ARTICLE IN PRESS

Discrete Mathematics xxx (xxxx) xxx

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the girth of two-dimensional real algebraically defined graphs

Allison J. Ganger [a], Shannon N. Golden [b], Brian G. Kronenthal [b,*], Carter A. Lyons [c]

[a] Allegheny College, Meadville, PA 16335, United States
[b] Kutztown University of Pennsylvania, Kutztown, PA 19530, United States
[c] Nebraska Wesleyan University, Lincoln, NE 68504, United States

ARTICLE INFO

Article history:
Received 14 February 2018
Accepted 31 October 2018
Available online xxxx

Keywords:
Algebraically defined graph
Cycle
Girth
Projective plane

ABSTRACT

For a polynomial $f \in \mathbb{R}[X, Y]$, we define a bipartite graph $\Gamma_{\mathbb{R}}(f)$ where each partite set is a copy of $\mathbb{R}^2$. Furthermore, (a_1, a_2) in the first partite set is adjacent to $[x_1, x_2]$ in the second if and only if $a_2 + x_2 = f(a_1, x_1)$. The main result of this paper is that every graph $\Gamma_{\mathbb{R}}(f)$ has girth 4 or 6, and moreover we classify infinite families of such graphs by girth.

1. Introduction

An algebraically defined graph $\Gamma_{\mathcal{R}}(f_2(X, Y), f_3(X, Y), \ldots, f_n(X, Y))$ is constructed using a ring $\mathcal{R}$ and functions $f_i(X, Y)$, where $2 \leq i \leq n$ for an integer $n \geq 2$. These graphs are bipartite where each partite set is a copy of $\mathcal{R}^n$. We label the vertices in the first partite set $(a_1, a_2, \ldots, a_n)$ and in the second $[x_1, x_2, \ldots, x_n]$. In order for two vertices to be adjacent, denoted $(a_1, a_2, \ldots, a_n) \sim [x_1, x_2, \ldots, x_n]$, their coordinates must satisfy the equations $a_i + x_i = f_i(a_1, x_1)$ for all i such that $2 \leq i \leq n$.

Considering $\Gamma_{\mathcal{R}}(f_2(X, Y), f_3(X, Y))$, Dmytrenko, Lazebnik, and Williford [2] studied the case where $\mathcal{R}$ is a finite field of odd order $\mathbb{F}_q$ and f_2 and f_3 are monomials. They conjectured that all such monomial graphs of girth at least 8 are isomorphic to $\Gamma_{\mathbb{F}_q}(XY, XY^2)$. This work was expanded upon by Kronenthal [4], and the conjecture was ultimately proven by Hou, Lappano, and Lazebnik [3]. In addition, Kronenthal and Lazebnik [5] and Kronenthal, Lazebnik, and Williford [6] studied graphs over algebraically closed fields of characteristic zero and applied some of their techniques to graphs over finite fields.

The study of two-dimensional algebraically defined graphs, and in particular their girth, can be motivated by the construction of projective planes. Indeed, it is known (see Dmytrenko [1] and Lazebnik and Thomason [7]) that every graph $\Gamma_{\mathbb{F}_q}(f)$ with girth greater than 4 can be completed to a projective plane of order q (although not all projective planes of order q can be constructed in this way). This construction further motivates the study of two-dimensional algebraically defined graphs $\Gamma_{\mathcal{R}}(f_2(X, Y))$ over $\mathcal{R} = \mathbb{R}$. For ease of notation, we call this graph $\Gamma_{\mathbb{R}}(f)$ with vertices of the form (a, a_2) and $[x, x_2]$. Our main result is as follows:

Theorem 1. *For all $f(X, Y) \in \mathbb{R}[X, Y]$, the girth of $\Gamma_{\mathbb{R}}(f)$ is either 4 or 6.*

* Corresponding author.
E-mail addresses: ganger.allison@huskers.unl.edu (A.J. Ganger), sgold940@live.kutztown.edu (S.N. Golden), kronenthal@kutztown.edu (B.G. Kronenthal), clyons@nebrwesleyan.edu (C.A. Lyons).

https://doi.org/10.1016/j.disc.2018.10.047
0012-365X/

Please cite this article as: A.J. Ganger, S.N. Golden, B.G. Kronenthal et al., On the girth of two-dimensional real algebraically defined graphs, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.10.047.

Moreover, we can explicitly state the girth of infinite families of algebraically defined graphs over $\mathbb{R}$. Before presenting our theorem, we need to make a few comments. We will use the notation $\mathbb{N}$ to represent the positive integers, with $2\mathbb{N} = \{2n \mid n \in \mathbb{N}\}$ and $2\mathbb{N}-1 = \{2n-1 \mid n \in \mathbb{N}\}$. Note that a polynomial $f(X,Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}X^iY^j \in \mathbb{R}[X,Y]$ can have only finitely many nonzero terms. In this paper, a term $\alpha_{i,j}X^iY^j$ is defined to be *mixed* when i and j are of opposite parity. Finally, a few conditions in Theorem 2 involve considering the minimum or maximum of a set; if the set is empty then that specific condition would not yield a girth 4 graph.

