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The anti-Ramsey problem for the Sidon equation

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ABSTRACT

For $n \geq k \geq 4$, let $AR^k_{X+Y=Z+T}(n)$ be the maximum number of rainbow solutions to the Sidon equation $X + Y = Z + T$ over all k-colorings $c : [n] \to [k]$. It can be shown that the total number of solutions in $[n]$ to the Sidon equation is $n^3/12 + O(n^2)$ and so, trivially, $AR^k_{X+Y=Z+T}(n) \leq n^3/12 + O(n^2)$. We improve this upper bound to

$$AR^k_{X+Y=Z+T}(n) \leq \left(\frac{1}{12} - \frac{1}{24k}\right) n^3 + O_k(n^2)$$

for all $n \geq k \geq 4$. Furthermore, we give an explicit k-coloring of $[n]$ with more rainbow solutions to the Sidon equation than a random k-coloring, and gives a lower bound of

$$\left(\frac{1}{12} - \frac{1}{3k}\right) n^3 - O_k(n^2) \leq AR^k_{X+Y=Z+T}(n).$$

When $k = 4$, we use a different approach based on additive energy to obtain an upper bound of $3n^3/96 + O(n^2)$, whereas our lower bound is $2n^3/96 - O(n^2)$ in this case.

1. Introduction

Most of the notation we use is standard. For a positive integer n, let $[n] = \{1, 2, \dots, n\}$. If X is a set and $m \geq 0$ is an integer, then $\binom{X}{m}$ is the set of all subsets of X of size m. A *k-coloring* of a set X is a function $c : X \to [k]$. The function c need not be onto. A subset $Y \subset X$ is *monochromatic* under c if $c(y) = c(y')$ for all $y, y' \in Y$. The set Y is *rainbow* if no two elements of Y have been assigned the same color.

The hypergraph Ramsey Theorem states that for any positive integers s, k, and m, there is an $N = N(s, k, m)$ such that for all $n \geq N$ the following holds: if c is any k-coloring of $\binom{[n]}{m}$, then there is a set $S \subset [n]$ such that $\binom{S}{m}$ is monochromatic under c. This theorem is one of the most important theorems in combinatorics. Today, Ramsey Theory is a cornerstone in combinatorics and there is a vast amount of literature on Ramsey type problems. Here we will focus on a Ramsey problem in the integers and recommend Landman and Robertson [5] for a more comprehensive introduction to this area. The problem we consider is inspired by the investigations of two recent papers.

In [7], Saad and Wolf introduced an arithmetic analog of some problems in graph Ramsey Theory. In particular, given a graph H, let $RM_k(H, n)$ be the minimum number of monochromatic copies of H over all k-colorings $c : E(K_n) \to [k]$. The parameter $RM_k(H, n)$ is the *Ramsey multiplicity* of H and has been studied for different graphs H. The arithmetic analog from [7] replaces graphs with linear equations, and sets up a general framework where these ideas from graph theory

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(Sidorenko property, Ramsey multiplicity) have natural counterparts. Fix an abelian group Γ. If L is a linear equation with integer coefficients, one can look at the minimum number of monochromatic solutions to L over all k-colorings $c : \Gamma \to [k]$. One of the first examples given in [7] (see Example 1.1) concerns the Sidon equation $X + Y = Z + T$. This famous equation has a rich history in combinatorics. A *Sidon set* in an abelian group Γ is a set having only trivial solutions to $X + Y = Z + T$. If $A \subset \Gamma$ is a Sidon set and Γ is finite, then a simple counting argument gives $|A| \le 2|\Gamma|^{1/2} + 1$. The constant 2 can be improved in many cases, but what concerns us here is that when A is much larger, say $|A| = \alpha|\Gamma|$ for some $\alpha > 0$, then A will certainly contain nontrivial solutions to the Sidon equation. Thus, a natural question is given a k-coloring $c : \Gamma \to [k]$, at least how many solutions to the Sidon equation must be monochromatic. This question, and several others including results on $X + Y = Z$ (Schur triples) and $X + Y = 2Z$ (3 term a.p.'s), is answered by the results of [7]. For more in this direction, we refer the reader to that paper.

Recently, De Silva, Si, Tait, Tunçbilek, Yang, and Young [2] studied a rainbow version of Ramsey multiplicity. Instead of looking at the minimum number of monochromatic copies of H over all k-colorings $c : E(K_n) \to [k]$, De Silva et al. look at the maximum number of rainbow copies of H. One must consider rainbow copies of H since giving every edge of K_n the same color clearly maximizes the number of monochromatic copies. Define $rb_k(H; n)$ to be the maximum number of rainbow copies of H over all k-colorings $c : E(K_n) \to [k]$. This parameter is called the *anti-Ramsey multiplicity* of H, and [2] investigates the behavior of this function for different graphs H.

In this paper, we consider an arithmetic analog of anti-Ramsey multiplicity thereby combining the problems raised in [2] with the arithmetic setting of [7]. We will focus entirely on the Sidon equation $X + Y = Z + T$. The Sidon equation measures the additive energy of a set. The additive energy of a set $A \subset \Gamma$ is the number of four tuples $(a, b, c, d) \in A^4$ such that $a + b = c + d$. Typically it is written as

$$E(A) = |\{(a, b, c, d) \in A^4 : a + b = c + d\}|.$$

This fundamental parameter measures the additive structure of A, and for more on additive energy, see Tao and Vu [8]. Additive energy is perhaps one of the reasons why the Sidon equation is used as a first example in [7]. We would also like to remark that rainbow solutions to the Sidon equation were studied by Fox, Mahdian, and Radoičić [3]. They proved that in every 4-coloring of $[n]$ where the smallest color classes has size at least $\frac{n+1}{6}$, there is at least one rainbow solution to the Sidon equation. This result is also discussed in [4] which surveys several problems on conditions ensuring a rainbow solution to an equation. Lastly, Bevilacqua et al. [1] investigated the minimum number of colors required to guarantee a rainbow solution to $x_1 + x_2 \equiv kx_3 (\text{mod } n)$ where k is a fixed integer.

