Improving bounds on the order of regular graphs of girth 5

E. Abajo [a], C. Balbuena [b,*], M. Bendala [a], X. Marcote [b]

[a] Departamento de Matemática Aplicada I, Universidad de Sevilla, Spain
[b] Departament d'Enginyeria Civil i Ambiental, Universitat Politècnica de Catalunya, E-08034 Barcelona, Spain

ABSTRACT

A (k, g)-graph is a k-regular graph with girth g and a (k, g)-cage is a (k, g)-graph with the fewest possible number of vertices $n(k, g)$. Constructing (k, g)-cages and determining the order are both very hard problems. For this reason, an intensive line of research is devoted to constructing smaller (k, g)-graphs than previously known ones, providing in this way new upper bounds to $n(k, g)$ each time such a graph is constructed.

The paper focuses on girth $g = 5$, where cages are known only for degrees $k \leq 7$. We construct $(k, 5)$-graphs using and extending techniques of amalgamation into the incidence graphs of elliptic semiplanes of type $\mathcal{L}$ introduced and exposed by Funk (2009). The order of these graphs provides better upper bounds on $n(k, 5)$ than those known so far, for values of k such that either $13 \leq k \leq 33$ or $k \geq 66$.

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1. Introduction

All considered graphs are undirected and simple. For undefined terminology and notations the reader may refer to [5]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If the vertex set is a multiplicative (additive) group, every edge $\{u, v\} \in E(G)$ has a *Cayley color*, defined as the pair $(uv^{-1})^{\pm 1}$ ($\pm(u - v)$, respectively). The *girth* of a graph G is the length g of a shortest cycle. The set of vertices adjacent to a vertex $v \in V$ is denoted $N(v)$, and the *degree* of v is the cardinality of $N(v)$.

A (k, g)-graph is a k-regular graph of girth g and a (k, g)-cage is a (k, g)-graph with the fewest possible number of vertices. Constructions of cages can be found in the complete survey [8]. The order of a (k, g)-cage is denoted by $n(k, g)$. A lower bound $n_0(k, g)$ on the number of vertices of a (k, g)-graph, known as the *Moore bound*, is given by:

$$n_0(k, g) = \begin{cases} 1 + k + k(k - 1) + \cdots + k(k - 1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k - 1) + \cdots + (k - 1)^{g/2 - 1}) & \text{if } g \text{ is even.} \end{cases}$$

In this paper we focus on *the cage problem* for $g = 5$. In this case $n_0(k, 5) = 1 + k^2$, and this bound is only attained for $k = 2, 3, 7$ and, perhaps, for $k = 57$ (cf. [11]). The only known $(k, 5)$-graphs of minimum order for $k = 4, 5, 6$ are cages (cf. [13,14,18–21]). No other $(k, 5)$-cage has been identified so far and most of the work carried out focus on constructing a new $(k, 5)$-graph with fewer vertices, for degrees $k > 7$. Following the notation in [8,9], the currently best known bound on $n(k, 5)$ is given by the order $rec(k, 5)$ of this graph.

The goal of this paper is to improve the bounds on $n(k, 5)$, that is, to find lower values of $rec(k, 5)$. The methods are constructive, and based on the idea of *amalgam*. As far as we now, we reach this goal for most of the integers k in the range $13, \ldots, 33$, for $k \in \{66, 67, 68\}$, and for every $k \geq 74$.

Table 1
Upper bounds on the order of a $(k, 5)$-graph.

k	Upper bound	Due to	Reference	New upper bound
8	80	Royle	[15]	
9	96	Jørgensen	[12]	
10	124	Exoo	[7]	
11	154	Exoo	[7]	
12	203	Exoo	[7]	
13	230	Exoo	[7]	**226**
14	284	Abreu et al.	[2]	**280**
15	310	Abreu et al.	[2]	
16	336	Jørgensen	[12]	
17	436	Abajo et al.	[1]	
18	468	Abajo et al.	[1]	
19	500	Abajo et al.	[1]	
20	564	Abajo et al.	[1]	
21	666	Abajo et al.	[1]	**658**
22	704	Abajo et al.	[1]	
23	880	Funk	[9]	**874**
24	924	Funk	[9]	**920**
25	968	Funk	[9]	**960**
26	1012	Funk	[9]	**1010**
27	1056	Funk	[9]	**1054**
28	1200	Funk	[9]	**1192**
29	1248	Funk	[9]	**1240**
30	1404	Funk	[9]	**1392**
31	1456	Funk	[9]	**1444**
32	1624	Abajo et al.	[1]	**1608**
33	1680	Abajo et al.	[1]	**1664**

In Section 2 we consider the incidence graph $\mathcal{L}_q$ of the elliptic semiplane of type L, and define a new amalgamation technique inspired by papers from Jørgensen [12], Funk [9] and Abreu et al. [2]. This technique requires both regular and bi-regular graphs to increase the degree of the resulting graph while removing vertices in flexible ways. We also recall a class of graphs introduced in [1] that will play a key role in our constructions.

In Section 3 we work with prime powers from $q = 11$ to $q = 29$. By using the new technique we construct $(k, 5)$-graphs, for $k \in \{13, 14, 21, 23, \dots, 33\}$, that provide new bounds on $n(k, 5)$. Table 1 shows the values of $rec(k, 5)$ for $8 \le k \le 33$ and highlights our contributions.

In the rest of the paper we restrict ourselves to amalgams of r_q-regular graphs into $\mathcal{L}_q$. This has been done in [1] (cf. [1], Table 2 and Theorems 1.1, 1.2) for $r_q = 5, 6$; and the achieved values of $rec(k, 5)$ for $34 \le k \le 65$ and $69 \le k \le 73$ remain untouched.

In Section 4, (cf. Theorem 4.1 and Corollary 4.1), we deal with $q = 61$ and prime powers $q \ge 71$. For each q we find the best possible amalgam with r_q-regular graphs to prove that

$$n(q + r_q, 5) \le 2(q^2 - 1)$$

for $7 \le r_q \le 12$. This way, we improve the value $rec(k, 5)$ for $k \in \{66, 67, 68\}$ and from $k = 74$ up to $k = 479 + 12 = 491$.

This process, which first fixes r_q and then determines the related prime powers q, could continue indefinitely. Now we search for an explicit expression of r_q as an increasing function of q, similar to Jørgensen's bound (cf. [12], Corollary 19):

$$n\left(q + \left\lfloor \frac{\sqrt{q-1}}{4} \right\rfloor, 5\right) \le 2(q^2 - 1) \text{ for an odd prime power } q.$$

In Section 5 (cf. Theorems 5.1–5.3) we improve this result and obtain better values of $rec(k, 5)$ for $k > 491$. The proof of these theorems is based on the *finite, relative* and *direct product difference sets* provided by Singer [17], Bose [6] and Ganley [10], respectively.

2. Constructions

Using position matrices, inspired by the ones introduced by Abreu et al. in [3], the adjacency matrix of a projective plane was directly obtained in [4]. Furthermore, using also the position matrices of a family of Latin Squares, the adjacency matrix of the following bipartite graph $\mathcal{L}_q = (\mathbb{L}, \mathbb{P})$ was also obtained in Theorem 4.4. of [4].

Definition 2.1 (*[4]*). Let $q \ge 2$ be a prime power and $\mathbb{F}_q$ a Galois field. Denote $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Consider the sets $\mathbb{L} = (\mathbb{F}_q \times \mathbb{F}_q^*) \cup (\{\infty\} \times \mathbb{F}_q^*)$ and $\mathbb{P} = (\mathbb{F}_q \times \mathbb{F}_q^*) \cup (\{\infty\} \times \mathbb{F}_q^*)$ and denote by $\ell[a, i], p(x, j)$, the elements of $\mathbb{L}$ and $\mathbb{P}$, respectively. The bipartite graph $\mathcal{L}_q = (\mathbb{L}, \mathbb{P})$ is defined as follows:

$$\text{For } i \in \mathbb{F}_q^* : N(\ell[a, i]) = \begin{cases} \{p(a + ij, j) : j \in \mathbb{F}_q^*\} \cup \{p(\infty, i)\} & \text{if } a \in \mathbb{F}_q; \\ \{p(x, i) : x \in \mathbb{F}_q\} & \text{if } a = \infty. \end{cases}$$

This graph $\mathcal{L}_q$ is in fact the incidence graph of an elliptic semiplane of type L. It is obtained by removing from the projective plane a line together with all its points, and another point not belonging to that line together with all the lines incident to it. Notice that the graph $\mathcal{L}_q$ inherits the duality principle from the projective plane: any valid sentence in $\mathcal{L}_q$ still holds true when lines $\ell[a, i]$ and points $p(x, j)$ are interchanged. In particular, the following properties of the graph $\mathcal{L}_q$ have been proved in [4] and they play a fundamental role throughout the paper.

