7, K has only white and gray edges. Denote the vertices of C_h^t by $\{1, \dots, h\}$ such that distinct i and j are adjacent if and only if $|i - j| \leq t \pmod h$.

Case *(a)*.

Note that Fact 12 establishes that the range of forbidden cycle lengths is nonempty.

Let $r \in \{0, \dots, t-1\}$ and $h \geq t^2 - t$. By way of contradiction, for some $k \in \{\lceil h/(t+r+1) \rceil, \dots, \lfloor h/t \rfloor\}$, let K have a gray cycle on a set of black vertices $\{v_1, \dots, v_k\}$ such that $v_i v_{i+1}$ is a gray edge, where the indices are taken modulo k. Note that $k \geq 1$ because $t \geq 1$, $0 \leq r \leq t-1$, and $h \geq \max\{t^2 - t, 2t+2\}$.

We will partition $V(C_h^t)$ into at most r independent sets and exactly k cliques $C_1, \dots, C_k$ such that there is no edge between nonconsecutive cliques. First partition $V(C_h^t)$ into k sets of consecutive vertices $S_1, \dots, S_k$, with each set S_i of size either $\lceil h/k \rceil$ or $\lfloor h/k \rfloor$.

If $r = 0$, then simply let $C_i = S_i$ for $i = 1, \dots, k$. Because we need the sets C_i to be cliques, each must be of size at most $t+1$. Because we need nonconsecutive sets C_i and $C_{i'}$ to have no edge between them, each must be of size t. Using Fact 14, we see that these conditions are satisfied because $\lceil h/(t+1) \rceil \leq k \leq \lfloor h/t \rfloor$.

Fact 14. *A set of size h can be partitioned into sets of size t or $t+1$ as long as $h \geq t(t-1)$. Moreover, for any $k \in \{\lceil h/(t+1)\rceil, \dots, \lfloor h/t\rfloor\}$, such a partition exists with exactly k parts.*

The proof of Fact 14 is in Section 6.

Now we will assume $r \geq 1$ and choose $r' \in \{\lceil h/k\rceil - (t+1), \lfloor h/k\rfloor - t\}$ provided $0 \leq r' \leq r$. Note that this consists of only one such value for r' if $k \nmid h$. Thus, it is required that both (i) $0 \leq \lfloor h/k\rfloor - t$ and (ii) $\lceil h/k\rceil - (t+1) \leq r$. As long as $k \leq \lfloor h/t\rfloor$, (i) is satisfied and as long as $k \geq \lceil h/(t+r+1)\rceil$, (ii) is satisfied. Thus, such a choice for r' is possible.

If $r' = 0$, then again let $C_i = S_i$ for $i = 1, \dots, k$.

If $r' \geq 1$, then for $j \in \{1, \dots, r'\}$, let A_j consist of the jth vertex of each of $S_1, \dots, S_k$ and let $C_i = S_i - \cup_{j=1}^{r'} A_j$. Observe that for $r' \geq 1$, we have $|S_i| \geq t+1$ and so there are at least t vertices between each pair of vertices in every A_j. Therefore, A_j is an independent set for $j = 1, \dots, r'$. We have $|C_i| \leq t+1$ so C_i is a clique for $i = 1, \dots, k$. In addition, $|C_i| \geq t$ and so there are no edges between C_i and $C_{i'}$ unless $|i - i'| = 1 \pmod{k}$.

We now have a contradiction because this partition shows that C_h^t embeds in $\tilde{K}$. The map is as follows: map each A_j to a different white vertex and C_i to v_i for $i = 1, \dots, k$. The only edges between parts of the given partition of $V(C_h^t)$ are incident to an A_j or are between C_i and $C_{i'}$, where $|i - i'| = 1 \pmod{k}$. Each such pair has a gray edge and so the mapping witnesses $C_h^t \mapsto \tilde{K}$.

Case *(b).*

In this case, we use a similar partition to that of Case *(a)*. Let $k = \lceil h/(2t+1)\rceil - 1$ and $\rho = h - (k-1)(2t+1)$. Since $h \geq 2t+2$, we have $k \geq 1$. Partition $V(C_h^t)$ into $k+1$ consecutive parts, $S_1, \dots, S_{k+1}$, where $|S_1| = \cdots = |S_{k-1}| = 2t+1$, $|S_k| = \lceil \rho/2\rceil$ and $|S_{k+1}| = \lfloor \rho/2\rfloor$. Note that $t+1 \leq |S_{k+1}| \leq |S_k| \leq 2t+1$.

For $j = 1, \dots, t$, let A_j consist of the jth vertex in each S_i and let $C_i = S_i - \cup_{j=1}^{t} A_j$ for $i = 1, \dots, k+1$.

For every C_i, there is a set of t vertices before and after C_i that belong to $\cup_{j=1}^{t} A_j$. Hence, there is no edge between any distinct C_i and $C_{i'}$.

Therefore, $\tilde{K}$ has at most $k = \lceil h/(2t+1)\rceil - 1$ black vertices; otherwise, $A_1, \dots, A_t$ can be mapped arbitrarily to each of the t white vertices and $C_1, \dots, C_{k+1}$ can be mapped arbitrarily to $k+1$ different black vertices in $\tilde{K}$.

Case *(c).*

If $(t+1) \mid h$, then $\chi(C_h^t) = t+1$ and $\tilde{K}$ having at least $t+1$ white vertices means that $C_h^t \mapsto \tilde{K}$, a contradiction.

If $(t+1) \nmid h$, then partition $V(C_h^t)$ into $k = \lfloor h/(t+1)\rfloor + 1$ parts $S_1, \dots, S_k$ of consecutive vertices with $|S_1| = \cdots = |S_{k-1}| = t+1$ and $|S_k| = h - (k-1)(t+1) \leq t$. For $j = 1, \dots, t+1$, let A_j consist of the jth vertex in each of $S_1, \dots, S_{k-1}$. The graph induced by $V(C_h^t) - \cup_{j=1}^{t+1} A_j$ forms a clique of size at most t in S_k.

Therefore, $\tilde{K}$ cannot have a black vertex; otherwise, $A_1, \dots, A_{t+1}$ can be mapped arbitrarily to each of the $t+1$ white vertices and $V(C_h^t) - \cup_{j=1}^{t+1} A_j$ can be mapped to the black vertex.

5. Proof of Theorem 3 : $\mathrm{ed}_{\mathcal{H}} = \gamma_{\mathcal{H}}$

We will use Lemma 13 to prove Theorem 3. Recall that $h \geq 2t(t+1)+1 \geq t(t+1)$. By Proposition 10, this means $\chi(C_h^t) = t+1$ if $(t+1) \mid h$ and $\chi(C_h^t) = t+2$ if $(t+1) \nmid h$.

