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Homeomorphically irreducible spanning trees in hexangulations of surfaces

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ABSTRACT

A homeomorphically irreducible spanning tree (HIST) of a connected graph is a spanning tree without vertices of degree two. The determination of the existence problem of a homeomorphically irreducible spanning tree in a plane cubic graph is NP-complete. A hexangulation of a surface is a cubic graph embedded on a surface such that every face is bounded by a hexagon. It is a problem asked by Hoffmann-Ostenhof and Ozeki that whether there are finitely or infinitely many hexangulations of torus with homeomorphically irreducible spanning trees. In this paper, we show that a family of hexangulations of surfaces, denoted by $H(m, n)$, have a homeomorphically irreducible spanning tree if and only if it has an odd number of faces, which answers the problem of Hoffmann-Ostenhof and Ozeki for hexangulations of surfaces.

1. Introduction

All graphs considered in this paper are finite and simple. A *cycle* is a connected 2-regular graph and a tree is a connected graph without a cycle. Let G be a connected graph. A *spanning tree* T of G is a connected subgraph containing all vertices of G but no cycles. A spanning tree T is *homeomorphically irreducible* if T does not have vertices of degree two.

Homeomorphically irreducible spanning tree (HIST) was first studied by Hill [7] who conjectured that every triangulation of the sphere with at least 4 vertices contains a HIST. A stronger conjecture was proposed by Malkevitch [13] that every near-triangulation of the sphere with at least 4 vertices has a HIST. In 1990, Albertson, Berman, Hutchinson and Thomassen [1] proved Malkevitch's conjecture, and they further showed that it is NP-complete to decide whether a graph G contains a HIST. It is an open problem raised by Albertson et al. [1] that whether a connected graph in which every edge is contained in at least two triangles contains a HIST, which was settled by Chen and Shan [5]. Chen, Ren and Shan proved that every connected and locally connected graph with at least 4 vertices has a HIST [4].

A *decomposition* of a graph G is a partition of G into edge-disjoint subgraphs whose union is the graph G. If a cubic graph G has a HIST, denoted by T, then $G - E(T)$ consists of disjoint cycles and some isolated vertices. So, a cubic graph has a HIST if and only if it admits a decomposition into a spanning tree and a family of cycles. It is known that it is NP-complete to determine whether a plane cubic graph has a HIST or not [6]. For cubic graphs on surfaces with bounded size of face, Malkevitch [13] asked for a characterization of fullerenes with a HIST, which are plane cubic graphs such that a face is either

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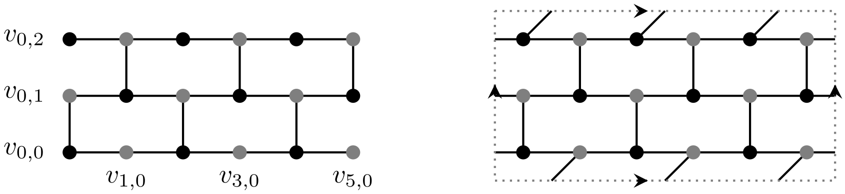

Fig. 1. A rectangular hexagon lattice $L(6, 3)$ and the hexangulation $T(6, 3)$ on the torus.

hexagonal or pentagonal. Recently, Hoffmann-Ostenhof, Noguchi and Ozeki [9] showed that there exist cyclically k-edge-connected cubic graphs without a HIST for every positive integer k, which settles a problem of Albertson et al. [1]. The result of Hoffmann-Ostenhof et al. [9] demonstrates that the cyclic edge-connectivity is not a sufficient condition for a cubic graph to have a HIST. In fact, so far, there is no known sufficient conditions for cubic graphs with a HIST. Hoffmann-Ostenhof and Ozeki proposed the following question in [10].

Problem 1.1. Are there finitely or infinitely many 3-regular hexangulations of the torus with a HIST?

A *hexangulation of a surface* is a cubic graph embedded in the surface such that every face is a hexagon. A simple calculation by applying Euler's formula to a hexangulation of a surface Σ shows that the surface Σ must be the torus or the Klein bottle. A k-prism ($k > 3$) is the Cartesian product of a cycle C_k and K_2, which does not have a HIST. If k is even, then a k-prism can be embedded in the torus such that every face is a hexagon. Hence, a hexangulation of the torus may not have a HIST. In [9], Hoffman-Ostenhof, Noguchi and Ozeki constructed infinitely many hexangulations of the torus with a HIST, which have small face-width. The *face-width* of a graph G embedded in a surface is the smallest number of closed faces (including boundaries) whose union contains a non-contractible curve.

