Nonlinear Analysis: Hybrid Systems 33 (2019) 195–210

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Stability analysis of a class of switched nonlinear systems using the time scale theory

Fatima Z. Taousser [a,*], Michael Defoort [b], Mohamed Djemai [b], Seddik M. Djouadi [a], Kevin Tomsovic [a]

[a] Department of Electrical Engineering and Computer Science, University of Tennessee, Knoxville, TN 37996, USA
[b] Universite Polytechnique. Hauts-De-France, 59313 Valenciennes, France

ARTICLE INFO

Article history:
Received 16 May 2018
Accepted 11 February 2019
Available online xxxx

Keywords:
Time scales
Switched systems
Stability analysis
Multi-agent systems
Consensus

ABSTRACT

This paper investigates stability for a class of switched nonlinear systems evolving on a non-uniform time domain. The studied class of systems switch between a nonlinear continuous-time and a nonlinear discrete-time subsystem. This problem is formulated using time scale theory where the latter is formed by a union of disjoint intervals with variable lengths and variable gaps. Using Gronwall's inequality and properties of the time scale exponential function, sufficient conditions are derived to ensure exponential stability. The results can be applied to cases where the continuous-time subsystem or the discrete-time subsystem is not necessarily stable, and the state matrices of each subsystem are not necessarily pairwise commuting. To illustrate the effectiveness of the proposed scheme, the consensus problem for general linear multi-agent systems under intermittent information transmissions is studied under this framework.

1. Introduction

Switched systems consists of a family of subsystems and a switching rule that orchestrates the switching between them. They can be found in many areas such as network control systems, computer science, electrical engineering, etc. Stability and stabilization of this class of systems are one of the most important studied problems. During the last two decades, a number of approaches have been proposed [1–4]. They usually consider either continuous-time subsystems [5] or discrete-time subsystem [6,7], but are studied separately.

However, there are many applications of switched systems where their temporal nature cannot be represented by the continuous line or a discrete uniform time domain. Indeed, a closed-loop system consisting of a continuous-time system and an intermittent controller is one application. The class of impulsive systems with non-instantaneous jumps can also be considered. The consensus problem under intermittent information due to communication obstacles and limitations of sensors is another example [8,9]. One can note that the network control systems with data packet dropout due to unreliable transmissions, can be modeled as a linear system with time-varying delay. For time delay systems, usually, based on Lyapunov–Krasovskii theory some LMI-based conditions are derived. However, the required communication time rate conditions are quite complex to compute [10,11]. To reduce such limitations, time scale theory was introduced in our previous works [9,12], where the consensus problem with intermittent information transmissions was converted to an asymptotic stabilization problem of switched systems which evolve on a mixed continuous/discrete time domain. In [9],

* Corresponding author.
E-mail addresses: ftaousse@tennessee.edu (F.Z. Taousser), michael.defoort@uphf.fr (M. Defoort), mohamed.djemai@uphf.fr (M. Djemai), mdjouadi@utk.edu (S.M. Djouadi), ktomsovi@utk.edu (K. Tomsovic).

https://doi.org/10.1016/j.nahs.2019.02.006
1751-570X/

the problem was studied assuming the commutativity condition of the matrices of the dynamical subsystems. In [12], it was studied using a common Lyapunov function, which cannot be easily computed, and requires that all subsystems are exponentially stable. Motivated by these results, in this paper, new conditions for exponential stability of this class of switched systems are derived by removing the commutativity condition of the matrices and without assuming exponential stability of all subsystems.

Time scale theory is a very useful theory, since it is an appropriate tool to study continuous-time and discrete-time systems in a unified framework [13–15]. In general, the continuous and the discrete dynamical systems are studied separately and most of the results have to be proved for each case (using continuous analysis or discrete analysis). Time scale theory can unify the continuous and the discrete analysis in one formula. Hence, we can prove a result only once and that will be true for the continuous case and the discrete one. In addition, this theory extends the study of dynamical systems on any non-uniform time domain (discrete non-uniform or a combination between discrete points and continuous intervals), which is very useful in the study of complex dynamical systems. It allows analyzing the stability of dynamical systems on non-uniform time domains which are closed subsets of $\mathbb{R}$. Using this theory, time scale dynamic equations become continuous differential equations when time domain $\mathbb{R}$ is considered and difference equations when time domain $h\mathbb{Z}$ $(h > 0)$ is considered. Recently, the stability of dynamic equations on time scale has been investigated in many works. For instance, in [16–19], the authors used the generalized exponential function to study linear systems on an arbitrary time domain. Uncertain dynamic equations [20,21] and a class of nonlinear systems [22] were also studied using time scale theory. A stochastic approach has been proposed in [23,24]. However, the extension of these results to the class of switched nonlinear systems is not trivial.

Using a common quadratic Lyapunov function, a class of switched normal linear systems, consisting of two subsystems evolving on the continuous-time and discrete uniform time domains was studied in [25]. Recently, using time scale theory, sufficient conditions were proposed for the analysis of a class of switched linear systems which consists of a set of linear continuous-time and discrete-time subsystems in [9,12,26–28]. In [29], necessary and sufficient conditions for exponential stability for this class of switched systems were derived by determining a region of exponential stability using the generalized exponential function properties. Nevertheless, it is required in these papers that the matrices of each subsystem are pairwise commuting. The case of simultaneously triangularizable switched systems evolving on an arbitrary time domain was considered in [30] using a common Lyapunov function. However, the approaches proposed in [12,26,30,31] fail if one subsystem is not exponentially stable. In [28], a class of switched systems where all subsystems are unstable was studied.

The objective of this paper is to extend the results in [12], where the stability analysis of a class of nonlinear switched systems consisting of an exponentially stable continuous-time and exponentially stable discrete-time subsystems, evolving on a time domain formed by a union of disjoint intervals with variable lengths and gaps is studied by introducing a common quadratic Lyapunov function (CQLF). The existence of a CQLF requires that all subsystems are stable and in several cases, the CQLF may not exist. In order to overcome these restrictions, existing results will be generalized to a large class of nonlinear systems, where the matrices of the nominal linear system are not commutative and/or unstable. Sufficient conditions are proposed to ensure stability using the solution of the system and by introducing the Gronwall's inequality and properties of the generalized exponential function. One should highlight that similarly to previous results [9,12,27], the switching times are supposed to be known. This means that the time scale is given. To illustrate the effectiveness of the proposed method, the consensus problem for general linear multi-agent systems under intermittent information transmissions is considered.

The rest of this paper is organized as follows. Section 2 recalls some preliminaries on time scale theory. The studied class of systems is described in Section 3. In Section 4, sufficient conditions which ensure the exponential stability of the nonlinear switched systems are introduced using the Gronwall's inequality and some properties of the time scale exponential function. To illustrate the effectiveness of the proposed scheme, the consensus problem for general linear multi-agent systems under intermittent information transmissions is studied in Section 5.

2. Preliminaries

We recall some basics on time scale theory (for more details see [14,30]). A *time scale* $\mathbb{T}$ is any closed subset of $\mathbb{R}$. The discrete set $h\mathbb{Z}$ $(h > 0)$ with fixed step h, $\mathbb{N}$, the real numbers $\mathbb{R}$, any discrete subset or any combination of discrete points and closed intervals, are examples of time scales. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. If $\sup(\mathbb{T}) = M < \infty$ (i.e, finite), $\sigma(M) = M$. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. The *graininess function* $\mu : \mathbb{T} \to \mathbb{R}^+$, is defined by $\mu(t) = \sigma(t) - t$ (μ measures the distance between two consecutive points). A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$, *left-scattered* if $\rho(t) < t$, *right-dense* if $\sigma(t) = t$ and *left-dense* if $\rho(t) = t$ (Fig. 1). If $\sup \mathbb{T} = m$ is left-scattered, then $\mathbb{T}^\kappa := \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. The *$\Delta$-derivative* of $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}^\kappa$ is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}. \tag{1}$$

The Δ-derivative generalizes the derivative in the continuous sense and the difference operator in the discrete sense. If $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$ and $f^{\Delta}(t) = \dot{f}(t)$. If $\mathbb{T} = h\mathbb{Z}$, $\sigma(t) = t + h$ and $f^{\Delta}(t) = \frac{f(t+h)-f(t)}{h}$. In particular, if $h = 1$, $f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t)$, the difference operator.

Fig. 1. (a) t is right-scattered and left-dense. (b) t is right-dense and left-scattered. (c) t is dense. (d) t is isolated.

A function $f : \mathbb{T} \to \mathbb{R}$ is *rd-continuous*, if it is continuous at every right-dense point in $\mathbb{T}$ and its left-hand limit exists at every left-dense points in $\mathbb{T}$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is regressive, if $1 + \mu(t)f(t) \neq 0,\ \forall t \in \mathbb{T}^\kappa$ and f is called positively regressive if it satisfies $1 + \mu(t)f(t) > 0,\ \forall t \in \mathbb{T}^\kappa$. The set of all rd-continuous and regressive (respectively. positively regressive) functions is denoted by $\mathcal{R}$ (resp. $\mathcal{R}^+$). Similarly, a matrix function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is called *regressive*, if $I + \mu(t)A(t)$ is invertible, $\forall t \in \mathbb{T}^\kappa$, where I is the identity matrix. Equivalently, $A(t)$ is regressive if and only if all its eigenvalues $\lambda_i(t)$ are regressive (i.e, $1 + \mu(t)\lambda_i(t) \neq 0,\ \forall 1 \le i \le n,\ \forall t \in \mathbb{T}$). The set $\mathcal{R}$ together with the *circle addition* $\oplus$ defined by $(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t)$, $p, q \in \mathcal{R}$, $t \in \mathbb{T}$ is an Abelian group. The neutral element is zero and the inverse element of p is $\ominus p(t) = \frac{-p(t)}{1+\mu(t)p(t)}$. The *circle subtraction* is defined by $(p \ominus q)(t) = p(t) \oplus (\ominus q(t)) = \frac{p(t)-q(t)}{1+\mu(t)q(t)}$. Note that if $p, q \in \mathcal{R}$, then $\ominus p$, $p \oplus q$, $p \ominus q$, $q \ominus p \in \mathcal{R}$.

The *generalized exponential function* of $p \in \mathcal{R}$ on time scale $\mathbb{T}$ is expressed by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad s,t \in \mathbb{T} \tag{2}$$

where $\xi_{\mu(t)}(z) = \begin{cases} \frac{\log(1+\mu(t)z)}{\mu(t)} & \text{if} \quad \mu(t) \neq 0 \\ z & \text{if} \quad \mu(t) = 0. \end{cases}$

The log is the principal logarithm function and the delta integral is used [32].

The generalized exponential function $e_p(t, t_0)$ generalizes the standard real exponential, since for $\mathbb{T} = \mathbb{R}$ and p constant, we have $e_p(t, t_0) = e^{p(t-t_0)}$ and for a non uniform discrete time scale with graininess function $\mu(t)$, the exponential function is given by $e_p(t_k, t_0) = \prod_{i=0}^{k}(1 + \mu(t_i)p(t_i))$.

