Sufficient Conditions for Temporal Logic Specifications in Hybrid Dynamical Systems ⋆

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Abstract: In this paper, we introduce operators, semantics, and conditions that, when possible, are solution-independent to guarantee basic temporal logic specifications for hybrid dynamical systems. Employing sufficient conditions for forward invariance and finite time attractivity of sets for such systems, we derive such sufficient conditions for the satisfaction of formulas involving temporal operators and atomic propositions. Furthermore, we present how to certify formulas that have more than one operator. Academic examples illustrate the results throughout the paper.

Keywords: Linear Temporal Logic, Model Checking, Hybrid Systems

1. INTRODUCTION

High-level languages are useful in formulating specifications for dynamical systems that go beyond classical asymptotic stability, where convergence to the desired point or set is typically certified to occur in the limit, that is, over an infinitely long time horizon; see, e.g., [Kloetzer and Belta, 2008, Kwon and Agha, 2008, Tabuada and Pappas, 2006]. Temporal logic employs operators and logic to define formulas that the solutions (or executions) to the systems should satisfy after some finite time, or during a particular amount of bounded time.

Linear temporal logic (LTL), as introduced in [Pnueli, 1977], permits to formulate specifications that involve temporal properties of computer programs; see also [Pnueli and Manna, 1992]. Numerous contributions pertaining to modeling, analysis, design, and verification of LTL specifications for dynamical systems have appeared in the literature in recent years. Without attempting to present a thorough review of the very many articles in such topic, it should be noted that in [Fainekos et al., 2009], the authors employ temporal logic to solve a problem involving multiple mobile robots. In [Dimitrova and Majumdar, 2014], the design of controllers to satisfy alternating-time temporal logic (ATL*), which is an expressive branching-time logic that allows for quantification over control strategies, is pursued using barrier and Lyapunov functions for a class of continuous-time systems. More recently, using similar programming tools, in [Saha and Julius, 2016], tools to design reactive controllers for mixed logical dynamical systems so as to satisfy high-level specifications given in the language of metric temporal logic are proposed. Promising extensions of these techniques to the case of specifications that need to hold over pre-specified bounded horizons, called signal temporal logic, have been recently pursued in several articles; see, e.g., [Raman et al., 2015], to just list a few.

Tools for the systematic study of temporal logic properties in dynamical systems that have solutions (or executions) changing continuously over intervals of ordinary continuous time and, at certain time instances, having jumps in their continuous-value and discrete-valued states, such as the frameworks proposed in [Collins, 2004, Goebel et al., 2012, Haddad et al., 2006, Lygeros et al., 2003, van der Schaft and Schumacher, 2000], are much less developed. In such hybrid dynamical systems, the study of temporal logic using discretization-based approaches may not be fitting as, in principle, the time at which a jump occurs is not known a priori and are likely to occur aperiodically.

In this paper, we present tools that permit guaranteeing high-level specifications for solutions to hybrid dynamical systems that neither require discretization of the dynamics or of the state space, nor the computation of the solutions themselves. Our approach consists of imposing mild properties on the data defining the system and requiring existence of solution-independent certificates, such as Lyapunov-like functions for the satisfaction of the given formula. For a broad class of hybrid dynamical systems, sufficient conditions for the satisfaction of temporal logic formulas using one temporal operator are first presented using forward (pre-)invariance and finite time attractivity tools in [Chai and Sanfelice, 2015, Li and Sanfelice, 2016]. Our approach allows us to provide an estimate of the (hybrid) time it takes for a temporal specification to be satisfied, with the estimate only depending on a Lyapunov function and the initial condition of the solution being considered, but not involving the solution itself.

While our most of results do not require computing solutions to the hybrid dynamical system, which is a key advantage when compared to methods for continuous-time, discrete-time, and mixed logic dynamical systems cited above and the method for hybrid traces in [Cimatti et al., 2015], the price to pay when using the results in this paper is finding a certificate for finite time attractivity,

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which is in terms of a Lyapunov-like function. It should be noted that though our conditions are weaker than those in [Dimitrova and Majumdar, 2014] finding such functions might be challenging at times. However, the same complexity is present in Lyapunov methods for certifying asymptotic stability of a point (or a set) in [Khalil, 2002] or for employing continuously differentiable barrier certificates and Lyapunov functions to certify temporal logic constraints for continuous-time systems.

The remainder of this paper is organized as follows. Basic notions are summarized in Section 2. Section 3 introduces LTL for hybrid systems. The sufficient conditions to guarantee the satisfaction of LTL formulas are presented in Section 4. Due to space constraints, some details and proofs are omitted and will be published elsewhere.

2. PRELIMINARIES

A hybrid system $\mathcal{H} = (C, F, D, G)$ can be described as follows [Goebel et al., 2012]:

$$\mathcal{H} \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{1}$$

where $x \in \mathcal{X}$ is the state and $\mathcal{X}$ is the state space, $F : \mathcal{X} \rightrightarrows \mathcal{X}$ is a set-valued map and denotes the flow map capturing the continuous dynamics on the flow set C, and $G : \mathcal{X} \rightrightarrows \mathcal{X}$ is a set-valued map and defines the jump map capturing the discrete dynamics on the jump set D. A solution ϕ to $\mathcal{H}$ has initial condition $\phi(0,0) \in \mathcal{X} \cap (\overline{C} \cup D)$ and is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t is the ordinary time variable, j is the discrete jump variable, $\mathbb{R}_{\geq 0} := [0, \infty)$, and $\mathbb{N} := \{0, 1, 2, \dots\}$. The domain $\operatorname{dom} \phi \subset \overline{\mathbb{R}}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \cap ([0, T] \times \{0, 1, \dots, J\})$ can be written as the union of sets $\bigcup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution is given by $(t, j) \mapsto \phi(t, j)$ and for each j, $t \mapsto \phi(t, j)$ is absolutely continuous. A function $\phi : E \to \mathbb{R}^n$ is a hybrid arc if E is a hybrid time domain and if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval $I_j = \{t : (t, j) \in E\}$. A hybrid arc ϕ is a solution to $\mathcal{H} = (C, F, D, G)$ if $\phi(0,0) \in \overline{C} \cup D$; for all $j \in \mathbb{N}$ such that I_j has nonempty interior, $\phi(t, j) \in C$ for all $t \in \operatorname{int} I_j$ and $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all $t \in I_j$; for all $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$, $\phi(t, j) \in D$ and $\phi(t, j + 1) \in G(\phi(t, j))$. A solution to $\mathcal{H}$ is called maximal if it cannot be further extended.

