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On a problem of Neumann

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ABSTRACT

A conjecture widely attributed to Neumann is that all finite non-desarguesian projective planes contain a Fano subplane. In this note, we show that any finite projective plane of even order which admits an orthogonal polarity contains many Fano subplanes. The number of planes of order less than n previously known to contain a Fano subplane was $O(\log n)$, whereas the number of planes of order less than n that our theorem applies to is not bounded above by any polynomial in n.

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1. Introduction

A fundamental question in incidence geometry is about the subplane structure of projective planes. There are relatively few results concerning when a projective plane of order k is a subplane of a projective plane of order n. Neumann [9] found Fano subplanes in certain Hall planes, which led to the conjecture that every finite non-desarguesian plane contains $PG(2, 2)$ as a subplane (this conjecture is widely attributed to Neumann, though it does not appear in her work).

Johnson [7] and Fisher and Johnson [4] showed the existence of Fano subplanes in many translation planes. Petrak [10] showed that Figueroa planes contain $PG(2, 2)$ and Caliskan and Petrak [3] showed that Figueroa planes of odd order contain $PG(2, 3)$. Caliskan and Moorhouse [2] showed that all Hughes planes contain $PG(2, 2)$ and that the Hughes plane of order q^2 contains $PG(2, 3)$ if $q \equiv 5 \pmod 6$. We prove the following.

Theorem 1. *Let Π be a finite projective plane of even order n which admits an orthogonal polarity. Then Π contains $\Omega\left(n^3\right)$ Fano subplanes.*

Ganley [5] showed that a finite semifield plane admits an orthogonal polarity if and only if it can be coordinatized by a commutative semifield. A result of Kantor [8] implies that the number of nonisomorphic planes of order n a power of 2 that can be coordinatized by a commutative semifield is not bounded above by any polynomial in n. Thus, Theorem 1 applies to many projective planes (in fact, almost all planes which have been described [8]).

2. Proof of Theorem 1

The proof of Theorem 1 is graph theoretic, and we collect some definitions and results first. Let $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a projective plane of order n. We write $p \in l$ or say p is on l if $(p, l) \in \mathcal{I}$. Let π be a polarity of Π. That is, π maps points to lines and lines to points, π^2 is the identity function, and π respects incidence. Then one may construct the polarity graph G_π^o as follows. $V(G_\pi^o) = \mathcal{P}$ and $p \sim q$ if and only if $p \in \pi(q)$. That is, the neighborhood of a vertex p is the line $\pi(p)$ that p gets mapped to under the polarity. If $p \in \pi(p)$, then p is an *absolute point* and the vertex p will have a loop on it. A polarity is

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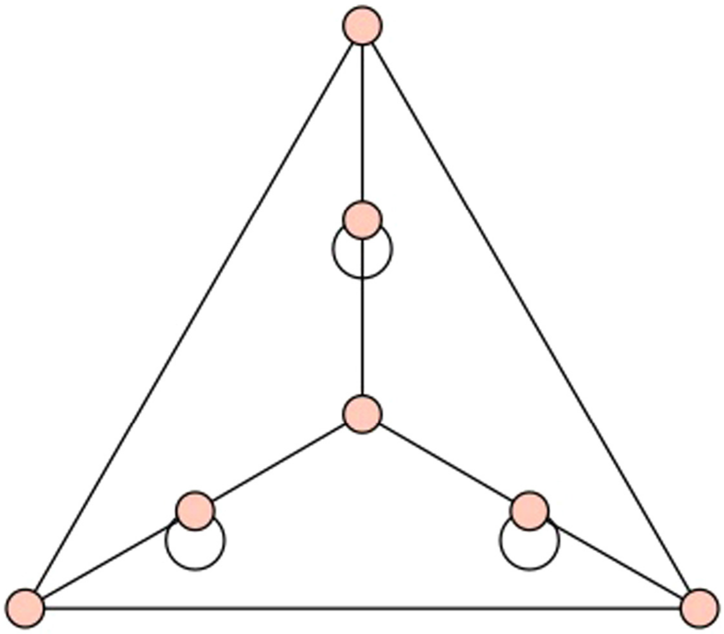

Fig. 1. ER_2^o.

orthogonal if exactly $n+1$ points are absolute. We note that as neighborhoods in the graph represent lines in the geometry, each vertex in G_π^o has exactly $n+1$ neighbors (if v is an absolute point, it has exactly n neighbors other than itself). We provide proofs of the following preliminary observations for completeness.