Since we are interested in the maximum number of rainbow solutions to $X + Y = Z + T$, we must take a moment to carefully describe how solutions are counted. First, since we are only counting rainbow solutions, we only care about solutions to $X + Y = Z + T$ in which all of the terms are distinct. Additionally, we want to count solutions that can be obtained by interchanging values on the same side of the equation as being the same. With this in mind, we define a set of four distinct integers $\{x_1, x_2, x_3, x_4\} \in \binom{[n]}{4}$ a *Sidon 4-set* if these integers form a solution to the Sidon equation $X + Y = Z + T$. Given a Sidon 4-set $\{x_1, x_2, x_3, x_4\}$, we can determine exactly which pairs appear on each side of $X + Y = Z + T$. Without loss of generality, we may assume that x_1 is the largest among the x_i's and x_4 is the smallest. It follows that $x_1 + x_4 = x_2 + x_3$ and again without loss of generality, we may assume $x_2 > x_3$ so $x_1 > x_2 > x_3 > x_4$. In short, given a Sidon 4-set $\{x_1, x_2, x_3, x_4\}$, the two extreme values appear on one side of $X + Y = Z + T$, and the two middle values appear on the other side.

Now we are ready to define the Ramsey function that is the focus of this work. Let $n \ge k \ge 4$ be integers. We define

$$AR^k_{X+Y=Z+T}(n)$$

to be the maximum number of rainbow Sidon 4-sets over all colorings $c : [n] \to [k]$. It can be shown that the total number of Sidon 4-sets in $[n]$ is exactly

$$\frac{n^3}{12} - \frac{3n^2}{8} + \frac{5n}{12} - \theta$$

where $\theta = 0$ if n is even, and $\theta = \frac{1}{8}$ if n is odd. This immediately implies the upper bound

$$AR^k_{X+Y=Z+T}(n) \le \frac{n^3}{12} - \frac{3n^2}{8} + \frac{5n}{12}$$

for $n \ge k \ge 4$. A First Moment Method argument gives a lower bound of

$$\left(\frac{1}{12} - \frac{1}{2k} + O\left(\frac{1}{k^2}\right)\right) n^3 - O_k(n^2) \le AR^k_{X+Y=Z+T}(n).$$

Our first theorem improves both of these bounds.

Theorem 1.1. *For integers $n \ge k \ge 4$,*

$$\left(\frac{1}{12} - \frac{1}{3k} + \frac{\theta}{k^2}\right) n^3 - O_k(n^2) \le AR^k_{X+Y=Z+T}(n) \le \left(\frac{1}{12} - \frac{1}{24k}\right) n^3 + O_k(n^2)$$

where $\theta = \frac{1}{3}$ if k is even, and $\theta = \frac{1}{4}$ if k is odd.

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When $k = 4$, we can improve the upper bound using a different argument.

Theorem 1.2. *For $n \geq 4$,*

$$\frac{2n^3}{96} - O(n^2) \leq AR^4_{X+Y=Z+T}(n) \leq \frac{3n^3}{96} + O(n^2).$$

The lower bound in Theorem 1.2 is a consequence of Theorem 1.1. Finding an asymptotic formula for $AR^k_{X+Y=Z+T}(n)$ is an open problem.

The rest of this paper is organized as follows. In Sections 2 and 3, we prove the upper bounds of Theorems 1.1 and 1.2, respectively. The lower bound is proved in Section 4. Some concluding remarks and further discussion are given in Section 5.

2. An upper bound for *k* colors

Key to our upper bound for $k > 4$ is the following lemma. It gives a lower bound for the number of Sidon 4-sets that contain a fixed pair. It will be applied to pairs that are monochromatic under a given coloring c.

Lemma 2.1. *Let n be a positive integer and let $\{b < a\} \in \binom{[n]}{2}$. Define $f_n(\{b < a\})$ to be the number of Sidon 4-sets $\{x_1, x_2, x_3, x_4\} \in \binom{[n]}{4}$ with $\{a, b\} \subset \{x_1, x_2, x_3, x_4\}$. Then f_n satisfies*

$$f_n(\{b < a\}) \geq \frac{n}{2} - 4.$$

Proof. We will consider two possibilities depending on the positioning of a and b within the equation $x_1 + x_4 = x_2 + x_3$.

Claim 1. *If $a + b = x_i + x_j$, then the number of $x_i, x_j \in [n]$ with $x_i < x_j$ that satisfy this equation is at least*

$$\begin{cases} \lfloor \frac{a+b-1}{2} \rfloor - 1 & \text{if } a + b \leq n + 1, \\ n - \lfloor \frac{a+b-1}{2} \rfloor - 1 & \text{if } a + b > n + 1. \end{cases}$$

Proof of Claim 1. First suppose $a + b \leq n + 1$. Let $x_i = m$ and $x_j = a + b - m$ where $1 \leq m \leq \lfloor \frac{a+b-1}{2} \rfloor$. Since $a + b \leq n + 1$, the integers x_i and x_j are in $[n]$ for all m in the specified range. Since we must exclude $x_i = b$, $x_j = a$ (a, b, x_i, and x_j must all be distinct to be a Sidon 4-set), we obtain that the total amount of possible values for x_i, x_j is at least $\lfloor \frac{a+b-1}{2} \rfloor - 1$.

Now suppose $a + b > n + 1$. Let $x_i = a + b - n + m$ and $x_j = n - m$ where $0 \leq m \leq n - \lfloor \frac{a+b-1}{2} \rfloor$. Since $a + b > n + 1$, we have $n - m \leq x_i < x_j \leq n$ for all m in the specified range. As before, the solution $x_i = b$ and $x_j = a$ must be excluded. Here we obtain that the total amount of possible values for x_i and x_j is at least

$$n - \left\lceil \frac{a+b-1}{2} \right\rceil \geq n - \left\lfloor \frac{a+b-1}{2} \right\rfloor - 1.$$

Claim 2. *If $a + x_i = b + x_j$, then the number of $x_i, x_j \in [n]$ with $x_i < x_j$ that satisfy this equation is at least $n - (a - b) - 3$.*

Proof of Claim 2. Note that $a + x_i = b + x_j$ implies $a - b = x_j - x_i$. Since $b < a$, we have that $a - b > 0$. Let $x_i = m$ and $x_j = a - b + m$. The range of m for which we have a valid solution is $1 \leq m \leq n - (a - b)$. However, we also require that $\{a, b\} \cap \{x_i, x_j\} = \emptyset$ and so the solutions $(x_i, x_j) = (2b - a, b)$, $(x_i, x_j) = (b, a)$, and $(x_i, x_j) = (a, 2a - b)$ must all be excluded. Thus, we obtain the number x_i, x_j that satisfy the equation $a + x_i = b + x_j$ and all other constraints is at least $n - (a - b) - 3$.

