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STABLE MANIFOLDS FOR A CLASS OF SINGULAR EVOLUTION EQUATIONS AND EXPONENTIAL DECAY OF KINETIC SHOCKS

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Abstract. We construct stable manifolds for a class of singular evolution equations including the steady Boltzmann equation, establishing in the process exponential decay of associated kinetic shock and boundary layers to their limiting equilibrium states. Our analysis is from a classical dynamical systems point of view, but with a number of interesting modifications to accomodate ill-posedness with respect to the Cauchy problem of the underlying evolution equation.

1. Introduction.

In this paper we study decay rates at infinity of (possibly) large-amplitude relaxation shocks

$$u(x,t)=u^*(x-st),\qquad \lim_{\tau\to\pm\infty}u(\tau)=u^\pm, \tag{1.1}$$

of kinetic-type relaxation systems

$$A_0u_t+Au_x=Q(u), \tag{1.2}$$

on a general Hilbert space $\mathbb{H}$, where A_0, A are given (constant) bounded linear operators and Q is a bounded bilinear map. More generally, we study existence and properties of stable/unstable manifolds for a class of singular evolution equations arising through the study of such profiles.

Making the change of variables $\tau=x-st$ we obtain that the profiles u^* satisfy the equation $(A-sA_0)u_\tau=Q(u)$. By frame-indifference, we may without loss of generality take $s=0$, yielding

$$Au_\tau=Q(u). \tag{1.3}$$

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We are interested in the singular case, as arises for example for Boltzmann's equation [5, 16, 17, 26], that the linear operator A is self-adjoint, one-to-one, but *not invertible*.

This is a crucial point of the current paper, since in the case when the linear operator A has a bounded inverse, one would reduce equation (1.3) to an evolution equation with bounded linear part, that can be treated similarly as in the case of nonlinear equations on finite dimensional spaces. The case at hand, when the linear operator A does not have a bounded inverse, requires a different approach, since (1.3) or its linearization along equilibria might not be well-posed; therefore it is not clear if one can use the usual variation of constants formula to look for mild solutions. Rather, we use the frequency domain reformulation of these equations following the approach in [14] and [15].

Other cases of non-well-posed equations in the sense that they do not generate an evolution family either in forward or backward time on the entire space, arise in the study of modulated waves on cylindrical domains (see [29, 35, 36]), Morse theory (see [1, 2, 33], the theory of PDE Hamiltonian systems (see [34]), and the theory of functional-differential equations of mixed type (see [18]). The particular form (1.3), however, in which the singularity arises through the coefficient of the τ-derivative with other terms bounded, does not seem to have been treated before, and does not appear to be amenable to the methods of these previous works; see the discussion of Section 1.4.1. This is the class of singular evolution equation to which we refer in the title of the paper.

The examples we are interested in arise in certain kinetic and discrete kinetic relaxation approximation models, *in particular, the Boltzman equation*

$$f_t + \xi_1 f_x = Q(f), \quad x \in \mathbb{R}^1,\, \xi \in \mathbb{R}^3, \tag{1.4}$$

where $f = f(t, x, \xi)$ denotes density at time t, spatial point x of particles with velocity ξ and Q is a bilinear collision operator (cf. [10]). After rescaling by $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$, (1.4) can be put in form (1.2), with A is equal to the operator of multiplication by the function $\xi_1/\langle \xi \rangle$ and $\mathbb{H}$ an appropriate weighted L^2 space[1] in the variable ξ. For details of this reduction, see [26]. In [24, 25, 26, 37], Métivier, Texier and Zumbrun obtained existence results for a somewhat larger class of models of shocks with small amplitude $\varepsilon := \|u^+ - u^-\|$, in particular yielding exponential decay rates as $\tau = (x - st) \to \pm\infty$; see also the earlier papers [5, 16] in the specific case of Boltzmann's equation. These results were obtained by fixed-point iteration on the whole line, using in an essential way the small-amplitude assumption to construct initial approximations based on a formal fluid-dynamical approximation by Chapman-Enskog expansion.

Here, our interest is in treating *large-amplitude* profiles, without a priori information on the shape of the profile, by *dynamical systems techniques* that would apply also in the case of boundary layers, where the solution is not necessarily defined on the whole line. Our larger goal is to develop dynamical systems tools analogous to those of [9, 21, 22, 23, 38, 39, 40, 41, 44, 45], sufficient to treat 1- and multi-D stability by the techniques of those papers. See in particular the discussion of [40, Remark 4.2.1(4), p. 55], proposing a path toward stability of Boltzmann shock profiles. For this program, the proof of exponential decay rates and the establishment

[1]Namely, the standard choice weighted by the square root of the Maxwellian at u^+ (resp. u^-) as in [10, 26].

of a stable manifold theorem are essential first steps. For a corresponding center manifold theorem, see [32].

1.1. **Assumptions.** In [26, Section 4] it is shown that the Boltzman equation with hard-sphere potential can be recast as an equation of form (1.2) where the linear operator A and the nonlinearity Q satisfy the hypotheses (H1)-(H2) below. Following [26], we assume these throughout.

Hypothesis (H1) The linear operator A is bounded and self-adjoint on the Hilbert space $\mathbb{H}$. There exists $\mathbb{V}$ a *proper*, closed subspace of $\mathbb{H}$ with $\dim \mathbb{V}^\perp < \infty$ and $B : \mathbb{H} \times \mathbb{H} \to \mathbb{V}$ is a bilinear, symmetric, continuous map such that $Q(u) = B(u, u)$.

Hypothesis (H2) At $u^\pm$ (necessarily equilibria, $Q(u^\pm) = 0$), linearized operators $Q'(u^\pm)$ satisfy (i) $Q'(u^\pm)$ is self-adjoint and $\ker Q'(u^\pm) = \mathbb{V}^\perp$, and (ii) There exists $\delta_\pm > 0$ such that $Q'(u^\pm)_{|\mathbb{V}} \le -\delta_\pm I_{\mathbb{V}}$.

We adjoin to (H1)-(H2) the following two hypotheses, also satisfied for Boltzmann's equation.

Hypothesis (H3) The linear operator A is one-to-one.

Hypothesis (H4) The linear operator $P_{\mathbb{V}^\perp} A_{|\mathbb{V}^\perp}$ is invertible on the finite dimensional space $\mathbb{V}^\perp$, where $P_{\mathbb{V}^\perp}$ denotes the operator of orthogonal projection onto $\mathbb{V}^\perp$.

Hypothesis (H3) for Boltzmann's and related kinetic equations reflects the fact that A is a multiplication operator on a weighted L^2 space, possessing only essential and no point spectrum. Hypothesis (H4) amounts to the assumption that the associated finite-dimensional linearized equilibrium flow $P_{\mathbb{V}^\perp}(A_0)_{|\mathbb{V}^\perp} h_t + P_{\mathbb{V}^\perp} A_{|\mathbb{V}^\perp} h_x = 0$ of (1.2) about $u^\pm$ be noncharacteristic, where $h := P_{\mathbb{V}^\perp} u$. It is readily seen to be the condition that the center subspace of the linearized flow of (1.3) about $u^\pm$ consist entirely of the trivial, equilibrium subspace $\mathbb{V}^\perp$, which is the condition under which we may expect exponential decay to equilibrium; see [17, 32] for further discussion.

1.2. **Results.** First, we show that linearized equation $Au' = Q'(u^\pm)u$ is equivalent to an equation of the form $u' = Su$, where S generates not a C^0-semigroup, but rather a *bi-semigroup* [3, 14].

Definition 1.1. The linear operator S is said to generate a *bi-semigroup* if it has the decomposition $S = S_1 \oplus (-S_2)$ on a direct sum decomposition of the entire space $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2$, where S_j, $j = 1, 2$, generate C^0-semigroups on $\mathbb{H}_j$, $j = 1, 2$. The bi-semigroup is called *stable* if the semigroups generated by S_j, $j = 1, 2$ are stable on $\mathbb{H}_j$, $j = 1, 2$.

We recall that the first order linear differential operator with constant coefficients $\partial_\tau - S$ is invertible on function spaces such as $L^2(\mathbb{R}, \mathbb{H})$ if and only if the equation $u' = Su$ has an exponential dichotomy on $\mathbb{R}$. We note that for any $u_0 \in \mathbb{V}^\perp$ the function $u(\tau) = u_0$ is a solution of equation $Au' = Q'(u^\pm)u$. Therefore, equation $Au' = Q'(u^\pm)u$ does not have an exponential dichotomy on the entire space $\mathbb{H}$; instead it exhibits an exponential dichotomy on a direct complement of the finite dimensional space $\mathbb{V}^\perp$. To prove this result, we reduce the equation by using the decomposition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \mathbb{V}^\perp \oplus \mathbb{V} \to \mathbb{V}^\perp \oplus \mathbb{V}, \quad Q'(u^\pm) = \begin{bmatrix} 0 & 0 \\ 0 & Q'_{22}(u^\pm) \end{bmatrix} : \mathbb{V}^\perp \oplus \mathbb{V} \to \mathbb{V}^\perp \oplus \mathbb{V}.$$

Indeed, if u is a solution of equation $Au' = Q'(u^{\pm})u$, then the pair (h, v) defined by $v = P_{\mathbb{V}}u$ and $h = P_{\mathbb{V}^{\perp}}u$, satisfies the system

$$(A_{22} - A_{21}A_{11}^{-1}A_{12})v' = Q'_{22}(u^{\pm})v, \quad h = -A_{11}^{-1}A_{12}v.$$

We introduce the linear operators $S_{\pm}^{\mathrm{r}} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}Q'_{22}(u^{\pm})$. These are densely defined, and indeed generate bi-semigroups. Our dichotomy results are summarized in the following theorem.

Theorem 1.2. *Assume Hypotheses (H1)-(H4). Then,*

(i) The bi-semigroup generated by $S_{\pm}^{\mathrm{r}} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}Q'_{22}(u^{\pm})$ is exponentially stable on $\mathbb{V}$;

(ii) The linear space $\mathbb{H}$ can be decomposed into linear stable, center and unstable subspaces, $\mathbb{H} = \mathbb{V}^{\perp} \oplus \mathbb{H}_{\pm}^{\mathrm{s}} \oplus \mathbb{H}_{\pm}^{\mathrm{u}}$ such that

- *(a) for any $u_0 \in \mathbb{V}^{\perp}$ the function $u(\tau) \equiv u_0$ is a solution of equation $Au' = Q'(u^{\pm})u$;*
- *(b) for any $u_0 \in \mathbb{H}_{\pm}^{\mathrm{s}}$ the solution of equation $Au' = Q'(u^{\pm})u$ on $\mathbb{R}_+$ with $u(0) = u_0$ decays exponentially at $+\infty$;*
- *(c) for any $u_0 \in \mathbb{H}_{\pm}^{\mathrm{u}}$ the solution of equation $Au' = Q'(u^{\pm})u$ on $\mathbb{R}_-$ with $u(0) = u_0$ decays exponentially at $-\infty$.*

From this point, we turn our attention towards our main goal, the existence of stable/unstable manifolds of solutions of equation (1.3) near the equilibria u^+/u^-, respectively. The first step is to show that this equation can be reduced to an equation of the form

$$\Gamma\mathbf{u}' = E\mathbf{u} + D(\mathbf{u}, \mathbf{u}), \tag{1.5}$$

where $D(\cdot, \cdot)$ is a bounded, bilinear map, Γ is a one-to-one, self-adjoint, bounded linear operator, E is a self-adjoint, bounded, negative definite and the linear operator $\Gamma^{-1}E$ generates a stable bi-semigroup $\{T_{\mathrm{s/u}}^{\Gamma,E}(\tau)\}_{\tau\geq 0}$ on a Hilbert space $\mathbb{X}$. To construct the manifolds, we introduce a notion of mild solutions of equation (1.5) on $\mathbb{R}_{\pm}$ using the results from Theorem 1.2. Next, we apply, formally, the Fourier transform in (1.5) and then we solve for $\mathscr{F}\mathbf{u}$. In this way we obtain that mild solutions of equation (1.5) on say $\mathbb{R}_+$ satisfy equation

$$\mathbf{u}(\tau) = T_{\mathrm{s}}^{\Gamma,E}(\tau)\mathbf{u}(0) + \big(\mathscr{K}_{\Gamma,E}D(\mathbf{u}, \mathbf{u})\big)(\tau), \ \tau \geq 0. \tag{1.6}$$

Here $\mathscr{K}_{\Gamma,E}$ is the Fourier multiplier defined by the operator-valued function defined by $R_{\Gamma,E}(\omega) = (2\pi\mathrm{i}\omega\Gamma - E)^{-1}$. The linear operator $\mathscr{K}_{\Gamma,E}$ is well-defined and bounded on $L^2(\mathbb{R}, \mathbb{X})$. To construct stable manifolds of evolution equations on finite-dimensional spaces, one uses a fixed point argument to solve equation (1.6) on the space $C_0(\mathbb{R}, \mathbb{X})$, of continuous functions decaying at $\pm\infty$, or on $L^{\infty}(\mathbb{R}, \mathbb{X})$. However, in our infinite-dimensional case such an argument does not seem to be possible, since the Fourier multiplier $\mathscr{K}_{\Gamma,E}$ cannot be extended to a bounded linear operator on $L^{\infty}(\mathbb{R}, \mathbb{X})$, see Example 3.4. Therefore, a crucial point of our construction is to find a proper subspace of $L^{\infty}(\mathbb{R}, \mathbb{X})$ that is invariant under $\mathscr{K}_{\Gamma,E}$. Since the operator-valued function $R_{\Gamma,E}$ is bounded, one can readily check that $H^1(\mathbb{R}, \mathbb{X})$ is invariant under $\mathscr{K}_{\Gamma,E}$. However, equation (1.6) is a functional equation on the half-line, not the full line. Moreover, since *not* every trajectory of the operator valued function $R_{\Gamma,E}$ belongs to $L^2(\mathbb{R}, \mathbb{X})$, it turns out that the space $H^1(\mathbb{R}_+, \mathbb{X})$ is *not* invariant under $\mathscr{K}_{\Gamma,E}$. To deal with this setback, we parameterize equation (1.6). In Section 3 we find solutions $\mathbf{u}$ such that $\mathbf{u}(0) = \mathbf{v}_0 - E^{-1}D(\mathbf{u}(0), \mathbf{u}(0))$ where $\mathbf{v}_0$ is a parameter in a dense subspace. Substituting in equation (1.6), we conclude that to construct our stable/unstable manifold it is enough to prove existence of solutions of equation

$$\mathbf{u} = T_{\mathrm{s}}^{\Gamma,E}(\cdot)P_{\mathrm{s}}^{\Gamma,E}\mathbf{v}_0 + (\mathscr{K}_{\Gamma,E}D(\mathbf{u},\mathbf{u}))_{|\mathbb{R}_+} - T_{\mathrm{s}}^{\Gamma,E}(\cdot)P_{\mathrm{s}}^{\Gamma,E}E^{-1}D(\mathbf{u}(0),\mathbf{u}(0)), \quad (1.7)$$

where $P_{\mathrm{s}}^{\Gamma,E}$ denotes the projection onto the stable subspace $\mathbb{X}_{\mathrm{s}}^{\Gamma,E}$ parallel to the unstable subspace $\mathbb{X}_{\mathrm{u}}^{\Gamma,E}$. An important step of our construction is to find an appropriate subspace of parameters $\mathbf{v}_0$ such that the trajectory $T_{\mathrm{s}}^{\Gamma,E}(\cdot)P_{\mathrm{s}}^{\Gamma,E}\mathbf{v}_0$ belongs to $H^1(\mathbb{R}_+,\mathbb{X})$. To achieve this goal, we use that the linear operators Γ and E are self-adjoint and bounded, hence the bi-semigroup generator $\Gamma^{-1}E$ is similar to a multiplication operator by a real valued function bounded from below on some L^2 space. See Section 3 for the details of this construction.

Theorem 1.3. *Assume Hypotheses (H1)-(H4). Then, for any integer $r \geq 1$ there exists a C^r local stable manifold $\mathscr{M}_{\mathrm{s}}^+$ near u^+ and a C^r local unstable manifold $\mathscr{M}_{\mathrm{u}}^-$ near u^-, expressible in $w^\pm = u - u^\pm$ as C^r embeddings $\mathscr{J}_{\mathrm{s}}^+$ and $\mathscr{J}_{\mathrm{u}}^-$ of $\mathbb{H}_{\mathrm{s}}^+ \cap \mathrm{dom}\big(|\tilde{A}^{-1}Q_{22}'(u^+)|^{\frac{1}{2}}\big)$ and $\mathbb{H}_{\mathrm{u}}^- \cap \mathrm{dom}\big(|\tilde{A}^{-1}Q_{22}'(u^-)|^{\frac{1}{2}}\big)$ with norms*

$$\|h\|_{\mathrm{dom}\left(|\tilde{A}^{-1}Q_{22}'(u^\pm)|^{\frac{1}{2}}\right)} = \left(\|h\|_{\mathbb{H}}^2 + \big\||\tilde{A}^{-1}Q_{22}'(u^\pm)|^{\frac{1}{2}}h\big\|_{\mathbb{H}}^2\right)^{\frac{1}{2}}$$

into $\mathbb{H}$ with the standard norm, that are locally invariant under the forward flow of equation $Au' = Q(u)$ and expressible as the union of orbits of all solutions $w^\pm \in H^1(\mathbb{R}_\pm,\mathbb{H})$ such that $w^\pm$ is sufficiently small in $H^1(\mathbb{R}_\pm,\mathbb{H})$ norm. (Recall that $\tilde{A} = A_{22} - A_{21}A_{11}^{-1}A_{12}$).

Finally, we use this result to prove that H^1 shock or boundary layer profiles decay exponentially.

Corollary 1. *Assume Hypotheses (H1)-(H4). Let $u^* \in H^1(\mathbb{R},\mathbb{H})$ be a solution of equation $Au_\tau = Q(u)$, H^1-convergent to $u^\pm$ in the sense that $u^* - u^\pm \in H^1(\mathbb{R}_\pm,\mathbb{H})$, and let $-\nu_\pm := -\nu\big(\tilde{A}, Q_{22}'(u^\pm)\big) < 0$ be the decay rate of the bi-semigroup generated by the pair $\big(\tilde{A}, Q_{22}'(u^\pm)\big)$. Then, there exist $\alpha \in (0, \min\{\nu_+,\nu_-\})$ such that $u^* - u^\pm \in H_\alpha^1(\mathbb{R}_\pm,\mathbb{H})$. In particular, there exists $\alpha > 0$ such that $\|u^*(\tau) - u^\pm\| \leq c(\alpha)e^{-\alpha|\tau|}$ for any $\tau \in \mathbb{R}_\pm$.*

1.3. Applications to Boltzmann's equation.

As mentioned above, the assumptions (H1)-(H4) of Section 1.1 are abstracted from, and satisfied by, the steady Boltzmann equation with hard sphere collision potential [26], after the change of coordinates $f \to \langle\xi\rangle^{1/2}f$, $Q \to \langle\xi\rangle^{-1/2}Q(\langle\xi\rangle^{-1/2}\cdot)$, $\langle\xi\rangle := \sqrt{1+|\xi|^2}$, with A equal to the operator of multiplication by the function $\xi_1/\langle\xi\rangle$. The Hilbert space $\mathbb{H}$ is determined by the slight strengthening of the classical square-root Maxwellian weighted norm $\|f\|_{\mathbb{H}} := \|\langle\cdot\rangle^{1/2}M_{u^+}^{-1/2}(\cdot)f(\cdot)\|_{L^2}$ (used in the construction of the stable manifold near u^+), and $\|f\|_{\mathbb{H}} := \|\langle\cdot\rangle^{1/2}M_{u^-}^{-1/2}(\cdot)f(\cdot)\|_{L^2}$ (used in the construction of the unstable manifold near u^-) [6, 10, 26], where

$$M_u(\xi) = \rho(4\pi e/3)^{-3/2}e^{-|\xi-v|^2(4e/3)^{-1}}$$

denotes the Maxwellian distribution indexed by the hydrodynamic moments,

$$u = (\rho, v^T, e)^T \in \mathbb{R}^5$$

with ρ corresponding to density, $v \in \mathbb{R}^3$ velocity, and e internal energy. See [6, 10] for further discussion, and [26] for a detailed treatment of the reduction to form (1.2) considered here.

Thus, Theorem 1.3 and Corollary 1 apply in particular to this fundamental case. More generally, they apply to Boltzmann's equation with any collision potential, or "cross-section", for which (H1)–(H4) are satisfied in the coordinates above, with the crucial aspects being boundedness of the nonlinear collision operator as a bilinear map and spectral gap of the linearized collision operator. This includes besides the hard-sphere potential also the hard cutoff potentials of Grad [5, 26].

For the class of admissible cross-sections defined implicitly by (H1)-(H4), Corollary 1 implies exponential decay of $H^1_{\rm loc}$ Boltzmann shock or boundary layer profiles *of arbitrary amplitude*, so long as such profiles (i) exist, (ii) are uniformly bounded, and (iii) converge to their endstates in the weak sense that $u^*-u^\pm$ lies in $H^1(\mathbb{R}_\pm,\mathbb{H})$. This fundamental property, a cornerstone of the dynamical systems approach to stability developed for viscous shock and relaxation waves, had previously been been established for kinetic shocks only in the small-amplitude limit [16, 26].

However, we do not here establish existence of large-amplitude profiles; indeed, the "structure problem," as discussed by Truesdell, Ruggeri, Boillat, and others [4], of existence and structure of large-amplitude Boltzmann shocks, is one of the fundamental open problems in the theory.

1.4. **Discussion and open problems.** In our analysis, the Hilbert structure of $\mathbb{H}$ and symmetry of A and $Q'(u^\pm)$ play an important role; see (H1)-(H2). This structure is implied, for example, by existence of a convex entropy for system (1.2) ([7]). In the case of the Boltzmann equation, it is related to increase of thermodynamical entropy and the Boltzmann H-Theorem; see [26, Notes on the proof of Proposition 3.5, point 2]. In the finite-dimensional setting, Hypotheses (H1) and (H2) reduce essentially to the *stability* and *Kawashima* conditions of [8] (see (h1)-(h4) of the reference).

Further insight may be gained using the invertible coordinate transformation $(-E)^{1/2}$ and spectral decomposition of $(-E)^{-1/2}\Gamma(-E)^{-1/2}$ to write the reduced system $\Gamma\mathbf{u}'=E\mathbf{u}+D(\mathbf{u},\mathbf{u})$ of (1.5) formally as a family of scalar equations

$$(\alpha_\lambda\partial_\tau-1)\mathbf{u}_\lambda=D_\lambda(\mathbf{u},\mathbf{u}),\tag{1.8}$$

indexed by λ, where $\mathbf{u}_\lambda$ is the coordinate of $\mathbf{u}$ associated with spectrum α_λ, real, in the eigendecomposition of $(-E)^{-1/2}\Gamma(-E)^{-1/2}$, with $\|\mathbf{u}\|^2_{\mathbb{X}}=\int|\mathbf{u}_\lambda|^2d\mu_\lambda$, where $d\mu(\lambda)$ denotes the spectral measure associated with $(-E)^{-1/2}\Gamma(-E)^{-1/2}$ and α_λ are bounded with an accumulation point at 0. In the first place, we see directly that $(\Gamma\partial_\tau-E)$ is boundedly invertible on $L^2(\mathbb{R},\mathbb{X})$, with resolvent kernel given in $\mathbf{u}_\lambda$ coordinates by the scalar resolvent kernel

$$R_\lambda(\tau,\theta)=\alpha_\lambda^{-1}e^{(\tau-\theta)\alpha_\lambda^{-1}}\text{ whenever }(\tau-\theta)\alpha_\lambda<0,\tag{1.9}$$

which is readily seen to be *integrable with respect to* τ, hence bounded coordinate-by-coordinate.

On the other hand, we see at the same time that the operator norm of the full kernel R with respect to $L^2(\mu)$ is

$$\|R(\tau,\theta)\|_{L^2(\mu)\to L^2(\mu)}=\sup_{\alpha_\lambda(\tau-\theta)<0}|\alpha_\lambda^{-1}e^{(\tau-\theta)\alpha_\lambda^{-1}}|\,,\tag{1.10}$$

yielding the upper bound $\|R(\tau,\theta)\|_{L^2(\mu)\to L^2(\mu)}\lesssim 1/|\tau-\theta|$ for all $\tau\neq\theta$. When Γ does not have a bounded inverse, i.e., there exists a sequence $\alpha_{\lambda_j}\to 0$ such that $[\alpha_{\lambda_j}/2,2\alpha_{\lambda_j}]$ has positive spectral measure we obtain also the lower bound

$$\|R(\tau,\theta)\|_{L^2(\mu)\to L^2(\mu)}\gtrsim 1/|\tau-\theta|\text{ for }\tau-\theta=\alpha_j\to 0,$$

showing that $\|R(\tau,\theta)\|_{L^2(\mu)\to L^2(\mu)}$ is unbounded.

Likewise, a construction as in Example 3.4 shows, when Γ does not have a bounded inverse, that $\Gamma\partial_\tau - E$ is *not* boundedly invertible on $L^\infty(\mathbb{R},\mathbb{X})$, motivating our choice of spaces $H^1(\mathbb{R},\mathbb{X})$, $H^1(\mathbb{R}_+,\mathbb{X})$ in the analysis, rather than the usual $L^\infty(\mathbb{R},\mathbb{X})$. This implies, by contradiction, that the operator norm of the resolvent kernel is not only unbounded but non-integrable (cf. [42]).