Theorem 2. *Let $f(X,Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}X^iY^j \in \mathbb{R}[X,Y]$. If i and j are odd for all nonzero $\alpha_{i,j}$, and all $\alpha_{i,j} \geq 0$ or all $\alpha_{i,j} \leq 0$, then $\Gamma_{\mathbb{R}}(f)$ has girth 6 (see Proposition 7). Moreover, $\Gamma_{\mathbb{R}}(f)$ has girth 4 in each of the following cases:*

1. *At least one of $\sum_{i,j\in\mathbb{N}} \alpha_{i,j}a^{i+j}$, $\sum_{i,j\in 2\mathbb{N}-1} \alpha_{i,j}a^{i+j}$, $\sum_{i,j\in\mathbb{N}} \alpha_{i,j}(-1)^i a^{i+j}$, $\sum_{i,j\in\mathbb{N}} \alpha_{i,j}(-1)^j a^{i+j}$, $\sum_{i\in 2\mathbb{N}-1}\sum_{j\in\mathbb{N}} \alpha_{i,j}a^{i+j}$, or $\sum_{i\in\mathbb{N}}\sum_{j\in 2\mathbb{N}-1} \alpha_{i,j}a^{i+j}$ is zero for some nonzero real number a.*
2. *At least one of $\max\{i \mid \alpha_{i,j} \neq 0$ for some $j\}$ or $\max\{j \mid \alpha_{i,j} \neq 0$ for some $i\}$ is even.*
3. *At least one of $\min\{i \mid \alpha_{i,j} \neq 0$ for some $j\}$ or $\min\{j \mid \alpha_{i,j} \neq 0$ for some $i\}$ is even.*
4. *At least one of $\min\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$, $\max\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$, $\min\left\{j \,\middle|\, \sum_{i\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$, or $\max\left\{j \,\middle|\, \sum_{i\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$ is even.*
5. (a) *Let $n = \min\{i \in 2\mathbb{N} \mid \alpha_{i,j} \neq 0$ for some $j\}$. There exists a nonzero term $\alpha_{n,j}X^nY^j$ that is mixed, and for $p < n$, all nonzero terms $\alpha_{p,j}X^pY^j$ are mixed.*
   (b) *Let $n = \min\{j \in 2\mathbb{N} \mid \alpha_{i,j} \neq 0$ for some $i\}$. There exists a nonzero term $\alpha_{i,n}X^iY^n$ that is mixed, and for $p < n$, all nonzero terms $\alpha_{i,p}X^iY^p$ are mixed.*
6. (a) *Let $k = \max\{i \mid \alpha_{i,j} \neq 0$ for some $j\}$ and $\ell = \min\{i \mid \alpha_{i,j} \neq 0$ for some $j\}$. Let $p = \max\{j \mid \alpha_{k,j} \neq 0\}$, $m = \min\{j \mid \alpha_{k,j} \neq 0\}$, $q = \max\{j \mid \alpha_{\ell,j} \neq 0\}$, and $n = \min\{j \mid \alpha_{\ell,j} \neq 0\}$. Either m and n are of opposite parity or p and q are of opposite parity.*
   (b) *Let $k = \max\{j \mid \alpha_{i,j} \neq 0$ for some $i\}$ and $\ell = \min\{j \mid \alpha_{i,j} \neq 0$ for some $i\}$. Let $p = \max\{i \mid \alpha_{i,k} \neq 0\}$, $m = \min\{i \mid \alpha_{i,k} \neq 0\}$, $q = \max\{i \mid \alpha_{i,\ell} \neq 0\}$, and $n = \min\{i \mid \alpha_{i,\ell} \neq 0\}$. Either m and n are of opposite parity or p and q are of opposite parity.*
7. (a) *The sums $\sum_{j\in\mathbb{N}} \alpha_{p,j}$ and $\sum_{j\in\mathbb{N}} \alpha_{q,j}$ have opposite signs, where $p = \min\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$ and $q = \max\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$.*
   (b) *The sums $\sum_{i\in\mathbb{N}} \alpha_{i,p}$ and $\sum_{i\in\mathbb{N}} \alpha_{i,q}$ have opposite signs, where $p = \min\left\{j \,\middle|\, \sum_{i\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$ and $q = \max\left\{j \,\middle|\, \sum_{i\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$.*

We will now briefly discuss how the rest of this paper is organized. In Section 2, we introduce important notation and tools that we will use throughout this paper. In Section 3, we prove Theorems 1 and 2. In Section 4, we apply some results to other rings and fields of characteristic zero. We end with a conjecture on the girth of algebraically defined graphs $\Gamma_{\mathbb{R}}(f)$ where $f \in \mathbb{R}[X,Y]$ is a trinomial.

2. Preliminary tools & notation

We will begin by discussing restrictions on the first coordinates in a 4-cycle. The following lemma is discussed on page 2 of [7].

Lemma 3. *Let $\Gamma_{\mathcal{R}}(f)$ contain a 4-cycle $(a,a_2) \sim [x,x_2] \sim (b,b_2) \sim [y,y_2] \sim (a,a_2)$. Then $a \neq b$ and $x \neq y$.*

We now state a necessary and sufficient condition for the existence of a 4-cycle $(a,a_2) \sim [x,x_2] \sim (b,b_2) \sim [y,y_2] \sim (a,a_2)$ in $\Gamma_{\mathcal{R}}(f)$.

Lemma 4 ([1]). *A 4-cycle exists in $\Gamma_{\mathcal{R}}(f)$ if and only if there exist $a,b,x,y \in \mathcal{R}$ such that $a \neq b$, $x \neq y$, and*

$$0 = f(a,x) - f(b,x) + f(b,y) - f(a,y). \quad (1)$$

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Since (1) appears repeatedly throughout this paper, we will introduce the following notation used, e.g., in [1,2,5,6]:

$$\Delta_2(f)(a,b;x,y) = f(a,x) - f(b,x) + f(b,y) - f(a,y).$$

Observe that when $f(X,Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j} X^i Y^j$,

$$\Delta_2(f)(a,b;x,y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}(a^i - b^i)(x^j - y^j).$$

Of particular interest, $\Delta_2(f)(a,b;x,y)$ depends only on the first coordinates of the vertices in the cycle. Moreover, note that there will be many 4-cycles with the same first coordinates as a given 4-cycle $(a,a_2) \sim (x,x_2) \sim (b,b_2) \sim (y,y_2) \sim (a,a_2)$, and we say that they are all of type $(a,b;x,y)$.

