Graph switching, 2-ranks, and graphical Hadamard matrices

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ABSTRACT

We study the behavior of the 2-rank of the adjacency matrix of a graph under Seidel and Godsil–McKay switching, and apply the result to graphs coming from graphical Hadamard matrices of order 4^m. Starting with graphs from known Hadamard matrices of order 64, we find (by computer) many Godsil–McKay switching sets that increase the 2-rank. Thus we find strongly regular graphs with parameters $(63, 32, 16, 16)$, $(64, 36, 20, 20)$, and $(64, 28, 12, 12)$ for almost all feasible 2-ranks. In addition we work out the behavior of the 2-rank for a graph product related to the Kronecker product for Hadamard matrices, which enables us to find many graphical Hadamard matrices of order 4^m for which the number of related strongly regular graphs with different 2-ranks is unbounded as a function of m. The paper extends results from the article 'Switched symplectic graphs and their 2-ranks' by the first and the last author.

1. Introduction

The *2-rank* of a graph is the rank of its adjacency matrix over $\mathbb{F}_2$. It is a well-studied and useful graph parameter (see for example [3,11]). Sometimes the 2-rank can be used to distinguish cospectral graphs, such as strongly regular graphs (SRGs) with the same parameters (and therefore the same spectrum). An important fact is that the 2-rank of a graph is an even number (see [3], or [6]).

Godsil–McKay switching (GM-switching) is an operation on graphs that does not change the spectrum of the adjacency matrix. For GM-switching to work, one needs a vertex subset with special properties, called a GM-set. However, GM-switching can change the 2-rank, in which case the switched graph is obviously non-isomorphic to the original one. This idea was a starting point of an earlier paper [1] by two of the present authors. They gave switching sets in the symplectic graph $Sp(2m, 2)$, a famous SRG with parameters

$$P_0(m) = (2^{2m} - 1, \ 2^{2m-1}, \ 2^{2m-2}, \ 2^{2m-2}),$$

which increase the 2-rank after switching. In addition, repeated GM-switching was applied for the case $m = 3$, and many new strongly regular graphs with parameters $P_0(3) = (63, 32, 16, 16)$ were found and the 2-ranks vary from 6 to 18. In this paper we use an improved computer search and obtain examples with 2-rank 20, 22 and 24. In addition we apply the same idea to SRGs with parameters

$$P_\pm(m) = (2^{2m}, \ 2^{2m-1} \pm 2^{m-1}, \ 2^{2m-2} \pm 2^{m-1}, \ 2^{2m-2} \pm 2^{m-1}).$$

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For $m = 3$ we found such SRGs for all 2-ranks in $\{8, 10, \ldots, 26\}$. SRGs with parameter sets $P_0(m)$ and $P_\pm(m)$ correspond to graphical Hadamard matrices of order 4^m. For these Hadamard matrices there is a recursive construction using Kronecker products. We find the behavior of the 2-rank of the corresponding graphs for this construction. Using this we obtain SRGs with parameters $P_0(m)$ and 2-ranks $2m, 2m + 2, \ldots, 2m + 18\lfloor m/3 \rfloor$ and SRGs with parameters $P_\pm(m)$ and 2-ranks $2m + 2, 2m + 4, \ldots, 2m + 2 + 18\lfloor m/3 \rfloor$. It is known that the 2-ranks of SRGs with parameter sets $P_0(m)$ and $P_\pm(m)$ lie in the intervals $\left[2m, 2^{2m-1} - 2^{m-1} - 2\right]$, and $\left[2m + 2, 2^{2m-1} - 2^{m-1}\right]$, respectively (see [6,1]). For $m = 2$, the upper and lower bounds coincide, and for $m = 3$ there are ten possible 2-ranks for each parameter set, of which only one value is still open (26 for $P_0(3)$, and 28 for $P_\pm(3)$).

We only consider undirected graphs without loops or multiple edges. For the relevant background on graphs and matrices we refer to [4]. The $m \times n$ all-ones matrix is denoted by $J_{m,n}$, or just J, and $\mathbf{1}$ is the all-ones vector. We denote the column space of a matrix M over $\mathbb{F}_2$ by $\mathrm{Col}_2(M)$. If G is a graph with adjacency matrix A, then we sometimes write $\mathrm{Col}_2(G)$ instead of $\mathrm{Col}_2(A)$.

2. Seidel switching

Consider a graph $G = (V, E)$ of order n and let X be a subset of V of cardinality m ($0 < m < n$). *Seidel switching* in G with respect to X is an operation on E defined as follows: All edges from E between X and $V \setminus X$ are deleted, and all possible edges between X and $V \setminus X$ which are not in E are inserted (edges with both vertices inside X, or outside X remain unchanged). If A is the adjacency matrix of G, then $S = J - 2A - I$ is the *Seidel matrix* of G. So the off-diagonal entries of S are ± 1, and $S_{i,j} = -1$ if and only if i and j are adjacent. In terms of the Seidel matrix, Seidel switching with respect to X means that the rows and columns corresponding to X are multiplied by -1. This implies that Seidel switching does not change the spectrum of the Seidel matrix S.