The exponential function has the properties, for $p, q \in \mathcal{R}$, $t, s \in \mathbb{T}$

$$e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t), \quad e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s).$$

If $p \in \mathcal{R}^+$ such that $p(t) \le q(t)$, then $e_p(t,s) \le e_q(t,s)$, $\forall t \ge s,\ t,s \in \mathbb{T}$.

Let A be a regressive matrix. The unique matrix-valued solution of

$$x^\Delta(t) = A(t)\,x(t), \quad x(t_0) = x_0, \quad t, t_0 \in \mathbb{T},$$

is the generalized exponential function denoted by $e_A(t, t_0)x_0$.

Let $\mathbb{T}$ be an arbitrary time scale. Since we are interested in an asymptotic theory, $\mathbb{T}$ is supposed to be unbounded above, and the graininess function μ is assumed to be bounded [33]. Consider the dynamical system

$$x^\Delta(t) = R(t,x), \quad x(t_0) = x_0, \quad t, t_0 \in \mathbb{T}, \tag{3}$$

with $R : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous function in the first argument with $R(t, 0) = 0$ and satisfies all conditions such that (3) has a unique solution $x(t)$ with $x(t_0) = x_0$ (see [33] for more details). System (3) is said to be exponentially stable, if there exists a constant $\beta = \beta(t_0) \ge 1$ and a negative constant α with $\alpha \in \mathcal{R}^+$, such that the corresponding solution satisfies

$$\|x(t)\| \le \beta\|x_0\|e_\alpha(t,t_0), \quad \forall t, t_0 \in \mathbb{T}, \quad t \ge t_0. \tag{4}$$

If β can be chosen independently of t_0, then system (3) is said to be uniformly exponentially stable [20]. This characterization is a generalization of the definition of exponential stability for dynamical systems defined on $\mathbb{R}$ or $h\mathbb{Z}$. More specifically, the condition that $\alpha < 0$ and $\alpha \in \mathcal{R}^+$ in this characterization is reduced to $\alpha < 0$ for $\mathbb{T} = \mathbb{R}$, and to $0 < 1 + \mu(t)\alpha < 1$, $\forall t \in \mathbb{T}$, for any discrete time scale $\mathbb{T}$ with graininess function $\mu(t)$. Since the generalized exponential function can be negative, the positive regressivity of α is needed (see [13]).

Remark 1. Note that, there are several definitions of exponential stability of dynamical systems on time scales in the literature. In [19] (respectively [16] and [34]), the authors have defined the exponential stability via the standard exponential function $e^{\alpha(t-t_0)}$ (respectively $e_{-\alpha}(t, t_0)$ and $e_{\ominus\alpha}(t, t_0)$). Nevertheless, we have the following relationship between the different exponential functions (see [34]),

$$e_{-\alpha}(t,t_0) \le e^{-\alpha(t-t_0)} \le e_{\ominus\alpha}(t,t_0), \qquad \text{for } t, t_0 \in \mathbb{T};\ t \ge t_0;\ \alpha > 0 \text{ and } -\alpha \in \mathcal{R}^+,$$

Hence, in this paper, we will use the exponential function $e_\alpha(t, t_0)$ for the exponential stability, which gives a better exponential decay estimate of $x(t)$.

For the stability analysis of the linear dynamic system on time scale $\mathbb{T}$,

$$x^{\Delta}(t) = Ax(t), \quad x(t_0) = x_0, \quad t, t_0 \in \mathbb{T}, \tag{5}$$

a set of exponential stability is determined in [19] (using the definition of the exponential stability via the standard exponential function $e^{\alpha(t-t_0)}$) as,

$$\mathcal{S}(\mathbb{T}) := \{\lambda \in \mathbb{C} : \limsup_{T\to\infty} \frac{1}{T-t_0} \int_{t_0}^{T} \lim_{s\to\mu(t)} \frac{\log|1+s\lambda|}{s} \Delta t < 0\}.$$

However, since $\mathcal{S}(\mathbb{T})$ can be difficult to compute on general time scales, other more tractable sufficient conditions for the exponential stability of (5) have been explored. We define the *Hilger disc* for all $t \in \mathbb{T}$ as

$$\mathcal{H}_{\mu(t)} := \left\{ z \in \mathbb{C} : \; |1 + z\mu(t)| < 1, \; z \neq -\frac{1}{\mu(t)} \right\}.$$

When $\mu(t) = 0$, we define $\mathcal{H}_0 = \{z \in \mathbb{C} : Re(z) < 0\} = \mathbb{C}^-$, the open left-half complex plane. The smallest Hilger disc (denoted $\mathcal{H}_{\min}$) is the Hilger disc associated with $\mu_{\max} = \sup_{t\in\mathbb{T}} \mu(t) < \infty$. A regressive constant matrix A is called *Hilger stable* if $spec(A) \subset \mathcal{H}_{\min}$ (i.e all eigenvalues of A are in $\mathcal{H}_{\min}$) [30]. In [35] it is shown that the Hilger disc $\mathcal{H}_{\min}$ is a subset of the region of exponential stability $\mathcal{S}(\mathbb{T})$. Notice that Hilger discs are not all required to be subset of $\mathcal{S}(\mathbb{T})$, but $\mathcal{H}_{\min} \subset \mathcal{S}(\mathbb{T})$.

Let us recall some basics on nonlinear dynamical systems on time scale. Let $\mathbb{T}$ be an arbitrary time scale. The function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be rd-continuous, if each component of f is rd-continuous. The function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be rd-continuous, if $g(t) = f(t, x(t))$ is rd-continuous for all continuous function $x : \mathbb{T} \to \mathbb{R}^n$. The function f is said to be Lipschitz continuous, if there exist a constant $L > 0$, such that, $|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$, for all $(t, x_1), (t, x_2) \in \mathbb{T} \times \mathbb{R}^n$.

Theorem 1 (*Variation of Constants [13]*)**.** *Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on $\mathbb{T}$. Suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then, the initial value problem*

$$x^{\Delta}(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0 \tag{6}$$

has a unique solution $x : \mathbb{T} \to \mathbb{R}^n$ given by

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(s))f(s)\Delta s. \tag{7}$$

We have the following property [13]:

$$\int_{t_k}^{\sigma(t_k)} f(s)\Delta s = \mu(t_k)f(t_k). \tag{8}$$

Let us present the Gronwall inequalities on time scale.

Theorem 2 (*Gronwall Inequality on Time Scale [36]*)**.** *Let $x,\ f,\ p$ be rd-continuous functions and $p \geq 0$. Then,*

$$x(t) \leq f(t) + \int_{t_0}^{t} x(s)p(s)\Delta s, \;\; \forall t, t_0 \in \mathbb{T}$$

implies that

$$x(t) \leq f(t) + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)p(s)\Delta s, \;\; \forall t, t_0 \in \mathbb{T}.$$

Corollary 1 (*[36]*)**.** *Let $x,\ p$ be rd-continuous functions, $x_0 \in \mathbb{R}$ and $p \geq 0$. Then,*

$$x(t) \leq x_0 + \int_{t_0}^{t} p(s)x(s)\Delta s, \;\; \forall t, t_0 \in \mathbb{T}$$

implies that

$$x(t) \leq x_0 \; e_p(t, t_0), \;\; \forall t, t_0 \in \mathbb{T}.$$

3. Problem statement

This paper is concerned with the stability analysis of a class of switched nonlinear systems evolving on the time scale, formed by a union of disjoint intervals with variable lengths and gaps. Let $\{t_k\}_{k\in\mathbb{N}}$ be a monotonically increasing sequence without accumulation points. Let $t_k < \tau_k < t_{k+1}$, $\forall k \in \mathbb{N}^*, (\mathbb{N}^* = \mathbb{N}/\{0\})$ and $\tau_0 = t_0$. Consider the time scale $\mathbb{T} = \cup_{k=0}^{\infty}[\tau_k, t_{k+1}]$. So, the forward jump operator $\sigma(.)$ and the graininess function $\mu(.)$ of $\mathbb{T}$ are given by, $\forall k \in \mathbb{N}$,

$$\sigma(t) = \begin{cases} t, & \text{for } \tau_k \le t < t_{k+1} \\ \tau_{k+1}, & \text{for } t = t_{k+1} \end{cases} \quad \text{and} \quad \mu(t) = \begin{cases} 0, & \text{for } \tau_k \le t < t_{k+1} \\ \tau_{k+1} - t_{k+1}, & \text{for } t = t_{k+1} \end{cases}$$

Let us rewrite $\mathbb{T}$ by using its forward jump operator $\sigma(.)$ as follows,

$$\mathbb{T} = \mathbb{P}_{\{\sigma(t_k), t_{k+1}\}} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}], \tag{9}$$

The system considered is described by

$$x^{\Delta}(t) = \begin{cases} A_c x(t) + f(t, x(t)) & \text{for} \quad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ A_d x(t) + g(t, x(t)) & \text{for} \quad t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases} \tag{10}$$

where $t_0 = \sigma(t_0) = 0$ is the initial time, $x(t) \in \mathbb{R}^n$ is the state, $A_c \in \mathbb{R}^{n\times n}$ and $A_d \in \mathbb{R}^{n\times n}$ are constant regressive matrices. One can notice that there are perturbations (or uncertainties) which act on the continuous and discrete parts of the switched system and are modeled by the rd-continuous functions $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$. Let $F_x : \mathbb{T} \to \mathbb{R}^n$, be defined by

$$F_x(t) = \begin{cases} f(t, x(t)) & \text{for} \quad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ g(t, x(t)) & \text{for} \quad t \in \cup_{k=0}^{\infty}\{t_{k+1}\}, \end{cases} \tag{11}$$

where x at the index of F is the state of the switched system (10), which indicates dependence of the perturbation function F on x. Suppose that the conditions of existence and uniqueness of the solution of each subsystem in (10) are satisfied (i.e, f and g are rd-continuous, bounded and Lipschitz continuous [13]).

The first equation of (10) describes the continuous-time dynamics of the system, and the second one describes the discrete-time dynamic of the non instantaneous state jumps. One should notice that at times $t = t_{k+1}$, $k \in \mathbb{N}$, the discrete-dynamic of the state is determined by the Δ-derivative, such that

$$x^{\Delta}(t_{k+1}) = \frac{x(\sigma(t_{k+1})) - x(t_{k+1})}{\sigma(t_{k+1}) - t_{k+1}} = \frac{x(\sigma(t_{k+1})) - x(t_{k+1})}{\mu(t_{k+1})} = A_d x(t) + g(t, x(t)), \quad \forall k \in \mathbb{N},$$

where $\mu(t_{k+1})$ corresponds to the time needed for the state to jump from $x(t_{k+1})$ to $x(\sigma(t_{k+1}))$. Hence, the dynamical system commutes between a continuous-time subsystem and a discrete-time subsystem. The switching times from the continuous subsystem to the discrete one are $\{t_{k+1}\}_{k\in\mathbb{N}}$ and from the discrete subsystem to the continuous one are $\{\sigma(t_{k+1})\}_{k\in\mathbb{N}}$.

