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Another construction of edge-regular graphs with regular cliques

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ABSTRACT

We exhibit a new construction of edge-regular graphs with regular cliques that are not strongly regular. The infinite family of graphs resulting from this construction includes an edge-regular graph with parameters $(24, 8, 2)$. We also show that edge-regular graphs with 1-regular cliques that are not strongly regular must have at least 24 vertices.

1. Introduction and definitions

We begin with various definitions of regularity. A v-vertex graph Γ is called ***k*-regular** if there exists a k such that each vertex of Γ has degree k. A v-vertex, k-regular non-empty graph is called **edge-regular** with parameters (v, k, λ) if every pair of adjacent vertices $x \sim y$ have λ common neighbours. A (v, k, λ)-edge-regular graph is called **strongly regular** with parameters (v, k, λ, μ) if every pair of distinct nonadjacent vertices $x \not\sim y$ have μ common neighbours. A clique $\mathcal{C}$ is called **regular** (or e-regular) if every vertex not in $\mathcal{C}$ is adjacent to a constant number $e > 0$ of vertices in $\mathcal{C}$.

Recently, the authors [7] provided an infinite family of non-strongly-regular, edge-regular graphs having regular cliques, thus answering a question of Neumaier [8, Page 248]. The smallest graph in this family has parameters $(28, 9, 2)$. In this note we offer a new construction that gives rise to a non-strongly-regular, edge-regular graph $\mathcal{G}$ with parameters $(24, 8, 2)$ having a 1-regular clique. In fact, $\mathcal{G}$ is isomorphic to one of the four examples found by Goryainov and Shalaginov [6]. Recently, however, Evans et al. [4] discovered an edge-regular, but not strongly regular graph on 16 vertices that has 2-regular cliques with order 4, and proved that, up to isomorphism, this is the unique edge-regular, but not strongly regular graph on at most 16 vertices having a regular clique.

A graph Γ of diameter d is called ***a*-antipodal** if the relation of being at distance d or distance 0 is an equivalence relation on the vertices of Γ with equivalence classes of size a. A set of cliques of a graph Γ that partition the vertex set of Γ is called a **spread** in Γ. Distance regular graphs and generalised quadrangles are edge-regular graphs satisfying further regularity conditions, for their definitions, see Brouwer and Haemers [3] (see below for the definition of a distance regular graph). Brouwer [1] gave a construction for antipodal distance regular graphs from generalised quadrangles having a spread of regular cliques. Inspired by Brouwer, our construction is a generalisation of the other direction, i.e., we construct graphs from antipodal distance-regular graphs of diameter three.

In Section 2 we present our construction and in Section 3 we show that non-strongly-regular, edge-regular graphs having 1-regular cliques must have at least 24 vertices.

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2. Construction

Let Γ be a graph with diameter d. We call Γ **distance-regular** if, for any two vertices $x, y \in V(\Gamma)$, the number of vertices at distance i from x and distance j from y depends only on i, j, and the distance from x to y. It is clear that distance regular graphs are edge-regular.

Let Γ be a graph of diameter d. For a given t, make t copies $\Gamma^{(1)}, \dots, \Gamma^{(t)}$ of Γ. For each vertex $x \in V(\Gamma)$, denote by $\Gamma_i(x)$ the set of vertices at distance i from x and denote by x_j the corresponding copy of x in the graph $\Gamma^{(j)}$. Define the sets

$$E_1(\Gamma) = \{\{x_i, y_j\} \; : \; x \in V(\Gamma),\; y \in \Gamma_d(x), \text{ and } i, j \in \{1, \dots, t\}\}$$
$$E_2(\Gamma) = \{\{x_i, x_j\} \; : \; x \in V(\Gamma) \text{ and } i \neq j\}.$$

Let $\hat{\Gamma}$ denote the disjoint union of the graphs $\Gamma^{(1)}, \dots, \Gamma^{(t)}$. Then define $F_t(\Gamma)$ to be the graph with vertex set $V(F_t(\Gamma)) = V(\hat{\Gamma})$ and edge set

$$E(F_t(\Gamma)) = E(\hat{\Gamma}) \cup E_1(\Gamma) \cup E_2(\Gamma).$$

Theorem 2.1. *Let Γ be an a-antipodal distance-regular graph of diameter 3 with edge-regular parameters (v, k, λ) such that a is a proper divisor of $\lambda + 2$. Then*

1. $F_{\frac{\lambda+2}{a}}(\Gamma)$ *has a spread of 1-regular cliques each of size $\lambda + 2$;*
2. $F_{\frac{\lambda+2}{a}}(\Gamma)$ *is $(v(\lambda + 2)/a, k + \lambda + 1, \lambda)$-edge-regular;*
3. $F_{\frac{\lambda+2}{a}}(\Gamma)$ *is not strongly regular.*

Proof. Let $t = (\lambda + 2)/a$.