Lemma 1. *Let Π be a projective plane with polarity π, and G_π^o be the associated polarity graph.*

(a) *For all $u, v \in V(G_\pi^o)$, u and v have exactly 1 common neighbor.*
(b) *G_π^o is C_4-free.*
(c) *If u and v are two absolute points of G_π^o, then $u \not\sim v$.*
(d) *If $v \in V(G_\pi^o)$, then the neighborhood of v induces a graph of maximum degree at most 1.*
(e) *Let $e = uv$ be an edge of G_π^o such that neither u nor v is an absolute point. Then e lies in a unique triangle in G_π^o.*

Proof. To prove (a), let u and v be an arbitrary pair of vertices in $V(G_\pi^o)$. Because Π is a projective plane, $\pi(u)$ and $\pi(v)$ meet in a unique point. This point is the unique vertex in the intersection of the neighborhood of u and the neighborhood of v. (b) and (c) follow from (a).

To prove (d), if there is a vertex of degree at least 2 in the graph induced by the neighborhood of v, then G_π^o contains a 4-cycle, a contradiction by (b).

Finally, let $u \sim v$ and neither u nor v an absolute point. Then by (a) there is a unique vertex w adjacent to both u and v. Now uvw is the purported triangle, proving (e). □

Proof of Theorem 1. We will now assume Π is a projective plane of even order n, that π is an orthogonal polarity, and that G_π^o is the corresponding polarity graph (including loops). Since n is even and π is orthogonal, a classical theorem of Baer [1], see also Theorem 12.6 in [6] says that the $n+1$ absolute points under π all lie on one line. Let $a_1, \ldots, a_{n+1}$ be the set of absolute points and let l be the line containing them. Then there is some $p \in \mathcal{P}$ such that $\pi(l) = p$. This means that in G_π^o, the neighborhood of p is exactly the set of points $\{a_1, \ldots, a_{n+1}\}$. For $1 \le i \le n+1$, let N_i be the neighborhood of a_i. Then by Lemma 1(b), $N_i \cap N_j = \{p\}$ if $i \ne j$. Further, counting gives that

$$V(G_\pi^o) = \left(\bigcup_{i=1}^{n+1}\{a_i\}\right) \cup \left(\bigcup_{i=1}^{n+1} N_i\right). \tag{1}$$

Let ER_2^o be the graph on 7 points which is the polarity graph (with loops) of $PG(2,2)$ under the orthogonal polarity (see Fig. 1).

Lemma 2. *If ER_2^o is a subgraph of G_π^o, then Π contains a Fano subplane.*

Proof. Let $v_1, \ldots, v_7$ be the vertices of a subgraph ER_2^o of G_π^o. Let $l_i = \pi(v_i)$ for $1 \le i \le 7$. Then the lines $l_1, \ldots, l_7$ in Π restricted to the points $v_1, \ldots, v_7$ form a point-line incidence structure, and one can check directly that it satisfies the axioms of a projective plane. $\square$

Thus, it suffices to find ER_2^o in G_π^o. To find ER_2^o it suffices to find distinct i, j, k such that there are $v_i \in N_i \setminus \{p, a_i\}$, $v_j \in N_j \setminus \{p, a_j\}$, and $v_k \in N_k \setminus \{p, v_k\}$ where $v_i v_j v_k$ forms a triangle in G_π^o, for then the points $p, a_i, a_j, a_k, v_i, v_j, v_k$ yield the subgraph ER_2^o. Now note that for all i, and for $v \in N_i \setminus \{p, a_i\}$, v has exactly n neighbors that are not absolute points. There are $n+1$ choices for i and $n-1$ choices for v. As each edge is counted twice, this yields

$$\frac{n(n-1)(n+1)}{2}$$

edges with neither end in $\{p, a_1, \ldots, a_{n+1}\}$. By Lemma 1(e), there are at least

$$\frac{n^3-n}{6}$$

triangles in G_π^o. By Lemma 1(c), there are no triangles incident with p. By Lemma 1(a) there are no triangles that have more than one vertex in N_i, and therefore there are no triangles incident with $\{p, a_1, \ldots, a_{n+1}\}$ (we note that this also shows that there are exactly $(n^3-n)/6$ triangles in G_π^o). Since the absolute points form a line, each vertex not equal to p has a unique absolute neighbor. Thus for any triangle, the triangle along with its absolute neighbors and p form a copy of ER_2^o. Therefore we have found

$$\frac{n^3-n}{6}$$

copies of ER_2^o in G_π^o. We note that this expression is positive for all even natural numbers n. $\square$

3. Concluding remarks

First, we note that Theorem 1 says that there are many copies of $PG(2, 2)$ in any plane satisfying the hypotheses, and echoing Petrak [10], perhaps one could find subplanes of order 4 for n large enough. We also note that it is crucial in the proof that the absolute points form a line. When n is odd, the proof fails (as it must, since our proof does not detect if Π is desarguesian or not and $PG(2, q)$ does not contain $PG(2, 2)$ when q is odd).

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