In this paper, we show that there are infinitely many hexangulations of the torus and the Klein bottle with arbitrarily large face-width which have a decomposition into a spanning tree and a cycle, and hence have a HIST. On the other hand, combining the results of [9] (Corollary 3) and [3,15], we can show that there are infinitely many hexangulations of the torus and the Klein bottle with arbitrarily large face-width without a HIST. These results settle Problem 1.1 for hexangulations of surfaces.

2. Constructions of hexangulations of surfaces

The characterization of hexangulations of the torus was obtained by Altschuler [3] and independently by Thomassen [15] who also presented a characterization for hexangulations of the Klein bottle. A simple description for the characterizations of hexangulations of the torus and Klein bottle can be found in [17].

Let $R(m, n)$ be the integer rectangle of a 2-dimensional Euclidean plane $\mathbb{R}^2$ consisting of all points in $\{(x, y) : x, y \in \mathbb{N}$ and $0 \le x \le m-1, 0 \le y \le n-1\}$ where $\mathbb{N}$ is the set of all non-negative integers. Let $v_{i,j}$ be a vertex corresponding to the point $(i, j) \in \mathbb{R}^2$. A *rectangular hexagon lattice* $L(m, n)$ is a graph with vertex set $V = \{v_{i,j} | (i, j) \in R(m, n)\}$ and edge set $E = \{v_{i,j}v_{i+1,j} | 0 \le i \le m-2, 0 \le j \le n-1\} \cup \{v_{i,j}v_{i,j+1} | 0 \le i \le m-1, 0 \le j \le n-2 \text{ and } i \equiv j \pmod 2\}$. For example, see Fig. 1 (left). Color all vertices $v_{i,j}$ by black if $i \equiv j \pmod 2$ and by gray if $i \not\equiv j \pmod 2$. Then a such coloring is a proper coloring of $L(m, n)$.

Take $L(m, n)$ such that $m \equiv 0 \pmod 2$. To construct a hexangulation of the torus, add edges to $L(m, n)$ to join $v_{0,i}$ and $v_{m-1,i}$ for all $i \in \{0, 1, \ldots, n-1\}$, and edges to join $v_{i,0}$ and $v_{i,n-1}$ if n is even or $v_{i-1,n-1}$ if n is odd for all $i \in \{1, 3, \ldots, m-1\}$. Denote a such hexangulation of the torus by $T(m, n)$ (see Fig. 1). To construct a hexangulation of the Klein bottle, add edges to join $v_{0,i}$ and $v_{m-1,i}$ for all $i \in \{0, 1, \ldots, n-1\}$ and edges to join $v_{i,0}$ to $v_{m-1-i,n-1}$ if n is odd or $v_{m-i,n-1}$ if n is even, for all $i \in \{1, 3, \ldots, m-1\}$. Denote this hexangulation by $K(m, n)$. A hexangulation $H(m, n)$ of a surface means either $T(m, n)$ for the torus or $K(m, n)$ for the Klein bottle. From the above constructions, to form a hexangulation, m is always an even integer. Since $H(m, n)$ is simple, $m \ge 4$ is even and $n \ge 2$. The 2-coloring of $L(m, n)$ given above is also a proper 2-coloring of $H(m, n)$. It has been proved in [17] (see Theorem 19) that the face width of $K(m, n)$ satisfies $\mathrm{fw}(K(m, n)) = \min\{m/2, n\}$, which has been generalized to $H(m, n)$ as follows.

Lemma 2.1. *A hexangulation $H(m, n)$ of a surface has* $\mathrm{fw}(H(m, n)) = \min\{m/2, n\}$.

Proof. By a result (Theorem 19) of [17], it suffices to show that $T(m, n)$ has $\mathrm{fw}(T(m, n)) = \min\{m/2, n\}$. Denote the torus by Σ, and let $h(x, y)$ be the closed hexagonal face centered at $(x+1, y+\frac{1}{2})$ with $x \equiv y \pmod 2$. Let $R(y)$ be the union of all $h(x, y)$ with the same y-coordinate, and let $C(x)$ be the union of all hexagons in $\{h(x, y) | y \in \{0, 1, \ldots, n-1\} \text{ is even}\} \cup \{h(x+1, y) | y \in \{0, 1, \ldots, n-1\} \text{ is odd}\}$ for $x \in \{0, 2, \ldots, m-2\}$. Then both $R(y)$ and $C(x)$ contain a non-contractible simple closed curve. So $\mathrm{fw}(T(m, n)) \le \min\{m/2, n\}$.

