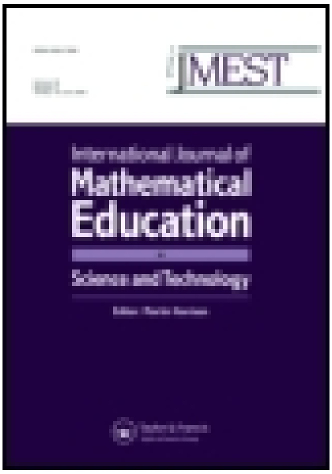

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‘Negative of my money, positive of her money’: secondary students’ ways of relating equations to a debt context

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'Negative of my money, positive of her money': secondary students' ways of relating equations to a debt context

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We interviewed 40 students each in grades 7 and 11 to investigate their integer-related reasoning. In one task, the students were asked to write and interpret equations related to a story problem about borrowing money from a friend. All the students solved the story problem correctly. However, they reasoned about the problem in different ways. Many students represented the situation numerically without invoking negative numbers, whereas others wrote equations involving negative numbers. When asked to interpret equations involving negative numbers in relation to the story, students did so in two ways. Their responses reflect distinct perspectives concerning the relationship between arithmetic equations and borrowing/owing. We discuss these findings and their implications regarding the role of contexts in integer instruction.

Keywords: integers; contexts; story problems; negative numbers; money

> You ever have negative money? That's depressing, isn't it? You look in your bank account: Negative ten dollars. That's how much I have now. Negative ten. That means I don't even have no money now! I wish I did. I wish I didn't have anything. I wish I just had nothing, but I have less than that. I don't have none. I have 'not ten.' If it's free, I can't. . . afford it. Someone could come up to me, 'Take this. It's free.'. . . [I'd say] that costs nothing! I can't afford that. That's more than I have. I gotta raise ten bucks to be broke. That's where I'm at. That's not good. That's bad. (Louis C.K., comedian)

1. Introduction

The use of contexts such as money is prevalent in integer instruction, and educators have conventional ways of relating equations to these contexts. Because of their widespread use, we assume that the use of these contexts is intended to be supportive of student learning or to reveal something of students' understanding of integers. But how do students think about the relationships between integer equations and contexts? What do students' responses reveal about their understanding of integers? In interviews with 7th and 11th graders in

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the United States, we investigated how the students reasoned about relationships between particular equations and a story problem concerning borrowing money from a friend. We found that students reasoned about these relationships in different ways. Awareness of these distinct ways of reasoning is important if teachers are to engage students in meaningful mathematical activity that involves relating integers to contexts.

In the next section, we describe the theoretical perspective that informs this research. We then review relevant research literature concerning the teaching and learning of integers. We also describe textbook approaches to integer instruction, based on a review conducted previously. We state our research question and describe the interview task and the methods of analysis that were used in this study. We then present the results of our analyses. In addition to reporting numbers of students who gave particular answers, we introduce two distinct ways of reasoning that we identified regarding the relationship between the money context and arithmetic equations. These ways of reasoning, which we refer to as perspectives, help to characterize an aspect of students' reasoning that is highly relevant to integer instruction. We conclude the paper by discussing the implications of our findings and inviting further research in this area.

2. Children's mathematical thinking perspective

In this paper, we introduce a distinction with regard to how students relate equations to a particular context. Thus, our focus is on *students*' ways of reasoning regarding that relationship. We approach our research from a children's mathematical thinking perspective. Children's mathematical thinking is distinct from that of adults in that children and adults tend to approach problems differently from one other. For example, whereas most adults tend to think of the following story problem as a subtraction problem, young children almost universally approach it as an addition problem (with a missing addend): *Carla has 7 dollars. How many more dollars does she have to earn so that she will have 11 dollars to buy a puppy?*.[1]

We value children's mathematics, and we take seriously the nature of that mathematics, whether or not it is correct from an expert perspective. We value seeing mathematics through children's eyes to better understand the sense that they make (e.g., [2,3]). This perspective is based on constructivist principles that children have existing knowledge and experiences that they bring with them into the classroom and upon which they continue to build (e.g., [4–8]). We take this view because the ultimate goal of our research is to find ways to better support children's learning of mathematics.[4,9,10]

In this paper, we bring a children's mathematical thinking perspective to the analysis of children's interview responses. By contrast, some authors explicitly superimpose their own views of mathematics on children's responses. For example, Bruno and Martinón [11] asserted that addition and subtraction are one and the same (i.e., that subtraction is simply adding a negative), and they evaluated the extent to which students' responses were in agreement with that assertion. Although we understand this stance from a mathematician's point of view, it does not accord with the thinking of children for whom addition and subtraction have decidedly different meanings (e.g., [4]). The distinctions that are salient for students are important to us and are the focus of this paper.