Using the finite-dimensional variation of constants formula scalar mode-by-scalar mode, we may, further, express (1.8) as the fixed point equation

$$\begin{aligned}\mathbf{u}_\lambda(\tau) &= e^{\alpha_\lambda^{-1}\tau}\Pi_S\mathbf{u}_\lambda(0) + \int_0^\tau \Pi_S\alpha_\lambda^{-1}e^{\alpha_\lambda^{-1}(\tau-\theta)}D_\lambda(\mathbf{u}(\theta),\mathbf{u}(\theta))\,\mathrm{d}\theta\\ &\quad - \int_\tau^{+\infty}\Pi_U\alpha_\lambda^{-1}e^{\alpha_\lambda^{-1}(\tau-\theta)}D_\lambda(\mathbf{u}(\theta),\mathbf{u}(\theta))\,\mathrm{d}\theta,\end{aligned}\tag{1.11}$$

where Π_U and Π_S denote projections onto the stable and unstable subspaces determined by $\mathrm{sgn}\alpha_\lambda$. In (1.11) and (1.12) below we denote the spectral components of $\Pi_{S/U}g$ by $\Pi_{S/U}g_\lambda$ for any $g\in L^2(\mu)$, slightly abusing the notation. From (1.11) we find after a brief calculation/integration by parts the derivative formula

$$\begin{aligned}\mathbf{u}'_\lambda(\tau) &= \alpha_\lambda^{-1}e^{\alpha_\lambda^{-1}\tau}\big(\Pi_S\mathbf{u}_\lambda(0) + D_\lambda(\mathbf{u}(0),\mathbf{u}(0))\big)\\ &\quad + \int_0^\tau \Pi_S\alpha_\lambda^{-1}e^{\alpha_\lambda^{-1}(\tau-\theta)}D'_\lambda(\mathbf{u}(\theta),\mathbf{u}(\theta))\,\mathrm{d}\theta\\ &\quad - \int_\tau^{+\infty}\Pi_U\alpha_\lambda^{-1}e^{\alpha_\lambda^{-1}(\tau-\theta)}D'_\lambda(\mathbf{u}(\theta),\mathbf{u}(\theta))\,\mathrm{d}\theta,\end{aligned}\tag{1.12}$$

which shows that $\mathbf{u}\in H^1(\mathbb{R}_+,L^2(\mu))$ only if $\alpha_\lambda^{-1}e^{\alpha_\lambda^{-1}\tau}\Pi_S\big(\mathbf{u}_\lambda(0)+D_\lambda(\mathbf{u}(0),\mathbf{u}(0))\big)\in L^2(\mathbb{R}_+,L^2(\mu))$, or

$$\Pi_S\big(\mathbf{u}_\lambda(0) + D_\lambda(\mathbf{u}(0),\mathbf{u}(0))\big)\in \mathrm{dom}((-E)^{-1/2}\Gamma(-E)^{-1/2})^{1/2}.$$

This is quite different from the usual finite-dimensional ODE or dynamical systems scenario, and explains why we need to take some care in setting up the $H^1(\mathbb{R}_+,\mathbb{X})$ contraction formulation. In particular, we find it necessary to parametrize not by $\Pi_S\mathbf{u}(0)$ as is customary in the finite-dimensional ODE case, but rather by $\Pi_S\mathbf{v}_0 := \Pi_S\big(\mathbf{u}_\lambda(0)+D_\lambda(\mathbf{u}(0),\mathbf{u}(0))\big)$ where $\mathbf{u}'_\lambda(0)=\alpha_\lambda^{-1}\mathbf{v}_0$.

1.4.1. *Relation to previous work.* The issue of noninvertibility of A for relaxation systems (1.2) originating from kinetic models and approximations was pointed out in [19, 20, 40]. This issue has been treated for finite-dimensional systems by Dressler and Yong [8] using singular perturbation techniques; see also [11, 12, 28]. These analyses concern the case that A has an *eigenvalue at zero*, and are of completely different character from the analysis carried out here of the case that A has *essential spectrum at zero*, i.e., an essential singularity; they are thus complementary to ours. In the present, semilinear setting, the case that A has a kernel is particularly simple, giving a *constraint* restricting solutions (under suitable nondegeneracy conditions) to a certain manifold, on which there holds a reduced relaxation system of standard, nonsingular, type. For Boltzmann's equation (1.4), Liu and Yu [17] have investigated existence of invariant manifolds in a weighted $L^\infty(x,\xi)$ *Banach space* setting, using time-regularization and detailed pointwise bounds.

As noted earlier, the treatment of ill-posed equations $\mathbf{u}'-S\mathbf{u}=f$, and derivation of resolvent bounds via generalized exponential dichotomies, has been carried out in a variety of contexts [1, 2, 29, 33, 34, 35, 36]. The essential difference here is that the corresponding resolvent equation $\Gamma\mathbf{u}' - E\mathbf{u} = f$ associated with (1.5) rewrites formally in the more singular form

$$\mathbf{u}' - S\mathbf{u} = \Gamma^{-1}f, \quad S = \Gamma^{-1}E,$$

for which the singularity Γ^{-1} enters not only in the generator S but also in the source. Thus, the solution operator is not the one $(\partial_\tau - S)^{-1}$ deriving from (generalized) exponential dichotomies of the homogeneous flow, but the more singular $(\Gamma\partial_\tau - E)^{-1}$ of (1.6), or, formally, the unbounded multiple $(\partial_\tau - S)^{-1}\Gamma^{-1}$. This explains the new features of unboundedness/nonintegrability of (the operator norm of) the resolvent, alluded to below (1.10).

1.4.2. *Open problems.* Our H^1 analysis suggests a number of interesting open questions. The first regards smoothing properties of the profile problem. In the finite-dimensional evolution setting, regularity of solutions is limited only by regularity of coefficients; here, however, that is not true even at the linear level. Certainly, for further (e.g., stability) analysis, we require profiles of *at least* regularity H^1, and likely higher. Our arguments can be modified to construct successively smaller stable manifolds in $H^s(\mathbb{R}_+,\mathbb{H})$, any $s \geq 1$, but for constructing profiles one would like to intersect unstable/stable manifolds that are as large as possible, thus in the weakest possible space. Hence, it is interesting to know, for H^1 *profiles* of (1.1) defined on the whole line, as opposed to decaying solutions defined on a half line, is further regularity enforced? For small-amplitude profiles, "Kawashima-type" energy estimates as in [25, 26] show that the answer is "yes." A very interesting open question is whether one can find similar energy estimates in the large-amplitude case yielding a similar conclusion. For related analysis in the finite-dimensional case, see [23].

A second question in somewhat opposite direction is "what is the minimal regularity needed to enforce exponential decay?" Specifically, we have shown that solutions of (1.3) that are sufficiently small in $H^1(\mathbb{R}_+,\mathbb{H})$ must decay pointwise at exponential rate; moreover, they lie on our constructed local H^1 stable manifold. What about solutions that are merely small in L^∞? A very interesting observation due to Fedja Nazarov [27] based on the indefinite Lyapunov functional relation $\langle u, Au\rangle' = \langle u, Q'(u^\pm)u\rangle - o(\|u\|^2_{\mathbb{H}})$ yields the L^2-exponential decay result $e^{\beta|\cdot|}\|u(\cdot)\| \in L^2(\mathbb{R}_+)$ for some $\beta > 0$, hence (by interpolation) in any L^p, $2 \leq p < \infty$. However, it is not clear what happens in the critical norm $p = \infty$; it would be very interesting to exhibit a counterexample or prove decay.

A glossary of notation. For $p \geq 1$, $J \subseteq \mathbb{R}$ and $\mathbb{X}$ a Banach space, $L^p(J,\mathbb{X})$ are the usual $\mathbb{X}$-valued Lebesgue spaces on J, associated with Lebesgue measure $d\tau$ on J. Similarly, $L^p(J,\mathbb{X}; w(\tau)\mathrm{d}\tau)$ are the weighted spaces with a weight $w \geq 0$. The respective spaces of bounded continuous functions on J are denoted by $C_\mathrm{b}(J,\mathbb{X})$ and $C_\mathrm{b}(J,\mathbb{X}; w(\tau))$. $H^s(\mathbb{R},\mathbb{X})$, $s > 0$, is the usual Sobolev space of $\mathbb{X}$ valued functions. In the sequel we also use the notation $H^s_\alpha(\mathbb{R},\mathbb{X}) = \{f : e^{\alpha|\cdot|}f \in H^s(\mathbb{R},\mathbb{X})\}$. The identity operator on a Banach space $\mathbb{X}$ is denoted by I (or by $I_\mathbb{X}$ if its dependence on $\mathbb{X}$ needs to be stressed). The set of bounded linear operators from a Banach space $\mathbb{X}$ to itself is denoted by $\mathscr{B}(\mathbb{X})$. For an operator T on a Hilbert space we use T^*, $\mathrm{dom}(T)$, $\ker T$, $\mathrm{im}T$, $\sigma(T)$, $\rho(T)$, $R(\lambda,T) = (\lambda - T)^{-1}$ and $T_{|\mathbb{Y}}$ to denote the adjoint, domain, kernel, range, spectrum, resolvent set, resolvent operator and the restriction of T to a subspace $\mathbb{Y}$ of $\mathbb{X}$. If $B : J \to \mathscr{B}(\mathbb{X})$ then M_B denotes the operator of multiplication by $B(\cdot)$ in $L^p(J,\mathbb{X})$ or $C_\mathrm{b}(J,\mathbb{X})$. If $\mathbb{X}_1$ and $\mathbb{X}_2$ are two subspaces of

$\mathbb{X}$, then $\mathbb{X}_1 \oplus \mathbb{X}_2$ denotes their direct (but not necessarily orthogonal) sum. The Fourier transform of a Borel measure μ is defined by $(\mathscr{F}\mu)(\omega) = \int_{\mathbb{R}} e^{-2\pi i x\omega} d\mu(x)$.

2. Stable bi-semigroups and exponential dichotomies of the linearization.

In this section we study the properties of the linearization of equation (1.3) (with $s = 0$) at the equilibria $u^{\pm}$:

$$\mathscr{L}^{\pm} = A\partial_\tau - Q'(u^{\pm}). \tag{2.1}$$

We can view the differential expression $\mathscr{L}^{\pm}$ as a densely defined, closed operator on $L^2(\mathbb{R}, \mathbb{H})$ with domain $\mathrm{dom}(\mathscr{L}^{\pm}) = \{u \in L^2(\mathbb{R}, \mathbb{H}) : Au' \in L^2(\mathbb{R}, \mathbb{H}))\}$. Throughout this section we assume Hypotheses (H1)-(H4).

It is well-known, see e.g. [13, 14], that the invertibility of $\mathscr{L}^{\pm}$ on $L^2(\mathbb{R}, \mathbb{H})$ is equivalent to the exponential dichotomy on $\mathbb{H}$ of equation

$$Au' = Q'(u^{\pm})u. \tag{2.2}$$

Remark 1. Under assumptions (H1) and (H2), the linear operator $\mathscr{L}^{\pm}$ is *not invertible* on $L^2(\mathbb{R}, \mathbb{H})$. Indeed, one can readily check that the linear operator $\mathscr{L}^{\pm}$ is invertible on $L^2(\mathbb{R}, \mathbb{H})$ with bounded inverse if and only if the operator of multiplication by the continuous, operator valued function $\widehat{\mathscr{L}^{\pm}}(\omega) = 2\pi \mathrm{i}\omega A - Q'(u^{\pm})$ is invertible on $L^2(\mathbb{R}, \mathbb{H})$ with bounded inverse. From the later we can infer that $Q'(u^{\pm})$ is invertible on $\mathbb{H}$, which contradicts Hypothesis (H2).

We note that for any $u_0 \in \mathbb{V}^{\perp}$ the constant function $u(\tau) = u_0$ is a solution of equation (2.2). Hence, (2.2) cannot exhibit an exponential dichotomy on the entire space $\mathbb{H}$. In this section we prove that equations (2.2) exhibit an exponential dichotomy on a direct complement of the finite dimensional space $\mathbb{V}^{\perp}$. Using the decomposition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \mathbb{V}^{\perp} \oplus \mathbb{V} \to \mathbb{V}^{\perp} \oplus \mathbb{V}, \quad Q'(u^{\pm}) = \begin{bmatrix} 0 & 0 \\ 0 & Q'_{22}(u^{\pm}) \end{bmatrix} : \mathbb{V}^{\perp} \oplus \mathbb{V} \to \mathbb{V}^{\perp} \oplus \mathbb{V} \tag{2.3}$$

and denoting by $v = P_{\mathbb{V}} u$ and $h = P_{\mathbb{V}^{\perp}} u$, one can readily check that equation (2.2) is equivalent to the system

$$\begin{cases} A_{11}h' + A_{12}v' = 0, \\ A_{21}h' + A_{22}v' = Q'_{22}(u^{\pm})v. \end{cases} \tag{2.4}$$

We note that Hypothesis (H4) holds if and only if the linear operator A_{11} is invertible on $\mathbb{V}^{\perp}$. Integrating the first equation, we obtain that solutions $u = (h, v)$ of (2.4) that decay to 0 at $\pm\infty$, satisfy the conditions

$$\begin{cases} h = -A_{11}^{-1}A_{12}v, \\ A_{21}h' + A_{22}v' = Q'_{22}(u^{\pm})v. \end{cases} \tag{2.5}$$

To prove that equation (2.2) has an exponential dichotomy on a complement of $\mathbb{V}^{\perp}$ it is enough to show that equation

$$(A_{22} - A_{21}A_{11}^{-1}A_{12})v' = Q'_{22}(u^{\pm})v \tag{2.6}$$

is equivalent to an equation of the form $u' = Su$, where the linear operator S generates a stable bi-semigroup on $\mathbb{V}$. We recall that a linear operator generates a bi-semigroup on a Banach or Hilbert space $\mathbb{X}$, if there exist two closed subspaces $\mathbb{X}_j$, $j = 1, 2$, of $\mathbb{X}$, *invariant* under S, such that $\mathbb{X} = \mathbb{X}_1 \oplus \mathbb{X}_2$ and $S_{|\mathbb{X}_1}$ and $-S_{|\mathbb{X}_2}$ generate C^0-semigroups on $\mathbb{X}_j$, $j = 1, 2$. We say that the bi-semigroup is exponentially stable if the two semigroups are exponentially stable. In the following lemma we collect some of the properties of the operator valued coefficients of the reduced equation (2.6).

Lemma 2.1. *Assume Hypotheses (H1)-(H4). Then,*

(i) The linear operator $\tilde{A} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ *is self-adjoint and one-to-one;*
(ii) The linear operator $Q'_{22}(u^{\pm})$ *is invertible with bounded inverse on* $\mathbb{V}$.

Proof. (i) Since the linear operator A is self-adjoint, from decomposition (2.3), we obtain that $A_{11}^* = A_{11}$, $A_{22}^* = A_{22}$ and $A_{12}^* = A_{21}$, which implies that $\tilde{A}$ is self-adjoint. To show that $\tilde{A}$ is one-to-one, we consider $v \in \ker\tilde{A}$ and denote by $h = -A_{11}^{-1}A_{12}v \in \mathbb{V}^{\perp}$. Using again the decomposition (2.3), one can readily check that $A(h+v) = 0$. From Hypothesis (H3) we infer that $h = -v$. Since $v \in \mathbb{V}$ and $h \in \mathbb{V}^{\perp}$ we conclude that $v = 0$, proving (i). Since $Q'_{22}(u^{\pm})$ is self-adjoint and $Q'_{22}(u^{\pm}) \leq -\delta_{\pm}I_{\mathbb{V}}$ by Hypothesis (H2), assertion (ii) follows shortly. □

Next, we note that equation (2.6) is of the form:

$$\Gamma u' = Eu, \tag{2.7}$$

where the linear operators Γ and E satisfy the following Hypothesis (S) below. In what follows we treat equation (2.7) which is more general than (2.6). In particular, we will show our bi-semigroup result *without assuming* that the linear operator Γ is obtained from the linear A satisfying Hypotheses (H1)–(H4) by the row-reduction method. Our goal is to prove that equation (2.7) is equivalent to an equation of the form $u' = S_{\Gamma,E}u$, where the linear operator $S_{\Gamma,E}$ generates a *exponentially stable bi-semigroup.*

Hypothesis (S) We assume that $\mathbb{X}$ is a Hilbert space and the bounded linear operators $\Gamma, E \in \mathscr{B}(\mathbb{X})$ satisfy the following conditions:

(i) Γ is self-adjoint and one-to-one;
(ii) The linear operator E is self-adjoint and $E \leq -\delta I_{\mathbb{X}}$, for some $\delta > 0$.

Since the linear operators Γ and E are bounded, one can readily check that the linear operator

$$S_{\Gamma,E} = \Gamma^{-1}E : \mathrm{dom}(S_{\Gamma,E}) = \{u \in \mathbb{X} : Eu \in \mathrm{im}\Gamma\} \to \mathbb{X}, \tag{2.8}$$

is closed on $\mathbb{X}$. In the next lemma we prove that $S_{\Gamma,E}$ is hyperbolic and the basic estimates satisfied by the norm of the resolvent operators. To formulate the lemma, we introduce the operator valued function $\mathscr{L}_{\Gamma,E} : \mathbb{R} \to \mathscr{B}(\mathbb{X})$ defined by $\mathscr{L}_{\Gamma,E}(\omega) = 2\pi\mathrm{i}\omega\Gamma - E$.

Lemma 2.2. *Assume Hypothesis (S). Then,*

(i) The linear operator $\mathscr{L}_{\Gamma,E}(\omega) = 2\pi\mathrm{i}\omega\Gamma - E$ *is invertible on* $\mathbb{X}$ *for any* $\omega \in \mathbb{R}$;
(ii) $\sup_{\omega\in\mathbb{R}} \|\mathscr{L}_{\Gamma,E}(\omega)^{-1}\| < \infty$;
(iii) $\mathrm{i}\mathbb{R} \subseteq \rho(S_{\Gamma,E})$ *and* $R(2\pi\mathrm{i}\omega, S_{\Gamma,E}) = (2\pi\mathrm{i}\omega - S_{\Gamma,E})^{-1} = \big(\mathscr{L}_{\Gamma,E}(\omega)\big)^{-1}\Gamma$ *for all* $\omega \in \mathbb{R}$;
(iv) There exists $c > 0$ *such that* $\|R(2\pi\mathrm{i}\omega, S_{\Gamma,E})\| \leq \frac{c}{1+|\omega|}$ *for all* $\omega \in \mathbb{R}$.

Proof. To prove (i) and (ii), we note that since Γ and E are self-adjoint operators we have that $\mathrm{Re}\mathscr{L}_{\Gamma,E}(\omega) = -E$ for any $\omega \in \mathbb{R}$. We obtain that

$$\mathrm{Re}\langle\mathscr{L}_{\Gamma,E}(\omega)\mathbf{x}, \mathbf{x}\rangle = -\langle E\mathbf{x}, \mathbf{x}\rangle \geq \delta\|\mathbf{x}\|^2 \quad \text{for any} \quad \omega \in \mathbb{R}, \mathbf{x} \in \mathbb{X}, \tag{2.9}$$

which implies that

$$\|\mathscr{L}_{\Gamma,E}(\omega)\mathbf{x}\| \geq \delta\|\mathbf{x}\| \quad \text{for any} \quad \omega \in \mathbb{R}, \mathbf{x} \in \mathbb{X}. \tag{2.10}$$

It follows that $\mathscr{L}_{\Gamma,E}(\omega)$ is one-to-one and $\mathrm{im}\mathscr{L}_{\Gamma,E}(\omega)$ is closed in $\mathbb{X}$ for any $\omega \in \mathbb{R}$. Moreover, from (2.9) we infer that $\ker \mathscr{L}_{\Gamma,E}(\omega)^* = \{0\}$ for any $\omega \in \mathbb{R}$, proving (i). Assertion (ii) is a consequence of (2.10).

Assertion (iii) follows from (i) and the definition of $S_{\Gamma,E}$ in (2.8). Indeed, since $\Gamma S_{\Gamma,E}u = Eu$ for any $u \in \mathrm{dom}(S_{\Gamma,E})$ one readily checks that $\big(\mathscr{L}_{\Gamma,E}(\omega)\big)^{-1}\Gamma(2\pi\mathrm{i}\omega - S_{\Gamma,E})u = u$ for any $u \in \mathrm{dom}(S_{\Gamma,E})$. Moreover, since the linear operators Γ and E are bounded, we have that $E(2\pi\mathrm{i}\omega\Gamma - E)^{-1} = 2\pi\mathrm{i}\omega\Gamma(2\pi\mathrm{i}\omega\Gamma - E)^{-1} - I$, which implies that

$$E\big(\mathscr{L}_{\Gamma,E}(\omega)\big)^{-1}\Gamma u = E(2\pi\mathrm{i}\omega\Gamma - E)^{-1}\Gamma u = 2\pi\mathrm{i}\omega\Gamma(2\pi\mathrm{i}\omega\Gamma - E)^{-1}\Gamma u - \Gamma u \in \mathrm{im}\Gamma \tag{2.11}$$

for any $u \in \mathbb{X}$. It follows that $\big(\mathscr{L}_{\Gamma,E}(\omega)\big)^{-1}\Gamma u \in \mathrm{dom}(S_{\Gamma,E})$ and

$$S_{\Gamma,E}\big(\mathscr{L}_{\Gamma,E}(\omega)\big)^{-1}\Gamma u = 2\pi\mathrm{i}\omega(2\pi\mathrm{i}\omega\Gamma - E)^{-1}\Gamma u - u$$

for any $u \in \mathbb{X}$, proving (iii).

Proof of (iv). Using the same argument used to prove the resolvent equation, one can show that

$$(2\pi\mathrm{i}\omega_1\Gamma - E)^{-1} - (2\pi\mathrm{i}\omega_2\Gamma - E)^{-1} = 2\pi\mathrm{i}(\omega_2 - \omega_1)(2\pi\mathrm{i}\omega_1\Gamma - E)^{-1}\Gamma(2\pi\mathrm{i}\omega_2\Gamma - E)^{-1} \tag{2.12}$$

for any $\omega_1, \omega_2 \in \mathbb{R}$. Setting $\omega_1 = \omega$, $\omega_2 = 0$ and multiplying the equation by E from the right we obtain that

$$(2\pi\mathrm{i}\omega\Gamma - E)^{-1}E + I = 2\pi\mathrm{i}\omega(2\pi\mathrm{i}\omega\Gamma - E)^{-1}\Gamma = 2\pi\mathrm{i}\omega R(2\pi\mathrm{i}\omega, S_{\Gamma,E}) \tag{2.13}$$

for any $\omega \in \mathbb{R}$. Assertion (iv) follows readily from (ii) and (2.13). □

Next, we prove that the linear operator $S_{\Gamma,E}$ generates a bi-semigroup by making use of the structure of the linear operators Γ and E, especially the fact that these operators are self-adjoint.

Lemma 2.3. *Assume Hypothesis (S). Then, the linear operator $S_{\Gamma,E}$ is similar to an operator of multiplication by some real-valued, measurable function $H_{\Gamma,E} : \Lambda \to \mathbb{R}$, such that $|H_{\Gamma,E}|$ is bounded from below, on $L^2(\Lambda, \mu)$, where (Λ, μ) is some measure space. Therefore, $S_{\Gamma,E}$ generates an exponentially stable bi-semigroup, having the representation:*

$$\mathbb{X}_{\mathrm{s}}^{\Gamma,E} = U_{\Gamma,E}^{-1}L^2(\Lambda_-, \mu), \quad \mathbb{X}_{\mathrm{u}}^{\Gamma,E} = U_{\Gamma,E}^{-1}L^2(\Lambda_+, \mu); \tag{2.14}$$

$$T_{\mathrm{s/u}}^{\Gamma,E}(\tau) = U_{\Gamma,E}^{-1}\widetilde{T}_{\mathrm{s/u}}^{\Gamma,E}(\tau)U_{\Gamma,E}\big|_{\mathbb{X}_{\mathrm{s/u}}^{\Gamma,E}}, \quad \textit{for any} \quad \tau \geq 0. \tag{2.15}$$

Here $U_{\Gamma,E} \in \mathscr{B}(\mathbb{X}, L^2(\Lambda, \mu))$ is invertible with bounded inverse, $\Lambda_\pm := \{\lambda \in \Lambda : \pm H_{\Gamma,E}(\lambda) > 0\}$ and the C^0-semigroups $\{\widetilde{T}_{\mathrm{s/u}}^{\Gamma,E}(\tau)\}_{\tau\geq 0}$ are defined by

$$\Big(\widetilde{T}_{\mathrm{s}}^{\Gamma,E}(\tau)\widetilde{f}\Big)(\lambda) = e^{\tau H_{\Gamma,E}(\lambda)}\widetilde{f}(\lambda), \quad \tau \geq 0,\ \lambda \in \Lambda_-,\ \widetilde{f} \in L^2(\Lambda_-, \mu);$$
$$\Big(\widetilde{T}_{\mathrm{u}}^{\Gamma,E}(\tau)\widetilde{f}\Big)(\lambda) = e^{-\tau H_{\Gamma,E}(\lambda)}\widetilde{f}(\lambda), \quad \tau \geq 0,\ \lambda \in \Lambda_+,\ \widetilde{f} \in L^2(\Lambda_+, \mu). \tag{2.16}$$

Proof. Since the linear operator E is bounded, self-adjoint, invertible and negative-definite, we have that $\widetilde{E} = (-E)^{\frac{1}{2}}$ is a bounded, self-adjoint, invertible linear operator on $\mathbb{X}$. One can readily check that

$$\widetilde{E}S_{\Gamma,E}\widetilde{E}^{-1} = \widetilde{E}\Gamma^{-1}E\widetilde{E}^{-1} = -\widetilde{E}\Gamma^{-1}\widetilde{E}. \tag{2.17}$$

Since the linear operator Γ and $\widetilde{E}$ are self-adjoint, we obtain that the linear operator $\widetilde{E}S_{\Gamma,E}\widetilde{E}^{-1}$ is self-adjoint. It follows that the linear operator $\widetilde{E}S_{\Gamma,E}\widetilde{E}^{-1}$ is unitarily equivalent to an operator of multiplication on some L^2 space. Therefore, there exists a measure space (Λ,μ), a real-valued, measurable function $H_{\Gamma,E}:\Lambda\to\mathbb{R}$ and a *unitary*, bounded, linear operator $V_{\Gamma,E}:\mathbb{X}\to L^2(\Lambda,\mu)$ such that $\widetilde{E}S_{\Gamma,E}\widetilde{E}^{-1}=V_{\Gamma,E}^{-1}M_{H_{\Gamma,E}}V_{\Gamma,E}$. It follows that

$$S_{\Gamma,E}=U_{\Gamma,E}^{-1}M_{H_{\Gamma,E}}U_{\Gamma,E},\quad\text{where}\quad U_{\Gamma,E}=V_{\Gamma,E}\widetilde{E}\in\mathscr{B}(\mathbb{X},L^2(\Lambda,\mu)).\tag{2.18}$$

Since $V_{\Gamma,E}$ is a unitary operator and $\tilde{E}$ is invertible, we immediately infer that $U_{\Gamma,E}$ is bounded with bounded inverse.

