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Pretty good quantum state transfer in asymmetric graphs via potential

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ABSTRACT

We construct infinite families of graphs in which pretty good state transfer can be induced by adding a potential to the nodes of the graph (i.e. adding a number to a diagonal entry of the adjacency matrix). Indeed, we show that given any graph with a pair of cospectral nodes, a simple modification of the graph, along with a suitable potential, yields pretty good state transfer between the nodes. This generalizes previous work, concerning graphs with an involution, to asymmetric graphs.

1. Introduction

Transfer of quantum information with high fidelity through networks of locally coupled spin particles is an important problem in quantum information processing. Information can be considered as excitation in the network initiated at an input node, which then spreads according to the action of a Hamiltonian. The quality of the transfer depends on how strongly the excitation can then be concentrated at a given target node. The transfer is *perfect* if there is a time t at which the probability of the excitation being at the target node is 1. Initiated by Bose [3], perfect state transfer has been extensively studied for various networks, both from the physical [13] and the mathematical [8] point of view. It turns out that perfect state transfer is notoriously difficult to achieve. All known constructions involve very special networks and/or very special, highly non-uniform coupling strengths. In particular, it has been shown in [14] that for uniformly coupled chains of length at least four there can never be perfect state transfer between endpoints, not even in the presence of magnetic fields.

There is a somewhat less restrictive notion of a *pretty good state transfer*, also referred to as "almost perfect state transfer". This requires the transfer probability to get arbitrarily close to 1 as time passes. While pretty good state transfer is a close approximation to perfect state transfer, it is somewhat easier to achieve. The first examples of spin chains admitting pretty good state transfer appeared in [17]. However, as demonstrated in [10,2,5,16], even pretty good state transfer is relatively rare in unmodulated spin chains with uniform couplings.

In this paper we study pretty good state transfer in the single-excitation subspace of a spin network with XX couplings, in the presence of a magnetic field. We will use graph theoretic terminology throughout the paper. We denote the network by G, the set of nodes (vertices) by $V(G)$, and the set of links (edges) by $E(G)$. The evolution of such a system is given by its Hamiltonian

$$H_{XX} = \frac{1}{2} \sum_{(i,j)\in E(G)} J_{ij}(X_iX_j + Y_iY_j) + \sum_{i\in V(G)} Q_i \cdot Z_i,$$

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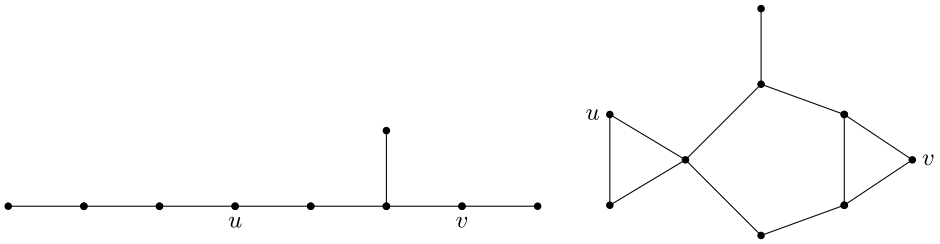

Fig. 1. Asymmetric, irregular graphs with cospectral nodes. In each graph, the cospectral vertices are labeled u and v.

where X_i, Y_i, Z_i denote the action of the standard Pauli matrices on the particle at node i, J_{ij} denotes the strength of the XX coupling between nodes i and j, and the Q_i's give the strength of the magnetic field yielding an energy potential at each node.

It has been shown [3,4] that the restriction of this Hamiltonian to the single-excitation subspace is modeled by a continuous-time quantum walk on a graph with transition matrix, $U(t)$, given by

$$U(t) = \exp(itA)$$

where A is the (possibly weighted) adjacency matrix of the graph.

Definition 1.1. Let G be a graph with vertices u and v.

1. We say that G admits *perfect state transfer* (PST) from u to v if there is some time $t > 0$ such that
$$|U(t)_{u,v}| = 1.$$
2. We say G admits *pretty good state transfer* (PGST) from u to v if, for any $\epsilon > 0$, there is a time $t > 0$ such that
$$|U(t)_{u,v}| > 1 - \epsilon.$$

Our primary focus in this paper will be the effect of adding a potential induced by a magnetic field (the Q_i above). Previous work in [15] showed that in graphs with an involutional symmetry, one can often induce pretty good state transfer between a pair of nodes by appropriately choosing a potential on the vertex set. In [9], it is shown that a potential can induce pretty good state transfer in strongly regular graphs as well. The contribution of this paper is to show how to construct asymmetric, non-regular graphs that admit PGST between a pair of nodes u, v if a suitably chosen potential is added to the adjacency matrix at u and v. The novelty of our constructions is that we do not require any symmetry or regularity in the graph. In addition, our results apply to arbitrary real symmetric matrices (not just adjacency matrices). We note that PST has been exhibited in asymmetric simple unweighted graphs in [1].

A necessary condition for both PST and PGST between vertices u and v of a graph is that u and v must be *cospectral* (see [8,2]), that is $G \setminus u$ and $G \setminus v$ have the same spectrum. One motivation for our previous work [15] is that symmetry in a graph always naturally leads to cospectral vertices. It also holds that pairs of vertices in strongly regular graphs are cospectral. However, cospectral pairs can arise in irregular graphs without any symmetry. Two relatively small examples are shown in Fig. 1.

Our constructions come in two types, based on the following two observations concerning cospectral vertices. First, in a graph with an equitable partition (defined in Section 4.1) with a part containing exactly two vertices, those two vertices are cospectral (see Fig. 2 for an example). Second, given two graphs with a cospectral pair, the vertices remain cospectral in the graph obtained by "gluing" the two graphs together along those vertices (see Lemma 4.6). We are able to show that given any graph with a pair of cospectral vertices, a simple modification of the graph, together with an appropriately chosen potential on the vertex set, yields PGST between those vertices. See Corollary 4.3, and Theorems 4.5, 4.12 and 4.13 for the precise details.

The key tool in our analysis is Theorem 2.11, which takes advantage of the fact that the characteristic polynomial of the adjacency matrix for a graph with cospectral nodes has a factorization. We give a simple, efficiently computable condition on the factors that implies PGST. Note, however, that the converse of Theorem 2.11 does not hold in general. Another critical piece in our proofs, of independent interest, is Lemma 3.2, which shows that adding a transcendental potential to a pair of cospectral nodes actually makes them *strongly* cospectral (see Section 2 for the definition).

2. Preliminaries

Let $\mathcal{F}$ be a subfield of the real numbers. Let M be a symmetric matrix with entries $\mathcal{F}$. We use $\phi_M \in \mathcal{F}[t]$ to denote its characteristic polynomial. The rows and columns of M will typically be indexed by the nodes of a finite graph. We will use $V(M)$ to denote the set of row/column indices of M, so we can think of $M \in \mathbb{R}^{V(M)\times V(M)}$. If $S \subset V(M)$, we write M_S for the symmetric submatrix obtained from M by *removing* the rows and columns indexed by S.

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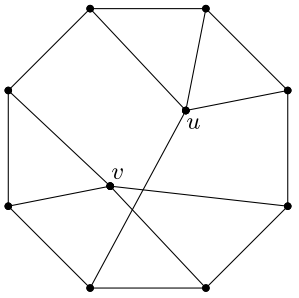

Fig. 2. A graph with an equitable partition with a part of size 2 (vertices u and v), and hence a cospectral pair (but no involution swapping u and v).