Proof (*Theorem* 3)**.** By definition $\mathrm{ed}_{\mathcal{H}}(p) \leq \gamma_{\mathcal{H}}(p)$ for all $p \in [0, 1]$, so we need to show equality.

Case 1: $p \in [1/2, 1]$.

First we will show that $\gamma_{\mathcal{H}}(p)$ is linear for $p \in [1/2, 1]$. Second, we show that $\mathrm{ed}_{\mathcal{H}}(1/2) = \gamma_{\mathcal{H}}(1/2)$ and $\mathrm{ed}_{\mathcal{H}}(1) = \gamma_{\mathcal{H}}(1)$. Finally, the continuity and concavity of the edit distance function establishes that $\mathrm{ed}_{\mathcal{H}}(p) = \gamma_{\mathcal{H}}(p)$ for all $p \in [1/2, 1]$.

Fact 15 establishes that $\gamma_{\mathcal{H}}(p) = \frac{1-p}{\ell_0 - 1}$ for $p \in [1/2, 1]$. Recall that $\ell_r = \left\lceil \frac{h}{t+r+1} \right\rceil$ for all $r \in \{0, 1, \dots, t\}$.

Fact 15. *Let h and t be positive integers and let $p \in [1/2, 1]$. If $h \geq (t+1)^2 + 1$, then*

$$\frac{1-p}{\ell_0 - 1} \leq \frac{p}{t+1}.$$

For $r \in \{1, \dots, t\}$ if $h \geq (t+1)(t+r)+1$, then

$$\frac{1-p}{\ell_0 - 1} \leq \frac{p(1-p)}{r(1-p) + (\ell_r - 1)p}.$$

The proof of Fact 15 is in Section 6. **Note:** The condition $h \geq 2t(t+1)+1$ suffices to achieve all of the conclusions in Fact 15.

A previous result establishes that, for $\mathcal{H} = \mathrm{Forb}(C_h^t)$, $\mathrm{ed}_{\mathcal{H}}(p) = \gamma_{\mathcal{H}}(p)$ for $p \in \{1/2, 1\}$.

Proposition 16 (*Balogh–Martin* [3]). *If $\mathcal{H}$ is a hereditary property, then $\mathrm{ed}_{\mathcal{H}}(1/2) = \gamma_{\mathcal{H}}(1/2)$. Moreover, if every complete graph is in $\mathcal{H}$, then $\mathrm{ed}_{\mathcal{H}}(1) = \gamma_{\mathcal{H}}(1) = 0$ and if every empty graph is in $\mathcal{H}$, then $\mathrm{ed}_{\mathcal{H}}(0) = \gamma_{\mathcal{H}}(0) = 0$.*

Finally, by Proposition 1, $\mathrm{ed}_{\mathcal{H}}(p)$ is continuous and concave down, so we may conclude that $\mathrm{ed}_{\mathcal{H}}(p) = \gamma_{\mathcal{H}}(p) = \frac{1-p}{\ell_0-1}$ for $p \in [1/2, 1]$. This concludes Case 1.

Case 2: $p \in [0, 1/2)$.

Proposition 16 gives $\mathrm{ed}_{\mathcal{H}}(0) = \gamma_{\mathcal{H}}(0) = 0$.

Now let $p \in (0, 1/2)$ and choose a p-core CRG $\tilde{K}$ such that $\mathrm{ed}_{\mathcal{H}}(p) = g_{\tilde{K}}(p)$ and $C_h^t \not\mapsto \tilde{K}$. (The existence of such a $\tilde{K}$ is guaranteed by Marchant and Thomason [5].) Recall that, by Proposition 7, each edge incident to white vertices is gray and the edges between black vertices are either white or gray.

By way of contradiction, assume that $g_{\tilde{K}}(p) < \gamma_{\mathcal{H}}(p)$. Suppose $\tilde{K}$ has r white vertices. Recall that for any $t \geq 1, h \geq 2t+2$. We consider several cases and show that we arrive at a contradiction in each case.

Case 2a: $p \in (0, 1/2)$ **and** $r \geq t+1$.

By Lemma 13(c), $(t+1) \not| \, h$ and $\tilde{K}$ has no black vertices. As long as $h \geq \max\{t(t+1), 3\}$, Proposition 10 gives that if $(t+1) \not| \, h$, then $\chi(C_h^t) = t+2$. Thus C_h^t embeds in $t+2$ white vertices and so this case reduces to $\tilde{K}$ having $r = t+1$ white vertices and no black vertices. Eq. (4) gives that $g_{\tilde{K}}(p) = p/(t+1)$, a contradiction to the assumption that $g_{\tilde{K}}(p) < \gamma_{\tilde{K}}(p)$. This concludes Case 2a.

Case 2b: $p \in (0, 1/2)$ and $r = t$.

Since $r = t$, Case *(b)* of Lemma 13 gives that $\tilde{K}$ has at most $\ell_t - 1$ black vertices. As a result, because K is the sub-CRG of $\tilde{K}$ induced by the black vertices, the smallest value $g_K(p)$ can achieve is when all edges are gray and the number of black vertices is as large as possible. From (4), $g_K(p) \geq (1-p)/(\ell_t - 1)$.

Using Proposition 6 we conclude that

$$g_{\tilde{K}}^{-1}(p) \leq tp^{-1} + \left(\frac{1-p}{\ell_t - 1}\right)^{-1}$$
$$g_{\tilde{K}}(p) \geq \frac{p(1-p)}{t(1-p) + (\ell_t - 1)p}$$

Hence, $\mathrm{ed}_{\mathcal{H}}(p) \geq \gamma_{\mathcal{H}}(p)$, a contradiction. This concludes Case 2b.

Case 2c: $p \in (0, 1/2)$ and $r \leq t-2$.

Since $\tilde{K}$ is a CRG with r white vertices, and K is the sub-CRG induced by the black vertices, Proposition 6 gives that $g_{\tilde{K}}(p)^{-1} = rp^{-1} + g_K^{-1}(p)$. Therefore,

$$\begin{aligned} g_K^{-1}(p) &= g_{\tilde{K}}^{-1}(p) - rp^{-1} \\ &> \left(\min_{r' \in \{0,1,\ldots,t\}} \left\{ \left(\frac{r'}{p} + \frac{\ell_{r'} - 1}{1-p}\right)^{-1} \right\}\right)^{-1} - \frac{r}{p} \\ g_K(p) &< \left(\max_{r' \in \{0,1,\ldots,t\}} \left\{ \frac{r' - r}{p} + \frac{\ell_{r'} - 1}{1-p} \right\}\right)^{-1} =: g_0(r, t; p). \end{aligned} \tag{5}$$

Given (5), Lemma 17 gives lower bounds on the gray degree of vertices and the codegree of pairs of vertices. Recall that $\deg_G(v)$ denotes the *number* of gray neighbors of $v \in V(K)$.