Next, we prove that the function $|H_{\Gamma,E}|$ is bounded from below. From (2.18) and Lemma 2.2(iii) we conclude that

$$\mathrm{i}\mathbb{R}\subseteq\rho(S_{\Gamma,E})=\rho(M_{H_{\Gamma,E}})\quad\text{and}\quad R(2\pi\mathrm{i}\omega,S_{\Gamma,E})=U_{\Gamma,E}^{-1}R(2\pi\mathrm{i}\omega,M_{H_{\Gamma,E}})U_{\Gamma,E}\tag{2.19}$$

for any $\omega\in\mathbb{R}$. From (2.19) and Lemma 2.2(iv) we obtain that

$$\operatorname{esssup}_{\lambda\in\Lambda}\frac{1}{|2\pi\mathrm{i}\omega-H_{\Gamma,E}(\lambda)|}=\|R(2\pi\mathrm{i}\omega,M_{H_{\Gamma,E}})\|\leq\frac{c}{1+|\omega|}\quad\text{for all}\quad\omega\in\mathbb{R},\tag{2.20}$$

which implies that there exists $\nu=\nu(\Gamma,E)>0$ such that

$$|H_{\Gamma,E}(\lambda)|\geq\nu\quad\text{for }\mu\text{ almost all }\lambda\in\Lambda.\tag{2.21}$$

The representation (2.18) holds true when we modify the function $H_{\Gamma,E}$ on a set of μ-measure 0, therefore we can assume from now on that the inequality (2.21) is true for any $\lambda\in\Lambda$.

From (2.18) we can immediately infer that $S_{\Gamma,E}$ generates a bi-semigroup. Defining $\Lambda_\pm:=\{\lambda\in\Lambda:\pm H_{\Gamma,E}(\lambda)>0\}$, from (2.21) we immediately conclude that

$$\Lambda_\pm:=\{\lambda\in\Lambda:\pm H_{\Gamma,E}(\lambda)\geq\nu\},\quad\Lambda=\Lambda_+\cup\Lambda_-,\quad\Lambda_+\cap\Lambda_-=\emptyset.\tag{2.22}$$

It follows that $L^2(\Lambda,\mu)=L^2(\Lambda_+,\mu)\oplus L^2(\Lambda_-,\mu)$. One can readily check that $M_{\pm\chi_{\Lambda_\mp}H_{\Gamma,E}}$, the operators of multiplication by the functions $\pm\chi_{\Lambda_\mp}H_{\Gamma,E}$ generate two C^0-semigroups on $L^2(\Lambda_\pm,\mu)$ given by (2.16). Here $\chi_{\Lambda_\mp}$ denotes the characteristic function of the set $\Lambda_\pm$. Assertions (2.14) and (2.15) are direct consequence of representation (2.18). Finally, from (2.16) and (2.21) we conclude that the C^0 semigroups $\{\widetilde{T}^{\Gamma,E}_{\mathrm{s/u}}(\tau)\}_{\tau\geq0}$, and thus $\{T^{\Gamma,E}_{\mathrm{s/u}}(\tau)\}_{\tau\geq0}$, are exponentially stable. □

We note that the main idea used to obtain the representation (2.18) is based on the unitary equivalence of self-adjoint operators to multiplication operators, which is spectral in nature. Thus, it is natural to refer to functions in $L^2(\Lambda,\mu)$ as spectral components of the generator $S_{\Gamma,E}$. In the next lemma we give a spectral representation of the operator valued function $R_{\Gamma,E}:\mathbb{R}\to\mathscr{B}(\mathbb{X})$ defined by $R_{\Gamma,E}(\omega)=(2\pi\mathrm{i}\omega\Gamma-E)^{-1}$.

Lemma 2.4. *Assume Hypothesis (S). Then,*

(i) The linear operators $U_{\Gamma,E}$ and E satisfy the identity

$$U_{\Gamma,E}E^{-1}U^*_{\Gamma,E}=-Id_{L^2(\Lambda,\mu)};\tag{2.23}$$

(ii) The operator-valued function $R_{\Gamma,E}$ has the following representation

$$R_{\Gamma,E}(\omega)=E^{-1}S^*_{\Gamma,E}R(2\pi\mathrm{i}\omega,S^*_{\Gamma,E})=E^{-1}U^*_{\Gamma,E}\widetilde{R}_{\Gamma,E}(\omega)(U^*_{\Gamma,E})^{-1}\tag{2.24}$$

for any $\omega \in \mathbb{R}$, *where* $\widetilde{R}_{\Gamma,E} : \mathbb{R} \to \mathscr{B}(L^2(\Lambda,\mu))$ *is given by*

$$\left(\widetilde{R}_{\Gamma,E}(\omega)\widetilde{f}\right)(\lambda) = \frac{H_{\Gamma,E}(\lambda)}{2\pi \mathrm{i}\omega - H_{\Gamma,E}(\lambda)}\widetilde{f}(\lambda),\ \omega \in \mathbb{R}, \lambda \in \Lambda, \widetilde{f} \in L^2(\Lambda,\mu). \tag{2.25}$$

Proof. (i) Since the linear operator $V_{\Gamma,E} \in \mathscr{B}(\mathbb{X}, L^2(\Lambda,\mu))$ is unitary, $\widetilde{E}^2 = -E$ and $U_{\Gamma,E} = V_{\Gamma,E}\widetilde{E}$ one can readily check that

$$\begin{aligned} U_{\Gamma,E}E^{-1}U^*_{\Gamma,E} &= V_{\Gamma,E}\widetilde{E}E^{-1}(V_{\Gamma,E}\widetilde{E})^* = -(V_{\Gamma,E}\widetilde{E}^{-1})\widetilde{E}V^*_{\Gamma,E} = -V_{\Gamma,E}V^{-1}_{\Gamma,E} \\ &= -Id_{L^2(\Lambda,\mu)}. \end{aligned} \tag{2.26}$$

Proof of (ii). From Lemma 2.2(iii), the definition of the linear operator $S_{\Gamma,E}$ in (2.8) and Hypothesis(S)(i)-(ii) we obtain that $R(2\pi\mathrm{i}\omega, S^*_{\Gamma,E}) = (2\pi\mathrm{i}\omega - S^*_{\Gamma,E})^{-1} = \Gamma R_{\Gamma,E}(\omega)$ for any $\omega \in \mathbb{R}$, which implies that

$$S^*_{\Gamma,E}(2\pi\mathrm{i}\omega - S^*_{\Gamma,E})^{-1} = S^*_{\Gamma,E}\Gamma R_{\Gamma,E}(\omega) = ER_{\Gamma,E}(\omega) \quad \text{for any} \quad \omega \in \mathbb{R}. \tag{2.27}$$

Moreover, from (2.18) we infer that

$$\begin{aligned} S^*_{\Gamma,E}(2\pi\mathrm{i}\omega - S^*_{\Gamma,E})^{-1} &= U^*_{\Gamma,E}M_{H_{\Gamma,E}}(2\pi\mathrm{i}\omega - M_{H_{\Gamma,E}})^{-1}(U^*_{\Gamma,E})^{-1} \\ &= U^*_{\Gamma,E}\widetilde{R}_{\Gamma,E}(\omega)(U^*_{\Gamma,E})^{-1} \end{aligned} \tag{2.28}$$

for any $\omega \in \mathbb{R}$. Since E is invertible by Hypothesis (S) (ii), assertion (2.24) follows from (2.26), (2.27) and (2.28). □

To conclude this section, we use Lemma 2.3 to prove Theorem 1.2. We recall the definition of the linear operators

$$S^{\mathrm{r}}_{\pm} = \tilde{A}^{-1}Q'_{22}(u^{\pm}). \tag{2.29}$$

Proof of Theorem 1.2. From Lemma 2.1 we have that the linear operators $\tilde{A} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ and $Q'_{22}(u^{\pm})$ satisfy Hypothesis (S). Assertion (i) follows directly from Lemma 2.3. Since equation (2.2) is equivalent to the system (2.5), we infer that assertion (ii) follows readily from (i). Moreover, if we denote the stable/unstable spaces of equation (2.6) by $\mathbb{V}^{\mathrm{s/u}}_{\pm}$, then the stable/unstable subspaces of equation (2.2) are given by the formula

$$\mathbb{H}^{\mathrm{s/u}}_{\pm} = \left\{(h,v) \in \mathbb{V}^{\perp} \oplus \mathbb{V} : h = -A_{11}^{-1}A_{12}v,\ v \in \mathbb{V}^{\mathrm{s/u}}_{\pm}\right\}. \tag{2.30}$$

One can readily check that $\mathbb{H} = \mathbb{V}^{\perp} \oplus \mathbb{H}^{\mathrm{s}}_{\pm} \oplus \mathbb{H}^{\mathrm{u}}_{\pm}$, proving the theorem.

3. **Solutions of general steady relaxation systems.** In this section we analyze the qualitative properties of solutions of the steady equation

$$Au_{\tau} = Q(u) \tag{3.1}$$

in $\mathbb{H}$ satisfying $\lim_{\tau\to\pm\infty} u(\tau) = u^{\pm}$ and its linearization along $u^{\pm}$. In particular, we are interested in describing the smoothness properties of these solutions. Also, it is interesting to consider all of these equations on $\mathbb{R}_{\pm}$, respectively. Making the change of variable $w^{\pm}(\tau) = u(\tau) - u^{\pm}$ in (3.1) we obtain the equations

$$Aw^{\pm}_{\tau}(\tau) = 2B(u^{\pm}, w^{\pm}(\tau)) + Q(w^{\pm}(\tau)). \tag{3.2}$$

Here, we recall that $Q(u) = B(u,u)$ is bilinear, symmetric, continuous on $\mathbb{H}$. Moreover, since the range of the bilinear map B is contained in $\mathbb{V}$, denoting by

$h^\pm = P_{\mathbb{V}^\perp} w^\pm$ and $v^\pm = P_{\mathbb{V}} w^\pm$, we obtain that equation (3.2) is equivalent to the system

$$\begin{cases} A_{11}h^\pm_\tau(\tau) + A_{12}v^\pm_\tau(\tau) = 0, \\ A_{21}h^\pm_\tau(\tau) + A_{22}v^\pm_\tau(\tau) = Q'_{22}(u^\pm)v^\pm(\tau) + Q(h^\pm(\tau) + v^\pm(\tau)). \end{cases} \tag{3.3}$$

Integrating the first equation and using that $\lim_{\tau\to\pm\infty} w^\pm(\tau) = 0$, we obtain that solutions $w^\pm = (h^\pm, v^\pm)$ of (3.3) satisfy the condition $h^\pm = -A_{11}^{-1}A_{12}v^\pm$. Plugging in the second equation of (3.3) we obtain that to prove the existence of a stable/unstable manifold around the equilibria u^+/u^-, respectively, it is enough to prove the existence of a stable/unstable manifold around equilibria $P_{\mathbb{V}}u^+/P_{\mathbb{V}}u^-$, respectively, of equation

$$(A_{22} - A_{21}A_{11}^{-1}A_{12})v^\pm_\tau(\tau) = Q'_{22}(u^\pm)v^\pm(\tau) + Q\big(v^\pm(\tau) - A_{11}^{-1}A_{12}v^\pm(\tau)\big). \tag{3.4}$$

We note that it is especially important to study the solutions of equations (3.3) and (3.4) close to $\pm\infty$, therefore we focus our attention on their solutions on $\mathbb{R}_\pm$, rather than the entire line. To study these equations we use the properties of exponentially stable bi-semigroups. We recall that if a linear operator S generates an exponentially stable bi-semigroup, then the linear operator $-S$ generates an exponentially stable bi-semigroup as well. Making the change of variables $\tau \to -\tau$ in (3.4), we obtain an equation that can be handled in the same way as the original equation, as shown in [14, Section 4]. Therefore, to understand the limiting properties of solutions of equations (3.4) at $\pm\infty$, we need to understand the limiting properties of solutions of equations of the form

$$\Gamma\mathbf{u}_\tau(\tau) = E\mathbf{u}(\tau) + D(\mathbf{u}(\tau), \mathbf{u}(\tau)), \quad \tau \in \mathbb{R}_+. \tag{3.5}$$

Here the pair of bounded linear operators (Γ, E) on a Hilbert space $\mathbb{X}$ satisfies Hypothesis (S) and $D : \mathbb{X}\times\mathbb{X} \to \mathbb{X}$ is a bounded, bilinear map.

In what follows the stable/unstable subspaces of $\mathbb{X}$ invariant under $\Gamma^{-1}E$ are denoted by $\mathbb{X}^{\Gamma,E}_{\mathrm{s/u}}$ and the exponentially stable bi-semigroup generated by $S_{\Gamma,E} = \Gamma^{-1}E$ on $\mathbb{X}$ is denoted by $\{T^{\Gamma,E}_{\mathrm{s/u}}(\tau)\}_{\tau\geq 0}$. Next, we introduce

$$\nu(\Gamma, E) = \operatorname{essinf}_{\lambda\in\Lambda}|H_{\Gamma,E}(\lambda)|. \tag{3.6}$$

From (2.15), (2.16) and (3.6) it follows that there exists $c(\Gamma, E) > 0$ such that

$$\|T^{\Gamma,E}_{\mathrm{s/u}}(\tau)\| \leq c(\Gamma, E)e^{-\nu(\Gamma,E)\tau} \quad \text{for any} \quad \tau \geq 0. \tag{3.7}$$

In addition, we denote by $P^{\Gamma,E}_{\mathrm{s/u}}$ the projections onto $\mathbb{X}^{\Gamma,E}_{\mathrm{s/u}}$ parallel to $\mathbb{X}^{\Gamma,E}_{\mathrm{u/s}}$, associated to the decomposition $\mathbb{X} = \mathbb{X}^{\Gamma,E}_{\mathrm{s}} \oplus \mathbb{X}^{\Gamma,E}_{\mathrm{u}}$ (direct sum, not necessarily orthogonal). From (2.12) and the definition of the function $R_{\Gamma,E}$ we have that

$$R_{\Gamma,E}(\omega_1) - R_{\Gamma,E}(\omega_2) = 2\pi\mathrm{i}(\omega_2 - \omega_1)R_{\Gamma,E}(\omega_1)\Gamma R_{\Gamma,E}(\omega_2) \quad \text{for all} \quad \omega_1, \omega_2 \in \mathbb{R}. \tag{3.8}$$

A first step towards understanding equation (3.5) is to study the perturbed equation

$$\Gamma\mathbf{u}_\tau(\tau) = E\mathbf{u}(\tau) + f(\tau), \quad \tau \in \mathbb{R}_+, \tag{3.9}$$

for some function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{X})$ or $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{X})$. For a function g defined on a proper subset of $\mathbb{R}$ we keep the same notation g to denote its extension to $\mathbb{R}$ by 0.

Definition 3.1. We say that

(i) The function $\mathbf{u} : [\tau_0, \tau_1] \to \mathbb{X}$ is a *smooth* solution of (3.9) on $[\tau_0, \tau_1]$ if $\mathbf{u} \in H^1([\tau_0, \tau_1], \mathbb{X})$ satisfies (3.9);

(ii) The function $\mathbf{u} : [\tau_0, \tau_1] \to \mathbb{X}$ is a *mild* solution of (3.9) on $[\tau_0, \tau_1]$ if it is square integrable on $[\tau_0, \tau_1]$ and satisfies

$$\widehat{\mathbf{u}}(\omega) = R(2\pi i\omega, S_{\Gamma,E})\big(e^{-2\pi i\omega\tau_0}\mathbf{u}(\tau_0) - e^{-2\pi i\omega\tau_1}\mathbf{u}(\tau_1)\big) + R_{\Gamma,E}(\omega)\widehat{f_{|[\tau_0,\tau_1]}}(\omega) \quad (3.10)$$

for almost all $\omega \in \mathbb{R}$;

(iii) The function $\mathbf{u} : \mathbb{R}_+ \to \mathbb{X}$ is a *mild* solution of (3.9) on $\mathbb{R}_+$ if $\mathbf{u}$ is a *mild* solution of (3.9) on $[0, \tau_1]$ for any $\tau_1 > 0$.

Our definition of mild solutions follows [14, Section 2], where it is shown that the frequency domain reformulation given in (3.10) is much easier to handle than the classical approach where one defines the mild solution by simply integrating equation (3.9). We note that by taking Fourier transform in (3.9) and integrating by parts, it is easy to verify that smooth solutions of equation are also mild solutions.

Remark 2. Denoting by $\mathscr{G}_{\Gamma,E} : \mathbb{R} \to \mathscr{B}(\mathbb{X})$ the Green function defined by

$$\mathscr{G}_{\Gamma,E}(\tau) = \begin{cases} T_{\mathrm{s}}^{\Gamma,E}(\tau)P_{\mathrm{s}}^{\Gamma,E} & \text{if } \tau \geq 0 \\ -T_{\mathrm{u}}^{\Gamma,E}(-\tau)P_{\mathrm{u}}^{\Gamma,E} & \text{if } \tau < 0 \end{cases}, \quad (3.11)$$

we have that (i) there exists a constant $c(\Gamma, E)$ such that $\|\mathscr{G}_{\Gamma,E}(\tau)\| \leq ce^{-\nu(\Gamma,E)|\tau|}$ for any $\tau \in \mathbb{R}$, and (ii) $\mathscr{F}\mathscr{G}_{\Gamma,E}(\cdot)\mathbf{x} = R(2\pi i\cdot, S_{\Gamma,E})\mathbf{x}$ for any $\mathbf{x} \in \mathbb{X}$.

Next, we define the linear operator $\mathscr{K}_{\Gamma,E} : L^2(\mathbb{R}, \mathbb{X}) \to L^2(\mathbb{R}, \mathbb{X})$ by $\mathscr{K}_{\Gamma,E}f = \mathscr{F}^{-1}M_{R_{\Gamma,E}}\mathscr{F}f$. Here we recall that $M_{R_{\Gamma,E}}$ denotes the multiplication operator on $L^2(\mathbb{R}, \mathbb{X})$ by the operator valued function $R_{\Gamma,E}$. From Lemma 2.2(ii) we have that $\sup_{\omega\in\mathbb{R}} \|R_{\Gamma,E}(\omega)\| < \infty$, which proves that $\mathscr{K}_{\Gamma,E}$ is well defined and bounded on $L^2(\mathbb{R}, \mathbb{X})$.

To prove our results we need to understand the properties of the Fourier multiplier defined by $\mathscr{K}_{\Gamma,E}$. Our first goal in this section is to show that the definition we use for mild solutions of equation (3.9) can be seen as an extension of the classical variation of constants formula. To prove such a result we need to understand some of the smoothing properties of $\mathscr{K}_{\Gamma,E}$.

Lemma 3.2. *Assume Hypothesis (S). Then,* $\Gamma(\mathscr{K}_{\Gamma,E}f)(\cdot) \in C_0(\mathbb{R}, \mathbb{X})$ *for any* $f \in L^2(\mathbb{R}, \mathbb{X})$.

Proof. Let $f \in L^2(\mathbb{R}, \mathbb{X})$ and $g = \mathscr{K}_{\Gamma,E}f$. To prove the lemma we note that it is enough to show that $\widehat{\Gamma g} \in L^1(\mathbb{R}, \mathbb{X})$. Using the definition of $\mathscr{K}_{\Gamma,E}$ we have that

$$\widehat{\Gamma g}(\omega) = \Gamma\widehat{g}(\omega) = \Gamma\widehat{\mathscr{K}_{\Gamma,E}f}(\omega) = \Gamma R_{\Gamma,E}(\omega)\widehat{f}(\omega) \quad \text{for all} \quad \omega \in \mathbb{R}. \quad (3.12)$$

From Lemma 2.2 and the definition of $S_{\Gamma,E}$ in (2.8) and its associated bi-semigroup, we have that

$$\|\Gamma R_{\Gamma,E}(\omega)\| = \|R_{\Gamma,E}(\omega)^*\Gamma^*\| = \|R_{\Gamma,E}(-\omega)\Gamma\| = \|R(-2\pi i\omega, S_{\Gamma,E})\| \leq \frac{c}{1 + |\omega|} \quad (3.13)$$

for all $\omega \in \mathbb{R}$. From (3.12) and (3.13) we conclude that $\widehat{\Gamma g} \in L^1(\mathbb{R}, \mathbb{X})$, proving the lemma. □

Now, we are ready to prove that (3.10) is a generalization of the variation of constants formula.

Lemma 3.3. *Assume Hypothesis (S) and let $f \in L^2(\mathbb{R}_+, \mathbb{X})$. Then, $\mathbf{u} : \mathbb{R}_+ \to \mathbb{X}$ is a mild solution of (3.9) on $\mathbb{R}_+$, square integrable on $\mathbb{R}_+$ and $\Gamma\mathbf{u} \in C_0(\mathbb{R}_+, \mathbb{X})$ if and only if*

$$\mathbf{u}(\tau) = T_{\mathrm{s}}^{\Gamma,E}(\tau)P_{\mathrm{s}}^{\Gamma,E}\mathbf{u}(0) + (\mathscr{K}_{\Gamma,E}f)(\tau) \quad \textit{for all} \quad \tau \geq 0; \tag{3.14}$$

Proof. First, we prove that any mild solution $\mathbf{u}$ of (3.9) on $\mathbb{R}_+$ which is square integrable on $\mathbb{R}_+$ satisfies equation (3.14), provided $\Gamma\mathbf{u} \in C_0(\mathbb{R}_+, \mathbb{X})$. Since $\chi_{[0,\tau_1]} \to \chi_{[0,\infty)}$ simple as $\tau_1 \to \infty$, from the Lebesgue dominated Convergence theorem we obtain that $\mathbf{u}\chi_{[0,\tau_1]} \to \mathbf{u}$ and $f\chi_{[0,\tau_1]} \to f$ in $L^2(\mathbb{R}_+, \mathbb{X}) \hookrightarrow L^2(\mathbb{R}, \mathbb{X})$ as $\tau_1 \to \infty$. Since the linear operators $\mathscr{F}$ and $\mathscr{K}_{\Gamma,E}$ are continuous on $L^2(\mathbb{R}, \mathbb{X})$ we conclude that

$$\mathscr{F}\big(\mathbf{u}\chi_{[0,\tau_1]} - \mathscr{K}_{\Gamma,E}(f\chi_{[0,\tau_1]})\big) \to \mathscr{F}(\mathbf{u} - \mathscr{K}_{\Gamma,E}f) \text{ in } L^2(\mathbb{R}, \mathbb{X}) \quad \text{as} \quad \tau_1 \to \infty. \tag{3.15}$$

Moreover, since $\mathbf{u}$ is a solution of (3.9) on $[0, \tau_1]$ for all $\tau_1 > 0$ we have that

$$\mathscr{F}\big(\mathbf{u}\chi_{[0,\tau_1]} - \mathscr{K}_{\Gamma,E}(f\chi_{[0,\tau_1]})\big)(\omega) = R_{\Gamma,E}(\omega)\big(\Gamma\mathbf{u}(0) - e^{-2\pi\mathrm{i}\omega\tau_1}\Gamma\mathbf{u}(\tau_1)\big) \tag{3.16}$$

for all $\omega \in \mathbb{R}$. Since $\Gamma\mathbf{u} \in C_0(\mathbb{R}_+, \mathbb{X})$ from (3.16) it follows that

$$\mathscr{F}\big(\mathbf{u}\chi_{[0,\tau_1]} - \mathscr{K}_{\Gamma,E}(f\chi_{[0,\tau_1]})\big)(\omega) \to R_{\Gamma,E}(\omega)\Gamma\mathbf{u}(0) \text{ as } \quad \tau_1 \to \infty, \quad \text{for all} \quad \omega \in \mathbb{R}. \tag{3.17}$$

From (3.15) and (3.17) we infer that

$$\mathscr{F}(\mathbf{u} - \mathscr{K}_{\Gamma,E}f)(\omega) = R_{\Gamma,E}(\omega)\Gamma\mathbf{u}(0) = R(2\pi\mathrm{i}\omega, S_{\Gamma,E})\mathbf{u}(0)$$

for almost all $\omega \in \mathbb{R}$. Taking inverse Fourier transform, from Remark 2(ii) we obtain that

$$\mathbf{u}(\tau) = T_{\mathrm{s}}^{\Gamma,E}(\tau)P_{\mathrm{s}}^{\Gamma,E}\mathbf{u}(0) + (\mathscr{K}_{\Gamma,E}f)(\tau) \quad \text{for almost all} \quad \tau \geq 0. \tag{3.18}$$

Next, we prove that equality (3.18) holds true for any $\tau \geq 0$. Indeed, multiplying the equation by Γ from the left, we obtain that $\Gamma\mathbf{u} = \Gamma T_{\mathrm{s}}^{\Gamma,E}(\cdot)P_{\mathrm{s}}^{\Gamma,E}\mathbf{u}(0) + \Gamma(\mathscr{K}_{\Gamma,E}f)(\cdot)$ almost everywhere on $\mathbb{R}_+$. Since $\Gamma\mathbf{u}$ is continuous on $\mathbb{R}_+$, $\{T_{\mathrm{s}}^{\Gamma,E}(\tau)\}_{\tau\geq 0}$ is a strongly continuous semigroup, and from Lemma 3.2 we have that $\Gamma(\mathscr{K}_{\Gamma,E})(\cdot)$ is continuous, we infer that the equality $\Gamma\mathbf{u} = \Gamma T_{\mathrm{s}}^{\Gamma,E}(\cdot)P_{\mathrm{s}}^{\Gamma,E}\mathbf{u}(0) + \Gamma(\mathscr{K}_{\Gamma,E}f)(\cdot)$ holds everywhere on $\mathbb{R}_+$. Since Γ is one-to-one on $\mathbb{X}$, by Hypothesis (S)(i), it follows that equation (3.14) holds true.