Lemmas 3 and 4 can be extended to a 6-cycle $(a,a_2) \sim [x,x_2] \sim (b,b_2) \sim [y,y_2] \sim (c,c_2) \sim [z,z_2] \sim (a,a_2)$, which is contained in $\Gamma_{\mathcal{R}}(f)$ if and only if there exist distinct a,b,c and distinct x,y,z such that

$$\Delta_3(f)(a,b,c;x,y,z) = f(a,x) - f(b,x) + f(b,y) - f(c,y) + f(c,z) - f(a,z) = 0.$$

Furthermore, when $f(X,Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j} X^i Y^j$,

$$\Delta_3(f)(a,b,c;x,y,z) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}\left[x^j(a^i - b^i) + y^j(b^i - c^i) + z^j(c^i - a^i)\right]. \tag{2}$$

Again, this expression depends only on the first coordinates of the vertices in the cycle. There will also be many 6-cycles with the same first coordinates, so we say that they are all of type $(a,b,c;x,y,z)$.

We end this section with the following isomorphisms of the graph $\Gamma_{\mathbb{F}}(f)$, where $\mathbb{F}$ is a field; see, e.g., [7] for proofs. First note that for a function $f = f(X,Y)$, we define $f^* = f(Y,X)$.

Lemma 5. *Let $\mathbb{F}$ be a field and $f \in \mathbb{F}[X,Y]$. Then*

$$\Gamma_{\mathbb{F}}(f) \cong \Gamma_{\mathbb{F}}(f^*), \tag{$\mathcal{I}_1$}$$

$$\Gamma_{\mathbb{F}}(f) \cong \Gamma_{\mathbb{F}}(cf), \text{ for all } c \in \mathbb{F}\setminus\{0\}, \text{ and} \tag{$\mathcal{I}_2$}$$

$$\Gamma_{\mathbb{F}}(f) \cong \Gamma_{\mathbb{F}}(f+g+h), \text{ for all } g \in \mathbb{F}[X] \text{ and } h \in \mathbb{F}[Y]. \tag{$\mathcal{I}_3$}$$

We will use ($\mathcal{I}_1$) to assume, when analyzing terms of a polynomial, a given condition applies to X instead of Y. By ($\mathcal{I}_2$), we can choose a leading coefficient by multiplying through by an appropriate c-value. Finally, as a consequence of ($\mathcal{I}_3$), we are able to assume that every term in our polynomial is a scalar multiple of X^iY^j for $i,j \in \mathbb{N}$.

3. Proofs of Theorems 1 and 2

We begin with a lemma used frequently throughout the remainder of this paper.

Lemma 6. *Let $f(X,Y) \in \mathbb{R}[X,Y]$. If there exists a real number $m \neq 0$ such that at least one of $f(X,m)$ and $f(m,Y)$ is not injective, then $\Gamma_{\mathbb{R}}(f)$ has girth 4.*

Proof. Let $f(X,Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j} X^i Y^j \in \mathbb{R}[X,Y]$. Without loss of generality by ($\mathcal{I}_1$), we will assume there exists a real number $m \neq 0$ such that $f(X,m)$ is not injective. Then we can fix $a \neq b$ such that $f(a,m) = f(b,m)$, and so

$$\Delta_2(f)(a,b;m,0) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}(a^i - b^i)m^j = f(a,m) - f(b,m) = 0.$$

Hence, by Lemma 4, a 4-cycle exists, and so $\Gamma_{\mathbb{R}}(f)$ has girth 4. □

In addition to Lemma 6, another tool we will use in this section is end behavior. In other words, for a univariate polynomial $f(X)$ of even degree (respectively odd degree), $\lim_{X\to\infty} f(X) = \lim_{X\to-\infty} f(X) = \pm\infty$ (respectively $\lim_{X\to\infty} f(X) = -\lim_{X\to-\infty} f(X) = \pm\infty$).

We will prove Theorem 2 by considering its seven parts individually; we will denote part 1 by Theorem 2.1, and so on. To begin, we will consider when particular combinations of the polynomial's coefficients sum to zero.

Theorem 2.1. *Let* $f(X,Y)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j\in\mathbb{R}[X,Y]$*. If at least one of* $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}a^{i+j}$, $\sum_{i,j\in 2\mathbb{N}-1}\alpha_{i,j}a^{i+j}$, $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^ia^{i+j}$, $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^ja^{i+j}$, $\sum_{i\in 2\mathbb{N}-1}\sum_{j\in\mathbb{N}}\alpha_{i,j}a^{i+j}$*, or* $\sum_{i\in\mathbb{N}}\sum_{j\in 2\mathbb{N}-1}\alpha_{i,j}a^{i+j}$ *is zero for some nonzero real number a, then* $\Gamma_{\mathbb{R}}(f)$ *has girth 4.*

Proof. Let $f(X,Y)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j\in\mathbb{R}[X,Y]$ such that $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}a^{i+j}=0$. Then, by Lemma 4, the graph $\Gamma_{\mathbb{R}}(f)$ contains a 4-cycle of type $(a,0;a,0)$ since

$$\Delta_2(f)(a,0;a,0)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(a^i-0^i)(a^j-0^j)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}a^{i+j}=0.$$

Similarly, in the cases where $f(X,Y)$ is a polynomial such that at least one of $\sum_{i,j\in 2\mathbb{N}-1}\alpha_{i,j}a^{i+j}$, $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^ia^{i+j}$, $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^ja^{i+j}$, $\sum_{i\in 2\mathbb{N}-1}\sum_{j\in\mathbb{N}}\alpha_{i,j}a^{i+j}$, or $\sum_{i\in\mathbb{N}}\sum_{j\in 2\mathbb{N}-1}\alpha_{i,j}a^{i+j}$ is zero for some nonzero real number a, the graphs $\Gamma_{\mathbb{R}}(f)$ contain a 4-cycle of type $(a,-a;a,-a)$, $(-a,0;a,0)$, $(a,0;-a,0)$, $(a,-a;a,0)$, or $(a,0;a,-a)$, respectively, and hence have girth 4. □