Lemma 17. *Let $t \geq 1$ be an integer, $r \in \{0, 1, \ldots, t-1\}$, and $p \in (0, 1/2)$. Let $p_t = \ell_t^{-1} = \left\lceil \frac{h}{2t+1} \right\rceil^{-1}$. Let K be a p-core CRG with all black vertices such that $g_K(p) < g_0(r, t; p)$. Then*

(a) for every $v \in V(K)$, we have $\deg_G(v) \geq \ell_{r+1}$, and
(b) for every $v, w \in V(K)$,

$$\deg_G(v, w) \geq \begin{cases} \ell_{r+2}, & \text{if } r \leq t-2; \\ 1, & \text{if } r = t-1 \text{ and } p \geq p_t. \end{cases}$$

The proof of Lemma 17 is in Section 6. **Note:** Since $h \geq 2t+2$, it is the case that $\ell_{r+1} \geq 2$ for $r \leq t-1$ and $\ell_{r+2} \geq 2$ for $r \leq t-2$.

Now we consider the derived graph F with vertex set $V(K)$ and edge set $\mathrm{EG}(K)$. From Lemma 17, we have a lower bound on both the minimum degree of F and the minimum codegree of F. From Lemma 13, the graph F has no cycle with lengths between $\ell_r = \left\lceil \frac{h}{t+r+1} \right\rceil$ and $L := \left\lfloor \frac{h}{t} \right\rfloor$.

Lemma 18 shows that F has no cycles with length larger than ℓ_r.

Lemma 18. *Let $t \geq 1$, $r \in \{0, 1, \ldots, t-1\}$ and $h \geq \max\{t(t-1), 2t+2\}$ be integers. Recall that $\ell_r = \lceil h/(t+r+1) \rceil$ and $L = \lfloor h/t \rfloor$.*

Let F be a graph with no cycle with length in $\{\ell_r, \ldots, L\}$ and every pair of vertices either has at least $\ell_{r+2} \geq 2$ common neighbors if $r \leq t-2$ or has at least 1 common neighbor if $r = t-1$.

Then F has no cycle of length more than $\ell_r - 1$.

The proof of Lemma 18 is in Section 6.

In the graph F, consider a maximum-length path. (In this paper, the length of a path is the number of vertices.) If any such a path can be made into a cycle, then Proposition 19 gives that F must be Hamiltonian. By Lemma 18, this means that $|V(K)| \leq \ell_r - 1$ and, as such, $g_K(p) \geq \frac{1-p}{\ell_r - 1}$, which is the g function for the CRG on $\ell_r - 1$ black vertices with all edges gray. This is a contradiction to our assumption in (5) by setting $r' = r$.

Proposition 19 is a common argument in proofs of Hamiltonian cycle results, including classical proofs of the theorems of Dirac [4] and Ore [9].

Proposition 19. *Let F be a connected graph. If some path of maximum length forms a cycle, then F is Hamiltonian.*

The proof of Proposition 19 is in Section 6.

So we may assume that every maximum-length path in F is not a cycle. Let $v_1 \cdots v_\ell$ be such a maximum length path. The common neighbors of v_1 and v_ℓ in F must be on this path, otherwise F has a longer path. From Lemma 17, it follows that v_1 and v_ℓ have at least $\ell_{r+2} \geq 2$ common neighbors on this path. However, Lemma 20 gives that there can only be one such neighbor, a contradiction.

Lemma 20. *Let $t \geq 1$, $r \in \{0, 1, \ldots, t-1\}$, and $h \geq 2t + 2$ be integers. Recall that $\ell_r = \lceil h/(t + r + 1) \rceil$. Let F be a graph with no cycle of length longer than $\ell_r - 1$, with every vertex having degree at least $\ell_{r+1} \geq 2$ and with every pair of vertices having at least one common neighbor. Furthermore, let F have the property that no maximum length path forms a cycle.*

Let $v_1 \cdots v_\ell$ be a path of maximum length in F. Then v_1 and v_ℓ have exactly one common neighbor v_c on this path. Furthermore, $N(v_1) \subseteq \{v_2, \ldots, v_c\}$ and $N(v_\ell) \subseteq \{v_c, \ldots, v_\ell\}$.

The proof of Lemma 20 is in Section 6. This concludes Case 2c.

Recall that $p_t = \ell_t^{-1} = \left\lceil \frac{h}{2t+1} \right\rceil^{-1}$.

Case 2d: $p \in [p_t, 1/2)$ and $r = t - 1$.

The CRG $\tilde{K}$ has $r = t - 1$ white vertices. By Proposition 6, $g_{\tilde{K}}^{-1}(p) = (t-1)p^{-1} + g_K^{-1}(p)$ and we arrive at a similar bound as in (5). That is,

$$\begin{aligned} g_K(p) < g_0(t-1, t; p) &= \left(\max_{r' \in \{0,1,\ldots,t\}} \left\{ \frac{r' - (t-1)}{p} + \frac{\ell_{r'} - 1}{1-p} \right\} \right)^{-1} \\ &\leq \frac{1-p}{\ell_{t-1} - 1}. \end{aligned}$$

Again, we consider the graph F with vertex set $V(K)$ and edge set $\mathrm{EG}(K)$. By Lemma 17, every vertex in F has degree at least ℓ_t and every pair of vertices has at least one common neighbor. By Lemma 18, F has no cycle of length more than $\ell_{t-1} - 1$. If there is a maximum-length path that is a cycle, then Proposition 19 gives that F is Hamiltonian, which means $|V(K)| \leq \ell_{t-1} - 1$. In that case $g_K(p) \geq \frac{1-p}{\ell_{t-1}-1}$, a contradiction.

So we may assume that every maximum-length path in F is not a cycle. Let $v_1 \ldots v_\ell$ be such a maximum-length path such that, in K, the sum $\mathbf{x}(v_1) + \mathbf{x}(v_\ell)$ is the largest among all such paths. Let v_c be the unique common neighbor of v_1 and v_ℓ as given by Lemma 20.