Proposition 2.1 ([4]). *Let $\mathcal{L}_q$ be the bipartite graph given in Definition 2.1 for a prime power $q \geq 2$. Denote $L_a = \{\ell[a, i] : i \in \mathbb{F}_q^*\}$ and $P_x = \{p(x, j) : j \in \mathbb{F}_q^*\}$, for $a, x \in \mathbb{F}_q \cup \{\infty\}$. The graph $\mathcal{L}_q$ has the following properties:*

- *(i) it is a q-regular graph of girth 6 with order $2(q^2 - 1)$;*
- *(ii) its vertex set admits a partition $\mathbb{L} \cup \mathbb{P}$, where $\mathbb{L} = \bigcup_{a \in \mathbb{F}_q \cup \{\infty\}} L_a$ and $\mathbb{P} = \bigcup_{x \in \mathbb{F}_q \cup \{\infty\}} P_x$;*
- *(iii) it has diameter 4 and any two distinct vertices either in L_a or in P_x are at distance 4;*
- *(iv) each block P_x is connected to each block L_a by a perfect matching for $a, x \in \mathbb{F}_q \cup \{\infty\}$ when $x \neq a$; this matching is obtained from the adjacency rule according to Definition 2.1.*

Property (iii) in Proposition 2.1 suggests the idea of adding new edges between vertices in the sets L_a and P_x of the graph $\mathcal{L}_q$ in order to increase its regularity, as long as the girth of the resulting graph is at least 5. The Cayley colors $(uv^{-1})^{\pm 1}$ of the edges $\{u, v\}$ of a graph with vertex set $\mathbb{F}_q^*$ play a central role [1,2,9,12]. Funk introduces the following amalgamation technique:

Definition 2.2 ([9]). *Given a prime power $q \geq 3$ and graphs G_L, G_P with vertex set $\mathbb{F}_q^*$, the amalgamation of these graphs into $\mathcal{L}_q$ is the graph $\mathcal{L}_q(G_L, G_P)$ obtained by adding to the graph $\mathcal{L}_q$ the sets of edges $\{\{\ell[a, i], \ell[a, i']\} : \{i, i'\} \in E(G_L)\}$ and $\{\{p(x, j), p(x, j')\} : \{j, j'\} \in E(G_P)\}$, with $a, x \in \mathbb{F}_q \cup \{\infty\}$.*

Definition 2.3 ([9]). *Given a prime power $q \geq 3$ and an integer $r > 0$, the pair of r-regular graphs G_L, G_P with vertex set $\mathbb{F}_q^*$ and girth $g \geq 5$ is said suitable for amalgamation into $\mathcal{L}_q$, or simple suitable, if they have disjoint sets of Cayley colors in $\mathbb{F}_q^*$.*

Jørgensen (cf. [12], Theorem 3) defines an equivalent construction and achieves the same result with a different terminology.

Theorem 2.1 ([9,12]). *Given a prime power $q \geq 3$ and a pair of r-regular graphs G_L, G_P suitable for amalgamation into $\mathcal{L}_q$, the graph $\mathcal{L}_q(G_L, G_P)$ is $(q + r)$-regular, has girth at least five and order $2(q^2 - 1)$.*

Next, we provide a refinement of this technique that exploits the particularities of the perfect matchings between L_∞ and P_x ($x \neq \infty$), and between P_∞ and L_a ($a \neq \infty$), that lead us to use specific graphs H_L, H_P in addition to the pair G_L, G_P. While in [9,12] $V(G_L) = V(G_P) = \mathbb{F}_q^*$, in the new technique $V(G_L), V(G_P), V(H_L), V(H_P) \subseteq \mathbb{F}_q^*$. This removal of vertices is the key to reduce the value $rec(k, 5)$.

Definition 2.4. *Given a prime power $q \geq 3$ and graphs G_L, G_P, H_L, H_P such that $V(G_L), V(G_P), V(H_L), V(H_P) \subseteq \mathbb{F}_q^*$, the amalgamation of these graphs into $\mathcal{L}_q$ is the graph denoted $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ with vertex set $V_L \cup V_P$ and edge set $E_L \cup E_P \cup E_q$, where $V_L = \bigcup_{a \in \mathbb{F}_q}\{\ell[a, i] : i \in V(G_L)\} \cup \{\ell[\infty, i] : i \in V(H_L)\}$, $V_P = \bigcup_{x \in \mathbb{F}_q}\{p(x, j) : j \in V(G_P)\} \cup \{p(\infty, j) : j \in V(H_P)\}$, and $E_L = \left(\bigcup_{a \in \mathbb{F}_q}\{\{\ell[a, i], \ell[a, i']\} : \{i, i'\} \in E(G_L)\}\right) \cup \{\{\ell[\infty, i], \ell[\infty, i']\} : \{i, i'\} \in E(H_L)\}$, $E_P = \left(\bigcup_{x \in \mathbb{F}_q}\{\{p(x, j), p(x, j')\} : \{j, j'\} \in E(G_P)\}\right) \cup \{\{p(\infty, j), p(\infty, j')\} : \{j, j'\} \in E(H_P)\}$, $E_q = E(\mathcal{L}_q[V_L \cup V_P])$, where $\mathcal{L}_q[V_L \cup V_P]$ denotes the subgraph of $\mathcal{L}_q$ induced by $V_L \cup V_P$.*

We determine conditions on the graphs G_L, G_P, H_L, H_P to ensure that the amalgam graph $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ is $(q + r)$-regular with girth at least five.

Definition 2.5. *Let $q \geq 3$ be a prime power, $r \geq 0$ be an integer and two partitions $\mathbb{F}_q^* = R_L \cup T_L \cup W_L = R_P \cup T_P \cup W_P$. The ordered set of graphs (G_L, G_P, H_L, H_P) is r-suitable for amalgamation into $\mathcal{L}_q$, or simply r-suitable, if it satisfies the following properties:*

- *(i) $V(G_L) = \mathbb{F}_q^* \setminus W_L = R_L \cup T_L$, where the vertices in R_L and T_L have degree $r + |W_P|$ and $r + |W_P| + 1$, respectively.*
  $V(G_P) = \mathbb{F}_q^* \setminus W_P = R_P \cup T_P$, where the vertices in R_P and T_P have degree $r + |W_L|$ and $r + |W_L| + 1$, respectively.
- *(ii) H_L and H_P are r-regular graphs with $V(H_L) = \mathbb{F}_q^* \setminus (W_P \cup T_P) = R_P$ and $V(H_P) = \mathbb{F}_q^* \setminus (W_L \cup T_L) = R_L$.*
- *(iii) The graphs G_L, G_P, H_L and H_P have girth at least 5.*
- *(iv) $E(H_L) \cap E(G_P) = \emptyset$, $E(H_P) \cap E(G_L) = \emptyset$ and G_L, G_P have disjoint sets of Cayley colors in $\mathbb{F}_q^*$.*

In Fig. 1 we illustrate the partitions of $\mathbb{F}_q^*$ and the vertex sets of the graphs G_L, G_P, H_L, H_P. The eliminated sets are shown in dashed lines.

We are interested in the properties of the amalgam graph $\mathcal{L}_q(G_L, G_P, H_L, H_P)$. Suitability guarantees that it is regular and free from triangles and quadrilaterals.
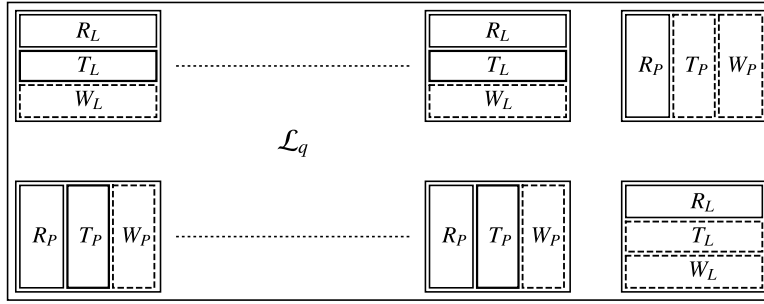
Fig. 1. $V(G_L) = R_L \cup T_L, V(G_P) = R_P \cup T_P, V(H_L) = R_P, V(H_P) = R_L.$

Theorem 2.2. *Let $q \geq 3$ be a prime power, $r \geq 0$ be an integer, and (G_L, G_P, H_L, H_P) be an ordered set of r-suitable graphs. The graph $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ is $(q + r)$-regular, has girth at least 5 and order $2(q^2 - 1) - (q + 1)(|W_L| + |W_P|) - (|T_L| + |T_P|)$.*

Proof. Denote $G = \mathcal{L}_q(G_L, G_P, H_L, H_P)$ and $G^* = \mathcal{L}_q[V_L \cup V_P]$. Let us see that G is $(q + r)$-regular by computing the degrees of the vertices of V_L, (the ones of V_P should be treated in the same way).

In G^* every vertex of $\bigcup_{a \in \mathbb{F}_q} \{\ell[a, i] : i \in V(G_L)\}$ has degree $q - 1 - |W_P|$ or $q - |W_P|$, depending on whether $i \in T_L$ or not, because

$$N_{G^*}(\ell[a, i]) = \begin{cases} \{p(a + ij, j) : j \in \mathbb{F}_q^* \setminus W_P\} & \text{if } i \in T_L; \\ \{p(a + ij, j) : j \in \mathbb{F}_q^* \setminus W_P\} \cup p(\infty, i) & \text{if } i \notin T_L, \end{cases}$$

and every vertex in $\{\ell[\infty, i] : i \in V(H_L)\}$ remains with degree q, due to $N_{G^*}(\ell[\infty, i]) = \{p(x, i) : x \in \mathbb{F}_q\}$. From the definition of G_L and H_L, the vertices of the set V_L have degree $q + r$ in G after amalgamation.

Let us check that G has girth at least five. Denote by C a shortest cycle of G and suppose, by contradiction, that $|C| \leq 4$. The graph G^* has girth at least 6 and the graphs G_L, G_P, H_L, H_P have girth at least 5, then the cycle C must have at least an edge contained in $E(G) \setminus E(G^*)$. By Proposition 2.1(iv), a triangle cannot exist in G, and C must be a cycle of length 4. Suppose that $\{\ell[a, i], \ell[a, i']\}$ is an edge of C, for $a \in \mathbb{F}_q, i, i' \in V(G_L)$ and $\{i, i'\} \in E(G_L)$. Let $p(x, j) \in V(C)$ be a neighbor of $\ell[a, i]$ and $p(x, j') \in V(C)$ be a neighbor of $\ell[a, i']$, for $x \in \mathbb{F}_q$ and $j, j' \in V(G_P)$. From Definition 2.1, it follows that $x = a + ij = a + i'j'$, yielding $ij = i'j'$ and $(j'j^{-1})^{\pm 1} = (ii'^{-1})^{\pm 1}$. Since by hypothesis G_L and G_P share no Cayley color, it follows that $\{j, j'\} \notin E(G_P)$; hence, $p(x, j)$ and $p(x, j')$ are not adjacent. Now assume that $\{\ell[\infty, i], \ell[\infty, i']\}$ is an edge of C, for $i, i' \in V(H_L)$. From Definition 2.1, edges $\{\ell[\infty, i], p(x, i)\}$ and $\{\ell[\infty, i'], p(x, i')\}$ belong to $E(G)$ for every $x \in \mathbb{F}_q$. Since $\{i, i'\} \in E(H_L)$ and $E(H_L) \cap E(G_P) = \emptyset$, vertices $p(x, i)$ and $p(x, i')$ are not adjacent. The same argument holds if we suppose that $\{p(\infty, j), p(\infty, j')\}$ is an edge of C with $j, j' \in V(H_P)$. In any case, there is no cycle C of length 4 in G. ∎

