STABLE RATIONALITY OF QUADRIC SURFACE BUNDLES OVER SURFACES

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ABSTRACT. We study rationality properties of quadric surface bundles over the projective plane. We exhibit families of smooth projective complex fourfolds of this type over connected bases, containing both rational and non-rational fibers.

1. INTRODUCTION

Is a deformation of a smooth rational (or irrational) projective variety still rational (or irrational)? The main goal of this paper is to show that rationality is not deformation invariant for families of smooth complex projective varieties of dimension four. Examples along these lines are known when singular fibers are allowed, e.g., smooth cubic threefolds (which are irrational) may specialize to cubic threefolds with ordinary double points (which are rational), while smooth cubic surfaces (which are rational) may specialize to cones over elliptic curves. Totaro shows that specializations of rational varieties need not be rational in higher dimensions if mild singularities are allowed [Tot16b]. However, de Fernex and Fusi [dFF13] show that the locus of rational fibers in a smooth family of projective complex threefolds is a countable union of closed subsets on the base.

Let S be a smooth projective rational surface over the complex numbers with function field $K = \mathbb{C}(S)$. A *quadric surface bundle* consists of a fourfold X and a flat projective morphism $\pi : X \to S$ such that the generic fiber Q/K of π is a smooth quadric surface. We assume that π factors through the projectivization of a rank four vector bundle on S such that the fibers are (possibly singular) quadric surfaces; see Section 3 for relevant background.

Theorem 1. *There exist smooth families of complex projective fourfolds $\phi : \mathcal{X} \to B$ over connected varieties B, such that for every $b \in B$*

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the fiber $\mathcal{X}_b = \phi^{-1}(b)$ is a quadric surface bundle over $\mathbb{P}^2$, and satisfying:

(1) *for very general $b \in B$ the fiber $\mathcal{X}_b$ is not stably rational;*
(2) *the set of points $b \in B$ such that $\mathcal{X}_b$ is rational is dense in B for the Euclidean topology.*

Concretely, we consider smooth hypersurfaces

$$X \subset \mathbb{P}^2 \times \mathbb{P}^3$$

of bidegree $(2,2)$; projection onto the first factor gives the quadric surface bundle.

Our approach has two key elements. First, we apply the technique of the decomposition of the diagonal [Voi15b, CTP16b, CTP16a, Tot16a] to show that very general $X \subset \mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2,2)$ fail to be stably rational. The point is to identify a degenerate quadric surface fibration, with non-trivial second unramified cohomology and mild singularities. The analogous degenerate conic bundles over $\mathbb{P}^2$ are the Artin-Mumford examples; deforming these allows one to show that very general conic bundles over $\mathbb{P}^2$ with large degeneracy divisor fail to be stably rational [HKT15]. Second, quadric surface bundles are rational over the base whenever they admit a section, indeed, whenever they admit a multisection of odd degree. If the base is rational then the total space is rational as well; this can be achieved over a dense set of the moduli space [Has99, Voi15a]. This technique also yields rationality for a dense family of cubic fourfolds containing a plane; no cubic fourfolds have been shown not to be stably rational.

Theorem 1 is proven in Section 7, which may serve as roadmap for the steps of our argument.

This paper is inspired by the approach of Voisin [Voi15a], who also considers fourfolds birational to quadric surface bundles. While our proof of rationality is similar, the analysis of unramified cohomology relies on work of Pirutka [Pir16] and Colliot-Thélène and Ojanguren [CTO89].

2. The specialization method

We recall implications of the "integral decomposition of the diagonal and specialization" method, following [Voi15b] [CTP16b], and [Pir16].

A projective variety X over a field k is *universally* CH_0-*trivial* if for all field extensions k'/k the natural degree homomorphism from the Chow group of zero-cycles

$$CH_0(X_{k'}) \to \mathbb{Z}$$

is an isomorphism. Examples include smooth k-rational varieties. More complicated examples arise as follows:

Example 2. [CTP16a, Lemma 2.3, Lemma 2.4] Let $X = \cup_i X_i$ be a projective, reduced, geometrically connected variety over a field k such that:

- Each irreducible component X_i is geometrically irreducible and k-rational, with isolated singularities.
- Each intersection $X_i \cap X_j$ is either empty or has a zero-cycle of degree 1.