Let v_1 have d neighbors in F. Since v_1 cannot have neighbors outside of this path, the sum of the weights, in K, of the neighbors of v_1 satisfy $d_G(v_1) \leq \mathbf{x}(v_2) + \cdots + \mathbf{x}(x_c)$. Notice that if $v_i \in \{v_1, \ldots, v_{c-1}\}$ is a predecessor of a neighbor of v_1, then it is an endpoint of a path containing the same ℓ vertices, namely $v_i v_{i-1} \cdots v_1 v_{i+1} v_{i+2} \cdots v_c \cdots v_\ell$. Hence all d predecessors of gray neighbors of v_1 (including v_1 itself) have weight at most $\mathbf{x}(v_1)$. From Proposition 9, all other vertices have weight at most $\frac{g_K(p)}{1-p}$. Proposition 8 gives

$$\begin{aligned} \frac{p - g_K(p)}{p} + \frac{1-p}{p}\mathbf{x}(v_1) = \mathbf{x}(v_1) + d_G(v_1) &\leq \mathbf{x}(v_1) + \cdots + \mathbf{x}(v_c) \\ &\leq d\mathbf{x}(v_1) + (c-d)\frac{g_K(p)}{1-p}. \end{aligned}$$

Rearranging the terms, we obtain

$$g_K(p)\left(\frac{c-d}{1-p} + \frac{1}{p}\right) \geq 1 - \mathbf{x}(v_1)\left(d - \frac{1-p}{p}\right).$$

Since $p^{-1} \leq p_t^{-1} = \ell_t$ and $\ell_t < d+1$, we may, by Lemma 17, lower bound the right-hand side by again using $\mathbf{x}(v_1) \leq \frac{g_K(p)}{1-p}$,

$$\begin{aligned} g_K(p)\left(\frac{c-d}{1-p} + \frac{1}{p}\right) &\geq 1 - \frac{g_K(p)}{1-p}\left(d - \frac{1-p}{p}\right) \\ g_K(p)\left(\frac{c}{1-p}\right) &\geq 1. \end{aligned}$$

Lemma 18 bounds the size of the longest cycle, so $c \leq \ell_{t-1} - 1$. Thus, $g_K(p) \geq \frac{1-p}{c} \geq \frac{1-p}{\ell_{t-1}-1} \geq g_0(t-1, t; p)$, a contradiction. This concludes Case 2d.

Case 2e: $p \in (0, p_t)$ and $r = t - 1$.

Because in the case of $p \in (0, p_t)$, the theorem only addresses the case where $(t+1) \nmid h$, we assume this is the case. Fact 21 establishes that, in the range $0 < p < p_t$, $\gamma_{\mathcal{H}}(p)$ is linear.

Fact 21. *Let $t \geq 1$ and $h \geq 2t+2$ be positive integers. Let $p_t = \ell_t^{-1} = \left\lceil \frac{h}{2t+1} \right\rceil^{-1}$ and recall that*

$$\gamma_{\mathcal{H}}(p) = \min_{r \in \{0,\dots,t\}} \left\{ \frac{p}{t+1}, \frac{p(1-p)}{r(1-p) + (\ell_r - 1)p} \right\}.$$

Then $\gamma_{\mathcal{H}}(p) = p/(t+1)$ for $p \in [0, p_t]$.

By Proposition 16, for $\mathcal{H} = \mathrm{Forb}(C_h^t)$, $\mathrm{ed}_{\mathcal{H}}(0) \leq \gamma_{\mathcal{H}}(0) = 0$ and by Case 2d, $\mathrm{ed}_{\mathcal{H}}(p_t) \leq \gamma_{\mathcal{H}}(p_t)$. By Fact 21, the function $\gamma_{\mathcal{H}}(p)$ is linear over $p \in [0, p_t]$ for $h \geq 2t+2$. By Proposition 1, $\mathrm{ed}_{\mathcal{H}}(p)$ is continuous and concave down, so we may conclude that $\mathrm{ed}_{\mathcal{H}}(p) = \gamma_{\mathcal{H}}(p) = \frac{p}{t+1}$ for $p \in [0, p_t]$. This concludes Case 2e and completes the proof of Theorem 3.

6. Proofs of lemmas and facts

Proof (*Corollary 5*)**.** The case of $t = 1$ is covered by Corollary 4. So, assume $t \geq 2$.

Let $r \in \{1, \dots, t-1\}$.

If $p \geq \dfrac{r}{r + \ell_0 - \ell_r}$, then $\dfrac{p(1-p)}{r(1-p) + (\ell_r - 1)p} \geq \dfrac{1-p}{\ell_0 - 1}$.

If $p \leq \dfrac{t-r}{t - r + \ell_r - \ell_t}$, then $\dfrac{p(1-p)}{r(1-p) + (\ell_r - 1)p} \geq \dfrac{p(1-p)}{t(1-p) + (\ell_t - 1)p}$.

Therefore, it suffices to show

$$\frac{t-r}{t-r+\ell_r-\ell_t} \geq \frac{r}{r+\ell_0-\ell_r}$$

$$(\ell_0 - \ell_r)(t-r) \geq (\ell_r - \ell_t)r. \tag{6}$$

To that end,

$$\begin{aligned}
(\ell_0 - \ell_r)(t-r) - (\ell_r - \ell_t)r &= (t-r)\ell_0 + r\ell_t - t\ell_r \\
&> \frac{(t-r)h}{t+1} + \frac{rh}{2t+1} - \frac{th}{t+r+1} - t \\
&= \frac{rt(t-r)h}{(t+1)(t+r+1)(2t+1)} - t \\
&\geq \frac{t(t-1)h}{(t+1)(2t)(2t+1)} - t.
\end{aligned}$$

If $h \geq 4t^2 + 10t + 12 + \frac{12}{t-1}$, then (6) is satisfied and the corollary follows.

Proof (*Fact 11*)**.** We only need to prove one direction because x and y are arbitrary. In both cases, we will prove the forward implication.

(a) Let $\lfloor h/x \rfloor \geq y$ and $h = qx + \rho$, where $\rho \in \{0, \dots, x-1\}$. Then $y \leq \lfloor h/x \rfloor = q$, so $h \geq xy + r\rho$. Thus $\lfloor h/y \rfloor \geq x + \lfloor \rho/y \rfloor \geq x$.
(b) Let $\lceil h/x \rceil \leq y$ and $h = qx - \rho$, where $\rho \in \{0, \dots, x-1\}$. Then $y \geq \lceil h/x \rceil = q$, so $h \leq yx - \rho$. Thus $\lceil h/y \rceil \leq x - \lfloor \rho/y \rfloor \leq x$.