Remark 2.1. Let $q \geq 3$ be a prime power and $r \geq 0$ be an integer. According to Theorem 2.2, the existence of r-suitable graphs (G_L, G_P, H_L, H_P), for certain sets $W_L, W_P, T_L, T_P \subseteq \mathbb{F}_q^*$, ensures that

$$n(q + r, 5) \leq 2(q^2 - 1) - (q + 1)(|W_L| + |W_P|) - (|T_L| + |T_P|). \tag{1}$$

When $W_L = W_P = \emptyset$, the removal of $\gamma = 1, \ldots, q$ sets $L_a \cup P_a$ from $\mathcal{L}_q(G_L, G_P, H_L, H_P)$, for $a \neq \infty$, provides

$$n(q + r - \gamma, 5) \leq 2(q + 1 - \gamma)(q - 1) - (|T_L| + |T_P|), \tag{2}$$

and if $W_L \neq \emptyset$ or $W_P \neq \emptyset$, the deletion of a set $L_a \cup P_a$ for $a \neq \infty$ from $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ produces a $(q + r - 1, q + r)$-regular graph.

When $T_L = T_P = \emptyset$, the deletion of the set $L_\infty \cup P_\infty$ generates a graph which ensures that

$$n(q + r - 1, 5) \leq 2q(q - 1) - q(|W_L| + |W_P|). \tag{3}$$

Remark 2.2. To simplify notation, we consider the natural isomorphism between the multiplicative group $\mathbb{F}_q^*$ and the additive group $\mathbb{Z}_{q-1}$, which allows us to identify $\xi^u \in \mathbb{F}_q^*$ with $u \in \mathbb{Z}_{q-1}$, for a primitive root $\xi \in \mathbb{F}_q^*$. Thus, we replace one group by the other in the following results. In particular, let us notice that the weight or Cayley color of an edge $\{u, v\}$ is $\pm(u - v)$, for vertices $u, v \in \mathbb{Z}_{q-1}$. By $\omega(G)$ we denote the set of Cayley colors of the edges of a graph G whose vertex set $V(G) \subseteq \mathbb{Z}_{q-1}$.

Next we describe some graphs of girth at least 5 introduced in [1], which are very useful for constructing suitable graphs for amalgamation.

Definition 2.6 ([1])**.** Let $N \geq 5$ be an integer, and let $k_1, \ldots, k_m$ be a sequence of different odd elements from $\mathbb{Z}_{2N}$.

(i) We denote by $S_{2N}(k_1, \ldots, k_m)$ the graph with vertex set $\mathbb{Z}_{2N}$ and edge set $\{\{2v, 2v + k_j\} : 0 \leq v \leq N - 1, \ 1 \leq j \leq m\}$ where the sum is performed modulo $2N$.

(ii) Let $0 < P, Q < N$ be two different even elements from $\mathbb{Z}_{2N}$. We denote by $S_{2N}(P, Q; k_1, \ldots, k_m)$ the graph obtained by adding to $S_{2N}(k_1, \ldots, k_m)$ the new edges $\{2v, 2v + P\}$ and $\{2v + 1, 2v + 1 + Q\}$ for $0 \leq v \leq N - 1$, where the sum is performed modulo 2N.

In the context of this paper, we recall some properties of these graphs

Lemma 2.1 ([1])**.** *Let $N \geq 5$ be an integer, and let $k_1, \ldots, k_m$ be a sequence of different odd elements from $\mathbb{Z}_{2N}$. The following statements hold:*

(i) *The graph $S_{2N}(k_1, \ldots, k_m)$ is m-regular, bipartite and has at most m Cayley colors in $\mathbb{Z}_{2N}$. Moreover, the girth of $S_{2N}(k_1, \ldots, k_m)$ is at least 6 iff all the numbers $k_i - k_j$ are different in $\mathbb{Z}_{2N}$, for $i \neq j$ and $1 \leq i, j \leq m$.*

(ii) *Let $0 < P, Q < N$ be two different even elements from $\mathbb{Z}_{2N}$. The graph $S_{2N}(P, Q; k_1, \ldots, k_m)$ is $(m + 2)$-regular and has at most $m + 2$ Cayley colors. Moreover, the girth of $S_{2N}(P, Q; k_1, \ldots, k_m)$ is at least 5 iff the following conditions hold in $\mathbb{Z}_{2N}$:*

(a) *The numbers $3P, 4P, 3Q, 4Q$ are different from 0.*
(b) *All the numbers $k_i - k_j$ are different for $i \neq j$ and $1 \leq i, j \leq m$.*
(c) *$k_i - k_j \neq a - a'$ for every pair $a, a' \in \{0, \pm P, \pm Q\}$.*

Lemma 2.2 ([1])**.** *Let $k_1, \ldots, k_m$ be a sequence of different odd integers such that all the numbers $k_i - k_j$ are different in $\mathbb{Z}$, for $i \neq j$ and $1 \leq i, j \leq m$; and let P, Q be two positive even integers such that $k_i - k_j \neq a - a'$ for every pair $a, a' \in \{0, \pm P, \pm Q\}$. The following statements hold:*

(i) *For every integer $N > \max\{|k_i|, k_i - k_j\}$, the graph $S_{2N}(k_1, \ldots, k_m)$ is m-regular, bipartite with girth at least 6 and has at most m Cayley colors in $\mathbb{Z}_{2N}$.*

(ii) *For every integer $N > \max\{|k_i|, k_i - k_j, a - a'\}$ where $a, a' \in \{0, \pm P, \pm Q\}$, the graph $S_{2N}(P, Q; k_1, \ldots, k_m)$ is $(m + 2)$-regular, with girth at least 5 and has at most $m + 2$ Cayley colors.*

The graphs described in Definition 2.6 are used in the rest of this paper. They will be given either by explicit constructions, or by using their relationship with finite difference sets.

3. Results for small values, $11 \leq q \leq 29$

We provide suitable graphs (G_L, G_P, H_L, H_P) with $V(G_L), V(G_P), V(H_L), V(H_P) \subseteq \mathbb{Z}_{q-1}$ to construct $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ and improve $rec(k, 5)$ for $k \in \{13, 14, 21, 23, 25, 27, 29, 31, 33\}$. Applying Remark 2.1 we also obtain new values of $rec(k, 5)$ for $k \in \{24, 26, 28, 30, 32\}$.

Theorem 3.1. *The upper bounds on $n(k, 5)$ listed in the last column of Table 1 for $k \in \{13, 14, 21, 23, 24, \ldots, 33\}$ provide better values of $rec(k, 5)$.*

Proof. For $q \in \{11, 13, 19, 23, 25, 27, 29\}$, let us consider the graph $\mathcal{L}_q$ given in Definition 2.1. Next we provide graphs (G_L, G_P, H_L, H_P), r-suitable for amalgamation into $\mathcal{L}_q$ with $r = k - q$. Theorem 2.2 guarantees that the graph $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ is k-regular and has girth at least 5.

▶ $q = 11 \rightarrow k = 13$:
Consider $q = 11$, $r = 2$, $W_L = W_P = \emptyset$, $T_L = \mathbb{Z}_{10}$ and $T_P = \{0, 4, 5, 9\}$. Let G_L be a Petersen graph with $V(G_L) = \mathbb{Z}_{10}$, two disjoint pentagons $(0, 2, 4, 6, 8, 0), (1, 5, 9, 3, 7, 1)$ and edges $\{0, 1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \{8, 9\}$ joining them. Hence, the set of Cayley colors of G_L (in $\mathbb{Z}_{10}$) is $\omega(G_L) = \pm\{1, 2, 4\}$. Let G_P be with $V(G_P) = \mathbb{Z}_{10}$ formed by $S_{10}(3, -3)$ and the chords $\{0, 5\}, \{4, 9\}$. The set of Cayley colors of G_P (in $\mathbb{Z}_{10}$) is $\omega(G_P) = \pm\{3, 5\}$, and clearly, $\omega(G_L) \cap \omega(G_P) = \emptyset$. By Theorem 2.2, $H_P = \emptyset$ and the vertices of H_L are the vertices of degree 2 in G_P, that is, $V(H_L) = \{1, 2, 3, 6, 7, 8\}$. Let H_L be the 6-cycle $(1, 2, 3, 8, 7, 6, 1)$ which satisfies the condition $E(H_L) \cap E(G_P) = \emptyset$. Since (G_L, G_P, H_L, H_P) is an ordered set of r-suitable graphs, the amalgam $\mathcal{L}_{11}(G_L, G_P, H_L, \emptyset)$ is 13-regular, has girth 5 and $2(11^2 - 1) - 10 - 4 = 226$ vertices. It states $rec(13, 5) = 226$.