Assumption 1. It is assumed throughout the paper that $\mathbb{T}$ is unbounded above and the graininess function is bounded as follows,

$$0 < \mu_{\min} \le \mu(t) \le \mu_{\max} < \infty, \;\; \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \tag{12}$$

and that

$$0 < \tau_{\min} \le \tau_k = t_{k+1} - \sigma(t_k) \le \tau_{\max} < \infty, \;\; \forall k \in \mathbb{N} \tag{13}$$

4. Stability analysis of the nonlinear switched system

Before studying the stability of the switched nonlinear system (10), the solution of (10) will be estimated. Then, the stability of the switched nominal system (i.e. with $F_x = 0$) will be studied. Based on this result, sufficient conditions are derived to ensure the exponential stability of the switched nonlinear systems (10).

4.1. Preliminary results

Consider the switched system (10). Using time scale theory, we can derive the general expression of the solution of (10) as follows:

For $t \in [\sigma(t_k), t_{k+1}[$, $k \in \mathbb{N}$,

$$\begin{aligned} x(t) &= e^{A_c(t-\sigma(t_k))}x(\sigma(t_k)) + \textstyle\int_{\sigma(t_k)}^{t} e^{A_c(t-s)}f(s, x(s))ds \\ &= e^{A_c(t-\sigma(t_k))}x(\sigma(t_k)) + \textstyle\int_{\sigma(t_k)}^{t} e^{A_c(t-\sigma(s))}f(s, x(s))\Delta s \end{aligned}$$

For $t = \sigma(t_k)$, $k \in \mathbb{N}$,

$$\begin{aligned} x(\sigma(t_k)) &= e_{A_d}(\sigma(t_k), t_k)x(t_k) + \textstyle\int_{t_k}^{\sigma(t_k)} e_{A_d}(\sigma(t_k), \sigma(s))g(s, x(s))\Delta s \\ &= (I + \mu(t_k)A_d)x(t_k) + \mu(t_k)g(t_k, x(t_k)) \end{aligned}$$

The solution of the nonlinear switched system (10) can be split as follows

- For $t_0 \leq t \leq t_1$, one gets

$$x(t) = e^{A_c(t-t_0)}x_0 + \int_{t_0}^{t} e^{A_c(t-s)}f(s, x(s))ds$$

Thus, for $t = t_1$

$$x(t_1) = e^{A_c(t_1-t_0)}x_0 + \int_{t_0}^{t_1} e^{A_c(t_1-s)}f(s, x(s))ds$$

- For $t = \sigma(t_1)$, one has

$$\begin{aligned} x(\sigma(t_1)) &= (I + \mu(t_1)A_d)x(t_1) + \mu(t_1)g(t_1, x(t_1)) \\ &= (I + \mu(t_1)A_d)\left[e^{A_c(t_1-t_0)}x_0 + \textstyle\int_{t_0}^{t_1} e^{A_c(t_1-s)}f(s, x(s))ds\right] + \mu(t_1)g(t_1, x(t_1)) \end{aligned}$$

- For $\sigma(t_1) \leq t \leq t_2$, we have

$$\begin{aligned} x(t) &= e^{A_c(t-\sigma(t_1))}x(\sigma(t_1)) + \textstyle\int_{\sigma(t_1)}^{t} e^{A_c(t-s)}f(s, x(s))ds \\ &= e^{A_c(t-\sigma(t_1))}(I + \mu(t_1)A_d)e^{A_c(t_1-t_0)}x_0 + \textstyle\int_{t_0}^{t_1} e^{A_c(t-\sigma(t_1))}(I + \mu(t_1)A_d)e^{A_c(t_1-s)}f(s, x(s))ds \\ &+ \mu(t_1)e^{A_c(t-\sigma(t_1))}g(t_1, x(t_1)) + \textstyle\int_{\sigma(t_1)}^{t} e^{A_c(t-s)}f(s, x(s))ds \\ &= e^{A_c(t-\sigma(t_1))}(I + \mu(t_1)A_d)e^{A_c(t_1-t_0)}x_0 + \textstyle\int_{t_0}^{t_1} e^{A_c(t-\sigma(t_1))}(I + \mu(t_1)A_d)e^{A_c(t_1-s)}f(s, x(s))ds \\ &\quad + \textstyle\int_{t_1}^{t} e^{A_c(t-\sigma(s))}F_x(s)\Delta s. \end{aligned}$$

By induction, one can show that for $\sigma(t_k) \leq t \leq t_{k+1}$, the solution of (10) can be expressed as

$$\begin{aligned} &x(t) = e^{A_c(t-\sigma(t_k))}\textstyle\prod_{i=0}^{k-1}(I + \mu(t_{k-i})A_d)e^{A_c(t_{k-i}-\sigma(t_{k-i-1}))}x_0 + e^{A_c(t-\sigma(t_k))}\times \\ &\textstyle\sum_{n=0}^{k-1}\int_{t_{k-n-1}}^{t_{k-n}}\prod_{i=0}^{n-1}(I + \mu(t_{k-i})A_d)e^{A_c(t_{k-i}-\sigma(t_{k-i-1}))}(I + \mu(t_{k-n})A_d)e^{A_c(t_{k-n}-\sigma(s))}\, F_x(s)\Delta s \\ &+ \textstyle\int_{t_k}^{t} e^{A_c(t-\sigma(s))}\, F_x(s)\Delta s \end{aligned} \tag{14}$$

In order to study the stability of the perturbed switched system (10), we will propose in the next subsection, sufficient stability conditions for the nominal linear switched system (i.e. $f = g = 0$). These conditions will be derived where the continuous-time subsystem or the discrete-time subsystem are not necessarily exponentially stable, and the state matrices of each subsystem are not necessarily pairwise commuting.

4.2. Stability analysis of the nominal switched system

Suppose that $f = g = 0$. The corresponding nominal switched linear system becomes

$$x^{\Delta}(t) = \begin{cases} A_c x(t), & \text{for} \quad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ A_d x(t), & \text{for} \quad t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases} \tag{15}$$

From (14), the explicit solution of (15), for $\sigma(t_k) \leq t \leq t_{k+1}$, $k \in \mathbb{N}$, is given by

$$x(t) \quad = e^{A_c(t-\sigma(t_k))}e_{A_d}(\sigma(t_k), t_k)\ \ldots\ e_{A_d}(\sigma(t_1), t_1)e^{A_c t_1}x_0 \tag{16}$$

Let us define the constants α_c and $\alpha_d \in \mathcal{R}^+$, with the corresponding constants $\beta_c, \beta_d \geq 1$ such that, $\forall k \in \mathbb{N}$, for $\sigma(t_k) \leq s \leq t \leq t_{k+1}$,

$$\begin{aligned} \|e_{A_c}(t, s)\| &\leq \beta_c\, e_{\alpha_c}(t, s) &&= \beta_c\, e^{\alpha_c(t-s)} \\ \|e_{A_d}(\sigma(t_k), t_k)\| &\leq \beta_d\, e_{\alpha_d}(\sigma(t_k), t_k) &&= \beta_d\,(1 + \mu(t_k)\alpha_d). \end{aligned} \tag{17}$$

Remark 2. The constants $\alpha_c, \alpha_d, \beta_c$ and β_d always exist. Indeed, they can be defined as $\alpha_c \geq \Re(\lambda_c) = \max_{1\leq j\leq n}\{\Re(\lambda_c^j),\ \lambda_c^j \in spec(A_c)\}$ and $(1 + \mu(t)\alpha_d) \geq \max_{1\leq j\leq n}\{|1 + \mu(t)\lambda_d^j|,\ \lambda_d^j \in spec(A_d)\},\ \forall\ t \in \cup_{k=1}^{\infty}\{t_k\}$, where $spec(A)$ is the spectrum of A and $\Re(\lambda)$ is the real part of λ (see [19]). The above two inequalities can be replaced by equalities, if A_c (resp. A_d) is diagonalizable. Note that, the constants β_c and β_d can be computed using the eigenvectors of A_c and A_d.

One can derive an upper bound for the solution of (16), as follows

$$\begin{aligned} \|x(t)\| &\le \|e^{A_c(t-\sigma(t_k))}\| \|e_{A_d}(\sigma(t_k), t_k)\| \|e^{A_c(t_k-\sigma(t_{k-1}))}\| \ \dots \ \|e_{A_d}(\sigma(t_1), t_1)\| \|e^{A_c t_1}\| \ \|x_0\| \\ &\le \beta_c^{k+1} \ \beta_d^k e^{\alpha_c(t-\sum_{i=1}^k \mu(t_i))} e_{\alpha_d}(t, t_0) \ \|x_0\| \\ &= \beta_c^{k+1} \ \beta_d^k e^{\alpha_c(t-\sum_{i=1}^k \mu(t_i))} \prod_{i=1}^k (1+\mu(t_i)\alpha_d) \ \|x_0\| \\ &\le \beta^{2k+1} e^{\alpha_c(t-\sum_{i=1}^k \mu(t_i))} \prod_{i=1}^k (1+\mu(t_i)\alpha_d) \ \|x_0\| \\ &= \beta \ (\beta^2)^k \ e^{\alpha_c(t-\sum_{i=1}^k \mu(t_i))} \ \prod_{i=1}^k (1+\mu(t_i)\alpha_d) \ \|x_0\|. \end{aligned} \tag{18}$$

with $\beta = \max\{\beta_c, \beta_d\} \ge 1$. In the following, we will derive sufficient conditions for exponential stability of the switched system (15) by investigating different cases depending on whether the continuous-time or the discrete-time subsystem are Hilger stable or unstable.

Remark 3. In this paper, we deal with exponential stability defined in (4). Notice that if $\alpha_c < 0$, then A_c is exponentially stable (which is equivalent to Hilger stable in the continuous case) and if $\alpha_d < 0$, with $\alpha_d \in \mathcal{R}^+$, then A_d is exponentially stable. This can be satisfied, if all the eigenvalues of A_d lies strictly in the Hilger disc $\mathcal{H}_{\min}$ (i.e A_d is Hilger stable). A_d is unstable if it has at least one eigenvalue with positive real part or if all the eigenvalues of A_d have negative real part, and there exists at least one eigenvalue λ_d^j such that $|1+\mu_{\min}\lambda_d^j| > 1$. In this case $\alpha_d > 0$.