Note that the sums in Theorem 2.1 are a few of many combinations we could have chosen; however, we did not discuss more general sums because in so doing, we would essentially be considering our original condition (1). Moreover, the results of Theorem 2.1 are especially easy to apply when we specify $a=\pm 1$. In particular, if f is chosen so that at least one of $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}$, $\sum_{i,j\in 2\mathbb{N}-1}\alpha_{i,j}$, $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^i$, $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^j$, $\sum_{i\in 2\mathbb{N}-1}\sum_{j\in\mathbb{N}}\alpha_{i,j}$, or $\sum_{i\in\mathbb{N}}\sum_{j\in 2\mathbb{N}-1}\alpha_{i,j}$ is zero, then $\Gamma_{\mathbb{R}}(f)$ has girth 4.

We next consider families of polynomials in which the largest (Theorem 2.2) or smallest (Theorem 2.3) exponent with respect to X or with respect to Y is even.

Theorem 2.2. *Let* $f(X,Y)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j\in\mathbb{R}[X,Y]$*. If* $\max\{i\mid\alpha_{i,j}\neq 0 \text{ for some } j\}$ *is even or* $\max\{j\mid\alpha_{i,j}\neq 0\text{ for some } i\}$ *is even, then* $\Gamma_{\mathbb{R}}(f)$ *has girth 4.*

Proof. Let $f(X,Y)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j\in\mathbb{R}[X,Y]$ where, without loss of generality by $(\mathcal{I}_1)$, $n=\max\{i\mid\alpha_{i,j}\neq 0\text{ for some } j\}$ is even. Note that there exists a natural number k such that $f(X,Y)$ contains the term $X^n\sum_{j=1}^{k}\alpha_{n,j}Y^j$ with not all $\alpha_{n,j}=0$ and $\alpha_{n,\ell}=0$ for all $\ell>k$. Then there exists $m\in\mathbb{R}$ such that $\sum_{j=1}^{k}\alpha_{n,j}m^j$ is nonzero. Since the degree of $f(X,m)$ is even, end behavior implies that $f(X,m)$ is not injective. Hence, $\Gamma_{\mathbb{R}}(f)$ has girth 4 by Lemma 6. □

Theorem 2.3. *Let* $f(X,Y)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j\in\mathbb{R}[X,Y]$*. If* $\min\{i\mid\alpha_{i,j}\neq 0\text{ for some } j\}$ *is even or* $\min\{j\mid\alpha_{i,j}\neq 0\text{ for some } i\}$ *is even, then* $\Gamma_{\mathbb{R}}(f)$ *has girth 4.*

Proof. Let $f(X,Y)\in\mathbb{R}[X,Y]$. Without loss of generality by $(\mathcal{I}_1)$, we will assume $m=\min\{i\mid\alpha_{i,j}\neq 0\text{ for some } j\}$ is even.

Let $n=\max\{i\mid\alpha_{i,j}\neq 0\text{ for some } j\}$; thus $f(X,Y)=\sum_{i=m}^{n}\sum_{j\in\mathbb{N}}\alpha_{i,j}X^iY^j$. Choose $x\neq 0$ to be a real number such that $\sum_{j\in\mathbb{N}}\alpha_{m,j}x^j$ is nonzero. Then, for that value of x,

$$\begin{aligned}\Delta_2(f)(a,b;x,0)&=\sum_{i=m}^{n}\sum_{j\in\mathbb{N}}\alpha_{i,j}(a^i-b^i)x^j\\&=g(a)-g(b),\end{aligned}$$

where we define $g(a)=\sum_{i=m}^{n}\sum_{j\in\mathbb{N}}\alpha_{i,j}a^ix^j$. Observe that

$$g'(a)=\sum_{i=m}^{n}\sum_{j\in\mathbb{N}}\alpha_{i,j}ia^{i-1}x^j=a^{m-1}\sum_{i=m}^{n}\sum_{j\in\mathbb{N}}\alpha_{i,j}ia^{i-m}x^j.$$

By our definition of x, $\sum_{j\in\mathbb{N}} \alpha_{m,j}a^m x^j$ and $\sum_{j\in\mathbb{N}} \alpha_{m,j}ma^{m-1}x^j$ are nonzero for all $a \neq 0$. Since m is even, $g'(a)$ has the root $a = 0$ of odd multiplicity. Therefore, $g(a)$ has a local extremum and is not injective. Thus, by Lemma 6, $\Gamma_{\mathbb{R}}(f)$ has girth 4. □

Theorems 2.2 and 2.3 immediately imply the following:

Theorem 2.4. *Let* $f(X, Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}X^iY^j \in \mathbb{R}[X, Y]$. *If at least one of* $\min\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$, $\max\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$, $\min\left\{j \,\middle|\, \sum_{i\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$, *or* $\max\left\{j \,\middle|\, \sum_{i\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$ *is even, then* $\Gamma_{\mathbb{R}}(f)$ *has girth* 4.

Proof. By $(\mathcal{I}_1)$, we may assume either $\min\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$ or $\max\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$ is even. Then this proof follows directly from Theorem 2.2 or Theorem 2.3 using a cycle of type $(a, b; 1, 0)$. □

Recall that we defined a term $\alpha_{i,j}X^iY^j$ to be *mixed* when i and j are of opposite parity.

Theorem 2.5. *Let* $f(X, Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}X^iY^j \in \mathbb{R}[X, Y]$. *If either of the following hold then* $\Gamma_{\mathbb{R}}(f)$ *has girth* 4.