▶ $q = 13 \rightarrow k = 14$:
Consider $q = 13$, $W_L = \{2, 7\}$, $W_P = \{7, 11\}$, $T_L = T_P = \emptyset$ and $r = 1$. Let G_L be a Petersen graph formed by the disjoint pentagons $(1, 6, 11, 4, 9, 1), (0, 5, 10, 3, 8, 0)$ and edges $\{1, 8\}, \{6, 5\}, \{11, 3\}, \{4, 0\}, \{9, 10\}$ joining them. The set of Cayley colors of G_L (in $\mathbb{Z}_{12}$) is $\omega(G_L) = \pm\{1, 4, 5\}$. Let G_P be a Petersen graph with disjoint pentagons $(0, 9, 6, 8, 10, 0), (2, 5, 3, 1, 4, 2)$, and edges $\{0, 2\}, \{9, 3\}, \{6, 4\}, \{8, 5\}, \{10, 1\}$ joining them. The set of Cayley colors of G_P (in $\mathbb{Z}_{12}$) is $\omega(G_P) = \pm\{2, 3, 6\}$, so $\omega(G_L) \cap \omega(G_P) = \emptyset$. By Theorem 2.2, $V(H_L) = \mathbb{Z}_{12} \setminus W_P$, $V(H_P) = \mathbb{Z}_{12} \setminus W_L$ and the degree of both H_L and H_P must be equal to $r = 1$. Let H_L be formed by the edges $\{0, 1\}, \{2, 3\}, \{4, 5\}, \{8, 9\}, \{6, 10\}$ and let H_P be formed by the edges $\{0, 3\}, \{1, 4\}, \{6, 9\}, \{8, 10\}, \{5, 11\}$. Notice that $E(H_P) \cap E(G_L) = \emptyset$ and $E(H_L) \cap E(G_P) = \emptyset$. Since the ordered set (G_L, G_P, H_L, H_P) is r-suitable, the graph $\mathcal{L}_{13}(G_L, G_P, H_L, H_P)$ is 14-regular, has girth 5 and order $2(13^2 - 1) - 56 = 280$. This provides $rec(14, 5) = 280$.

▶ $q = 19 \rightarrow k = 21$:

For $q = 19$, consider $r = 2$, $W_L = W_P = \{6\}$, $T_L = \{0, 2, 4, 5, 7, 8, 10, 13, 14, 15, 16\}$ and $T_P = \{0, 1, 3, 4, 5, 8, 11, 12, 13, 16, 17\}$. Let G_L be the $(3, 4)$-regular graph formed by the hamiltonian cycle $(0, 5, 12, 13, 2, 16, 3, 8, 7, 11, 15, 1, 14, 10, 9, 4, 17, 0)$ and the paths $(1, 0, 7, 2, 9)$, $(3, 14, 13, 17)$, $(11, 10, 5, 16, 15, 4, 8, 12)$. Let G_P be the $(3, 4)$-regular graph formed by the hamiltonian cycle $(0, 9, 11, 13, 10, 8, 2, 12, 4, 7, 17, 1, 16, 14, 5, 3, 15, 0)$ and the paths $(2, 17, 11, 14)$, $(9, 12, 3, 1, 10)$, $(7, 5, 8, 0, 16, 4, 13, 15)$. Since $\omega(G_L) = \pm\{1, 4, 5, 7\}$ and $\omega(G_P) = \pm\{2, 3, 6, 8, 9\}$ in $\mathbb{Z}_{18}$, we have $\omega(G_L) \cap \omega(G_P) = \emptyset$. Graphs H_L and H_P with $V(H_L) = \mathbb{Z}_{18} \setminus (W_P \cup T_P)$ and $V(H_P) = \mathbb{Z}_{18} \setminus (W_L \cup T_L)$ are formed by the cycles $(2, 7, 14, 15, 10, 9, 2)$ and $(1, 3, 9, 11, 17, 12, 1)$, respectively. Notice that $E(H_L) \cap E(G_P) = E(H_P) \cap E(G_L) = \emptyset$. Since all the requirements of suitability are satisfied, the graph $\mathcal{L}_{17}(G_L, G_P, H_L, H_P)$ has degree 21 and girth 5. Hence $rec(21, 5) = 2(19^2 - 1) - 40 - 22 = 658$.

▶ $q = 23 \rightarrow k = 23$:

Consider $q = 23$, $r = 0$, $W_L = \{5, 10, 20\}$, $W_P = \{1, 10, 17\}$, $T_L = \mathbb{Z}_{22} \setminus W_L$ and $T_P = \mathbb{Z}_{22} \setminus W_P$. Since G_L and G_P must have degree 4 and girth 5, both must be isomorphic to the unique $(4, 5)$-cage [14,16]. Consider the graph G_L with vertex set $V(G_L) = T_L$ formed by the two hamiltonian cycles $(0, 18, 4, 11, 7, 3, 8, 9, 13, 12, 19, 15, 14, 21, 16, 1, 6, 2, 17, 0)$, $(0, 7, 12, 16, 17, 9, 4, 21, 3, 2, 19, 18, 1, 8, 15, 11, 6, 13, 14, 0)$ and the graph G_P with $V(G_P) = T_P$ formed by the two cycles $(0, 16, 13, 2, 12, 21, 5, 7, 4, 6, 3, 14, 20, 11, 8, 18, 15, 9, 19, 0)$, $(0, 12, 6, 8, 19, 7, 13, 11, 21, 9, 3, 16, 18, 5, 14, 2, 15, 4, 20, 0)$. According to Definition 2.5 (ii), $V(H_L) = V(H_P) = \emptyset$. Since $\omega(G_L) = \pm\{1, 4, 5, 7, 8\}$ and $\omega(G_P) = \pm\{2, 3, 6, 9, 10, 11\}$ in $\mathbb{Z}_{22}$, the suitability condition $\omega(G_L) \cap \omega(G_P) = \emptyset$ is satisfied and the graph $\mathcal{L}_{23}(G_L, G_P, \emptyset, \emptyset)$ has degree 23 and girth 5 with $2(23^2 - 1) - 24 \cdot 6 - 38 = 874$ vertices. Then, $rec(23, 5) = 874$.

▶ $q = 23 \rightarrow k = 25, 24$:

To construct a $(25, 5)$-regular graph, consider $q = 23$, $r = 2$, $T_L = T_P = \emptyset$, $W_L = \{2, 7\}$ and $W_P = \{7, 14\}$. Let G_L be the 4-regular graph formed by the cycles $(4, 11, 6, 13, 12, 19, 4)$, $(0, 17, 21, 3, 8, 1, 18, 14, 10, 9, 16, 15, 20, 5, 0)$, $(0, 4, 9, 1, 6, 5, 10, 3, 11, 15, 14, 13, 21, 20, 19, 18, 17, 16, 12, 8, 0)$, and let G_P be the 4-regular graph formed by the cycles $(0, 9, 6, 4, 13, 16, 18, 12, 15, 17, 20, 11, 21, 19, 3, 1, 10, 8, 5, 2, 0)$, $(9, 15, 13, 11, 5, 3, 9)$, $(0, 10, 21, 12, 2, 4, 1, 20, 18, 6, 8, 17, 19, 16, 0)$. Since in $\mathbb{Z}_{22}$, $\omega(G_L) = \pm\{1, 4, 5, 7, 8\}$ and $\omega(G_P) = \pm\{2, 3, 6, 9, 10, 11\}$, we have $\omega(G_L) \cap \omega(G_P) = \emptyset$. Let H_L and H_P be the cycles $(0, 4, 3, 8, 12, 16, 20, 2, 6, 10, 18, 11, 15, 19, 1, 5, 13, 9, 17, 21, 0)$ and $(0, 3, 6, 9, 12, 15, 18, 5, 21, 1, 4, 10, 13, 16, 19, 8, 11, 14, 17, 20, 0)$, respectively. Clearly, $V(H_L) = \mathbb{Z}_{22} \setminus W_P$, $V(H_P) = \mathbb{Z}_{22} \setminus W_L$. Also, $E(H_L) \cap E(G_P) = \emptyset$ and $E(H_P) \cap E(G_L) = \emptyset$. Since the suitability conditions are satisfied, the graph $\mathcal{L}_{23}(G_L, G_P, H_L, H_P)$ has girth 5, is 25-regular and has $2(23^2 - 1) - 4 \cdot 24 = 960$ vertices which gives the new value $rec(25, 5) = 960$. Moreover, from Remark 2.1 (3) we have the new bound $rec(24, 5) = 920$.

▶ $q = 23 \rightarrow k = 27, 26$:

To construct a $(27, 5)$-regular graph, consider $q = 23$, $r = 4$, $W_L = W_P = T_L = \emptyset$ and $T_P = \{6, 17\}$. Let $G_L = S_{22}(4, 8; 1, -1)$, which is a 4-regular graph of girth 5 with $\omega(G_L) = \pm\{1, 4, 8\}$. Let G_P be formed by the cycle $S_{22}(7, -7)$ together with the three paths $(6, 16, 3, 20, 8, 21, 11, 17)$, $(6, 12, 18, 15, 2, 13, 4, 7, 5, 17)$ and $(6, 0, 10, 19, 9, 14, 5, 17)$. Hence, G_P is a $(4, 5)$-regular graph of girth 5 with $\omega(G_P) = \pm\{3, 5, 6, 7, 9, 10, 11\}$. Clearly $\omega(G_L) \cap \omega(G_P) = \emptyset$. Let H_L be the graph with $V(H_L) = \mathbb{Z}_{22} \setminus T_P$ formed by the cycles $(0, 9, 8, 4, 5, 11, 3, 2, 20, 16, 12, 13, 14, 10, 1, 21, 15, 7, 18, 19, 0)$, $(0, 13, 5, 18, 10, 9, 11, 16, 15, 14, 3, 19, 21, 4, 20, 0)$, $(1, 2, 7, 8, 12, 1)$ and $H_P = S_{22}(6, 10; 7, -7)$. It is easy to check that H_L and H_P are 4-regular graphs of girth 5 such that $E(H_L) \cap E(G_P) = \emptyset$, $E(H_P) \cap E(G_L) = \emptyset$. By Theorem 2.2, $\mathcal{L}_{23}(G_L, G_P, H_L, H_P)$ has girth 5, is 27-regular and has $2(23^2 - 1) - 2 = 1054$ vertices. This provides $rec(27, 5) = 1054$. Remark 2.1 (2) states $rec(26, 5) = 1010$.