Theorem 3. *Let the constants β, α_c and α_d be defined as in (18). Suppose that Assumption 1 is satisfied and one of the following assumptions holds*

(i) A_c and A_d are Hilger stable, with

$$\max_{i\in\mathbb{N}}(1+\mu(t_i)\alpha_d) < \frac{1}{\beta^2}, \quad (i.e, (1+\mu_{\min}\alpha_d) < \frac{1}{\beta^2}). \tag{19}$$

(ii) A_c and A_d are Hilger stable and

$$\alpha_c < \frac{-\log(\beta^2)}{\tau_{\min}} \tag{20}$$

(iii) A_c is Hilger stable and A_d can be Hilger stable or unstable and

$$\max_{i\in\mathbb{N}}(1+\mu(t_i)\ \alpha_d) < e^{-[\alpha_c\ \tau_{\min}\ +\log(\beta^2)]} \tag{21}$$

(iv) A_c is unstable, A_d is Hilger stable and

$$\max_{i\in\mathbb{N}}(1+\mu(t_i)\alpha_d) \le e^{[-\alpha_c\ \tau_{\max}-\log(\beta^2)]}, \quad (i.e, (1+\mu_{\min}\alpha_d) < e^{[-\alpha_c\ \tau_{\max}-\log(\beta^2)]}). \tag{22}$$

Then, the switched system (15) is exponentially stable.

Proof. To study the stability of the linear switched system (15), let us determine, $\alpha < 0$ with $\alpha \in \mathcal{R}^+$, such that, the solution $x(t)$, for $\sigma(t_k) \le t \le t_{k+1}$, $\forall k \in \mathbb{N}$, is bounded as follows,

$$\|x(t)\| \le \beta \ e^{\alpha(t-\sum_{i=1}^k \mu(t_i))} \prod_{i=1}^k (1+\mu(t_i)\alpha) \ \|x_0\| = \beta e_\alpha(t, t_0) \ \|x_0\|. \tag{23}$$

(i) Suppose that A_c and A_d are Hilger stable and that condition (19) is satisfied. From (18), we have

$$\|x(t)\| \le \beta \ e^{\alpha_c(t-\sum_{i=1}^k \mu(t_i))} \prod_{i=1}^k \beta^2(1+\mu(t_i)\alpha_d) \ \|x_0\|. \tag{24}$$

with $\alpha_c < 0$ and $0 > \alpha_d \in \mathcal{R}^+$. We have $\forall t_i, \ i \in \mathbb{N}$,

$$0 < \beta^2 \ (1+\mu(t_i)\alpha_d) = \beta^2 + \mu(t_i)\beta^2\alpha_d = 1 + \beta^2 - 1 + \mu(t_i)\beta^2\alpha_d \le 1 + \mu(t_i)\left[\frac{\beta^2-1}{\mu_{\min}} + \beta^2\alpha_d\right].$$

From condition (19), we get $\left[\frac{\beta^2-1}{\mu_{\min}} + \beta^2\alpha_d\right] < 0$, and from the above inequality, we have $\left[\frac{\beta^2-1}{\mu_{\min}} + \beta^2\alpha_d\right] \in \mathcal{R}^+$. Let $\alpha = \max\{\alpha_c, \left[\frac{\beta^2-1}{\mu_{\min}} + \beta^2\alpha_d\right]\} < 0$, which yields (23) and implies the exponential stability of system (15).
Note that, if $\alpha = \alpha_c$, we get $0 < \beta^2 \ (1+\mu_{\min}\alpha_d) < 1 + \mu_{\min}\alpha_c < 1$, then α is negative and positively regressive.

(ii) Let A_c and A_d be Hilger stable, such that condition (19) is not satisfied but condition (20) holds. From (18), we get

$$\|x(t)\| \leq \beta\ e^{k\log(\beta^2)+\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))}\prod_{i=1}^{k}(1+\mu(t_i)\alpha_d)\ \|x_0\|.$$

We have $k \leq \frac{t-\sum_{i=1}^{k}\mu(t_i)}{\tau_{\min}}$, and one can derive the following inequality

$$\|x(t)\| \leq \beta\ e^{\left[\frac{\log(\beta^2)}{\tau_{\min}}+\alpha_c\right](t-\sum_{i=1}^{k}\mu(t_i))}\prod_{i=1}^{k}(1+\mu(t_i)\alpha_d)\ \|x_0\|.$$

From (20), we get $\left[\frac{\log(\beta^2)}{\tau_{\min}}+\alpha_c\right] < 0$. Let $\alpha = \max\left\{\left[\frac{\log(\beta^2)}{\tau_{\min}}+\alpha_c\right], \alpha_d\right\} < 0$, and the above inequality becomes

$$\|x(t)\| \leq \beta\ e^{\alpha(t-\sum_{i=1}^{k}\mu(t_i))}\prod_{i=1}^{k}(1+\mu(t_i)\alpha)\ \|x_0\| = \beta e_\alpha(t,t_0)\ \|x_0\|,$$

which implies that the switched system (15) is exponentially stable.
Note that, if $\alpha = \left[\frac{\log(\beta^2)}{\tau_{\min}}+\alpha_c\right]$, we get, $1 > 1+\mu(t_i)\left[\frac{\log(\beta^2)}{\tau_{\min}}+\alpha_c\right] > 1+\mu(t_i)\alpha_d > 0$, which means that α is negative and positively regressive.

(iii) Suppose that A_c is Hilger stable and A_d can be Hilger stable or unstable. From condition (21) we have

$$\log(\beta^2\max_{1\leq i\leq k}(1+\mu(t_i)\ \alpha_d))+\alpha_c\ \tau_{\min} < 0,\quad \forall t_i,\ i\in\mathbb{N}.$$

with $\alpha_c < 0$. Let $\gamma_1 > 0$ and $\gamma_2 > 0$, satisfy the following equality

$$\log(\beta^2\max_{1\leq i\leq k}(1+\mu(t_i)\ \alpha_d))+\alpha_c\ \tau_{\min} = -\gamma_1-\gamma_2+\log(\beta^2\max_{1\leq i\leq k}(1+\mu(t_i)\ \alpha_d)) < 0,$$

such that, $-\gamma_2+\log(\beta^2\max_{1\leq i\leq k}(1+\mu(t_i)\ \alpha_d)) < 0$, which means that, $0 < \beta^2 e^{-\gamma_2}(1+\mu(t_i)\ \alpha_d) < 1$, $\forall t_i$, and by considering $\alpha_c\ \tau_{\min} = -\gamma_1-\gamma_2$. So we get from (18), the following,

$$\begin{aligned}\|x(t)\| &\leq \beta\ e^{\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))}\textstyle\prod_{i=1}^{k}\beta^2(1+\mu(t_i)\alpha_d)\ \|x_0\|\\ &= \beta\ e^{\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))}\textstyle\prod_{i=1}^{k}\beta^2 e^{-\gamma_2}(1+\mu(t_i)\alpha_d)e^{\gamma_2}\ \|x_0\|\\ &= \beta\ e^{\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))}e^{k\gamma_2}\textstyle\prod_{i=1}^{k}\beta^2 e^{-\gamma_2}(1+\mu(t_i)\alpha_d)\ \|x_0\|.\end{aligned}$$

Since $k \leq \frac{t-\sum_{i=1}^{k}\mu(t_i)}{\tau_{\min}}$, it follows that:

$$\begin{aligned}\|x(t)\| &\leq \beta\ e^{\left[\alpha_c+\frac{\gamma_2}{\tau_{\min}}\right](t-\sum_{i=1}^{k}\mu(t_i))}\textstyle\prod_{i=1}^{k}\beta^2 e^{-\gamma_2}(1+\mu(t_i)\alpha_d)\ \|x_0\|\\ &\leq \beta\ e^{\left[\frac{-\gamma_1}{\tau_{\min}}\right](t-\sum_{i=1}^{k}\mu(t_i))}\textstyle\prod_{i=1}^{k}\left(1+\mu(t_i)\left[\frac{\beta^2 e^{-\gamma_2}-1}{\mu(t_i)}+\beta^2 e^{-\gamma_2}\alpha_d\right]\right)\ \|x_0\|.\end{aligned} \tag{25}$$

As previously, let $\alpha = \max\left\{\frac{-\gamma_1}{\tau_{\min}}, \left[\frac{\beta^2 e^{-\gamma_2}-1}{\mu_{\min}}+\beta^2 e^{-\gamma_2}\alpha_d\right]\right\} < 0$, if $\beta^2 e^{-\gamma_2} > 1$, or $\alpha = \max\left\{\frac{-\gamma_1}{\tau_{\min}}, \left[\frac{\beta^2 e^{-\gamma_2}-1}{\mu_{\max}}+\beta^2 e^{-\gamma_2}\alpha_d\right]\right\} < 0$, if $\beta^2 e^{-\gamma_2} < 1$.
Notice that, if $\alpha = \frac{-\gamma_1}{\tau_{\min}}$, so $1 > \left(1+\mu(t_i)\left(\frac{-\gamma_1}{\tau_{\min}}\right)\right) > \beta^2 e^{-\gamma_2}(1+\mu(t_i)\alpha_d) > 0$, and if $\alpha = \left[\frac{\beta^2 e^{-\gamma_2}-1}{\mu_{\min}}+\beta^2 e^{-\gamma_2}\alpha_d\right]$, we have, $1 > \left(1+\mu(t_i)\left[\frac{\beta^2 e^{-\gamma_2}-1}{\mu(t_i)}+\beta^2 e^{-\gamma_2}\alpha_d\right]\right) > 0$, which implies that α negative and positively regressive. So inequality (25) yields (23), and implies the exponential stability of the switched system (15).