3. Background

In this section, we situate our study by presenting a review of mathematics education research literature related to the teaching and learning of integers. We focus on the use of contexts and on students' reasoning about such contexts. We also discuss the use of contexts in textbook approaches to integer instruction. After situating our study in this way, we formally state our research questions.

3.1. *What sense do students make of signed numbers?*

In his discussion of mathematical intuition, Fischbein [12] remarks: 'It is evident that one cannot spend 5 dollars when one has only 3, but mathematically one may write 3 - 5 and ask for a reasonable solution' (p.10). Whereas it is relatively easy to relate whole numbers and operations to contexts, doing so with signed numbers presents challenges. This was the case for mathematicians over centuries, and students today likewise struggle to make sense of signed numbers.[13,14]

The mathematics education research literature has documented difficulties that students have with making sense of negative numbers (e.g., [15–17]). Researchers have also studied various instructional approaches intended to ameliorate this situation (e.g., [18–21]). In teaching experiments and other interventions, researchers have investigated the use of a variety of instructional models, tools, or contexts for teaching students about integers. These include using computer microworlds [22] and animations,[23] scores and forfeits,[24] number lines,[25] balloons and weights,[19] and net worth.[26] Common among these approaches is an attempt to support students' learning by encouraging them to draw an analogy between a context or model and the integers in abstract. Contexts involving money are especially popular for this purpose.[2]

Stephan and Akyuz [26] used net worth as a context for addition and subtraction of integers in a teaching experiment with seventh graders. As the class progressed through the instructional sequence, students reasoned about the effects of various transactions on a person's net worth, and they represented these transactions with equations involving integer arithmetic. The class generalized from these problems to arrive at the sign rules for addition and subtraction of integers. Along the way, they developed strategies such as referencing zero to add and subtract integers. The authors' findings indicate that a money-related context could be productive in supporting students' meaningful learning of integer arithmetic.

Not a great deal is known about the nuances of students' integer-related reasoning. Although student thinking and learning in the domains of whole numbers and non-negative rational numbers have been researched substantially, few researchers have investigated *how* students think about integers.[27,28] Vlassis [29] studied the learning of a group of eighth graders in Belgium. Forty students were taught to solve linear equations with one unknown using the balance model, and five of these students were interviewed eight months later. Although the purpose of this study was to investigate the effectiveness of the balance model, the findings were focused on students' difficulties in dealing with negative integers, as both constants and coefficients. Vlassis reported differences in the interview participants' reasoning about equations involving negative versus positive integers:

> The negatives place the equation ('arithmetical' or 'non-arithmetical') on an abstract level. It is no longer possible to refer back to a concrete model or to arithmetic. The "didactical cut" does not seem to depend upon the structure of the equation (unknown on both sides of the equation), but upon the degree to which the equation has been made abstract by the negatives. Arithmetical equations with negatives therefore also represent an obstacle for those students who are unable to give them a concrete meaning. (p.350)

Vlassis reported that the use of the balance model was a viable approach to teaching students the formal method for solving linear equations. However, she noted that students' difficulties with negatives were not addressed by the model and would have to be overcome by other means.

In a series of experiments, Christou and Vosniadou [30] found that secondary students aged 12.5–14.5 years had a 'natural number bias' such that they tended to interpret symbols for real variables as representing natural numbers. The authors identified this bias as an important impediment in students' transition from arithmetic to algebra.

Chiu [31] reported metaphors that children and adults used when reasoning about integer arithmetic problems in a stock-market context. The most common metaphors identified involved motion, opposing objects, and social transactions. These metaphors were not necessarily related to the stock market, but Chiu found that they helped the participants to reason about the arithmetic required to solve tasks in the stock-market context.

Peled and Carraher [32] present evidence in favour of an algebraic approach to the teaching of integers. Part of their argument is that the contexts that are often used in integer instruction do not actually help students understand integers. With regard to money contexts in particular, the authors give two examples that illustrate the phenomenon that students have different ways of relating money contexts to numerical representations. They describe a classroom episode focused on two children who were using clothespins positioned on a clothesline to represent 'how much money they have' (p.313). When it came to one of the children borrowing money and going into debt, it was unclear to the class how this situation should be represented. On the one hand, he was receiving money. On the other hand, he would have to pay that money back. Also, once he used the borrowed money, he would be without money. The class discussed whether \$2 borrowed should result in the student's clothespin being positioned at 2, 0, or -2 on the number line.