Assume that the subset X corresponds to the first rows and columns of A, and let A_X denote the adjacency matrix of the switched graph G_X. Then

$$A_X = A + K \;(\mathrm{mod}\ 2), \quad \text{where } K = \begin{bmatrix} O & J_{m,n-m} \\ J_{n-m,m} & O \end{bmatrix}.$$

We know that 2-rank(A) is even, and since rank$(K) = 2$ (over any field), it follows that 2-rank$(A_X) \in \{$2-rank$(A) - 2$, 2-rank(A), 2-rank$(A) + 2\}$.

Lemma 2.1. *Suppose G_X is obtained from G by Seidel switching with respect to the set X of neighbors of a vertex x of G. Then x is an isolated vertex of G_X, and 2-rank$(G_X) =$ 2-rank$(G) - 2$ if $\mathbf{1} \in \mathrm{Col}_2(G)$, and 2-rank$(G_X) =$ 2-rank(G) otherwise.*

Proof. The first claim is obvious. Let $\mathbf{x}$ be the characteristic vector of X. Then clearly $\mathbf{x} \in \mathrm{Col}_2(A)$, and since $\mathbf{1}$ and $\mathbf{x}$ span $\mathrm{Col}_2(K)$, we have

$$\mathrm{Col}_2([A_X\ \mathbf{1}\ \mathbf{x}]) = \mathrm{Col}_2([A\ \mathbf{1}\ \mathbf{x}]) = \mathrm{Col}_2([A\ \mathbf{1}]).$$

Suppose $\mathbf{1} \in \mathrm{Col}_2(A)$. Then $\mathrm{Col}_2(A) = \mathrm{Col}_2([A\ \mathbf{1}]) = \mathrm{Col}_2([A_X\ \mathbf{1}\ \mathbf{x}])$. The switched graph G_X has an isolated vertex, therefore $\mathbf{1} \notin \mathrm{Col}_2(A_X)$. Hence $\mathrm{Col}_2(A_X)$ is a proper subspace of $\mathrm{Col}_2(A)$, from which it follows that 2-rank$(A) =$ 2-rank$(A_X) + 2$.

If 2-rank$(A_X) =$ 2-rank$(A) - 2$, then $\mathrm{Col}_2([A_X\ \mathbf{1}\ \mathbf{x}]) = \mathrm{Col}_2([A\ \mathbf{1}\ \mathbf{x}])$ implies that $\mathbf{x}, \mathbf{1} \notin \mathrm{Col}_2(A_X)$ and $\mathbf{x}, \mathbf{1} \in \mathrm{Col}_2(A)$. $\square$

Note that Lemma 2.1 also holds if X is the empty set, in which case $G_X = G$.

3. Godsil–McKay switching

Godsil and McKay introduced the following switching operation that leaves the spectrum of the adjacency matrix invariant.

Theorem 3.1. *Let G be a graph and let W be a subset of the vertex set of G which induces a regular subgraph. Assume that each vertex outside W is adjacent to $|W|$, $\frac{1}{2}|W|$ or 0 vertices of W. Make a new graph G_W from G as follows. For each vertex v outside W with $\frac{1}{2}|W|$ neighbors in W, delete the $\frac{1}{2}|W|$ edges between v and W, and join v instead to the $\frac{1}{2}|W|$ other vertices in W. Then G and G_W have the same adjacency spectrum.*

The operation that changes G into G_W is called *Godsil–McKay switching* (GM-switching). Notice that if all vertices outside W have $\frac{1}{2}|W|$ neighbors in W, then GM-switching is a special case of Seidel switching.

It is well-known that if a graph G_W has the same spectrum as a SRG G, then G_W is also strongly regular with the same parameters as G. Therefore GM-switching provides a tool to construct new SRGs from known ones. However, G_W may be isomorphic with G, but if GM-switching changes the 2-rank, this is obviously not the case.

Similar to Seidel switching, GM-switching can be described in terms of the adjacency matrices A and A_W of G and G_W. Indeed, $A_W = A + L \;(\mathrm{mod}\ 2)$, where L is defined by $L_{i,j} = 1$ if $i \in W, j \notin W$ and j has $\frac{1}{2}|W|$ neighbors in W; otherwise $L_{i,j} = 0$. Then 2-rank$(L) = 2$ and therefore 2-rank$(G_W) \in \{$2-rank$(G) - 2$, 2-rank(G), 2-rank$(G) + 2\}$ (see also [1]). Moreover, in the case 2-rank(G) increases, we have $\mathrm{Col}_2(A) \subset \mathrm{Col}_2(A_W)$, and therefore $\mathbf{1} \in \mathrm{Col}_2(A)$ implies $\mathbf{1} \in \mathrm{Col}_2(A_W)$.