These two possibilities ($a + b = x_i + x_j$ and $a + x_i = b + x_j$) are disjoint and cover all possible positions for a and b. A lower bound on the number of Sidon 4-sets $\{x_1, x_2, x_3, x_4\} \in \binom{[n]}{4}$ with $\{a, b\} \subset \{x_1, x_2, x_3, x_4\}$ is obtained by combining these two cases. So we have that if $a + b \leq n + 1$, then the amount of Sidon 4-sets that contain a and b is at least

$$\left\lfloor \frac{a+b-1}{2} \right\rfloor - 1 + n - (a - b) - 3 \geq n - \frac{a}{2} + \frac{3b}{2} - \frac{11}{2} \geq \frac{n}{2} - 4.$$

If $a + b > n + 1$, then the amount of Sidon 4-sets that contain a and b is at least

$$n - \left\lfloor \frac{a+b-1}{2} \right\rfloor - 1 + n - (a - b) - 3 \geq 2n - \frac{3a}{2} + \frac{b}{2} - \frac{9}{2} \geq \frac{n}{2} - 4. \quad \blacksquare$$

Theorem 2.2. *For integers $n \geq k \geq 4$,*

$$AR^k_{X+Y=Z+T}(n) \leq \left(\frac{1}{12} - \frac{1}{24k} \right) n^3 + O_k(n^2).$$

Proof. Let $c : [n] \to [k]$ be a k-coloring of $[n]$. Let X_i be the integers assigned color i by c. Let $\mathcal{M} \subset \binom{[n]}{2}$ be the set of all pairs $\{b < a\}$ which are monochromatic under c, i.e., $c(a) = c(b)$. Then

$$|\mathcal{M}| = \sum_{i=1}^{k} \binom{|X_i|}{2} = \frac{1}{2}\sum_{i=1}^{k} |X_i|^2 - \frac{1}{2}\sum_{i=1}^{k} |X_i| \geq \frac{1}{2}k\left(\frac{n}{k}\right)^2 - \frac{n}{2} = \frac{n^2}{2k} - \frac{n}{2}. \tag{1}$$

Let $f_n(\{b < a\})$ be the number of Sidon 4-sets in $[n]$ that contain $\{b < a\}$. By Lemma 2.1,

$$f_n(\{b < a\}) \geq \frac{n}{2} - 4. \tag{2}$$

The sum $\sum_{\{b<a\}\in\mathcal{M}} f_n(\{b < a\})$ counts the number of Sidon 4-sets that contain at least one monochromatic pair. A given Sidon 4-set is counted at most six times by this sum since there are $\binom{4}{2}$ ways to choose a pair from a Sidon 4-set. In fact, the only Sidon 4-sets that will be counted six times in this sum are those which are monochromatic under c. All others will be counted at most three times. Regardless, we have that the number of Sidon 4-sets that are not rainbow under c is at least

$$\frac{1}{6}\sum_{\{b<a\}\in\mathcal{M}} f_n(\{b < a\}) \geq \frac{1}{6}\sum_{\{b<a\}\in\mathcal{M}} \left(\frac{n}{2} - 4\right) \geq \frac{1}{6}\left(\frac{n^2}{2k} - \frac{n}{2}\right)\left(\frac{n}{2} - 4\right) = \frac{n^3}{24k} - O_k(n^2)$$

where we have used both (1) and (2). ■

3. An upper bound for four colors

For $k = 4$, the upper bound of Theorem 2.2 gives

$$AR^4_{X+Y=Z+T}(n) \leq \left(\frac{1}{12} - \frac{1}{96}\right)n^3 + O(n^2).$$

In the special case that $k = 4$, we can obtain a better upper bound with a different argument based on additive energy.

Let $A_1, A_2, \dots, A_t$ be finite sets of integers and define

$$E_t(A_1, A_2, \dots, A_t) = |\{(a_1, a_2, \dots, a_t) \in A_1 \times A_2 \times \cdots \times A_t : a_1 + a_2 + \cdots + a_t = 0\}|.$$

For integers $n \leq m$, write $[n, m]$ for the interval

$$\{n, n+1, n+2, \dots, m\}.$$

For a finite set $J \subset \mathbb{Z}$ with j elements, let

$$I(J) = [-\lceil j/2 \rceil, \lceil j/2 \rceil].$$

Note that $I(J)$ depends only on the cardinality of J.

A key ingredient in the proof of our upper bound is the following result of Lev [6].

Theorem 3.1 (*Lev [6]*). *Let $t \geq 2$ be an integer. For any finite sets $A_1, A_2, \dots, A_t \subset \mathbb{Z}$,*

$$E_t(A_1, A_2, \dots, A_t) \leq E_t(I(A_1), I(A_2), \dots, I(A_t)).$$

The main idea is to apply Theorem 3.1 with $t = 4$, where A_1, A_2, A_3, A_4 are color classes of a coloring $c : [n] \to [4]$. Before using Theorem 3.1, we need a few lemmas.

For finite sets $A, B \subset \mathbb{Z}$ and an integer m, let

$$r_{A+B}(m) = |\{(a, b) \in A \times B : a + b = m\}|.$$

Lemma 3.2. *Let $1 \leq \alpha \leq \beta$ be integers. If $A = [-\alpha, \alpha]$ and $B = [-\beta, \beta]$, then*

$$r_{A+B}(m) = \begin{cases} 2\alpha + 1 & \text{if } |m| \leq \beta - \alpha, \\ \beta + \alpha + 1 - |m| & \text{if } \beta - \alpha \leq |m| \leq \alpha + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $r_{A+B}(m) \leq \beta + \alpha + 1 - |m|$ whenever $|m| \leq \alpha + \beta$.

Proof. For any m with $|m| \leq \beta - \alpha$, we can write $m = j + (m - j)$ where $j \in [-\alpha, \alpha]$. The term $m - j$ is in B since if $|m| \leq \beta - \alpha$ and $|j| \leq \alpha$, then

$$|m - j| \leq |m| + |j| \leq \beta - \alpha + \alpha = \beta.$$

This shows that $r_{A+B}(m) = 2\alpha + 1$ whenever $|m| \leq \beta - \alpha$.