▶ $q = 25 \rightarrow k = 29, 28$:

For $q = 25$, consider $r = 4$, $W_L = W_P = \emptyset$ and $T_L = T_P = \{7, 11, 19, 23\}$. Let G_L be the graph $S_{24}(2, 10; 3, -3)$ together with the edges $\{7, 19\}$, $\{11, 23\}$. Let G_P be formed by the cycles $(2, 17, 18, 11, 10, 14, 20, 19, 2)$, $(1, 5, 6, 22, 3, 9, 8, 21, 1)$, $(0, 4, 23, 16, 12, 13, 7, 15, 0)$, $(0, 1, 17, 23, 8, 19, 12, 3, 14, 7, 6, 11, 0)$, $(2, 15, 9, 18, 13, 21, 10, 16, 5, 20, 4, 22, 2)$, and the edges $\{7, 23\}$, $\{11, 19\}$. Let H_L be formed by the cycles $(6, 13, 5, 12, 10, 9, 17, 16, 15, 4, 21, 2, 3, 8, 0)$, $(0, 16, 20, 13, 4, 10, 1, 3, 15, 14, 12, 2, 6, 9, 22, 8, 5, 17, 21, 18, 0)$, $(1, 18, 14, 22, 20)$, and H_P by the cycles $(0, 10, 14, 20, 9, 3, 4, 5, 1, 8, 17, 2, 13, 21, 15, 0)$, $(0, 12, 8, 14, 13, 18, 3, 10, 22, 2, 9, 1, 21, 16, 4, 17, 15, 20, 6, 5, 0)$ and $(18, 6, 22, 16, 12)$. The graphs G_L, G_P, H_L, H_P have girth 5, $E(H_L) \cap E(G_P) = \emptyset$, $E(H_P) \cap E(G_L) = \emptyset$ and $\omega(G_L) \cap \omega(G_P) = \emptyset$ because $\omega(G_L) = \pm\{2, 3, 10, 12\}$ and $\omega(G_P) = \pm\{1, 4, 5, 6, 7, 8, 9, 11\}$. Since the ordered set (G_L, G_P, H_L, H_P) is r-suitable, the graph $\mathcal{L}_{25}(G_L, G_P, H_L, H_P)$ is 29-regular, has girth 5 and order $2(25^2 - 1) - 8 = 1240$, proving that $rec(29, 5) = 1240$. Moreover, Remark 2.1 (2) states $rec(28, 5) = 1192$.

▶ $q = 27 \rightarrow k = 31, 30$:

To construct a $(31, 5)$-regular graph, consider $q = 27$, $r = 4$, $W_L = W_P = \emptyset$ and $T_L = T_P = \{5, 7, 9, 19, 21, 23\}$. Let G_L be the $(4, 5)$-regular graph with $V(G_L) = \mathbb{Z}_{26}$ and $E(G_L) = E(S_{26}(4, 8, 1, -1)) \cup \{\{5, 19\}, \{7, 21\}, \{9, 23\}\}$, and let G_P be formed by the cycles $(0, 10, 3, 16, 22, 13, 4, 11, 14, 12, 1, 17, 24, 0)$, $(0, 2, 11, 8, 10, 5, 14, 9, 3, 23, 12, 18, 16, 25, 1, 6, 22, 20, 17, 7, 13, 19, 24, 21, 4, 15, 0)$, $(2, 23, 20, 15, 9, 6, 21, 5, 25, 19, 8, 18, 7, 2)$, and the edges $\{5, 7\}$, $\{9, 19\}$, $\{21, 23\}$. Since $\omega(G_L) = \pm\{1, 4, 8, 12\}$ and $\omega(G_P) = \pm\{2, 3, 5, 6, 7, 9, 10, 11, 13\}$ in $\mathbb{Z}_{26}$, it follows that $\omega(G_L) \cap \omega(G_P) = \emptyset$. Let H_L be the 4-regular graph with $V(H_L) = \mathbb{Z}_{26} \setminus T_P$ and cycles $(0, 16, 12, 13, 17, 10, 11, 3, 15, 22, 14, 2, 1, 24, 8, 4, 20, 6, 18, 25, 0)$, $(0, 1, 13, 20, 14, 10, 24, 6, 16, 15, 17, 18, 2, 3, 4, 0)$, $(11, 25, 22, 8, 12, 11)$, and H_P be the 4-regular graph with $V(H_P) = \mathbb{Z}_{26} \setminus T_L$ with cycles $(0, 11, 8, 25, 13, 18, 3, 10, 22, 2, 12, 17, 24, 16, 1, 20, 15, 6, 4, 14, 0)$, $(0, 10, 25, 6, 12, 3, 1, 14, 17, 8, 20, 2, 13, 24, 15, 0)$, $(18, 4, 22, 16, 11, 18)$. Since all the suitability conditions are satisfied, bounds (1) and (2) in Remark 2.1 establish $rec(31, 5) = 1444$ and $rec(30, 5) = 1392$, respectively.

▶ $q = 29 \rightarrow k = 33, 32$:

Consider $q = 29, r = 4, T_L = T_P = \{2, 4, 7, 11, 16, 18, 21, 25\}$ and $W_L = W_P = \emptyset$. Let G_L be the graph with $E(G_L) = E(S_{28}(4, 8; 1, -1)) \cup \{\{2, 16\}, \{4, 18\}, \{7, 21\}, \{11, 25\}\}$ and let G_P be the graph formed by the hamiltonian cycle $(0, 22, 12, 2, 20, 9, 11, 6, 8, 24, 15, 17, 14, 21, 27, 10, 3, 16, 19, 25, 7, 5, 26, 23, 4, 1, 13, 18, 0)$, the 28-length-path $(1, 22, 9, 3, 5, 18, 8, 25, 10, 13, 11, 23, 2, 14, 19, 26, 24, 21, 0, 16, 6, 17, 7, 20, 27, 4, 15, 12)$ and the edges $\{1, 7\}, \{2, 18\}, \{4, 16\}, \{11, 21\}, \{12, 25\}$. Since $\omega(G_L) = \pm\{1, 4, 8, 14\}$ and $\omega(G_P) = \pm\{2, 3, 5, 6, 7, 9, 10, 11, 12, 13\}$ in $\mathbb{Z}_{28}$, it follows that $\omega(G_L) \cap \omega(G_P) = \emptyset$. Consider the 4-regular graph H_L with cycles $(0, 23, 12, 20, 14, 10, 8, 3, 15, 27, 6, 26, 1, 9, 13, 24, 5, 17, 22, 19, 0)$, $(8, 19, 27, 13, 12, 8)$, $(0, 1, 20, 5, 6, 10, 9, 17, 23, 15, 14, 22, 26, 3, 24, 0)$; and the isomorphic graph H_P formed by the cycles $(3, 17, 20, 22, 12, 3)$, $(0, 10, 23, 26, 8, 1, 27, 9, 5, 14, 24, 6, 13, 19, 15, 0)$, $(0, 3, 14, 23, 13, 17, 1, 10, 22, 6, 8, 5, 19, 12, 27, 24, 15, 26, 20, 9, 0)$. The graphs G_L, G_P, H_L and H_P are r-suitable, and bounds (1) and (2) state $rec(33, 5) = 1664$ and $rec(32, 5) = 1608$, respectively. ∎

4. Results for greater values, $61 \le q \le 479$

In the section above, new values of $rec(k, 5)$ are obtained by amalgams into $\mathcal{L}_q$, for prime powers $11 \le q \le 29$. In the next theorem we focus on $q = 61$ and $q \ge 71$ and amalgamate regular graphs G_L, G_P with degrees $r_q > 6$; similarly as in [9] for $r_q = 4$ and as in [1] for $r_q = 5, 6$.

Theorem 4.1. *Let $q \ge 61$ be an odd prime power. Then $n(q + r_q, 5) \le 2(q^2 - 1)$ where r_q is the following integer:*

$$
r_q = \begin{cases}
7 & \text{if } q \in \{61, 71, \dots, 89\}; \\
8 & \text{if } q \in \{101, \dots, 109\}; \\
9 & \text{if } q \in \{97, 113, \dots, 139\} - \{127, 131\}; \\
10 & \text{if } q \in \{127, 131, 149, \dots, 181\}; \\
11 & \text{if } q \in \{191, \dots, 223\}; \\
12 & \text{if } q \ge 227 \text{ and } q \ne 241; \\
13 & \text{if } q = 241.
\end{cases}
$$

Proof. For each odd prime power $q \ge 61$ (except $q = 67$) we construct a definite pair of suitable r_q-regular graphs G_L and G_P. Theorem 2.1 proves that the amalgam $\mathcal{L}_q(G_L, G_P)$ is $(q + r_q)$-regular with girth at least 5 and order $2(q^2 - 1)$. Next, we provide these two graphs for $q \in \{61, 97, 127, 131\}$ and $q \ge 263$. For the rest of values q the graphs G_L, G_P are indicated in Table 2.

▶ $q = 61 \rightarrow k = 68$:

We consider the 7-regular graph $G_L = S_{60}(12, 24; 1, -1, 5, 19, 27)$. To construct another 7-regular graph G_P, we add to the 5-regular bipartite graph $S_{60}(3, 7, 15, -29, 29)$ the edges $\{\{2i, 2i + 10\} : 0 \le i \le 29\}$, and the edges of the five cycles $(1 + 2i, 31 + 2i, 11 + 2i, 51 + 2i, 21 + 2i, 41 + 2i, 1 + 2i)$, for $0 \le i \le 4$. Since both G_L and G_P have girth 5 and $\omega(G_L) \cap \omega(G_P) = \emptyset$, due to $\omega(G_L) = \pm\{12, 24, 1, 5, 19, 27\}$ and $\omega(G_P) = \pm\{10, 20, 30, 3, 7, 15, 29\}$, the graph $\mathcal{L}_{61}(G_L, G_P)$ yields $n(61 + 7, 5) \le 2(61^2 - 1) = 7440$.

▶ $q = 97 \rightarrow k = 106$:

Let G_L be the graph with $V(G_L) = \mathbb{Z}_{96}$ and edges the ones in $S_{96}(1, -1, 7, 19, -23, -27, -37)$ (isomorphic to $\mathcal{L}_7$) together with the edges $\{2i, 2i+16\}$, for $0 \le i \le 47$, and the ones in the cycles $(1 + 2i, 49 + 2i, 17 + 2i, 81 + 2i, 33 + 2i, 65 + 2i, 1 + 2i)$, for $0 \le i \le 7$. The graph G_L is 9-regular, has girth 5 and set of Cayley colors $\omega(G_L) = \pm\{16, 32, 48, 1, 7, 19, 23, 27, 37\}$. Let G_P be the isomorphic graph obtained from G_L by the permutation on $V(G_L) = \mathbb{Z}_{96}$ expressed as the following product of cycles:

(1 29 62 37 19 90 55 77 76 46 27 18 11 8 4 2) · (3 48 30 17 39 63 68 42 25 66 41 64 40 24 14 7 38 23 67 70 43 72 44 26 15 28 16 10 5 69 86 53 89 84 52 32 20 12 6) · (9 83 92 56 34 21 79 58 35 82 51 88 54 33 60 36 22 13) · (31 71 87 78 47 73 74 45 93 95 85 61 80 50) · (49 59) · (81 91 94 57).