For convenience, we define the range of a solution ϕ to a hybrid system $\mathcal{H}$ as $\operatorname{rge} \phi = \{\phi(t, j) : (t, j) \in \operatorname{dom} \phi\}$. We also define the set of maximal solutions to $\mathcal{H}$ from the set K as $\mathcal{S}_{\mathcal{H}}(K) := \{\phi : \phi \text{ is maximal solution to } \mathcal{H} \text{ with } \phi(0,0) \in K\}$. See [Goebel et al., 2012] for more details about hybrid dynamical systems.

Given $x \in \mathbb{R}^n$ and a closed set $K \subset \mathbb{R}^n$, $|x|_K := \inf_{y \in K} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, denoted by $\alpha \in \mathcal{K}$, if it is zero at zero, continuous, and strictly increasing; α is a class-$\mathcal{K}_\infty$ function, denoted by $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and is unbounded. For any $x \in \mathbb{R}$, $\operatorname{ceil}(x)$ denotes the next larger integer of x.

3. LINEAR TEMPORAL LOGIC FOR HYBRID DYNAMICAL SYSTEMS

Linear Temporal Logic (LTL) provides a framework to specify desired properties such as *safety*, i.e., "something bad never happens," and *liveness*, i.e., "something good eventually happens." In this section, for a given hybrid system $\mathcal{H}$, we define operators and specify properties of $\mathcal{H}$ with LTL formulas [Sanfelice, 2015]. We first introduce atomic propositions.

Definition 3.1. (Atomic Proposition) An atomic proposition p is a statement on the system state x that, for each x, p is either `True` (1 or $\top$) or `False` (0 or $\bot$).

A proposition p will be treated as a single-valued function of x, that is, it will be a function $x \mapsto p(x)$. The set of all possible atomic propositions will be denoted by $\mathcal{P}$.

Logical and temporal operators are defined as follows:

Definition 3.2. (Logic Operators)

- $\neg$ is the *negation* operator
- $\vee$ is the *disjuction* operator
- $\wedge$ is the *conjunction* operator
- $\Rightarrow$ is the *implication* operator
- $\Leftrightarrow$ is the *equivalence* operator

Definition 3.3. (Temporal Operators)

- $\bigcirc$ is the *next* operator
- $\Diamond$ is the *eventually* operator
- $\Box$ is the *always* operator
- $\mathcal{U}$ is the *until* operator

Given a hybrid system $\mathcal{H}$, the semantics of LTL are defined as follows. For simplicity, we consider the case of no inputs and state-dependent atomic propositions. When a proposition p is `True` at $(t, j) \in \operatorname{dom} \phi$, i.e., $p(\phi(t, j)) = 1$, it is denoted by

$$\phi(t, j) \Vdash p, \tag{2}$$

whereas if p is `False` at $(t, j) \in \operatorname{dom} \phi$, it is written as

$$\phi(t, j) \nVdash p. \tag{3}$$

An LTL formula is a sentence that consists of atomic propositions and operators of LTL. An LTL formula f being satisfied by a solution $(t, j) \mapsto \phi(t, j)$ at some time (t, j) is given by

$$(\phi, (t, j)) \vDash f, \tag{4}$$

while f not satisfied by a solution $(t, j) \mapsto \phi(t, j)$ at some time (t, j) is denoted by [1]

$$(\phi, (t, j)) \nvDash f. \tag{5}$$

Let $p, q \in \mathcal{P}$ be atomic propositions. The semantics of LTL are defined as follows: given a solution ϕ to $\mathcal{H}$ and $(t, j) \in \operatorname{dom} \phi$

$$(\phi, (t, j)) \vDash p \Leftrightarrow \phi(t, j) \Vdash p \tag{6a}$$

$$(\phi, (t, j)) \vDash \neg p \Leftrightarrow (\phi, (t, j)) \nvDash p \tag{6b}$$

$$(\phi, (t, j)) \vDash p \vee q \Leftrightarrow (\phi, (t, j)) \vDash p \text{ or } (\phi, (t, j)) \vDash q \tag{6c}$$

$$(\phi, (t, j)) \vDash \bigcirc p \Leftrightarrow (\phi, (t, j + 1)) \vDash p \tag{6d}$$

$$\begin{aligned} (\phi, (t, j)) \vDash p \mathcal{U} q \Leftrightarrow\ & \exists (t', j') \in \operatorname{dom} \phi, \\ & t' + j' \geq t + j \text{ s.t. } (\phi, (t', j')) \vDash q, \\ & \text{and } \forall (t'', j'') \in \operatorname{dom} \phi \text{ s.t.} \\ & t + j \leq t'' + j'' < t' + j', (\phi, (t'', j'')) \vDash p \end{aligned} \tag{6e}$$

[1] Note that to be compatible with the literature, instead of $\Vdash$, we use $\vDash$ for a formula.

$$(\phi,(t,j)) \vDash p \wedge q \;\Leftrightarrow\; (\phi,(t,j)) \vDash p \text{ and } (\phi,(t,j)) \vDash q \quad (6f)$$

$$(\phi,(t,j)) \vDash \Box p \;\Leftrightarrow\; (\phi,(t',j')) \vDash p \quad (6g)$$
$$\forall\, t'+j' \geq t+j,\ (t',j') \in \text{dom } \phi$$

$$(\phi,(t,j)) \vDash \Diamond p \;\Leftrightarrow\; \exists (t',j') \in \text{dom } \phi, \quad (6h)$$
$$t'+j' \geq t+j \text{ s.t. } (\phi,(t',j')) \vDash p.$$

The same semantics of LTL are used for formulas. For example, a formula $f = \bigcirc p$ implies that f will be `True` at the next hybrid time so that $\phi(t, j+1) \Vdash p$ for all $(t,j) \in \text{dom } \phi$ and $(t, j+1) \in \text{dom } \phi$. With the above semantics, we propose sufficient conditions that, when possible are solution independent, to check whether a given solution satisfies a formula at hybrid time $(0,0)$ or at each hybrid time $(t,j) \in \text{dom } \phi$.