4. Hadamard matrices

A square $(+1, -1)$-matrix H of order n is a *Hadamard matrix* whenever $HH^\top = nI$. If a row or a column of a Hadamard matrix is multiplied by -1, it remains a Hadamard matrix. We can apply this operation a number of times such that the first row and column consist of all ones. Such a Hadamard matrix is called *normalized*. A Hadamard matrix H is said to be *graphical* if H is symmetric and it has constant diagonal, and H is *regular* if all row and column sums are equal. We assume that the diagonal entries of a graphical Hadamard matrix H are equal to 1 (otherwise consider $-H$). Then $A_H = \frac{1}{2}(J - H)$ is the adjacency matrix of a graph, say G_H. Note that $H - I$ is the Seidel matrix of G_H. If H is normalized, then G_H has an isolated vertex, and it is well-known that for $n > 4$ the graph on the remaining $n - 1$ vertices is strongly regular with parameters $(n - 1, \frac{n}{2}, \frac{n}{4}, \frac{n}{4})$. If H is graphical and regular, then the row and column sums are equal to $\epsilon\sqrt{n}$ where $\epsilon = \pm 1$, and G_H is strongly regular graph with parameters $(n, \frac{n}{2} - \frac{\epsilon}{2}\sqrt{n}, \frac{n}{4} - \frac{\epsilon}{2}\sqrt{n}, \frac{n}{4} - \frac{\epsilon}{2}\sqrt{n})$. Conversely, any strongly regular graph with one of the above parameters comes from a Hadamard matrix in the described way.

It is well known that if H_1 and H_2 are Hadamard matrices, then so is the Kronecker product $H_1 \otimes H_2$. Moreover, if H_1 and H_2 are normalized, then so is $H_1 \otimes H_2$, if H_1 and H_2 are graphical, then so is $H_1 \otimes H_2$, and if H_1 and H_2 are regular then so is $H_1 \otimes H_2$. For example

$$H_1 = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

are regular graphical Hadamard matrices, and so are $H_1 \otimes H_1$, $H_1 \otimes H_2$, and $H_2 \otimes H_2$. Note that $G_{H_1} = 2K_2$ and $G_{H_2} = K_4$. The SRGs $G_{H_1 \otimes H_1}$ and $G_{H_2 \otimes H_2}$ are isomorphic with parameters $P_-(2)$. The graph is known as the lattice graph $L(4)$. The SRG $G_{H_1 \otimes H_2}$ has parameters $P_+(2)$, and is known as the Clebsch graph. For later use we define $G_-(3) = G_{H_1 \otimes H_1 \otimes H_1}$, and $G_+(3) = G_{H_1 \otimes H_1 \otimes H_2}$, which are SRGs with parameters $P_-(3)$ and $P_+(3)$, respectively.

For a recent survey on graphical Hadamard matrices, we refer to [2].

5. A graph product and its 2-rank behavior

Inspired by the Kronecker product for Hadamard matrices we define the graph product denoted by $\otimes$ as follows. For $i = 1, 2$ let G_i be a graph of order n_i with vertex set V_i, Seidel matrix S_i and adjacency matrix A_i. Then $G_1 \otimes G_2$ is the graph with vertex set $V_1 \times V_2$, where two vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if either $\{x_1, y_1\}$ is an edge in G_1 and $\{x_2, y_2\}$ is not an edge in G_2, or $\{x_2, y_2\}$ is an edge in G_2 and $\{x_1, y_1\}$ is not an edge in G_1. Thus the Seidel matrix of $G_1 \otimes G_2$ equals $(S_1 + I) \otimes (S_2 + I) - I$. So if H_1 and H_2 are graphical Hadamard matrices, then $G_{H_1} \otimes G_{H_2} = G_{H_1 \otimes H_2}$.

The graph product $G_1 \otimes G_2$ is closely related to the modular product (see [8]), which gives the complement of $G_1 \otimes G_2$.