Then X is universally CH_0-trivial.

A projective morphism

$$\beta : \tilde{X} \to X$$

of k-varieties is *universally* CH_0-*trivial* if for all extensions k'/k the push-forward homomorphism

$$\beta_* : CH_0(\tilde{X}_{k'}) \to CH_0(X_{k'})$$

is an isomorphism.

Proposition 3. [CTP16b, Proposition 1.8] *Let*

$$\beta : \tilde{X} \to X$$

be a projective morphism such that for every schematic point x of X, the fiber $\beta^{-1}(x)$, considered as a variety over the residue field $\kappa(x)$, is universally CH_0-trivial. Then β is universally CH_0-trivial.

For example, if X is a smooth projective variety and

$$\beta : \mathrm{Bl}_Z(X) \to X$$

is a blowup of a smooth subvariety $Z \subset X$, then β is a universally CH_0-trivial morphism, since all fibers over (schematic) points are projective spaces. More interesting examples arise as resolutions of singularities of certain singular projective varieties.

Examples of failure of universal CH_0-triviality are given by smooth projective varieties X with nontrivial Brauer group $Br(X)$, or more generally, by varieties with nontrivial higher unramified cohomology [CTP16b, Section 1]. The following specialization argument is the key to recent advances in investigations of stable rationality:

Theorem 4. [Voi15b, Theorem 2.1], [CTP16b, Theorem 2.3] *Let*
$$\phi : \mathcal{X} \to B$$
be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber
$$X := \phi^{-1}(b)$$
satisfies the following conditions:
  - *the group $H_{nr}^2(\mathbb{C}(X)/\mathbb{C}, \mathbb{Z}/2)$ is nontrivial;*
  - *X admits a desingularization*
$$\beta : \tilde{X} \to X$$
    such that the morphism β is universally CH_0-trivial.
Then a very general fiber of ϕ is not stably rational.

3. QUADRIC SURFACE BUNDLES

Let S be a smooth projective variety over $\mathbb{C}$. Suppose that $\pi : X \to S$ is a quadric surface bundle, i.e., a flat projective morphism from a variety such that the generic fiber Q is a smooth quadric surface. We assume it admits a factorization
$$X \hookrightarrow \mathbb{P}(V) \to S,$$
where $V \to S$ is a rank four vector bundle and the fibers of π are expressed as quadric surfaces in the fibers of $\mathbb{P}(V) \to S$. There is a well-defined degeneracy divisor $D \subset S$ corresponding to where the associated quadratic form drops rank.

Trivializing V over an open cover of S, X may be expressed using a symmetric 4×4 matrix (a_{ij}):
$$\sum a_{ij} x_i x_j = 0.$$
The local equation for D is the determinant $\det((a_{ij}))$. Note that D has multiplicity ≥ 2 where the rank of fibers is less than three. Indeed, the hypersurface
$$\{\det(a_{ij}) = 0\} \subset \mathbb{P}^9_{(a_{ij})}$$
is singular precisely where all the 3×3 minors vanish.

3.1. **Rationality of quadric bundles.** It is well known that Q is rational over $K = \mathbb{C}(S)$ if and only if $Q(K) \neq \emptyset$, i.e., when π admits a rational section. A theorem of Springer [Spr52] implies that $Q(K) \neq \emptyset$ provided $Q(K') \neq \emptyset$ for some extension K'/K of odd degree, i.e., when π admits a rational multisection of odd degree. Thus we obtain

Proposition 5. *Let $\pi : X \to S$ be a quadric surface bundle as above, with S rational. Then X is rational provided π admits a multisection of odd degree.*

Our next step is to recast this in Hodge-theoretic terms:

Proposition 6. *Let $\pi : X \to S$ be a quadric surface bundle as above, with X smooth and S rational. Then X is rational if it admits an integral $(2,2)$-class meeting the fibers of π in odd degree.*

Remark 7. See [CTV12, Cor. 8.2] for results on the integral Hodge conjecture for quadric bundles over surfaces; these suffice for our application to quadric surface bundles over $\mathbb{P}^2$.