Definition 2.1. For any vector z, let $W(M, z) = \langle z, Mz, M^2z, \ldots \rangle$ denote the M-invariant subspace generated by z. Let us denote by $\rho_z = \rho_{z,M} \in \mathcal{F}[t]$ the minimal polynomial of M relative to z, that is, the smallest degree monic polynomial such that $\rho_z(M)z = 0$. It is well-known that ρ_z divides the usual minimal polynomial of M and that the degree of ρ_z equals the dimension of $W(M, z)$.

2.1. Cospectral nodes

Definition 2.2. Let M be a symmetric matrix. Two indices $u, v \in V(M)$ are *cospectral* if $\phi(M_u) = \phi(M_v)$.

Lemma 2.3 (*Theorem 3.1 of [11]*)**.** *Let M be a symmetric matrix, and let $u, v \in V(M)$. Let $M = \sum_\lambda \lambda E_\lambda$ be the spectral decomposition of M. Here E_λ denotes the orthogonal projection onto the eigenspace of M corresponding to the eigenvalue λ. We denote the characteristic vectors of u and v by e_u, e_v respectively. The following are equivalent:*

1. *u and v are cospectral.*
2. *$(E_\lambda)_{u,u} = (E_\lambda)_{v,v}$ for all λ.*
3. *$M^k(u, u) = M^k(v, v)$ for all k.*
4. *$W(M, e_u + e_v)$ is orthogonal to $W(M, e_u - e_v)$.*

Definition 2.4. We define P_+ to be the minimal polynomial of M relative to $e_u + e_v$, and P_- to be the minimal polynomial of M relative to $e_u - e_v$.

Lemma 2.5. *Given a symmetric matrix M and cospectral indices $u, v \in V(M)$, the characteristic polynomial of M decomposes as*

$$\phi_M = P_+ \cdot P_- \cdot P_0,$$

where P_+ and P_- have no multiple roots, and there is an orthonormal basis of eigenvectors of M such that:

1. *for each root λ of P_+ the basis contains a unique eigenvector φ with eigenvalue λ and $\varphi(u) = \varphi(v) \neq 0$,*
2. *for each root λ of P_- the basis contains a unique eigenvector φ with eigenvalue λ and $\varphi(u) = -\varphi(v) \neq 0$,*
3. *for each root λ of P_0 with multiplicity k the basis contains exactly k eigenvectors with eigenvalue λ all of which vanish on both u and v.*

In particular the degree of $P_\pm$ is the same as the dimension of the space $W(M, e_u \pm e_v)$.

Proof. Since M is diagonalizable, its minimal polynomial does not have multiple roots, and hence neither does P_+ nor P_-. The roots of P_+ are exactly those eigenvalues λ for which $E_\lambda(e_u + e_v) \neq 0$, and for such λ the eigenvector $\varphi = E_\lambda(e_u + e_v)$ satisfies that $\varphi(u) = \varphi(v)$. Similarly for P_-. By cospectrality of u and v, the eigenvectors obtained for P_+ and for P_- are pairwise orthogonal. Finally, extending to an orthogonal basis for M, it is clear that each remaining eigenvector satisfies $\varphi(u) = \varphi(v) = 0$. □

Remark 2.6. Since the coefficients of P_+ give the unique linear dependency among $e_u + e_v, M(e_u + e_v), \ldots, M^k(e_u + e_v)$, they belong to the same field as the entries of M. The same is true for P_-, and thus for P_0.

Definition 2.7. The indices $u, v \in V(M)$ are *strongly cospectral* if $\varphi(u) = \pm\varphi(v)$ for every eigenvector φ of M.

Lemma 2.8. *The following are equivalent:*

1. *u and v are strongly cospectral.*
2. *u and v are cospectral, and P_+ and P_- do not have any common roots.*
3. *$E_\lambda e_u = \pm E_\lambda e_v$ for all λ.*

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2.2. Pretty good state transfer

The discrete Schrödinger equation, for an $n \times n$ matrix M, is given by

$$\partial_t \psi_t = iM\psi_t,$$

where $\psi_t \in \mathbb{C}^n$ for $t \in [0, \infty)$. The solution of this equation can be written in the form

$$\psi_t = e^{itM}\psi_0.$$

Definition 2.9. We say that M has PGST from u to v if $\psi_0 = e_u$ implies that $\limsup_{t\to\infty} |\psi_t(v)| = 1$, or equivalently, that $\limsup_{t\to\infty} |e^{itM}(u, v)| = 1$.

The following is a characterization of PGST (see Theorem 2 in [2]).

Lemma 2.10. *Let $u, v \in V(M)$ for the symmetric matrix M. Then pretty good state transfer from u to v occurs if and only if the following two conditions are satisfied:*

1. *The indices u and v are strongly cospectral.*
2. *Let $\{\lambda_i\}$ be the roots of P_+, and $\{\mu_j\}$ the roots of P_-. Then for any choice of integers ℓ_i, m_j such that*

$$\sum_i \ell_i\lambda_i + \sum_j m_j\mu_j = 0$$
$$\sum_i \ell_i + \sum_j m_j = 0,$$

we have

$$\sum_i m_i \text{ is even.}$$

Note that the first condition could be weakened to just cospectral, since the second condition implies that P_+ and P_- do not share any roots, so this implies strongly cospectral if the nodes are cospectral.

The following theorem generalizes a result from [15] and a lemma from [9].

Theorem 2.11. *Let M be a symmetric matrix with entries in $\mathcal{F}$ with strongly cospectral indices $u, v \in V(M)$, and assume that P_+ and P_- are irreducible polynomials over $\mathcal{F}$. Then if*

$$\frac{\mathrm{Tr}(P_+)}{deg(P_+)} \neq \frac{\mathrm{Tr}(P_-)}{deg(P_-)},$$

where Tr *denotes the trace (i.e. the sum of roots) of a polynomial, then there is PGST from u to v.*

Proof. Our proof uses a technique from [15].

Suppose we have integers ℓ_i, m_j satisfying

$$\sum_i \ell_i\lambda_i + \sum_j m_j\mu_j = 0$$
$$\sum_i \ell_i + \sum_j m_j = 0.$$

To use Lemma 2.10, we wish to show $\sum_i \ell_i$ is even.

We will use a tool from Galois theory called the *field trace* of a field extension. For a Galois field extension $\mathcal{K}$ of $\mathcal{F}$, we define $\mathrm{Tr}_{\mathcal{K}/\mathcal{F}} : \mathcal{K} \to \mathcal{F}$ by

$$\mathrm{Tr}_{\mathcal{K}/\mathcal{F}}(\alpha) = \sum_{g\in Gal(\mathcal{K}/\mathcal{F})} g(\alpha).$$

The field trace is the trace of the linear map taking $x \mapsto \alpha x$. In Lemma A.1 of the Appendix we record a few basic facts about the field trace that we will use.

Now, let $\mathcal{F}$ be the base field (the field containing all the entries of M), let $\mathcal{L}/\mathcal{F}$ be the splitting field for P_+, $\mathcal{J}/\mathcal{F}$ the splitting field for P_-, and $\mathcal{K}/\mathcal{F}$ the smallest field extension containing both $\mathcal{L}$ and $\mathcal{J}$. Let us denote $r = deg(P_+)$ and $s = deg(P_-)$. Since P_+ and P_- are irreducible, then by Lemma A.1,

$$\mathrm{Tr}_{\mathcal{L}/\mathcal{F}}(\lambda_i) = \frac{[\mathcal{L} : \mathcal{F}]}{r} \sum_k \lambda_k.$$

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Note further that $\sum_k \lambda_k = \mathrm{Tr}(P_+)$, so we have shown

$$\mathrm{Tr}_{\mathcal{L}/\mathcal{F}}(\lambda_i) = \frac{[\mathcal{L}:\mathcal{F}]}{r}(\mathrm{Tr}(P_+))$$

for any i. In a similar way, by examining P_- we obtain

$$\mathrm{Tr}_{\mathcal{J}/\mathcal{F}}(\mu_j) = \frac{[\mathcal{J}:\mathcal{F}]}{s}(\mathrm{Tr}(P_-))$$

for any j.