Let α be a non-contractible simple closed curve in $C(x)$. Then $\Sigma - \alpha$ is a tube and every non-contractible simple closed curve of Σ contained in $\Sigma - \alpha$ is homotopic to α, which implies that a non-contractible simple closed curve ℓ not homotopic

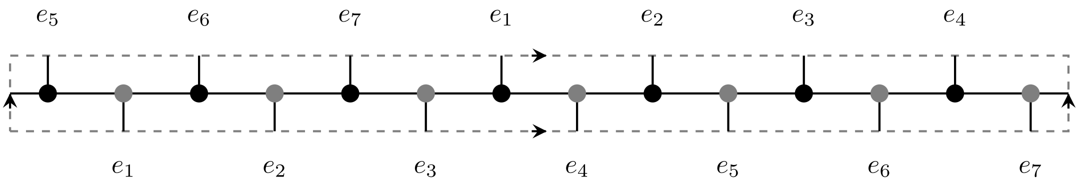

Fig. 2. A hexangulation of the torus with an odd number of faces but no HIST.

to α intersects with every $C(x)$ for each $x \in \{0, 2, \dots, m-2\}$. Therefore, the minimum number of hexagons of $T(m, n)$ required to contain a simple closed curve not homotopic to α is at least $m/2$. On the other hand, a non-contractible simple closed curve homotopic to α intersects with at least n hexagons, i.e., at least each from $R(y)$ for every $y \in \{0, 1, \dots, n-1\}$. So the number of hexagons of $T(m, n)$ whose union contains a non-contractible simple closed curve homotopic to α is at least n. Since the fundamental group of the torus is $\mathbb{Z}^2$, a non-contractible simple closed curve is either homotopic to α or non-homotopic to α. Therefore, $\text{fw}(T(m, n)) \geq \min\{m/2, n\}$. This completes the proof. $\square$

The following is our main result.

Theorem 2.2. *A hexangulation $H(m, n)$ of a surface has a homeomorphically irreducible spanning tree if and only if it has an odd number of faces.*

By the characterization of hexangulations of the Klein bottle [15] (see also [17]), we have the following direct corollary.

Corollary 2.3. *A bipartite hexangulation of the Klein bottle has a homeomorphically irreducible spanning tree if and only if it has an odd number of faces.*

As $m \to \infty$ and $n \to \infty$, then $\text{fw}(H(m, n)) \to \infty$. So the following result is a direct corollary of Lemma 2.1 and Theorem 2.2, which provides an answer to Problem 1.1.

Corollary 2.4. *Let Σ be either the torus or Klein bottle. Then there are infinitely many hexangulations of Σ with arbitrarily large face-width which have a homeomorphically irreducible spanning tree.*

Note that a bipartite hexangulation of the Klein bottle is isomorphic to $K(m, n)$ for some integers m and n [17]. Hence Theorem 2.2 provides a necessary and sufficient condition for bipartite hexangulations of the Klein bottle to have a HIST. However, it does not hold for the hexangulations of the torus. For example, the graph in Fig. 2 is a hexangulation of the torus but has no HIST because it has 14 vertices but no cycles of length four or eight. But we are able to show the following result for hexangulations of the torus.

Theorem 2.5. *Let $k \geq 3$ be an integer. If k is odd, then there exists a k-face hexangulation of the torus with a HIST. If k is even, then there is no k-face hexangulation of the torus with a HIST.*

The proofs of Theorems 2.2 and 2.5 are given in the next section.

3. Proof of main results

Let G be a cubic graph with a HIST T. Then $E(G-T)$ induces a family of cycles and $|E(G-T)| = |V(G)|/2 + 1$ since G has $3|V(G)|/2$ edges and T has $|V(G)| - 1$ edges. If G is a bipartite graph, every cycle in $G - E(T)$ has even length, which implies $|V(G)|/2 + 1$ is even, or equivalently, $|V(G)| \equiv 2 \pmod 4$. Hence we have the following useful lemma, which provides a necessary condition for a cubic bipartite graph to have a HIST.

Lemma 3.1 ([9]). *Let G be a cubic bipartite graph with a HIST. Then $|V(G)| \equiv 2 \pmod 4$.*

Let H be a bipartite hexangulation of a surface and let x be the number of faces. Then H has $2x$ vertices because every face of H is a hexagon and every vertex belongs to the boundaries of three different faces. If H has a HIST, then it follows from Lemma 3.1 that x is odd. Hence, we have the following proposition, which establishes the necessity of Theorem 2.2.