Peled and Carraher [32] also report asking 15 preservice teachers how they would record the transaction of borrowing \$20 from one's parents. Eight of the participants suggested recording it as -20, while six thought it should be recorded as 20, and one of them acknowledged both of these possibilities. The authors do not provide an in-depth analysis of these different ways of relating integers to a money context – that was not their purpose – but they do introduce intriguing examples of a phenomenon that arises in the teaching and learning of integers.

In summary, the research literature has focused primarily on documenting students' difficulties and on studying instructional approaches. There is insufficient research that helps to bridge these bodies of work by investigating students' thinking, especially in relation to contexts that typically arise in integer instruction and applications. A context of particular interest is that of financial transactions. As discussed in the following section, this context is used prominently in integer instruction.

3.2. *Textbook approaches to integer instruction*

Curriculum and standard documents internationally emphasize relating integers to contexts (e.g., [33–35]). Thus, the ability to relate integers to contexts is considered a part of understanding integers. In the United States, the use of story-problem contexts in integer instruction is commonplace, as evinced by Author's [36] review of 18 fifth- and sixth-grade textbooks adopted by the state of California. We found that 94% of these used the context of money in relation to integers. Elevation (89%) and temperature (89%) were also popular contexts. Story problems involving these contexts may be used to motivate integer lessons or as application problems. One theme that emerged from our analysis of textbook approaches to integer instruction was that the story problems often seemed accessible and easy to solve without the use of negative numbers. As an example, the following story problem and solution appeared in the integer-addition section of a sixth-grade textbook:

The Debate Club's income from a car wash was \$300, including tips. Supply expenses were \$25. Use integer addition to find the club's total profit or loss:

$300 + (-25)$	*Use negative for the expenses.*
$300 - 25$	*Find the difference of the absolute values.*
275	*The answer is positive.*

The club earned \$275.[37, p.77]

In the above story problem, the reader is told that the debate club earned \$300 from a car wash and spent \$25 on supplies, making their profit \$300 − \$25 = \$275. Why a person would 'use integer addition' to solve this problem, except for the instruction to do so, is unclear. In fact, the same story problem would be identified as a subtraction problem if it appeared in an elementary textbook. Furthermore, even when the problem is framed as involving integer addition, the way that this 'addition' is performed is to subtract. The 'use' of integer addition in this instance does not appear to be useful. It results only in an additional step prior to the subtraction.

3.3. *Research questions*

The prevalence of story problems in textbook approaches to integer instruction, together with our observations about the nature of these, makes students reasoning about such problems a topic of interest. Money problems, in particular, stand out due to their prevalence and to the findings of other researchers. Stephan and Akyuz [26] found that the context of net worth could support students' learning of integer arithmetic. However, net worth is not a money context that typically is made explicit in textbooks. Rather, simpler money contexts feature prominently. Peled and Carraher [32] have found that story problems that are intended to relate to integer arithmetic often do not invoke negative numbers for students. They also reported that students have different ways of relating integers to money contexts.

In this study, we build on the previous research by investigating how secondary students reasoned about a simple story problem that involved borrowing money from a friend. Specifically, we asked the following research questions: *In what ways do secondary students relate arithmetic equations to the context of borrowing money? How frequent are the different ways of reasoning?*

4. Methods

We interviewed 40 children each in Grades 7 and 11 during the spring of 2011. The interviews were conducted at eight public middle and high schools in an urban area in California. Schools were selected that were below, about, and above average in performance on standardized tests so that we would likely see a range of student thinking and have a representative sample of students. Seventh graders came from the mainstream populations in each school. We selected 11th graders who were identified as being on a successful mathematics track. Specifically, they were enrolled in either pre-calculus or calculus in 11th grade. The interviews were conducted near the end of the school year.

All Grades 5, 6, and 7 mathematics textbooks adopted at the time by the California Department of Education contained material concerning integers and integer arithmetic. Thus, we expected that the students we interviewed would have received substantial instruction on integers. Indeed, we know from the interviews that all 80 students were familiar with positive and negative integers and were able to solve (at least some) problems involving integer arithmetic. We selected seventh graders because they would have had the bulk of their instruction on integer arithmetic, and we were interested in how they reasoned about relationships between equations and a debt context post-instruction. We also interviewed 11th graders who were enrolled in or had completed pre-calculus because we were interested in what those data might reveal about the reasoning of students who had been successful in high school algebra. As we note below, the story problem itself was not expected to be challenging for either group of students. Rather, it provided an occasion to investigate how the students thought about relating arithmetic equations to the context.