Now apply the field trace to our linear combination of the λ_i and μ_j, and using the properties above we have,

$$\begin{aligned}
0 &= \mathrm{Tr}_{\mathcal{K}/\mathcal{F}}\left(\sum \ell_i\lambda_i + \sum m_j\mu_j\right)\\
&= \mathrm{Tr}_{\mathcal{K}/\mathcal{F}}\left(\sum \ell_i\lambda_i\right) + \mathrm{Tr}_{\mathcal{K}/\mathcal{F}}\left(\sum m_j\mu_j\right)\\
&= [\mathcal{K}:\mathcal{L}]\,\mathrm{Tr}_{\mathcal{L}/\mathcal{F}}\left(\sum \ell_i\lambda_i\right) + [\mathcal{K}:\mathcal{J}]\,\mathrm{Tr}_{\mathcal{J}/\mathcal{F}}\left(\sum m_j\mu_j\right)\\
&= [\mathcal{K}:\mathcal{L}]\sum \ell_i\,\mathrm{Tr}_{\mathcal{L}/\mathcal{F}}(\lambda_i) + [\mathcal{K}:\mathcal{J}]\sum m_j\,\mathrm{Tr}_{\mathcal{J}/\mathcal{F}}(\mu_j)\\
&= \frac{[\mathcal{K}:\mathcal{L}][\mathcal{L}:\mathcal{F}]}{r}\,\mathrm{Tr}(P_+)\sum \ell_i + \frac{[\mathcal{K}:\mathcal{J}][\mathcal{J}:\mathcal{F}]}{s}\,\mathrm{Tr}(P_-)\sum m_j\\
&= [\mathcal{K}:\mathcal{F}]\left(\frac{\mathrm{Tr}(P_+)}{r}\sum \ell_i + \frac{\mathrm{Tr}(P_-)}{s}\sum m_j\right)
\end{aligned}$$

This, along with our assumption at the beginning gives us

$$\begin{aligned}
\frac{\mathrm{Tr}(P_+)}{r}\sum \ell_i + \frac{\mathrm{Tr}(P_-)}{s}\sum m_j &= 0\\
\sum \ell_i + \sum m_j &= 0.
\end{aligned}$$

This is a system of two equations in the variable $\sum \lambda_i$, $\sum \mu_j$, and so if

$$\frac{\mathrm{Tr}(P_+)}{r} \neq \frac{\mathrm{Tr}(P_-)}{s},$$

then these two equations are linearly independent, and we obtain

$$\sum \lambda_i = \sum m_j = 0.$$

In particular, each sum is even, so Lemma 2.10 implies that we get pretty good state transfer. □

3. Diagonal perturbation

In this section we investigate how, given a symmetric matrix M with cospectral indices $u, v \in V(M)$, adding a diagonal matrix D to M can be used to achieve *strong* cospectrality of u, v and *irreducibility* of P_+ and P_-. We are going to do this by choosing D to have two non-zero values only. To establish notation, for any set of indices $S \subset V(M)$, let D_S denote the diagonal matrix with 1s in the positions belonging to S and 0s elsewhere. Let Q be a real number and $D = Q \cdot D_{\{u,v\}}$.

Lemma 3.1. *If $u, v \in V(M)$ are cospectral indices for M, then they are also cospectral for $M + D$.*

Proof.

$$\phi_{(M+D)_u} = \phi_{M_u} - Q \cdot \phi_{M_{uv}} = \phi_{M_v} - Q \cdot \phi_{M_{uv}} = \phi_{(M+D)_v}. \quad \square$$

3.1. Achieving strong cospectrality

The benefit of adding such a diagonal perturbation is that we can actually turn a pair of cospectral indices into strongly cospectral ones.

Let us say that a matrix M has *connected support* if for any two indices $u, v \in V(M)$ there is a $k \geq 0$ such that $M^k(u, v) \neq 0$. In case of a matrix with non-negative entries this is equivalent to the underlying graph being connected.

Lemma 3.2. *Let M be a symmetric matrix with connected support whose entries are in a field $\mathcal{F} \leq \mathbb{R}$, and assume $u, v \in V(M)$ are cospectral. Suppose Q is transcendental over $\mathcal{F}$, and $D = Q \cdot D_{\{u,v\}}$, then u and v are strongly cospectral for $M + D$.*

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Proof. By simple expansion, and using that cospectrality means $\phi_{M_u} = \phi_{M_v}$, we can write

$$\phi_{M+D} = \phi_M + 2Q\phi_{M_v} + Q^2\phi_{M_{uv}}. \tag{1}$$

By [11, Lemma 8.4] it is sufficient to show that $\phi_{M_{uv}}/\phi_{M+D}$ has only simple poles. We prove this by contradiction. First, we can assume that ϕ_{M+D} has multiple roots, and thus it is reducible over $\mathcal{F}(Q)$. However, since Q is transcendental over $\mathcal{F}$, it has to factor over $\mathcal{F}[Q]$. So we can either write

$$\phi_{M+D} = h \cdot (Qf_1 + f_0) \cdot (Qg_1 + g_0),$$

or possibly

$$\phi_{M+D} = h \cdot (Q^2 f_2 + Qf_1 + f_0),$$

where $f_i, g_j, h \in \mathcal{F}[x]$, and all the factors but h are irreducible over $\mathcal{F}[Q]$. In both cases it follows from (1) that h is a factor $\phi_{M_{uv}}$, thus there is a polynomial $\psi \in \mathcal{F}[x]$ such that

$$\frac{\phi_{M_{uv}}}{\phi_{M+D}} = \frac{\psi}{(Qf_1 + f_0) \cdot (Qg_1 + g_0)}$$

or

$$\frac{\phi_{M_{uv}}}{\phi_{M+D}} = \frac{\psi}{(Q^2 f_2 + Qf_1 + f_0)}.$$

The second option immediately implies that all poles are simple since the denominator is irreducible and thus has simple roots. In order to have non-simple poles in the first case, the two irreducible factors in the denominator must have a common root, but then they must have a common factor. However, they are irreducible over $\mathcal{F}(Q)$, so the only factors they can have are themselves, and since they are monic, they must coincide. This means that $\phi_M = hf_0^2$, $\phi_{M_u} = \phi_{M_v} = hf_1f_0$, $\phi_{M_{u,v}} = hf_1^2$. Thus $\psi_{u,v} = \phi_{M_u}\phi_{M_v} - \phi_M\phi_{M_{u,v}} = 0$ so, by [7, Lemma 1.1, Chapter 4.1], we get $(E_\lambda)_{u,v} = 0$ for all λ, so $M^k(u, v) = 0$ for all k. This contradicts the assumption that M has connected support, so u, v must be strongly cospectral for M. □

3.2. Trace

We see from Theorem 2.11 that the trace of P_+ and P_- can be useful in proving that there is PGST between two cospectral nodes. In this section we prove some important properties of $\operatorname{Tr} P_\pm$ under diagonal perturbation.