(a) *Let* $n = \min\{i \in 2\mathbb{N} \mid \alpha_{i,j} \neq 0 \text{ for some } j\}$. *There exists a nonzero term* $\alpha_{n,j}X^nY^j$ *that is mixed, and for* $p < n$, *all nonzero terms* $\alpha_{p,j}X^pY^j$ *are mixed.*
(b) *Let* $n = \min\{j \in 2\mathbb{N} \mid \alpha_{i,j} \neq 0 \text{ for some } i\}$. *There exists a nonzero term* $\alpha_{i,n}X^iY^n$ *that is mixed, and for* $p < n$, *all nonzero terms* $\alpha_{i,p}X^iY^p$ *are mixed.*

Proof. Without loss of generality by $(\mathcal{I}_1)$, we will only prove (a). To satisfy the criteria in the theorem, we represent f as the sum of three polynomials such that the degree of X in f is greater than, equal to, or less than $n = \min\{i \in 2\mathbb{N} \mid \alpha_{i,j} \neq 0 \text{ for some } j\}$, respectively. Thus, ordering from largest to smallest by the degree of X, we write

$$f(X, Y) = X^\ell \sum_{j\in\mathbb{N}} \alpha_{\ell,j}Y^j + \cdots + X^{n+1}\sum_{j\in\mathbb{N}} \alpha_{n+1,j}Y^j + X^n\sum_{j\in\mathbb{N}} \alpha_{n,j}Y^j + X^{n-1}\sum_{j\in 2\mathbb{N}} \alpha_{n-1,j}Y^j + \cdots + X\sum_{j\in 2\mathbb{N}} \alpha_{1,j}Y^j.$$

Note that $j \in 2\mathbb{N}$ in all terms $\alpha_{p,j}X^pY^j$ with $p < n$ due to (a).

Choose a real value $x \neq 0$ such that $\sum_{j\in 2\mathbb{N}-1} \alpha_{n,j}x^j$ is nonzero. Observe,

$$\begin{aligned}
\Delta_2(f)(a, b; x, -x) &= \left(a^\ell - b^\ell\right)\left(\sum_{j\in 2\mathbb{N}-1} \alpha_{\ell,j}\left(x^j - (-x)^j\right) + \sum_{j\in 2\mathbb{N}} \alpha_{\ell,j}\left(x^j - (-x)^j\right)\right) \\
&\quad + \cdots + \left(a^n - b^n\right)\left(\sum_{j\in 2\mathbb{N}-1} \alpha_{n,j}\left(x^j - (-x)^j\right) + \sum_{j\in 2\mathbb{N}} \alpha_{n,j}\left(x^j - (-x)^j\right)\right) \\
&\quad + \cdots + (a - b)\sum_{j\in 2\mathbb{N}} \alpha_{1,j}\left(x^j - (-x)^j\right) \\
&= \left(a^\ell - b^\ell\right)\left(\sum_{j\in 2\mathbb{N}-1} \alpha_{\ell,j}\left(2x^j\right) + \sum_{j\in 2\mathbb{N}} \alpha_{\ell,j}\,(0)\right) \\
&\quad + \cdots + \left(a^n - b^n\right)\left(\sum_{j\in 2\mathbb{N}-1} \alpha_{n,j}\left(2x^j\right) + \sum_{j\in 2\mathbb{N}} \alpha_{n,j}\,(0)\right) \\
&\quad + \cdots + (a - b)\sum_{j\in 2\mathbb{N}} \alpha_{1,j}\,(0) \\
&= 2\left(\left(a^\ell - b^\ell\right)\sum_{j\in 2\mathbb{N}-1} \alpha_{\ell,j}x^j + \cdots + \left(a^n - b^n\right)\sum_{j\in 2\mathbb{N}-1} \alpha_{n,j}x^j\right) \\
&= g(a) - g(b),
\end{aligned}$$

where $g(a) = 2\left(a^{\ell}\sum_{j\in 2\mathbb{N}-1}\alpha_{\ell,j}x^j + \cdots + a^n \sum_{j\in 2\mathbb{N}-1}\alpha_{n,j}x^j\right)$. Since n is even, the root $a = 0$ of $g(a)$ is of even multiplicity. Thus, $g(a)$ has a local extremum and is not injective. Therefore, by Lemmas 4 and 6, $\Gamma_{\mathbb{R}}(f)$ has girth 4. □

An example of a polynomial covered by the previous theorem is $f(X, Y) = 7X^9Y^{11} + 3X^8Y - 4X^6Y^3 + 5X^5Y^4$. This polynomial satisfies both conditions from Theorem 2.5, although that is not necessary: we only require that a polynomial satisfy *one* of these conditions. If we consider (a), then $n = 6$, whereas if we consider (b), then $n = 4$.

The following theorem accounts for polynomials such as $f(X, Y) = X^3Y^7+X^3Y^5+X^2Y^{11}+XY^6$. Note, the largest exponent with respect to X is 3. Furthermore, considering the X^3 terms, the largest Y exponent is 7. Also, the lowest exponent with respect to X is 1 with a corresponding largest Y exponent of 6. Since these Y exponents have opposite parity, Theorem 2.6 will imply that $\Gamma_{\mathbb{R}}(f)$ has girth 4.

Theorem 2.6. *If either m and n or p and q are of opposite parity in the cases below, then $\Gamma_{\mathbb{R}}(f)$ has girth 4.*

(a) Let $f(X, Y) = X^k\left(\alpha_{k,p}Y^p + \cdots + \alpha_{k,m}Y^m\right) + \cdots + X^{\ell}\left(\alpha_{\ell,q}Y^q + \cdots + \alpha_{\ell,n}Y^n\right) \in \mathbb{R}[X, Y]$ where $k = \max\{i \mid \alpha_{i,j} \neq 0 \text{ for some } j\}$ and $\ell = \min\{i \mid \alpha_{i,j} \neq 0 \text{ for some } j\}$. Define $p = \max\{j \mid \alpha_{k,j} \neq 0\}$, $m = \min\{j \mid \alpha_{k,j} \neq 0\}$, $q = \max\{j \mid \alpha_{\ell,j} \neq 0\}$, and $n = \min\{j \mid \alpha_{\ell,j} \neq 0\}$.