Theorem 5.1. *For two graphs G_1 and G_2 the following hold:*

(i) $\mathbf{1} \in \mathrm{Col}_2(G_1 \otimes G_2)$ *if and only if* $\mathbf{1} \in \mathrm{Col}_2(G_1)$ *or* $\mathbf{1} \in \mathrm{Col}_2(G_2)$,

(ii) *if* $\mathbf{1} \in \mathrm{Col}_2(G_1)$ *and* $\mathbf{1} \in \mathrm{Col}_2(G_2)$ *then*

$$2\text{-rank}(G_1 \otimes G_2) = 2\text{-rank}(G_1) + 2\text{-rank}(G_2) - 2,$$

(iii) *if* $\mathbf{1} \notin \mathrm{Col}_2(G_1)$ *or* $\mathbf{1} \notin \mathrm{Col}_2(G_2)$ *then*

$$2\text{-rank}(G_1 \otimes G_2) = 2\text{-rank}(G_1) + 2\text{-rank}(G_2).$$

Proof. Let n_i be the number of vertices of G_i for $i = 1, 2$, and let A_1, A_2, and $A_{1,2}$ be the adjacency matrix of G_1, G_2 and $G_1 \otimes G_2$, respectively. Then over $\mathbb{F}_2$ the matrix $A_{1,2}$ satisfies

$$A_{1,2} = A_1 \otimes J_{n_2,n_2} + J_{n_1,n_1} \otimes A_2. \tag{1}$$

(i) Assume $\mathbf{1} \in \mathrm{Col}_2(A_1)$, then $A_1\mathbf{v} = \mathbf{1}$ for some $\mathbf{v}$ in $\mathbb{F}_2^{n_1}$. The weight of $\mathbf{v}$ is equal to $\mathbf{1}^\top\mathbf{v} = \mathbf{v}^\top A_1\mathbf{v} = 0 \pmod 2$, because A_1 is symmetric with zero diagonal. If $\mathbf{e}$ is a unit vector, and $\mathbf{v}' = \mathbf{v} \otimes \mathbf{e}$, then (1) implies that (over $\mathbb{F}_2$)

$$A_{1,2}\mathbf{v}' = (A_1 \otimes J)(\mathbf{v} \otimes \mathbf{e}) + (J \otimes A_2)(\mathbf{v} \otimes \mathbf{e}) = A_1\mathbf{v} \otimes \mathbf{1} + J\mathbf{v} \otimes A_2\mathbf{e} = \mathbf{1} + \mathbf{0}.$$

Therefore $\mathbf{1} \in \mathrm{Col}_2(A_{1,2})$. Conversely, assume $\mathbf{1} \in \mathrm{Col}_2(A_{1,2})$. Then (1) implies that there exist $\mathbf{v}_1 \in \mathbb{F}_2^{n_1}$ and $\mathbf{v}_2 \in \mathbb{F}_2^{n_2}$ such that $\mathbf{1} = A_1\mathbf{v}_1 \otimes \mathbf{1} + \mathbf{1} \otimes A_2\mathbf{v}_2$. Therefore $A_i\mathbf{v}_i = \alpha_i\mathbf{1}$ with $\alpha_i \in \mathbb{F}_2$ for $i = 1, 2$. Clearly α_1 or α_2 is nonzero, so $\mathbf{1} \in \mathrm{Col}_2(A_1)$ or $\mathbf{1} \in \mathrm{Col}_2(A_2)$.

To prove (ii) and (iii), we first assume that G_1 and G_2 both have an isolated vertex. Then clearly $\mathbf{1} \notin \mathrm{Col}_2(A_1)$ and $\mathbf{1} \notin \mathrm{Col}_2(A_2)$. For $i = 1, 2$, let r_i be $2\text{-rank}(A_i)$, and let V_i be a $n_i \times r_i$ submatrix of A_i, such that its columns are a basis for $\mathrm{Col}_2(A_i)$. Consider the matrix

$$V_{1,2} = \begin{bmatrix} V_1 \otimes J_{n_2,r_2} & | & J_{n_1,r_1} \otimes V_2 \end{bmatrix}.$$

Since A_1 and A_2 have a zero column, the columns of $V_{1,2}$ are columns of $A_{1,2}$, and by (1) they span $\mathrm{Col}_2(A_{1,2})$. Also the columns of $V_{1,2}$ are independent, since $\mathrm{Col}_2(V_1 \otimes J_{n_2,r_2})$ and $\mathrm{Col}_2(J_{n_1,r_1} \otimes V_2)$ have no nonzero vector in common. Therefore 2-rank$(A_{1,2}) = r_1 + r_2$.

If G_1 or G_2 has no isolated vertex, we apply Seidel switching. Suppose that for $i = 1, 2$ G'_i is obtained from G_i by Seidel switching with respect to the neighbors of a vertex x_i. Then x_i is an isolated vertex of G'_i, and (x_1, x_2) is isolated in $G'_1 \otimes G'_2$. We claim that $G'_1 \otimes G'_2 = (G_1 \otimes G_2)'$, where $(G_1 \otimes G_2)'$ is obtained from $G_1 \otimes G_2$ by Seidel switching with respect to the neighbors of (x_1, x_2). Indeed, if S_1, S_2, S'_1, S'_2 and $S'_{1,2}$ are the Seidel matrices of G_1, G_2, G'_1, G'_2 and $(G_1 \otimes G_2)'$, respectively, then there exist diagonal matrices D_1 and D_2 with diagonal entries ± 1, such that $S'_1 + I = D_1(S_1 + I)D_1$ and $S'_2 + I = D_2(S_2 + I)D_2$. By use of the mixed-product property of the Kronecker product we get