Lemma 3.3. *Let M be a symmetric matrix whose elements are in a field $\mathcal{F} \leq \mathbb{R}$. Suppose $u, v \in V(M)$ are cospectral for M and let $D = Q \cdot D_{\{u,v\}}$ where $Q \in \mathbb{R}$ is transcendental over $\mathcal{F}$. Let $\phi_{M+D} = P_+ \cdot P_- \cdot P_0$ as in Lemma 2.5. Then $\operatorname{Tr} P_+ - Q \in \mathcal{F}$ and $\operatorname{Tr} P_- - Q \in \mathcal{F}$.*

Proof. Let us recall that $P_\pm$ are the minimal polynomials of $M + D$ relative to $e_u \pm e_v$, and that $\phi_{M+D} = P_+ \cdot P_- \cdot P_0$. Since Q is transcendental over $\mathcal{F}$, we have to have $P_+, P_-, P_0 \in \mathcal{F}[Q, t]$.

First we show that both P_+ and P_- are at least linear in Q. To see this, observe that the u and v coordinates of $(M + D)^k(e_u + e_v)$ contain a single term Q^k and no higher power of Q shows up elsewhere in $(M + D)^k(e_u + e_v)$, nor in $(M+D)^j(e_u+e_v)$ for any $j < k$. Hence if $P_+(t) = t^k + c_{k-1}t^{k-1} + \cdots$ where $c_j \in \mathcal{F}[Q]$, then we have $0 = P_+(M+D)(e_u+e_v) = (M + D)^k(e_u + e_v) + \sum_{j=0}^{k-1} c_j(M + D)^j(e_u + e_v)$. This can only happen if the Q^k coming from the first term is canceled by something. This can only be if at least some of the c_j coefficients are in $\mathcal{F}[Q] \setminus \mathcal{F}$. Thus P_+ needs to be at least linear in Q. The same argument shows this for P_- as well.

Note that ϕ_{M+D} is quadratic in Q, which implies that both P_+ and P_- have to be exactly linear in Q. Going back to the cancellation of the Q^k term in $(M + D)^k(e_u + e_v)$, we see that since the coefficients c_j are at most linear in Q, only the $c_{k-1}(M + D)^{k-1}(e_u + e_v)$ has a chance to cancel the Q^k term, and for this it has to be that $c_{k-1} + Q \in \mathcal{F}$. But $-c_{k-1} = \operatorname{Tr} P_+$, so this implies $\operatorname{Tr} P_+ - Q \in \mathcal{F}$. That the same holds for P_- follows the exact same way. □

We can prove a similar result for diagonal perturbations at a single vertex. This will be useful in some of our constructions in Section 4.

Lemma 3.4. *Let M be a symmetric matrix with connected support whose elements are in a field $\mathcal{F} \leq \mathbb{R}$. Suppose $u, v \in V(M)$ are cospectral for M, and let $w \in V(M)$ be another index such that there is an integer $d \geq 0$ for which $\langle e_w, M^d(e_u + e_v)\rangle \neq 0$. Let $D = Q \cdot D_{\{w\}}$ where $Q \in \mathbb{R}$ is transcendental over $\mathcal{F}$, and suppose u and v are also cospectral for $M + D$. Let $\phi_{M+D} = P_+ \cdot P_- \cdot P_0$ as in Lemma 2.5. Then $\operatorname{Tr} P_+ - Q \in \mathcal{F}$.*

Proof. The argument is almost identical to the previous one. Let d be the smallest power for which $\langle e_w, M^d(e_u + e_v)\rangle \neq 0$. From the setup it follows that $\phi_{M+D} \in \mathcal{F}[Q, t]$ is linear in Q, thus the polynomial $P_+ \in \mathcal{F}[Q, t]$ can be at most linear. But it also has to be at least linear, since in $(M+D)^k(e_u+e_v)$ there will be a Q^{k-d} term appearing that would only cancel if one of the c_j coefficients of P_+ contains Q. This coefficient then can only be c_{k-1} and, as previously, it can only happen if $c_{k-1} + Q \in \mathcal{F}$, and thus $\operatorname{Tr} P_+ - Q \in \mathcal{F}$. □

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Remark 3.5. Note that since ϕ_{M+D} is only linear in Q, then Q does not show up in P_-. This implies in particular that, $\langle e_w, M^d(e_u - e_v)\rangle = 0$ for all d.

3.3. Achieving irreducibility

Lemma 3.6. *Let M be a symmetric matrix whose entries are in a field $\mathcal{F} \leq \mathbb{R}$. Assume $u, v \in V(M)$ are* strongly cospectral *indices for M. If Q is transcendental over $\mathcal{F}$ and $D = Q \cdot D_{\{u,v\}}$, then the polynomials $P_+(M + D)$ and $P_-(M + D)$ are irreducible over $\mathcal{F}(Q)$.*

Proof. By Lemma 3.2 the indices u, v are also strongly cospectral for $M + D$. Then ϕ_{M+D} factors as $\phi_{M+D} = P_+ \cdot P_- \cdot P_0$ by Lemma 2.5. Since such a factorization exists for all values of Q, it has to be a factorization in $\mathcal{F}[Q, t]$. By expanding the determinant we also have

$$\phi_{M+D} = \phi_M - 2Q\phi_{M_u} + Q^2\phi_{M_{uv}}. \tag{2}$$

Thus we see that ϕ_{M+D} is quadratic in Q, and thus P_+, P_-, and P_0 can all be at most quadratic in Q. By Lemma 3.3, $\operatorname{Tr} P_+ - Q \in \mathcal{F}$ and $\operatorname{Tr} P_- - Q \in \mathcal{F}$. Thus there are non-zero polynomials $S_\pm, R_\pm \in \mathcal{F}[t]$ such that $P_\pm = S_\pm + Q \cdot R_\pm$, and hence by comparing the degrees in Q, we get that $P_0 \in \mathcal{F}[t]$.

Comparing to (2) we see that

$$\begin{aligned}\phi_M &= P_0 \cdot S_+ \cdot S_-\\ \phi_{M_u} &= P_0 \cdot (S_+R_- + R_+S_-)\\ \phi_{M_{uv}} &= P_0 \cdot R_+ \cdot R_-.\end{aligned}$$

Now suppose that P_+ is not irreducible. For transcendental Q, this implies that P_+ factors in $\mathcal{F}[t, Q]$, but since it is linear in Q, the only way for this to happen is that there is some factor $T \in \mathcal{F}[t]$ that divides both S_+ and R_+. If this is the case, then $T \cdot P_0$ divides all three of $\phi_M, \phi_{M_u}, \phi_{M_{uv}}$.

Let $(t - \lambda)^k$ be a factor of $T \cdot P_0$. Then by Lemma A.2, there are k eigenvectors that vanish on u. By strong cospectrality of u and v, these must vanish on v simultaneously, so there are k eigenvectors for λ that vanish on both u and v. This means that $(t - \lambda)^k$ is already a factor of P_0. Since this holds for all factors of $T \cdot P_0$ and hence $T = 1$. Thus P_+ is irreducible. The same argument gives that P_- is also irreducible. □

4. Constructions

In this section we explain how to obtain graphs with a pair of cospectral nodes u, v where adding a potential Q at nodes u and v, and possibly at a third node w results in PGST between u and v. The significance of these constructions is that they yield examples without symmetries, in particular without an involution mapping u to v.

4.1. Equitable partitions

Our first construction is based on equitable partitions. These can be thought of as direct generalizations of graphs with an involution.