Proposition 3.2. *Let H be a bipartite hexangulation of a surface. If H has a HIST, then H has an odd number of faces.*

In the following, we are going to show the sufficiency of Theorem 2.2, which follows from the following result.

Theorem 3.3. *A hexangualation $H(m, n)$ of a surface with an odd number of faces has a decomposition into a spanning tree and a cycle.*

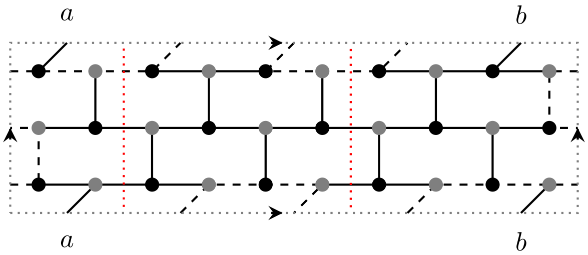

Fig. 3. A decomposition of $T(m, 3)$ into a spanning tree and a cycle.

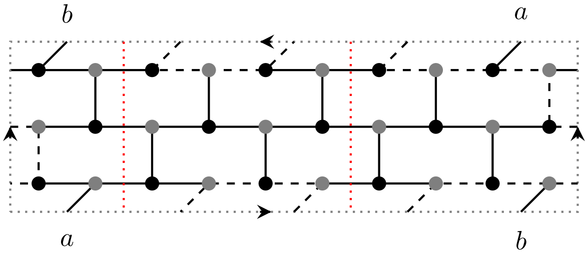

Fig. 4. A decomposition of $K(m, 3)$ into a spanning tree and a cycle.

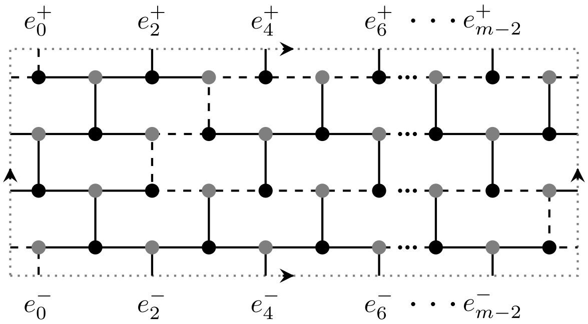

Fig. 5. A row-section R with width m.

The above theorem will be proved according to different cases, which are handled in following three lemmas. For the convenience, an edge e is sometimes treated as two half-edges e^+ and e^-, each of which is incident with one of the endvertices of e.

Lemma 3.4. *A hexangulation $H(m, n)$ of a surface with $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$ has a decomposition into a spanning tree and a cycle.*

Proof. First, note that both $T(m, 3)$ and $K(m, 3)$ with $m = 4k + 2$ have a decomposition into a spanning tree (HIST) and a cycle C.

For $T(4k+2, 3)$, the cycle $C = v_{0,0}v_{m-1,0}v_{m-2,0}v_{m-3,0}v_{m-4,2}\cdots v_{4i+1,2}v_{4i,2}v_{4i+1,0}v_{4i,0}v_{4i-1,0}v_{4i-2,2}\cdots v_{1,2}v_{0,2}v_{m-1,2}v_{m-1,1}$ $v_{0,1}v_{0,0}$ where i runs through every integer in order from $k - 1 = (m - 2)/4 - 1$ to 1, as shown in Fig. 3 (the cycle is demonstrated in dashed lines).

For $K(4k+2, 3)$, the cycle C is slightly different from the cycle of $T(4k+2, 3)$ because of the connections of the half edges crossing the top and bottom boundaries of the rectangular representation, which is $v_{0,0}v_{m-1,0}v_{m-2,0}v_{m-3,0}\cdots v_{4i+2,2}v_{4i+3,2}$ $v_{4i+4,2}v_{m-4i-5,0}v_{m-4i-6,0}v_{m-4i-7,0}\cdots v_{m-4,2}v_{m-3,2}v_{m-2,2}v_{m-1,2}v_{m-1,1}v_{0,1}v_{0,0}$ where i runs through every integer in order from 0 to $k - 2 = (m - 2)/4 - 2$, as shown in Fig. 4 (the cycle is demonstrated in dashed lines).