Proof (*Fact 12*)**.** Clearly, if $r \in \{0, \dots, t-1\}$, then $\left\lceil \frac{h}{t+r+1} \right\rceil \leq \left\lceil \frac{h}{t+1} \right\rceil$ so it suffices to prove this fact for $r = 0$. Let $h = qt + \rho$ with $\rho \in \{0, \dots, t-1\}$. Since $h \geq t(t-1)$, we have $q \geq t - 1 \geq \rho$. Then

$$\left\lceil \frac{h}{t+1} \right\rceil = q + \left\lceil \frac{\rho - q}{t+1} \right\rceil \leq q = \left\lfloor \frac{h}{t} \right\rfloor,$$

proving the fact.

Proof (*Fact 14*)**.** As long as $tk \leq h \leq (t+1)k$, the set $\{1, \dots, k\}$ can be partitioned into k sets, each of which has size t or $t+1$. Thus, we need

$$\left\lceil \frac{h}{t+1} \right\rceil \leq k \leq \left\lfloor \frac{h}{t} \right\rfloor.$$

To ensure that such a k exists, it suffices to find values of h for which $\lceil h/(t+1)\rceil \le \lfloor h/t\rfloor$. Setting $h = qt+\rho$ with $\rho \in \{0,1,\ldots,t-1\}$, this simplifies to

$$\left\lceil \frac{qt+\rho}{t+1}\right\rceil \le q \qquad \Longrightarrow \qquad 0 \le \left\lfloor \frac{q-\rho}{t+1}\right\rfloor .$$

If $q \ge t-1 \ge \rho$ (hence $h \ge t(t-1)$), then this inequality holds and there is a $k \in \{\lceil h/(t+1)\rceil, \ldots, \lfloor h/t\rfloor\}$ that admits the desired partition.

Proof (*Fact* 15)**.** If $h \ge (t+1)^2+1$, then $t+2 \le \lceil h/(t+1)\rceil = \ell_0$. Then,

$$t+1 \le \frac{1}{2}(\ell_0+t) \le p(\ell_0+t)$$

and so $\frac{1-p}{\ell_0-1} \le \frac{p}{t+1}$.

For $r \in \{1,\ldots,t\}$, let $h = q(t+1)+\rho$, where $\rho \in \{1,\ldots,t+1\}$. The bound $h \ge (t+1)(t+r)+1$ ensures $q \ge t+r$. Then,

$$\begin{aligned} r + \left\lceil \frac{h}{t+r+1}\right\rceil &= r + \left\lceil \frac{q(t+r+1)+\rho-qr}{t+r+1}\right\rceil \\ &= q + \left\lceil \frac{r(t+r+1)+\rho-qr}{t+r+1}\right\rceil \\ &\le q + \left\lceil \frac{r(t+r+1)+t+1-(t+r)r}{t+r+1}\right\rceil \\ &\le q+1 = \left\lceil \frac{h}{t+1}\right\rceil \end{aligned}$$

and so $\frac{1-p}{\ell_0-1} \le \frac{p(1-p)}{r(1-p)+(\ell_r-1)p}$. This proves the fact.

Proof (*Lemma* 17)**.**

(a) Let $v \in V(K)$. Using Proposition 8,

$$\begin{aligned} \deg_G(v) &\ge \left\lceil \frac{d_G(v)}{\max\{\mathbf{x}(u)\}}\right\rceil \ge \left\lceil \frac{\frac{p-g_K(p)}{p} + \frac{1-2p}{p}\mathbf{x}(v)}{\frac{g_K(p)}{1-p}}\right\rceil \\ &\ge \frac{(p-g_K(p))(1-p)}{p\,g_K(p)} = \frac{1-p}{g_K(p)} - \frac{1-p}{p} \\ &> \max_{r'\in\{0,1,\ldots,t\}} \left\{ \frac{(r'-r)(1-p)+(\ell_{r'}-1)\,p}{p} - \frac{1-p}{p}\right\} \\ &= \max_{r'\in\{0,1,\ldots,t\}} \left\{ \frac{(r'-r-1)(1-p)}{p} + \ell_{r'} - 1\right\} \\ &\ge \ell_{r+1} - 1. \end{aligned}$$

The last inequality is obtained by choosing $r' = r+1$.

(b) By inclusion–exclusion,

$$\begin{aligned} d_G(v,w) &\ge d_G(v) + d_G(w) - 1 \\ &= 2\,\frac{p-g_K(p)}{p} + \frac{1-2p}{p}(\mathbf{x}(v)+\mathbf{x}(w)) - 1 > \frac{p-2g_K(p)}{p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \deg_G(v,w) &\ge \left\lceil \frac{d_G(v,w)}{\max\{\mathbf{x}(u)\}}\right\rceil \ge \left\lceil \frac{\frac{p-2g_K(p)}{p}}{\frac{g_K(p)}{1-p}}\right\rceil = \left\lceil \frac{1-p}{g_K(p)} - \frac{2(1-p)}{p}\right\rceil \\ &> \max_{r'\in\{0,1,\ldots,t\}} \left\{ \frac{(r'-r)(1-p)+(\ell_{r'}-1)\,p}{p} - \frac{2(1-p)}{p}\right\} \\ &= \max_{r'\in\{0,1,\ldots,t\}} \left\{ \frac{(r'-r-2)(1-p)}{p} + \ell_{r'} - 1\right\}. \end{aligned}$$

If $r \le t-2$, then we choose $r' = r+2$. Then $\deg_G(v,w) > \ell_{r+2}-1$, and because $\deg_G(v,w)$ is an integer, $\deg_G(v,w) \ge \ell_{r+2}$.

If $r = t-1$, then we choose $r' = t$. Then $\deg_G(v, w) > -\frac{1-p}{p} + \ell_t - 1 = \ell_t - p^{-1} \geq 0$, since $p \geq p_t = \ell_t^{-1}$. Because $\deg_G(v, w)$ is an integer, $\deg_G(v, w) \geq 1$.

Proof (*Lemma* 18)**.** Recall $L = \lfloor h/t \rfloor$ and $\ell_r = \lceil h/(t+r+1) \rceil$ for $r \in \{0, 1, \ldots, t-1\}$. The condition $h \geq t(t-1)$ is sufficient to ensure that $\ell_r \leq L$ and so the range of excluded cycles is not empty. The condition $h \geq 2t+2$ is sufficient to ensure that $\ell_{r+2} \geq 2$ in the case where $r \leq t-2$. We leave verification of these to the reader.

We say that a *long cycle* is a cycle of length at least $L+1$. The objective of this proof is to show that there are no long cycles. Let $v_1 \cdots v_\ell$ be a smallest cycle in F among all those of length at least $L+1$.

Case 1: $0 \leq r \leq t-2$.