To finish the proof of lemma, we prove that under the assumption that $f \in L^2(\mathbb{R}_+, \mathbb{X})$, any function $\mathbf{u} : \mathbb{R}_+ \to \mathbb{X}$ satisfying equation (3.14) is square integrable on $\mathbb{R}_+$, $\Gamma\mathbf{u} \in C_0(\mathbb{R}_+, \mathbb{X})$ and is a mild solution of (3.9) on $[0, \tau_1]$ for any $\tau_1 > 0$. Indeed, since $\{T_{\mathrm{s}}^{\Gamma,E}(\tau)\}_{\tau\geq 0}$ is an exponentially stable C^0-semigroup on $\mathbb{X}$ and $\mathscr{K}_{\Gamma,E}$ is well-defined and bounded on $L^2(\mathbb{R}, \mathbb{X})$, one can readily check that $\mathbf{u}$ is square integrable on $\mathbb{R}_+$. Moreover, from Lemma 3.2 and (3.14) we conclude that $\Gamma\mathbf{u} \in C_0(\mathbb{R}_+, \mathbb{X})$.

Let $\varphi \in C_0^\infty(\mathbb{R})$ be a smooth, scalar function with compact support. Using the elementary properties of the Fourier transform and convolution, from (3.8) and (3.14) we obtain that

$$\begin{aligned}\widehat{\varphi\mathbf{u}}(\omega) - R_{\Gamma,E}(\omega)\widehat{\varphi'\Gamma\mathbf{u}}(\omega) &= (\widehat{\varphi} * \widehat{\mathbf{u}})(\omega) - R_{\Gamma,E}(\omega)(\widehat{\varphi'} * \widehat{\Gamma\mathbf{u}})(\omega) \\ &= \int_{\mathbb{R}} \widehat{\varphi}(\omega - \theta)\widehat{\mathbf{u}}(\theta)\mathrm{d}\theta - R_{\Gamma,E}(\omega)\int_{\mathbb{R}} 2\pi\mathrm{i}(\omega - \theta)\widehat{\varphi}(\omega - \theta)\Gamma\widehat{\mathbf{u}}(\theta)\mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \widehat{\varphi}(\omega-\theta)\Big(I - 2\pi\mathrm{i}(\omega-\theta)R_{\Gamma,E}(\omega)\Gamma\Big)\widehat{\mathbf{u}}(\theta)\mathrm{d}\theta \\
&= \int_{\mathbb{R}} \widehat{\varphi}(\omega-\theta)\Big(I - 2\pi\mathrm{i}(\omega-\theta)R_{\Gamma,E}(\omega)\Gamma\Big)R_{\Gamma,E}(\theta)\Big(\Gamma\mathbf{u}(0)+\widehat{f}(\theta)\Big)\mathrm{d}\theta \\
&= \int_{\mathbb{R}} \widehat{\varphi}(\omega-\theta)\Big(R_{\Gamma,E}(\theta) - 2\pi\mathrm{i}(\omega-\theta)R_{\Gamma,E}(\omega)\Gamma R_{\Gamma,E}(\theta)\Big)\Big(\Gamma\mathbf{u}(0)+\widehat{f}(\theta)\Big)\mathrm{d}\theta \\
&= R_{\Gamma,E}(\omega)\Big(\int_{\mathbb{R}} \widehat{\varphi}(\omega-\theta)\mathrm{d}\theta\Big)\Gamma\mathbf{u}(0) + R_{\Gamma,E}(\omega)\int_{\mathbb{R}} \widehat{\varphi}(\omega-\theta)\widehat{f}(\theta)\mathrm{d}\theta \\
&= \varphi(0)R_{\Gamma,E}(\omega)\Gamma\mathbf{u}(0) + R_{\Gamma,E}(\omega)\widehat{\varphi f}(\omega) \quad \text{for any} \quad \omega\in\mathbb{R}.
\end{aligned} \tag{3.19}
$$

Fix $\tau_1 > 0$ and let $\{\varphi_n\}_{n\geq 1}$ be a sequence of functions in $C_0^\infty(\mathbb{R})$ with the following properties: $0 \leq \varphi_n \leq 1$, $\|\varphi_n'\|_\infty \leq cn$, $\varphi_n(\tau) = 1$ for any $\tau \in [0, \tau_1 - 1/n]$ and $\varphi_n(\tau) = 0$ for any $\tau \notin (-1/n, \tau_1)$. Since the function $\mathbf{u}$ is defined on $\mathbb{R}_+$ and is extended to $\mathbb{R}$ by 0, we conclude that

$$
\begin{aligned}
\widehat{\varphi_n'\Gamma\mathbf{u}}(\omega) + e^{-2\pi\mathrm{i}\tau_1\omega}\Gamma\mathbf{u}(\tau_1) &= \int_{\mathbb{R}_+} e^{-2\pi\mathrm{i}\tau\omega}\varphi_n'(\tau)\Gamma\mathbf{u}(\tau)\mathrm{d}\tau + e^{-2\pi\mathrm{i}\tau_1\omega}\Gamma\mathbf{u}(\tau_1) \\
&= \int_{\tau_1-1/n}^{\tau_1} \varphi_n'(\tau)\Big(e^{-2\pi\mathrm{i}\tau\omega}\Gamma\mathbf{u}(\tau) - e^{-2\pi\mathrm{i}\tau_1\omega}\Gamma\mathbf{u}(\tau_1)\Big)\mathrm{d}\tau
\end{aligned} \tag{3.20}
$$

for any $n \geq 1$ and $\omega \in \mathbb{R}$. Hence, the following estimate holds

$$
\|\widehat{\varphi_n'\Gamma\mathbf{u}}(\omega) + e^{-2\pi\mathrm{i}\tau_1\omega}\Gamma\mathbf{u}(\tau_1)\| \leq nc\int_{\tau_1-1/n}^{\tau_1} \Big\|e^{-2\pi\mathrm{i}\tau\omega}\Gamma\mathbf{u}(\tau) - e^{-2\pi\mathrm{i}\tau_1\omega}\Gamma\mathbf{u}(\tau_1)\Big\|\mathrm{d}\tau \tag{3.21}
$$

for any $n \geq 1$ and $\omega \in \mathbb{R}$. Since $\Gamma\mathbf{u}$ is continuous on $\mathbb{R}_+$, from (3.21) we infer that $\widehat{\varphi_n'\Gamma\mathbf{u}}(\omega) \to -e^{-2\pi\mathrm{i}\tau_1\omega}\Gamma\mathbf{u}(\tau_1)$ as $n \to \infty$ for any $\omega \in \mathbb{R}$. Since $\varphi_n \to \chi_{[0,\tau_1)}$ pointwise as $n \to \infty$ and $0 \leq \varphi_n \leq 1$, for any $n \geq 1$, from the Lebesgue Dominated Convergence Theorem we obtain that $\varphi_n\mathbf{u} \to \chi_{[0,\tau_1]}\mathbf{u}$ and $\varphi_n f \to \chi_{[0,\tau_1]}f$ in $L^2(\mathbb{R}_+, \mathbb{X}) \hookrightarrow L^2(\mathbb{R}, \mathbb{X})$ as $n \to \infty$. Passing to the limit in (3.19) with $\varphi = \varphi_n$ we infer that

$$
\widehat{\chi_{[0,\tau_1]}\mathbf{u}}(\omega) + e^{-2\pi\mathrm{i}\tau_1\omega}R_{\Gamma,E}(\omega)\Gamma\mathbf{u}(\tau_1) = R_{\Gamma,E}(\omega)\Gamma\mathbf{u}(0) + R_{\Gamma,E}(\omega)\widehat{\chi_{[0,\tau_1]}f}(\omega) \tag{3.22}
$$

for any $\omega \in \mathbb{R}$, which implies that (3.9) holds, proving the lemma. □

To better understand the solutions of equation (3.5) we need to further study the Fourier multiplier $\mathscr{K}_{\Gamma,E}$: in particular we are interested in finding suitable subspaces of $L^2(\mathbb{R}, \mathbb{X})$ that are invariant under $\mathscr{K}_{\Gamma,E}$. We note that the operator-valued function $R_{\Gamma,E}$ is differentiable, and from Lemma 2.2(ii) we have that

$$
\sup_{\omega\in\mathbb{R}} \|R_{\Gamma,E}(\omega)\| < \infty, \ \sup_{\omega\in\mathbb{R}} |\omega|\|R_{\Gamma,E}'(\omega)\| < \infty. \tag{3.23}
$$

From the Mikhlin-Hormander multiplier theorem we conclude that the Fourier multiplier $\mathscr{K}_{\Gamma,E}$ is well-defined and bounded on $L^p(\mathbb{R}, X)$ for any $p \in (1,\infty)$. In the case of first-order differential equations on finite dimensional spaces one proves the existence of the stable manifold by using a fixed point argument on $L^\infty(\mathbb{R}_+, \mathbb{X})$ or $C_0(\mathbb{R}_+, \mathbb{X})$. In the example below, we prove that the Fourier multiplier $\mathscr{K}_{\Gamma,E}$ is not a bounded, linear operator on $L^\infty(\mathbb{R}_+, \mathbb{X})$. Therefore, to prove the existence result of a stable manifold of solutions of equation (3.1), we need to find a proper subspace of $L^\infty(\mathbb{R}_+, \mathbb{X})$ invariant under $\mathscr{K}_{\Gamma,E}$.

Example 3.4. Let $\mathbb{X} = \ell^2$, $\Gamma : \ell^2 \to \ell^2$ defined by $\Gamma\mathbf{x} = (-\frac{1}{e^n}\mathbf{x}_n)_{n\geq 1}$ for any $\mathbf{x} = (\mathbf{x}_n)_{n\geq 1} \in \ell^2$, $E = -Id_{\ell^2}$. Then, one can readily check that the pair (Γ, E) satisfies Hypothesis (S). Moreover, the Fourier multiplier $\mathscr{K}_{\Gamma,E} = \mathscr{F}^{-1}M_{R_{\Gamma,E}}\mathscr{F}$ does not map $L^2(\mathbb{R}_+, \ell^2) \cap L^\infty(\mathbb{R}_+, \ell^2)$ into $L^\infty(\mathbb{R}_+, \ell^2)$.

Indeed, we have that

$$R_{\Gamma,E}(\omega)\mathbf{x} = (\frac{e^n}{e^n - 2\pi\mathrm{i}\omega}\mathbf{x}_n)_{n\geq 1} = (\widehat{F}_n(\omega)\mathbf{x}_n)_{n\geq 1} \tag{3.24}$$

for any $\omega \in \mathbb{R}$ and $\mathbf{x} = (\mathbf{x}_n)_{n\geq 1} \in \ell^2$. Here, the sequence of functions $F_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$F_n(\tau) = \begin{cases} 0 & \text{if } \tau \geq 0 \\ e^n e^{e^n\tau} & \text{if } \tau < 0 \end{cases} . \tag{3.25}$$

It follows that the following representation holds true:

$$(\mathscr{K}_{\Gamma,E}f)(\tau) = \Big((F_n * f_n)(\tau)\Big)_{n\geq 1} \text{ for any } f = (f_n)_{n\geq 1} \in L^2(\mathbb{R}_+, \ell^2) \cap L^\infty(\mathbb{R}_+, \ell^2) \tag{3.26}$$

and $\tau \in \mathbb{R}$. Let $g : \mathbb{R}_+ \to \ell^2$ be defined by $g(\tau) = (g_n(\tau))_{n\geq 1}$, where $g_n = \chi_{[e^{-(n+1)},e^{-n})}$, $n \geq 1$. Here we recall that χ_J denotes the characteristic function of the set $J \subseteq \mathbb{R}$. We compute

$$\|g(\tau)\|^2_{\ell^2} = \sum_{n=1}^{\infty} \chi^2_{[e^{-(n+1)},e^{-n})}(\tau) = \sum_{n=1}^{\infty} \chi_{[e^{-(n+1)},e^{-n})}(\tau) = \chi_{(0,e^{-1})}(\tau) \tag{3.27}$$

for any $\tau \geq 0$. We conclude that $g \in L^2(\mathbb{R}_+, \ell^2) \cap L^\infty(\mathbb{R}_+, \ell^2)$. Moreover, from (3.25), we obtain that

$$(F_n * g_n)(\tau) = \int_{\mathbb{R}} F_n(\tau - s)g_n(s)\mathrm{d}s = \int_{e^{-(n+1)}}^{e^{-n}} e^n e^{e^n(\tau-s)}\mathrm{d}s = e^{e^n\tau}(e^{-1} - e^{-e}) \tag{3.28}$$

for any $\tau \in [0, e^{-(n+1)})$ and $n \geq 1$. Therefore, for any $m \in \mathbb{N}$ and any $\tau \in [0, e^{-(m+1)}]$ we have that

$$\|(\mathscr{K}_{\Gamma,E}g)(\tau)\|^2_{\ell^2} = \sum_{n=1}^{\infty} |(F_n * g_n)(\tau)|^2 \geq \sum_{n=1}^{m} e^{2e^n\tau}(e^{-1} - e^{-e})^2 \geq m(e^{-1} - e^{-e})^2. \tag{3.29}$$

Assume for a contradiction that $\mathscr{K}_{\Gamma,E}g \in L^\infty(\mathbb{R}_+, \ell^2)$. From (3.29) we infer that $\|\mathscr{K}_{\Gamma,E}g\|_\infty \geq \sqrt{m}(e^{-1} - e^{-e})$ for any $m \in \mathbb{N}$, which is a contradiction.

Next, we study if the Sobolev space $H^1(\mathbb{R}_+, \mathbb{X})$ is invariant under $\mathscr{K}_{\Gamma,E}$. First, we note that $g \in H^1(\mathbb{R}_+, \mathbb{X})$ if and only if $g \in L^2(\mathbb{R}_+, \mathbb{X})$ and the function $\omega \to 2\pi\mathrm{i}\omega\widehat{g}(\omega) - g(0)$ belongs to $L^2(\mathbb{R}, \mathbb{X})$. Here, we recall that if a function g is defined on a proper subset of $\mathbb{R}$, we use the same notation to denote its extension by 0 to the whole line. Using Lemma 2.2(ii) we can show that the space $H^1(\mathbb{R}, \mathbb{X})$ is invariant under $\mathscr{K}_{\Gamma,E}$. However, by using the same argument, we can check that $H^1(\mathbb{R}_+, \mathbb{X})$ is not invariant under $\mathscr{K}_{\Gamma,E}$ since $R_{\Gamma,E}(\cdot)\mathbf{x} \notin L^2(\mathbb{R}, \mathbb{X})$ for any $\mathbf{x} \in \mathbb{X} \setminus \mathrm{dom}(|S_{\Gamma,E}|^{1/2})$. Our goal is to prove the existence of an H^1 stable manifold by using a fixed point argument on equation (3.14) for $f = D(\mathbf{u}, \mathbf{u})$. Since $H^1(\mathbb{R}_+, \mathbb{X})$ is not invariant under $\mathscr{K}_{\Gamma,E}$, we need to rearrange the equation first by adding a correction term to $\mathscr{K}_{\Gamma,E}$. We parameterize equation (3.14) as follows: we look for solutions $\mathbf{u}$ satisfying

$\mathbf{u}(0) = \mathbf{v}_0 - E^{-1}f(0)$ for some $\mathbf{v}_0$ to be chosen later. Therefore, equation (3.14) is equivalent to

$$\mathbf{u} = T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}\mathbf{v}_0 + (\mathscr{K}_{\Gamma,E}f)_{|\mathbb{R}_+} - T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}E^{-1}f(0). \tag{3.30}$$

For any $f \in H^1(\mathbb{R}_+, \mathbb{X})$ we define the function

$$\mathscr{K}_{\Gamma,E}^{\rm mod}f := (\mathscr{K}_{\Gamma,E}f)_{|\mathbb{R}_+} - T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}E^{-1}f(0).$$

Clearly, $\mathscr{K}_{\Gamma,E}^{\rm mod}$ is a linear operator from $H^1(\mathbb{R}_+, \mathbb{X})$ to $L^2(\mathbb{R}_+, \mathbb{X})$. In what follows we prove that $\mathscr{K}_{\Gamma,E}^{\rm mod}f \in H^1(\mathbb{R}_+, \mathbb{X})$ for any $f \in H^1(\mathbb{R}_+, \mathbb{X})$ and compute its derivative.

Lemma 3.5. *Assume Hypothesis (S). Then,* $\mathscr{K}_{\Gamma,E}^{\rm mod}f \in H^1(\mathbb{R}_+, \mathbb{X})$ *for any* $f \in H^1(\mathbb{R}_+, \mathbb{X})$ *and* $(\mathscr{K}_{\Gamma,E}^{\rm mod}f)' = (\mathscr{K}_{\Gamma,E}f')_{|\mathbb{R}_+}$*. Moreover, there exists* $c(\Gamma, E) > 0$ *such that*

$$\|\mathscr{K}_{\Gamma,E}^{\rm mod}f\|_{H^1(\mathbb{R}_+,\mathbb{X})} \le c(\Gamma, E)\|f\|_{H^1(\mathbb{R}_+,\mathbb{X})}. \tag{3.31}$$

Proof. To prove our general result, we prove it for functions in a dense subset of $H^1(\mathbb{R}_+, \mathbb{X})$. We introduce the subspace $\mathscr{H}_\Gamma^1 = \{f : \mathbb{R}_+ \to \mathbb{X} : \text{there exists } g \in H^1(\mathbb{R}_+, \mathbb{X}) \text{ such that } f(\tau) = \Gamma g(\tau) \text{ for any } \tau \ge 0\}$. Since $\Gamma \in \mathscr{B}(\mathbb{X})$ is one-to-one and self-adjoint, one can readily check that $\mathscr{H}_\Gamma^1$ is a dense subspace of $H^1(\mathbb{R}_+, \mathbb{X})$. To prove the lemma we need to compute $\widehat{\mathscr{K}_{\Gamma,E}^{\rm mod}f}$.

Let $g \in H^1(\mathbb{R}_+, \mathbb{X})$ and $f = \Gamma g$. From the definition of the Fourier multiplier $\mathscr{K}_{\Gamma,E}$, from Remark 2(ii) we obtain that

$$\begin{aligned}\widehat{\mathscr{K}_{\Gamma,E}f}(\omega) &= R_{\Gamma,E}(\omega)\Gamma\widehat{g}(\omega) = (2\pi i\omega\Gamma - E)^{-1}\Gamma\widehat{g}(\omega) = R(2\pi i\omega, S_{\Gamma,E})\widehat{g}(\omega)\\ &= \widehat{\mathscr{G}_{\Gamma,E} * g}(\omega),\end{aligned} \tag{3.32}$$

for any $\omega \in \mathbb{R}$, which implies that $\mathscr{K}_{\Gamma,E}f = \mathscr{G}_{\Gamma,E} * g$. It follows that

$$\begin{aligned}(\mathscr{K}_{\Gamma,E}f)(\tau) &= \int_{-\infty}^{\tau} T_{\rm s}^{\Gamma,E}(\tau - s)P_{\rm s}^{\Gamma,E}g(s){\rm d}s - \int_{\tau}^{\infty} T_{\rm u}^{\Gamma,E}(s - \tau)P_{\rm u}^{\Gamma,E}g(s){\rm d}s\\ &= -T_{\rm u}^{\Gamma,E}(-\tau)\int_0^{\infty} T_{\rm u}^{\Gamma,E}(s)P_{\rm u}^{\Gamma,E}g(s){\rm d}s \quad \text{for any} \quad \tau < 0.\end{aligned} \tag{3.33}$$

We infer that $\chi_{\mathbb{R}_+}(\mathscr{K}_{\Gamma,E}f) = \mathscr{K}_{\Gamma,E}f + F_1$, where $F_1 : \mathbb{R} \to \mathbb{X}$ is the function defined by $F_1(\tau) = 0$ for $\tau \ge 0$ and $F_1(\tau) = T_{\rm u}^{\Gamma,E}(-\tau)\int_0^\infty T_{\rm u}^{\Gamma,E}(s)P_{\rm u}^{\Gamma,E}g(s){\rm d}s$ for $\tau < 0$. We recall that the generator of the C^0-semigroup $\{T_{\rm u}^{\Gamma,E}(\tau)\}_{\tau\ge0}$ is $S_{\rm u}^{\Gamma,E} = -(S_{\Gamma,E})_{|\mathbb{X}_{\rm u}^{\Gamma,E}}$. Therefore, we obtain that

$$\begin{aligned}\widehat{F_1}(\omega) &= \int_{-\infty}^0 e^{-2\pi i\omega\tau}T_{\rm u}^{\Gamma,E}(-\tau)\mathbf{x}_{\rm u}{\rm d}\tau = -\big(2\pi i\omega + S_{\rm u}^{\Gamma,E}\big)^{-1}\mathbf{x}_{\rm u} = -R(2\pi i\omega, S_{\Gamma,E})\mathbf{x}_{\rm u}\\ &= -R_{\Gamma,E}(\omega)\Gamma\mathbf{x}_{\rm u}\end{aligned} \tag{3.34}$$

for any $\omega \in \mathbb{R}$, where $\mathbf{x}_{\rm u} = \int_0^\infty T_{\rm u}^{\Gamma,E}(s)P_{\rm u}^{\Gamma,E}g(s){\rm d}s$. Next, we define the function $F_2 : \mathbb{R} \to \mathbb{X}$ by $F_2(\tau) = T_{\rm s}^{\Gamma,E}(\tau)\mathbf{x}_{\rm s}$ for $\tau \ge 0$ and $F_2(\tau) = 0$ for $\tau < 0$, where $\mathbf{x}_{\rm s} = P_{\rm s}^{\Gamma,E}E^{-1}f(0)$. Similarly, since the generator of the C^0-semigroup $\{T_{\rm s}^{\Gamma,E}(\tau)\}_{\tau\ge0}$ is $S_{\rm s}^{\Gamma,E} = (S_{\Gamma,E})_{|\mathbb{X}_{\rm s}^{\Gamma,E}}$, we have that

$$\begin{aligned}\widehat{F_2}(\omega) &= \int_0^\infty e^{-2\pi i\omega\tau}T_{\rm s}^{\Gamma,E}(\tau)\mathbf{x}_{\rm s}{\rm d}\tau = \big(2\pi i\omega - S_{\rm s}^{\Gamma,E}\big)^{-1}P_{\rm s}^{\Gamma,E}E^{-1}f(0)\\ &= R_{\Gamma,E}(\omega)\Gamma P_{\rm s}^{\Gamma,E}S_{\Gamma,E}^{-1}g(0) = R_{\Gamma,E}(\omega)\Gamma S_{\Gamma,E}^{-1}P_{\rm s}^{\Gamma,E}g(0)\\ &= R_{\Gamma,E}(\omega)\Gamma E^{-1}\Gamma P_{\rm s}^{\Gamma,E}g(0)\end{aligned} \tag{3.35}$$

for any $\omega\in\mathbb{R}$. From (3.33), (3.34) and (3.35) we conclude that

$$\begin{aligned}\widehat{\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f}(\omega)&=\chi_{\mathbb{R}_+}\widehat{(\mathscr{K}_{\Gamma,E}f)}(\omega)-\widehat{F_2}(\omega)=\widehat{\mathscr{K}_{\Gamma,E}f}(\omega)+\widehat{F_1}(\omega)-\widehat{F_2}(\omega)\\&=R_{\Gamma,E}(\omega)\widehat{f}(\omega)-R_{\Gamma,E}(\omega)\Gamma\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s\\&\quad-R_{\Gamma,E}(\omega)\Gamma E^{-1}\Gamma P_{\mathrm{s}}^{\Gamma,E}g(0)\end{aligned}\tag{3.36}$$

for any $\omega\in\mathbb{R}$. In addition, from (3.33) we have that

$$(\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f)(0)=-\mathbf{x}_{\mathrm{u}}-P_{\mathrm{s}}^{\Gamma,E}S^{-1}_{\Gamma,E}g(0)=-\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s-S^{-1}_{\Gamma,E}P_{\mathrm{s}}^{\Gamma,E}g(0).\tag{3.37}$$