The set $\omega(G_P) = \mathbb{Z}_{96} \setminus (\{0\} \cup \omega(G_L))$, and the graph $\mathcal{L}_{96}(G_L, G_P)$ yields $n(96 + 9, 5) \le 2(97^2 - 1)$.

▶ $q = 127 \rightarrow k = 137$:

We start with the graph $S_{126}(1, -1, 5, 13, 29, 39, -45, 61)$ which is isomorphic to $\mathcal{L}_8$, and consider the graph $G_L = S_{126}(18, 36; 1, -1, 5, 13, 29, 39, -45, 61)$. It is 10 regular, has girth 5 and $\omega(G_L) = \pm\{18, 36, 1, 5, 13, 29, 39, 45, 61\}$. The isomorphic graph G_P is obtained from G_L by the following permutation on $V(G_L) = \mathbb{Z}_{126}$:

(1 106 78 59 23 123 81 27 39 55 11 54 42 31 96 79 67 117 121 115 12 6 3 103 53 112 86 63 99 105 102 76 58 44 32 21 41 9 82 61 94 72 56 43 124 92 71 116 88 64 47 37 52 36 28 19 109 69 80 60 45 125 98 74 57 51 108 84 62 46 33 65 25 118 89 10 5 114 87 104 77 107 122 91 13 100 75 101 110 85 93 97 11 26 18 14 7 120 90 70 50 35 66 48 34 22 16 8 4 2) · (15 38 29 40 30 20) · (17 68 49 24) · (113 119).

Since $\omega(G_P) \cap \omega(G_L) = \emptyset$, the graph $\mathcal{L}_{127}(G_L, G_P)$ establishes $n(127 + 10, 5) \le 2(127^2 - 1)$.

▶ $q = 131 \rightarrow k = 141$:

Consider the 10-regular graph $G_L = S_{130}(26, 52; 1, -1, 7, -13, -31, -35, 55, -59)$ and the isomorphic graph G_P obtained by the following permutation on $V(G_L) = \mathbb{Z}_{130}$:

(1 34 24 12 6 3 110 83 79 77 76 52 40 27 35 111 99 118 87 121 126 91 95 54 41 56 42 28 21 58 43 39 117 103 123 122 89 115 102 72 50 32 23 33 18 9 38 26 20 10 5 13 73 74 51 17 116 86 64 46 30 22 11 94 68 48 31 37 36 25 14 7 119 120 88

Table 2
The r_q-regular graphs G_L and G_P are suitable for amalgamation into $\mathcal{L}_q$.

q	r_q	G_L	G_P
61	7	$S_{60}(12, 24; 1, -1, 5, 19, 27)$	*See proof of Theorem 4.1*
71	7	$S_{70}(10, 20; 1, -1, 5, 13, -23)$	$S_{70}(2, 4; 3, -7, 15, -21, 31)$
73	7	$S_{72}(16, 28; 1, -1, 5, -9, 25)$	$S_{72}(2, 4; 3, -7, 15, -23, 35)$
79	7	$S_{78}(12, 24; 1, -1, 5, -9, -27)$	$S_{78}(2, 4; 3, -7, 15, -21, 31)$
81	7	$S_{80}(12, 24; 1, -1, 5, -9, -29)$	$S_{80}(2, 4; 3, -7, 15, -21, -39)$
83	7	$S_{82}(12, 24; 1, -1, 5, -9, -27)$	$S_{82}(2, 4; 3, -7, 15, -21, 31)$
89	7	$S_{88}(12, 24; 1, -1, 5, -9, -27)$	$S_{88}(2, 4; 3, -7, 15, -21, 33)$
97	9	*See proof of Theorem 4.1*	*See proof of Theorem 4.1*
101	8	$S_{100}(16, 32; 1, -1, 5, -9, -21, -39)$	$S_{100}(2, 4; 3, -7, 15, -23, 33, -43)$
103	8	$S_{102}(16, 32; 1, -1, 5, -9, -21, 35)$	$S_{102}(2, 4; 3, -7, 15, -23, -43, 45)$
107	8	$S_{106}(16, 32; 1, -1, 5, -9, -21, 29)$	$S_{106}(2, 4; 3, -7, 15, -23, 33, 47)$
109	8	$S_{108}(16, 32; 1, -1, 5, -9, -21, 29)$	$S_{108}(2, 4; 3, -7, 15, -23, 33, -43)$
113	9	$S_{112}(18, 36; 1, -1, 27, 31, 43, -47, 51)$	$S_{112}(2, 4; 3, -15, 25, -29, 37, -39, 53)$
121	9	$S_{120}(18, 36; 1, -1, 5, -11, -25, -33, 45)$	$S_{120}(6, 12; 3, -7, -29, -37, 41, 55, 57)$
125	9	$S_{124}(18, 36; 1, -1, 5, 15, -25, -33, -45)$	$S_{124}(2, 4; 3, -7, -19, 21, -39, 47, -53)$
127	10	$S_{126}(18, 36; 1, -1, 5, 13, 29, 39, -45, 61)$	*See proof of Theorem 4.1*
131	10	$S_{130}(26, 52; 1, -1, 7, -13, -31, -35, 55, -59)$	*See proof of Theorem 4.1*
137	9	$S_{136}(18, 36; 1, -1, 5, -9, 21, 49, 61)$	$S_{136}(2, 4; 3, -7, 15, -23, 33, -43, 65)$
139	9	$S_{138}(18, 36; 1, -1, 5, -9, 21, 33, -47)$	$S_{138}(2, 4; 3, -7, 15, -23, 35, 49, -59)$
149	10	$S_{148}(18, 36; 1, -1, 7, -21, 31, 41, 45, 57)$	$S_{148}(2, 4; 3, -9, 17, -25, -43, 53, -63, -73)$
151	10	$S_{150}(18, 36; 1, -1, 5, 15, -23, 45, 57, 65)$	$S_{150}(2, 4; 3, -7, 19, -21, 37, 49, -59, 69)$
157	10	$S_{156}(18, 36; 1, -1, 5, 13, -29, -39, -61, 69)$	$S_{156}(2, 4; 3, -7, 19, -21, -43, -55, 63, -73)$
163	10	$S_{162}(18, 36; 1, -1, 5, -9, -25, -47, -59, 75)$	$S_{162}(2, 4; 3, -7, 15, -21, 31, -41, -71, 73)$
167	10	$S_{166}(18, 36; 1, -1, 5, -9, 21, -41, 49, -65)$	$S_{166}(2, 4; 3, -7, 15, -23, 33, -55, 69, -69)$
169	10	$S_{168}(18, 36; 1, -1, 5, -9, -21, 47, -59, 81)$	$S_{168}(2, 4; 3, -7, 15, -23, 43, -53, 67, -67)$
173	10	$S_{172}(18, 36; 1, -1, 5, -9, 21, 49, 61, -77)$	$S_{172}(2, 4; 3, -7, 15, -23, 39, 57, -67, -81)$
179	10	$S_{178}(18, 36; 1, -1, 5, -9, 21, 33, -61, 79)$	$S_{178}(2, 4; 3, -7, 15, -23, -59, -73, 75)$
181	10	$S_{180}(18, 36; 1, -1, 5, -9, 21, 33, -61, 79)$	$S_{180}(2, 4; 3, -7, 15, -23, 35, -37, 59, -85)$
191	11	$S_{190}(18, 36; 1, -1, 5, 15, -25, 43, 63, 75, -81)$	$S_{190}(2, 4; 3, -7, 17, -23, -41, 49, -61, 79, -89)$
193	11	$S_{192}(18, 36; 1, -1, 5, 13, 29, 39, -45, 61, 81)$	$S_{192}(4, 8; 3, -3, 17, -25, -35, 43, 73, -83, -85)$
197	11	$S_{196}(18, 36; 1, -1, 5, -9, -33, 49, -61, 79, 95)$	$S_{196}(2, 4; 3, -7, 15, 35, -41, 51, 65, -67, 89)$
199	11	$S_{198}(18, 36; 1, -1, 5, -9, -21, 39, -51, 67, 83)$	$S_{198}(2, 4; 3, -7, 15, -23, -43, 55, 69, 87, -87)$
211	11	$S_{210}(18, 36; 1, -1, 5, -9, 21, 33, -57, 85, -101)$	$S_{210}(2, 4; 3, -7, 15, -23, -37, 47, -71, -95, 97)$
223	11	$S_{222}(18, 36; 1, -1, 5, -9, 21, 33, -47, 91, -107)$	$S_{222}(2, 4; 3, -7, 15, -23, 35, 49, -51, -75, 97)$
227	12	$S_{226}(18, 36; 1, -1, 5, -9, 33, 49, -65, 79, -85, 101)$	$S_{226}(2, 4; 3, -7, 15, 35, -37, 51, 69, -93, 95, -107)$
229	12	$S_{228}(12, 30; 1, -1, 5, -9, -45, 57, 73, -79, 95, -107)$	$S_{228}(18, 42; 3, -3, 11, 31, 41, 43, -61, 93, 97, -109)$
233	12	$S_{232}(36, 72; 1, -1, 5, 25, -39, 47, 59, 75, -95, -105)$	$S_{232}(4, 8; 3, -7, -35, 43, 57, 63, -69, 87, -87, 89)$
239	12	$S_{238}(18, 36; 1, -1, -5, 11, -29, 33, 59, -85, 103, 111)$	$S_{238}(2, 4; 3, -7, 19, -37, 51, 63, -71, 87, -89, -109)$
241	13	$S_{240}(24, 48; 1, -1, -5, 9, 27, 39, -55, 61, -75, -91, 107)$	$S_{240}(72, 96; -3, 15, -15, 23, 25, 29, -37, 79, -83, 99, 115)$
243	12	$S_{242}(18, 36; 1, -1, -5, 9, -21, 43, 75, -89, 101, 113)$	$S_{242}(2, 4; 3, -7, 15, -23, 33, -59, 79, -91, 103, -119)$
251	12	$S_{250}(18, 36; 1, -1, 5, -9, 25, 47, -59, 87, 99, 115)$	$S_{250}(2, 4; 3, -7, 15, -21, -39, 43, 63, -65, -95, -111)$
257	12	$S_{256}(18, 36; 1, -1, 5, -9, 21, -43, 61, 87, 99, -111)$	$S_{256}(2, 4; 3, -7, 15, -23, 33, -55, 57, -69, -89, 93)$
263	12	$S_{262}(28, 56; 1, -1, -13, 61, 65, 91, 97, 21, 107, 115)$	$S_{262}(20, 40; 3, -15, 27, -47, 49, 55, 57, -59, -63, -73)$
⋮	⋮	⋮	⋮
q	12	$S_{q-1}(28, 56; 1, -1, -13, 61, 65, 91, 97, 21, 107, 115)$	$S_{q-1}(20, 40; 3, -15, 27, -47, 49, 55, 57, -59, -63, -73)$