Proof. Let $F_1(X/S) \to S$ denote the relative variety of lines of π. Let $S_\circ \subset S$ denote the largest open subset such that $S_\circ \cap D$ is smooth and $X_\circ = X \times_S S_\circ$. Then $F_1(X_\circ/S_\circ) \to S_\circ$ factors

$$F_1(X_\circ/S_\circ) \xrightarrow{p} T_\circ \to S_\circ,$$

where the second morphism is a double cover branched along $S_\circ \cap D$ and the first morphism is an étale $\mathbb{P}^1$-bundle. In particular $F_1(X_\circ/S_\circ)$ is non-singular. Let $\alpha \in \mathrm{Br}(T_\circ)[2]$ denote the Brauer class arising from p.

Let F be a resolution of the closure of $F_1(X_\circ/S_\circ)$ in $F_1(X/S)$ obtained by blowing up over the complement of $S_\circ$. The incidence correspondence between X and $F_1(X/S)$

$$\Gamma' \subset X \times_S F_1(X/S)$$

induces a correspondence Γ between X and F and a homomorphism

$$\Gamma_* : \mathrm{CH}^2(X) \to \mathrm{Pic}(F).$$

Let η denote the generic point of S; there is a quadratic map

$$\Xi : \mathrm{Pic}(F_\eta) \to \mathrm{CH}^2(X_\eta)$$

given by

$$\Xi\left(\sum a_i x_i\right) = \frac{1}{2}\left(\sum a_i \ell(x_i)\right)^2$$

where $\ell(x_i) \subset X_\eta$ is the line which corresponds to the point $x_i \in F_\eta$. In geometric terms, consider $Z \subset F_\eta$ a finite reduced subscheme with support on each component of F_η, e.g., a choice of n lines from each ruling. Take the union of the corresponding rulings in X_η and set $\Xi(Z) \subset X_\eta$ to be the n^2 points where the rulings cross. This is compatible with rational equivalence and yields the desired mapping. Thus a divisor with odd degree on each geometric component of F_η gives rise to a rational multisection of odd degree.

The correspondence Γ and mapping Ξ guarantee the following conditions are equivalent:

- $\alpha = 0$;
- F admits a divisor intersecting the generic fiber F_η with odd degree on each component;
- X admits a rational multisection of odd degree.

As the correspondence Γ also acts at the level of Hodge classes we obtain:

> If X admits an integral $(2, 2)$-class intersecting the fibers of π with odd degree then F admits an integral $(1, 1)$-class intersecting the generic fiber F_η with odd degree on each component.

Applying the Lefschetz $(1, 1)$ Theorem to F and Proposition 5 we obtain the result. □

3.2. **A key example.** The generic fiber of π is a quadric surface, hence admits a diagonal form

(3.1) $$Q = <1, a, b, abd>,$$

i.e., is given by the equation

$$s^2 + at^2 + bu^2 + abdv^2 = 0$$

where $a, b, d \in K^\times$ and (s, t, u, v) are homogeneous coordinates in $\mathbb{P}^3$. Note that since $k := \mathbb{C} \subset K$, this form is equivalent to the form $<1, -a, -b, abd>$.

Theorem 3.17 in [Pir16] gives a general formula for the unramified H^2 of the field $K(Q)$, in terms of the divisor of rational functions $a, b, d \in K^\times$, under the assumption that d is not a square.