In the following, we are going to extend the decomposition of $H(m, 3)$ to $H(m, 4l + 3)$ where $m = 4k + 2$ for some integer $k \geq 1$. A *section R of width m* is a tubular graph obtained from $T(m, 4)$ by cutting each edge in $S = \{v_{i,0}v_{i,1} | i \in \{0, 2, \ldots, m-2\}\}$ into two half edges as shown in Fig. 5. These half edges are e_i^+ and e_i^- for even integers $i \in \{0, 2, \ldots, m-2\}$. Note that R has a path P (as shown in Fig. 5 in dashed lines) such that $R - E(P)$ is acyclic.

Note that, $T(m, 4l+3)$ (resp. $K(m, 4l+3)$) can be constructed from $T(m, 3)$ (resp. $K(m, 3)$) by adding l sections R of width m in the following way: (1) Cut each edge in $\{v_{i,0}v_{i,1} : 0 \leq i \leq 4k$ and $i \equiv 0 \pmod 2\}$ of $T(m, 3)$ (resp. $K(m, 3)$) into two

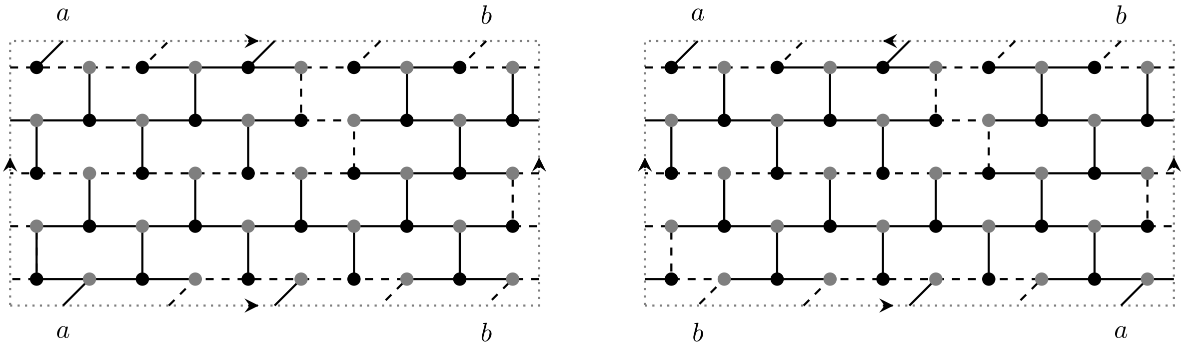

Fig. 6. A decomposition of $T(10, 5)$ (left) and $K(10, 5)$ (right) into a spanning tree and a cycle.

half-edges: one incident with $v_{i,0}$ and the other incident with $v_{i,1}$; (2) Take l copies of the section R, denoted by $R_1, \ldots, R_l$ and merge them into a big graph Q by joining the half edges e_i^+ of the αth copy and the half edge e_i^- of the $(\alpha+1)$th copy for $\alpha \in \{1, \ldots, l-1\}$; (3) Join the half edge e_i^- of R_1 in Q to the half edge of $T(m, 3)$ (resp. $K(m, 3)$) incident with $v_{i,0}$, and join the half edge e_i^+ of R_l in Q to the half edge of $T(m, 3)$ (resp. $K(m, 3)$) incident with $v_{i,1}$.

From the above construction, the paths P in the l copies of R are joined together to a long path P' in Q such that $Q - E(P')$ is acyclic and the vertices except these incident with the half edges e_i^+ of R_l in Q for $i \in \{4, 6, \ldots, m-2\}$ belong to the same component of $Q - E(P')$. In the resulting $T(m, 4l+3)$ (resp $K(m, 4l+3)$), the path P' is inserted to the cycle C of $T(m, 3)$ (resp. $K(m, 3)$) to generate a new cycle C' by connecting the half edge incident with $v_{0,0}$ to e_0^- and connecting the half edge incident with $v_{1,0}$ to e_0^+, respectively. It is not hard to see that $T(m, 4l+3) - E(C')$ (resp. $K(m, 4l+3) - E(C')$) is connected and acyclic, which is a spanning tree. Therefore, $T(m, 4l+3)$ (resp. $K(m, 4l+3)$) has a decomposition into a cycle C' and a spanning tree. This completes the proof of Lemma 3.4. □

Now, we turn to the case $H(m, n)$ with $n \equiv 1 \pmod 4$. First, we show a similar result for $H(m, n)$ with $n \equiv 1 \pmod 4$ and $m \geq 10$ as Lemma 3.4 for $H(m, n)$ with $n \equiv 3 \pmod 4$.