Now suppose $\beta - \alpha \leq m \leq \alpha + \beta$, say $m = \beta - \alpha + l$ for some $l \in \{0, 1, \ldots, 2\alpha\}$. Then

$$m = \beta - \alpha + l = (-\alpha + l + t) + (\beta - t) \tag{3}$$

for $t \in \{0, 1, \ldots, 2\alpha - l\}$. We now check that for each such t, we have $-\alpha + l + t \in A$ and $\beta - t \in B$. Since $\alpha \leq \beta$ and $0 \leq l \leq 2\alpha$,

$$-\beta \leq \beta - 2\alpha \leq \beta - 2\alpha + l = \beta - (2\alpha - l) \leq \beta - t \leq \beta$$

so $|\beta - t| \leq \beta$ hence $\beta - t \in B$. Similarly,

$$-\alpha \leq -\alpha + l + t \leq -\alpha + l + 2\alpha - l = \alpha$$

so $-\alpha + l + t \in A$. Therefore, in (3), the term $-\alpha + l + t$ belongs to A and $\beta - t$ belongs to B. Furthermore, this is all of the ways to write m as a sum of an integer in A and an integer in B. We conclude that for $\beta - \alpha \leq m \leq \alpha + \beta$, $r_{A+B}(m) = \beta + \alpha + 1 - m$. The proof is completed by noting that if $m > \alpha + \beta$, then $r_{A+B}(m) = 0$, and A and B are symmetric about 0 so that $r_{A+B}(m) = r_{A+B}(-m)$.

As for the assertion that $r_{A+B}(m) \leq \beta + \alpha + 1 - |m|$ for $|m| \leq \alpha + \beta$, it is enough to check that $\beta + \alpha + 1 - |m| \geq 2\alpha + 1$ for $|m| \leq \beta - \alpha$. An easy computation shows that these two inequalities are equivalent. ■

Lemma 3.3. *If α is a positive integer and $J = [-\alpha, \alpha]$, then*

$$r_{J+J}(m) = \begin{cases} 2\alpha + 1 - |m| & \text{if } |m| \leq 2\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply Lemma 3.2 with $\alpha = \beta$. ■

Lemma 3.4. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be positive integers such that $\alpha := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ is divisible by 4. If $A_i = [-\alpha_i, \alpha_i]$ and $J = [-\alpha/4, \alpha/4]$, then for any integer m with $|m| \leq \alpha/2$,*

$$r_{A_1+A_2}(m) + r_{A_3+A_4}(m) \leq 2r_{J+J}(m).$$

Proof. Let m be an integer with $|m| \leq \frac{\alpha}{2}$. By Lemma 3.2,

$$\begin{aligned} r_{A_1+A_2}(m) + r_{A_3+A_4}(m) &\leq \alpha_1 + \alpha_2 + 1 - |m| + \alpha_3 + \alpha_4 + 1 - |m| \\ &= 2(\alpha/2 + 1 - |m|) = 2r_{J+J}(m). \end{aligned}$$

For the last equality, we have used Lemma 3.3 with $J = [-\alpha/4, \alpha/4]$. ■

Lemma 3.5. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be positive integers such that $\alpha := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ is divisible by 4. If $A_i = [-\alpha_i, \alpha_i]$ and $J = [-\alpha/4, \alpha/4]$, then*

$$\sum_{m \in \mathbb{Z}} r_{A_1+A_2}(m) r_{A_3+A_4}(m) \leq \sum_{m=-\frac{\alpha}{2}}^{\frac{\alpha}{2}} r_{J+J}(m)^2.$$

Proof. First we show that if $|m| > \frac{\alpha}{2}$, then the product

$$r_{A_1+A_2}(m) r_{A_3+A_4}(m)$$

must be 0. If $r_{A_1+A_2}(m) \neq 0$ and $r_{A_3+A_4}(m) \neq 0$, then by Lemma 3.2,

$$|m| \leq \alpha_1 + \alpha_2 \quad \text{and} \quad |m| \leq \alpha_3 + \alpha_4.$$

Adding the two inequalities together gives $|m| \leq \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{2}$ and so $|m| \leq \frac{\alpha}{2}$. Thus,

$$\sum_{m \in \mathbb{Z}} r_{A_1+A_2}(m) r_{A_3+A_4}(m) = \sum_{m=-\frac{\alpha}{2}}^{\frac{\alpha}{2}} r_{A_1+A_2}(m) r_{A_3+A_4}(m).$$

By Lemma 3.4, for any m with $|m| \leq \frac{\alpha}{2}$, we have $r_{A_1+A_2}(m) + r_{A_3+A_4}(m) \leq 2r_{J+J}(m)$. Thus, the product $r_{A_1+A_2}(m) r_{A_3+A_4}(m)$ is at most $r_{J+J}(m)^2$. Since this holds for all m with $|m| \leq \frac{\alpha}{2}$,

$$\sum_{m \in \mathbb{Z}} r_{A_1+A_2}(m) r_{A_3+A_4}(m) = \sum_{m=-\frac{\alpha}{2}}^{\frac{\alpha}{2}} r_{A_1+A_2}(m) r_{A_3+A_4}(m) \leq \sum_{m=-\frac{\alpha}{2}}^{\frac{\alpha}{2}} r_{J+J}(m)^2$$

which completes the proof of the lemma. ■

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Lemma 3.6. *If α is a positive integer and $J=[-\alpha,\alpha]$, then*

$$E_4(J,J,J,J)=\frac{16\alpha^3}{3}+8\alpha^2+\frac{14\alpha}{3}+1.$$

Proof. We must count the number of 4-tuples (x_1,x_2,x_3,x_4) with $-\alpha\le x_i\le\alpha$ and

$$x_1+x_2+x_3+x_4=0.$$

For an integer m with $0\le|m|\le 2\alpha$, we have $r_{J+J}(m)=2\alpha+1-|m|$ by Lemma 3.3. The number of 4-tuples (x_1,x_2,x_3,x_4) with $-\alpha\le x_i\le\alpha$ and $x_1+x_2+x_3+x_4=0$ is

$$\begin{aligned}\sum_{m=-2\alpha}^{2\alpha} r_{J+J}(m)r_{J+J}(-m) &= r_{J+J}(0)^2+2\sum_{m=1}^{2\alpha} r_{J+J}(m)^2\\ &= (2\alpha+1)^2+2\sum_{m=1}^{2\alpha}(2\alpha+1-m)^2\\ &= \frac{16\alpha^3}{3}+8\alpha^2+\frac{14\alpha}{3}+1. \qquad \blacksquare\end{aligned}$$

Theorem 3.7. *The function $AR^4_{X+Y=Z+T}(n)$ satisfies*

$$AR^4_{X+Y=Z+T}(n)\le\frac{3n^3}{96}+O(n^2).$$

Proof. First we assume that n is divisible by 8. An easy monotonicity argument will complete the proof for all n.