(b) Let $f(X, Y) = Y^k\left(\alpha_{p,k}X^p + \cdots + \alpha_{m,k}X^m\right) + \cdots + Y^{\ell}\left(\alpha_{q,\ell}X^q + \cdots + \alpha_{n,\ell}X^n\right) \in \mathbb{R}[X, Y]$ where $k = \max\{j \mid \alpha_{i,j} \neq 0 \text{ for some } i\}$ and $\ell = \min\{j \mid \alpha_{i,j} \neq 0 \text{ for some } i\}$. Define $p = \max\{i \mid \alpha_{i,k} \neq 0\}$, $m = \min\{i \mid \alpha_{i,k} \neq 0\}$, $q = \max\{i \mid \alpha_{i,\ell} \neq 0\}$, and $n = \min\{i \mid \alpha_{i,\ell} \neq 0\}$.

Proof. By $(\mathcal{I}_1)$, we will only prove (a). Let $f(X, Y) = X^k\left(\alpha_{k,p}Y^p + \cdots + \alpha_{k,m}Y^m\right)+\cdots+X^{\ell}\left(\alpha_{\ell,q}Y^q + \cdots + \alpha_{\ell,n}Y^n\right) \in \mathbb{R}[X, Y]$ where k, ℓ, p, m, q, and n are as defined in (a). Furthermore, suppose that either m and n or p and q are of opposite parity. By Theorems 2.2 and 2.3, if k or ℓ is even, then $\Gamma_{\mathbb{R}}(f)$ has girth 4. Thus, we will consider when k and ℓ are odd. Note that

$$\Delta_2(f)(a, 0; x, 0) = a^{\ell}\left(a^{k-\ell}\sum_{j=m}^{p}\alpha_{k,j}x^j + \cdots + \sum_{j=n}^{q}\alpha_{\ell,j}x^j\right).$$

Therefore, if

$$a^{k-\ell}\sum_{j=m}^{p}\alpha_{k,j}x^j + \cdots + \sum_{j=n}^{q}\alpha_{\ell,j}x^j = 0,$$

then by Lemma 4, a 4-cycle of type $(a, 0; x, 0)$ exists.

We claim that there exists a real value $x \neq 0$ such that $\sum_{j=m}^{p}\alpha_{k,j}x^j$ and $\sum_{j=n}^{q}\alpha_{\ell,j}x^j$ are of opposite signs. Indeed, when p and q are of opposite parity, the existence of such a value x follows by end behavior. When m and n are of opposite parity, then $x = 0$ is a root with multiplicities of opposite parity in $\sum_{j=m}^{p}\alpha_{k,j}x^j$ and $\sum_{j=n}^{q}\alpha_{\ell,j}x^j$, so there exists some $x \in \mathbb{R}$ such that the sums are of opposite signs. Hence, our claim is proven.

Choose x as described in the previous paragraph. Now let $g(a) = a^{k-\ell}\sum_{j=m}^{p}\alpha_{k,j}x^j + \cdots + \sum_{j=n}^{q}\alpha_{\ell,j}x^j$. Note that $\lim_{a\to\infty} g(a)$ and $g(0) = \sum_{j=n}^{q}\alpha_{\ell,j}x^j$ are of opposite signs. Therefore, by the Intermediate Value Theorem there exists an $a \neq 0$ such that $g(a) = 0$, and so a 4-cycle exists. Thus, $\Gamma_{\mathbb{R}}(f)$ has girth 4. □

The following theorem addresses the final family of girth 4 algebraically defined graphs in Theorem 2.

Theorem 2.7. *Let $f(X, Y) = \sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j \in \mathbb{R}[X, Y]$. If either of the following hold, then $\Gamma_{\mathbb{R}}(f)$ has girth 4.*

(a) For $p = \min\left\{i \,\middle|\, \sum_{j\in\mathbb{N}}\alpha_{i,j} \neq 0\right\}$ and $q = \max\left\{i \,\middle|\, \sum_{j\in\mathbb{N}}\alpha_{i,j} \neq 0\right\}$, $\sum_{j\in\mathbb{N}}\alpha_{p,j}$ and $\sum_{j\in\mathbb{N}}\alpha_{q,j}$ have opposite signs.

(b) For $p = \min\left\{j \,\middle|\, \sum_{i\in\mathbb{N}}\alpha_{i,j} \neq 0\right\}$ and $q = \max\left\{j \,\middle|\, \sum_{i\in\mathbb{N}}\alpha_{i,j} \neq 0\right\}$, $\sum_{i\in\mathbb{N}}\alpha_{i,p}$ and $\sum_{i\in\mathbb{N}}\alpha_{i,q}$ have opposite signs.

Please cite this article as: A.J. Ganger, S.N. Golden, B.G. Kronenthal et al., On the girth of two-dimensional real algebraically defined graphs, Discrete Mathematics (2018), https://doi.org/10.1016/j.disc.2018.10.047.