Observe that this case requires $t \geq 2$. Consider the path $v_1 \cdots v_{\ell_r - 1}$ on the cycle $v_1 \cdots v_\ell v_1$. There is no cycle of length ℓ_r and so the common neighbors of v_1 and v_{ℓ_r-1} are all in $\{v_2, \ldots, v_{\ell_r-2}\}$. Note that Lemma 17 establishes that v_1 and v_{ℓ_r-1} have at least $\ell_{r+2} \geq 2$ common neighbors.

Since all common neighbors of v_1 and v_{ℓ_r-1} are in $\{v_2, \ldots, v_{\ell_r-2}\}$, we have $\ell_r - 3 \geq \ell_{r+2}$. Hence,

$$\frac{h}{t+r+3} \leq \left\lceil \frac{h}{t+r+3} \right\rceil \leq \left\lceil \frac{h}{t+r+1} \right\rceil - 3 < \frac{h}{t+r+1} - 2$$

and so $h > (t+r+1)(t+r+3)$.

This gives that the number of common neighbors of v_1 and v_{ℓ_r-1} is at least $\ell_{r+2} = \left\lceil \frac{h}{t+r+3} \right\rceil \geq t+r+2 \geq 4$.

Therefore, v_1 and v_{ℓ_r-1} have at least two common neighbors in $\{v_3, \ldots, v_{\ell_r-3}\}$. Let $i > 2$ and $j < \ell_r - 2$ be, respectively, the smallest and largest indices of vertices in $\{v_3, \ldots, v_{\ell_r-3}\}$ that are common neighbors of v_1 and v_{ℓ_r-1}. That is, $3 \leq i \leq j \leq \ell_r - 3$. The cycle $v_1 v_i v_{i+1} \cdots v_{\ell-1} v_\ell v_1$ has length $\ell - i + 2$. The cycle $v_1 v_2 \cdots v_{j-1} v_j v_{\ell_r-1} v_{\ell_r} \cdots v_{\ell-1} v_\ell$ has length $\ell + j - \ell_r + 2$.

Since these two cycles have length strictly less than ℓ, they cannot be long cycles. Hence, their length is at most $\ell_r - 1$, giving

$$\begin{aligned} \ell - i + 2 &\leq \ell_r - 1 \\ \ell + j - \ell_r + 2 &\leq \ell_r - 1. \end{aligned}$$

We can add these inequalities and rearrange the terms, $3\ell_r - 2\ell - 5 \geq j - i + 1$. Because the cycle is a long cycle, $\ell \geq L+1$. Because there are at least $\ell_{r+2} - 2$ common neighbors of v_1 and v_{ℓ_r-1} in $\{v_3, \ldots, v_{\ell_r-3}\}$, $j - i + 1 \geq \ell_{r+2} - 2$. Consequently,

$$3\ell_r - 2L - 7 \geq 3\ell_r - 2\ell - 5 \geq j - i + 1 \geq \ell_{r+2} - 2. \tag{7}$$

To verify there are no long cycles, we must show that (7) produces a contradiction. Since $0 \leq r \leq t-2$,

$$\begin{aligned} 3\ell_r - 2L - 7 &= 3\left\lceil \frac{h}{t+r+1} \right\rceil - 2\left\lfloor \frac{h}{t} \right\rfloor - 7 \\ &< 3\left(\frac{h}{t+r+1} + 1\right) - 2\left(\frac{h}{t} - 1\right) - 7 \\ &= \frac{h}{t+r+3} - 2 - \frac{2h(rt + r^2 + 4r + 3)}{t(t+r+1)(t+r+3)} \\ &< \left\lceil \frac{h}{t+r+3} \right\rceil - 2 = \ell_{r+2} - 2, \end{aligned}$$

a contradiction for all $t \geq 2$, $r \leq t-2$, and $h \geq 2t + 2r + 3$. Therefore, for $0 \leq r \leq t-2$, F has no cycle of length longer than $\ell_r - 1$.

Case 2: $r = t-1$.

Consider the path $v_1 \cdots v_{\ell-2}$ on the cycle $v_1 \cdots v_\ell v_1$. There is no cycle of length $\ell - 1 \geq L > \ell_{t-1}$ and so the common neighbors of v_1 and $v_{\ell-2}$ are all in $\{v_2, \ldots, v_{\ell-3}\}$. Note that Lemma 17 establishes that v_1 and $v_{\ell-2}$ have at least 1 common neighbor.

Let $i \in \{2, \ldots, \ell-3\}$ be an index such that v_i is a common neighbor of v_1 and $v_{\ell-2}$. If $i = 2$, then the cycle $v_2 v_3 \cdots v_{\ell-3} v_{\ell-2} v_2$ has length $\ell - 3$. If $i = \ell - 3$, then the cycle $v_1 v_2 \cdots v_{\ell-4} v_{\ell-3} v_1$ has length $\ell - 3$. In either case, this cycle must be of length less than ℓ_{t-1} but this is a contradiction because

$$\ell - 3 \geq L - 2 = \left\lfloor \frac{h}{t} \right\rfloor - 2 = \left\lceil \frac{h}{2t} \right\rceil + \left(\left\lfloor \frac{h}{t} \right\rfloor - \left\lceil \frac{h}{2t} \right\rceil - 2\right) \geq \ell_{t-1}.$$

Verifying that $\lfloor h/t \rfloor - \lceil h/(2t) \rceil - 2 \geq 0$ for $h \geq 5t$ can be found via setting $h = (2t)q - \rho$ with $0 \leq \rho \leq 2t-1$ and studying the cases $q \geq 4$ and $q = 3$. Since $5t \leq 2t(t+1) + 1$ for all $t \geq 1$, the conditions on h verify the contradiction.

If $3 \leq i \leq \ell - 4$, then the cycle $v_1 v_2 \cdots v_i v_1$ has length i and the cycle $v_1 v_i v_{i+1} \cdots v_{\ell-2} v_{\ell-1} v_\ell v_1$ has length $\ell - i + 2$. Since these two cycles have length strictly less than ℓ, they cannot be long cycles. Hence, their length is at most $\ell_{t-1} - 1$, giving

$$i \leq \ell_{t-1} - 1$$

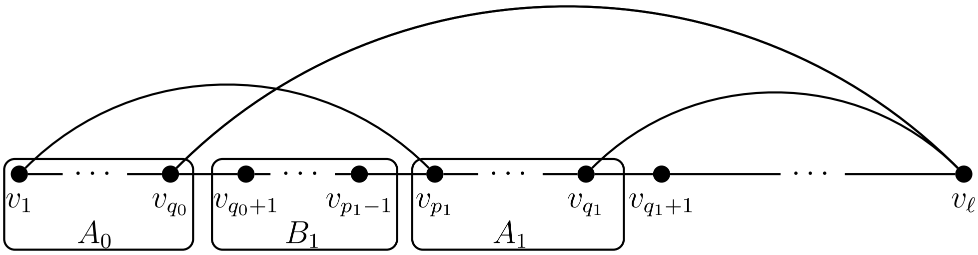

Fig. 1. Partition of vertices of the path. Sets A_i are iteratively constructed so that they contain consecutive vertices of this path starting with a neighbor of v_1 and ending with the last neighbor of v_ℓ so that no neighbor of v_1 appears after neighbors of v_ℓ in each set. Sets B_i contain consecutive vertices between sets A_{i-1} and A_i, if there are any. The first vertex is placed in A_0 and the last vertex v_ℓ in A_s.

$$\ell - i + 2 \leq \ell_{t-1} - 1.$$

We can add these inequalities and rearrange the terms, $2\ell_{t-1} - \ell - 4 \geq 0$.