65 127 129 125 128 92 67 53 55 19 106 81 78 60 44 29 112 84 63 114 85 168 4 2) · (100 71 97 108 82 62 45 15 113 101 109 105 124 90 66 47) · (61 59 107 104 80) · (69 98 70 49 93 96).
Since $\omega(G_P) \cap \omega(G_L) = \emptyset$, the graph $\mathcal{L}_{131}(G_L, G_P)$ establishes $n(131 + 10, 5) \leq 2(131^2 - 1)$.

▶ $q \geq 263 \rightarrow k = q + 12$:

Consider the graphs $G_L = S_{q-1}(28, 56; 1, -1, -13, 61, 65, 91, 97, 21, 107, 115)$ and $G_P = S_{q-1}(20, 40; 3, -15, 27, -47, 49, 55, 57, -59, -63, -73)$. The integers in their definitions verify the conditions in Lemma 2.2, so both of them are 12-regular graphs with girth 5 when $q - 1 = 2N > 260$, that is, when $q \geq 263$ is an odd prime power. As their Cayley colors are clearly disjoint, G_L and G_P form a suitable pair. ■

Let us notice that Theorem 4.1 is sharp when $q \in \{61, 71, 73, 79, 97, 113, 121, 127, 131, 149,$ 151, 191, 193, 197, 199\}, because $|V(G_L)| = |V(G_P)| = q - 1 < rec(r_q + 1, 5)$, so r_q is the maximum degree that the chosen suitable graphs G_L and G_P can reach.

In [1] the authors consider some prime powers $31 \leq q \leq 59$ and $q = 64, 67$ to obtain bounds on $n(k, 5)$ for degrees $34 \leq k \leq 65$ and $69 \leq k \leq 73$. These values of $rec(k, 5)$ are sharp and remain untouched. Now, we consider greater values of k.

Corollary 4.1. *Given an integer $k \in \{66, 67, 68\}$ and $k \geq 74$, let q be the lowest odd prime power such that $k \leq q + r_q$, where r_q is given in Table 2. Then, $n(k, 5) \leq 2(k - r_q + 1)(q - 1)$. In particular, $n(k, 5) \leq 2(k - 11)(q - 1)$ when $235 \leq k \leq q + 12$.*

Proof. Let $q \in \{61, 71, \ldots, \}$ be an odd prime power, and let G_L, G_P be the suitable pair of graphs provided in the proof of Theorem 4.1. The graph $\mathcal{L}_q(G_L, G_P)$ is $(q + r_q)$-regular, has girth at least five and order $2(q + 1)(q - 1)$. Deleting iteratively from this graph γ blocks of lines and γ blocks of points (cf. Remark 2.1 (2)), we have $n(q + r_q - \gamma, 5) \leq 2(q + 1 - \gamma)(q - 1)$, or equivalently, $n(k, 5) \leq 2(k - r_q + 1)(q - 1)$ for $k \leq q + r_q$. ∎

In the following section we establish bounds better than the last one $n(k, 5) \leq 2(k - 11)(q - 1)$ for large odd prime powers q and degrees k.

5. Results for $q \in \{313, 337, 367\}$ and large values $q > 479$

In this section, thanks to a simple relationship between *difference sets* and graphs $S_{2N}(k_1, \ldots, k_m)$, we construct three distinct families of graphs which are suitable for amalgamation into $\mathcal{L}_q$. This way, we obtain new bounds on $n(k, 5)$ for large values of k. In particular, we improve the known bound

$$n\left(q + \left\lfloor \frac{\sqrt{q - 1}}{4} \right\rfloor, 5\right) \leq 2(q^2 - 1) \text{ for an odd prime power } q$$

proved by Jørgensen (cf. [12], Corollary 19) using also difference sets. We begin with a simple auxiliar lemma.

Lemma 5.1. *Let m, N be two positive integers and $D = \{d_1, \ldots, d_m\} \subset \mathbb{Z}_N$ such that all the numbers $d_i - d_j$ with $i \neq j$ are different in $\mathbb{Z}_N$. Define $K = \{k_1, \ldots, k_m\}$ where $k_i = 2d_i + 1 \in \mathbb{Z}_{2N}$ and consider an even non-zero $\delta \in \mathbb{Z}_{2N}$. Then:*

(i) $S_{2N}(k_1, \ldots, k_m)$ and $S_{2N}(k_1 - \delta, \ldots, k_m - \delta)$ are m-regular graphs with girth at least 6.
(ii) $V = K \cap (K - \delta) \neq \emptyset$ iff $\delta = k_i - k_j$ for some $1 \leq i, j \leq m$. In this case, $V = \{k_i - \delta\}$.
(iii) $(-K) \cap (K - \delta) \neq \emptyset$ iff $\delta = k_i + k_j$ for some $1 \leq i, j \leq m$.

Proof. The assertion (*i*) follows from Lemma 2.1. Items (*ii*) and (*iii*) follow from direct calculations. First, if $\alpha \in K \cap (K - \delta)$, there is an ordered pair (k_i, k_j) with $\alpha = k_j = k_i - \delta$ and $\delta = k_i - k_j$ in $\mathbb{Z}_{2N}$. Since (k_i, k_j) is unique, we have $V = \{k_i - \delta\}$. In a similar way, if $\alpha \in (-K) \cap (K - \delta)$ then $\alpha = -k_i = k_j - \delta$ and $\delta = k_i + k_j$. ∎

Roughly speaking, a subset $D \subset H$ of an group H (written in additive notation) is a *finite difference set* if every non-zero element of H can be represented by the same number λ of differences $d - d'$ with $d, d' \in D$. We only need the first and simplest class of difference sets, that is, $\lambda = 1$ and H cyclic, defined and studied by Singer [17] in 1938.

Lemma 5.2 (cf. [17], Theorem 2). *Given a prime power p, there is a subset $D = \{d_0, \ldots, d_p\} \subset \mathbb{Z}_{p^2+p+1}$ such that each non-zero element $h \in \mathbb{Z}_{p^2+p+1}$ is represented in a unique way as $h = d_i - d_j$.*

We use these finite difference sets to obtain the following result:

Theorem 5.1. *Let p, q be prime powers such that $q = 2(p^2 + p + 1) + 1$. Then*

$$n\left(q + \frac{\sqrt{2q - 5} - 1}{2}, 5\right) \leq 2q^2 - q - 1.$$

Proof. Denote $N = p^2 + p + 1$ and let $\{d_0, \ldots, d_p\} \subset \mathbb{Z}_N$ be the difference set given by Lemma 5.2. Consider $K = \{k_0, \ldots, k_p\} \subset \mathbb{Z}_{2N} = \mathbb{Z}_{q-1}$ with $k_i = 2d_i + 1$.

Since the set $B = \{k_i + k_j : 0 \leq i, j \leq p\}$ has only $(p + 2)(p + 1)/2$ elements, take an even non-zero $\delta \in \mathbb{Z}_{2N} \setminus B$ and its unique representation $\delta = k_{i_\delta} - k_{j_\delta}$. By Lemma 5.1 and the election of δ, the $(p + 1)$-regular graph $G_L = S_{q-1}(k_0, \ldots, k_p)$ and the p-regular graph $G_P = S_{q-1}(k_0 - \delta, \ldots, k_{i_\delta - 1} - \delta, k_{i_\delta + 1} - \delta, \ldots, k_p - \delta)$ have girth al least 6 and disjoint sets of Cayley colors.

In order to apply Theorem 2.2 for $r = p$, $W_L = W_P = \emptyset$, $T_L = \mathbb{Z}_{q-1}$ and $T_P = \emptyset$, we define $H_L = S_{q-1}(k_0, \ldots, k_{p-1})$ and $H_P = \emptyset$. Clearly, $E(H_L) \cap E(G_P) = \emptyset$ because $\omega(H_L) \cap \omega(G_P) = \emptyset$. The graph $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ has girth at least 5, order $2(q^2 - 1) - (q - 1) = 2q^2 - q - 1$ and degree $q + p = q + \frac{1}{2}(\sqrt{2q - 5} - 1)$. ∎

Let us notice that the graph $S_{2(p^2+p+1)}(k_0, \ldots, k_p)$ is isomorphic to the incidence graph $\mathcal{P}_p$ of the projective plane. Actually, Singer proves the existence of finite difference sets as a consequence of a geometric property of the projective planes.

Now we turn our attention to *difference sets* of a group H *relative* to a subgroup H_1. We are only interested in the differences defined by Bose [6] in 1942.