Since $g\in H^1(\mathbb{R}_+,\mathbb{X})$ we have that $2\pi\mathrm{i}\omega\widehat{g}(\omega)=g(0)+\widehat{g'}(\omega)$ for any $\omega\in\mathbb{R}$. From (3.36) and (3.37) it follows that

$$\begin{aligned}&2\pi\mathrm{i}\omega(\widehat{\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f})(\omega)-(\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f)(0)\\&=2\pi\mathrm{i}\omega R_{\Gamma,E}(\omega)\widehat{f}(\omega)-2\pi\mathrm{i}\omega R_{\Gamma,E}(\omega)\Gamma\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s\\&\qquad-2\pi\mathrm{i}\omega R_{\Gamma,E}(\omega)\Gamma E^{-1}\Gamma P_{\mathrm{s}}^{\Gamma,E}g(0)+\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s+S^{-1}_{\Gamma,E}P_{\mathrm{s}}^{\Gamma,E}g(0)\\&=R_{\Gamma,E}(\omega)\Gamma\big(2\pi\mathrm{i}\omega\widehat{g}(\omega)\big)+\Big(I_{\mathbb{X}}-2\pi\mathrm{i}\omega R_{\Gamma,E}(\omega)\Gamma\Big)\Big(\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s\\&\qquad\qquad+E^{-1}\Gamma P_{\mathrm{s}}^{\Gamma,E}g(0)\Big)\\&=2\pi\mathrm{i}\omega R_{\Gamma,E}(\omega)\Gamma\widehat{g'}(\omega)+R_{\Gamma,E}(\omega)\Gamma g(0)-R_{\Gamma,E}(\omega)E\Big(\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s\\&\qquad\qquad+E^{-1}\Gamma P_{\mathrm{s}}^{\Gamma,E}g(0)\Big)\\&=R_{\Gamma,E}(\omega)\widehat{f'}(\omega)+R_{\Gamma,E}(\omega)\Big(\Gamma P_{\mathrm{u}}^{\Gamma,E}g(0)-E\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s\Big)\end{aligned}\tag{3.38}$$

for any $\omega\in\mathbb{R}$. Since $g\in H^1(\mathbb{R}_+,\mathbb{X})$ we obtain that $\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s\in\mathrm{dom}(S_{\mathrm{u}}^{\Gamma,E})\subseteq\mathrm{dom}(S_{\Gamma,E})$ and

$$S_{\mathrm{u}}^{\Gamma,E}\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s=-P_{\mathrm{u}}^{\Gamma,E}g(0)-\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g'(s)\mathrm{d}s.\tag{3.39}$$

Since $S_{\mathrm{u}}^{\Gamma,E}=-(S_{\Gamma,E})_{|\mathbb{X}_{\mathrm{u}}^{\Gamma,E}}$, from (3.39) we obtain that

$$E\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g(s)\mathrm{d}s=\Gamma P_{\mathrm{u}}^{\Gamma,E}g(0)+\Gamma\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g'(s)\mathrm{d}s.\tag{3.40}$$

Using Remark 2(ii), (3.38) and (3.40) we infer that

$$\begin{aligned}&2\pi\mathrm{i}\omega(\widehat{\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f})(\omega)-(\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f)(0)\\&=R_{\Gamma,E}(\omega)\widehat{f'}(\omega)-R_{\Gamma,E}(\omega)\Gamma\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g'(s)\mathrm{d}s\\&=\mathscr{F}\Big(\mathscr{K}_{\Gamma,E}f'-\mathscr{G}_{\Gamma,E}(\cdot)\int_0^\infty T_{\mathrm{u}}^{\Gamma,E}(s)P_{\mathrm{u}}^{\Gamma,E}g'(s)\mathrm{d}s\Big)\end{aligned}\tag{3.41}$$

for any $\omega \in \mathbb{R}$. Arguing similarly as in (3.32), we have that $\mathscr{K}_{\Gamma,E}f' = \mathscr{G}_{\Gamma,E} * g'$, which implies that

$$\begin{aligned}(\mathscr{K}_{\Gamma,E}f')(\tau) &= \int_{-\infty}^{\tau} T_{\rm s}^{\Gamma,E}(\tau-s)P_{\rm s}^{\Gamma,E}g'(s){\rm d}s - \int_{\tau}^{\infty} T_{\rm u}^{\Gamma,E}(s-\tau)P_{\rm u}^{\Gamma,E}g'(s){\rm d}s \\ &= -T_{\rm u}^{\Gamma,E}(-\tau)\int_0^{\infty} T_{\rm u}^{\Gamma,E}(s)P_{\rm u}^{\Gamma,E}g'(s){\rm d}s \\ &= \mathscr{G}_{\Gamma,E}(\tau)\int_0^{\infty} T_{\rm u}^{\Gamma,E}(s)P_{\rm u}^{\Gamma,E}g'(s){\rm d}s \end{aligned} \tag{3.42}$$

for any $\tau < 0$. Since $\mathscr{G}_{\Gamma,E}(\tau)\mathbf{x} = 0$ for any $\tau \geq 0$ and any $\mathbf{x} \in \mathbb{X}_{\rm u}^{\Gamma,E}$, from (3.41) and (3.42) we conclude that

$$2\pi{\rm i}\omega(\widehat{\mathscr{K}_{\Gamma,E}^{\rm mod}f})(\omega) - (\mathscr{K}_{\Gamma,E}^{\rm mod}f)(0) = \mathscr{F}(\chi_{\mathbb{R}_+}\mathscr{K}_{\Gamma,E}f')(\omega) \quad \text{for any} \quad \omega \in \mathbb{R}. \tag{3.43}$$

Since $f' \in L^2(\mathbb{R}_+,\mathbb{X}) \hookrightarrow L^2(\mathbb{R},\mathbb{X})$ and $\mathscr{K}_{\Gamma,E}$ is bounded on $L^2(\mathbb{R},\mathbb{X})$ from (3.43) we infer that $\mathscr{K}_{\Gamma,E}^{\rm mod}f \in H^1(\mathbb{R}_+,\mathbb{X})$ and $(\mathscr{K}_{\Gamma,E}^{\rm mod}f)' = (\mathscr{K}_{\Gamma,E}f')_{|\mathbb{R}_+}$ for any $f \in \mathscr{H}_\Gamma^1$.

Next, we fix $f \in H^1(\mathbb{R}_+,\mathbb{X})$ and let $\{f_n\}_{n\geq 1}$ be a sequence of functions in $\mathscr{H}_\Gamma^1$ such that $f_n \to f$ as $n \to \infty$ in $H^1(\mathbb{R}_+,\mathbb{X})$. From Remark 2(i) we obtain that

$$\|T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}E^{-1}f_n(0) - T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}E^{-1}f(0)\|_2 \leq c\|f_n(0)-f(0)\| \leq c\|f_n - f\|_{H^1} \tag{3.44}$$

for any $n \geq 1$. Since the Fourier multiplier $\mathscr{K}_{\Gamma,E}$ is bounded on $L^2(\mathbb{R},\mathbb{X})$, from (3.44) we conclude that

$$(\mathscr{K}_{\Gamma,E}f_n)_{|\mathbb{R}_+} - T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}E^{-1}f_n(0) \to (\mathscr{K}_{\Gamma,E}f)_{|\mathbb{R}_+} - T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}E^{-1}f(0)$$

as $n \to \infty$ in $L^2(\mathbb{R}_+,\mathbb{X})$. Hence $\mathscr{K}_{\Gamma,E}^{\rm mod}f_n \to \mathscr{K}_{\Gamma,E}^{\rm mod}f$ as $n \to \infty$ in $L^2(\mathbb{R}_+,\mathbb{X})$. Moreover, $(\mathscr{K}_{\Gamma,E}^{\rm mod}f_n)' = (\mathscr{K}_{\Gamma,E}f_n')_{|\mathbb{R}_+} \to (\mathscr{K}_{\Gamma,E}f')_{|\mathbb{R}_+}$ as $n \to \infty$ in $L^2(\mathbb{R}_+,\mathbb{X})$. It follows that $\mathscr{K}_{\Gamma,E}^{\rm mod}f \in H^1(\mathbb{R}_+,\mathbb{X})$ and $(\mathscr{K}_{\Gamma,E}^{\rm mod}f)' = (\mathscr{K}_{\Gamma,E}f')_{|\mathbb{R}_+}$.

To finish the proof of lemma, we need to prove (3.31). Indeed, since $\mathscr{K}_{\Gamma,E}$ is bounded on $L^2(\mathbb{R},\mathbb{X})$ we have that

$$\begin{aligned}\|\mathscr{K}_{\Gamma,E}^{\rm mod}f\|_{H^1}^2 &= \|(\mathscr{K}_{\Gamma,E}f)_{|\mathbb{R}_+} - T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}E^{-1}f(0)\|_2^2 + \|(\mathscr{K}_{\Gamma,E}f')_{|\mathbb{R}_+}\|_2^2 \\ &\leq 4\|\mathscr{K}_{\Gamma,E}f\|_2^2 + 4\|T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}E^{-1}f(0)\|_2^2 + c\|f'\|_2^2 \\ &\leq c\|f\|_2^2 + c\|f'\|_2^2 + c\|f(0)\|^2 \leq c\|f\|_{H^1}^2. \end{aligned} \tag{3.45}$$

for any $f \in H^1(\mathbb{R}_+,\mathbb{X})$, proving the lemma. □

Remark 3. Assume $\mathscr{W} : \mathbb{R} \to \mathscr{B}(\mathbb{X})$ is a piecewise strongly continuous operator valued function such that $\|\mathscr{W}(\tau)\| \leq ce^{-\nu|\tau|}$ for all $\tau \in \mathbb{R}$. Then, for any $\alpha \in [0,\nu)$ we have that $\mathscr{W} * f \in L_\alpha^2(\mathbb{R},\mathbb{X})$ and

$$\|\mathscr{W} * f\|_{L_\alpha^2} \leq c\|f\|_{L_\alpha^2} \quad \text{for any} \quad f \in L_\alpha^2(\mathbb{R},\mathbb{X}). \tag{3.46}$$

Proof. Fix $\alpha \in [0,\nu)$. Since $\mathscr{W}$ decays exponentially, one can readily check that

$$\begin{aligned}\|(\mathscr{W} * f)(\tau)\|^2 &\leq \Big(\int_{\mathbb{R}} e^{-\frac{\nu-\alpha}{2}|\tau-s|}e^{-\frac{\nu+\alpha}{2}|\tau-s|}\,\|f(s)\|{\rm d}s\Big)^2 \\ &\leq \frac{1}{\nu-\alpha}\int_{\mathbb{R}} e^{-(\nu+\alpha)|\tau-s|}\,\|f(s)\|^2{\rm d}s \end{aligned} \tag{3.47}$$

for any $\tau\in\mathbb{R}$. Since $\int_{\mathbb{R}} e^{-(\nu+\alpha)|\tau-s|}e^{2\alpha|\tau|}\mathrm{d}\tau \le \frac{1}{\nu-\alpha}e^{2\alpha|s|}$ for any $s\in\mathbb{R}$, from (3.47) we obtain that

$$\int_{\mathbb{R}} e^{2\alpha|\tau|}\|(\mathscr{W}*f)(\tau)\|^2\mathrm{d}\tau \le \frac{1}{\nu-\alpha}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-(\nu+\alpha)|\tau-s|}e^{2\alpha|\tau|}\mathrm{d}\tau\Big)\|f(s)\|^2\mathrm{d}s$$
$$\le \frac{1}{(\nu-\alpha)^2}\int_{\mathbb{R}} e^{2\alpha|s|}\|f(s)\|^2\mathrm{d}s = \frac{1}{(\nu-\alpha)^2}\|f\|^2_{L^2_\alpha}, \tag{3.48}$$

proving the remark. □

Next, we analyze the invariance properties of weighted spaces under $\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}$. In particular, we are interested in checking whether the weighted Sobolev space $H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ is invariant under $\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}$. To prove this result we need the following lemma:

Lemma 3.6. *Assume Hypothesis (S) and let $\psi\in H^2(\mathbb{R})$ be a smooth scalar function. Then,*

$$\mathscr{K}_{\Gamma,E}\Big(\psi f+\psi'(\mathscr{G}^*_{\Gamma,E}*f)\Big)=\psi\mathscr{K}_{\Gamma,E}f \quad \textit{for any} \quad f\in L^2(\mathbb{R},\mathbb{X}). \tag{3.49}$$

Proof. To start the proof of lemma, we first justify that the left hand side of equation (3.49) is well defined. Indeed, since $\psi\in H^2(\mathbb{R})$ we have that $\psi,\psi'\in L^\infty(\mathbb{R})$. Moreover, from Remark 2(i) it follows that $\mathscr{G}_{\Gamma,E}$ and thus $\mathscr{G}^*_{\Gamma,E}$ are exponentially decaying, operator valued functions, which implies that $\mathscr{G}^*_{\Gamma,E}*f\in L^2(\mathbb{R},\mathbb{X})$ for any $f\in L^2(\mathbb{R},\mathbb{X})$. We conclude that $\psi f+\psi'(\mathscr{G}^*_{\Gamma,E}*f)\in L^2(\mathbb{R},\mathbb{X})$ for any $f\in L^2(\mathbb{R},\mathbb{X})$.

From Remark 2(ii) we have that $\mathscr{F}\mathscr{G}_{\Gamma,E}(\cdot)\mathbf{x}=R(2\pi\mathrm{i}\cdot,S_{\Gamma,E})\mathbf{x}$ for any $\mathbf{x}\in\mathbb{X}$. From Lemma 2.2 we obtain that

$$\mathscr{F}\mathscr{G}^*_{\Gamma,E}(\cdot)\mathbf{x}=\big(\mathscr{F}\mathscr{G}_{\Gamma,E}(-\cdot)\big)^*\mathbf{x}=R(-2\pi\mathrm{i}\cdot,S_{\Gamma,E})^*\mathbf{x}=\big(R_{\Gamma,E}(-\cdot)\Gamma\big)^*\mathbf{x}=\Gamma R_{\Gamma,E}(\cdot)\mathbf{x} \tag{3.50}$$

for any $\mathbf{x}\in L^2(\mathbb{R},\mathbb{X})$. Since $\psi\in H^2(\mathbb{R})$ we have that $\widehat{\psi},\widehat{\psi'}\in L^1(\mathbb{R})$. Taking Fourier transform, from (3.8) and (3.50) we obtain that

$$\begin{aligned}
\widehat{\psi\mathscr{K}_{\Gamma,E}f}(\omega)&=(\widehat{\psi}*\widehat{\mathscr{K}_{\Gamma,E}f})(\omega)=\int_{\mathbb{R}}\widehat{\psi}(\omega-\theta)\widehat{\mathscr{K}_{\Gamma,E}f}(\theta)\mathrm{d}\theta\\
&=\int_{\mathbb{R}}\widehat{\psi}(\omega-\theta)R_{\Gamma,E}(\theta)\widehat{f}(\theta)\mathrm{d}\theta\\
&=\int_{\mathbb{R}}\widehat{\psi}(\omega-\theta)\Big(R_{\Gamma,E}(\omega)+2\pi\mathrm{i}(\omega-\theta)R_{\Gamma,E}(\omega)\Gamma R_{\Gamma,E}(\theta)\Big)\widehat{f}(\theta)\mathrm{d}\theta\\
&=R_{\Gamma,E}(\omega)\int_{\mathbb{R}}2\pi\mathrm{i}(\omega-\theta)\widehat{\psi}(\omega-\theta)\Gamma R_{\Gamma,E}(\theta)\widehat{f}(\theta)\mathrm{d}\theta\\
&\qquad+R_{\Gamma,E}(\omega)\int_{\mathbb{R}}\widehat{\psi}(\omega-\theta)\widehat{f}(\theta)\mathrm{d}\theta\\
&=R_{\Gamma,E}(\omega)(\widehat{\psi}*\widehat{f})(\omega)+R_{\Gamma,E}(\omega)\int_{\mathbb{R}}\widehat{\psi'}(\omega-\theta)\Gamma R_{\Gamma,E}(\theta)\widehat{f}(\theta)\mathrm{d}\theta\\
&=R_{\Gamma,E}(\omega)\widehat{\psi f}(\omega)+R_{\Gamma,E}(\omega)\int_{\mathbb{R}}\widehat{\psi'}(\omega-\theta)\widehat{\mathscr{G}_{\Gamma,E}}^*(\theta)\widehat{f}(\theta)\mathrm{d}\theta\\
&=R_{\Gamma,E}(\omega)\widehat{\psi f}(\omega)+R_{\Gamma,E}(\omega)\int_{\mathbb{R}}\widehat{\psi'}(\omega-\theta)\widehat{\mathscr{G}^*_{\Gamma,E}*f}(\theta)\mathrm{d}\theta\\
&=R_{\Gamma,E}(\omega)\Big(\widehat{\psi f}(\omega)+\widehat{\psi'(\mathscr{G}^*_{\Gamma,E}*f)}(\omega)\Big)\quad\text{for any}\quad\omega\in\mathbb{R}.
\end{aligned}\tag{3.51}$$

Assertion (3.49) follows readily from the definition of $\mathscr{K}_{\Gamma,E}$. □

Lemma 3.7. *Assume Hypothesis (S). Then,* $\mathscr{K}^{\rm mod}_{\Gamma,E}f \in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ *for any* $f \in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ *and* $\alpha \in (0,\nu(\Gamma,E))$*, where* $\nu(\Gamma,E)$ *is defined in* (3.6)*. Moreover, there exists* $c(\Gamma,E,\alpha)>0$ *such that*

$$\|\mathscr{K}^{\rm mod}_{\Gamma,E}f\|_{H^1_\alpha(\mathbb{R}_+,\mathbb{X})} \le c(\Gamma,E,\alpha)\|f\|_{H^1_\alpha(\mathbb{R}_+,\mathbb{X})}. \tag{3.52}$$

Proof. Let $\{\phi_n\}_{n\ge1}$ be a sequence of scalar functions in $C^\infty(\mathbb{R})$ such that $0\le \phi_n \le 1$, $\phi_n(\tau)=1$ whenever $|\tau|\le n$, $\phi_n(\tau)=0$ whenever $|\tau|\ge n+1$ and $\sup_{n\ge1}\|\phi_n'\|_\infty<\infty$. We define the sequence of scalar functions $\{\psi_n\}$ by the formula $\psi_n(\tau)=e^{\alpha\langle\tau\rangle}\phi_n(\tau)$, where $\langle\tau\rangle=\sqrt{1+|\tau|^2}$. Since $\alpha>0$ one can readily check that $\psi_n\in H^2(\mathbb{R})$ for any $n\ge1$. Moreover, there exists a constant $c_0>0$, independent of Γ, E and α such that

$$|\psi_n(\tau)|\le e^{\alpha\tau},\quad |\psi_n'(\tau)|\le c_0e^{\alpha\tau},\quad \text{for any}\quad \tau\in\mathbb{R}. \tag{3.53}$$

Let $f\in H^1_\alpha(\mathbb{R}_+,\mathbb{X})\hookrightarrow H^1(\mathbb{R}_+,\mathbb{X})$. First, we prove that $\mathscr{K}^{\rm mod}_{\Gamma,E}f\in L^2_\alpha(\mathbb{R}_+,\mathbb{X})$. We note that $(\psi_n)_{|\mathbb{R}_+}\mathscr{K}^{\rm mod}_{\Gamma,E}f=(\psi_n\mathscr{K}_{\Gamma,E}f)_{|\mathbb{R}_+}-(\psi_n)_{|\mathbb{R}_+}T^{\Gamma,E}_{\rm s}(\cdot)P^{\Gamma,E}_{\rm s}E^{-1}f(0)$ for any $n\ge1$. Using Lemma 3.6, Remark 3 and (3.53) we estimate

$$\begin{aligned}\|(\psi_n\mathscr{K}_{\Gamma,E}f)_{|\mathbb{R}_+}\|_2 &\le \|\psi_n\mathscr{K}_{\Gamma,E}f\|_2=\|\mathscr{K}_{\Gamma,E}(\psi_nf+\psi_n'(\mathscr{G}^*_{\Gamma,E}*f))\|_2\\ &\le c\|\psi_nf+\psi_n'(\mathscr{G}^*_{\Gamma,E}*f)\|_2\\ &\le c(\Gamma,E)\|f\|_{L^2_\alpha}+c(\Gamma,E)\|\mathscr{G}^*_{\Gamma,E}*f\|_{L^2_\alpha}\\ &\le c(\Gamma,E,\alpha)\|f\|_{L^2_\alpha}\end{aligned} \tag{3.54}$$

for any $n\ge1$. Moreover, since $\alpha\in(0,\nu(\Gamma,E))$ one can readily check that

$$\|(\psi_n)_{|\mathbb{R}_+}T^{\Gamma,E}_{\rm s}(\cdot)P^{\Gamma,E}_{\rm s}E^{-1}f(0)\|_2\le c(\Gamma,E,\alpha)\|f(0)\|\le c(\Gamma,E,\alpha)\|f\|_{H^1_\alpha} \tag{3.55}$$

for any $n\ge1$. From (3.54) and (3.55) we conclude that $(\psi_n)_{|\mathbb{R}_+}\mathscr{K}^{\rm mod}_{\Gamma,E}f\in L^2(\mathbb{R}_+,\mathbb{X})$ and $\|(\psi_n)_{|\mathbb{R}_+}\mathscr{K}^{\rm mod}_{\Gamma,E}f\|_2\le c(\Gamma,E,\alpha)\|f\|_{H^1_\alpha}$ for any $n\ge1$. Passing to the limit as $n\to\infty$ we conclude that $\|\mathscr{K}^{\rm mod}_{\Gamma,E}f\|_{L^2_\alpha}\le c(\Gamma,E,\alpha)\|f\|_{H^1_\alpha}$.

From Lemma 3.5 we have that

$$\mathscr{K}^{\rm mod}_{\Gamma,E}f\in H^1(\mathbb{R}_+,\mathbb{X})\quad\text{and}\quad (\mathscr{K}^{\rm mod}_{\Gamma,E}f)'=(\mathscr{K}_{\Gamma,E}f')_{|\mathbb{R}_+}.$$

Using again Lemma 3.6, Remark 3 and (3.53) we have that

$$\begin{aligned}\|(\psi_n\mathscr{K}_{\Gamma,E}f')_{|\mathbb{R}_+}\|_2 &\le \|\psi_n\mathscr{K}_{\Gamma,E}f'\|_2=\|\mathscr{K}_{\Gamma,E}(\psi_nf'+\psi_n'(\mathscr{G}^*_{\Gamma,E}*f'))\|_2\\ &\le c\|\psi_nf'+\psi_n'(\mathscr{G}^*_{\Gamma,E}*f')\|_2\\ &\le c(\Gamma,E)\|f'\|_{L^2_\alpha}+c(\Gamma,E)\|\mathscr{G}^*_{\Gamma,E}*f'\|_{L^2_\alpha}\\ &\le c(\Gamma,E,\alpha)\|f'\|_{L^2_\alpha}\le c(\Gamma,E,\alpha)\|f\|_{H^1_\alpha}\end{aligned} \tag{3.56}$$

for any $n\ge1$. It follows that $\|(\psi_n)_{|\mathbb{R}_+}(\mathscr{K}^{\rm mod}_{\Gamma,E}f)'\|_2\le c(\Gamma,E,\alpha)\|f\|_{H^1_\alpha}$ for any $n\ge1$. Passing to the limit as $n\to\infty$ we infer that $(\mathscr{K}^{\rm mod}_{\Gamma,E}f)'\in L^2_\alpha(\mathbb{R}_+,\mathbb{X})$ and $\|(\mathscr{K}^{\rm mod}_{\Gamma,E}f)'\|_{L^2_\alpha}\le c(\Gamma,E,\alpha)\|f\|_{H^1_\alpha}$, proving the lemma. □

4. **Stable manifolds of general steady relaxation systems.** In this section we prove the existence of a local stable manifold of equation (3.1), by reducing the equation to (3.4), as explained in Section 3. To prove this result, we prove it for the general case of equation (3.5). First, we recall that from Lemma 3.3 we have that mild solutions of equation (3.5) on $\mathbb{R}_+$ satisfy the equation

$$\mathbf{u}(\tau)=T^{\Gamma,E}_{\rm s}(\tau)P^{\Gamma,E}_{\rm s}\mathbf{u}(0)+\big(\mathscr{K}_{\Gamma,E}D(\mathbf{u}(\cdot),\mathbf{u}(\cdot))\big)(\tau),\ \tau\ge0. \tag{4.1}$$

Using the parametrization $\mathbf{u}(0) = \mathbf{v}_0 - E^{-1}D(\mathbf{u}(0), \mathbf{u}(0))$, as in equation (3.30), to prove our result it is enough to prove the existence of a stable manifold of equation

$$\mathbf{u} = T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}\mathbf{v}_0 + \mathscr{K}_{\Gamma,E}^{\rm mod}D(\mathbf{u}(\cdot), \mathbf{u}(\cdot)). \tag{4.2}$$

In this section we prove that for $\mathbf{v}_0$ in a certain subspace of $\mathbb{X}_{\rm s}^{\Gamma,E}$ with $\|\mathbf{v}_0\|$ small enough there exists a unique solution of equation of (4.2), in the space $H^1_\alpha(\mathbb{R}_+, \mathbb{X})$, with $\alpha \in [0, \nu(\Gamma, E))$. Throughout this section we assume Hypothesis (S). Next, we study the trajectories of the semigroup $\{T_{\rm s}^{\Gamma,E}(\tau)\}_{\tau\geq 0}$ that belong to the space $H^1_\alpha(\mathbb{R}_+, \mathbb{X})$. First, we introduce the subspace

$$\mathbb{X}_{\frac{1}{2}}^{\Gamma,E} = \Big\{\mathbf{x} \in \mathbb{X} : \int_\Lambda |H_{\Gamma,E}(\lambda)|\,|(U_{\Gamma,E}\mathbf{x})(\lambda)|^2 \mathrm{d}\mu(\lambda) < \infty\Big\}. \tag{4.3}$$

The space $\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}$ is a Banach space when endowed with the norm

$$\|\mathbf{x}\|_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}} = \big\||M_{H_{\Gamma,E}}|^{1/2}U_{\Gamma,E}\mathbf{x}\big\|_{L^2(\Lambda,\mu)}, \tag{4.4}$$

where $U_{\Gamma,E}$ is defined in (2.18) and $M_{H_{\Gamma,E}}$ is the operator of multiplication by the function $H_{\Gamma,E}$ introduced in Lemma 2.3. From (2.18) it follows that $\mathbb{X}_{\frac{1}{2}}^{\Gamma,E} = \mathrm{dom}(|S_{\Gamma,E}|^{\frac{1}{2}})$ and that the norm $\|\cdot\|_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}}$ is equivalent to the graph norm on $\mathrm{dom}(|S_{\Gamma,E}|^{\frac{1}{2}})$.