4. SUFFICIENT CONDITIONS FOR LTL FORMULAS WITH ONE TEMPORAL OPERATOR

In this section, we present sufficient conditions to guarantee atomic propositions involving the temporal operators *always*($\Box$), *eventually*($\Diamond$), and *until*($\mathcal{U}$). Due to space constraints, conditions satisfying the *next* operator are omitted. We first build a set K on which the atomic proposition is satisfied. Then, the satisfaction of the formula is assured by guaranteeing particular properties of the solutions to the hybrid system relative to the set K.

4.1 Conditions to guarantee $\Box$

According to the definition of the $\Box$ operator, given an atomic proposition p, a solution $(t,j) \mapsto \phi(t,j)$ to a hybrid system $\mathcal{H} = (C,F,D,G)$ satisfies the formula

$$f = \Box p \quad (7)$$

at $(t,j) \in \text{dom } \phi$ when we have that $\phi(t',j')$ satisfies p for all $t'+j' \geq t+j$ such that $(t',j') \in \text{dom } \phi$. The set of points in $\mathcal{X}$ satisfying an atomic proposition p is given by

$$K := \{x \in \mathcal{X} : p(x) = 1\}. \quad (8)$$

To guarantee that every solution ϕ to $\mathcal{H}$ satisfies f in (7) at each $(t,j) \in \text{dom } \phi$, each solution needs to start and stay in the set K. For this purpose, we recall the definition of forward pre-invariance and then present sufficient conditions guaranteeing f in (7). Our result relies on an extension of a result on forward pre-invariance in [Chai and Sanfelice, 2015].

Definition 4.1. (Forward pre-Invariance) Consider a hybrid system $\mathcal{H}$ on $\mathcal{X}$. A set $K \subset \mathcal{X}$ is said to be forward pre-invariant for $\mathcal{H}$ if for every $x \in K$ there exists at least one solution, and every solution $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ satisfies $\text{rge}\, \phi \subset K$.

The conditions given below provide sufficient conditions to verify that $\mathcal{H}$ is such that every solution ϕ to $\mathcal{H}$ with $\phi(0,0) \Vdash p$ satisfies $f = \Box p$. Sufficient conditions in terms of Lyapunov-like functions as in [Chai and Sanfelice, 2015] and barrier functions in [Maghenem and Sanfelice, 2018] can also be formulated. Below, $T_{K \cap C}(x)$ denotes the tangent cone of $\{x \in C : p(x) = 1\}$ at a point $x \in \mathcal{X}$; see [Goebel et al., 2012, Definition 5.12].

Assumption 4.2. Suppose C is closed in $\mathcal{X}$, $C \subset \text{dom } F$, and $D \subset \text{dom } G$, and

- The state space $\mathcal{X}$ and the atomic proposition p are such that K in (8) is closed; and
- For every $x \in \mathcal{X}$ such that $p(x) = 1$, $x \in C \cup D$; and
- The map $F : \mathcal{X} \rightrightarrows \mathcal{X}$ is outer semicontinuous, locally bounded relative to $\{x \in C : p(x) = 1\}$, and $F(x)$ is convex nonempty for every $x \in \{x \in C : p(x) = 1\}$. Additionally, the map F is locally Lipschitz on C.

Theorem 4.3. Consider a hybrid system $\mathcal{H} = (C,F,D,G)$ on $\mathcal{X}$ satisfying Assumption 4.2. Then, the formula $f = \Box p$ is satisfied for all solutions ϕ to $\mathcal{H}$ (and for all $(t,j) \in \text{dom } \phi$) if $\phi(0,0) \Vdash p$ and the following properties hold:

1) for each $x \in \mathcal{X}$ such that $p(x) = 1$ and $x \in D$, every $\xi \in G(x)$ satisfies $p(\xi) = 1$; and
2) for each $x \in \mathcal{X}$ such that $p(x) = 1$, $x \in C$, and $x \notin L$, $F(x) \subset T_{K \cap C}(x)$, where $L = \{x \in C : F(x) \cap T_C(x) = \emptyset\}$. ■

Remark 4.4. Note that $\Box p$ is satisfied for all solutions ϕ to $\mathcal{H}$ if $\phi(0,0) \Vdash p$ and $\phi(t,j) \Vdash p$ for all future hybrid time $(t,j) \in \text{dom } \phi$. Under the conditions in Theorem 4.3, solutions with $\phi(0,0) \nVdash p$ may satisfy p after some time if ϕ reaches the set $\{x \in \mathcal{X} : p(x) = 1\}$ in finite time. Convergence to such set in finite hybrid time is presented in the next section.