Lemma 5.3 (cf. [6], Theorem 1). *Consider a prime power $p \geq 3$, the cyclic group $H = \mathbb{Z}_{p^2-1}$ and the subgroup $H_1 = (p + 1)$ formed by the multiples of $p + 1$. Then there exists a set $D = \{d_1, \ldots, d_p\} \subset H$ such that each element $h \in H \setminus H_1$ is represented as $d_i - d_j$ in a unique way, and no difference $d_i - d_j$ with $i \neq j$ lies in H_1.*

Bose's relative difference sets let us obtain the following bounds. Let us notice that item (*ii*) establishes a result valid for every large odd prime power q.

Theorem 5.2. *The following upper bounds hold:*

(i) *Let $p \geq 7$ and q be prime powers such that $q = 2(p^2 - 1) + 1$. Then*

$$n\left(q + \sqrt{\frac{q+1}{2}} + 2, 5\right) \leq 2(q^2 - 1).$$

(ii) *Given q an odd prime power, let $p \geq 9$ be the highest prime power such that $q \geq 4p^2 - 3$. Then*

$$n(q + p + 2, 5) \leq 2(q^2 - 1).$$

In particular, if $q = 4p^2 - 3$, we have

$$n\left(q + \frac{\sqrt{q+3}}{2} + 2, 5\right) \leq 2(q^2 - 1).$$

Proof (*i*). Assume that $q = 2(p^2 - 1) + 1$ is a prime power for a prime power $p \geq 7$. Notice that $p = \sqrt{\frac{q+1}{2}}$. From Theorem 4.1 it follows that $n(q + p + 2, 5) \leq 2(q^2 - 1)$ for $p \in \{7, 8, 11\}$ and $q \in \{97, 127, 241\}$, while for $p \in \{9, 16\}$ the values of q are not prime powers.

For $p = 13, q = 337$ and $q+p+2 = 352$ we give the following explicit construction: Let $G_L = S_{336}(28, 56; 1, -1, 5, -13, 21, -43, 59, 67, 91, -95, -121, -131, 165)$ and let G_P be the graph with $V(G_P) = \mathbb{Z}_{336}$ and whose edges are the ones in the graph $S_{336}(-27, -29, -23, -41, -7, -71, 31, 39, 63, -123, -149, -159, 137)$ together with the edges $\{2i, 2i + 140\}$, for $0 \leq i \leq 167$, and the ones in the cycles $(1 + 2i, 85 + 2i, 197 + 2i, 309 + 2i, 57 + 2i, 169 + 2i, 281 + 2i, 29 + 2i, 253 + 2i, 141 + 2i, 225 + 2i, 113 + 2i, 1 + 2i)$, for $i = 0, \ldots, 13$. Since $\omega(G_P) \cap \omega(G_L) = \emptyset$, and since the rest of suitability conditions of Theorem 2.1 are fulfilled, the graph $\mathcal{L}_{337}(G_L, G_P)$ yields $n(337 + 15, 5) \leq 2(337^2 - 1) = 227136$. Hence, item (*i*) holds for $p \in \{7, 8, 11, 13\}$.

From now on we assume that $p \geq 17$ and denote $N = p^2 - 1$. Let $\{d_1, \ldots, d_p\} \subset \mathbb{Z}_{p^2-1}$ be the relative difference set given in Lemma 5.3 and define $K = \{k_1, \ldots, k_p\}$ where $k_i = 2d_i + 1 \in \mathbb{Z}_{2(p^2-1)}$. Consider the sets $A = \{k_i - k_j : 1 \leq i, j \leq p, i \neq j\}$, $B = \{k_i + k_j : 1 \leq i, j \leq p\}$ and $C = (2(p+1))$. Define the odd elements $m_i \in \mathbb{Z}_{2(p+1)}$ such that $k_i \equiv m_i \mod 2(p+1)$. By Lemma 5.3, $C \cap A = \emptyset$ and the values m_i are all different.

Our first step is to prove that $C \setminus B$ contains an element $\delta \neq 0$. If $\alpha = k_i + k_j \in B \cap C$ then $m_i + m_j = 0$, and α requires a unique pair $\{m_i, -m_i\}$. If p is even it could be $-m_i = m_i = p + 1$, but, in any case $|B \cap C| \leq \lfloor |K|/2 \rfloor = \lfloor p/2 \rfloor$. Let us take a non-zero $\delta \in C \setminus B$.

Consider the elements $P_0 = 2(p+1)$, $Q_0 = 4(p+1)$, $P_1 = 6(p+1)$, $Q_1 = 8(p+1)$ in C (in case $p = 17$, choose $Q_1 = 10(p+1)$) and define the graphs $G_L = S_{2(p^2-1)}(P_0, Q_0; k_1, \ldots, k_p)$ and $G_P = S_{2(p^2-1)}(P_1, Q_1; k_1 - \delta, \ldots, k_p - \delta)$. By Lemmas 5.1 and 2.1 this pair of graphs is suitable for amalgamation into $\mathcal{L}_q$ and Theorem 2.1 proves (*i*).

(*ii*) Consider a prime power $p \geq 9$. As in the proof of (*i*), define k_i, B, C, δ and $k'_i = k_i - \delta$ as elements of $\mathbb{Z}_{2(p^2-1)}$. With abuse of language, we also denote by $k_i, k'_i \in \mathbb{Z}$ their representatives such that

$$-(p^2 - 1) < k_i, k'_i \leq p^2 - 1 \tag{4}$$

and define the integers $P_0 = 2(p+1)$, $Q_0 = 4(p+1)$, $P_1 = 6(p+1)$, $Q_1 = 8(p+1)$.

Now consider a prime power $q \geq 4p^2 - 3$ and the graphs $G_L = S_{q-1}(P_0, Q_0; k_1, \ldots, k_p)$ and $G_P = S_{q-1}(P_1, Q_1; k'_1, \ldots, k'_p)$. From the inequalities (4),

$$\max(k_i - k_j) < 2(p^2 - 1) \leq \frac{q-1}{2}.$$

In order to apply Lemma 2.2, it remains to see that $2Q_1 = 16(p+1) < \frac{q-1}{2}$. A simple computation shows that this inequality holds just for $p \geq 9$ (and $q \geq 321$).

Moreover, from the construction of k_i, k'_i, Lemma 5.1 and the inequalities (4), it follows that G_L and G_P share no Cayley color in $\mathbb{Z}_{q-1}$. They form a suitable pair of graphs and Theorem 2.1 provides the conclusion (*ii*).   ∎

With the notations of the proof of Theorem 5.2, item (*i*), let us notice that the graph $S_{2(p^2-1)}(k_1, \ldots, k_p)$ is isomorphic to the incidence graph $\mathcal{L}_p$. In consequence, the graphs G_L, G_P are actually amalgams of 2-regular graphs into $\mathcal{L}_p$ and the construction in the proof has a recursive character.

Another family, defined by Ganley in [10], is the following one: Given an additive group H and two subgroups H_1, H_2 such that $H = H_1 \oplus H_2$, a *direct product difference set* is a subset $D = \{d_1, \ldots, d_m\} \subset H$ such that each $h \in H \setminus (H_1 \cup H_2)$ is represented in a unique way as $h = d_i - d_j$, and no difference $d_i - d_j$ with $i \neq j$ lies in $H_1 \cup H_2$.

Lemma 5.4 (cf. [10], Theorem 2.4). *For a prime $p \geq 3$, there exists a direct product difference set $D = \{d_1, \ldots, d_{p-1}\}$ in the additive group $\mathbb{Z}_{p(p-1)}$ relative to the subgroups $H_1 = (p - 1)$ and $H_2 = (p)$.*

We use these difference sets for our last result, in which we establish another bound on $n(k, 5)$.

Theorem 5.3. *Let $p \geq 5$ be a prime and q be a prime power such that $q = 2p(p - 1) + 1$. Then*

$$n\Big(q + \frac{\sqrt{2q - 1} + 3}{2}, 5\Big) \leq 2(q^2 - 1).$$

Proof. For $p = 5$ and $q = 2p(p - 1) + 1 = 41$, the bound $n(41 + 6, 5) \leq 2(41^2 - 1)$ is proved in [1]. For $p = 7$ and $p = 11$, the result cannot be applied because $q = 85$ and $q = 221$ are not prime powers. Assume $p \geq 13$ is a prime. Let $d_1, \ldots, d_{p-1}$ be the integers provided by Lemma 5.4 and consider $k_i = 2d_i + 1$. The bipartite $(p - 1)$-regular graph $S_{2p(p-1)}(k_1, \ldots, k_{p-1})$ has girth $g \geq 6$.

Define sets $B = \{k_i + k_j : 0 \leq i, j \leq p - 2\}$ and $C = (2(p - 1)) \cup (2p)$. Let $P_0 = 2(p - 1)$, $Q_0 = 4(p - 1)$, $P_1 = 2p$, $Q_1 = 4p$. Choosing a non-zero $\delta \in C \setminus B$ and following the same reasoning as in Theorem 5.2, we have that the graphs $G_L = S_{2p(p-1)}(P_0, Q_0; k_1, \ldots, k_{p-1})$ and $G_P = S_{2p(p-1)}(P_1, Q_1; k_1 - \delta, \ldots, k_{p-1} - \delta)$ are suitable for amalgamation into $\mathcal{L}_q$. Hence the graph $\mathcal{L}_q(G_L, G_P)$ has girth at least 5, order $2(q^2 - 1)$ and degree $q + p - 1 + 2 = q + \frac{\sqrt{2q-1}+3}{2}$. ∎

Remark 5.1. On the one hand, the results of this section can be applied to $p = 13$ and $q = 313$ (Theorem 5.3), $q = 337$ (Theorem 5.2, item (i)), and $q = 367$ (Theorem 5.1). On the other hand, given an odd prime power $q > 479$, the highest prime power p such that $q \geq 4p^2 - 3$ verifies $p \geq 11$. In this case, the bound $n(q + p + 2, 5) \leq 2(q^2 - 1)$ given by Theorem 5.2 is better than the result $n(q + 12, 5) \leq 2(q^2 - 1)$ provided by Theorem 4.1.

The graphs constructed in this article are available in several formats. Readers can request them by email.

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