Lemma 4.1. *Assume Hypothesis (S). Then $T_{\rm s}^{\Gamma,E}(\cdot)\mathbf{v}_0 \in H^1_\alpha(\mathbb{R}_+, \mathbb{X})$ for any $\alpha \in [0, \nu(\Gamma, E))$. Moreover, there exists $c = c(\Gamma, E) > 0$ such that*

$$\|T_{\rm s}^{\Gamma,E}(\cdot)\mathbf{v}_0\|_{H^1_\alpha(\mathbb{R}_+,\mathbb{X})} \leq c(\Gamma, E, \alpha)\|\mathbf{v}_0\|_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}} \tag{4.5}$$

for any $\mathbf{v}_0 \in \mathbb{X}_{\rm s}^{\Gamma,E} \cap \mathbb{X}_{\frac{1}{2}}^{\Gamma,E}$.

Proof. Fix $\mathbf{v}_0 \in \mathbb{X}_{\rm s}^{\Gamma,E} \cap \mathbb{X}_{\frac{1}{2}}^{\Gamma,E}$, $\alpha \in [0, \nu(\Gamma, E))$ and let $\widetilde{g}_0 = U_{\Gamma,E}\mathbf{v}_0 \in L^2(\Lambda, \mu)$. From (2.14) we have that $\widetilde{g}_0 \in L^2(\Lambda_-, \mu)$, that is $\widetilde{g}_0(\lambda) = 0$ for any $\lambda \in \Lambda_+$. Since $\mathbf{v}_0 \in \mathbb{X}_{\frac{1}{2}}^{\Gamma,E}$ we infer that

$$\int_\Lambda |H_{\Gamma,E}(\lambda)||\widetilde{g}_0(\lambda)|^2 \mathrm{d}\mu(\lambda) = \big\||M_{H_{\Gamma,E}}|^{1/2}\widetilde{g}_0\big\|^2_{L^2(\Lambda,\mu)} = \|\mathbf{v}_0\|^2_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}} < \infty. \tag{4.6}$$

From (2.15) it follows that $T_{\rm s}^{\Gamma,E}(\tau)\mathbf{v}_0 = U_{\Gamma,E}^{-1}\widetilde{T}_{\rm s}^{\Gamma,E}(\tau)\widetilde{g}_0$ for any $\tau \geq 0$. We introduce the function $\widetilde{h}_0 : \mathbb{R}_+ \to L^2(\Lambda, \mu)$ defined by $\big(\widetilde{h}_0(\tau)\big)(\lambda) := e^{\alpha\tau}\big(\widetilde{T}_{\rm s}^{\Gamma,E}(\tau)\widetilde{g}_0\big)(\lambda)$. From (2.16) we have that $\big(\widetilde{h}_0(\tau)\big)(\lambda) = e^{\tau(\alpha+H_{\Gamma,E}(\lambda))}\chi_{\Lambda_-}(\lambda)\widetilde{g}_0(\lambda)$ for any $\tau \geq 0$ and $\lambda \in \Lambda$. Since $H_{\Gamma,E}(\lambda) \leq -\nu$ for any $\lambda \in \Lambda_-$, from (4.6) we obtain that $\widetilde{h}_0 \in H^1(\mathbb{R}_+, L^2(\Lambda, \mu))$ and

$$\begin{aligned}
\|\widetilde{h}_0\|^2_{H^1} &\leq \int_0^\infty e^{2\tau\alpha}\|\widetilde{T}_{\rm s}^{\Gamma,E}(\tau)\widetilde{g}_0\|^2_{L^2(\Lambda,\mu)}\mathrm{d}\tau \\
&\qquad + \int_0^\infty \int_{\Lambda_-} |(\alpha + H_{\Gamma,E}(\lambda))e^{\tau(\alpha+H_{\Gamma,E}(\lambda))}\widetilde{g}_0(\lambda)|^2 \mathrm{d}\mu(\lambda)\mathrm{d}\tau \\
&\leq \int_0^\infty e^{2\tau(\alpha-\nu)}\mathrm{d}\tau\,\|\widetilde{g}_0\|^2_{L^2(\Lambda,\mu)} \\
&\qquad + \int_{\Lambda_-} |\alpha + H_{\Gamma,E}(\lambda)|^2|\widetilde{g}_0(\lambda)|^2\Big(\int_0^\infty e^{2\tau(\alpha+H_{\Gamma,E}(\lambda))}\mathrm{d}\tau\Big)\mathrm{d}\mu(\lambda)
\end{aligned}$$

$$\leq \frac{1}{2(\nu-\alpha)}\|U_{\Gamma,E}\|^2\|\mathbf{v}_0\|^2 + \frac{1}{2}\int_{\Lambda_-}|\alpha + H_{\Gamma,E}(\lambda)||\widetilde{g}_0(\lambda)|^2\mathrm{d}\mu(\lambda)$$
$$\leq \frac{1}{2(\nu-\alpha)}\|U_{\Gamma,E}\|^2\|\mathbf{v}_0\|^2 + \frac{1}{2}|\alpha|\|\widetilde{g}_0\|^2_{L^2(\Lambda,\mu)} + \frac{1}{2}\int_{\Lambda}|H_{\Gamma,E}(\lambda)||\widetilde{g}_0(\lambda)|^2\mathrm{d}\mu(\lambda)$$
$$\leq \frac{1}{2}\Big(|\alpha| + \frac{1}{(\nu-\alpha)}\Big)\|U_{\Gamma,E}\|^2\|\mathbf{v}_0\|^2 + \frac{1}{2}\|\mathbf{v}_0\|^2_{\mathbb{X}^{\Gamma,E}_{\frac{1}{2}}} \leq c(\Gamma,E,\alpha)\|\mathbf{v}_0\|^2_{\mathbb{X}^{\Gamma,E}_{\frac{1}{2}}}. \tag{4.7}$$

We conclude that $T^{\Gamma,E}_{\mathrm{s}}(\cdot)\mathbf{v}_0 \in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ and (4.5) holds true, proving the lemma. □

We introduce the function $\Psi_{\Gamma,E} : \mathbb{X}^{\Gamma,E}_{\mathrm{s}} \cap \mathbb{X}^{\Gamma,E}_{\frac{1}{2}} \times H^1_\alpha(\mathbb{R}_+,\mathbb{X}) \to H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ defined by

$$\Psi_{\Gamma,E}(\mathbf{v}_0, f) = T^{\Gamma,E}_{\mathrm{s}}(\cdot)P^{\Gamma,E}_{\mathrm{s}}\mathbf{v}_0 + \mathscr{K}^{\mathrm{mod}}_{\Gamma,E}D(f(\cdot), f(\cdot)). \tag{4.8}$$

To prove that equation (4.1) has a unique solution on $H^1_\alpha(\mathbb{R}_+,\mathbb{X})$, we show that the function $\Psi_{\Gamma,E}$ satisfies the conditions of the Contraction Mapping Theorem. In what follows we will use the following notation to denote the closed balls of $H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ and $\mathbb{X}^{\Gamma,E}_{\mathrm{s}} \cap \mathbb{X}^{\Gamma,E}_{\frac{1}{2}}$ centered at the origin

$$\Omega_\alpha(\varepsilon) = \big\{f \in H^1_\alpha(\mathbb{R}_+,\mathbb{X}) : \|f\|_{H^1_\alpha(\mathbb{R}_+,\mathbb{X})} \leq \varepsilon\big\},$$
$$\overline{B}_{\mathrm{s}}(0,\varepsilon) = \big\{\mathbf{v}_0 \in \mathbb{X}^{\Gamma,E}_{\mathrm{s}} \cap \mathbb{X}^{\Gamma,E}_{\frac{1}{2}} : \|\mathbf{v}_0\|_{\mathbb{X}^{\Gamma,E}_{\frac{1}{2}}} \leq \varepsilon\big\}. \tag{4.9}$$

Lemma 4.2. *Assume Hypothesis (S). If $\widetilde{\nu} \in (0, \nu(\Gamma,E))$, where $\nu(\Gamma,E)$ is defined in (3.6), then*

(i) There exist $\varepsilon_1 = \varepsilon_1(\Gamma,E,\widetilde{\nu}) > 0$ and $\varepsilon_2 = \varepsilon_2(\Gamma,E,\widetilde{\nu}) > 0$ such that $\Psi_{\Gamma,E}$ maps $\overline{B}_{\mathrm{s}}(0,\varepsilon_1) \times \Omega_\alpha(\varepsilon_2)$ to $\Omega_\alpha(\varepsilon_2)$ and

$$\|\Psi_{\Gamma,E}(\mathbf{v}_0,f) - \Psi_{\Gamma,E}(\mathbf{v}_0,g)\|_{H^1_\alpha(\mathbb{R}_+,\mathbb{X})} \leq \frac{1}{2}\|f-g\|_{H^1_\alpha(\mathbb{R}_+,\mathbb{X})} \tag{4.10}$$

for any $\mathbf{v}_0 \in \overline{B}_{\mathrm{s}}(0,\varepsilon_1)$, $f,g \in \Omega_\alpha(\varepsilon_2)$ and $\alpha \in [0,\widetilde{\nu}]$;

(ii) For any $\mathbf{v}_0 \in \overline{B}_{\mathrm{s}}(0,\varepsilon_1)$ equation $\mathbf{u} = \Psi_{\Gamma,E}(\mathbf{v}_0,\mathbf{u})$ has a unique, local solution denoted $\overline{\mathbf{u}}(\cdot;\mathbf{v}_0) \in \Omega_\alpha(\varepsilon_2) \subset H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ for any $\alpha \in [0,\widetilde{\nu}]$;

(iii) The function $\Sigma_{\Gamma,E} : \overline{B}_{\mathrm{s}}(0,\varepsilon_1) \to \Omega_\alpha(\varepsilon_2)$ defined by $\Sigma_{\Gamma,E}(\mathbf{x}_0) = \overline{\mathbf{u}}(\cdot,\mathbf{v}_0)$ is of class C^r for any $\alpha \in [0,\widetilde{\nu}]$.

Proof. (i) First, we introduce the function

$$\mathscr{D}_\alpha : H^1_\alpha(\mathbb{R}_+,\mathbb{X}) \times H^1_\alpha(\mathbb{R}_+,\mathbb{X}) \to H^1_\alpha(\mathbb{R}_+,\mathbb{X}) \text{ defined by } \mathscr{D}_\alpha(f,g) = D(f(\cdot),g(\cdot)).$$

Since $D(\cdot,\cdot)$ is a bounded bilinear map on $\mathbb{X}$, we infer that for any $f,g \in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ we have that $\mathscr{D}_\alpha(f,g) \in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ and $\big(\mathscr{D}_\alpha(f,g)\big)' = D(f(\cdot),g'(\cdot)) + D(g(\cdot),f'(\cdot))$, proving that $\mathscr{D}_\alpha$ is well-defined. Also, one can readily check that $\mathscr{D}_\alpha$ is a bilinear map. Moreover, since $H^1_\alpha(\mathbb{R}_+,\mathbb{X}) \hookrightarrow L^\infty_\alpha(\mathbb{R}_+,\mathbb{X})$ we have that

$$e^{\alpha\tau}\|f(\tau)\| \leq \|f\|_{L^\infty_\alpha(\mathbb{R}_+,\mathbb{X})} \leq \|f\|_{H^1_\alpha(\mathbb{R}_+,\mathbb{X})} \text{ for any } \tau \geq 0. \tag{4.11}$$

Using again that $D(\cdot,\cdot)$ is a bounded bilinear map on $\mathbb{X}$, from (4.11) it follows that

$$\|\mathscr{D}_\alpha(f,g)\|^2_{H^1_\alpha} = \int_0^\infty e^{2\alpha\tau}\|D(f(\tau),g(\tau))\|^2\mathrm{d}\tau$$
$$+ \int_0^\infty e^{2\alpha\tau}\|D(f(\tau),g'(\tau)) + D(g(\tau),f'(\tau))\|^2\mathrm{d}\tau$$

$$
\begin{aligned}
&\leq \int_0^\infty e^{2\alpha\tau}\|D\|^2\,\|f(\tau)\|^2\|g(\tau)\|^2\mathrm{d}\tau\\
&\qquad + 2\int_0^\infty e^{2\alpha\tau}\|D\|^2\Big(\|f(\tau)\|^2\,\|g'(\tau)\|^2 + g(\tau)\|^2\,\|f'(\tau)\|^2\Big)\mathrm{d}\tau\\
&\leq \|D\|^2\|f\|^2_{L^\infty_\alpha}\|g\|^2_{L^2_\alpha} + 2\|D\|^2\Big(\|f\|^2_{L^\infty_\alpha}\|g'\|^2_{L^2_\alpha} + \|g\|^2_{L^\infty_\alpha}\|f'\|^2_{L^2_\alpha}\Big)\\
&\leq 5\|D\|^2\|f\|^2_{H^1_\alpha}\|g\|^2_{H^1_\alpha}, \qquad (4.12)
\end{aligned}
$$

which proves that the bilinear map $\mathscr{D}_\alpha$ is bounded on $H^1_\alpha(\mathbb{R},\mathbb{X})$. From Lemma 3.7, Lemma 4.1 and (4.12) we infer that $\Psi_{\Gamma,E}(\mathbf{v}_0,f)\in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ and

$$
\begin{aligned}
\|\Psi_{\Gamma,E}(\mathbf{v}_0,f)\|_{H^1_\alpha} &\leq \|T_{\mathrm{s}}^{\Gamma,E}(\cdot)\mathbf{v}_0\|_{H^1_\alpha} + \|\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}D(f,f)\|_{H^1_\alpha}\\
&\leq c\,\|\mathbf{v}_0\|_{\mathbb{X}^{\Gamma,E}_{\frac12}} + c\,\|\mathscr{D}_\alpha(f,f)\|_{H^1_\alpha}\\
&\leq c\,\|\mathbf{v}_0\|_{\mathbb{X}^{\Gamma,E}_{\frac12}} + 2c\|D\|\,\|f\|^2_{H^1_\alpha} \leq c\,(\varepsilon_1+\varepsilon_2^2), \qquad (4.13)
\end{aligned}
$$

for any $\mathbf{v}_0\in\overline{B}_{\mathrm{s}}(0,\varepsilon_1)$ and $f\in\Omega_\alpha(\varepsilon_2)$. Here the constant $c=c(\Gamma,E,\alpha)$ depends on the constants from (3.52) and (4.5), therefore it can be it chosen such that

$$
\sup_{\alpha\in[0,\widetilde{\nu}]} c(\Gamma,E,\alpha)<\infty \text{ for any } \widetilde{\nu}\in(0,\nu(\Gamma,E)). \qquad (4.14)
$$

It follows that for any $\widetilde{\nu}\in(0,\nu(\Gamma,E))$ there exist $\varepsilon_1=\varepsilon_1(\Gamma,E,\widetilde{\nu})>0$ and $\varepsilon_2=\varepsilon_2(\Gamma,E,\widetilde{\nu})>0$ such that

$$
c(\Gamma,E,\alpha)\,\varepsilon_2(\Gamma,E,\widetilde{\nu})\leq\frac{1}{16} \text{ and } c(\Gamma,E,\alpha)\,\varepsilon_1(\Gamma,E,\widetilde{\nu})\leq\frac{\varepsilon_2(\Gamma,E,\widetilde{\nu})}{2} \qquad (4.15)
$$

for any $\alpha\in[0,\widetilde{\nu}]$. From (4.13) and (4.15) we conclude that

$$
\Psi_{\Gamma,E} \text{ maps } \overline{B}_{\mathrm{s}}(0,\varepsilon_1(\Gamma,E,\widetilde{\nu}))\times\Omega_\alpha(\varepsilon_2(\Gamma,E,\widetilde{\nu})) \text{ to } \Omega_\alpha(\varepsilon_2(\Gamma,E,\widetilde{\nu})) \qquad (4.16)
$$

for any $\alpha\in[0,\widetilde{\nu}]$ and $\widetilde{\nu}\in(0,\nu(\Gamma,E))$. To finish the proof of (i) we prove (4.10). We note that

$$
\begin{aligned}
\Big\|\Psi_{\Gamma,E}(\mathbf{v}_0,f)-\Psi_{\Gamma,E}(\mathbf{v}_0,g)\Big\|_{H^1_\alpha} &= \Big\|\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}\big(\mathscr{D}_\alpha(f,f)-\mathscr{D}_\alpha(g,g)\big)\Big\|_{H^1_\alpha}\\
&\leq c(\Gamma,E,\alpha)\|\mathscr{D}_\alpha(f,f)-\mathscr{D}_\alpha(g,g)\|_{H^1_\alpha}. \qquad (4.17)
\end{aligned}
$$

To estimate the H^1_α-norm of the right hand side of (4.17), we use that $\mathscr{D}_\alpha(\cdot,\cdot)$ is bilinear and bounded on $H^1_\alpha(\mathbb{R}_+,\mathbb{X})$, which implies that $\mathscr{D}_\alpha(f,f)-\mathscr{D}_\alpha(g,g)=\mathscr{D}_\alpha(f-g,f-g)+2\mathscr{D}_\alpha(g,f-g)$. Since $\mathscr{D}_\alpha(\cdot,\cdot)$ is a bounded bilinear map on $H^1_\alpha(\mathbb{R}_+,\mathbb{X})$, we have that $\mathscr{D}_\alpha(g,f-g)\in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ and $\big(\mathscr{D}_\alpha(g,f-g)\big)'=\mathscr{D}_\alpha(g',f-g)+\mathscr{D}_\alpha(g,f'-g')$ for any $f,g\in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$. It follows that

$$
\begin{aligned}
\|\mathscr{D}_\alpha(g,f-g)\|^2_{H^1_\alpha} &= \int_0^\infty e^{2\alpha\tau}\|D(g(\tau),f(\tau)-g(\tau))\|^2\mathrm{d}\tau\\
&\qquad + \int_0^\infty e^{2\alpha\tau}\|\big(D(g,f-g)\big)'(\tau)\|^2\mathrm{d}\tau\\
&\leq \|D\|^2\int_0^\infty e^{2\alpha\tau}\|g(\tau)\|^2\|f(\tau)-g(\tau))\|^2\mathrm{d}\tau\\
&\qquad + 2\|D\|^2\int_0^\infty e^{2\alpha\tau}\|g'(\tau)\|^2\|f(\tau)-g(\tau))\|^2\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\|D\|^2 \int_0^\infty e^{2\alpha\tau} \|g(\tau)\|^2 \|f'(\tau) - g'(\tau))\|^2 \mathrm{d}\tau \\
&\leq 2\|D\|^2 \|g\|^2_{L^\infty_\alpha} \|f-g\|^2_{H^1_\alpha} + 2\|D\|^2 \|f-g\|^2_{L^\infty_\alpha} \|g'\|_{L^2_\alpha} \\
&\leq 2\|D\|^2 \|g\|^2_{H^1_\alpha} \|f-g\|^2_{H^1_\alpha}. \qquad (4.18)
\end{aligned}
$$

From (4.12) and (4.18) we obtain that

$$
\begin{aligned}
\|\mathscr{D}_\alpha(f,f) - \mathscr{D}_\alpha(g,g)\|_{H^1_\alpha} &\leq \|\mathscr{D}_\alpha(f-g,f-g)\|_{H^1_\alpha} + 2\|\mathscr{D}_\alpha(g,f-g)\|_{H^1_\alpha} \\
&\leq 2\|D\| \|f-g\|^2_{H^1_\alpha} + 2\|D\| \|g\|_{H^1_\alpha} \|f-g\|_{H^1_\alpha} \\
&\leq 4\|D\| \big(\|f\|_{H^1_\alpha} + \|g\|_{H^1_\alpha}\big) \|f-g\|_{H^1_\alpha} \qquad (4.19)
\end{aligned}
$$

Therefore, from (4.15), (4.17) and (4.19) have that

$$
\|\Psi_{\Gamma,E}(\mathbf{v}_0,f) - \Psi_{\Gamma,E}(\mathbf{v}_0,g)\|_{H^1_\alpha} \leq 8c(\Gamma,E,\alpha)\varepsilon_2(\Gamma,E,\widetilde{\nu}) \|f-g\|_{H^1_\alpha} \leq \frac{1}{2}\|f-g\|_{H^1_\alpha} \qquad (4.20)
$$

for any $\mathbf{v}_0 \in \overline{B}_{\mathrm{s}}(0,\varepsilon_1(\Gamma,E,\widetilde{\nu}))$, $f,g \in \Omega_\alpha(\varepsilon_2(\Gamma,E,\widetilde{\nu}))$, $\alpha \in [0,\widetilde{\nu}]$, $\widetilde{\nu} \in (0,\nu(\Gamma,E))$, proving (i).

Assertion (ii) follows shortly from (i) by applying the Contraction Mapping Theorem. By smooth dependence on parameters of solutions of fixed point mappings (see, e.g., Lemma A.1), to prove (iii) it is enough to show that $\Psi_{\Gamma,E}$ is of class C^r on $\mathbb{X}^{\Gamma,E}_{\mathrm{s}} \cap \mathbb{X}^{\Gamma,E}_{\frac{1}{2}} \times H^1_\alpha(\mathbb{R}_+,\mathbb{X})$. We note that

$$
\Psi_{\Gamma,E}(\mathbf{v}_0,f) = T^{\Gamma,E}_{\mathrm{s}}(\cdot)P^{\Gamma,E}_{\mathrm{s}}\mathbf{v}_0 + \mathscr{K}^{\mathrm{mod}}_{\Gamma,E}\mathscr{D}_\alpha(f,f) \quad \text{for any} \quad f \in H^1_\alpha(\mathbb{R},\mathbb{X}). \qquad (4.21)
$$

Since $\Psi_{\Gamma,E}$ is affine in $\mathbf{v}_0$, $\mathscr{K}^{\mathrm{mod}}_{\Gamma,E} \in \mathscr{B}(H^1_\alpha(\mathbb{R}_+,\mathbb{X}))$ by Lemma 3.7 and the bilinear map $\mathscr{D}_\alpha$ is bounded on $H^1_\alpha(\mathbb{R},\mathbb{X})$ by (4.12), from (4.21) we infer that $\Psi_{\Gamma,E}$ is of class C^r on $\mathbb{X}^{\Gamma,E}_{\mathrm{s}} \cap \mathbb{X}^{\Gamma,E}_{\frac{1}{2}} \times H^1_\alpha(\mathbb{R}_+,\mathbb{X})$, proving the lemma. □

Lemma 4.3. *Assume Hypothesis (S). Then,*

(i) $(\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f)(0) + P^{\Gamma,E}_{\mathrm{s}}E^{-1}f(0) \in \mathbb{X}^{\Gamma,E}_{\mathrm{u}}$ *for any* $f \in H^1(\mathbb{R}_+,\mathbb{X})$*;*

(ii) $\overline{\mathbf{u}}(\cdot;\mathbf{v}_0)$ *is a mild solution of equation (3.5) on* $\mathbb{R}_+$ *satisfying the condition*

$$
P^{\Gamma,E}_{\mathrm{s}}\overline{\mathbf{u}}(0;\mathbf{v}_0) = \mathbf{v}_0 - P^{\Gamma,E}_{\mathrm{s}}E^{-1}D(\overline{\mathbf{u}}(0;\mathbf{v}_0),\overline{\mathbf{u}}(0;\mathbf{v}_0)) \qquad (4.22)
$$

for any $\mathbf{v}_0 \in \overline{B}_{\mathrm{s}}(0,\varepsilon_1(\Gamma,E,\widetilde{\nu}))$.