1. Since Γ is a-antipodal, its vertex set can be partitioned into v/a a-subsets of $V(\Gamma)$ such that each subset contains a vertex and all its antipodes. For each part P in the partition, take a vertex $x \in P$ and define the set
$$\mathcal{C}_x = \{x_i \; : \; i \in \{1, \dots, t\}\} \cup \bigcup_{y \in \Gamma_3(x)} \{y_i \; : \; i \in \{1, \dots, t\}\} \subset V(F(\Gamma)).$$
It is clear that each $\mathcal{C}_x$ is a clique in $F(\Gamma)$ of size $\lambda + 2$ and that these cliques partition the vertex set of $F(\Gamma)$. To see that these cliques are 1-regular, consider a vertex z not in the clique $\mathcal{C}_x$. Let $\Gamma^{(i)}$ be the copy of Γ containing z. Since Γ is distance-regular and has diameter 3, the vertex z must be adjacent to precisely one vertex in the set $\{x_i\} \cup \{y_i \; : \; y \in \Gamma_3(x)\}$. Therefore $\mathcal{C}_x$ is 1-regular.
2. It is clear that $F(\Gamma)$ has $vt = v(\lambda + 2)/a$ vertices. Let x be a vertex of $F(\Gamma)$ inside the ith copy $\Gamma^{(i)}$ of Γ. Then x is adjacent to $k + a - 1$ vertices inside $\Gamma^{(i)}$ and x is adjacent to the vertices x_j and y_j for all $j \neq i$ and $y \in \Gamma_3(x)$. Hence x has valency $k + a - 1 + a(t - 1) = k + \lambda + 1$. Now suppose that y is adjacent to x. If y is in the clique $\mathcal{C}_x$ then, since $\mathcal{C}_x$ is 1-regular, y and x have λ common neighbours. Otherwise y must be a vertex of $\Gamma^{(i)}$, where the number of common neighbours of y and x are equal to the number of common neighbours of adjacent vertices in Γ, which is λ.
3. Let x be a vertex of $F_t(\Gamma)$ inside the ith copy $\Gamma^{(i)}$ of Γ. Form the set $\mu(\Gamma) = \{\nu_{y,z} \; : \; y, z \in V(\Gamma) \text{ and } y \not\sim z\}$, where $\nu_{y,z}$ denotes the number of common neighbours of y and z. Since Γ is not strongly regular, the set $\mu(\Gamma)$ must have at least 2 elements. Furthermore, since Γ has diameter 3, we see that $0 \in \mu(\Gamma)$. Let $\eta \in \mu(\Gamma)$ with $\eta \neq 0$. Consider the vertices y and z, where y is in $\Gamma^{(i)}$ such that $y \not\sim x$ and $\nu_{x,y} = \eta$ and z is in $\Gamma^{(j)}$ with $j \neq i$ and $z \not\sim x$. The number of common neighbours of x and y is $\eta + 2$ and the number of common neighbours of x and z is 2. Hence $F_t(\Gamma)$ is not strongly regular. □

A **Taylor graph** is a 2-antipodal distance-regular graph of diameter 3 (for a proper definition, see Brouwer, Cohen, and Neumaier [2, Page 13]).

Example 1. Let Γ be a Taylor graph with edge-regular parameters (v, k, λ). It is known [2, Theorem 1.5.3] that λ is even. By Theorem 2.1, the graph $F_{\lambda/2+1}(\Gamma)$ is a non-strongly-regular $(v(\lambda+2)/2, k+\lambda+1, \lambda)$-edge-regular graph having a 1-regular clique. The smallest example of this family is the icosahedral graph $\mathcal{P}$, which has parameters $(12, 5, 2)$. The graph $F_2(\mathcal{P})$ is a non-strongly-regular $(24, 8, 2)$-edge-regular graph having a 1-regular clique, furthermore, $F_2(\mathcal{P})$ is isomorphic to one of the four examples of Goryainov and Shalaginov [6].

We can also use other constructions of antipodal distance-regular graphs of diameter 3 due to Brouwer, Hensel, and Mathon (see Godsil and Hensel [5] or Brouwer, Cohen, and Neumaier [2, Page 385]). These constructions produce a-antipodal edge-regular graphs satisfying $\lambda + 2 \equiv 0 \pmod{a}$ with $a \geqslant 3$.