This is, however, a contradiction because

$$2\ell_{t-1} - \ell - 4 \leq 2\left\lceil \frac{h}{2t} \right\rceil - \left\lfloor \frac{h}{t} \right\rfloor - 3 < 2\left(\frac{h}{2t} + 1\right) - \left(\frac{h}{t} - 1\right) - 3 = 0.$$

Proof (*Proposition* 19)**.** Let $v_1 \cdots v_\ell$ be a longest path in F such that $v_1 v_\ell \in E(F)$. If F is not Hamiltonian, there exists a $w \in V(F) - \{v_1, \ldots, v_\ell\}$. Because F is connected, there exists $i \in \{1, \ldots, \ell\}$ and $w' \in V(F) - \{v_1, \ldots, v_\ell\}$ such that v_i is adjacent to w'. Then there is a longer path: $v_{i+1} \cdots v_\ell v_1 \cdots v_i w'$, a contradiction.

Proof (*Lemma* 20)**.** Because $v_1 \cdots v_\ell$ is a longest path in F, neither v_1 nor v_ℓ can have neighbors off this path, as that would yield a longer path. Thus $N(v_1) \cup N(v_\ell) \subseteq \{v_1, \ldots, v_\ell\}$ in F.

Case 1: $\ell \leq \ell_r$.

If v_i is adjacent to v_1, then v_{i-1} cannot be adjacent to v_ℓ. Thus, the predecessors of $N(v_1)$ and the neighbors of v_ℓ are disjoint subsets in $\{v_1, \ldots, v_{\ell-1}\}$. Since both v_1 and v_ℓ have degree at least ℓ_{r+1},

$$2\ell_{r+1} \leq \ell - 1 \leq \ell_r - 1.$$

However,

$$\begin{aligned} \ell_r - 2\ell_{r+1} - 1 &= \left\lceil \frac{h}{t+r+1} \right\rceil - 2\left\lceil \frac{h}{t+r+2} \right\rceil - 1 \\ &< \frac{h}{t+r+1} - \frac{2h}{t+r+2} = -\frac{h(t+r)}{(t+r+1)(t+r+2)} < 0. \end{aligned} \tag{8}$$

Case 2: $\ell \geq \ell_r + 1$.

Partition the vertices of this path into $2s + 1$ consecutive sets $A_0, B_1, A_1, \ldots, A_s, B_s$ with $s \geq 0$, constructed so that, in each set A_i, neighbors of v_1 appear before neighbors of v_ℓ as follows:

We let neighbors of v_1 be denoted with v_{p_i} and neighbors of v_ℓ be denoted with v_{q_i} in this construction. Let A_0 contain v_1 and add consecutive vertices of this path until we arrive at a neighbor of v_ℓ. From this point forward we do not allow another neighbor of v_1 to be in A_0, i.e. we continue adding consecutive vertices until we reach the last neighbor v_{q_0} of v_ℓ before another neighbor v_{p_1} of v_1. Then $A_0 = \{v_1, \ldots, v_{q_0}\}$, and we define $B_1 = \{v_{q_0+1}, \ldots, v_{p_1-1}\}$. Note that this definition does not preclude B_1 being an empty set. Continuing with this algorithm, we define sets $A_1 = \{v_{p_1}, \ldots, v_{q_1}\}$ and $B_2 = \{v_{q_1+1}, \ldots, v_{p_2-1}\}$, where v_{p_1} is a neighbor of v_1 on this path, v_{q_1} is the last neighbor of v_ℓ in A_1 before another neighbor v_{p_2} of v_1 as shown in Fig. 1. We continue in this way and define sets $A_i = \{v_{p_i}, \ldots, v_{q_i}\}$ and $B_i = \{v_{q_{i-1}+1}, \ldots, v_{p_i-1}\}$ for $i \in \{1, \ldots, s\}$, adding the last vertex v_ℓ into the set A_s.

Now we analyze this partition:

- We call the sets $B_i, i \in \{1, \ldots, s\}$, *gaps* as they do not contain any neighbors of either v_1 or v_ℓ, but only contain vertices that succeed a given neighbor of v_ℓ and precede a given neighbor of v_1. According to the definition, gaps may be empty, but we will see below that this is not possible in this case.
- Each set $A_i, i \in \{0, \ldots, s\}$, contains at most one common neighbor of v_1 and v_ℓ.
- By construction, neighbors of v_1 (other than a common neighbor, if exists) precede neighbors of v_ℓ in each A_i, $i \in \{0, \ldots, s\}$.

It will suffice to show that $s = 0$. This will imply that no neighbor of v_1 follows the first neighbor of v_ℓ on this path, which further implies that $N(v_1)$ entirely precedes $N(v_\ell)$, except possibly for a single common vertex. Since v_1 and v_ℓ have at least one common neighbor, the lemma will follow.

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Notice that $v_1 \cdots v_{q_0} v_\ell v_{\ell-1} \cdots v_{p_1} v_1$ is a cycle as seen in Fig. 1. In fact, for any $i \geq 1$, removing the gap B_i from vertices $\{v_1, \dots, v_\ell\}$ forms a cycle, so by assumption, $\ell - |B_i| \leq \ell_r - 1$ and none of the gaps can be empty. Therefore, $\sum_{i=1}^{s} |B_i| \geq s(\ell - \ell_r + 1)$.