Proof. (i) First, we recall the definition of the subspace $\mathscr{H}^1_\Gamma = \{f : \mathbb{R}_+ \to \mathbb{X} :$ there exists $g \in H^1(\mathbb{R}_+,\mathbb{X})$ such that $f(\tau) = \Gamma g(\tau)$ for any $\tau \geq 0\}$, which is dense in $H^1(\mathbb{R}_+,\mathbb{X})$. Let $g \in H^1(\mathbb{R}_+,\mathbb{X})$ and $f = \Gamma g$. From (3.37) we obtain that

$$
\begin{aligned}
(\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f)(0) + P^{\Gamma,E}_{\mathrm{s}}E^{-1}f(0) &= -\int_0^\infty T^{\Gamma,E}_{\mathrm{u}}(s)P^{\Gamma,E}_{\mathrm{u}}g(s)\mathrm{d}s - S^{-1}_{\Gamma,E}P^{\Gamma,E}_{\mathrm{s}}g(0) \\
&\quad + P^{\Gamma,E}_{\mathrm{s}}E^{-1}\Gamma g(0) \\
&= -\int_0^\infty T^{\Gamma,E}_{\mathrm{u}}(s)P^{\Gamma,E}_{\mathrm{u}}g(s)\mathrm{d}s \in \mathbb{X}^{\Gamma,E}_{\mathrm{u}}. \qquad (4.23)
\end{aligned}
$$

From Lemma 3.5 we infer that the operator $f \to (\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f)(0) + P^{\Gamma,E}_{\mathrm{s}}E^{-1}f(0) :$ $H^1(\mathbb{R}_+,\mathbb{X}) \to \mathbb{X}$ is linear and bounded. Since $\mathscr{H}^1_\Gamma$ is dense in $H^1(\mathbb{R}_+,\mathbb{X})$, from (4.23) we conclude that $(\mathscr{K}^{\mathrm{mod}}_{\Gamma,E}f)(0)+P^{\Gamma,E}_{\mathrm{s}}E^{-1}f(0) \in \mathbb{X}^{\Gamma,E}_{\mathrm{u}}$ for any $f \in H^1(\mathbb{R}_+,\mathbb{X})$.

Proof of (ii) Fix $\mathbf{v}_0 \in \overline{B}_{\rm s}(0,\varepsilon_1(\Gamma,E,\widetilde{\nu}))$. Since $\overline{\mathbf{u}}(\cdot;\mathbf{v}_0)$ is a solution of equation (4.2), from (i) we obtain that

$$\begin{aligned} P_{\rm s}^{\Gamma,E}\overline{\mathbf{u}}(0;\mathbf{v}_0) &= P_{\rm s}^{\Gamma,E}\Big(\mathbf{v}_0 + (\mathscr{K}_{\Gamma,E}^{\rm mod}D(\overline{\mathbf{u}}(\cdot;\mathbf{v}_0),\overline{\mathbf{u}}(\cdot;\mathbf{v}_0))(0)\Big) \\ &= \mathbf{v}_0 - P_{\rm s}^{\Gamma,E}E^{-1}D(\overline{\mathbf{u}}(0;\mathbf{v}_0),\overline{\mathbf{u}}(0;\mathbf{v}_0)), \end{aligned}$$

proving (4.22). Moreover,

$$\begin{aligned} &\overline{\mathbf{u}}(\tau;\mathbf{v}_0) - T_{\rm s}^{\Gamma,E}(\tau)P_{\rm s}^{\Gamma,E}\overline{\mathbf{u}}(0;\mathbf{v}_0) \\ &= T_{\rm s}^{\Gamma,E}(\tau)P_{\rm s}^{\Gamma,E}(\mathbf{v}_0 - \overline{\mathbf{u}}(0;\mathbf{v}_0)) + \big(\mathscr{K}_{\Gamma,E}^{\rm mod}D(\overline{\mathbf{u}}(\cdot;\mathbf{v}_0),\overline{\mathbf{u}}(\cdot;\mathbf{v}_0)\big)(\tau) \\ &= T_{\rm s}^{\Gamma,E}(\tau)P_{\rm s}^{\Gamma,E}E^{-1}D(\overline{\mathbf{u}}(0;\mathbf{v}_0),\overline{\mathbf{u}}(0;\mathbf{v}_0)) + \big(\mathscr{K}_{\Gamma,E}^{\rm mod}D(\overline{\mathbf{u}}(\cdot;\mathbf{v}_0),\overline{\mathbf{u}}(\cdot;\mathbf{v}_0))\big)(\tau) \\ &= \big(\mathscr{K}_{\Gamma,E}D(\overline{\mathbf{u}}(\cdot;\mathbf{v}_0),\overline{\mathbf{u}}(\cdot;\mathbf{v}_0))\big)(\tau) \quad \text{for any} \quad \tau \geq 0. \end{aligned}$$

From Lemma 3.3 we conclude that $\overline{\mathbf{u}}(\cdot;\mathbf{v}_0)$ is a mild solution of equation (3.5) on $\mathbb{R}_+$, proving the lemma. ☐

Next, we define $\mathscr{J}_{\rm s}^{\Gamma,E} : \overline{B}_{\rm s}(0,\varepsilon_1) \to \mathbb{X}$ by

$$\mathscr{J}_{\rm s}^{\Gamma,E}(\mathbf{v}_0) = P_{\rm u}^{\Gamma,E}\overline{\mathbf{u}}(0;\mathbf{v}_0) - P_{\rm s}^{\Gamma,E}E^{-1}D(\overline{\mathbf{u}}(0;\mathbf{v}_0),\overline{\mathbf{u}}(0;\mathbf{v}_0)) \tag{4.24}$$

and introduce the manifold

$$\mathscr{M}_{\rm s}^{\Gamma,E} = \big\{\mathbf{v}_0 + \mathscr{J}_{\rm s}^{\Gamma,E}(\mathbf{v}_0) : \mathbf{v}_0 \in B_{\rm s}(0,\varepsilon_1)\big\}. \tag{4.25}$$

Next, we are going to show that the manifold $\mathscr{M}_{\rm s}^{\Gamma,E}$ is invariant under the forward flow of equation (3.5). To prove this result we need to study the time translations of solutions $\overline{\mathbf{u}}(\cdot;\mathbf{v}_0)$ of equation (4.2). To achieve this goal we need the results of the next lemma.

Lemma 4.4. *Assume Hypothesis (S). Then,*

(i) The modified Fourier multiplier $\mathscr{K}_{\Gamma,E}^{\rm mod}$ satisfies the translation formula

$$(\mathscr{K}_{\Gamma,E}^{\rm mod}f)(\tau+\tau_0) = \big(\mathscr{K}_{\Gamma,E}^{\rm mod}f(\cdot+\tau_0)\big)(\tau) + T_{\rm s}^{\Gamma,E}(\tau)\int_0^{\tau_0} T_{\rm s}^{\Gamma,E}(\tau_0-s)P_{\rm s}^{\Gamma,E}E^{-1}f'(s){\rm d}s \tag{4.26}$$

for any $f \in H^1(\mathbb{R}_+,\mathbb{X})$, $\tau,\tau_0 \geq 0$;

(ii) $T_{\rm s}^{\Gamma,E}(\tau)\mathbf{v}_0 \in \mathbb{X}_{\rm s}^{\Gamma,E} \cap \mathbb{X}_{\frac12}^{\Gamma,E}$ for any $\tau \geq 0$ and $\mathbf{v}_0 \in \mathbb{X}_{\rm s}^{\Gamma,E} \cap \mathbb{X}_{\frac12}^{\Gamma,E}$. Moreover,

$$\|T_{\rm s}^{\Gamma,E}(\tau)\mathbf{v}_0\|_{\mathbb{X}_{\frac12}^{\Gamma,E}} \leq e^{-\nu(\Gamma,E)\tau}\|\mathbf{v}_0\|_{\mathbb{X}_{\frac12}^{\Gamma,E}}, \quad \lim_{\tau\to0+}\|T_{\rm s}^{\Gamma,E}(\tau)\mathbf{v}_0 - \mathbf{v}_0\|_{\mathbb{X}_{\frac12}^{\Gamma,E}} = 0; \tag{4.27}$$

(iii) $\int_0^{\tau_0} T_{\rm s}^{\Gamma,E}(\tau_0-s)P_{\rm s}^{\Gamma,E}E^{-1}f(s){\rm d}s \in \mathbb{X}_{\frac12}^{\Gamma,E}$ for any $f \in L^2([0,\tau_0],\mathbb{X})$ and

$$\Big\|\int_0^{\tau_0} T_{\rm s}^{\Gamma,E}(\tau_0-s)P_{\rm s}^{\Gamma,E}E^{-1}f(s)ds\Big\|_{\mathbb{X}_{\frac12}^{\Gamma,E}} \leq c(\Gamma,E)\Big(\int_0^{\tau_0}\|f(s)\|^2{\rm d}s\Big)^{\frac12} \tag{4.28}$$

for any $f \in L^2([0,\tau_0],\mathbb{X})$.

Proof. (i) Let $g \in H^1(\mathbb{R}_+,\mathbb{X})$ and $f = \Gamma g$. From (3.32) we have that $\mathscr{K}_{\Gamma,E}f = \mathscr{G}_{\Gamma,E} * g$ and $\mathscr{K}_{\Gamma,E}f(\cdot+\tau_0) = \mathscr{G}_{\Gamma,E} * g(\cdot+\tau_0)$. It follows that

$$\begin{aligned} (\mathscr{K}_{\Gamma,E}f)(\tau+\tau_0) = &\int_0^{\tau+\tau_0} T_{\rm s}^{\Gamma,E}(\tau+\tau_0-s)P_{\rm s}^{\Gamma,E}g(s){\rm d}s \\ &- \int_{\tau+\tau_0}^{\infty} T_{\rm u}^{\Gamma,E}(s-\tau-\tau_0)P_{\rm u}^{\Gamma,E}g(s){\rm d}s \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\tau_0}^{\tau} T_{\mathrm{s}}^{\Gamma,E}(\tau-\theta)P_{\mathrm{s}}^{\Gamma,E} g(\theta+\tau_0)\mathrm{d}\theta \\
&\qquad - \int_{\tau}^{\infty} T_{\mathrm{u}}^{\Gamma,E}(\theta-\tau)P_{\mathrm{u}}^{\Gamma,E} g(\theta+\tau_0)\mathrm{d}\theta \\
&= \big(\mathscr{G}_{\Gamma,E} * g(\cdot+\tau_0)\big)(\tau) + \int_{-\tau_0}^{0} T_{\mathrm{s}}^{\Gamma,E}(\tau-\theta)P_{\mathrm{s}}^{\Gamma,E} g(\theta+\tau_0)\mathrm{d}\theta \\
&= \big(\mathscr{K}_{\Gamma,E} f(\cdot+\tau_0)\big)(\tau) + \int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} g(s)\mathrm{d}s \qquad (4.29)
\end{aligned}
$$

for any $\tau \geq 0$. Since $S_{\mathrm{s}}^{\Gamma,E} = (S_{\Gamma,E})_{|\mathbb{X}_{\mathrm{s}}^{\Gamma,E}}$ is the generator of the C^0-semigroup $\{T_{\mathrm{s}}^{\Gamma,E}(\tau)\}_{\tau\geq 0}$, we infer that the function $\tau \to T_{\mathrm{s}}^{\Gamma,E}(\tau)(S_{\mathrm{s}}^{\Gamma,E})^{-1}\mathbf{x} : \mathbb{R}_+ \to \mathbb{X}$ is of class C^1 for any $\mathbf{x} \in \mathbb{X}$. Integrating by parts we obtain that

$$
\begin{aligned}
&\int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f'(s)ds \\
&= \int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} S_{\Gamma,E}^{-1} g'(s)ds \\
&= \int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} g(s)\mathrm{d}s + T_{\mathrm{s}}^{\Gamma,E}(\tau)P_{\mathrm{s}}^{\Gamma,E} S_{\Gamma,E}^{-1} g(\tau_0) \\
&\qquad - T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0)P_{\mathrm{s}}^{\Gamma,E} S_{\Gamma,E}^{-1} g(0) \\
&= \int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} g(s)\mathrm{d}s + T_{\mathrm{s}}^{\Gamma,E}(\tau)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f(\tau_0) \\
&\qquad - T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f(0). \qquad (4.30)
\end{aligned}
$$

From (4.29) and (4.30) we conclude that

$$
\begin{aligned}
&(\mathscr{K}_{\Gamma,E}^{\mathrm{mod}} f)(\tau+\tau_0) \\
&= (\mathscr{K}_{\Gamma,E} f)(\tau+\tau_0) - T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f(0) \\
&= \big(\mathscr{K}_{\Gamma,E} f(\cdot+\tau_0)\big)(\tau) + \int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} g(s)\mathrm{d}s \\
&\qquad - T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f(0) \\
&= \big(\mathscr{K}_{\Gamma,E}^{\mathrm{mod}} f(\cdot+\tau_0)\big)(\tau) + \int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} g(s)\mathrm{d}s \\
&\qquad - T_{\mathrm{s}}^{\Gamma,E}(\tau+\tau_0)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f(0) + T_{\mathrm{s}}^{\Gamma,E}(\tau)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f(\tau_0) \\
&= \big(\mathscr{K}_{\Gamma,E}^{\mathrm{mod}} f(\cdot+\tau_0)\big)(\tau) + T_{\mathrm{s}}^{\Gamma,E}(\tau)\int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f'(s)\mathrm{d}s \qquad (4.31)
\end{aligned}
$$

for any $\tau \geq 0$, $f \in \mathscr{H}_{\Gamma}^{1}$. Since $\mathscr{H}_{\Gamma}^{1}$ is dense in $H^1(\mathbb{R}_+, \mathbb{X})$, for any $f \in H^1(\mathbb{R}_+, \mathbb{X})$ there exists $\{f_n\}_{n\geq 1}$ a sequence of functions in $\mathscr{H}_{\Gamma}^{1}$ such that $f_n \to f$ as $n \to \infty$ in $H^1(\mathbb{R}_+, \mathbb{X})$. It follows that $f_n(\cdot+\tau_0) \to f(\cdot+\tau_0)$ as $n \to \infty$ in $H^1(\mathbb{R}_+, \mathbb{X})$. Since $\mathscr{K}_{\Gamma,E}^{\mathrm{mod}}$ is a bounded linear operator on $H^1(\mathbb{R}_+, \mathbb{X})$ by Lemma 3.5, we infer that

$$
(\mathscr{K}_{\Gamma,E}^{\mathrm{mod}} f_n) \to (\mathscr{K}_{\Gamma,E}^{\mathrm{mod}} f) \text{ and } \mathscr{K}_{\Gamma,E}^{\mathrm{mod}} f_n(\cdot+\tau_0) \to \mathscr{K}_{\Gamma,E}^{\mathrm{mod}} f(\cdot+\tau_0) \qquad (4.32)
$$

as $n \to \infty$ in $H^1(\mathbb{R}_+, \mathbb{X})$. Moreover, one can readily check that

$$
\Big\| \int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f_n'(s)\mathrm{d}s - \int_{0}^{\tau_0} T_{\mathrm{s}}^{\Gamma,E}(\tau_0-s)P_{\mathrm{s}}^{\Gamma,E} E^{-1} f'(s)ds \Big\|
$$

$$\leq c(\Gamma,E)\Big(\int_0^{\tau_0} e^{-2\nu(\Gamma,E)(\tau_0-s)}\mathrm{d}s\Big)^{1/2}\|f_n'-f'\|_2 \leq c(\Gamma,E)\|f_n-f\|_{H^1} \quad (4.33)$$

for any $n \geq 1$. Assertion (i) follows from (4.31), (4.32) and (4.33).

Proof of (ii). Let $\mathbf{v}_0 \in \mathbb{X}_{\mathrm{s}}^{\Gamma,E}\cap\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}$. We recall that from (2.14) we obtain that $\widetilde{g}_0 = U_{\Gamma,E}\mathbf{v}_0 \in L^2(\Lambda,\mu)$ and $\widetilde{g}_0(\lambda)=0$ for any $\lambda\in\Lambda_+$. From (2.15) we infer that

$$\int_\Lambda |H_{\Gamma,E}(\lambda)|\,|(U_{\Gamma,E}T_{\mathrm{s}}^{\Gamma,E}(\tau)\mathbf{v}_0)(\lambda)|^2\mathrm{d}\mu(\lambda) = \int_\Lambda |H_{\Gamma,E}(\lambda)|\,|(\widetilde{T}_{\mathrm{s}}^{\Gamma,E}(\tau)\widetilde{g}_0)(\lambda)|^2\mathrm{d}\mu(\lambda)$$
$$= \int_{\Lambda-} e^{2\tau H_{\Gamma,E}(\lambda)}|H_{\Gamma,E}(\lambda)|\,|\widetilde{g}_0(\lambda)|^2\mathrm{d}\mu(\lambda) \leq e^{-2\nu\tau}\|\mathbf{v}_0\|^2_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}} \text{ for any } \tau\geq 0. \quad (4.34)$$

From (4.34) we conclude that $T_{\mathrm{s}}^{\Gamma,E}(\tau)\mathbf{v}_0 \in \mathbb{X}_{\mathrm{s}}^{\Gamma,E}\cap\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}$ and $\|T_{\mathrm{s}}^{\Gamma,E}(\tau)\mathbf{v}_0\|_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}} \leq e^{-\nu(\Gamma,E)\tau}\|\mathbf{v}_0\|_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}}$ for any $\tau\geq 0$. Moreover, using (2.15) again we obtain that

$$\big\|T_{\mathrm{s}}^{\Gamma,E}(\tau)\mathbf{v}_0-\mathbf{v}_0\big\|^2_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}} = \int_\Lambda |H_{\Gamma,E}(\lambda)|\,\Big|\Big(U_{\Gamma,E}\big(T_{\mathrm{s}}^{\Gamma,E}(\tau)\mathbf{v}_0-\mathbf{v}_0\big)\Big)(\lambda)\Big|^2\mathrm{d}\mu(\lambda)$$
$$= \int_{\Lambda-}\big(1-e^{2\tau H_{\Gamma,E}(\lambda)}\big)|H_{\Gamma,E}(\lambda)|\,|\widetilde{g}_0(\lambda)|^2\mathrm{d}\mu(\lambda) \quad (4.35)$$

Passing to the limit as $\tau\to 0$ in (4.35), from Lebesgue Dominated Convergence Theorem, it follows that $\lim_{\tau\to 0+}\|T_{\mathrm{s}}^{\Gamma,E}(\tau)\mathbf{v}_0-\mathbf{v}_0\|_{\mathbb{X}_{\frac{1}{2}}^{\Gamma,E}}=0$, proving (ii).

Proof of (iii). First, we introduce the function $\widetilde{f}:\mathbb{R}_+\to L^2(\Lambda,\mu)$ by $\widetilde{f}(\tau) = U_{\Gamma,E}P_{\mathrm{s}}^{\Gamma,E}E^{-1}f(\tau)$ and let $\widetilde{h}_0 = U_{\Gamma,E}\big(\int_0^{\tau_0}T_{\mathrm{s}}^{\Gamma,E}(\tau_0-s)P_{\mathrm{s}}^{\Gamma,E}E^{-1}f(s)\mathrm{d}s\big)$. To simplify the notation, in what follows we denote by $\widetilde{f}(\tau,\lambda) = \big(\widetilde{f}(\tau)\big)(\lambda)$. Since $f\in L^2([0,\tau_0],\mathbb{X})$ we infer that $\widetilde{f}\in L^2([0,\tau_0],L^2(\Lambda,\mu))$, thus

$$\int_0^{\tau_0}\int_\Lambda |\widetilde{f}(\tau,\lambda)|^2\mathrm{d}\mu(\lambda)\mathrm{d}\tau = \|\widetilde{f}\|^2_{L^2([0,\tau_0],L^2(\Lambda,\mu))} \leq c(\Gamma,E)\int_0^{\tau_0}\|f(s)\|^2\mathrm{d}s \quad (4.36)$$

From (2.15) one can readily check that

$$\widetilde{h}_0(\lambda) = \int_0^{\tau_0}\big(\widetilde{T}_{\mathrm{s}}^{\Gamma,E}(\tau_0-s)\widetilde{f}(s)\big)(\lambda)\mathrm{d}s = \int_0^{\tau_0} e^{(\tau_0-s)H_{\Gamma,E}(\lambda)}\widetilde{f}(s,\lambda)\mathrm{d}s \quad (4.37)$$

for any $\lambda\in\Lambda$. From (4.37) we obtain that

$$|\widetilde{h}_0(\lambda)|^2 \leq \int_0^{\tau_0} e^{2(\tau_0-s)H_{\Gamma,E}(\lambda)}\mathrm{d}s\int_0^{\tau_0}|\widetilde{f}(s,\lambda)|^2\mathrm{d}s = \frac{1-e^{2\tau_0 H_{\Gamma,E}(\lambda)}}{2|H_{\Gamma,E}(\lambda)|}\int_0^{\tau_0}|\widetilde{f}(s,\lambda)|^2\mathrm{d}s$$
$$\leq \frac{1}{2|H_{\Gamma,E}(\lambda)|}\int_0^{\tau_0}|\widetilde{f}(s,\lambda)|^2\mathrm{d}s \quad \text{for any} \quad \lambda\in\Lambda. \quad (4.38)$$

From (4.36) and (4.38) it follows that

$$\int_\Lambda |H_{\Gamma,E}(\lambda)|\,|\widetilde{h}_0(\lambda)|^2\mathrm{d}\mu(\lambda) \leq \frac{1}{2}\int_\Lambda\int_0^{\tau_0}|\widetilde{f}(\tau,\lambda)|^2\mathrm{d}\tau\mathrm{d}\mu(\lambda)$$
$$\leq c(\Gamma,E)\int_0^{\tau_0}\|f(s)\|^2\mathrm{d}s < \infty. \quad (4.39)$$

We conclude that $\widetilde{h}_0\in\mathrm{dom}(|M_{H_{\Gamma,E}}|^{1/2})$ and

$$\|\widetilde{h}_0\|_{\mathrm{dom}(|M_{H_{\Gamma,E}}|^{1/2})} \leq c(\Gamma,E)\|f\|_{L^2([0,\tau_0],\mathbb{X})},$$

proving the lemma. □

Lemma 4.5. *Assume Hypothesis (S). Then, the manifold $\mathscr{M}_{\rm s}^{\Gamma,E}$ is locally invariant under the forward flow of equation (3.5).*

Proof. Let $\widetilde{\nu}\in(0,\nu(\Gamma,E))$, $\alpha\in[0,\widetilde{\nu}]$ and assume $\mathbf{u}_0$ is a H^1_α solution of equation (3.5) such that $\mathbf{u}_0(0)\in\mathscr{M}_{\rm s}^{\Gamma,E}$. Then, there exists $\mathbf{v}_0\in\overline{B}(0,\varepsilon_1)$ such that $\mathbf{u}_0(0)=\mathbf{v}_0+\mathscr{J}_{\rm s}^{\Gamma,E}(\mathbf{v}_0)$. From (4.22) and (4.24) we obtain that $\mathbf{u}_0(0)=\overline{\mathbf{u}}(0;\mathbf{v}_0)$. Since $\mathbf{u}_0$ is a H^1_α solution of equation (3.5), we have that $\mathbf{u}_0$ satisfies equation (4.1). It follows that

$$
\begin{aligned}
\mathbf{u}_0(\tau)&=T_{\rm s}^{\Gamma,E}(\tau)P_{\rm s}^{\Gamma,E}\mathbf{u}_0(0)+\big(\mathscr{K}_{\Gamma,E}D(\mathbf{u}_0,\mathbf{u}_0)\big)(\tau)\\
&=T_{\rm s}^{\Gamma,E}(\tau)\big(P_{\rm s}^{\Gamma,E}\mathbf{u}_0(0)+P_{\rm s}^{\Gamma,E}E^{-1}D(\mathbf{u}_0(0),\mathbf{u}_0(0))\big)+\big(\mathscr{K}_{\Gamma,E}^{\rm mod}D(\mathbf{u}_0,\mathbf{u}_0)\big)(\tau)\\
&=T_{\rm s}^{\Gamma,E}(\tau)\big(P_{\rm s}^{\Gamma,E}\overline{\mathbf{u}}(0;\mathbf{v}_0)+P_{\rm s}^{\Gamma,E}E^{-1}D(\overline{\mathbf{u}}(0;\mathbf{v}_0),\overline{\mathbf{u}}(0;\mathbf{v}_0))\big)\\
&\quad+\big(\mathscr{K}_{\Gamma,E}^{\rm mod}D(\mathbf{u}_0,\mathbf{u}_0)\big)(\tau)\\
&=T_{\rm s}^{\Gamma,E}(\tau)P_{\rm s}^{\Gamma,E}\mathbf{v}_0+\big(\mathscr{K}_{\Gamma,E}^{\rm mod}D(\mathbf{u}_0,\mathbf{u}_0)\big)(\tau)\quad\text{for any}\quad\tau\geq0. \qquad (4.40)
\end{aligned}
$$