Example 4.5. Consider a hybrid system $\mathcal{H} = (C,F,D,G)$ with the state $x = (x_1, x_2) \in \mathbb{R}^2$ given by

$$F(x) := \begin{bmatrix} x_2 \\ -x_1 x_2 \end{bmatrix} \quad \forall x \in C := \{x \in \mathbb{R}^2 : |x| \leq 1, x_2 \geq 0\},$$

$$G(x) := \begin{bmatrix} -0.9 x_1 \\ x_2 \end{bmatrix} \quad \forall x \in D := \{x \in \mathbb{R}^2 : x_1 \geq -1, x_2 = 0\}.$$

Define an atomic proposition p as follows: for every $x \in \mathcal{X} := \mathbb{R}^2$, $p(x) = 1$ when $|x| \leq 1$ and $x_2 \geq 0$; $p(x) = 0$ otherwise. Let $K = \{x \in \mathcal{X} : p(x) = 1\}$. It is clear that for each $x \in D \cap K$, every $\xi \in G(x)$ satisfy $p(\xi) = 1$. For every $x \in \text{int } C$, $T_{K \cap C}(x) = \mathbb{R} \times \mathbb{R}$; for every x in the boundary of $K \cap C$, $T_{K \cap C}(x)$ is the set of tangent vectors to the unit circle or $T_{K \cap C}(x)$ includes all vectors that point inward; for every $x \in K$ such that $x_2 = 0$, $F(x) = 0$. That is, for each $x \in \mathcal{X}$ such that $x \in (C \cap K) \backslash L$, $F(x) \subset T_{K \cap C}(x)$. Therefore, via Theorem 4.3, the formula $f = \Box p$ is satisfied for each solution ϕ to $\mathcal{H}$ from K and at each $(t,j) \in \text{dom } \phi$. △

4.2 Conditions to guarantee $\Diamond$

A solution $(t,j) \mapsto \phi(t,j)$ to a hybrid system $\mathcal{H}$ satisfies the formula

$$f = \Diamond p \quad (9)$$

at $(t,j) \in \text{dom } \phi$ when there exists $(t',j') \in \text{dom } \phi$ such that $t'+j' \geq t+j$, and $\phi(t',j')$ satisfies p. The same set K introduced in (8) is used in this section.

To guarantee that every solution ϕ to $\mathcal{H}$ satisfies f in (9) at each $(t,j) \in \text{dom } \phi$, the distance of each solution to K should become zero at some finite $(t,j) \in \text{dom } \phi$. For this purpose, we recall the definition of finite time attractivity (FTA) for hybrid systems. After that, we will present sufficient conditions guaranteeing the formula f in (9). Our result relies on an extension of a result on FTA in [Li and Sanfelice, 2016].

We consider a hybrid system $\mathcal{H}$ on $\mathcal{X}$, a closed set $K \subset \mathcal{X}$, and an open neighborhood [2] $\mathcal{N}$ of K. The amount of time required to converge to the set K is captured by the settling-time function $\mathcal{T}$.

Definition 4.6. (Finite Time Attractivity) A closed set K is said to be finite time attractive (FTA) for $\mathcal{H}$ if there

[2] The set $\mathcal{N}$ can be chosen as $\mathcal{N} \supset \mathcal{X}$ for the global version of FTA.

exists an open neighborhood U of K such that every solution $\phi \in \mathcal{S}_{\mathcal{H}}(U)$, $\sup_{(t,j)\in \mathrm{dom}\ \phi} t+j \geq \mathcal{T}(\phi(0,0))$ and

$$\lim_{(t,j)\in \mathrm{dom}\ \phi\, :\, t+j \nearrow \mathcal{T}(\phi(0,0))} |\phi(t,j)|_K = 0. \tag{10}$$

As stated above, the satisfaction of the formula $f = \diamondsuit p$ is assured at the hybrid time $(0,0)$ by conditions that guarantee that the set K in (8) is FTA for $\mathcal{H}$.

In the following, we propose sufficient conditions to satisfy the formula $f = \diamondsuit p$. Using Clarke generalized derivative, we define the functions u_C and u_D as follows: $u_C(x) := \max_{v\in F(x)} \max_{\zeta \in \partial V(x)} \langle \zeta, v\rangle$ for each $x \in C$, and $-\infty$ otherwise; $u_D(x) := \max_{\zeta \in G(x)} V(\zeta) - V(x)$ for each $x \in D$, and $-\infty$ otherwise, where ∂V is the generalized gradient of V in the sense of Clarke; see, e.g. [Sanfelice et al., 2007].

Theorem 4.7. Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathcal{X}$. Suppose the state space $\mathcal{X}$ and the atomic proposition p are such that K in (8) is closed. Suppose there exists an open set $\mathcal{N}$ that defines an open neighborhood of K such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathcal{X}$. Then, if either

1) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c_1 > 0, c_2 \in [0,1)$ such that
   1.1) for every $x \in \mathcal{N} \cap (\overline{C} \cup D)$ such that $p(x) = 0$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ satisfies
   $$\frac{V^{1-c_2}(x)}{c_1(1-c_2)} \leq \sup_{(t,j)\in \mathrm{dom}\ \phi} t; \tag{11}$$
   1.2) for all $x \in (C \cup D \cup G(D)) \cap \mathcal{N}$, there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that
   $$\alpha_1(|x|_K) \leq V(x) \leq \alpha_2(|x|_K), \tag{12}$$
      1.2a) for each $x \in \mathcal{X}$ such that $x \in C \cap \mathcal{N}$ and $p(x) = 0$, $u_C(x) + c_1 V^{c_2}(x) \leq 0$;
      1.2b) for each $x \in \mathcal{X}$ such that $x \in D \cap \mathcal{N}$ and $p(x) = 0$, $u_D(x) \leq 0$.

or

2) there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c > 0$ such that
   2.1) for every $x \in \mathcal{N} \cap (\overline{C} \cup D)$ such that $p(x) = 0$, each $\phi \in \mathcal{S}_{\mathcal{H}}(x)$ satisfies
   $$\mathrm{ceil}\left(\frac{V(x)}{c}\right) \leq \sup_{(t,j)\in \mathrm{dom}\ \phi} j; \tag{13}$$
   2.2) for all $x \in (C \cup D \cup G(D)) \cap \mathcal{N}$, there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying (12) and
      2.2a) for each $x \in \mathcal{X}$ such that $x \in C \cap \mathcal{N}$ and $p(x) = 0$, $u_C(x) \leq 0$;
      2.2b) for each $x \in \mathcal{X}$ such that $x \in D \cap \mathcal{N}$ and $p(x) = 0$, $u_D(x) \leq -\min\{c, V(x)\}$.