Definition 4.1. An equitable partition of a symmetric matrix M is a partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of its index set $V(M)$ such that for any $P_i, P_j \in \mathcal{P}$ and any $v_1, v_2 \in P_i$, one has

$$\sum_{u \in P_j} M(v_1, u) = \sum_{u \in P_j} M(v_2, u).$$

Theorem 4.2. *Let M be a symmetric matrix with connected support whose elements are in a field $\mathcal{F} \leq \mathbb{R}$. Suppose M admits an equitable partition $\mathcal{P}$ such that $P_1 = \{u, v\}$ and $P_2 = \{w\}$, then for algebraically independent numbers Q_1, Q_2 that are transcendental over $\mathcal{F}$ and for $D = Q_1 \cdot D_{\{u,v\}} + Q_2 \cdot D_{\{w\}}$ the matrix $M + D$ admits PGST between u and v.*

Corollary 4.3. *If a connected graph has an equitable partition with a part consisting of u, v and another part consisting of w only, then by adding suitable potentials at u, v, and w one can guarantee PGST between u and v.*

Proof. We proceed step-by-step as follows: first we show that u and v are cospectral in both M and $M + Q_2 \cdot D_{\{w\}}$. Then we show that u and v are strongly cospectral in $M + D$, and furthermore that the corresponding P_+ and P_- are irreducible. Finally, we show that $\operatorname{Tr} P_+/\deg P_+ \neq \operatorname{Tr} P_-/\deg P_-$ hence by Theorem 2.11 there is PGST between u and v.

Let us start by proving cospectrality of u and v. First, let $\Pi_{\mathcal{P}}$ denote the *partition matrix* corresponding to $\mathcal{P}$. That is, the columns of $\Pi_{\mathcal{P}}$ are indexed by $1, 2, \ldots, k$, and the rows are indexed by $V(M)$, and $\Pi_{\mathcal{P}}(j, x)$ is 1 if $x \in P_j$ and 0 otherwise.

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Second, let $M_{\mathcal{P}}$ denote the *quotient* matrix, given as $M_{\mathcal{P}}(i,j) = \sum_{y\in P_j} M(x,y)$ for some fixed $x \in P_i$. As $\mathcal{P}$ is equitable, the value $M_{\mathcal{P}}(i,j)$ does not depend on the particular choice of x. Note, that $M_{\mathcal{P}}$ is a $k\times k$ matrix, though not necessarily symmetric.

A simple computation shows that $M\cdot\Pi_{\mathcal{P}} = \Pi_{\mathcal{P}}\cdot M_{\mathcal{P}}$. Note that, since $P_1 = \{u, v\}$, we can write e_u+e_v as $\Pi_{\mathcal{P}}(1,0,\ldots,0)^T$. Now we can compute

$$\langle e_u - e_v, M^m(e_u+e_v)\rangle = (e_u-e_v)^T M^m \Pi_{\mathcal{P}}(1,0,\ldots,0)^T = (e_u-e_v)^T\Pi_{\mathcal{P}} M_{\mathcal{P}}^m(1,0,\ldots,0)^T = 0,$$

since $(e_u-e_v)^T\Pi_{\mathcal{P}} = 0$. This shows that $W(M, e_u+e_v)$ is orthogonal to $W(M, e_u-e_v)$ and hence u and v are cospectral.

As $\mathcal{P}$ is also an equitable partition for $M+Q_2\cdot D_{\{w\}}$, it follows that u and v are also cospectral for $M+Q_2\cdot D_{\{w\}}$. Let us write $Q_1 = A+B$ where A, B, Q_2 are all algebraically independent of each other and of $\mathcal{F}$. This can be done by choosing A to be independent of Q_1, Q_2 and transcendental over $\mathcal{F}$ and then setting $B = Q_1 - A$. Then, by Lemma 3.2 and the assumption that A is transcendental over $\mathcal{F}(Q_2)$, we find that u and v are strongly cospectral for $M+Q_2\cdot D_{\{w\}}+A\cdot D_{\{u,v\}}$. Then, by Lemmas 3.6 and 3.2, u and v are not only strongly cospectral for $M+D = (M+Q_2\cdot D_{\{w\}}+A\cdot D_{\{u,v\}})+B\cdot D_{\{u,v\}}$, but also the corresponding P_+ and P_- are irreducible.

Now, by Lemmas 3.3 and 3.4, we find that $\operatorname{Tr} P_+ - Q_2 \in \mathcal{F}(Q_1)$ and since ϕ_{M+D} is linear in Q_2 this implies $\operatorname{Tr} P_- \in \mathcal{F}(Q_1)$. Then surely $\operatorname{Tr} P_+/\deg P_+$ cannot equal $\operatorname{Tr} P_-/\deg P_-$ since that would imply $Q_2 \in \mathcal{F}(Q_1)$, a contradiction.

Finally, by Theorem 2.11 we get that there is PGST between u and v. □

Remark 4.4. Given any graph with an equitable partition with a part of size two (and thus a cospectral pair) it is straightforward to add a single vertex and attach it to the vertices of one of the parts of the partition to produce a graph satisfying the conditions of the corollary.

4.2. Gluing

Our second construction is based on an arbitrary graph G with a pair of cospectral nodes $u, v \in V(G)$. We will show that either simply adding a transcendental potential Q at the nodes u and v induces PGST between them, or else one can modify G in a relatively simple way: by gluing a long path to G with u and v being its endpoints, and then adding a transcendental potential Q at u and v we get PGST between u and v.

Theorem 4.5. *Let G be a graph with $u, v \in V(G)$ cospectral, and such that 0 is not an eigenvalue of the adjacency matrix of $G\setminus\{u,v\}$. Fix an integer $q \geq 0$. Construct G_q by gluing a path of length q to G by attaching its endpoints to u and v. In other words, by adding $q-1$ new nodes $x_1, x_2, \ldots, x_{q-1}$ to G together with the edges $ux_1, x_1x_2, x_2x_3, \ldots, x_{q-2}x_{q-1}, x_{q-1}v$. (For $q = 0$ we simply take $G_0 = G$.)*

Let $Q \in \mathbb{R}$ be a transcendental number, and put a potential Q at the nodes u and v in G_q. Then either the potential induces PGST between u and v in G or there is an infinite set of integers $S \subset \mathbb{Z}$ such that this potential induced PGST between u and v in G_q for all $q \in S$.

Again, the main novelty of this construction is that it does not require the graph to admit any kind of symmetry. In fact, one can start from any graph with a pair of cospectral nodes, of which many examples have been described in the literature. We give the proof at the end of this section.

We begin by describing a general gluing construction that preserves cospectrality. This has been independently discovered by Godsil [6]. As we have done so far, we will prove everything in the general context of symmetric matrices, but we are still primarily interested in the case where the matrices in question are the adjacency matrices of graphs.

Let M_1 and M_2 be symmetric matrices such that $V(M_1)\cap V(M_2) = \{u, v\}$. We can extend them to matrices $\tilde{M}_1, \tilde{M}_2$ on $V(M_1)\cup V(M_2)$ by declaring them to be 0 wherever they were not previously defined. Then we define their sum $M_1\oplus M_2 = \tilde{M}_1 + \tilde{M}_2$, in particular $V(M_1\oplus M_2) = V(M_1)\cup V(M_2)$. When M_i is the adjacency matrix of the graph G_i $(i = 1,2)$, then $M = M_1\oplus M_2$ is the adjacency matrix of $G = G_1\cup_{uv} G_2$ sometimes referred to as the 2-sum of G_1 and G_2, that is obtained by gluing the two u nodes together and the two v nodes together. Note that G may have multiple edges.