Interviews were videotaped and typically lasted 60–90 minutes. The interviews consisted of a range of tasks related to negative integers, including open number sentences, number comparisons, and story problems. Interviews were conducted at the students' school sites, during the school day. This report focuses on one task, which we refer to as the Money Problem. In other papers, we present the results of different interview tasks.[2,3]

4.1. *The Money Problem*

Students were given a simple story problem that involved borrowing money from a friend. We used easy numbers and an uncomplicated context because our focus was on students' reasoning in relating equations to the context, not on their abilities to perform complicated arithmetic or to solve multi-step story problems. We expected all students to understand the story problem itself and to solve it quickly, probably without the need for written work. The story problem was merely a starting point for the follow-up tasks of more interest to us. The problem appeared in written form on a sheet of paper that was presented to the student. It read as follows:

> Yesterday, you borrowed $8 from a friend to buy a school t-shirt. Today, you borrowed another $5 from the same friend to buy lunch. What's the situation now?

The problem was read aloud, and interviewers checked for students' understanding of the problem before moving forward. The interviewer often clarified the question by asking, 'Do you owe your friend money? Does your friend owe you money? How much money?' Students were asked to solve the problem and to explain their thinking. Each of the 80 students correctly answered that she or he would owe the friend $13. Thus, students seemed to understand the story and the question being asked.

Students were asked to write an equation that would represent the problem, including its solution, and to explain how this equation related to the story. Students were also asked if they could write additional equations that would also represent the story and to explain those as well. Next, they were presented with three equations, one by one. They were told that these equations had been written by other students to represent the same story. Students were asked to tell whether they thought that each equation matched the story and to explain their decision. The equations shown to students were the following:

(i)	$-8 + -5 = -13$
(ii)	$-8 - 5 = -13$
(iii)	$8 + 5 = 13$

The order of presentation depended on the equations that the student had written. If one of the equations (i)–(iii) matched an equation the student had written, it was shown before equations that the student had not written.

The Money Problem itself is similar to story problems used in integer-arithmetic sections of many US middle school textbooks. However, we structured the tasks differently.[1] Rather than instructing students to 'use integer addition' or otherwise suggesting how they should approach the problem, interviewers simply asked them to answer the question and to write one or more equations that they saw as representing the situation. When interviewers presented equations (i)–(iii), they did not suggest that these were correct or incorrect. They presented these equations as the responses of other students and said that they were interested in what the student being interviewed thought about each equation. Students were free to say 'yes' that the equation did or 'no' that it did not represent the story for each equation and to explain their thinking. We designed the task in this way because we wanted to know how students would reason of their own accord about a story problem involving borrowing money and whether and how they would relate negative numbers to the story.

4.2. *Analysis of students' responses*

We first analysed a subset of the data by open coding, attempting to understand and describe students' ways of reasoning. We used constant comparative analysis to identify emergent, distinguishing themes in students' reasoning.[38] After a set of codes that fit this subset of the data had been generated, they were used to code the remainder of the data. In particular, our analysis led to two prevalent codes, which we call *perspectives*. All but two students used one or the other of these perspectives in his or her responses, and only one student used both perspectives. Thus, the 80 students interviewed almost exclusively exhibited one perspective or the other. The perspective codes apply to students' explanations of their own equations as well as to their interpretations of equations (i)–(iii).

The lead author coded the 80 students' responses. As a reliability check, his coding decisions were compared with those of independent coders for the responses of 25% of the 80 students. Coders agreed on perspective codes for 9 of 10 seventh graders and 8 of 10 eleventh graders (85% overall). Disagreements were resolved by revisiting definitions, comparing these to the data, and coming to consensus regarding the most appropriate code.

5. Results

We offer examples of students' responses and describe in detail our interpretations of their reasoning. We then label and define the two perspectives that we identified. Finally, we report on frequencies of correct responses and occurrences of each perspective among the 40 seventh graders and the 40 eleventh graders who were interviewed.

5.1. *Students' responses*

Many students initially wrote $8 + 5 = 13$ as their equation. As an example, a seventh grader named Evelyn[2] wrote $8 + 5 = 13$ and explained her thinking as follows:

> *Interviewer*: Okay. And can you explain to me how this equation matches the story?
>
> *Evelyn*: Okay. So, yesterday I borrowed 8 dollars from my friend to buy a school t-shirt. So, I have 8 dollars from my friend [points to *8* in equation]. And then today I borrowed 5 from the

same friend, and I bought a lunch. And then plus 5 that I borrowed from her [circles + *5* in equation] equals 13 dollars that I borrowed from her [circles *13* in equation]. And what's the situation? I owe her 13 dollars.

Evelyn gave a very sensible response to the Money Problem. She seemed to understand the scenario described, and she correctly answered that she would owe her friend \$13. In Evelyn's explanation of her equation, she attended to number magnitude (i.e., absolute value). She articulated a correspondence between the \$8 in the story and the 8 in her equation, the \$5 in the story and the 5 in her equation, and the \$13 that was her answer and the 13 in her equation. Evelyn did not talk about the signs of the numbers or say anything to indicate that she had made a deliberate choice with regard to sign. Although she knew who would owe the money to whom, she gave no indication that she intended to use her equation to convey directional information. She indicated the direction verbally.