$$(S'_1 + I) \otimes (S'_2 + I) = (D_1 \otimes D_2)((S_1 + I) \otimes (S_2 + I))(D_1 \otimes D_2) = S'_{1,2} + I,$$

which proves the claim. Now we use Lemma 2.1. If $\mathbf{1} \in \mathrm{Col}_2(A_1)$ and $\mathbf{1} \in \mathrm{Col}_2(A_2)$ then $\mathbf{1} \in \mathrm{Col}_2(A_{1,2})$, 2-rank$(G'_1) =$ 2-rank$(G_1) - 2$, 2-rank$(G'_2) =$ 2-rank$(G_2) - 2$ and 2-rank$(G_1 \otimes G_2) =$ 2-rank$(G_1 \otimes G_2)' + 2$. Therefore 2-rank$(G_1 \otimes G_2) =$ 2-rank$(G_1) +$ 2-rank$(G_2) - 2$, which proves (ii). The cases of statement (iii) go similarly. □

6. SRGs with parameters $P_0(3)$ and $P_\pm(3)$

In this section, we report the result of a computer search for GM-switching sets in SRGs with parameters $P_0(3) = (63, 32, 16, 16), P_+(3) = (64, 36, 20, 20),$ and $P_-(3) = (64, 28, 12, 12)$. We start with known SRGs with the smallest possible 2-rank and search for GM-switching sets of size 4 that increase the 2-rank after switching. We switch, and then continue the search with the newly obtained SRGs. However, unlike in the preceding paper [1], we do not stop if we find no switching set that increases the 2-rank. Instead, we also consider switching sets that do not change the 2-rank, switch and then continue the search. A complete search considering all suitable switching sets of size 4 in each step is far out of reach, so we stop the search if we have not found a switching set that increases the 2-rank in several thousand iterations.

For more details about the computational aspects, see the SAGE worksheet[1] or,[2] where graph strings and series of switching sets (following SAGE vertex labeling) are provided in order to reproduce the results shown in this section.

A SRG with parameters $P_0(3)$ has a minimal possible 2-rank of 6 and there is a unique such SRG (see [11]): the symplectic graph $Sp(6, 2)$. The vertex set V of $Sp(6, 2)$ consists of the nonzero vectors in $\mathbb{F}_2^6$, and two vertices $\mathbf{x} = (x_1, \ldots, x_6)$ and $\mathbf{y} = (y_1, \ldots, y_6)$ are adjacent if $x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3 + x_5 y_6 + x_6 y_5 = 1$. In Table 1, the first row gives a GM-switching set in $Sp(6, 2)$, and each subsequent row gives a GM-switching set in the SRG corresponding to the resulting graph from carrying out GM-switching on the previous row. The last column gives the 2-rank after switching. Note that at some stages we use switching sets that do not increase the 2-rank. Here the upper bound for the 2-rank is 26. Unfortunately our search found no such graph, so the existence of a SRG with 2-rank 26 and parameters $P_0(3)$ remains open.

Table 1
Increasing 2-ranks by repeated GM-switching in $Sp(6, 2)$.

GM-switching set	2-rank
$\{(100000), (010000), (101000), (011000)\}$	8
$\{(000010), (000001), (001010), (001001)\}$	10
$\{(100010), (101010), (110011), (111011)\}$	12
$\{(000100), (010100), (001111), (011111)\}$	14
$\{(000110), (000101), (010110), (010101)\}$	16
$\{(001000), (100001), (110010), (011011)\}$	18
$\{(110100), (111100), (100111), (101111)\}$	18
$\{(110100), (111100), (110101), (111101)\}$	20
$\{(010100), (110110), (101101), (001111)\}$	20
$\{(100100), (110100), (101100), (111100)\}$	22
$\{(000011), (110001), (001011), (111001)\}$	22
$\{(000001), (001001), (110001), (111001)\}$	22
$\{(010000), (000001), (010010), (000011)\}$	24

In this regard, Ihringer's recent construction [7] also uses GM-switching. We did test computationally the graphs from [7] with parameters $P_0(3)$, but they did not provide new 2-ranks. In fact, Ihringer's graphs with parameters $P_0(3)$ can also be constructed by our approach. Another construction of SRGs with parameters $P_0(m)$ by Kantor [9] uses a symplectic spread in the projective geometry $PG(2m - 1, 2)$. However, only if the spread is non-Desarguesian, Kantor's graph is nonisomorphic with $Sp(2m, 2)$, and according to [10], there exists no non-Desarguesian spread in $PG(5, 2)$.