Lemma 4.6. *If $u, v \in V(M_i)$; $(i = 1, 2)$ are cospectral pairs for both M_1 and M_2, then they are also cospectral in $M_1\oplus M_2$.*

Proof. Let $M = M_1\oplus M_2$. We can compute ϕ_{M_u} by expanding the determinant along the column corresponding to v:

$$\phi_{M_u} = \phi_{M_{1u}}\phi_{M_{2uv}} + \phi_{M_{2u}}\phi_{M_{1uv}} - t\phi_{M_{1uv}}\phi_{M_{2uv}}.$$

By cospectrality $\phi_{M_{j_u}} = \phi_{M_{j_v}}$, and thus the right hand side does not change when exchanging the roles of u and v. Hence $\phi_{M_u} = \phi_{M_v}$ as claimed. □

In what follows we assume that u and v are indeed cospectral in M_1 and in M_2 and let $M = M_1\oplus M_2$. Let us introduce the notation $\phi_{M_j} = P_+^j\cdot P_-^j\cdot P_0^j$ $(j = 1, 2)$ and $\phi_M = P_+\cdot P_-\cdot P_0$, according to Lemma 2.5.

Lemma 4.7. $\deg P_+ \leq \deg P_+^1 + \deg P_+^2 - 1$ *and* $\deg P_- \leq \deg P_-^1 + \deg P_-^2 - 1$

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Proof. We know, by Definitions 2.4 and 2.1 that $\deg P_+ = \dim W(M, e_u + e_v)$ and $\deg P_+^j = \dim W(\tilde{M}_j, e_u + e_v)$. We will show that $W(M, e_u + e_v) \leq W(\tilde{M}_1, e_u + e_v) + W(\tilde{M}_2, e_u + e_v)$. From this, the first part of the lemma will follow since $\langle e_u + e_v \rangle \leq W(\tilde{M}_1, e_u + e_v) \cap W(\tilde{M}_2, e_u + e_v)$.

Let us denote by Π_1, Π_2, and Π_0 the "natural" projection operators from $\mathbb{R}^G$ to $\mathbb{R}^{G_1}$, $\mathbb{R}^{G_2}$, and $\mathbb{R}^{\{u,v\}}$ respectively. First note that, by cospectrality, $e_u M^k(e_u + e_v) = e_v M^k(e_u + e_v)$ for any k. In other words, $e_u + e_v$ is an eigenvector of $\Pi_0 M^k$ for any k. The same is true with $\tilde{M}_1$ or $\tilde{M}_2$ in place of M. Also note that a simple computation gives $\tilde{M}_1\tilde{M}_2 = \tilde{M}_1\Pi_0\tilde{M}_2$ and $\tilde{M}_2\tilde{M}_1 = \tilde{M}_2\Pi_0\tilde{M}_1$.

It is then sufficient to prove that $\Pi_1 M^k(e_u + e_v) \in W(\tilde{M}_1, e_u + e_v)$ and $\Pi_2 M^k(e_u + e_v) \in W(\tilde{M}_2, e_u + e_v)$. Without loss of generality it is sufficient to prove the first one. Using $M = \tilde{M}_1 + \tilde{M}_2$ we can compute

$$M^k = \sum_{j=0}^{k} \left(\tilde{M}_1^j \sum_{\substack{0<j_1,j_2,\dots \\ j+j_1+j_2+\dots=k}} \tilde{M}_2^{j_1}\tilde{M}_1^{j_2}\tilde{M}_2^{j_3}\tilde{M}_1^{j_4}\dots \right)$$

and so

$$\Pi_1 M^k(e_u + e_v) = \sum_{j=0}^{k} \left(\Pi_1\tilde{M}_1^j \sum_{\substack{0<j_1,j_2,\dots \\ j+j_1+j_2+\dots=k}} \Pi_0\tilde{M}_2^{j_1}\Pi_0\tilde{M}_1^{j_2}\Pi_0\tilde{M}_2^{j_3}\Pi_0\tilde{M}_1^{j_4}\dots(e_u + e_v) \right).$$

Here, each term in the sum is just a multiple of $(e_u + e_v)$ since it is an eigenvector of each $\Pi_0\tilde{M}_\epsilon^j : \epsilon = 1,2$. Hence there are constants c_j depending only on j and k such that

$$\Pi_1 M^k(e_u + e_v) = \sum_{j=0}^{k} c_j \Pi_1 \tilde{M}_1^j(e_u + e_v) = \sum_{j=0}^{k} c_j \left(\tilde{M}_1^j(e_u + e_v) - \Pi_0\tilde{M}_1^j(e_u + e_v) \right) \in W(\tilde{M}_1, e_u + e_v),$$

and this is what we wanted to show.

The argument for P_- is analogous. □

Remark 4.8. Any eigenvector of M_1 or M_2 that vanishes on u, v can be extended to $V(M)$ by zeros to obtain an eigenvector of M with the same eigenvalue. Thus P_0 is divisible by $P_0^1 \cdot P_0^2$.

Lemma 4.9. *Let $k_j \geq 0$ denote the multiplicity of λ in P_0^j $(j = 1, 2)$. Suppose the multiplicity of λ in P_0 is strictly bigger than $k_1 + k_2$. Then λ is an eigenvalue of M_{1uv} and M_{2uv}.*

Proof. By the assumption on the multiplicity there has to be an eigenvector of M with eigenvalue λ vanishing on both u and v that is not identically zero on either M_1 or M_2. The restriction of this vector to $V(M_1) \setminus \{u, v\}$ and to $V(M_2) \setminus \{u, v\}$ then yield eigenvectors showing that λ is indeed an eigenvalue of both of these matrices. □

Corollary 4.10. *If M_{1uv} and M_{2uv} do not share any eigenvalues, then $P_0 = P_0^1 \cdot P_0^2$.*

Theorem 4.11. *If M_{1uv} and M_{2uv} do not share any eigenvalues, then $\deg P_+ = \deg P_+^1 + \deg P_+^2 - 1$ and $\deg P_- = \deg P_-^1 + \deg P_-^2 - 1$.*

Proof. By Corollary 4.10 and by Lemma 4.7 we have

$$\begin{aligned} |V(M)| &= \deg P_0 + \deg P_+ + \deg P_- \leq \\ &\leq \deg P_0^1 + \deg P_0^2 + \deg P_+^1 + \deg P_+^2 - 1 + \deg P_-^1 + \deg P_-^2 - 1 = |V(M_1)| + |V(M_2)| - 2 = |V(M)| \end{aligned}$$

Since the left and right hand sides are equal, there must be equality in the middle, finishing the proof. □

Proof of Theorem 4.5. Let A denote the adjacency matrix of G. By assumption u and v are cospectral for A. The matrix $H_G = A + Q \cdot D_{\{u,v\}}$ is the Hamiltonian for the graph G together with the potential. By Lemmas 3.2 and 3.6 we know that u and v are strongly cospectral for H_G and the corresponding $P_+^{H_G}$ and $P_-^{H_G}$ polynomials are irreducible, and by Lemma 3.3 we know that $\mathrm{Tr}\, P_+^{H_G} - Q$ and $\mathrm{Tr}\, P_-^{H_G} - Q$ are both rational. (To show irreducibility we need to apply the same trick as in the proof of Theorem 4.2: adding the potential in two steps, first ensuring strong cospectrality, then irreducibility.) So by Theorem 2.11, the only way there could be no PGST between u and v is if

$$\deg P_+^{H_G} = \deg P_-^{H_G}.$$

Let now $\mathbb{P}_q$ denote the path graph on $q+1$ nodes, and let A_q denote its adjacency matrix. Let us call the endpoints u and v. It is clear that u and v are cospectral in $\mathbb{P}_q$, for instance because $\mathbb{P}_q$ admits an equitable partition, each part consisting of a pair of symmetric nodes, or the single node in the middle.