Suppose $c:[n]\to\{1,2,3,4\}$ is a 4-coloring of $[n]$. Let X_i be the integers assigned color i by c and $|X_i|=c_in$. The number of rainbow solutions to $X+Y=Z+T$ is exactly

$$N(c):=E_4(X_1,X_2,-X_3,-X_4)+E_4(X_1,X_3,-X_2,-X_4)+E_4(X_1,X_4,-X_2,-X_3).$$

By Theorem 3.1,

$$\begin{aligned}N(c) \le\ & E_4(I(X_1),I(X_2),I(-X_3),I(-X_4))+E_4(I(X_1),I(X_3),I(-X_2),I(-X_4))\\ &+E_4(I(X_1),I(X_4),I(-X_2),I(-X_3)).\end{aligned}$$

We will show that each of the terms on the right hand side is at most $\frac{n^3}{96}+O(n^2)$.

For $1\le i\le 4$,

$$I(\pm X_i)=[-\lceil c_in/2\rceil,\lceil c_in/2\rceil].$$

We also have that $c_1+c_2+c_3+c_4\le 1$. Assume that each $\frac{c_in}{2}$ is an integer. Let $A_1=I(X_1)$, $A_2=I(X_2)$, $A_3=I(-X_3)$, $A_4=I(-X_4)$, and $J=[-n/8,n/8]$. By Lemmas 3.5 and 3.6

$$\begin{aligned}E_4(A_1,A_2,A_3,A_4) &= \sum_{m\in\mathbb{Z}} r_{A_1+A_2}(m)r_{A_3+A_4}(-m)=\sum_{m\in\mathbb{Z}} r_{A_1+A_2}(m)r_{A_3+A_4}(m)\\ &\le \sum_{m=-n/4}^{n/4} r_{J+J}(m)^2=E_4(J,J,J,J)=\frac{n^3}{96}+O(n^2).\end{aligned}$$

We apply this same estimate to $E_4(X_1,X_3,-X_2,-X_4)$ and $E_4(X_1,X_4,-X_2,-X_3)$ to obtain

$$N(c)\le\frac{3n^3}{96}+O(n^2).$$

If the $\frac{c_in}{2}$ are not integers, we can still apply the above argument but now J must be replaced with $J=[-n/8-1,n/8+1]$. Nevertheless, we still have $E_4(J,J,J,J)\le\frac{n^3}{96}+O(n^2)$ as $E_4(J,J,J,J)$ increases by $O(n^2)$ when the interval J increases from $[-n/8,n/8]$ to $[-n/8-1,n/8+1]$.

If n is not divisible by 8, then let l be the smallest integer for which $n+l$ is divisible by 8 (so $1\le l\le 7$). By monotonicity,

$$AR^4_{X+Y=Z+T}(n)\le AR^4_{X+Y=Z+T}(n+l)\le\frac{3(n+l)^3}{96}+O((n+l)^2)=\frac{3n^3}{96}+O(n^2). \qquad \blacksquare$$

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4. A lower bound for k colors

In this section we prove a lower bound on $AR^k_{X+Y=S+T}(n)$ for $k \geq 4$. We will need two lemmas before proving the lower bound, and continue to write

$$r_{A+B}(m) = |\{(a,b) \in A \times B : a + b = m\}|.$$

Lemma 4.1. *Let $1 \leq i < j \leq k$ be integers and let n be a positive integer that is divisible by k. If $X_i = \{m \in [n] : m \equiv i (\text{mod } k)\}$ and $X_j = \{m \in [n] : m \equiv j (\text{mod } k)\}$, then*

$$r_{X_i+X_j}(i+j+tk) \geq \begin{cases} t+1 & \text{if } 0 \leq t \leq \frac{n}{k} - 1, \\ \frac{2n}{k} - 1 - t & \text{if } \frac{n}{k} \leq t \leq \frac{2n}{k} - 2. \end{cases}$$

If $l \not\equiv i + j (\text{mod } k)$, then $r_{X_i+X_j}(l) = 0$.

Proof. First note that since k divides n,

$$X_i = \{i, i+k, i+2k, \ldots, i+n-k\} \quad \text{and} \quad X_j = \{j, j+k, j+2k, \ldots, j+n-k\}.$$

If $l = a + b$ for some $a \in X_i$ and $b \in X_j$, then $l \equiv i + j (\text{mod } k)$. Thus, $r_{X_i+X_j}(l) = 0$ whenever $l \not\equiv i + j (\text{mod } k)$. This proves the last assertion of the lemma.

Let t be an integer with $0 \leq t \leq \frac{n}{k} - 1$. We claim that for each $\alpha \in \{0, 1, \ldots, t\}$, we get

$$i + j + tk = (i + \alpha k) + (j + (t - \alpha)k)$$

where $i + \alpha k \in X_i$ and $j + (t-\alpha)k \in X_j$. The inequality

$$i \leq i + \alpha k \leq i + tk \leq i + \left(\frac{n}{k} - 1\right) k = i + n - k$$

shows that $i + \alpha k \in X_i$ for each $\alpha \in \{0, 1, \ldots, t\}$. Similarly,

$$j \leq j + (t-\alpha)k \leq j + tk \leq j + \left(\frac{n}{k} - 1\right) k = j + n - k$$

shows that $j + \alpha k \in X_j$ for each $\alpha \in \{0, 1, \ldots, t\}$. Consequently,

$$r_{X_i+X_j}(i+j+tk) \geq t + 1$$

whenever $t \in \{0, 1, \ldots, \frac{n}{k} - 1\}$.

Now let t be an integer with $\frac{n}{k} \leq t \leq \frac{2n}{k} - 2$. Write $t = \frac{2n}{k} - \beta$ where $2 \leq \beta \leq \frac{n}{k}$. For each $\alpha \in \{1, 2, \ldots, \beta - 1\}$, we can write

$$i + j + tk = i + j + \left(\frac{2n}{k} - \beta\right) k = \left(i + \left(\frac{n}{k} - \alpha\right) k\right) + \left(j + \left(\frac{n}{k} - (\beta - \alpha)\right) k\right).$$