On the other hand, by the degree assumption and since each set A_i contains at most one common neighbor of v_1 and v_ℓ, we obtain $2\ell_{r+1} \leq |N(v_1)| + |N(v_\ell)| \leq \left(\sum_{i=0}^{s} |A_i|\right) + (s+1) - 2$. Combining these two inequalities we have

$$\ell = \sum_{i=0}^{s} |A_i| + \sum_{i=1}^{s} |B_i| \geq 2\ell_{r+1} - (s+1) + 2 + s(\ell - \ell_r + 1)$$
$$= s(\ell - \ell_r) + 2\ell_{r+1} + 1.$$

If $s \geq 1$, then we have $\ell \geq \ell - \ell_r + 2\ell_{r+1} + 1$ which simplifies to $\ell_r - 2\ell_{r+1} - 1 \geq 0$, which is contradicted by (8). Therefore $s = 0$ and the lemma follows.

Proof (*Fact* 21)**.** We need to show that $\gamma_{\mathcal{H}}(p_t) = p_t/(t+1)$. Since

$$\gamma_{\mathcal{H}}(p_t) = p_t \cdot \min_{r \in \{0,\dots,t\}} \left\{ \frac{1}{t+1}, \frac{1-p_t}{r(1-p_t) + (\ell_r - 1)\, p_t} \right\},$$

we need to show that $\frac{\ell_r - 1}{\ell_t - 1} \leq t - r + 1$ for all $r \in \{0, \dots, t-1\}$.

To do this, let $h = q(2t+1) - \rho$ where $\rho \in \{0, \dots, 2t\}$ and $q \geq 2$ (because $h \geq 2t+2$). Then,

$$\frac{\ell_r - 1}{\ell_t - 1} = \frac{1}{q-1}\left(q - 1 + \left\lceil \frac{q(t-r) - \rho}{t+r+1} \right\rceil\right)$$
$$\leq \frac{1}{q-1}\left(q - 1 + \frac{q(t-r) + t + r}{t+r+1}\right)$$
$$= t - r + 1 + \frac{t^2 - r^2 + 2t - q(t^2 - r^2)}{(q-1)(t+r+1)},$$

which is at most $t - r + 1$ if $q \geq 3$ or if $r \leq t-2$ and $q = 2$. In the case where $r = t-1$ and $q = 2$, then $\frac{\ell_r - 1}{\ell_t - 1} = 1 + \left\lceil \frac{2-r}{2t} \right\rceil \leq 2 = t - r + 1$.

7. Conclusion and open questions

We have obtained the edit distance function over all of its domain for C_h^t when $t+1$ does not divide h and $h \geq 2t(t+1)+1$. When $t+1$ divides h and $h \geq 2t(t+1)+1$, we have obtained the function for $p \in [p_t, 1]$, where $p_t = \left\lceil \frac{h}{2t+1} \right\rceil^{-1}$. The function, however, is not known when $h \leq 2t(t+1)$ or when $t+1$ divides h and $p \in [0, p_t)$.

Small h: In reducing the lower bound required of h, we note that in the proof of Theorem 3, we required $h \geq 2t(t+1)+1$ in Fact 15. This ensured that the $\gamma_{\mathcal{H}}$ function for $p \in [1/2, 1]$ was linear and by the concavity and continuity of the edit distance function (see Proposition 1), this ensures that $\mathrm{ed}_{\mathcal{H}}(p) = \gamma_{\mathcal{H}}(p)$ in that interval. So, more careful analysis of the case $p \geq 1/2$ may enable one to reduce the lower bound on h, but these arguments are very different from the case where $p < 1/2$.

Proposition 10 required a bound of $h \geq t(t+1)$ in order to give a simple expression for $\chi(C_h^t)$. A more careful analysis of the chromatic number and of the $\gamma_{\mathrm{Forb}(C_h^t)}(p)$ function may be possible. The bound $h \geq \max\{t(t-1), 2t+2\}$ was required in Facts 12 and 14 and Lemmas 13 and 18 in order to have a forbidden cycles condition on the CRG. Fact 14, in particular, may be able to be avoided in some cases, because it is a special case of the Frobenius number. Though it seems possible to reduce the lower bound of $2t(t+1)+1$ on h, it seems unlikely that there is a general argument that does not require a quadratic lower bound on h in terms of t.

Small p: As to the case of $p < p_t$, $(t+1) \mid h$ and h sufficiently large, we showed in Section 5 that if $K \in \mathcal{K}(\mathrm{Forb}(C_h^t))$ is a p-core CRG with $p < 1/2$ which has $r \neq t-1$ white vertices, then $g_K(p) = \gamma_{\mathrm{Forb}(C_h^t)}(p)$. Therefore, to solve the problem for the remaining case when $t+1$ divides h, and p is small, one only needs to consider CRGs with exactly $t-1$ white vertices and with no gray cycle with lengths in $\{\lceil h/(2t) \rceil, \dots, \lfloor h/t \rfloor\}$ in the sub-CRG induced by the black vertices.

Observe that, in the case where $(t+1) \nmid h$, the fact that $\chi(C_h^t) = t+2$ means that $\gamma_{\mathrm{Forb}(C_h^t)}(p)$ includes the linear function $p/(\chi - 1) = p/(t+1)$. Indeed, we need only to prove that $\mathrm{ed}_{\mathrm{Forb}(C_h^t)}(p) = p/(t+1)$ for $p \in \{0, p_t\}$ and then the result for $p \in [0, p_t]$ follows from the continuity and concavity of the edit distance function.

In the case where $(t+1) \mid h$, however, $\chi(C_h^t) = t+1$ would give the function $p/(\chi - 1) = p/t$. Unfortunately, $\frac{p}{t} \geq \frac{p(1-p)}{t(1-p)+(\ell_t - 1)p}$ for all $p \in [0, 1]$ and so the function p/t is useless. Indeed, when h sufficiently large,

$$\gamma_{\mathrm{Forb}(C_h^t)}(p) = \frac{p(1-p)}{t(1-p) + (\ell_t - 1)p}, \qquad \text{for } p \in [0, p_t].$$

However, it may be the case that $\mathrm{ed}_{\mathcal{H}}(p) < \gamma_{\mathcal{H}}(p)$ for small p. That is the case for $\mathcal{H} = \mathrm{Forb}(K_{3,3})$ [5] and for $\mathcal{H} = \mathrm{Forb}(K_{2,t})$ ($t \geq 9$) [8]. The fact that the edit distance function is less than the gamma function is witnessed by an infinite sequence of CRGs derived from constructions that produce lower bounds for a certain bipartite Turán problem.

In particular, future work may focus on bipartite C_h, i.e., when $2 \mid h$. Any p-core CRG in $K \in \mathcal{K}(\mathrm{Forb}(C_h))$ on black vertices has no gray cycle with length in $\{\lceil h/2 \rceil, \ldots, h\}$. There may be such a CRG that demonstrates $\mathrm{ed}_{\mathrm{Forb}(C_h)}(p) < \gamma_{\mathrm{Forb}(C_h)}(p)$ for small p.

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