(iv) Suppose that A_c is unstable and A_d is Hilger stable. From condition (22), we have,

$$\log(\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d))+\alpha_c\ \tau_{\max}+\log(\beta^2) < 0$$

with $\alpha_c > 0$ and $0 < \max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d) < 1$. Let the constants $\gamma_1' > 0$, $\gamma_2' > 0$ and $\gamma_3' > 0$, satisfy the following equality

$$\log(\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d))+\alpha_c\ \tau_{\max}+\log(\beta^2) = -\gamma_1'-\gamma_2'+\alpha_c\ \tau_{\max}+\log(\beta^2) < 0$$

such that, $-\gamma_2' + \alpha_c\ \tau_{\max} + \log(\beta^2) = -\gamma_3' < 0$, and by considering $\log(\max_{1\le i\le k}(1+\mu(t_i)\alpha_d)) = -\gamma_1' - \gamma_2'$, we get from (18) the following,

$$
\begin{aligned}
\|x(t)\| &\le \beta\ e^{k\log(\beta^2)+\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))}\prod_{i=1}^{k}(1+\mu(t_i)\alpha_d)\ \|x_0\|\\
&= \beta\ e^{-k\gamma_2'+k\log(\beta^2)+\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))+k\gamma_2'}\prod_{i=1}^{k}(1+\mu(t_i)\alpha_d)\ \|x_0\|\\
&= \beta\ e^{k(-\gamma_2'+\log(\beta^2))+\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))}e^{k\gamma_2'}\prod_{i=1}^{k}(1+\mu(t_i)\alpha_d)\ \|x_0\|\\
&= \beta\ e^{k(-\gamma_2'+\log(\beta^2))+\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))}\prod_{i=1}^{k}e^{\gamma_2'}(1+\mu(t_i)\alpha_d)\ \|x_0\|\\
&\le \beta\ e^{k(-\alpha_c\tau_{\max}-\gamma_3')+\alpha_c(t-\sum_{i=1}^{k}\mu(t_i))}\prod_{i=1}^{k}e^{\gamma_2'}(1+\mu(t_i)\alpha_d)\ \|x_0\|.
\end{aligned}
$$

The inequalities, $k \ge \frac{t-\sum_{i=1}^{k}\mu(t_i)}{\tau_{\max}}$ and $(-\alpha_c\tau_{\max} - \gamma_3') < 0$, yield:

$$
\begin{aligned}
\|x(t)\| &\le \beta\ e^{\left[-\alpha_c-\frac{\gamma_3'}{\tau_{\max}}\right](t-\sum_{i=1}^{k}\mu(t_i))}\prod_{i=1}^{k}e^{\gamma_2'}(1+\mu(t_i)\alpha_d)\ \|x_0\|\\
&\le \beta\ e^{\left[-\alpha_c-\frac{\gamma_3'}{\tau_{\max}}\right](t-\sum_{i=1}^{k}\mu(t_i))}\prod_{i=1}^{k}(1+\mu(t_i)\left[\frac{e^{\gamma_2'}-1}{\mu_{\min}}+e^{\gamma_2'}\alpha_d\right])\ \|x_0\|.
\end{aligned}
$$

Let $\alpha = \max\left\{\left[-\alpha_c - \frac{\gamma_3'}{\tau_{\max}}\right], \left[\frac{e^{\gamma_2'}-1}{\mu_{\min}} + e^{\gamma_2'}\alpha_d\right]\right\} < 0$, which gives $\|x(t)\| \le \beta\ e_\alpha(t,t_0)$, and implies the exponential stability of the switched system (15).
Note that, if $\alpha = \left[-\alpha_c - \frac{\gamma_3'}{\tau_{\max}}\right]$, then $1 > \left(1+\mu(t_i)\left[-\alpha_c - \frac{\gamma_3'}{\tau_{\max}}\right]\right) > e^{\gamma_2'}(1+\mu(t_i)\alpha_d) > 0$, and if $\alpha = \left[\frac{e^{\gamma_2'}-1}{\mu_{\min}} + e^{\gamma_2'}\alpha_d\right]$, we have $0 < \left(1+\mu(t_i)\left[\frac{e^{\gamma_2'}-1}{\mu_{\min}} + e^{\gamma_2'}\alpha_d\right]\right) < e^{\gamma_2'}(1+\mu(t_i)\alpha_d) < e^{\gamma_2'}e^{-\gamma_1'-\gamma_2'} = e^{-\gamma_1'} < 1$. This shows that α is negative and positively regressive. ■

Remark 4. The exponential functions of the upper bound of $x(t)$ given by (18), depend on α_c and α_d, which are different. So it is easier to study the exponential stability of the switched system (15) by using the exponential function $e_\alpha(t,t_0)$ as in (23) than $e^{\alpha(t-t_0)}$.

Remark 5. Condition (21) means that the dynamics of the discrete-time subsystem (stable or unstable) are less significant than the continuous-time subsystem dynamics to guarantee exponential stability of the switched system. In assumption (iv), α_c is strictly positive. Inequality (22) means that the unstable continuous-time subsystem dynamics are less dominant than the stable discrete-time subsystem to guarantee exponential stability of the switched system.

Note that, the conditions of Theorem 3 can be checked easily, since we suppose that the time scale $\mathbb{T}$ is given, so $\tau_{\min}, \tau_{\max}, \mu_{\min}$ and $\mu_{\max}$ are known, and the terms α_c, α_d and β are derived from the eigenvalues and eigenvectors of the matrices A_c and A_d.

Example 1. To illustrate the effectiveness of Theorem 3, the following examples are presented.

First, let us consider the linear switched system described by

$$
x^\Delta(t) = \begin{cases} \begin{pmatrix} \frac{-3}{2} & 1 \\ 1 & -1 \end{pmatrix} x(t), & t \in \cup_{k=0}^{\infty}\left[2k + \frac{1.5k}{k+1.25},\ 2(k+1)\right[\\ \begin{pmatrix} \frac{-1}{2} & \frac{1}{10} \\ 0 & -1 \end{pmatrix} x(t), & t \in \cup_{k=0}^{\infty}\{2(k+1)\} \end{cases} \tag{26}
$$

evolving in time scale $\mathbb{T} = \bigcup_{k=0}^{\infty}\left[2k + \frac{1.5k}{k+1.25}\ ,\ 2(k+1)\right]$. It can be expressed as (15) with $t_k = 2k$, $\sigma(t_k) = 2k + \frac{1.5k}{k+1.25}$, $\frac{2}{3} \le \mu(t_k) = \sigma(t_k) - t_k = \frac{1.5k}{k+1.25} \le \frac{3}{2}$, $\forall k \in \mathbb{N}^*$, and $\frac{1}{2} \le (t_{k+1} - \sigma(t_k)) \le 2$, $\forall k \in \mathbb{N}$. Note that A_c and A_d are Hilger stable. Here one can easily check condition (19).

Let us now consider the same system on the time scale $\mathbb{T} = \bigcup_{k=0}^{\infty}\left[\frac{5}{2}k + \frac{3k}{2k+7},\ \frac{5}{2}(k+1)\right]$, with $t_k = \frac{5}{2}k$, $\sigma(t_k) = \frac{5}{2}k + \frac{3k}{2k+7}$, $\frac{1}{3} \le \mu(t_k) = \sigma(t_k) - t_k = \frac{3k}{2k+7} \le \frac{3}{2}$ and $1 \le (t_{k+1} - \sigma(t_k)) \le \frac{5}{2}$, $\forall k \in \mathbb{N}^*$. The discrete subsystem is still Hilger stable on this time scale, but conditions (19) and (20) are not satisfied whereas condition (21) is verified. Hence exponential stability holds.

Let the same time scale $\mathbb{T} = \bigcup_{k=0}^{\infty}\left[\frac{5}{2}k + \frac{3k}{2k+7},\ \frac{5}{2}(k+1)\right]$ with $A_c = \begin{pmatrix} -6 & 4 \\ 4 & -4 \end{pmatrix}$ and $A_d = \begin{pmatrix} \frac{1}{6} & \frac{-1}{30} \\ 0 & \frac{1}{3} \end{pmatrix}$. Here, the continuous-time system is exponentially stable whereas the discrete one is unstable. Nevertheless, Assumption (iii) is satisfied.

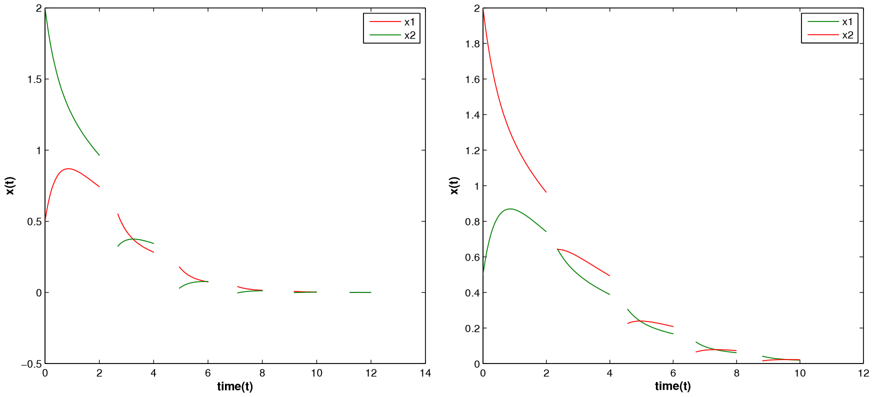

Fig. 2. Trajectories of the switched system (15) with Hilger stable subsystems. Left: Condition (19) is satisfied. Right: Condition (21) is satisfied.

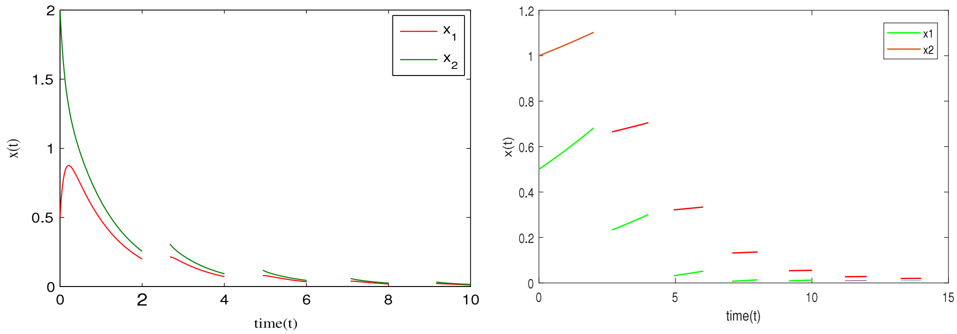

Fig. 3. Trajectories of the switched system (15). Left: A_c is exponentially stable and A_d is unstable and condition (21) is satisfied. Right: A_c is unstable and A_d is Hilger stable and condition (22) is satisfied.

Lastly, consider the time scale $\mathbb{T} = \bigcup_{k=0}^{\infty}\left[2k + \frac{1.5k}{k+1.25}, \ 2(k+1)\right]$ with $A_c = \begin{pmatrix} 0.0643 & 0.0468 \\ 0.0331 & 0.0243 \end{pmatrix}$ and $A_d = \begin{pmatrix} -1 & 0 \\ 0 & -0.6 \end{pmatrix}$. Here, the system switches between an unstable continuous-time subsystem and an Hilger stable discrete-time subsystem. Furthermore, condition (22) is satisfied.

The corresponding trajectories are given in Figs. 2 and 3 where the initial state is $x_0 = [0.5 \ \ 2]^T$. One can see the exponential stability of the switched system on the corresponding time scales $\mathbb{T}$.

Based on Theorem 3, we derive sufficient conditions for exponential stability of the switched system (15), where nonlinearities act on both the continuous-time and discrete-time subsystems.

4.3. Stability analysis of the nonlinear switched system

In this section, we will consider the nonlinear switched system (10).

Theorem 4. *Suppose that Assumption 1 is satisfied and the following assumptions hold:*

(i) The nominal switched system (15) is exponentially stable such that its solution satisfies

$$\|x(t)\| \leq \beta \ e_{\alpha}(t, t_0) \ \|x_0\| \tag{27}$$

with $\beta \geq 1, \alpha < 0$ with $\alpha \in \mathcal{R}^+$.

(ii) There exists a constant $L \geq 0$ such that

$$\|F_x(t)\| \leq L\|x(t)\|,$$

where $F_x(t)$ is defined as in (11).