hold, then, the formula $f = \diamondsuit p$ is satisfied for every solution ϕ to $\mathcal{H}$ from $L_V(r) \cap (\overline{C} \cup D)$ at $(t,j) = (0,0)$ where $L_V(r) = \{x \in \mathcal{X} : V(x) \leq r\}$, $r \in [0,\infty]$, is a sublevel set of V contained in $\mathcal{N}$. Moreover, for each $\phi \in \mathcal{S}_{\mathcal{H}}(L_V(r) \cap (\overline{C} \cup D))$, defining $\xi = \phi(0,0)$, the first time $(t',j') \in \mathrm{dom}\ \phi$ such that $\phi(t',j') \vDash \diamondsuit p$ satisfies

$$t' + j' = \mathcal{T}(\xi), \tag{14}$$

and an upper bound on that hybrid time is given as follows:

a) if 1) holds, then $\mathcal{T}(\xi)$ is upper bounded by $\mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi)$, where $\mathcal{T}^\star(\xi) = \frac{V^{1-c_2}(\xi)}{c_1(1-c_2)}$ and $\mathcal{J}^\star(\xi)$ is such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \mathrm{dom}\ \phi$.

b) if 2) holds, then $\mathcal{T}(\xi)$ is upper bounded by $\mathcal{T}^\star(\xi) + \mathcal{J}^\star(\xi)$, where $\mathcal{J}^\star(\xi) = \mathrm{ceil}\left(\frac{V(\xi)}{c}\right)$ and $\mathcal{T}^\star(\xi)$ is such that $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi)) \in \mathrm{dom}\ \phi$ and $(\mathcal{T}^\star(\xi), \mathcal{J}^\star(\xi) - 1) \in \mathrm{dom}\ \phi$. ■

Remark 4.8. Under condition 1.2) or 2.2) in Theorem 4.7, given a solution ϕ to $\mathcal{H}$, there exists some time $(t',j') \in \mathrm{dom}\ \phi$ such that ϕ satisfies p. Furthermore, we have this satisfaction in finite time (t',j'), obtained by the settling-time function $\mathcal{T}$, for which an upper bound depends on the Lyapunov function and the initial condition only. Note that a settling-time function $\mathcal{T}$ does not need to be computed. However, we provide an estimate of when convergence happens using an upper bound that depends on V and the constants involved in items 1) and 2) only.

Remark 4.9. Note that conditions (11) and (13) hold for free for complete solutions unbounded in t or/and j in their domain. Moreover, maximal solutions are complete when the conditions in [Goebel et al., 2012, Proposition 2.10 and Proposition 6.10] hold.

Example 4.10. Inspired from [Li and Sanfelice, 2016, Example 3.3], consider the hybrid system $\mathcal{H} = (C, F, D, G)$ with state $x = (z, \tau) \in \mathbb{R} \times [0,1]$ given by [3]

$$F(x) := \begin{bmatrix} -k|z|^\alpha \mathrm{sgn}(z) \\ 1 \end{bmatrix} \quad \forall x \in C := \mathbb{R} \times [0,1],$$

$$G(x) := \begin{bmatrix} -z \\ 0 \end{bmatrix} \quad \forall x \in D := \mathbb{R} \times \{1\},$$

where $\alpha \in (0,1)$ and $k > 0$. Consider the function $V : \mathbb{R} \times [0,1] \to \mathbb{R}_{\geq 0}$ given by $V(x) = \frac{1}{2}x^2$ for each $x \in C$. Moreover, each $x \in \mathbb{R} \times [0,1]$ satisfies p only when $x \in \{0\} \times [0,1]$. Then, the set K is given by $\{x \in C : p(x) = 1\}$. We have that $\langle \nabla V(x), F(x)\rangle = -k|z|^{1+\alpha} = -2^{\frac{1+\alpha}{2}} k V(x)^{\frac{1+\alpha}{2}}$ for each $x \in C \setminus K$. Furthermore, for all $x \in D \setminus K$, $V(G(x)) - V(x) = 0$. Therefore, condition 1.2) in Theorem 4.7 is satisfied with $\mathcal{N} = \mathbb{R} \times \mathbb{R}$, $c_1 = 2^{\frac{1+\alpha}{2}} k > 0$ and $c_2 = \frac{1+\alpha}{2} \in (0,1)$. By applying [Goebel et al., 2012, Proposition 6.10], item 1.1) in Theorem 4.7 holds since every maximal solution to $\mathcal{H}$ is complete with its domain of definition unbounded in the t direction. Thus, the formula $f = \diamondsuit p$ is satisfied for all solutions ϕ to $\mathcal{H}$ at $(t,j) = (0,0)$. △

Next, the bouncing ball example in [Goebel et al., 2012, Example 1.1] illustrates Lyapunov conditions for verifying that $\diamondsuit p$ is satisfied for all solutions to $\mathcal{H}$ at $(t,j) = (0,0)$.