We claim that $i + \left(\frac{n}{k} - \alpha\right) k \in X_i$ and $j + \left(\frac{n}{k} - (\beta-\alpha)\right) k \in X_j$. Now

$$i + k = i + n - \left(\frac{n}{k} - 1\right) k \leq i + n - (\beta - 1)k \leq i + \left(\frac{n}{k} - \alpha\right) k \leq i + n - k$$

where we have used the inequalities $\beta \leq \frac{n}{k}$, $\alpha \leq \beta - 1$, and $\alpha \geq 1$. We conclude that for each $\alpha \in \{1, 2, \ldots, \beta - 1\}$, the term $i + \left(\frac{n}{k} - \alpha\right) k$ is in X_i. Similarly,

$$\begin{aligned} j + k = j + \left(\frac{n}{k} - \left(\frac{n}{k} - 1\right)\right) k &\leq j + \left(\frac{n}{k} - (\beta - 1)\right) k \leq j + \left(\frac{n}{k} - (\beta - \alpha)\right) k \\ &\leq j + \left(\frac{n}{k} - 1\right) k = j + n - k \end{aligned}$$

shows that $j + \left(\frac{n}{k} - (\beta - \alpha)\right) k$ is in X_j for each $\alpha \in \{1, 2, \ldots, \beta - 1\}$. Therefore,

$$r_{X_i+X_j}(i+j+tk) \geq \beta - 1.$$

Since $t = \frac{2n}{k} - \beta$, we have $\beta - 1 = \frac{2n}{k} - t - 1$ and this completes the proof of the lemma. ■

For the next lemma we will count Sidon 4-sets in $\mathbb{Z}_k$. A *Sidon 4-set in* $\mathbb{Z}_k$ is a set of four distinct elements $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_k$ such that $\alpha + \beta \equiv \gamma + \delta (\text{mod } k)$. We will denote such a 4-set by $\{\alpha + \beta \equiv \gamma + \delta\}$. The reason we cannot simply write $\{\alpha, \beta, \gamma, \delta\}$ is that in $\mathbb{Z}_k$, four distinct residues may lead to more than one solution to the Sidon equation. For example, in $\mathbb{Z}_4$,

$$1 + 2 \equiv 3 + 4 (\text{mod } 4) \quad \text{and} \quad 1 + 4 \equiv 2 + 3 (\text{mod } 4).$$

This does not occur in $\mathbb{Z}$ because of the ordering of the integers. Let $\mathcal{S}(k)$ be the collection of all Sidon 4-sets in $\mathbb{Z}_k$. Finishing off the example of $k = 4$, it is easily seen that

$$\mathcal{S}(4) = \{\{1 + 2 \equiv 3 + 4\}, \{1 + 4 \equiv 2 + 3\}\} \tag{4}$$

and so $|\mathcal{S}(4)| = 2$.

We are now ready to state and prove the next lemma.

Lemma 4.2. *Let $k \geq 4$ be an integer. If $\mathcal{S}(k)$ is the family of all Sidon 4-sets in $\mathbb{Z}_k$, then*

$$|\mathcal{S}(k)| = \frac{k^3}{8} - \frac{k^2}{2} + \theta k$$

where $\theta = \frac{1}{2}$ if k is even, and $\theta = \frac{3}{8}$ if k is odd.

Proof. For this lemma, we will write $a \equiv b$ for $a \equiv b (\text{mod } k)$.

Let us first assume that k is even. Where this will come into play is that when k is even, the congruence $2X \equiv b$ will have exactly two solutions when b is even, and no solutions when b is odd. First we choose a pair $\{x_1, x_2\} \in \binom{\mathbb{Z}_k}{2}$. This can be done in $\binom{k}{2}$ ways and this pair will be one side of the equation $X + Y \equiv Z + T$. Our counting from this point forward depends on if $x_1 + x_2$ is even or odd when viewed as an integer.

Case 1: $x_1 + x_2$ is even

If $x_1 + x_2$ is even, then the congruence $2X \equiv x_1 + x_2$ has exactly two solutions, say y_1 and y_2. Note that no y_i can be the same as an x_i for if, say $y_1 \equiv x_1$, then from $y_1 + y_1 \equiv x_1 + x_2$ we get $x_2 \equiv y_1 \equiv x_1$ contradicting the way x_1 and x_2 have been chosen. Therefore, in the case that $x_1 + x_2$ is even, there are $k - 4$ choices for x_3 for which the unique x_4 satisfying

$$x_1 + x_2 \equiv x_3 + x_4$$

will have the property that all of x_1, x_2, x_3, and x_4 are distinct. We conclude that

$$\{x_1 + x_2 \equiv x_3 + x_4\}$$

is indeed a Sidon 4-set. This Sidon 4-set is counted exactly four times in this way: we could have chosen x_3 or x_4 after having chosen the pair $\{x_1, x_2\}$, and we could have also started by choosing the pair $\{x_3, x_4\}$ instead. When k is even, the number of pairs $\{x_1, x_2\}$ for which $x_1 + x_2$ is even is exactly $\sum_{t=1}^{\frac{k}{2}-1} 2t = \frac{k^2}{4} - \frac{k}{2}$ (this can be seen by looking at the diagonals in a Cayley table for $\mathbb{Z}_k$). Altogether, we have a count of

$$\frac{(\frac{k^2}{4} - \frac{k}{2})(k-4)}{4}$$

Sidon 4-sets $\{x_1 + x_2 \equiv x_3 + x_4\}$ where $x_1 + x_2$ is even.

Case 2: $x_1 + x_2$ is odd

If $x_1 + x_2$ is odd, then $2X \equiv x_1 + x_2$ has no solution since $\gcd(2, k)$ does not divide $x_1 + x_2$. Now there will be $k - 2$ choices for x_3 and the unique x_4 satisfying $x_1 + x_2 \equiv x_3 + x_4$ will have the property that $\{x_1, x_2, x_3, x_4\}$ is a Sidon 4-set. There are $\sum_{t=1}^{\frac{k}{2}} (2t - 1) = \frac{k^2}{4}$ pairs $\{x_1, x_2\}$ for which $x_1 + x_2$ is odd. This gives a count of

$$\frac{(\frac{k^2}{4})(k-2)}{4}$$

Sidon 4-sets $\{x_1 + x_2 \equiv x_3 + x_4\}$ where $x_1 + x_2$ is odd.

Combining the two cases, there are exactly

$$\frac{(\frac{k^2}{4} - \frac{k}{2})(k-4)}{4} + \frac{(\frac{k^2}{4})(k-2)}{4} = \frac{k^3}{8} - \frac{k^2}{2} + \frac{k}{2}$$

Sidon 4-sets in $\mathbb{Z}_k$ when k is even.