(iii) $\alpha + \beta L < 0$

Then, the perturbed switched system (10) is uniformly exponentially stable.

Proof. Suppose that the nominal switched system (15) is exponentially stable, such that, from assumption (i), one has

$$\|e^{A_c(t-\sigma(t_k))}\prod_{i=0}^{k}(I+\mu(t_{k-i})A_d)e^{A_c(t_{k-i}-\sigma(t_{k-i-1}))}\| \leq \beta\ e_\alpha(t,t_0),$$

with $\beta \geq 1, \alpha < 0$ and $\alpha \in \mathcal{R}^+$. From (14), one can derive an upper bound for the solution of the perturbed system (10), as follows:

$$\|x(t)\| \quad \leq \beta e_\alpha(t,t_0)\|x_0\| + \int_{t_0}^{t} \beta e_\alpha(t,\sigma(s))\ \|F_x(s)\|\Delta s.$$

It implies that

$$\frac{\|x(t)\|}{e_\alpha(t,t_0)} \leq \beta\|x_0\| + \int_{t_0}^{t}\beta\frac{e_\alpha(t,\sigma(s))}{e_\alpha(t,t_0)}L\|x(s)\|\Delta s$$

We have

$$e_\alpha(t,\sigma(s)) = \frac{1}{e_\alpha(\sigma(s),t)} = \frac{1}{(1+\mu(s)\alpha)e_\alpha(s,t)} = \frac{1}{(1+\mu(s)\alpha)}\ e_\alpha(t,s)$$

Hence,

$$\begin{aligned}\frac{\|x(t)\|}{e_\alpha(t,t_0)} &\leq \beta\|x_0\| + \int_{t_0}^{t}\frac{\beta L}{(1+\mu(s)\alpha)}\ \frac{e_\alpha(t,s)}{e_\alpha(t,t_0)}\|x(s)\|\Delta s\\ &= \beta\|x_0\| + \int_{t_0}^{t}\frac{\beta L}{(1+\mu(s)\alpha)}\ e_\alpha(t,s)e_\alpha(t_0,t)\|x(s)\|\Delta s\\ &= \beta\|x_0\| + \int_{t_0}^{t}\frac{\beta L}{(1+\mu(s)\alpha)}\ e_\alpha(t_0,s)\|x(s)\|\Delta s\\ &= \beta\|x_0\| + \int_{t_0}^{t}\frac{\beta L}{(1+\mu(s)\alpha)}\ \frac{\|x(s)\|}{e_\alpha(s,t_0)}\Delta s\end{aligned}$$

Using Gronwall's inequality given in Corollary 1, we obtain

$$\frac{\|x(t)\|}{e_\alpha(t,t_0)} \leq \beta\|x_0\|\ e_{\frac{\beta L}{(1+\mu(.)\alpha)}}(t,t_0),$$

which implies that

$$\begin{aligned}\|x(t)\| &\leq \beta\|x_0\|\ e_\alpha(t,t_0)\ e_{\frac{\beta L}{(1+\mu(.)\alpha)}}(t,t_0)\\ &\leq \beta\|x_0\|\ e_{\alpha\oplus\frac{\beta L}{(1+\mu(.)\alpha)}}(t,t_0)\end{aligned}$$

We have

$$\alpha\oplus\frac{\beta L}{(1+\mu(.)\alpha)} = \alpha + \frac{\beta L}{1+\mu(.)\alpha} + \frac{\mu(.)\alpha\beta L}{1+\mu(.)\alpha} = \alpha+\beta L.$$

Therefore,

$$\|x(t)\| \leq \beta\|x_0\|\ e_{\alpha+\beta L}(t,t_0)$$

Hence, from assumption (iii), the nonlinear switched system (10) is uniformly exponentially stable. ■

Remark 6. Note that, if $\alpha+\beta L < 0$, then $(\alpha+\beta L)$ is positively regressive because, since $\alpha < 0$ and $\alpha \in \mathcal{R}^+$, we have $0 < 1+\mu(t)\alpha < 1$. It implies that $0 < \mu(t)\beta L < 1+\mu(t)(\alpha+\beta L) < 1+\mu(t)\beta L$, which means that $(\alpha+\beta L) \in \mathcal{R}^+$.

Example 2. To illustrate Theorem 4, the following example is presented.

Let us consider the first switched system given in Example 1, where condition (19) is satisfied. From Theorem 3, the nominal switched system is exponentially stable with $\alpha_c = -0.2192,\ \alpha_d = -0.5,\ \beta = 1.2198$ and $\mu_{\min} = \frac{2}{3}$. We have $\left[\frac{\beta^2-1}{\mu_{\min}}+\beta^2\alpha_d\right] = -0.0121$ and $\alpha = \max\{-0.0121, -0.2192\} = -0.0121$.

Consider now the nonlinear switched system

$$x^\Delta(t) = \begin{cases}\begin{pmatrix}\frac{-3}{2} & 1\\ 1 & -1\end{pmatrix}x(t)+0.0075\sin(x(t)), & t\in\cup_{k=0}^{\infty}\left[2k+\frac{1.5k}{k+1.25},\ 2(k+1)\right[\\ \begin{pmatrix}\frac{-1}{2} & \frac{1}{10}\\ 0 & -1\end{pmatrix}x(t)+0.0098\sin(x(t)), & t\in\cup_{k=0}^{\infty}\{2(k+1)\}\end{cases} \tag{28}$$

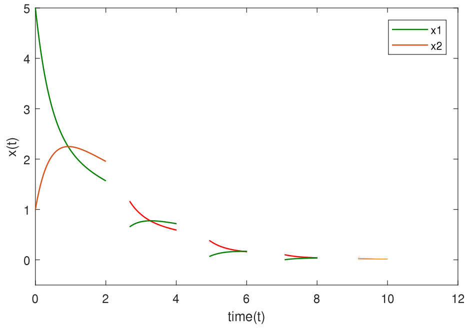

Fig. 4. Convergence of the trajectory of switched system (28) with initial condition $x_0 = [5\ \ 1]^T$.

The conditions of Theorem 4 are satisfied since $\alpha+\beta L = -1.45.10^{-4} < 0$. Therefore the exponential stability of perturbed switched system (28) is established. The trajectory converges to zero as it is shown in Fig. 4 where the initial condition is $x_0 = [5\ \ 1]^T$.

5. Application to consensus of multi-agent

To illustrate the effectiveness of the theoretical results, we consider in this section the consensus problem for linear multi-agent systems (MASs) under intermittent information transmissions.

5.1. Problem formulation

The MAS consists of N agents whose model is described by the following linear dynamics,

$$\dot{x}_i = Ax_i + Bu_i, \qquad i \in \{1, \ldots, N\} \tag{29}$$

where $x_i \in \mathbb{R}^n$ (resp. $u_i \in \mathbb{R}^m$) is the state (resp. control input) of agent i and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant real matrices.

The communication network among the N agents is described by a graph $\mathcal{G}$ which consists of a nonempty node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$. Each edge $(i, j) \in \mathcal{E}$ in the directed graph [37] corresponds to an information link from agent i to agent j, which means that agent j can receive information from agent i. The graph $\mathcal{G}$ is represented by the adjacency matrix $G = (a_{ij}) \in \mathbb{R}^{N\times N}$ defined by $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The Laplacian matrix of $\mathcal{G}$ is defined as $M = (m_{ij}) \in \mathbb{R}^{N\times N}$ with $m_{ii} = \sum_{j=1}^{N} a_{ij}$ and $m_{ij} = -a_{ij}$ for $i \neq j$. The graph $\mathcal{G}$ is connected if there is a path between every two nodes of $\mathcal{G}$.

It is assumed that local information is exchanged between neighboring agents through a communication channel (which is subject to uncertainty, modeling quantization errors, noise) over some disconnected time intervals because of possible sensor failures or communication obstacles.

Hereafter, the following hypotheses are made:

- The graph $\mathcal{G}$ is fixed and undirected.
- Each communication failure has a bounded duration, denoted by $b \in \mathbb{R}^+$.
- There is no more than one communication failure over each time interval of length b.
- The communication channel is modeled with an additive uncertainty $\delta(t) \in \mathbb{R}$, bounded as:

  $$|\delta(t)| \leq \delta_{max}, \tag{30}$$

  where δ_{max} is a known positive constant.
- The pair (A, B) is stabilizable.

Since the graph $\mathcal{G}$ is undirected, there is a nonsingular matrix P such that $M = PDP^{-1}$ where D is the diagonal matrix containing the eigenvalues of M (which are non-negative and real).

Let $\{t_0, t_1, t_2, t_3, \ldots\}$ be a monotonically increasing sequence without finite accumulation points. Suppose that the communication between agents will be interrupted at time instants t_k, $k \in \mathbb{N}^*$. Based on the available local information, the following decentralized intermittent controller is proposed, $\forall i \in \{1, \ldots, N\}$,

$$u_i(t) = \begin{cases} \sum_j a_{ij}(1 + \delta(t))K(x_j(t) - x_i(t)) & \text{if } t \in ([t_0, t_1[) \cup (\cup_{k=1}^{\infty}[t_k + b, t_{k+1}[) \\ \sum_j a_{ij}(1 + \delta(t_{k+1}))K(x_j(t_{k+1}) - x_i(t_{k+1})) & \text{if } t \in \cup_{k=0}^{\infty}[t_{k+1}, t_{k+1} + b[\end{cases} \tag{31}$$

with $K \in \mathbb{R}^{m\times n}$. Here, agents can communicate according to the communication topology, during $([t_0, t_1[) \cup (\cup_{k=1}^{\infty}[t_k + b, t_{k+1}[)$. It should be noted that during the remaining intervals, the control (31) remains constant since a communication failure has occurred at times $t_k,\ k \in \mathbb{N}^*$.

The control objective is to design a matrix K in the protocol (31), using time scale theory, such that the consensus problem is solved, i.e.

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \qquad \forall i, j = \{1, \ldots, N\}, \tag{32}$$

Let us denote the error between agents by $e_i(t) = x_i(t) - x_j(t)$ for $i, j = \{1, \ldots, N\}$. Let the vector $e = (e_1^T, \ldots, e_N^T)^T$ and $u = (u_1^T, \ldots, u_N^T)^T$. The dynamic of the state error $e(t)$ can be written in a compact form as follows:

$$\dot{e}(t) = \begin{cases} (I_N \otimes A)e(t) - (1 + \delta(t))(M \otimes BK)e(t) & t \in ([t_0, t_1[) \cup (\cup_{k=1}^{\infty}[t_k + b, t_{k+1}[) \\ (I_N \otimes A)e(t) - (1 + \delta(t_{k+1}))(M \otimes BK)e(t_{k+1}), & t \in \cup_{k=0}^{\infty}[t_{k+1}, t_{k+1} + b[, \end{cases}$$

where $\otimes$ is the Kronecker product.