Then if $G_q = G \cup_{u,v} \mathbb{P}_q$ then $M = A \oplus A_q$ is the adjacency matrix of G_q. Finally let $H = M + Q \cdot D_{\{u,v\}}$ denote the Hamiltonian of G_q together with the potential. Then $H = M_1 \oplus M_2$ where $M_1 = A + Q \cdot D_{\{u,v\}}$ and $M_2 = A_q$. It is well-known that the eigenvalues of $A_{q_{uv}} = A_{q-1}$ are $2\cos(j\pi/q)$ $(j = 1, 2, \ldots, q-1)$. It is also not hard to show that $\deg P_+^2 = \lceil (q+1)/2 \rceil$ and $\deg P_-^2 = \lfloor (q+1)/2 \rfloor$.

For any non-zero real number λ there is at most one prime p such that $\lambda = 2\cos(j\pi/(2p))$ for some $1 \le j \le 2p-1$, and since 0 is not an eigenvalue of A_{uv} by assumption. Thus if p is a sufficiently large prime number and $q = 2p$, then $A_{q_{uv}}$ and A_{uv} do not share any eigenvalues. Then, by Theorem 4.11 we find that

$$\deg P_+^H = \deg P_+^{M_1} + \deg P_+^{M_2} - 1 = \deg P_+^{H_G} + \lceil (2p+1)/2 \rceil - 1 = \deg P_+^{H_G} + p$$

and

$$\deg P_-^H = \deg P_-^{M_1} + \deg P_-^{M_2} - 1 = \deg P_-^{H_G} + \lfloor (2p+1)/2 \rfloor - 1 = \deg P_-^{H_G} + p - 1,$$

so $\deg P_+^H \neq \deg P_-^H$. At the same time $\operatorname{Tr} P_+^H - Q$ and $\operatorname{Tr} P_-^H - Q$ are both rational, and u and v are strongly cospectral and P_+^H and P_-^H are irreducible, as before. So by Theorem 2.11 there is PGST between u and v in G_q.

We can in fact remove the condition of Theorem 4.5 that 0 not be an eigenvalue of A_{uv} if we allow potential to be placed on vertices other than u and v (the two cospectral vertices). This is the content of the next two theorems.

Theorem 4.12. *Let G be a graph with $u, v \in V(G)$ cospectral. Let k be any odd integer and let $\mathbb{P}_k$ denote the path on q nodes, and call its endpoints u, v. Add a suitably chosen potential to every vertex of $\mathbb{P}_k$ so that $G \setminus u, v$ shares no eigenvalues with $\mathbb{P}_k \setminus u, v$. Create G' by gluing the path with potential to the nodes u and v. Then putting a transcendental potential Q on u and v induces PGST from u to v in G'.*

Proof. Adding a potential to every vertex of $\mathbb{P}_k$ simply adds a multiple of the identity to its adjacency matrix, so the eigenvalues shift by the amount of the potential. Thus clearly a potential can be chosen so that $G \setminus \{u, v\}$ and $\mathbb{P}_k \setminus \{u, v\}$ do not share any eigenvalues. Then the proof proceeds exactly as in the proof of Theorem 4.5 to show that there is PGST. □

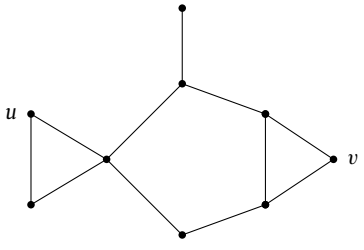
Theorem 4.13. *Let G be a graph with $u, v \in V(G)$ cospectral. Let k be any odd integer and let $\mathbb{P}_k$ denote the path on q nodes, and call its endpoints u, v. Denote its central vertex by w. Add a transcendental potential Q' to w and then create G' by gluing the path with this potential to the nodes u and v. Then putting a transcendental potential Q algebraically independent from Q' on u and v induces PGST from u to v in G'.*

Proof. By Lemma 3.4, Q' appears in $Tr(P_+)$ but not in $Tr(P_-)$, but Q' is algebraically independent from any other terms that could show up in the trace, so it must be that $Tr(P_+)$ and $Tr(P_-)$ are distinct. Then the theorem follows from Theorem 2.11. □

5. Examples, discussion, and further questions

Our results succeed in giving infinite families of graphs for which we can put a potential on the vertices to induce PGST between two vertices. Furthermore, the potential required can be assumed to be zero on most vertices of the graph. In addition, the examples produced do not require the symmetry condition of [15] and do not require the graph to be strongly regular as in [9]. We will examine some examples, including the graphs shown in the introduction.

Example 5.1. Let G be the graph below.



Direct computation can show that vertices u and v are cospectral in G (but *not* strongly cospectral). Putting a transcendental potential Q on u and v makes these vertices strongly cospectral by Lemma 3.2, and in fact P_+ and P_- have different degrees in this case, so this potential is enough to obtain pretty good state transfer. Gluing paths with an even number of vertices give an infinite family of graphs for which the potential induces pretty good state transfer, and each graph in this family does not have an automorphism mapping u to v.

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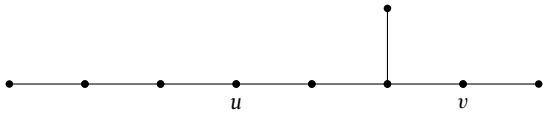
Note that we chose paths of even length simply because we know that these change the degree of P_+ and P_- by the same amount, and this graph already has $deg(P_+) \neq deg(P_-)$. We could in fact glue *any* graph with a pair of cospectral vertices as long as the resulting graph has P_+ and P_- with distinct degree or trace, and achieve a graph for which the potential induces pretty good state transfer.

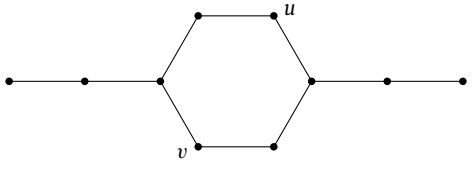
Example 5.2. Let G be the graph shown below.



Here, by direct computation, we have $deg(P_+) = deg(P_-)$ and $Tr(P_+) = Tr(P_-) = 0$, so in order for our results to give PGST, we need to use Theorem 4.12 or 4.13.

We pose the natural question: given any pair of cospectral vertices u and v, can we always induce PGST by a potential placed only on vertices u and v? We can answer this question in the negative with the following example.

Example 5.3. Consider the graph pictured below, with the vertices u, v as labeled.



Computation shows that $deg(P_+) = deg(P_-) = 5$ and $Tr(P_+) = Tr(P_-) = Q$, where Q is the value of the potential on u and v. So Theorem 2.11 is uninformative. But using Lemma 2.10 directly, since the degrees of P_+ and P_- are odd, we can simply take $\ell_i = 1$ for each i and $m_j = -1$ for each j, and we will have an integer linear combination of the eigenvalues equal to 0 with $\sum \ell_i$ and $\sum m_j$ odd. Thus, no matter what value of potential we put at u and v, there cannot be PGST between u and v.

The question remains open if we can induce PGST by putting potential on other vertices as well, since this could in theory change the degrees of P_+ and P_-.

To create an infinite family of graphs in which PGST occurs, we can glue paths to this graph via Theorem 4.5.

Note that this graph has an involution swapping u and v that fixes no vertices or edges (see [15]) and with an odd number of orbits. This is the only situation we are aware of where there is a cospectral pair, and we can prove that no potential on u and v can induce PGST. It is an open question if this is the only kind of such graphs.