Lemma 3.5. *Let $H(m, n)$ be a hexangulation of a surface with $m \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 4$. If $m \geq 10$, then $H(m, n)$ has a decomposition into a spanning tree and a cycle.*

Proof. First, note that both $T(m, 5)$ and $K(m, 5)$ with $m \equiv 2 \pmod 4$ have a decomposition into a spanning tree and a cycle as shown in Fig. 6.

For $T(m, 5)$, let $C = v_{0,0}v_{m-1,0}v_{m-2,4}v_{m-1,4}\cdots v_{4i,4}v_{4i+1,4}v_{4i+2,4}v_{4i+3,0}v_{4i+4,0}v_{4i+5,0}\cdots v_{m-4,0}\ v_{m-3,0}v_{m-4,4}v_{m-5,4}v_{m-5,3}$ $v_{m-4,3}v_{m-4,2}\cdots v_{0,2}v_{m-1,2}v_{m-1,1}v_{0,1}v_{0,0}$ where i runs over all integers from 0 to $k-2 = (m-2)/4 - 2$ in order. (For example, see the dashed-line cycle of $T(10, 5)$ in Fig. 6.) Then $T(m, 5) - E(C)$ is a tree. Hence, $T(m, 5)$ with $m = 4k+2$ has a decomposition into a cycle and a spanning tree.

Similarly, for $K(m, 5)$, let $C = v_{0,0}v_{1,0}v_{m-2,4}v_{m-1,4}\cdots v_{4i,4}v_{4i+1,4}v_{4i+2,4}v_{m-4i-3,0}v_{m-4i-4,0}\ v_{m-4i-5,0}\cdots v_{4,0}v_{3,0}v_{m-4,4}$ $v_{m-5,4}v_{m-5,3}v_{m-4,3}v_{m-4,2}\cdots v_{0,2}v_{m-1,2}v_{m-1,1}v_{0,1}v_{0,0}$, where i runs over all integers from 0 to $k-2 = (m-2)/4 - 2$ in order. (For example, see the dashed-line cycle of $K(10, 5)$ in Fig. 6.) Then $K(m, 5) - E(C)$ is a tree. It follows that $K(m, 5)$ has a decomposition into a cycle and a spanning tree.

To obtain a desired decomposition of $T(m, n)$ (resp. $K(m, n)$) with $m = 4k+2$ and $n = 4l+1$ for $k \geq 2$ and $l \geq 2$, we use a similar construction as shown in the proof of Lemma 3.4. First, take $l-1$ copies of the section R (see Fig. 5) and merge them together to build a large graph Q by joining the half edges e_i^+ of the αth copy R and the corresponding half edges e_i^- of the $(\alpha+1)$th copy of R for $i \in \{0, 2, \ldots, m-2\}$ and $\alpha \in \{1, \ldots, l-2\}$. Then Q has m half edges, denoted by e_i^+ (from the $(l-1)$th copy of R) and e_i^- (from the first copy of R). Then the dashed-line paths in each copy of R are merged together to a long path P' of Q. Then, cut every edge in $S = \{v_{i,0}v_{i,1} | i \in \{0, 2, 4, \ldots, m-2\}\}$ into two half edges and then insert Q into $T(m, n)$ (resp. $K(m, n)$) by connecting the half edge e_i^- of Q to the half edge incident with $v_{i,0}$, and connecting the half edge e_i^+ with the half edge incident with $v_{i,1}$. Hence, the cycle C of $T(m, 5)$ (resp. $K(m, 5)$) is extended into a cycle C' of $T(m, n)$ (resp. $K(m, n)$) by merging C and P' together. Note that C' is non-separating and intersects with every hexagon of $T(m, n)$ (resp. $K(m, n)$), and the spanning tree of $T(m, 5)$ (resp. $K(m, 5)$) is extended into a spanning tree of $T(m, n)$ (resp. $K(m, n)$).

Therefore, both $T(m, n)$ and $K(m, n)$ with $m = 4k+2$ and $n = 4l+1$ for $k \geq 2$ and $l \geq 2$ have a decomposition into a cycle and a spanning tree. In other words, $H(m, n)$ with $m = 4k+2$ and $n = 4l+1$ for $k \geq 2$ and $l \geq 2$ have a desired decomposition. This completes the proof of Lemma 3.5. □

Now, we consider the remaining case $H(6, n)$ with $n = 4l+1$ for $l \geq 1$.