Defining $y(t) = (P^{-1} \otimes I_n)e(t)$, one has

$$\dot{y}(t) = \begin{cases} ((I_N \otimes A) - (1 + \delta(t))(D \otimes BK))y(t), & t \in ([t_0, t_1[) \cup (\cup_{k=1}^{\infty}[t_k + b, t_{k+1}[) \\ (I_N \otimes A)y(t) - (1 + \delta(t_{k+1}))(D \otimes BK)y(t_{k+1}), & t \in \cup_{k=0}^{\infty}[t_{k+1}, t_{k+1} + b[. \end{cases} \tag{33}$$

Following [38,39], the consensus problem for system (29) with controller (31) can be reached iff the following $N - 1$ subsystems are asymptotically stable:

$$\dot{z}^i(t) = \begin{cases} (A - \lambda_i(1 + \delta(t))BK)z^i(t), & t \in ([t_0, t_1[) \cup (\cup_{k=1}^{\infty}[t_k + b, t_{k+1}[) \\ Az^i(t) - \lambda_i(1 + \delta(t_{k+1}))BKz^i(t_{k+1}), & t \in \cup_{k=0}^{\infty}[t_{k+1}, t_{k+1} + b[, \end{cases} \tag{34}$$

where λ_i are the nonzero eigenvalues of M and $i = 2, \ldots, N$.

Let us state the consensus problem using time scale theory since system (34) combines a discrete variable (i.e, $z^i(t_{k+1})$) and a continuous variable (i.e, $z^i(t)$). To facilitate the controller design, during communication failures, only the behavior of the solution of (34) at times $t_k,\ k \in \mathbb{N}^*$ is considered. Using the definition of the Δ-derivative and considering the specific time scale $\mathbb{T} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$, with

$$\begin{cases} \sigma(t_k) = t_k + b, & k \in \mathbb{N}^* \\ \sigma(t_0) = t_0 = 0. \end{cases} \tag{35}$$

The $N - 1$ subsystems (34) $(i = 2, \ldots, N)$ can be expressed as,

$$(z^i)^{\Delta}(t) = \begin{cases} (A - \lambda_i BK)z^i(t) - \lambda_i\delta(t)BKz^i(t), & t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ A_d z^i(t) - \dfrac{1}{b}[\lambda_i\delta(t)\int_0^b e^{As}dsBK]z^i(t), & t \in \cup_{k=0}^{\infty}\{t_{k+1}\}, \end{cases} \tag{36}$$

with $A_d = \dfrac{1}{b}\left[e^{Ab} - I_n - \lambda_i \int_0^b e^{As}dsBK\right]$. Indeed, from the second equation of (34), we have for $t \in [t_{k+1}, \sigma(t_{k+1})[,\ k \in \mathbb{N}$,

$$z^i(t) = e^{A(t-t_{k+1})}z^i(t_{k+1}) - \int_{t_{k+1}}^{t} e^{A(t-s)}\lambda_i(1 + \delta(t_{k+1}))BKz^i(t_{k+1})ds.$$

Hence, for $t = \sigma(t_{k+1})$, we get

$$\begin{aligned} z^i(\sigma(t_{k+1})) &= e^{A(\sigma(t_{k+1})-t_{k+1})}z^i(t_{k+1}) - \lambda_i(1 + \delta(t_{k+1}))\int_{t_{k+1}}^{\sigma(t_{k+1})} e^{A(\sigma(t_{k+1})-s)}BKds\ z^i(t_{k+1}) \\ &= e^{Ab}z^i(t_{k+1}) - \lambda_i(1 + \delta(t_{k+1}))\int_0^b e^{As}BKds\ z^i(t_{k+1}). \end{aligned}$$

If we consider the behavior of z^i only at times t_{k+1} and $\sigma(t_{k+1})$, we get the Δ-derivative of $z^i(t_{k+1})$,

$$\begin{aligned} (z^i)^{\Delta}(t_{k+1}) &= \frac{z^i(\sigma(t_{k+1})) - z^i(t_{k+1})}{\sigma(t_{k+1}) - t_{k+1}} \\ &= \frac{\left[e^{Ab} - I_n - \lambda_i\int_0^b e^{As}BKds\right]z^i(t_{k+1}) - \left[\lambda_i\delta(t_{k+1})\int_0^b e^{As}BKds\right]z^i(t_{k+1})}{b}. \end{aligned}$$

System (36) switches between a continuous-time and a discrete-time subsystem due to the presence of interruption of communication.

Remark 7. Using time scale theory, the consensus problem for single-integrator MASs has been analyzed in [40]. Some sufficient conditions have been recently derived to solve the consensus problem [9,12] for linear MASs. However, in these works, it is required that A and BK commute with each other. Here, we remove this restrictive condition to design the controller gain.

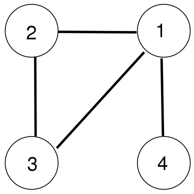

Fig. 5. Communication network $\mathcal{G}$.

5.2. Numerical results

We consider a group of four agents ($N = 4$). The communication network is depicted in Fig. 5. From this graph, one can get the Laplacian matrix $M = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ and deduce $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$ and $\lambda_4 = 4$. The upper bound of the additive uncertainty on the communication channel is $\delta_{\max} = 0.0027$. The agent dynamics are described by (29) with $A = \begin{pmatrix} 0 & 1 \\ 0.1 & 0.05 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It is assumed that the neighboring agents exchange information only when $t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[$ with:

$$\begin{cases} \sigma(t_0) = t_0 = 0 \\ \sigma(t_k) = t_k + b, \qquad k \in \mathbb{N}^* \\ t_k \quad = 2(2k-1) + 0.1\log k, \qquad k \in \mathbb{N}^* \end{cases} \tag{37}$$

An interruption of communication occurs at each time t_k, $k \in \mathbb{N}^*$ and is bounded by $b = \frac{1}{2}$. The control gain of the consensus protocol (31) is set as $K = \begin{pmatrix} 1 & 0.5 \end{pmatrix}$. The objective is to verify that the decentralized control law (31) solves the consensus problem under intermittent information transmission. According to the previous section, the three subsystems (36) with $i = 2, 3, 4$ should be uniformly exponentially stable.

The subsystem 1, i.e. with $i = 2$ becomes

$$(z^2)^{\Delta}(t) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -0.9 & -0.45 \end{pmatrix} z^2(t) + \delta(t) \begin{pmatrix} 0 & 0 \\ -1 & -0.5 \end{pmatrix} z^2(t), & t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ \begin{pmatrix} -0.2274 & 0.8905 \\ -0.9151 & -0.4323 \end{pmatrix} z^2(t) + \delta(t) \begin{pmatrix} 0.2526 & 0.1263 \\ 1.0138 & 0.5084 \end{pmatrix} z^2(t), & t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases}$$

For $\delta = 0$, the associated eigenvalues of A_c and A_d are $\lambda_{c,1} = -0.225 + 0.92j$, $\lambda_{c,2} = -0.225 - 0.92j$, $\lambda_{d,1} = -0.3298 + 0.897j$ and $\lambda_{d,2} = -0.3298 - 0.897j$. Here $\alpha_c = -0.225$ and $\max_{i,j} |1 + \mu(t_i)\lambda_{d,j}| = 0.9479 < 1$, so $\alpha_d = -0.1042$. We conclude that the continuous and discrete subsystems are Hilger stable. We have $\beta_c = 1.2809$, $\beta_d = 1.1217$, so $\beta = 1.2809$. One can verify that condition (ii) of Theorem 3 is fulfilled since $\tau_{\min} = 2$ and $\alpha_c = -0.225 < \frac{-\log(\beta^2)}{2} = -0.1205$. It implies that the nominal subsystem is exponentially stable with $\alpha = \max\{\frac{\log(\beta^2)}{\tau_{\min}} + \alpha_c, \alpha_d\} = -0.1045$. The perturbation term satisfies condition (ii) of Theorem 4 with $L = 0.003$. Therefore, condition (iii) of Theorem 4 is verified. From Theorem 4, one can conclude that the uncertain switched subsystem 1 is uniformly exponentially stable.

For the subsystem 2, we have

$$(z^3)^{\Delta}(t) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -2.9 & -1.45 \end{pmatrix} z^3(t) + \delta(t) \begin{pmatrix} 0 & 0 \\ -3 & -1.5 \end{pmatrix} z^3(t), & t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ \begin{pmatrix} -0.7326 & 0.6379 \\ -2.9488 & -1.4491 \end{pmatrix} z^3(t) + \begin{pmatrix} 0.7579 & 0.3789 \\ 3.0505 & 1.5252 \end{pmatrix} z^3(t), & t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases}$$

For $\delta = 0$, the associated eigenvalues of A_c and A_d are $\lambda_{c,1} = -0.725 + 1.54j$, $\lambda_{c,2} = -0.725 - 1.54j$, $\lambda_{d,1} = -1.09 + 1.32j$ and $\lambda_{d,2} = -1.09 + 1.32j$. Here $\alpha_c = -0.725$, $\max_{i,j} |1 + \mu(t_i)\lambda_{d,j}| = 0.803 < 1$ and $\alpha_d = -0.394$, so as the subsystem 1, the continuous and discrete subsystems are Hilger stable. We have $\beta_c = 2.041$, $\beta_d = 2.2684$, so $\beta = 2.2684$. One can verify that condition (iii) of Theorem 3 is fulfilled since $\tau_{\min} = 2$ and $\max_i(1 + \mu(t_i)\alpha_d) = 0.803 < \exp[-(-0.725 \times 2 + \log(2.2684^2))] = 0.8285$. It implies that the nominal subsystem is exponentially stable with $\gamma_1 = 0.02$, $\gamma_2 = 1.43$ and $\alpha = -0.0224$. The perturbation term satisfies condition (ii) of Theorem 4 with $L = 0.0099$. Therefore, condition (iii) of Theorem 4 is verified. From Theorem 4, one can conclude that, the uncertain switched subsystem 2 is uniformly exponentially stable.

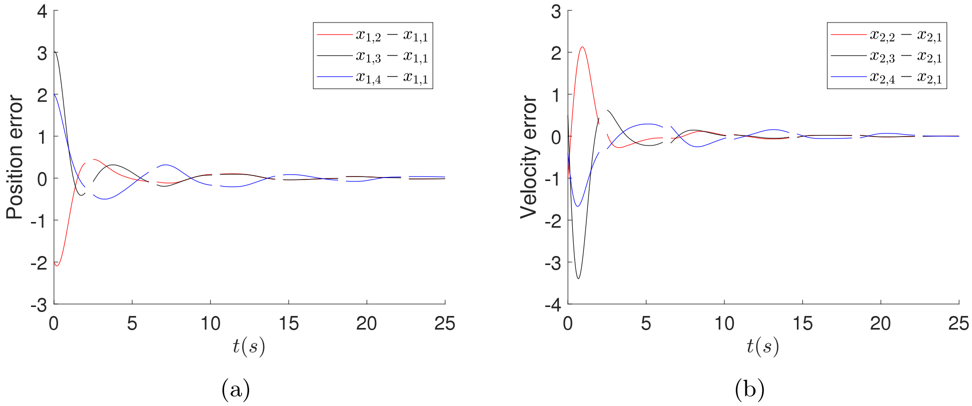

Fig. 6. Trajectories of (a) position error $(x_{1,2}-x_{1,1}, x_{1,3}-x_{1,1}, x_{1,4}-x_{1,1})^T$; (b) velocity error $(x_{2,2}-x_{2,1}, x_{2,3}-x_{2,1}, x_{2,4}-x_{2,1})^T$ on the time scale $\mathbb{T}=\bigcup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$.