From Lemma 4.2 we infer that equation $\mathbf{u}=\Psi_{\Gamma,E}(\mathbf{v}_0,\mathbf{u})$ has a unique solution, which implies that $\mathbf{u}_0=\overline{\mathbf{u}}(\cdot;\mathbf{v}_0)$. Fix $\tau_0\geq0$ and let $\mathbf{u}_{\tau_0}:\mathbb{R}_+\to\mathbb{X}$ be the function defined by $\mathbf{u}_{\tau_0}(\tau)=\mathbf{u}_0(\tau+\tau_0)=\overline{\mathbf{u}}(\tau+\tau_0;\mathbf{v}_0)$. Since $\overline{\mathbf{u}}(\cdot;\mathbf{v}_0)\in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ for any $\alpha\in[0,\widetilde{\nu}]$, it follows that $u_{\tau_0}\in H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ for any $\alpha\in[0,\widetilde{\nu}]$. From Lemma 4.4(i), (4.22), (4.40) and since $\mathbf{u}_0=\overline{\mathbf{u}}(\cdot;\mathbf{v}_0)$ we conclude that

$$
\begin{aligned}
\mathbf{u}_{\tau_0}(\tau)=\mathbf{u}_0(\tau+\tau_0)&=T_{\rm s}^{\Gamma,E}(\tau+\tau_0)\mathbf{v}_0+\Big(\mathscr{K}_{\Gamma,E}^{\rm mod}D(\mathbf{u}_0,\mathbf{u}_0)\Big)(\tau+\tau_0)\\
&=T_{\rm s}^{\Gamma,E}(\tau)\int_0^{\tau_0}T_{\rm s}^{\Gamma,E}(\tau_0-s)P_{\rm s}^{\Gamma,E}E^{-1}\big(D(\mathbf{u}_0(s),\mathbf{u}_0(s))\big)'{\rm d}s\\
&\quad+T_{\rm s}^{\Gamma,E}(\tau)T_{\rm s}^{\Gamma,E}(\tau_0)\mathbf{v}_0+\big(\mathscr{K}_{\Gamma,E}^{\rm mod}D(\mathbf{u}_{\tau_0},\mathbf{u}_{\tau_0})\big)(\tau)\\
&=T_{\rm s}^{\Gamma,E}(\tau)\mathbf{v}_1+\big(\mathscr{K}_{\Gamma,E}^{\rm mod}D(\mathbf{u}_{\tau_0},\mathbf{u}_{\tau_0})\big)(\tau)\quad\text{for any}\quad\tau\geq0, \qquad (4.41)
\end{aligned}
$$

where $\mathbf{v}_1=T_{\rm s}^{\Gamma,E}(\tau_0)\mathbf{v}_0+\int_0^{\tau_0}T_{\rm s}^{\Gamma,E}(\tau_0-s)P_{\rm s}^{\Gamma,E}E^{-1}\big(D(\mathbf{u}_0(s),\mathbf{u}_0(s))\big)'{\rm d}s$. From Lemma 4.4(ii) and (iii) we infer that $\mathbf{v}_1\in\mathbb{X}_{\rm s}^{\Gamma,E}\cap\mathbb{X}_{\frac12}^{\Gamma,E}$. Moreover,

$$
\begin{aligned}
\|\mathbf{v}_1-\mathbf{v}_0\|_{\mathbb{X}_{\frac12}^{\Gamma,E}}&\leq\|T_{\rm s}^{\Gamma,E}(\tau_0)\mathbf{v}_0-\mathbf{v}_0\|_{\mathbb{X}_{\frac12}^{\Gamma,E}}+c(\Gamma,E)\Big(\int_0^{\tau_0}\|D(\mathbf{u}_0(s),\mathbf{u}_0'(s))\|^2{\rm d}s\Big)^{\frac12}\\
&\leq\|T_{\rm s}^{\Gamma,E}(\tau_0)\mathbf{v}_0-\mathbf{v}_0\|_{\mathbb{X}_{\frac12}^{\Gamma,E}}+c(\Gamma,E)\|\mathbf{u}_0\|_{H^1}\Big(\int_0^{\tau_0}\|\mathbf{u}_0'(s)\|^2{\rm d}s\Big)^{\frac12}.
\end{aligned}
\qquad (4.42)
$$

From Lemma 4.4(ii) and (iii) we infer that there exists $\tau_1>0$ such that $\mathbf{v}_1=\mathbf{v}_1(\tau_0)\in B_{\rm s}(0,\varepsilon_1)$ for any $\tau_0\in[0,\tau_1]$. Since equation $\mathbf{u}=\Psi_{\Gamma,E}(\mathbf{v}_1,\mathbf{u})$ has a unique solution in $\Omega_\alpha(\varepsilon_2)$ for any $\alpha\in[0,\widetilde{\nu}]$, from (4.41) we conclude that $\mathbf{u}_{\tau_0}=\overline{\mathbf{u}}(\cdot;\mathbf{v}_1)$. From Lemma 4.3(ii) it follows that

$$
\begin{aligned}
\mathbf{u}_0(\tau_0)=\mathbf{u}_{\tau_0}(0)=\overline{\mathbf{u}}(0;\mathbf{v}_1)&=P_{\rm s}^{\Gamma,E}\overline{\mathbf{u}}(0;\mathbf{v}_1)+P_{\rm u}^{\Gamma,E}\overline{\mathbf{u}}(0;\mathbf{v}_1)\\
&=\mathbf{v}_1-P_{\rm s}^{\Gamma,E}E^{-1}D(\overline{\mathbf{u}}(0;\mathbf{v}_1),\overline{\mathbf{u}}(0;\mathbf{v}_1))+P_{\rm u}^{\Gamma,E}\overline{\mathbf{u}}(0;\mathbf{v}_1)\\
&=\mathbf{v}_1+\mathscr{J}_{\rm s}^{\Gamma,E}(\mathbf{v}_1)\in\mathscr{M}_{\rm s}^{\Gamma,E}\quad\text{for any}\quad\tau_0\in[0,\tau_1], \qquad (4.43)
\end{aligned}
$$

proving the lemma. □

In the next lemma we prove that the nonlinear manifold $\mathscr{M}_{\rm s}^{\Gamma,E}$ is tangent to the corresponding linear stable subspace $\mathbb{X}_{\rm s}^{\Gamma,E}$.

Lemma 4.6. *Assume Hypothesis (S). Then,* $\big(\mathscr{J}_{\rm s}^{\Gamma,E}\big)'(0)=0$*, i.e.,* $\mathscr{M}_{\rm s}^{\Gamma,E}$ *is tangent to* $\mathbb{X}_{\rm s}^{\Gamma,E}$*.*

Proof. First, we compute $\Sigma'_{\Gamma,E}(0)$, where $\Sigma_{\Gamma,E}$ is defined in Lemma 4.2(iii). Differentiating with respect to $\mathbf{v}_0$ in the fixed point equation (4.2), we obtain that

$$\Sigma'_{\Gamma,E}(\mathbf{v}_0)=\partial_{\mathbf{v}_0}\Psi_{\Gamma,E}(\mathbf{v}_0,\Sigma_{\Gamma,E}(\mathbf{v}_0))+\partial_{\mathbf{u}}\Psi_{\Gamma,E}(\mathbf{v}_0,\Sigma_{\Gamma,E}(\mathbf{v}_0))\Sigma'_{\Gamma,E}(\mathbf{v}_0) \tag{4.44}$$

for any $\mathbf{v}_0\in\overline{B}_{\rm s}(0,\varepsilon_1)$. Since equation $\mathbf{u}=\Psi_{\Gamma,E}(0,\mathbf{u})$ has a unique solution, and 0 trivially satisfies the equation, we infer that $\Sigma_{\Gamma,E}(0)=0$. Moreover, since $\Psi_{\Gamma,E}(\mathbf{v}_0,0)=T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}\mathbf{v}_0$ for any $\mathbf{v}_0\in\overline{B}_{\rm s}(0,\varepsilon_1)$ we have that

$$\big(\partial_{\mathbf{v}_0}\Psi_{\Gamma,E}(0,0)\big)\mathbf{v}_1=T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}\mathbf{v}_1\quad\text{for any}\quad \mathbf{v}_1\in\mathbb{X}_{\rm s}^{\Gamma,E}\cap\mathbb{X}_{\frac12}^{\Gamma,E}.$$

From (4.12) we have that

$$\|\Psi_{\Gamma,E}(0,\mathbf{u})\|_{H^1_\alpha}=\|\mathscr{K}^{\rm mod}_{\Gamma,E}D(\mathbf{u},\mathbf{u})\|_{H^1_\alpha}\le c(\Gamma,E)\|D(\mathbf{u},\mathbf{u})\|_{H^1_\alpha}\le c(\Gamma,E)\|\mathbf{u}\|^2_{H^1_\alpha} \tag{4.45}$$

for any $\mathbf{u}\in\Omega_\alpha(\varepsilon_2)$, which implies that $\partial_{\mathbf{u}}\Psi_{\Gamma,E}(0,0)=0$. From (4.44) it follows that $\big(\Sigma'_{\Gamma,E}(0)\big)\mathbf{v}_1=T_{\rm s}^{\Gamma,E}(\cdot)P_{\rm s}^{\Gamma,E}\mathbf{v}_1$ for any $\mathbf{v}_1\in\mathbb{X}_{\rm s}^{\Gamma,E}\cap\mathbb{X}_{\frac12}^{\Gamma,E}$. Since the linear operator $\mathrm{Tr}_0:H^1_\alpha(\mathbb{R}_+,\mathbb{X})\to\mathbb{X}$ defined by $\mathrm{Tr}_0f=f(0)$ is bounded, we have that

$$\big(\partial_{\mathbf{v}_0}\overline{\mathbf{u}}(0;0)\big)\mathbf{v}_1=\mathrm{Tr}_0\Sigma'_{\Gamma,E}(0)\mathbf{v}_1=P_{\rm s}^{\Gamma,E}\mathbf{v}_1\quad\text{for any}\quad \mathbf{v}_1\in\mathbb{X}_{\rm s}^{\Gamma,E}\cap\mathbb{X}_{\frac12}^{\Gamma,E}. \tag{4.46}$$

Since the $D(\cdot,\cdot)$ is a bounded, bilinear map on $\mathbb{X}$, from (4.24) and (4.46) we obtain that

$$\begin{aligned}\Big(\big(\mathscr{J}_{\rm s}^{\Gamma,E}\big)'(0)\Big)\mathbf{v}_1&=P_{\rm u}^{\Gamma,E}\big(\partial_{\mathbf{v}_0}\overline{\mathbf{u}}(0;0)\big)\mathbf{v}_1-2P_{\rm s}^{\Gamma,E}E^{-1}D\Big(\overline{\mathbf{u}}(0;0),\big(\partial_{\mathbf{v}_0}\overline{\mathbf{u}}(0;0)\big)\mathbf{v}_1\Big)\\&=P_{\rm u}^{\Gamma,E}P_{\rm s}^{\Gamma,E}\mathbf{v}_1-2P_{\rm s}^{\Gamma,E}E^{-1}D(0,P_{\rm s}^{\Gamma,E}\mathbf{v}_1)=0\end{aligned} \tag{4.47}$$

for any $\mathbf{v}_1\in\mathbb{X}_{\rm s}^{\Gamma,E}\cap\mathbb{X}_{\frac12}^{\Gamma,E}$. From (4.25) it follows immediately that the manifold $\mathscr{M}_{\rm s}^{\Gamma,E}$ is tangent to $\mathbb{X}_{\rm s}^{\Gamma,E}\cap\mathbb{X}_{\frac12}^{\Gamma,E}$ at $\mathbf{v}_0=0$. □

Theorem 4.7. *Assume Hypothesis (S). Then, for any integer* $r\ge1$ *there exists a local* C^r *stable manifold* $\mathscr{M}_{\rm s}^{\Gamma,E}$ *tangent to* $\mathbb{X}_{\rm s}^{\Gamma,E}\cap\mathbb{X}_{\frac12}^{\Gamma,E}$ *at the origin, expressible as* C^r *embeddings* $\mathscr{J}_{\rm s}^{\Gamma,E}$ *of* $\mathbb{X}_{\rm s}^{\Gamma,E}\cap\mathbb{X}_{\frac12}^{\Gamma,E}$ *with norm* $\|\cdot\|_{\mathbb{X}_{\frac12}^{\Gamma,E}}$ *into* $\mathbb{X}$ *with the standard norm, locally invariant under the forward flow of equation* $\Gamma\mathbf{u}'=E\mathbf{u}+D(\mathbf{u},\mathbf{u})$ *and expressible as the union of orbits of all mild solutions* $\mathbf{u}\in H^1(\mathbb{R}_+,\mathbb{X})$ *such that* $\mathbf{u}$ *is sufficiently small in* $H^1(\mathbb{R}_+,\mathbb{X})$ *norm.*

Proof. The theorem follows from Lemma 4.3(ii), Lemma 4.5 and the fact that equation $\mathbf{u}=\Psi_{\Gamma,E}(\mathbf{v}_0,\mathbf{u})$ has a unique solution on $H^1_\alpha(\mathbb{R}_+,\mathbb{X})$ for any $\mathbf{v}_0\in\overline{B}_{\rm s}(0,\varepsilon_1)$, $\alpha\in[0,\tilde\nu]$ and $\widetilde\nu\in(0,\nu(\Gamma,E))$. □

When proving results on existence of nonlinear stable/unstable manifolds in the case of first-order differential equations on finite dimensional spaces, the manifolds can be expressed as graphs of C^r functions from $\mathbb{H}_{\rm s/u}$ to $\mathbb{H}_{\rm u/s}\oplus\mathbb{H}_{\rm c}$, where $\mathbb{H}_{\rm s}$, $\mathbb{H}_{\rm u}$ and $\mathbb{H}_{\rm c}$ are the linear stable, unstable and center subspaces of the linearization along the equilibria at $+\infty$. In our case we can prove a similar result by combining the definitions of the function $\mathscr{J}_{\rm s}^{\Gamma,E}$ in (4.24) and of the manifold $\mathscr{M}_{\rm s}^{\Gamma,E}$ in (4.25).

Corollary 2. *Assume Hypothesis (S). Then,*

$$\mathscr{M}_{\rm s}^{\Gamma,E}=\big\{\mathbf{v}_0-P_{\rm s}^{\Gamma,E}E^{-1}D(\overline{\mathbf{u}}(0;\mathbf{v}_0),\overline{\mathbf{u}}(0;\mathbf{v}_0))+P_{\rm u}^{\Gamma,E}\overline{\mathbf{u}}(0;\mathbf{v}_0):\mathbf{v}_0\in B_{\rm s}(0,\varepsilon_1)\big\}. \tag{4.48}$$

Proof. To prove the corollary, we use the Inverse Function Theorem to solve for $\mathbf{v}_0$ in the $\mathbb{X}_{\rm s}^{\Gamma,E}$ component of elements of the manifold. Since the $D(\cdot,\cdot)$ is a bounded, bilinear map on $\mathbb{X}$, from (4.46) we obtain that the Fréchet derivative of the function $\mathscr{Y}_{\rm s}^{\Gamma,E}:\{\mathbf{v}_0\in\mathbb{X}_{\rm s}^{\Gamma,E}:\|\mathbf{v}_0\|\le\varepsilon_1\}\to\mathbb{X}_{\rm s}^{\Gamma,E}$ defined by $\mathscr{Y}_{\rm s}^{\Gamma,E}(\mathbf{v}_0)=\mathbf{v}_0-P_{\rm s}^{\Gamma,E}E^{-1}D(\overline{\mathbf{u}}(0;\mathbf{v}_0),\overline{\mathbf{u}}(0;\mathbf{v}_0))$ is $\big(\mathscr{Y}_{\rm s}^{\Gamma,E}\big)'(0)=I_{X_{\rm s}^{\Gamma,E}}$. It follows that ε_1 can be chosen small enough such that the function $\mathscr{Y}_{\rm s}^{\Gamma,E}$ is invertible on $\{\mathbf{v}_0\in\mathbb{X}_{\rm s}^{\Gamma,E}:\|\mathbf{v}_0\|\le\varepsilon_1\}$. From (4.48) we obtain that

$$\mathscr{M}_{\rm s}^{\Gamma,E}=\big\{\mathbf{v}_1+P_{\rm u}^{\Gamma,E}\overline{\mathbf{u}}\big(0;(\mathscr{Y}_{\rm s}^{\Gamma,E})^{-1}(\mathbf{v}_1)\big)\in\mathbb{X}_{\rm s}^{\Gamma,E}\oplus\mathbb{X}_{\rm u}^{\Gamma,E}:\mathbf{v}_1\in\mathscr{Y}_{\rm s}^{\Gamma,E}\big(B_{\rm s}(0,\varepsilon_1)\big)\big\}. \tag{4.49}$$

Thus, $\mathscr{M}_{\rm s}^{\Gamma,E}=\mathrm{Image}(\tilde{\mathscr{J}}_{\rm s}^{\Gamma,E})$, where the function $\tilde{\mathscr{J}}_{\rm s}^{\Gamma,E}:\mathscr{Y}_{\rm s}^{\Gamma,E}\big(B_{\rm s}(0,\varepsilon_1)\big)\to\mathbb{X}_{\rm u}^{\Gamma,E}$ is defined by $\tilde{\mathscr{J}}_{\rm s}^{\Gamma,E}(\mathbf{v}_1)=P_{\rm u}^{\Gamma,E}\overline{\mathbf{u}}\big(0;(\mathscr{Y}_{\rm s}^{\Gamma,E})^{-1}(\mathbf{v}_1)\big)$. □

Using Theorem 4.7 we can now prove the main result of this paper, the existence of stable and unstable manifolds of equation $Au_\tau=Q(u)$ near the equilibria $u^\pm$ at $\pm\infty$. We recall that the linear operator $S_\pm^{\rm r}=\big(A_{22}-A_{21}A_{11}^{-1}A_{12}\big)^{-1}Q'_{22}(u^\pm)$ generates an exponentially stable bi-semigroup on $\mathbb{V}$ (Theorem 1.2(i)) and that equation $Au'=Q'(u^\pm)u$ has an exponential trichotomy on $\mathbb{H}$, with stable/unstable subspaces denoted $\mathbb{H}_\pm^{\rm s/u}$ and center subspace $\mathbb{V}^\perp$ (Theorem 1.2(ii)). Moreover, we recall that the pair $(\Gamma,E)=(A_{22}-A_{21}A_{11}^{-1}A_{12},Q'_{22}(u^\pm))$ satisfies Hypothesis (S) by Lemma 2.1. In this case we have that $\mathbb{X}_{\frac12}^{\Gamma,E}=\mathrm{dom}(|S_{\Gamma,E}|^{\frac12})=\mathrm{dom}\big(|(A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1}Q'_{22}(u^\pm)|^{\frac12}\big)$. Finally, we introduce

$$-\nu_\pm:=-\nu\big(A_{22}-A_{21}A_{11}^{-1}A_{12},Q'_{22}(u^\pm)\big)<0$$

the decay rate of the exponentially stable bi-semigroup generated by the pair $\big(A_{22}-A_{21}A_{11}^{-1}A_{12},Q'_{22}(u^\pm)\big)$.

Proof of Theorem 1.3. Making the change of variables $w^\pm=u-u^\pm$ in equation $Au_\tau=Q(u)$ and denoting by $h^\pm=P_{\mathbb{V}^\perp}w^\pm$ and $v=P_{\mathbb{V}}w^\pm$, we obtain the system $\tilde{A}v_\tau^\pm=Q'_{22}(u^\pm)v^\pm+D(v^\pm,v^\pm)$, $h^\pm=-A_{11}^{-1}A_{12}v^\pm$, as shown in Section 3. Here $\tilde{A}=A_{22}-A_{21}A_{11}^{-1}A_{12}$ and the bilinear map $D:\mathbb{V}\times\mathbb{V}\to\mathbb{V}$ is defined by $D(v_1,v_2)=B(v_1-A_{11}^{-1}A_{12}v_1,v_2-A_{11}^{-1}A_{12}v_2)$, is bilinear and bounded on $\mathbb{V}$. Since the linear operator $S_\pm^{\rm r}$ generates an exponentially stable bi-semigroup on $\mathbb{V}$, the theorem follows from Theorem 1.2, Theorem 4.7 and Corollary 2.

Next, we show how we can use Theorem 4.7 to prove that any solution u^* of equation $Au_\tau=Q(u)$ satisfying the condition $u^*-u^\pm\in H^1(\mathbb{R}_+,\mathbb{X})$ converges exponentially to the equilibria $u^\pm$ at $\pm\infty$.

Proof of Corollary 1.4. First, we note that since u^* is a solution of equation $Au_\tau=Q(u)$, we have that $\tilde{A}(v^*-v^\pm)'=Q'_{22}(u^\pm)(v^*-v^\pm)+D(v^*-v^\pm,v^*-v^\pm)$ and $h^*=-A_{11}^{-1}A_{12}v^*$. Using the uniqueness property of solutions along the manifold $\mathscr{M}_{\rm s/u}^\pm$ given by Theorem 4.7, we conclude that $v^*-v^\pm\in H_\alpha^1(\mathbb{R}_\pm,\mathbb{V})$ for any $\alpha\in[0,\widetilde{\nu}]$, for some $\widetilde{\nu}\in(0,\min\{\nu_+,\nu_-\})$. Since $h^*=-A_{11}^{-1}A_{12}v^*$ we obtain that $u^*-u^\pm=$

$(v^* - v^\pm) - A_{11}^{-1}A_{12}(v^* - v^\pm)$, which implies that $u^* - u^\pm \in H^1_\alpha(\mathbb{R}_\pm, \mathbb{H})$, proving the corollary.

Appendix A. **Smooth dependence on parameters of fixed point solutions.** We include for completeness the following standard result, together with its proof, of smooth dependence on parameters of solutions of fixed point mappings.

Lemma A.1 (Lemma 2.4, [43])**.** *Assume $\mathbb{Y}$ and $\mathbb{Z}$ are two Banach spaces, and $\Psi : \mathbb{Y} \times \mathbb{Z} \to \mathbb{Z}$ is a continuous function such that $\Psi(y, \cdot)$ is (locally) contractive for any $y \in \mathbb{Y}$, defining a fixed point map $\mathbf{z} : \mathbb{Y} \to \mathbb{Z}$. If Ψ is C^r (Fréchet sense) on $\mathbb{Y} \times \mathbb{Z}$, $r \geq 1$, then $\mathbf{z}$ is C^k from $\mathbb{Y}$ to $\mathbb{Z}$.*

Proof of Lemma A.1. (following [43]) Expanding, we have

$$\begin{aligned}\|\mathbf{z}(y_2) - \mathbf{z}(y_1)\|_\mathbb{Z} &= \|\Psi(y_2, \mathbf{z}(y_2)) - \Psi(y_1, \mathbf{z}(y_1))\|_\mathbb{Z} \\ &\leq \|\Psi(y_2, \mathbf{z}(y_2)) - \Psi(y_2, \mathbf{z}(y_1))\|_\mathbb{Z} + \|\Psi(y_2, \mathbf{z}(y_1)) - \Psi(y_1, \mathbf{z}(y_1))\|_\mathbb{Z} \\ &\leq \theta\|\mathbf{z}(y_2) - \mathbf{z}(y_1)\|_\mathbb{Z} + L\|y_2 - y_1\|_\mathbb{Y},\end{aligned} \tag{A.1}$$

where $0 < \theta < 1$ and $0 < L$ are contraction and Lipschitz coefficients, yielding after rearrangement $\|\mathbf{z}(y_2) - \mathbf{z}(y_1)\|_\mathbb{Z} \leq \frac{L}{1-\theta}\|y_2 - y_1\|_\mathbb{Y}$. Applying Taylor's Theorem, we thus have

$$\begin{aligned}\mathbf{z}(y_2) - \mathbf{z}(y_1) &= \Psi_y(y_2 - y_1) + \Psi_z(\mathbf{z}(y_2) - \mathbf{z}(y_1)) + o(\|y_2 - y_1\|_\mathbb{Y} \\ &\qquad + \|\mathbf{z}(x_2) - \mathbf{z}(y_1)\|_\mathbb{Z}) \\ &= \Psi_y(y_2 - y_1) + \Psi_y(\mathbf{z}(y_2) - \mathbf{z}(y_1)) + o(\|y_2 - y_1\|_\mathbb{Y}),\end{aligned} \tag{A.2}$$

where all derivatives are evaluated at $(x_1, \mathbf{z}(x_1))$. Noting that the operator norm $\|\Psi_z\|$ is bounded by the contraction coefficient $0 < \theta < 1$, we have by Neumann series expansion that $(\mathrm{Id} - \Psi_z)$ is invertible with uniformly bounded inverse and $\|(\mathrm{Id} - \Psi_z)^{-1}\| \leq (1 - \theta)^{-1}$. Thus, rearranging, we have

$$\mathbf{z}(y_2) - \mathbf{z}(y_1) = (\mathrm{Id} - \Psi_z)^{-1}\Psi_y(y_2 - y_1) + (\|y_2 - y_1\|_\mathbb{Y}),$$

yielding the result for $r = 1$ by definition of (Frechet) derivative, with

$$\mathbf{z}_y = (\mathrm{Id} - \Psi_z)^{-1}\Psi_y(y, \mathbf{z}(y)). \tag{A.3}$$

The results for $r \geq 1$ then follow by induction upon differentiation of (A.3). $\square$

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