When k is odd, a similar counting argument can be done. The key difference is that for any pair $\{x_1, x_2\}$, the congruence $2X \equiv x_1 + x_2$ has exactly one solution since $\gcd(k, 2) = 1$ always divides $x_1 + x_2$. This unique solution must be avoided when choosing x_3 and so there will be $k - 3$ choices for x_3. The rest of the counting is similar to as before and we obtain

$$\frac{\binom{k}{2}(k-3)}{4} = \frac{k^3}{8} - \frac{k^2}{2} + \frac{3k}{8}$$

Sidon 4-sets in $\mathbb{Z}_k$ when k is odd. ■

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Theorem 4.3. *Let $n \geq k \geq 4$ be integers and assume that n is divisible by k. If $\mathcal{S}(k)$ is the family of all Sidon 4-sets in $\mathbb{Z}_k$, then*

$$AR^k_{X+Y=Z+T}(n) \geq 2|\mathcal{S}(k)|\left(\frac{n^3}{3k^3} - O_k(n^2)\right).$$

Proof. Let $n \geq k \geq 4$ be integers where k divides n. Define the coloring $c : [n] \to [k]$ by $c(i) = i(\text{mod } k)$ where we use residues in the set $\{1, 2, \ldots, k\}$. The number of rainbow Sidon 4-sets under c is

$$\sum_{l=1}^{2n} \sum_{1\leq i<j<s<t\leq k} \left(r_{X_i+X_j}(l)r_{X_s+X_t}(l) + r_{X_i+X_s}(l)r_{X_j+X_t}(l) + r_{X_i+X_t}(l)r_{X_j+X_s}(l)\right) \tag{5}$$

where $X_i = \{m \in [n] : m \equiv i(\text{mod } k)\}$. To see this, observe that if $x_1 + x_2 = x_3 + x_4$ is a Sidon 4-set that is rainbow, then there are distinct colors $1 \leq i < j < s < t \leq k$ with

$$\{c(x_1), c(x_2), c(x_3), c(x_4)\} = \{i, j, s, t\}.$$

This rainbow Sidon 4-set is counted exactly once by the sum (5) precisely when $l = x_1 + x_2$, and by only one of the terms in the sum

$$r_{X_i+X_j}(l)r_{X_s+X_t}(l) + r_{X_i+X_s}(l)r_{X_j+X_t}(l) + r_{X_i+X_t}(l)r_{X_j+X_s}(l). \tag{6}$$

The unique nonzero term depends on which two colors appear on the same side of the equation $x_1 + x_2 = x_3 + x_4$. For instance, if colors i and j appear on the same side, then the first term in (6) is the one that counts $\{x_1, x_2, x_3, x_4\}$.

Fix an $l \in [n]$ and four distinct colors i, j, s, t. By Lemma 4.1, the product

$$r_{X_i+X_j}(l)r_{X_s+X_t}(l)$$

is not zero if and only if $l \equiv i+j(\text{mod } k)$ and $l \equiv s+t(\text{mod } k)$. This clearly implies $i+j \equiv s+t(\text{mod } k)$ and so $\{i+j \equiv s+t\}$ is Sidon 4-set in $\mathbb{Z}_k$. For $u \in \{0, 1, \ldots, k-1\}$, let

$$\mathcal{S}(k, u)$$

be the Sidon 4-sets $\{\alpha + \beta \equiv \gamma + \delta\} \in \mathcal{S}(k)$ for which $\alpha + \beta \equiv u(\text{mod } k)$. The collection $\{\mathcal{S}(k, u) : 0 \leq u \leq k-1\}$ forms a partition of $\mathcal{S}(k)$. Since $r_{X_i+X_j}(l) \neq 0$ if only if $l \equiv i+j(\text{mod } k)$, (5) can be rewritten as

$$S := \sum_{l=0}^{\frac{2n}{k}-1} \sum_{u=1}^{k} \sum_{\{\alpha+\beta\equiv\gamma+\delta\}\in\mathcal{S}(k,u)} r_{X_\alpha+X_\beta}(u+kl)r_{X_\gamma+X_\delta}(u+kl).$$

In order to use Lemma 4.1, we split this sum into two sums S_1 and S_2 where $S \geq S_1 + S_2$. Define

$$S_1 := \sum_{l=0}^{\frac{n}{k}-1} \sum_{u=1}^{k} \sum_{\{\alpha+\beta\equiv\gamma+\delta\}\in\mathcal{S}(k,u)} r_{X_\alpha+X_\beta}(u+kl)r_{X_\gamma+X_\delta}(u+kl)$$

and

$$S_2 := \sum_{l=\frac{n}{k}}^{\frac{2n}{k}-2} \sum_{u=1}^{k} \sum_{\{\alpha+\beta\equiv\gamma+\delta\}\in\mathcal{S}(k,u)} r_{X_\alpha+X_\beta}(u+kl)r_{X_\gamma+X_\delta}(u+kl).$$

By Lemma 4.1,

$$S_1 \geq \sum_{l=0}^{\frac{n}{k}-1} \sum_{u=1}^{k} \sum_{\{\alpha+\beta\equiv\gamma+\delta\}\in\mathcal{S}(k,u)} (l+1)^2 = \sum_{l=0}^{\frac{n}{k}-1} |\mathcal{S}(k)|(l+1)^2 = |\mathcal{S}(k)|\left(\frac{n^3}{3k^3} - O_k(n^2)\right). \tag{7}$$

A similar application of Lemma 4.1 gives

$$S_2 \geq |\mathcal{S}(k)|\left(\frac{n^3}{3k^3} - O_k(n^2)\right). \tag{8}$$

Combining (7) and (8), we have

$$S \geq S_1 + S_2 \geq 2|\mathcal{S}(k)|\left(\frac{n^3}{3k^3} - O_k(n^2)\right) \tag{9}$$

which tells us that the number of rainbow Sidon 4-sets under the coloring c is at least the right hand side of (9). ∎

Corollary 4.4. *For integers $n \geq k \geq 4$, the function $AR^k_{X+Y=Z+T}(n)$ satisfies*

$$AR^k_{X+Y=Z+T}(n) \geq \left(\frac{1}{12} - \frac{1}{3k} + \frac{\theta}{k^2}\right) n^3 - O_k(n^2)$$

where $\theta = \frac{1}{3}$ if k is even, and $\theta = \frac{1}{4}$ if k is odd.

Proof. First assume that n is divisible by k. By Theorem 4.3 and Lemma 4.2,

$$AR^k_{X+Y=Z+T}(n) \geq 2\left(\frac{k^3}{8} - \frac{k^2}{2} + \gamma k\right)\left(\frac{n^3}{3k^3} - O_k(n^2)\right).$$

where $\gamma = \frac{1}{2}$ if k is even and $\gamma = \frac{3}{8}$ if k is odd.