The same conclusion holds for the subsystem 3, such that,

$$(z^4)^{\Delta}(t)=\begin{cases}\begin{pmatrix}0 & 1\\ -3.9 & -1.95\end{pmatrix}z^4(t)+\delta(t)\begin{pmatrix}0 & 0\\ -4 & -2\end{pmatrix}z^4(t), & t\in\cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ \begin{pmatrix}-0.9852 & 0.5116\\ -3.9656 & -1.9576\end{pmatrix}z^4(t)+\delta(t)\begin{pmatrix}1.0105 & 0.5052\\ 4.0673 & 2.0337\end{pmatrix}z^4(t), & t\in\cup_{k=0}^{\infty}\{t_{k+1}\}\end{cases}$$

For $\delta=0$, the eigenvalues of A_c and A_d are $\lambda_{c,1}=-0.975+1.717j$, $\lambda_{c,2}=-0.975-1.717j$, $\lambda_{d,1}=-1.47+1.33j$ and $\lambda_{d,2}=-1.47-1.33j$. So, $\alpha_c=-0.975$, $\max_{i,j}|1+\mu(t_i)\lambda_{d,j}|=0.719<1$ and $\alpha_d=-0.562$, so as the previous subsystems, the continuous and discrete subsystems are Hilger stable. We have $\beta_c=3.0122$ and $\beta_d=2.2684$, so $\beta=3.0122$. The condition (iii) of Theorem 3 is fulfilled since $\tau_{\min}=2$ and $\max_i(1+\mu(t_i)\alpha_d)=0.719<\exp[-(-0.975\times 2+\log(3.0122^2))]=0.7748$, which implies that the nominal subsystem is exponentially stable with $\gamma_1=0.05$, $\gamma_2=1.9$ and $\alpha=-0.00485$. The perturbation term satisfies condition (ii) of Theorem 4 with $L=0.012$. Therefore, condition (iii) of Theorem 4 is verified. From Theorem 4, one can conclude the uniform exponential stability of the uncertain switched subsystem 3.

Since all the subsystems are exponentially stable, one can conclude that using the decentralized controller (31), the consensus is achieved in spite of intermittent information transmission.

For each agent i, the state x_i is denoted as $x_i=(x_{1,i}, x_{2,i})$. The trajectories of the position error $(x_{1,2}-x_{1,1}, x_{1,3}-x_{1,1}, x_{1,4}-x_{1,1})^T$ and the velocity error $(x_{2,2}-x_{2,1}, x_{2,3}-x_{2,1}, x_{2,4}-x_{2,1})^T$ on the time scale $\mathbb{T}=\bigcup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$ are depicted in Fig. 6.

6. Conclusion

In this paper, the exponential stability of perturbed switched systems on the important time scale $\mathbb{T}=\cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$ is studied. The system considered switches between linear continuous-time and linear discrete-time subsystems, in the presence of nonlinear uncertainties. First, the nominal system is considered and sufficient conditions for exponential stability are derived by considering the cases where unstable subsystems may be present. Using results on the exponential stability of linear perturbed systems on time scales, sufficient conditions are derived to ensure the stability of the switched uncertain system. Several illustrative examples are presented together with an application to multi-agent consensus problem with intermittent communication, demonstrating the method's validity.

Acknowledgments

This work was supported by the International Campus on Safety and Intermodality in Transportation, the European Community, the Regional Delegation for research and Technology, the Nord-Pas-de-Calais Region, the Ministry of Higher Education and Research France, the National Center for Scientific Research France under the UVHC BI-CFNes and PHC NUSANTARA project, and by the National Science Foundation USA under the Grant ECCS-1711432, and the Engineering Research Center Program of the National Science Foundation USA and the Department of Energy under NSF USA Award Number EEC-1041877.

References

[1] D. Liberzon, Switching in Systems and Control, Birkhauser, Boston, 2003.
[2] D. Liberzon, A. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag. 19 (1999) 59–70.
[3] H. Lin, P. Antsaklis, Stability and stabilizability of switched linear systems: A survey of recent results, IEEE Trans. Automat. Control 54 (2009) 308–332.
[4] Z. Sun, S. Ge, Analysis and synthesis of switched linear control systems, Automatica 41 (2005) 181–195.
[5] J. Geromel, P. Colaneri, Stability and stabilization of continuous-time switched linear systems, SIAM J. Control Optim. 45 (2006) 1915–1930.
[6] J. Geromel, P. Colaneri, Stability and stabilization of discrete-time switched systems, Internat. J. Control 79 (2006) 719–728.
[7] W. Zhang, L. Yu, Stability analysis for discrete-time switched time-delay systems, Automatica 45 (2009) 2265–2271.
[8] G. Guo, L. Ding, Q. Han, A distributed event-triggered transmission strategy for sampled-data consensus of multi-agent systems, Automatica 50 (2014) 1489–1496.
[9] F.Z. Taousser, M. Defoort, M. Djemai, Consensus for linear multi-agent system with intermittent information transmissions using the time-scale theory, Internat. J. Control 89 (2016) 210–220.
[10] M. Yu, L. Wang, T. Chu, An LMI approach to networked control systems with data packet 1dropout and 1transmission delay, in: 43rd IEEE Conference on Decision and Control (CDC), 04 (2004) 14–17.
[11] C. Fiter, L. Hetel, W. Perruquetti, J.P. Richard, A state dependent sampling for linear state feedback, Automatica 48 (8) (2012) 1860–1867.
[12] F.Z. Taousser, M. Defoort, M. Djemai, Stability analysis of a class of uncertain switched systems on time scale using Lyapunov functions, Nonlinear Anal. Hybrid Syst. 16 (2015) 13–23.
[13] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhauser, boston, 2001.
[14] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.
[15] S. Hilger, Analysis on measure chains- A unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[16] J. DaCuhna, Stability for time varying linear dynamic systems on time scales, J. Comput. Appl. Math. 176 (2005) 381–410.
[17] T. Doan, A. Kalauch, S. Siegmund, F. Wirth, Stability radii for positive linear time-invariant systems on time scales, Systems Control Lett. 59 (2010) 173–179.
[18] A. Peterson, Y. Raffoul, Exponential stability of dynamic equations on time scales, Adv. Difference Equ. 2 (2005) 133–144.
[19] C. Potzsche, S. Siegmund, F. Wirth, A spectral characterization of exponential stability for linear time-invariant systems on time scales, Discrete Contin. Dyn. Syst. 9 (2003) 1223–1241.
[20] N.H. Du, L.H. Tien, On the exponential stability of dynamic equations on time scales, J. Math. Anal. Appl. 2 (331) (2007) 1159–1174.
[21] N.H. Du, D.D. Thuan, N.C. Liem, Stability radius of implicit dynamic equations with constant coefficients on time scales, Systems Control Lett. 60 (2011) 596–603.
[22] Z. Bartosiewicz, E. Piotrowska, On stabilisability of nonlinear systems on time scales, Internat. J. Control 86 (2013) 139–145.
[23] D.R. Poulsen, J.M. Davis, I.A. Gravagne, Stochastic time scales: quadratic Lyapunov functions and probabilistic regions of stability, in: IEEE Southeastern Symposium on System Theory, 2013.
[24] D.R. Poulsen, J.M. Davis, I.A. Gravagne, Optimal Control on Stochastic Time Scales, in: IFAC World Congress, Toulouse, 2017.
[25] G. Zhai, X. Xu, H. Lin, A. Michel, Analysis and design of switched normal systems, Nonlinear Anal. 65 (2006) 2248–2259.
[26] J.M. Davis, I.A. Gravagne, A.A. Ramos, Stability of Switched Linear Systems on non-uniform Time Domains, in :IEEE Southeastern Symp. Systems Theory, Texas, 2010.
[27] F.Z. Taousser, M. Defoort, M. Djemai, Stability analysis of a class of switched linear systems on non-uniform time domains, Systems Control Lett. 74 (2014) 24–31.
[28] F.Z. Taousser, M. Defoort, M. Djemai, S. Nicaise, Stability analysis of switched linear systems on non-uniform time domains with unstable subsystems, in: IFAC World Congress, Toulouse, 2017.
[29] F.Z. Taousser, M. Defoort, M. Djemai, Region of exponential stability of switched linear systems on time scales, in: 5th IFAC Conference on Analysis and Design of Hybrid Systems, Atlanta, USA, 2015.
[30] G. Eisenbarth, J. Davis, I. Gravagne, Stability of simultaneously triangularizable switched systems on hybrid domains, Electron. J. Differential Equations 63 (2014) 1–19.
[31] I.A. Gravagne, E. Johns, J.M. Davisi, Switched linear systems on time scales with relaxed commutativity constraints, in: IEEE Southeastern Symposium on Systems Theory, Auburn University, 2011.
[32] G. Guseinov, Integration on time scales, J. Math. Anal. Appl. 285 (2003) 107–127.
[33] C. Potzsche, Faserbunder dynamischer Gleichungen auf Maßketten (Ph.D. thesis), Logos Verlag, 2002.
[34] A.L. Liu, Boundedness and exponential stability of solutions to dynamic equations on time scales, Electron. J. Differential Equations 12 (2006) 1–14.
[35] T. Gard, J. Hoffacker, Asymptotic behavior of natural growth on time scales, Dynam. Systems Appl. 12 (2003) 131–147.
[36] E. Akin-Bohner, M. Bohner, F. Akin, Pachpatte inequalities on time scales, J. Inequal. Pure Appl. Math. 6 (2005) 1–23.
[37] W. Ren, R. Beard, Consensus seeking in multi-agent systems under dynamically changing interaction topologies, IEEE Trans. Autom. Control 50 (2005) 655–661.
[38] W. Yu, G. Chen, M. Cao, Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems, Automatica 46 (2010) 1089–1095.
[39] Z. Yu, H. Jiang, C. Hu, Second-order consensus for multiagent systems via intermittent sampled data control, IEEE Trans. Syst. Man Cybern. A 46 (2018) 1089–1095.
[40] J. Shen, J. Cao, Consensus of multi-agent systems on time scales, IMA J. Math. Cont. Inf. 50 (2012) 507–517.