Proof. By $(\mathcal{I}_1)$ we will only prove (a). Let $f(X,Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}X^iY^j \in \mathbb{R}[X,Y]$ where $p = \min\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$ and $q = \max\left\{i \,\middle|\, \sum_{j\in\mathbb{N}} \alpha_{i,j} \neq 0\right\}$. Also let $\sum_{j\in\mathbb{N}} \alpha_{p,j}$ and $\sum_{j\in\mathbb{N}} \alpha_{q,j}$ have opposite signs. Note that

$$\begin{aligned}\Delta_2(f)(a,b;1,0) &= \sum_{i,j\in\mathbb{N}} \alpha_{i,j}(a^i - b^i)\\ &= \sum_{i=p}^{q}\sum_{j\in\mathbb{N}} \alpha_{i,j}(a^i - b^i).\end{aligned}$$

If p or q is even, by Theorem 2.4, $\Gamma_{\mathbb{R}}(f)$ has girth 4. So, we will consider when p and q are odd. Now, let $b = -a$. Then

$$\begin{aligned}\Delta_2(f)(a,-a;1,0) &= \sum_{i=p}^{q}\sum_{j\in\mathbb{N}} \alpha_{i,j}(a^i - (-a)^i)\\ &= 2\sum_{\substack{p\le i\le q\\ i \text{ odd}}}\sum_{j\in\mathbb{N}} \alpha_{i,j}a^i.\end{aligned}$$

Hence, by Lemma 4, if there exists a real value $a \neq 0$ such that

$$\sum_{\substack{p\le i\le q\\ i \text{ odd}}}\sum_{j\in\mathbb{N}} \alpha_{i,j}a^{i-p} = 0,$$

then $\Gamma_{\mathbb{R}}(f)$ contains a 4-cycle of type $(a,-a;1,0)$. Let

$$g(a) = \sum_{\substack{p\le i\le q\\ i \text{ odd}}}\sum_{j\in\mathbb{N}} \alpha_{i,j}a^{i-p}.$$

Note, $\lim_{a\to\infty} g(a)$ and $g(0) = \sum_{j\in\mathbb{N}} \alpha_{p,j}$ are of opposite signs. Therefore, by the Intermediate Value Theorem, there exists some real value $a \neq 0$ such that $g(a) = 0$. Thus, the graph $\Gamma_{\mathbb{R}}(f)$ has girth 4. □

Previously, all of our theorems addressed algebraically defined graphs of girth 4. Now, we will present a family of graphs having girth 6.

Proposition 7. *Let $f(X,Y) = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}X^iY^j \in \mathbb{R}[X,Y]$ be a nonzero polynomial such that i and j are odd for all nonzero $\alpha_{i,j}$. If all $\alpha_{i,j} \geq 0$ or all $\alpha_{i,j} \leq 0$, then $\Gamma_{\mathbb{R}}(f)$ has girth 6.*

Proof. Let $f(X,Y) = \sum_{i,j\in 2\mathbb{N}-1} \alpha_{i,j}X^iY^j \in \mathbb{R}[X,Y]$ be a nonzero polynomial. We will assume without loss of generality by $(\mathcal{I}_2)$ that every nonzero $\alpha_{i,j}$ is positive.

First, we will prove $\Gamma_{\mathbb{R}}(f)$ does not contain any 4-cycles. Note that

$$\Delta_2(f)(a,b;x,y) = \sum_{i,j\in 2\mathbb{N}-1} \alpha_{i,j}(a^i - b^i)(x^j - y^j).$$

Since $i,j \in 2\mathbb{N}-1$, if $a > b$ and $x > y$ or $a < b$ and $x < y$, then every $\alpha_{i,j}(a^i - b^i)(x^j - y^j)$ is positive. Likewise, if $a < b$ and $x > y$ or $a > b$ and $x < y$, then every $\alpha_{i,j}(a^i - b^i)(x^j - y^j)$ is negative. Thus, $\Delta_2(f)(a,b;x,y) = \sum_{i,j\in 2\mathbb{N}-1} \alpha_{i,j}(a^i - b^i)(x^j - y^j) \neq 0$ for all $a \neq b$ and $x \neq y$, and so $\Gamma_{\mathbb{R}}(f)$ does not contain a 4-cycle.

Now, note the graph $\Gamma_{\mathbb{R}}(f)$ contains a 6-cycle of type $(-1,1,0;0,1,-1)$, and therefore has girth 6. □

Combined, the above results prove Theorem 2. We will now prove Theorem 1.

Theorem 1. *For all $f(X,Y) \in \mathbb{R}[X,Y]$, the girth of $\Gamma_{\mathbb{R}}(f)$ is either 4 or 6.*

Proof. Let $f(X,Y)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j\in\mathbb{R}[X,Y]$. Note by (2),

$$\begin{aligned}\Delta_3(f)(0,1,-1;x,0,1)&=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}\left[x^j\left(0^i-1^i\right)+0^j\left(1^i-(-1)^i\right)+1^j\left((-1)^i-0^i\right)\right]\\&=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}\left[-x^j+(-1)^i\right].\end{aligned}$$

Therefore, by Lemma 4, if we can find some real value $x\notin\{0,1\}$ such that

$$\sum_{i,j\in\mathbb{N}}\alpha_{i,j}\left[-x^j+(-1)^i\right]=0,$$

or equivalently $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}x^j=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^i$, then a 6-cycle of type $(0,1,-1;x,0,1)$ exists. Note that $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}x^j=\sum_{j=m}^{n}\sum_{i\in\mathbb{N}}\alpha_{i,j}x^j$, where $m=\min\left\{j\,\middle|\,\sum_{i\in\mathbb{N}}\alpha_{i,j}\neq 0\right\}$ and $n=\max\left\{j\,\middle|\,\sum_{i\in\mathbb{N}}\alpha_{i,j}\neq 0\right\}$. If n is even, then $\Gamma_{\mathbb{R}}(f)$ has girth 4 by Theorem 2.4.

Therefore, we will only consider when n is odd, in which case end behavior and the Intermediate Value Theorem imply the existence of a real number x such that

$$\sum_{i,j\in\mathbb{N}}\alpha_{i,j}x^j=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^i.$$

If $x\neq 0$ and $x\neq 1$, then this x produces a cycle of type $(0,1,-1;x,0,1)$. We will now demonstrate that $\Gamma_{\mathbb{R}}(f)$ has girth 4 when $x=0$ or $x=1$.

First consider when $x=0$. Then

$$0=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^i.$$

By Theorem 2.1, graphs with a polynomial satisfying this condition have girth 4.