Example 4.11. Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ modeling a ball bouncing vertically on the ground, with the state $x = (x_1, x_2) \in \mathcal{X} := \mathbb{R}^2$ given by

$$F(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \quad \forall x \in C := \{x \in \mathcal{X} : x_1 \geq 0\},$$

$$G(x) := \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} \quad \forall x \in D := \{x \in \mathcal{X} : x_1 = 0, x_2 \leq 0\},$$

where x_1 denotes the height above the surface and x_2 is the vertical velocity. The parameter $\gamma > 0$ is the gravity coefficient and $\lambda \in (0,1)$ is the restitution coefficient. Every maximal solution to this system is Zeno. Define an atomic proposition p as follows: for each $x \in \mathcal{X}$, $p(x) = 1$ when $x_2 \leq 0$, and $p(x) = 0$ otherwise. With K in (8) and $\mathcal{N} = \mathcal{X}$, let $V(x) = |x_2|$ for all $x \in \mathcal{X}$. This function is continuously differentiable on the open set $\mathcal{X} \setminus (\mathbb{R} \times \{0\})$ and

[3] The function sgn : $\mathbb{R} \to \{-1, 1\}$ is defined as $\mathrm{sgn}(z) = 1$ if $z \geq 0$, and $\mathrm{sgn}(z) = -1$ otherwise.

it is Lipschitz on $\mathcal{X}$. It follows that $\langle \nabla V(x), F(x)\rangle = -\gamma$ for each $x \in (C \cap \mathcal{N}) \setminus K$, and $u_C(x) + c_1 V^{c_2}(x) \leq 0$ holds with $c_1 = \gamma$ and $c_2 = 0$. Therefore, condition 1.2) in Theorem 4.7 is satisfied since $(D \cap \mathcal{N}) \setminus K = \emptyset$. Note that by applying [Goebel et al., 2012, Proposition 6.10], every maximal solution is complete and condition 1.1) in Theorem 4.7 holds with the chosen constants c_1 and c_2 due to the properties of the hybrid time domain of each maximal solution. Therefore, the formula $f = \diamondsuit p$ is satisfied for all maximal solutions to $\mathcal{H}$ at $(t, j) = (0, 0)$. Since every solution from K, after some time, jumps from K and then converges to K again in finite time, we have that $f = \diamondsuit p$ holds for every (t, j) in the domain of each solution. △

Note that Theorem 4.7 guarantees that $\diamondsuit p$ is satisfied for all solutions ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$. These conditions can be extended to guarantee that $\diamondsuit p$ is satisfied for all (t, j) in the domain of any solution if the set K is forward pre-invariant or when only jumps are allowed from points in K and the jump map maps points in K into $\mathcal{N}$.

Theorem 4.12. Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathcal{X}$. Suppose the state space $\mathcal{X}$ and the atomic proposition p are such that K in (8) is closed and that there exists an open set $\mathcal{N}$ that defines an open neighborhood of K such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathcal{X}$. Then, if there exists a continuous function $V : \mathcal{N} \to \mathbb{R}_{\geq 0}$, locally Lipschitz on an open neighborhood of $C \cap \mathcal{N}$, and $c, c_1 > 0$, $c_2 \in [0, 1)$ such that each $\phi \in \mathcal{S}_{\mathcal{H}}(L_V(r) \cap (\overline{C} \cup D))$ is complete, $G(D \cap K) \subset L_V(r) \cap (\overline{C} \cup D)$, and at least one among items 1.2) and 2.2) in Theorem 4.7 holds, then, the formula $f = \diamondsuit p$ is satisfied for every solution ϕ to $\mathcal{H}$ from $L_V(r) \cap (\overline{C} \cup D)$ and for all (t, j) in the domain of each solution, where $L_V(r) = \{x \in \mathcal{X} : V(x) \leq r\}$, $r \in [0, \infty]$ is a sublevel set of V contained in $\mathcal{N}$. ■

Example 4.13. Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ modeling two impulsive oscillators capturing the dynamics of two fireflies. This system has the state $x = (x_1, x_2) \in \mathbb{R}^2$ and the data given by

$$F(x) := \begin{bmatrix} \gamma \\ \gamma \end{bmatrix} \qquad \forall x \in C := [0, 1] \times [0, 1],$$

$$G(x) := \begin{bmatrix} g((1 + \tilde{\varepsilon})x_1) \\ g((1 + \tilde{\varepsilon})x_2) \end{bmatrix} \forall x \in D := \{x \in C : \max\{x_1, x_2\} = 1\},$$

where $\gamma > 0$ and the parameter $\tilde{\varepsilon} > 0$ denotes the effect on the timer of a firefly when the timer of the other firefly expires, and the set-valued map g is given by $g(z) = z$ when $z < 1$; $g(z) = 0$ when $z > 1$; $g(z) = \{0, 1\}$ when $z = 1$. Define p as follows: for each $x \in \mathbb{R}^2$, $p(x) = 1$ when $x \in C$ and $x_1 = x_2$, and $p(x) = 0$ otherwise. Then, the set K is $\{x \in C : p(x) = 1\}$. Let $k = \frac{\tilde{\varepsilon}}{2+\tilde{\varepsilon}}$ and note that $\frac{1+\tilde{\varepsilon}}{2+\tilde{\varepsilon}} = \frac{1+k}{2}$. Define $V(x) := \min\{|x_1 - x_2|, 1 + k - |x_1 - x_2|\}$ for all $x \in \mathcal{X} := \{x \in \mathbb{R}^2 : V(x) < \frac{1+k}{2}\} = \{x \in \mathbb{R}^2 : |x_1 - x_2| \neq \frac{1+k}{2}\}$. This function is continuously differentiable on the open set $\mathcal{X} \setminus K$ and it is Lipschitz on $\mathcal{X}$. Let $m^\star = \frac{1+k}{2}$ and $m \in (0, m^\star)$. Consider $C_m = C \cap M$ and $D_m = D \cap M$, where $M := \{z \in C \cup D : V(x) \leq m\}$. By the definition of V, it follows that $\langle \nabla V(x), F(x)\rangle = 0$ for each $x \in C_m \setminus K$. Since V is symmetric, without loss of generality, consider $x = (1, x_2) \in D_m \setminus K$ where $x_2 \in [0, 1] \setminus \{\frac{1}{2+\tilde{\varepsilon}}\}$.[4] Then, we obtain $V(x) = \min\{1 - x_2, k + x_2\}$. When $g((1+\tilde{\varepsilon})x_2) = 0$, it follows that $V(G(x)) = 0$; when $g((1 + \tilde{\varepsilon})x_2) = (1 + \tilde{\varepsilon})x_2$, it can be shown that $V(x) \geq V(G(x))$. Thus, $V(G(x)) - V(x) \leq 0$ for all $x \in D_m \setminus K$. By applying [Goebel et al., 2009, Proposition 6.10], every maximal solution to a hybrid system $\mathcal{H}_m = (C_m, F, D_m, G)$ is complete. Moreover, given $\tilde{\varepsilon} > 0$, for $\varepsilon = \frac{\tilde{\varepsilon}}{1+\tilde{\varepsilon}}$ and m such that $(K + \varepsilon\mathbb{B}) \cap C \subset C_m$, we have that for all $x \in D_m \cap (K + \varepsilon\mathbb{B})$, $G(x) = 0 \in K$. By applying Theorem 4.7, K is FTA for $\mathcal{H}_m$ with $\mathcal{N} := \{x \in C \cup D : V(x) < m\}$. Thus, the formula $f = \diamondsuit p$ is satisfied for all solutions to $\mathcal{H}_m$, or equivalently, for each solution ϕ to $\mathcal{H}$ from $\mathcal{N}$; f is satisfied all $(t, j) \in \text{dom } \phi$. △