Lemma 3.6. *A hexangulation $H(6, n)$ of a surface with $n \equiv 1 \pmod 4$ and $n \geq 5$ has a decomposition into a spanning tree and a cycle.*

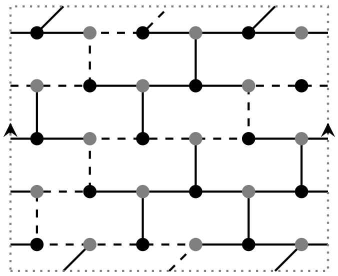

Fig. 7. A decomposition of $H(6, 5)$.

Proof. First, both $T(6, 5)$ and $K(6, 5)$ have a decomposition into a spanning tree and a non-contractible cycle $C = v_{0,0}v_{0,1}v_{1,1}v_{1,2}v_{2,2}v_{3,2}v_{4,2}v_{4,3}v_{5,3}v_{0,3}v_{1,3}v_{1,4}v_{2,4}v_{3,0}v_{2,0}v_{1,0}v_{0,0}$. (See Fig. 7: identify the top dotted-line and the bottom dotted-line in the same direction for $T(6, 5)$ and in reversed directions for $K(6, 5)$. In both cases, $v_{2,4}v_{3,0}$ is an edge of $H(6, 5)$.)

Now we extend the decomposition of $H(6, 5)$ to a decomposition of $H(6, 4l + 1)$ for $l \geq 2$ by adding l sections R of width 5 as the same approach to extend the decomposition of $H(m, 3)$ to $H(m, 4l + 3)$ in the proof of Lemma 3.4. Merge the cycle C of $H(6, 5)$ and the path P', obtained by joining the paths P in the l copies of the row-section R of width 5, to generate a non-contractible and non-separating cycle C' of $H(6, 4l + 1)$. So $H(6, 4l + 1) - E(C')$ is a connected plane graph. Note that the cycle C' intersects with every hexagon of $H(6, 4l + 1)$. Hence $H(6, 4l + 1) - E(C')$ is a tree because a non-hexagonal cycle of $H(6, 4l + 1)$ is either non-contractible or separating. So $H(6, 4l + 1)$ has a decomposition into a spanning tree and a cycle C'. This completes the proof. □

Combining Lemmas 3.4–3.6, the proof of Theorem 3.3 is completed. Hence Theorem 2.2 follows.

In the following, we show Theorem 2.5 by constructing a new family of hexangulations of the torus.

Proof of Theorem 2.5. First, for any integer $k \geq 3$, we construct a bipartite hexangulation of the torus with k faces: (i) take a rectangular hexagon lattice $L(m, 1)$ with $m = 2k$; (ii) first identify $v_{0,0}$ with $v_{m,0}$ and then join $v_{i,0}$ and $v_{i+1+2t,0}$ for $i \in \{1, 3, \ldots, m - 1\}$ and $t \in \{0, 1, \ldots, k\}$ to generate a hexangulation of the torus (all subscripts taken modulo m). Denote a hexangulation of the torus generated this way by $T(m, 1, t)$. (For example, see $T(14, 1, 2)$ in Fig. 2.) By the construction, a vertex $v_{i,0}$ of $T(m, 1, t)$ is adjacent to $v_{j,0}$ if and only if $|i - j| \in \{1, 2t + 1\}$.

First, assume that k is an odd integer. Then the first part of Theorem 2.5 follows from the following claim because $T(m, 1, t)$ has k faces when $m = 2k$.

Claim. *The hexangulation $T(m, 1, t)$ with $m = 2k$ and $t = (k - 1)/2$ has a HIST.*

Proof of Claim. Note that $v_{i,0}$ is adjacent to $v_{i+k,0}$. So $C = v_{0,0}v_{1,0} \ldots v_{k,0}v_{0,0}$ is a non-contractible cycle of $T(m, 1, t)$ of length $k + 1$. In $G - E(C)$, the vertex $v_{i,0}$ with $1 \leq i \leq k - 1$ is adjacent to a vertex $v_{j,0}$ with $k + 1 \leq j \leq 2m - 1$ because $|j - i| = k = 2t + 1$. Therefore $T(m, 1, t) - E(C)$ is connected and has exactly $3m/2 - |E(C)| = 3m/2 - (k + 1) = m - 1$ edges. Hence $T(m, 1, t) - E(C)$ is a HIST of $T(m, 1, t)$. This completes the proof of the Claim.