Now, consider when $x=1$. Then

$$\sum_{i,j\in\mathbb{N}}\alpha_{i,j}=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^i.$$

Since all $\alpha_{i,j}$ with i even cancel, this yields $\sum_{i\in 2\mathbb{N}-1}\sum_{j\in\mathbb{N}}\alpha_{i,j}=0$. Algebraically defined graphs with polynomials satisfying this condition have girth 4 by Theorem 2.1.

Hence, in all cases, $\Gamma_{\mathbb{R}}(f)$ has girth either 4 or 6. $\square$

4. Concluding remarks

Some results from Section 3 can be extended to $\Gamma_{\mathcal{R}}(f)$ where $\mathcal{R}\neq\mathbb{R}$. In fact, we can extend Theorem 2.1 and Proposition 7 to the following two corollaries.

Corollary 8. *Let $\mathcal{R}$ be $\mathbb{Q}$ or $\mathbb{Z}$. Let $f(X,Y)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j\in\mathcal{R}[X,Y]$.*

1. *If at least one of $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}a^{i+j}$, $\sum_{i,j\in 2\mathbb{N}-1}\alpha_{i,j}a^{i+j}$, $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^ia^{i+j}$, $\sum_{i,j\in\mathbb{N}}\alpha_{i,j}(-1)^ja^{i+j}$, $\sum_{i\in 2\mathbb{N}-1}\sum_{j\in\mathbb{N}}\alpha_{i,j}a^{i+j}$, or $\sum_{i\in\mathbb{N}}\sum_{j\in 2\mathbb{N}-1}\alpha_{i,j}a^{i+j}$ is zero for some nonzero $a\in\mathcal{R}$, then $\Gamma_{\mathcal{R}}(f)$ has girth 4.*
2. *If $i,j\in 2\mathbb{N}-1$ for all nonzero $\alpha_{i,j}$, and all $\alpha_{i,j}\geq 0$ or all $\alpha_{i,j}\leq 0$, then $\Gamma_{\mathcal{R}}(f)$ has girth 6.*

Corollary 9. *Let $f(X,Y)=\sum_{i,j\in\mathbb{N}}\alpha_{i,j}X^iY^j\in k\mathbb{Z}[X,Y]$ for $k\in\mathbb{N}$.*

1. *If at least one of i or j is even for all nonzero $\alpha_{i,j}$, then $\Gamma_{k\mathbb{Z}}(f)$ has girth 4.*
2. *If $i,j\in 2\mathbb{N}-1$ for all nonzero $\alpha_{i,j}$, and all $\alpha_{i,j}\geq 0$ or all $\alpha_{i,j}\leq 0$, then $\Gamma_{k\mathbb{Z}}(f)$ has girth 6.*

Note that Theorems 2.2, 2.3, 2.7 and Proposition 7 account for all algebraically defined graphs when $f(X,Y)$ is a binomial with real coefficients. In other words, given any binomial $f(X,Y)$, we can determine whether $\Gamma_{\mathbb{R}}(f)$ has girth 4 or girth 6. However, these results do not account for every trinomial $f(X,Y)$. Investigating trinomials using techniques similar to those employed previously in this paper (see e.g., Theorem 2.3) informs the following conjecture:

Conjecture 10. *Let* $f(X,Y) = \alpha_{k,\ell}X^kY^\ell + \alpha_{m,n}X^mY^n + \alpha_{p,q}X^pY^q \in \mathbb{R}[X,Y]$ *be a trinomial where* $\ell \geq n \geq q$*. There are three cases to consider.*

1. *Let* $\ell \neq q$ *and* $n \neq q$*. If there exists some* $a \in \mathbb{R}$ *such that*

$$\ell a^{\left(\frac{m-k}{\ell-q}+k\right)}\delta^{\ell-q} + n\alpha_{m,n}a^{\left(\frac{m-k}{n-q}+m\right)}\delta^{n-q} + q\alpha_{p,q}a^p < 0,$$

where

$$\delta = \frac{-n(n-q)}{\ell(\ell-q)},$$

then $\Gamma_{\mathbb{R}}(f)$ *has girth* 4*; otherwise it has girth* 6.

2. *Let* $\ell \neq q$, $n = q$*. If there exists some* $a \in \mathbb{R}$ *such that*

$$a^k\delta^\ell + (\alpha_{m,n}a^m + \alpha_{p,q}a^p)\delta^n < 0,$$

where

$$\delta = \left(\frac{-n\left(\alpha_{m,n}a^{m-k} + \alpha_{p,q}a^{p-k}\right)}{\ell}\right)^{\frac{1}{\ell-n}},$$

then $\Gamma_{\mathbb{R}}(f)$ *has girth* 4*; otherwise it has girth* 6.

3. *Let* $\ell = n = q$*. Now* $f(X,Y) = Y^\ell(\alpha_{k,\ell}X^k + \alpha_{m,n}X^m + \alpha_{p,q}X^p)$ *where* $k \geq m \geq p$ *and* $k, p \in 2\mathbb{N}-1$*. If there exists some* $a \in \mathbb{R}$ *such that*

$$ka^{k-q} + m\alpha_{m,n}a^{m-q} + p\alpha_{p,q} < 0,$$

then $\Gamma_{\mathbb{R}}(f)$ *has girth* 4*; otherwise it has girth* 6.

Notice that in part 3, we restricted $k, p \in 2\mathbb{N}-1$. This is because if either k or p is even, these cases were already covered in Theorems 2.2 and 2.3, respectively.

Acknowledgments

We are grateful to Felix Lazebnik for insightful conversations and input on the topic of this paper. We also thank the reviewers for their thoughtful comments and suggestions. This project has been supported by NSF #1560222, REU Site: Undergraduate Research in Mathematics, Applied Mathematics, and Statistics at Lafayette College.

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