[4] $\left(1, \frac{1}{2+\tilde{\varepsilon}}\right) \in \left\{x \in \mathcal{X} : V(x) = \frac{1+k}{2}\right\}$.

4.3 Conditions to guarantee $\mathcal{U}$

According to the definition of the $\mathcal{U}$ operator, a solution $(t, j) \mapsto \phi(t, j)$ to a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfies the formula

$$f = p \,\mathcal{U}\, q \tag{15}$$

when there exists $(t', j') \in \text{dom } \phi$ and $t' + j' \geq t + j$ such that $\phi(t', j')$ satisfies q and for all $(t'', j'') \in \text{dom } \phi$ such that $t + j \leq t'' + j'' < t' + j'$ and $\phi(t'', j'')$ satisfies p. The set of points in $\mathcal{X}$ satisfying the atomic proposition p and the set of points satisfying the atomic proposition q are, respectively, given by

$$K = \{x \in \mathcal{X} : p(x) = 1\} \quad \text{and} \quad M = \{x \in \mathcal{X} : q(x) = 1\}. \tag{16}$$

To guarantee that a solution ϕ to $\mathcal{H}$ satisfies f in (15) at $(t, j) = (0, 0)$, if q is ever satisfied, the solution needs to start and stay in the set K at least until convergence to the set M happens; or the solution needs to start from the set M. Below, we present sufficient conditions guaranteeing f in (15) by applying the results in Sections 4.1 and 4.2.

The following result is immediate.

Theorem 4.14. Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathcal{X}$. Suppose every $x \in \mathcal{X}$ satisfies either $p(x) = 1$ or $q(x) = 1$, and that every solution $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{X})$ is complete. Then, the formula $f = p\mathcal{U}q$ is satisfied for every solution ϕ at every $(t, j) \in \text{dom } \phi$ to $\mathcal{H}$. ■

Furthermore, if the conditions for FTA in Theorem 4.7 with p therein replaced by q hold and there exists an open set $\mathcal{N}$ defining an open neighborhood of M in (16) such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathcal{X}$, then, under the assumptions in Theorem 4.7, solutions to $\mathcal{H}$ from $L_V(r)$ are guaranteed to satisfy q in finite time where $L_V(r) = \{x \in \mathcal{X} : V(x) \leq r\}$, $r \in [0, \infty]$ is a sublevel set of V contained in $\mathcal{N}$.

The following result relaxes the covering of $\mathcal{X}$ in Theorem 4.14 by requiring that K contains a subset of the basin for finite-time attractivity of M. It provides conditions for the formula $f = p\mathcal{U}q$ to be satisfied for all solutions ϕ to $\mathcal{H}$, both at $(t, j) = (0, 0)$ and any $(t, j) \in \text{dom } \phi$.

Theorem 4.15. Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathcal{X}$, $C \subset \text{dom } F$, and $D \subset \text{dom } G$. Suppose the state space $\mathcal{X}$ and the atomic propositions p and q are such that K and M in (16). Suppose there exists an open set $\mathcal{N}$ defining an open neighborhood of M such that $G(\mathcal{N}) \subset \mathcal{N} \subset \mathcal{X}$. Then, the formula $f = p\mathcal{U}q$ is satisfied for every solution ϕ to $\mathcal{H}$ at $(t, j) = (0, 0)$ if

1) M is closed;
2) at least one among condition 1) and 2) in Theorem 4.7 with p therein replaced by q is satisfied with some function V as required therein;
3) $\phi(0, 0) \in (K \cap L_V(r)) \cup M$;
4) $(L_V(r) \cap (\overline{C} \cup D)) \setminus M \subset K$,

where $L_V(r)$ is a sublevel set of V contained in $\mathcal{N}$. Moreover, the upper bound of the settling-time function $\mathcal{T}(\phi(0,0))$ is given in item a) or b) in Theorem 4.7, respectively. Furthermore, if the following holds:

5) For each $x \in M \cap D$, $G(x) \subset L_V(r) \cap (\overline{C} \cup D)$ where $L_V(r)$ as above,

then the formula $f = p\mathcal{U}q$ is satisfied for every solution ϕ to $\mathcal{H}$ at every $(t,j) \in \text{dom } \phi$. ■

Though at times might be more restrictive, condition 4) in Theorem 4.15 can be replaced by forward invariance of K when C and F satisfy condition 2) in Theorem 4.3.

The bouncing ball example in Example 4.11 is used to illustrate Theorem 4.15.