The interviewer then introduced the next task of interpreting equations written by other students. She first showed Evelyn the equation $8 + 5 = 13$, the same equation Evelyn had written. Evelyn wrote 'yes' next to it to indicate that she thought it matched the story. When the interviewer showed Evelyn equation (i), she was able to interpret it. She described a meaning for the negative numbers in relation to the story:

Interviewer: Would you read that one out loud, Evelyn? [Int. reveals $-8 + -5 = -13$].

Evelyn: Okay. Negative 8 plus negative 5 equals negative 13. Okay, well [pause], that makes sense! [Evelyn nods her head.]

Interviewer: It does?

Evelyn: Yeah, 'cause you're owing 5—well, this [points to $8 + 5 = 13$] is really telling you the total of how much you owe. And this [points to $-8 + -5 = -13$] is telling you that you're owing negative—minus—negative—well, you're owing 8 dollars, and then you're also owing 5, and you're subtracting, from like your money, 13.

Interviewer: Okay. That was a nice explanation.

Evelyn: I wasn't really thinking about negatives in that moment.

Although Evelyn had not written an equation involving negative numbers herself, she was able to make sense of $-8 + -5 = -13$ by reasoning about negatives as representing money owed.

Other students thought differently about the meaning of negative numbers in relation to the story. Like Evelyn, another seventh grader named Anh initially wrote $8 + 5 = 13$ to represent the story. When she was shown $-8 + -5 = -13$, Anh read the equation aloud and gave her interpretation of it:

Negative 8 plus negative 5 equals negative 13. I would think that's sort of right because you subtracted 8 dollars from your friend, and then you subtracted 5 again. So, you took—so, negative's like you took—so, you took 13 dollars from your friend.

Anh decided that $-8 + -5 = -13$ did describe the story, and she wrote 'yes' next to it. Anh interpreted negative numbers as indicating money taken from the lender, or lost by the lender. By borrowing \$8 and \$5 from the friend, the borrower had taken a total of \$13 from her friend. The net result of these transactions was that the friend had lost \$13.

5.2. *Students' perspectives*

Students' responses, like those presented above, led to the identification of two distinct perspectives regarding the relationship between equations and the context of borrowing money. We describe these as *Negatives as Debt* and *Negatives as Loss*.

5.2.1. *Negatives-as-Debt perspective*

The student views negative numbers as representing debts. For example, $-8 + -5 = -13$ is seen as matching the story about borrowing money from a friend because -8 represents a debt of \$8, -5 represents an additional debt of \$5, and -13 represents the total debt of \$13. From this perspective, positive numbers make sense to represent the lender's situation. That is, (positive) 8 + (positive) 5 = (positive) 13 would not describe the situation from the borrower's side, but it could be used to describe the situation from the lender's side. Note that we use *Negatives as Debt* as shorthand for *Negative as Debt, Positives as Credit*. This perspective code applies to reasoning about both positive and negative numbers.

5.2.2. *Negatives-as-Loss perspective*

The student views negative numbers as representing money that the lender lost by lending it. Thus, $-8 + -5 = -13$ does not match the story from the borrower's side because the borrower gained money. Rather, it represents the story from the lender's side because the lender lost money. From this perspective, positive numbers are seen as representing money that the borrower gained (even though the money was borrowed). Thus, $8 + 5 = 13$ (interpreted as involving positive numbers) matches the story from the borrower's side because 8 represents \$8 given to the borrower by the friend, 5 represents an additional \$5 given to the borrower by that friend, and 13 represents the total of \$13 that the borrower gained. Like Negatives as Debt, *Negatives as Loss* refers to a way of reasoning about both positive and negative numbers (*Negatives as Loss, Positives as Gain*).

5.2.3. *Clarification: perspectives versus sides*

To clarify, by *perspective* we do not mean looking at the borrowing/lending situation from one side or the other. Rather, we mean to refer to students' ways of relating the signs of the numbers to the story. A student's perspective, together with a focus on one side of the situation or the other, affected whether the student gave a yes or no response to an equation. If a student took a Negatives-as-Debt perspective, then $-8 + -5 = -13$ made sense to describe the story from the side of the borrower, not the lender. If a student took a Negatives-as-Loss perspective, then $-8 + -5 = -13$ made sense to describe the story from the side of the lender, not the borrower. Our descriptions of students' perspectives are not descriptions of from which side of the situation the students viewed it. We found that perspective, not side, tended to be consistent across students' interpretations of equations. To make sense of an equation involving negatives, having written and explained $8 + 5 = 13$, students often switched sides. They did not switch perspectives.