We know two nonisomorphic SRGs with parameters $P_-(3)$ and 2-rank 8. One is $G_-(3) = 2K_2 \otimes 2K_2 \otimes 2K_2$, which was defined in Section 4. We easily have 2-rank$(2K_2) = 4$, and $\mathbf{1} \in \mathrm{Col}_2(2K_2)$, so Theorem 5.1(ii) gives 2-rank$(G_-(3)) = 8$. Let $\{1, 2, 3, 4\}$ be the vertex set of $2K_2$, and let $\{1, 2\}$ and $\{3, 4\}$ be the edges. Then each vertex of $G_-(3)$ can be represented by a triple in $\{1, 2, 3, 4\}^3$. With this notation the GM-switching sets that lead to a SRG with parameter set $P_-(3)$ and 2-rank 26 are given in the left part of Table 2.

Table 2
Increasing 2-ranks by repeated GM-switching in $G_-(3)$ (left) and $G'_-(3)$ (right).

GM-switching sets	2-rk	GM-switching sets	2-rk
{(1, 1, 1), (1, 1, 3), (2, 2, 1), (2, 2, 3) }	10	{(1, 1, 1), (3, 2, 2), (4, 4, 1), (2, 3, 2)}	10
{(1, 1, 1), (1, 1, 3), (2, 1, 1), (2, 1, 3) }	12	{(3, 3, 1), (2, 2, 1), (3, 3, 2), (2, 2, 2)}	12
{(1, 1, 2), (2, 2, 4), (4, 2, 1), (3, 1, 3) }	14	{(2, 2, 4), (3, 3, 4), (2, 3, 4), (3, 2, 4)}	14
{(4, 4, 4), (1, 2, 2), (2, 3, 2), (3, 1, 4) }	16	{(3, 2, 2), (2, 3, 2), (4, 4, 4), (1, 1, 4)}	16
{(4, 3, 4), (3, 3, 2), (3, 2, 4), (4, 2, 2) }	18	{(1, 4, 2), (1, 1, 2), (4, 1, 3), (4, 4, 3)}	18
{(4, 4, 2), (3, 4, 2), (3, 1, 1), (4, 1, 1) }	20	{(4, 2, 2), (2, 4, 2), (3, 1, 3), (1, 3, 3)}	20
{(1, 1, 2), (1, 1, 4), (2, 2, 2), (2, 2, 4) }	20	{(1, 3, 2), (4, 3, 2), (4, 2, 3), (1, 2, 3)}	20
{(4, 4, 1), (3, 3, 3), (1, 3, 2), (2, 4, 4) }	22	{(1, 2, 1), (2, 4, 1), (4, 3, 4), (3, 1, 4)}	22
{(1, 3, 2), (1, 3, 4), (2, 4, 2), (2, 4, 4) }	22	{(4, 3, 1), (2, 4, 1), (1, 2, 4), (3, 1, 4)}	22
{(4, 3, 3), (3, 3, 1), (4, 2, 3), (3, 2, 1) }	24	{(4, 2, 1), (3, 4, 2), (1, 3, 1), (2, 1, 2)}	24
{(2, 1, 2), (4, 3, 2), (3, 2, 2), (1, 4, 2) }	24	{(1, 4, 1), (4, 1, 1), (2, 2, 4), (3, 3, 4)}	24
{(3, 4, 4), (1, 2, 4), (2, 3, 4), (4, 1, 4) }	26	{(3, 2, 2), (3, 3, 1), (2, 2, 4), (2, 3, 3)}	26

Table 3
Increasing 2-ranks by repeated GM-switching in $G_+(3)$ (left) and $G'_+(3)$ (right).