Now, we assume that k is even. By the construction, $T(m, 1, t)$ has no edges joining $v_{i,0}$ and $v_{j,0}$ if $i \equiv j \pmod 2$. So $T(m, 1, t)$ is bipartite. It follows from Proposition 3.2 that $T(m, 1, t)$ has no HIST. This completes the proof of Theorem 2.5. □

4. Concluding remarks

Theorem 2.2 provides a necessary and sufficient condition for bipartite hexangulations of the Klein bottle. However, the example in Fig. 2 shows that the necessary condition in Lemma 3.1 is not sufficient for bipartite hexangulations of the torus. It is interesting to ask for a characterization of bipartite cubic graphs for which, the necessary condition of Lemma 3.1 is also sufficient.

An *even-subgraph* of a graph G is a subgraph in which every vertex has even degree. If G is cubic, then an even-subgraph of a cubic graph consists of a family of cycles. A connected cubic graph has a HIST if and only if the graph can be decomposed into a spanning tree and an even-subgraph. A 3-decomposition of a cubic graph is a decomposition of the graph into three subgraphs: a spanning tree, a matching and an even-subgraph. A cubic graph may not have a decomposition into a spanning tree and an even-subgraph. However, it was conjectured by Hoffmann-Ostenhof that every connected cubic graph always admits a 3-decomposition, the so-called 3-Decomposition Conjecture. The 3-Decomposition Conjecture has been verified for planar graphs [8,14], and cubic graphs with a Hamiltonian path [12]. Note that, all hexangulations of surfaces are Hamiltonian [2,16]. Therefore, a hexangulation of a surface does have a 3-decomposition. As shown in [6,11], it is NP-complete to determine whether a plane cubic graph has a decomposition into a spanning tree and a family of cycles. But, it is polynomial-time to find a 3-decomposition for a 3-connected plane cubic graph [14].

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References

[1] M.O. Albertson, D.M. Berman, J.P. Hutchinson, C. Thomassen, Graphs with homeomorphically irreducible spanning trees, J. Graph Theory 14 (1990) 247–258.
[2] B. Alspach, C.-Q. Zhang, Hamilton cycles in cubic Cayley graphs on dihedral groups, Ars Combin. 28 (1989) 101–108.
[3] A. Altschuler, Construction and enumeration of regular maps on the torus, Discrete Math. 4 (1973) 201–217.
[4] G. Chen, H. Ren, S. Shan, Homeomorphically irreducible spanning trees in locally connected graphs, Combin. Probab. Comput. 21 (2012) 107–111.
[5] G. Chen, S. Shan, Homeomorphically irreducible spanning trees, J. Combin. Theory Ser. B 103 (2013) 409–414.
[6] R.J. Douglas, NP-Completeness and degree restricted spanning trees, Discrete Math. 105 (1992) 41–47.
[7] A. Hill, Graphs with homeomorphically irreducible spanning trees, in: London Math. Soc. Lecture Note Ser., vol. 13, Cambridge Univ. Press, London, 1974, pp. 61–68.
[8] A. Hoffmann-Ostenhof, T. Kaiser, K. Ozeki, Decomposing planar cubic graphs, J. Graph Theory 88 (4) (2018) 631–640.
[9] A. Hoffmann-Ostenhof, K. Noguchi, K. Ozeki, On homeomorphically irreducible spanning trees in cubic graphs, J. Graph Theory 89 (2) (2018) 93–100.
[10] A. Hoffmann-Ostenhof, K. Ozeki, On HISTs in Cubic Graphs, arXiv:1507.07689v1.
[11] P. Lemke, The Maximum Leaf Spanning Tree Problem for Cubic Graphs Is NP-Complete, in: Preprint series, vol. 428, IMA, Univ. of Minn, 1988.
[12] W. Liu, P. Li, Decomposition of cubic traceable graphs, Discuss. Math. Graph Theory (2018) http://dx.doi.org/10.7151/dmgt.2132.
[13] J. Malkevitch, Spanning trees in polytopal graphs, Ann. New York Acad. Sci. 319 (1979) 362–367.
[14] K. Ozeki, D. Ye, Decomposing plane cubic graphs, European J. Combin. 52 (2016) 40–46.
[15] C. Thomassen, Tilings of the torus and the klein bottle and vertex-transitive graphs on a fixed surface, Trans. Amer. Math. Soc. 323 (1991) 605–635.
[16] D. Ye, Hamilton cycles in cubic polyhex graphs on the klein bottle, Ars Combin. 112 (2013) 205–212.
[17] D. Ye, H. Zhang, Face-width of pfaffian braces and polyhex graphs on surfaces, Electron. J. Combin. 21 (4) (2014) P4.37: 1–19.