As an illustration of the distinction between side and perspective, consider the response of Tommy. Like Anh, Tommy wrote $8 + 5 = 13$ to represent the story. Also like Anh, Tommy expressed a Negatives-as-Loss perspective. However, Tommy said 'no' to $-8 + -5 = -13$, whereas Anh said 'yes.' As shown in Tommy's response below, the difference between their answers results from the side of the situation on which they focused – the borrower's or the lender's:

> *Interviewer*: Can you read that one for me? [Interviewer reveals $-8 + -5 = -13$ on paper.]
>
> *Tommy*: Negative 8 plus negative 5 equals negative 13.
>
> *Interviewer*: Okay, and what do you think about that? Do you think that that describes the situation, or the story?
>
> *Tommy*: Um, no. Because it's like, it's basically like saying that they owe you because it's like, it's like you're not taking any money. It's like they're taking your money. Because it's negative, which means it's like, it's kind of lower; it's lower than zero. Like, so then it's like they're owing you money, instead of you owing them.

Like Anh, Tommy interpreted negative numbers as indicating someone 'taking' money. However, he viewed the situation from his side (as the borrower), and so he said that the equation would not fit because the negatives would mean that money had been taken from him, rather than given to him. Tommy considered whether the equation would make sense to describe the borrower's situation. On the basis of his way of relating negative numbers to the context, it did not: the borrower had gained \$13, not lost \$13. Anh reasoned about the meaning of negative numbers similarly to Tommy, but she said that the equation could represent the story because she viewed it as describing the lender's situation: \$13 was taken from the lender, so it made sense to use negative numbers to describe that aspect of the situation.

Whereas Anh and Tommy thought about the situation from one side, some students – especially 11th graders – switched sides to view different equations as describing the story (and answer 'yes'). The 11th graders also tended to be more articulate than the 7th graders in explaining their reasoning. A particularly clear explanation, given by an 11th grader named Sarah, illustrates a student switching sides to interpret an equation involving negatives. It also illustrates how the perspectives may be applied to the interpretation of both negative and positive numbers. Like most students, Sarah initially wrote $8 + 5 = 13$. When she was shown $-8 + -5 = -13$, she explained that this equation could also represent the story:

> *Interviewer*: How about this one? [Interviewer reveals $-8 + -5 = -13$ on paper.] Would you read it out loud?
>
> *Sarah*: Negative 8 plus negative 5 equals negative 13. Um, I was thinking about doing it with negatives at first, like, just like that problem. Like I *owed* her 13 dollars. So, it was like negative of my money. But I thought I would just leave it easier and keep it positive—with, like, positive numbers.
>
> *Interviewer*: Okay. So, what do you think about this one? Do you think that it could describe that story?
>
> *Sarah*: Yeah, it still could.
>
> *Interviewer*: Okay.

> *Sarah*: Because this [pointing to -8] could be like negative of my money, [points to -5] negative of my money, and [points to -13] negative of my money. Whereas, this one [points to 8 in her original number sentence] is positive of her money, [points to 5] positive of her money, and [points to 13] positive of her money.

In Sarah's response to $-8 + -5 = -13$, she articulated a Negatives-as-Debt perspective. That is, she viewed negative numbers as an appropriate representation of her debt. She then returned to her original equation and superimposed on it a Negatives as Debt, Positives as Credit perspective: if $8 + 5 = 13$ represents the friend's situation, then positive numbers are appropriate. For Sarah, the action/result of borrowing money could be regarded as either 'negative of my money' (negative from the borrower's side) or 'positive of her money' (positive from the lender's side). Both views reflect the perspective that we call Negative as Debt, Positives as Credit.

5.3. *Frequencies of perspectives*

We report on the perspectives exhibited by the 40 seventh graders and the 40 eleventh graders interviewed. We include in these results the frequencies of correct answers to the story problem, the equations that students wrote, and the perspective(s) that students expressed.

5.3.1. *Seventh graders' responses and perspectives*

All 40 seventh graders solved the money story problem correctly. That is, they said something like 'I owe my friend \$13.' They were then asked to write one or more equations to describe the story. Of the 40 students, 33 (82.5%) wrote $8 + 5 = 13$ as one of their equations.

More specifically, 37 of the 40 seventh graders wrote only one equation. Of these, 31 wrote $8 + 5 = 13$. Five of the students who wrote only one equation wrote $-8 + -5 = -13$, and one wrote $-8 - 5 = -13$. Three students wrote two distinct equations each. One wrote $-8 - 5 = -13$ first, and then $8 + 5 = 13$. Another wrote $8 + 5 = 13$ first, and then $-8 - 5 = -13$. The third wrote unique equations: first L8 + L5 = M (where L stood for his friend Liam and M stood for money) and then L8 + L5 = L13.