GM-switching sets	2-rk	GM-switching sets	2-rk
{(1, 1, 1), (1, 1, 3), (1, 2, 1), (1, 2, 3)}	10	{(1, 1, 1), (2, 1, 2), (4, 4, 4), (3, 4, 3)}	10
{(2, 1, 3), (1, 3, 1), (3, 4, 3), (4, 2, 1)}	12	{(1, 1, 2), (2, 1, 1), (2, 2, 2), (1, 2, 1)}	12
{(2, 1, 1), (3, 3, 4), (2, 4, 2), (3, 2, 3)}	14	{(1, 2, 3), (4, 4, 1), (1, 4, 1), (4, 2, 3)}	14
{(2, 1, 2), (2, 1, 4), (4, 4, 2), (4, 4, 4)}	16	{(1, 2, 4), (3, 3, 2), (1, 1, 3), (3, 4, 1)}	16
{(2, 3, 3), (4, 3, 2), (1, 4, 4), (3, 4, 1)}	18	{(3, 3, 1), (2, 1, 3), (3, 2, 2), (2, 4, 4)}	18
{(3, 4, 4), (4, 4, 3), (2, 4, 1), (1, 4, 2)}	20	{(2, 2, 3), (4, 4, 2), (2, 3, 4), (4, 1, 1)}	20
{(3, 1, 2), (4, 3, 2), (1, 4, 4), (2, 2, 4)}	20	{(3, 1, 3), (4, 2, 1), (1, 4, 3), (2, 3, 1)}	20
{(1, 1, 4), (4, 3, 1), (1, 4, 3), (4, 2, 2)}	20	{(2, 4, 1), (4, 2, 2), (1, 4, 4), (3, 2, 3)}	22
{(4, 1, 1), (3, 3, 3), (1, 4, 1), (2, 2, 3)}	22	{(2, 2, 4), (3, 4, 2), (4, 2, 1), (1, 4, 3)}	22
{(4, 1, 3), (3, 1, 1), (4, 3, 3), (3, 3, 1)}	22	{(1, 1, 2), (3, 4, 4), (2, 2, 2), (4, 3, 4)}	22
{(2, 1, 1), (4, 1, 4), (3, 2, 3), (1, 2, 2)}	24	{(4, 3, 2), (1, 1, 4), (2, 3, 3), (3, 1, 1)}	22
{(2, 3, 3), (4, 3, 2), (1, 4, 4), (3, 4, 1)}	24	{(1, 1, 2), (4, 3, 2), (4, 4, 3), (2, 1, 3)}	24
{(1, 3, 3), (3, 3, 2), (2, 4, 4), (4, 4, 1)}	24	{(1, 1, 1), (3, 4, 3), (2, 2, 1), (4, 3, 3)}	24
{(2, 4, 4), (3, 4, 3), (4, 2, 1), (3, 2, 3)}	24	{(2, 2, 4), (3, 4, 2), (1, 3, 4), (4, 1, 2)}	24
{(2, 1, 2), (2, 1, 4), (4, 3, 4), (1, 4, 4)}	26	{(4, 2, 1), (2, 3, 3), (1, 4, 1), (1, 3, 4)}	26

As mentioned before, $2K_2 \otimes 2K_2$ is an SRG with parameters $P_-(2)$ known as the lattice graph $L(4)$. However there is one other SRG with parameters $P_-(2)$, known as the Shrikhande graph, which can be obtained from $L(4)$ by Seidel switching with respect to any 4-coclique (in this particular case, Seidel switching and GM-switching are the same). We easily have 2-rank(Shrikhande) $= 6$ and $\mathbf{1} \in \mathrm{Col}_2$(Shrikhande). Define $G'_-(3) =$ Shrikhande $\otimes 2K_2$. Then $G'_-(3)$ is another SRG with parameters $P_-(3)$, 2-rank($G'_-(3)$) $= 8$ and $\mathbf{1} \in \mathrm{Col}_2(G'_-(3))$. We have also searched for GM-switching sets in $G'_-(3)$. The outcome is given in the right part of Table 2, where we use the same vertex set as for $G_-(3)$, but replaced $2K_2 \otimes 2K_2$ by the Shrikhande graph obtained by switching with respect to {(1, 1), (2, 2), (3, 3), (4, 4)}.

We also considered two nonisomorphic SRGs with parameter sets $P_+(3)$ and 2-rank 8. The first one is $G_+(3) = 2K_2 \otimes 2K_2 \otimes K_4$, which was defined in Section 4. The other one is $G'_+(3) =$ Shrikhande $\otimes K_4$. Again the vertex set is given by $\{1, 2, 3, 4\}^3$. The sequence of GM-switching sets leading to SRGs with parameters $P_+(3)$ and 2-rank 26 is given in Table 3.

The upper bound for the 2-rank of a graph with parameters $P_\pm(3)$ is 28. So only the existence of one with 2-rank 28 is unsolved. If G is a SRG with parameters $P_\pm(3)$ with 2-rank(G) $= 26$, and $\mathbf{1} \notin \mathrm{Col}_2(G)$, then from Lemma 2.1 it follows that isolating a vertex by Seidel switching gives an SRG G' with parameter set $P_0(3)$ and 2-rank(G') $= 26$, the only open case for $P_0(3)$. Unfortunately, it turns out that $\mathbf{1} \in \mathrm{Col}_2(G)$ for every graph G in Tables 2 and 3. This is not very surprising, since we know that $\mathbf{1} \in \mathrm{Col}_2(G_\pm(3))$ and $\mathbf{1} \in \mathrm{Col}_2(G'_\pm(3))$, and in Section 3 we observed that $\mathbf{1}$ remains in the column space of the adjacency matrix if GM-switching increases the 2-rank.

7. SRGs with parameters $P_0(m)$ and $P_\pm(m)$

The computer result from the previous section and the graph product introduced in Section 5 lead to the following result.

Theorem 7.1.