If n is not divisible by k, then choose $r \in [k-1]$ so that $n-r$ is divisible by k. We then have by monotonicity,

$$AR^k_{X+Y=Z+T}(n) \geq AR^k_{X+Y=Z+T}(n-r) \geq 2\left(\frac{k^3}{8} - \frac{k^2}{2} + \gamma k\right)\left(\frac{(n-r)^3}{3k^3} - O_k(n^2)\right).$$

The lower order term can be absorbed into the $O_k(n^2)$ error term so we get

$$\begin{aligned} AR^k_{X+Y=Z+T}(n) &\geq 2\left(\frac{k^3}{8} - \frac{k^2}{2} + \gamma k\right)\left(\frac{n^3}{3k^3} - O_k(n^2)\right) \\ &= \left(\frac{1}{12} - \frac{1}{3k} + \frac{\theta}{k^2}\right) n^3 - O_k(n^2) \end{aligned}$$

in either case. Here $\theta = \frac{1}{3}$ if k is even, and $\theta = \frac{1}{4}$ if k is odd. ∎

5. Concluding remarks

In this paper we studied the anti-Ramsey function $AR^k_{X+Y=Z+T}(n)$ which concerns colorings of $[n]$. One could also consider colorings of $\mathbb{Z}_n$. Write $AR^k_{X+Y\equiv Z+T}(\mathbb{Z}_n)$ for the maximum number of rainbow solutions to $X+Y \equiv Z+T (\text{mod } n)$ over all k-colorings $c : \mathbb{Z}_n \to [k]$. As in the case of $[n]$, we count solutions that only differ by ordering as the same. This is discussed in detail prior to Lemma 4.2. Now by Lemma 4.2,

$$AR^k_{X+Y\equiv Z+T}(\mathbb{Z}_n) \leq \frac{n^3}{8} - \frac{n^2}{2} + \theta n$$

where $\theta = \frac{1}{2}$ if n is even, and $\theta = \frac{3}{8}$ if n is odd. When $k = 4$, it is easy to improve this upper bound as follows. Let $c : \mathbb{Z}_n \to [4]$ be a coloring of $\mathbb{Z}_n$ and let X_i be the elements of $\mathbb{Z}_n$ assigned color i by c. The number of rainbow solutions to the Sidon equation $X+Y \equiv Z+T (\text{mod } n)$ where colors 1 and 2 appear on the same side is at most

$$\min\{|X_1||X_2||X_3|, |X_1||X_2||X_4|, |X_1||X_3||X_4|, |X_2||X_3||X_4|\}. \tag{10}$$

Indeed, once we have chosen three values for the four variables X, Y, Z, and T, the last variable is uniquely determined. Since $|X_1| + |X_2| + |X_3| + |X_4| = n$, (10) is at most $\frac{n^3}{64}$. There are two other possible ways to obtain a rainbow solution to $X+Y \equiv Z+T (\text{mod } n)$. One is where colors 1 and 3 appear on the same side, and the other is where colors 1 and 4 appear on the same side. This gives the upper bound

$$AR^4_{X+Y\equiv Z+T}(\mathbb{Z}_n) \leq \frac{3n^3}{64}.$$

As for a lower bound, a natural idea is to try the same coloring that is used to prove Theorem 4.3. It turns out that this is not more difficult if we consider arbitrary $k \geq 4$, nevertheless we restrict to $k = 4$ for simplicity. Define the coloring $c : \mathbb{Z}_n \to [4]$ by $c(i) = i (\text{mod } 4)$ where we use residues in $\{1, 2, 3, 4\}$ for the colors. If n is not divisible by 4, then this coloring may not be well defined! A simple example is when $n = 5$ where $c(5) = 1$, and $c(10) = 2$, however, 5 and 10 are the same element of $\mathbb{Z}_5$. An obvious way to fix this is to fix equivalence class representatives, say $\mathbb{Z}_n = \{1, 2, \ldots, n\}$. Unfortunately this does not solve the problem as we still require the arithmetic in $\mathbb{Z}_n$ when finding solutions to $X+Y \equiv Z+T (\text{mod } n)$. To proceed further, let us now assume that n is divisible by 4 and so the coloring c will be well defined and will not depend on how we represent the elements of $\mathbb{Z}_n$. It is now straightforward to adapt Lemma 4.1 to the $\mathbb{Z}_n$ case. For $1 \leq i < j \leq 4$, we would have

$$r_{X_i+X_j}(i+j+4t) = \frac{n}{4}$$

for all $t \in \{0, 1, \ldots, \frac{n}{4} - 1\}$, and $r_{X_i+X_j}(l) = 0$ if $l \not\equiv i+j (\text{mod } 4)$. The proof of this follows along the same lines as the proof of Lemma 4.1, except now

$$i+j+4t \equiv (i+4\alpha) + (j+4(t-\alpha)) (\text{mod } n)$$

for all $\alpha \in \{1, \ldots, \frac{n}{4}\}$. One then obtains the lower bound

$$\begin{aligned} AR^4_{X+Y\equiv Z+T}(\mathbb{Z}_n) &\geq \sum_{l=0}^{\frac{n}{4}-1}\sum_{u=1}^{4}\sum_{\{\alpha+\beta\equiv\gamma+\delta\}\in\mathcal{S}(4,u)} r_{X_\alpha+X_\beta}(u+4l)r_{X_\gamma+X_\delta}(u+4l) \\ &= \sum_{l=0}^{\frac{n}{4}-1} r_{X_1+X_2}(3+4l)r_{X_3+X_4}(3+4l) + r_{X_1+X_4}(1+4l)r_{X_2+X_3}(1+4l) \\ &= \sum_{l=0}^{\frac{n}{4}-1}\left(\left(\frac{n}{4}\right)^2+\left(\frac{n}{4}\right)^2\right) = \frac{n^3}{32} \end{aligned}$$

again, assuming n is divisible by 4.

When $k = 4$, determining an asymptotic formula for the number of rainbow solutions to the Sidon equation in $[n]$ or $\mathbb{Z}_n$ would certainly be interesting. Additionally, improving the upper bound

$$AR^k_{X+Y=Z+T}(n) \leq \left(\frac{1}{12} - \frac{1}{24k}\right)n^3 + O_k(n^2)$$

seems possible. Using the methods of this paper, one might be able to improve the $\frac{1}{24k}$ to $\frac{1}{12k}$, but we believe the lower bound is closer to the truth and so any significant improvement may require some new ideas.

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