In total, only 8 (20%) of the 40 seventh graders wrote equations involving negative numbers. Six students wrote $-8 + -5 = -13$, and two wrote $-8 - 5 = -13$. Students' explanations for these equations attended to the meaning of negative numbers in the story context. That is, they explicitly represented the direction of borrowing/owing symbolically, and they explained their representation. Including the student who wrote the equations with L for Liam, a total of nine students wrote an equation other than $8 + 5 = 13$. Of these, seven expressed a Negatives-as-Debt perspective, and two expressed a Negatives-as-Loss perspective in explaining these equations.

The students were then shown equations (i)–(iii). We report here on the perspective reflected in their interpretations of each of these equations. We include the responses of all the 40 students, regardless of the equation that they wrote initially. Often students' responses explicitly addressed perspective for either (i) or (ii), but not both. In cases for which the perspective was explicitly addressed for both, the perspective was always consistent. For these reasons, we group students' responses to (i) and (ii). Of the 40 seventh graders, 19 (47.5%) articulated a Negatives-as-Debt perspective in interpreting equations with negative numbers in relation to the story. They interpreted the negative numbers in the given equation as representing debt. For them, the given equation was appropriate for describing the borrower's situation. An additional 19 students (47.5%) expressed a Negatives-as-Loss perspective in interpreting equations with negative numbers. These students viewed negative numbers as representing the money lost or taken away. For these students, the given equation described the lender's situation. Two students (5%) were not given any perspective code for equation (i) or (ii). Although they were familiar with negative numbers and able to make sense of the story, they were not able to interpret negative numbers in relation to the story.

When equation (iii) ($8 + 5 = 13$) was shown to the students, most of them simply noted that it matched an equation they had written. Therefore, for students who had written $8 + 5 = 13$, little was learned from this question. However, 7 of the 40 students did give an interpretation of $8 + 5 = 13$ that invoked a signed-number perspective. Five students interpreted $8 + 5 = 13$ in terms of Negatives as Debt (in this case, Positives as Credit). Two students interpreted $8 + 5 = 13$ in terms of Negatives as Loss (Positives as Gain). These seven students had written an equation other than $8 + 5 = 13$ initially, and the perspective reflected in their interpretation of $8 + 5 = 13$ was consistent with an explanation they had given previously. For example, a student who had written $-8 + -5 = -13$ and given a Negatives-as-Debt explanation then interpreted $8 + 5 = 13$ in terms of Positives as Credit. He had written $-8 + -5 = -13$ to describe the situation of the borrower, with negative numbers used to represent money owed. He said that $8 + 5 = 13$ could be used to describe the situation of the lender, who would eventually be paid back (i.e., positive 13 represented a credit of $13).

5.3.2. *Eleventh graders' responses and perspectives*

All 40 eleventh graders solved the story problem correctly, saying something like 'I would owe my friend $13.' Of these 40 students, 19 (47.5%) wrote $8 + 5 = 13$ or wrote both $8 + 5 = 13$ and $5 + 8 = 13$. Ten of the students (25%) wrote only equations involving negative numbers. Eleven (27.5%) wrote a mix: $8 + 5 = 13$, as well as at least one equation involving negative numbers.

For equations with negatives, 25 (62.5%) of the 40 eleventh graders interpreted these in terms of Negatives as Debt, whereas 14 (35%) interpreted them in terms of Negatives as Loss. One 11th grader (2.5%) did not seem to have a sensible interpretation of negative numbers in relation to the story. When describing how $8 + 5 = 13$ related to the story, 30 (75%) of the 40 eleventh graders addressed magnitude only, so they did not get a perspective code for that response. Of the remaining 10 students, 7 of them (17.5% of the 11th graders) interpreted the equation in terms of Negatives as Debt, and 3 (7.5% of the 11th graders) interpreted the equation in terms of Negatives as Loss.

5.3.3. *Comparison of seventh- and eleventh-grade results*

Table 1 summarizes the results for both groups of students. There are similarities and differences in the results for seventh and eleventh grades. For both groups, all students solved the story problem correctly. These groups are distinguished by the frequency with which they used negative numbers spontaneously in their equations and the relative frequencies of the Negatives-as-Debt and Negatives-as-Loss perspectives. Only 20% of the seventh graders wrote an equation involving negative numbers. By contrast, more than half (52.5%) of the 11th graders wrote an equation involving negatives. Of the 38 seventh graders who were able to interpret equations involving negative numbers in relation to the story,

Table 1. Summary of results for 7th and 11th graders.