3. At least 24 vertices for 1-regular cliques

In the remainder of this note, we prove the following result.

Theorem 3.1. *Let Γ be an edge-regular graph with a* 1*-regular clique that is not strongly regular. Then Γ has at least* 24 *vertices.*

First we establish a lower bound on the vertex degree. For a graph Γ and a vertex $x \in V(\Gamma)$, let $\Gamma(x)$ denote the set of neighbours of x. The $q \times q$ **grid** (also known as the *square lattice graph*) is defined to be the Cartesian product of two complete graphs of order q. It is well-known [2] that the $q \times q$ grid is strongly regular with parameters $(q^2, 2(q-1), q-2, 2)$.

Lemma 3.2. *Let Γ be a non-complete k-regular edge-regular graph having a* 1*-regular clique of order c. Then $k \geqslant 2(c-1)$. In the case of equality, Γ is the $c \times c$ grid and is thus strongly regular.*

Proof. Set $m = k/(c-1)$. Since Γ is not complete, we have $m > 1$. Let $\mathcal{C}$ be a regular clique of order c. Let x be a vertex in $\mathcal{C}$. Since there are no edges between $\Gamma(x) \cap \mathcal{C}$ and $\Gamma(x) \backslash \mathcal{C}$, we find that $\lambda = c - 2$.

Now suppose y is a vertex adjacent to x but not in $\mathcal{C}$. Note that x has $k - (c-1) = (m-1)(c-1)$ neighbours outside of $\mathcal{C}$. Hence the number of common neighbours of x and y is at most $(m-1)(c-1) - 1$. Therefore $c - 2 \leqslant (m-2)(c-1) - 1$. Again, since Γ is not complete, we have $c - 1 \geqslant 1$. Hence we must have $m \geqslant 2$.

In the case of equality, we see that y is adjacent to every neighbour of x outside $\mathcal{C}$. Furthermore, the subgraph induced on the neighbourhood of x is the disjoint union of two complete graphs each of order $c - 1$. Therefore, Γ is the Cartesian product of two complete graphs of order c, i.e., the $c \times c$ grid. □

Next, we need a lower bound on the size of a regular clique.

Proposition 3.3 ([7, Proposition 5.2])**.** *Let Γ be an edge-regular graph having a regular clique. Suppose that Γ is not strongly regular. Then Γ has a regular clique of order at least* 4*.*

The final ingredient is a nonexistence result for a graph on 20 vertices.

Proposition 3.4. *There does not exist an edge-regular graph with parameters* $(20, 7, 2)$ *and a* 1*-regular clique.*

Proof. Suppose, for a contradiction, there does exist such a graph Γ. By Proposition 3.3, Γ must have a regular clique $\mathcal{C}$ of size at least 4. Moreover, since $\lambda = 2$, the clique $\mathcal{C}$ must have size 4. Let $x \in \mathcal{C}$. The subgraph induced on $\Gamma(x)$ is the disjoint union of a 3-cycle T and a 4-cycle C. Set $K = \{x\} \cup V(T)$ and let Δ be the subgraph induced on the vertices $V(\Gamma) \backslash K$. Observe that, since K is a 1-regular clique of Γ, the subgraph Δ is 6-regular. Furthermore, observe that each of the 16 pairs of adjacent vertices in $\bigcup_{x \in K} \Gamma(x) \backslash K$ has a common vertex in K. Hence there are 16 edges in Δ that are each contained in precisely one triangle and the remaining 32 edges are in precisely 2 triangles. Therefore Δ has $(16 + 2 \cdot 32)/3$ triangles. Since this number is not an integer, we establish a contradiction. □

Now the proof of Theorem 3.1.

Proof of Theorem 3.1. By Proposition 3.3, Γ must have a regular clique $\mathcal{C}$ of size $c \geqslant 4$. Furthermore, by Lemma 3.2, the degree of the vertices of Γ is at least 7. If $k \geqslant 8$ then, since $\mathcal{C}$ is 1-regular, Γ must have at least 24 vertices. It therefore suffices to consider the case when $k = 7$ and $c = 4$. In this case, Γ must be edge-regular with parameters $(20, 7, 2)$. But by Proposition 3.4, no such graph exists. □

Note added in proof

Rhys Evans and Sergey Goryainov observed that one can generalise the construction given in Section 2. One does not need an isomorphism between the antipodal graphs, a bijection between the fibre classes suffices.

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