  (i) *There exist SRGs with parameter set $P_0(m)$ and 2-rank r for every even $r \in [2m, 2(m + 9\lfloor \frac{m}{3} \rfloor)]$.*

  (ii) *There exist SRGs with parameter set $P_+(m)$ and 2-rank r for every even $r \in [2(m + 1), 2(m + 1 + 9\lfloor \frac{m}{3} \rfloor)]$.*

  (iii) *There exist SRGs with parameter set $P_-(m)$ and 2-rank r for every even $r \in [2(m + 1), 2(m + 1 + 9\lfloor \frac{m}{3} \rfloor)]$.*

Proof. Put $\ell = \lfloor \frac{m}{3} \rfloor$, and let $G_1, \ldots, G_\ell$ be graphs coming from normalized graphical Hadamard matrices of order 64, which are given in Table 1 (so G_i is a SRG with parameters $P_0(3)$ extended with an isolated vertex), and let G_0 be the graph of a

normalized Hadamard matrix of order $4^{m-3\ell}$. Put $r_i = 2\text{-rank}(G_i)$ for $i = 0, \ldots, \ell$, and define $G = G_0 \otimes G_1 \otimes \cdots \otimes G_\ell$. Then G is a SRG with parameters $P_0(m)$, extended with an isolated vertex, and Theorem 5.1(iii) implies that $2\text{-rank}(G) = r_0 + r_1 + \cdots + r_\ell$. Now by the results in the previous section, for $i = 1, \ldots, \ell$, we can choose for r_i any even number in $[6, 24]$. This proves (i).

The proofs of (ii) and (iii) go similarly. Let $G_1, \ldots, G_\ell$ be SRGs with parameters $P_\pm(3)$ given in Tables 2 and 3 (so G_i comes from a regular graphical Hadamard matrices of order 64). For G_0 we take K_1 if $m - 3\ell = 0$, $2K_2$ if $m - 3\ell = 1$, and $2K_2 \otimes 2K_2$ if $m - 3\ell = 2$. Again $G = G_0 \otimes G_1 \otimes \cdots \otimes G_\ell$, and $r_i = 2\text{-rank}(G_i)$ for $i = 0, \ldots, \ell$. Then G is a SRG with parameters $P_\pm(m)$, and for each of $r_1, \ldots, r_\ell$ we can take any even value in $[8, \ldots, 26]$. We have seen that $\mathbf{1} \in \text{Col}_2(G_i)$ for each G_i, unless $i = 0$ and $m = 3\ell$. Therefore Theorem 5.1 gives $2\text{-rank}(G) = r_0 + r_1 + \cdots + r_\ell - 2\ell$ if $m > 3\ell$, and $2\text{-rank}(G) = r_1 + \cdots + r_\ell - 2\ell + 2$ if $m = 3\ell$. So $2\text{-rank}(G)$ can become any even number in $[2(m + 1), 2(m + 1 + 9\lfloor \frac{m}{3} \rfloor)]$. If an odd number of graphs $G_1, \ldots, G_\ell$ have parameters $P_+(3)$, then G has parameters $P_+(m)$, otherwise G has parameters $P_-(m)$. $\square$

By Lemma 2.1, isolating by Seidel switching a vertex of a SRG G with parameters $P_\pm(m)$ and $2\text{-rank } r$, gives a SRG with parameters $P_0(m)$ and $2\text{-rank } r - 2$ if $\mathbf{1} \in \text{Col}_2(G)$, and r otherwise. Since each graph G from Tables 2 and 3 has $\mathbf{1} \in \text{Col}_2(G)$, case (i) of Theorem 7.1 can also be obtained from case (ii), or (iii).

Two Hadamard matrices are *equivalent* if one can be obtained from the other by row and column permutation and multiplication of rows and columns by -1. Clearly each graphical Hadamard matrix is equivalent to a normalized graphical Hadamard matrix, and by Lemma 2.1, the SRGs from equivalent normalized graphical Hadamard matrices have the same 2-rank. So case (i) of Theorem 7.1 gives:

Corollary 7.2. *The function $f(m)$ which gives the number of nonequivalent graphical Hadamard matrices of order 4^m is unbounded.*

Lemma 2.1 implies that the 2-ranks of graphs from equivalent graphical Hadamard matrices differ by at most 2. Therefore Theorem 7.1 also implies that the statement of Corollary 7.2 remains true if we restrict to regular graphical Hadamard matrices. We remark that for Corollary 7.2 we do not need the full strength of Theorem 7.1. It already follows from the weaker version in [1], and allows a simple direct proof. Also the result is not very surprising. For several values of m there exist a large number of nonequivalent (regular graphical) Hadamard matrices of order 4^m, and by taking Kronecker products this leads to many different constructions. However, we are not aware of another result that proves the nonequivalence of an unbounded number of these constructions.

SRGs with parameters $P_+(m)$ are known as max energy graphs, see [5]. So Theorem 7.1(ii) implies that the function giving the number of nonisomorphic max energy graphs of order 4^m is unbounded.

Acknowledgments

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