Example 4.16. Consider $\mathcal{H} = (C, F, D, G)$ in Example 4.11. Define p as 1 when $x_2 \geq 0$, and 0 otherwise. Define q as 1 when $x_2 \leq 0$, and 0 otherwise. With the sets K and M in (16), as shown in Example 4.11, item 2) in Theorem 4.15 is satisfied with $\mathcal{N} = \mathbb{R}^2$. Thus, every solution from M, after some time, jumps from M to K and then converges to M again in finite time. Moreover, from the definition of M and K in (16), if a solution does not belong to M, then it belongs to K. Furthermore, K satisfies item 4) in Theorem 4.15 since $(\overline{C} \cup D) \setminus M \subset K$. Thus, every solution that has not converged to M remains in K at least until it converges to M, which is guaranteed to occur in finite hybrid time. △

5. FINAL REMARKS

Section 4 provides sufficient conditions for formulas that involve a single temporal operator. Table 1 summarizes the conditions for each temporal operator. As indicated therein, all that is needed is either a certificate for finite-time convergence in terms of a Lyapunov-like function, or the data of the hybrid system and the set of points where the proposition is true to satisfy conditions for invariance. The latter can be actually certified using Lyapunov-like functions or barrier functions as in [Chai and Sanfelice, 2015], which for space reasons is not pursued here.

Moreover, the case of logic operators can be treated similarly by using intersections, unions, and complements of the sets where the propositions hold. For instance, sufficient conditions for $\Box(p \wedge q)$ can immediately be derived from the sufficient conditions already given in Section 4.1 with $\{x \in \mathcal{X} : p(x) = 1\} \cap \{x \in \mathcal{X} : q(x) = 1\}$ in place of $\{x \in \mathcal{X} : p(x) = 1\}$. Thus, the conditions in Table 1 can be combined to certify more involved formulas. A systematic methodology to satisfy general formulas within the considered language is part of current research.

	Sufficient Conditions
$\Box p$	a) $G(x) \subset \{x \in \mathcal{X} : p(x) = 1\}$ $\quad \forall x \in D, p(x) = 1$ b) $F(x) \subset T_{\{x \in C : p(x) = 1\}}(x)$ $\quad \forall x \in C, p(x) = 1,$ $F(x) \cap T_C(x) \neq \emptyset$ or Lyapunov-like/Barrier functions
$\Diamond p$	c) Lyapunov function for FTA
$p\mathcal{U}q$	d) Combination of a), b), and c)
$\bigcirc p$	e) $G(D) \subset D \cap \{x \in \mathcal{X} : p(x) = 1\}$

Table 1. Sufficient conditions for $\Box$, $\Diamond$, $\mathcal{U}$, $\bigcirc$.

REFERENCES

Chai, J. and Sanfelice, R.G. (2015). On notions and sufficient conditions for forward invariance of sets for hybrid dynamical systems. In *Decision and Control (CDC), 54th Annual Conference on*, 2869–2874. IEEE.

Cimatti, A., Roveri, M., and Tonetta, S. (2015). HRELTL: A temporal logic for hybrid systems. *Information and Computation*, 245, 54–71.

Collins, P. (2004). A trajectory-space approach to hybrid systems.

Dimitrova, R. and Majumdar, R. (2014). Deductive control synthesis for alternating-time logics. In *Proceedings of the 14th International Conference on Embedded Software*, 14. ACM.

Fainekos, G.E., Girard, A., Kress-Gazit, H., and Pappas, G.J. (2009). Temporal logic motion planning for dynamic robots. *Automatica*, 45(2), 343–352.

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2009). Hybrid dynamical systems. *IEEE Control Systems*, 29(2).

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2012). *Hybrid Dynamical Systems: modeling, stability, and robustness.* Princeton University Press.

Haddad, W.M., Chellaboina, V., and Nersesov, S.G. (2006). *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control.* Princeton University.

Khalil, H. (2002). *Nonlinear Systems.* Prentice Hall.

Kloetzer, M. and Belta, C. (2008). A fully automated framework for control of linear systems from temporal logic specifications. *IEEE Transactions on Automatic Control*, 53(1), 287–297.

Kwon, Y. and Agha, G. (2008). LTLC: Linear temporal logic for control. *Hybrid Systems: Computation and Control*, 316–329.

Li, Y. and Sanfelice, R.G. (2016). Results on finite time stability for a class of hybrid systems. In *American Control Conference (ACC), 2016*, 4263–4268. IEEE.

Lygeros, J., Johansson, K., Simić, S., Zhang, J., and Sastry, S.S. (2003). Dynamical properties of hybrid automata. 48(1), 2–17.

Maghenem, M. and Sanfelice, R.G. (2018). Barrier function certificates for invariance in hybrid inclusions (submitted).

Pnueli, A. (1977). The temporal logic of programs. In *18th Annual Symposium on Foundations of Computer Science, 1977*, 46–57. IEEE.

Pnueli, A. and Manna, Z. (1992). The temporal logic of reactive and concurrent systems. *Springer*, 16, 12.

Raman, V., Donzé, A., Sadigh, D., Murray, R.M., and Seshia, S.A. (2015). Reactive synthesis from signal temporal logic specifications. In *Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control*, 239–248. ACM.

Saha, S. and Julius, A.A. (2016). An MILP approach for real-time optimal controller synthesis with metric temporal logic specifications. In *American Control Conference (ACC), 2016*, 1105–1110. IEEE.

Sanfelice, R.G. (2015). *Analysis and Design of Cyber-Physical Systems: A Hybrid Control Systems Approach*, 3–31. CRC Press. doi:10.1201/b19290-3.

Sanfelice, R.G., Goebel, R., and Teel, A.R. (2007). Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52(12), 2282–2297.

Tabuada, P. and Pappas, G.J. (2006). Linear time logic control of discrete-time linear systems. *IEEE Transactions on Automatic Control*, 51(12), 1862–1877.

van der Schaft, A. and Schumacher, H. (2000). *An Introduction to Hybrid Dynamical Systems.* Lecture Notes in Control and Information Sciences, Springer.