Grade level	Number of students interviewed	Percentage who correctly solved the story problem (%)	Percentage who used negatives in an equation (%)	Percentage who invoked Negatives-as-Debt (%)	Percentage who invoked Negatives-as-Loss (%)
7	40	100	20	47.5	47.5
11	40	100	52.5	62.5	35

half interpreted these in terms of Negatives as Debt and half interpreted them in terms of Negatives as Loss. By contrast, of the 39 eleventh graders who were able to interpret equations involving negatives in relation to the story, 62.5% interpreted these in terms of Negatives as Debt, and 35% interpreted them in terms of Negatives as Loss. Thus, the 11th graders (who were on a successful mathematics track) were more likely than the 7th graders to use negative numbers to represent the situation, and they were more likely to interpret the relationship between negative numbers and the story problem from a Negatives-as-Debt perspective.

6. Conclusions

We have contributed findings concerning secondary students' mathematical thinking related to integers. Specifically, we found differences in the ways in which 7th and 11th graders related equations involving integers to a money context. All the 80 participants were able to solve the Money Problem. Most initially wrote $8 + 5 = 13$ and focused only on number magnitude in their explanations. However, almost all of the students were able to interpret equations involving negative numbers in relation to the story when asked to do so. Students' responses showed that the use of negative numbers was unnecessary for them to solve a problem about borrowing money, yet most were capable of relating signed numbers to the context. Thus, we agree with Peled and Carraher [32] that the typical use of such contexts in instruction may not help students understand integers.

Peled and Carraher [32] reported that students had different ways of relating integers to money contexts. We investigated this phenomenon in a systematic way, and we found two perspectives – Negatives as Debt and Negatives as Loss – to be prevalent among 7th and 11th graders. These perspectives were equally common among the sample of 7th graders, whereas Negatives as Debt was more common than Negatives as Loss among the 11th graders. (As a reminder, the 11th graders were on what we considered to be a successful mathematics track, whereas the 7th graders belonged to the general student population.) We also found that students tended to reason one way or the other in relating equations to the Money Problem. To relate different equations to the same situation, they switched sides rather than perspectives.

The major instructional implication from this work is that it is important to be sensitive to nuanced distinctions in how people relate signed numbers to contexts. Often, students and teachers refer to negative numbers as being related to owing. It is easy to do this without making explicit the details of the representational relationship that one has in mind. Our findings concerning students' perspectives remind us of the importance of eliciting the details of students' thinking and of having explicit discussions of different ways of reasoning. If teachers are unaware of students' distinct perspectives, their abilities to support students' learning are limited. The distinction between Negatives as Debt and

Negatives as Loss may go overlooked unless the teacher and students are precise about how they see numbers and equations as being related to contexts. Thus, the distinctions introduced in this paper have the potential to be powerful tools for teachers, giving them a lens and a language to aid in communicating with students about challenging ideas that arise in integer instruction.

The Negatives-as-Debt perspective seems to be more conventional among mathematically literate adults (such as the authors of this paper). It was also the more common perspective among the 11th graders in our study. Note, however, that we do not view these perspectives as being correct or incorrect. Rather, they involve matters of convention, and this reminds us of the importance of clear communication in the doing of mathematics. Indeed, we are capable of making sense of and adopting either perspective, and we believe that the ability to recognize and switch perspectives is a desirable aspect of understanding signed numbers. We elaborate on this point in another paper [39].

This paper contributes to the mathematics education research literature related to integers primarily by investigating students' perspectives concerning the relationship between arithmetic equations and a debt context. This research is important and relevant because debt contexts are often used in integer instruction, and there has been little research in this area. Stephan and Akyuz [26] found evidence that the more complicated context of net worth could be a viable one for supporting students' learning of integer arithmetic – in a mathematics class taught by a highly skilled teacher who was sensitive to students' thinking and to the shared meanings that developed during classroom activities. Whatever contexts are used in integer instruction, we believe that this latter aspect is of utmost importance. Knowing the ways of reasoning that students may bring when relating integers and equations to such contexts can better equip teachers to support students' learning.

Although this study contributes to the literature, additional questions remain. We believe that it would be worthwhile to further investigate students' perspectives and the implications of these for integer instruction. We are pleased to contribute our findings to the literature, and we hope that others will make use of this work.

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Notes

1. To be clear, we did not select this story problem for instructional purposes. The interviews were not intended to teach students anything about integers, and we are not suggesting that this context should be used in integer instruction. We investigated how students thought about the money problem because it is the kind of problem that *is* often used in integer instruction.
2. Student names are pseudonyms.

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