A further question concerns the algebraic complexity of the potential necessary to induce PGST. In all of our results, we have used transcendental values of potential. This accomplishes two things: first, we can turn any pair of cospectral vertices into a strongly cospectral pair (Lemma 3.2), and further, this guarantees that P_+ and P_- are irreducible polynomials (Lemma 3.6; note that irreducibility is necessary to apply Theorem 2.11). However, the assumption of a transcendental potential is a drawback in terms of practical considerations. It is of interest to determine if simpler (algebraic, ideally rational) potentials might do as well. Another question of note is to determine if any of the examples where we can get PGST can also have PST.

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Appendix

Here we prove some lemmas used in the paper.

Lemma A.1. *The field trace map* $\mathrm{Tr}_{\mathcal{K}/\mathcal{F}} : \mathcal{K} \to \mathcal{F}$ *defined in the proof of Theorem 2.11 satisfies the following properties:*

- $\mathrm{Tr}_{\mathcal{K}/\mathcal{F}}$ *is an* $\mathcal{F}$*-linear map.*
- *For* $\alpha \in \mathcal{F}$, $\mathrm{Tr}_{\mathcal{K}/\mathcal{F}}(\alpha) = [\mathcal{K} : \mathcal{F}]\alpha$.

- *For $\mathcal{K} \geq \mathcal{L} \geq \mathcal{F}$ we have $\mathrm{Tr}_{\mathcal{K}/\mathcal{F}} = \mathrm{Tr}_{\mathcal{L}/\mathcal{F}} \circ \mathrm{Tr}_{\mathcal{K}/\mathcal{L}}$.*
- *If $\mathcal{L}/\mathcal{F}$ is the splitting field of a degree r irreducible polynomial over $\mathcal{F}$ with roots $\lambda_1, \ldots, \lambda_r$, then for any i*

$$Tr_{\mathcal{L}/\mathcal{F}}(\lambda_i) = \frac{[\mathcal{L} : \mathcal{F}]}{r} \sum_j \lambda_k.$$

Proof. For a field extension $\mathcal{K}$ of $\mathcal{F}$, recall the definition of the field trace, for $\alpha \in \mathcal{K}$ is

$$\mathrm{Tr}_{\mathcal{K}/\mathcal{F}}(\alpha) = \sum_{g \in Gal(\mathcal{K}/\mathcal{F})} g(\alpha).$$

The linearity over $\mathcal{F}$ is clear from the definition.

The second property follows since any automorphism in $\mathcal{K}/\mathcal{F}$ fixes any element of $\mathcal{F}$.

Recall from the Galois correspondence that $Aut(\mathcal{K}/\mathcal{L})$ is a subgroup of $Aut(\mathcal{K}/\mathcal{F})$. The third property follows from splitting the sum $\sum_{g \in Gal(\mathcal{K}/\mathcal{F})} g(\alpha)$ over cosets of $Aut(\mathcal{K}/\mathcal{F})$.

Finally, $\mathcal{L}$ is a Galois extension of $\mathcal{F}$ since it is the splitting field of an irreducible polynomial. For an individual root λ_i, we have

$$\mathrm{Tr}_{\mathcal{L}/\mathcal{F}}(\lambda_i) = \sum_{g \in Gal(\mathcal{K}/\mathcal{F})} g(\lambda_i)$$

and since $\mathcal{L}$ is a Galois extension, the group acts transitively on $\lambda_1, \ldots, \lambda_r$. Let $H \leq Gal(\mathcal{K}/\mathcal{F})$ be the subgroup that fixes λ_i. By the orbit–stabilizer theorem, the cosets of H are in bijection with elements of the orbit of λ_i, which is all of $\lambda_1, \ldots, \lambda_r$ since the action of the Galois group is transitive. This implies $|H| = |Gal(\mathcal{L}/\mathcal{F})|/r$. Let $g_1, \ldots, g_r$ be a set of coset representatives for H, then we can break up the sum as

$$\sum_{j=1}^{r} g_j \sum_{h \in H} h(\lambda_i) = \sum_{j=1}^{r} g_j(\lambda_i)|H| = \frac{[\mathcal{L} : \mathcal{F}]}{r} \sum_j \lambda_k. \quad \square$$

Lemma A.2. *Let M be any real symmetric $n \times n$ matrix, and let u be an index for M. Suppose λ is an eigenvalue of multiplicity at least k of both M and M_u. Then there are k linearly independent eigenvectors of M corresponding to λ that vanish at u.*

Proof. If λ has multiplicity strictly larger than k as an eigenvalue of M, then it is easy to see that we can adjust a basis for the eigenspace so that at least k of the corresponding eigenvectors vanish at u.

So let us suppose that the multiplicity of λ as an eigenvalue of M is exactly k, and as an eigenvalue of M_u is at least k. Let us denote by $\lambda_1 \leq \cdots \leq \lambda_n$ the eigenvalues of M, and by $\mu_1 \leq \cdots \leq \mu_{n-1}$ the eigenvalues of M_u. Then the interlacing theorem for symmetric matrices (see for example Theorem 4.3.8 of [12]) says we have

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

Then, given the assumption on the multiplicity of λ above, we have $\lambda_{j-1} < \lambda = \lambda_j = \lambda_{j+1} = \cdots \lambda_{j+k-1} < \lambda_{j+k}$. There are two possibilities for the μ-s:

$$\lambda = \mu_j = \cdots = \mu_{j+k-1} \qquad \text{or} \qquad \lambda = \mu_{j-1} = \cdots = \mu_{j+k-2}$$

We will consider the first possibility, the second one can be dealt with in a similar fashion. Let us choose an orthonormal basis $(\varphi_k)_{k=1}^n$ of eigenvectors of M in such a way that $\varphi_j, \varphi_{j+1}, \ldots, \varphi_{j+k-2}$ all vanish on u. This can be done since the multiplicity of λ is k and we are only asking for the first $k-1$ corresponding eigenvectors to vanish on u. Then, by the min–max principle, we have

$$\begin{aligned}
\lambda_{j+k-1} &= \min_{\substack{x \neq 0, x \in \mathbb{R}^n \\ x \perp \varphi_1, \ldots, \varphi_{j+k-2}}} \frac{x^T M x}{x^T x} \\
&\leq \min_{\substack{x \neq 0, x \in \mathbb{R}^n \\ x \perp \varphi_1, \ldots, \varphi_{j+k-2} \\ x(u)=0}} \frac{x^T M x}{x^T x} = \min_{\substack{x \neq 0, x \in \mathbb{R}^{n-1} \\ x \perp \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{j+k-2}}} \frac{x^T M_u x}{x^T x} \\
&\leq \max_{y_1, \ldots, y_{j+k-2} \in \mathbb{R}^{n-1}} \min_{\substack{x \neq 0, x \in \mathbb{R}^{n-1} \\ x \perp y_1, \ldots, y_{j+k-2} \\ x(u)=0}} \frac{x^T M_u x}{x^T x} = \mu_{j+k-1} = \lambda_{j+k-1}.
\end{aligned}$$

This implies that the first inequality has to be equality, so there is an x attaining the minimum that is orthogonal to $\varphi_1, \ldots, \varphi_{j+k-2}$ and for which $x(u) = 0$. This x then has to be an eigenvector with eigenvalue $\lambda_{j+k-1} = \lambda$, so we exhibited k pairwise orthogonal eigenvectors for λ vanishing on u.

The case when $\lambda = \mu_{j-1} = \cdots = \mu_{j+k-2}$ is done similarly, except we use the characterization of λ_j as a maximum, and we fix $\varphi_{j+1}, \ldots, \varphi_{j+k-1}$ to vanish on u. $\square$

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