In Section 4 we will analyze the following special case:

Example 8. Consider the fourfold $X \subset \mathbb{P}^2 \times \mathbb{P}^3$ given by

(3.2) $$yzs^2 + xzt^2 + xyu^2 + F(x, y, z)v^2 = 0,$$

where
$$F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz).$$

Dehomogenize by setting $z = 1$ to obtain a quadric surface over $k(\mathbb{P}^2)$:
$$ys^2 + xt^2 + xyu^2 + F(x, y, 1)v^2 = 0.$$

Multiplying through by xy and absorbing squares into the variables yields
$$xS^2 + yT^2 + U^2 + xyF(x, y, 1)V^2 = 0,$$

which is of the form (3.1).

We compute the divisor $D \subset \mathbb{P}^2$ parametrizing singular fibers of $\pi : X \to \mathbb{P}^2$. This is reducible, consisting of the coordinate lines (with multiplicty two) and a conic tangent to each of the lines:
$$D = \{x^2 y^2 z^2 (x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0\}.$$

Remark 9. Hypersurfaces of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ may also be regarded as conic bundles over the second factor. The degeneracy surface in $\mathbb{P}^3$ has degree six and at least eight nodes, corresponding to rank-one fibers. As a byproduct of the proof of Theorem 1, we obtain failure of stable rationality for very general conic bundles of this type.

4. The Brauer group of the special fiber

We refer the reader to [CTO89, Sect. 1] and [CT95] for basic properties of unramified cohomology.

Let K be a field. We write
$$H^n(K) = H^n(K, \mathbb{Z}/2)$$

for its n-th Galois cohomology with constant coefficients $\mathbb{Z}/2$. Let $K = k(X)$ be the function field of an algebraic variety X over $k = \mathbb{C}$, and let ν be a rank one discrete valuation of K, trivial on k. For $n \geq 1$, we have a natural homomorphism
$$\partial_\nu^n : H^n(K) \to H^{n-1}(\kappa(\nu)),$$

where $\kappa(\nu)$ is the residue field of ν. The group
$$H_{nr}^n(K/k) := \cap_\nu \mathrm{Ker}(\partial_\nu^n)$$

is called the n-th unramified cohomology of K. It is a stable birational invariant [CTO89, Prop. 1.2] and vanishes if X is stably rational [CTO89, Cor. 1.2.1]. Recall that for a smooth projective X we have
$$\mathrm{Br}(X)[2] = H_{nr}^2(k(X)/k).$$

The following proposition is similar to the examples in [Pir16, Section 3.5].

Proposition 10. *Let $K = k(x,y) = k(\mathbb{P}^2)$, $X \to \mathbb{P}^2$ the quadric surface bundle defined in Example 8,*

$$\alpha = (x,y) \in \mathrm{Br}(K)[2],$$

and α' its image in $H^2(k(X))$. Then α' is contained in $H^2_{nr}(k(X)/k)$ and is nontrivial; in particular,

$$H^2_{nr}(k(X)/k) \neq 0.$$

Proof. Let Q be the generic fiber of the natural projection $\pi : X \to \mathbb{P}^2$. Since the discriminant of Q is not a square, the homomorphism

$$H^2(K) \to H^2(K(Q))$$

is injective [Ara75, p. 469], [Kah08, 6.4.13]. Note that $\alpha \neq 0$ as the conic $xS^2 + yT^2 = U^2$ has no rational points over $k(x,y)$; it follows that α' is also nontrivial. It remains to show that for every rank one discrete valuation ν on $K(Q)$ that is trivial on k, we have $\partial_\nu(\alpha') = 0$. (For simplicity, we write ∂_ν for ∂^2_ν.) We use standard coordinates x and y (resp. y and z, resp. x and z) for the open charts of the projective plane. Let us first investigate the ramification of α on $\mathbb{P}^2$; from the definition, we only have the following nontrivial residues:

- $\partial_x(\alpha) = y$ at the line $L_x : x = 0$, where we write y for its class in the residue field $k(y)$ modulo squares;
- $\partial_y(\alpha) = x$ at the line $L_y : y = 0$;
- $\partial_z(\alpha) = \partial_z(z, zy) = y$ at the line $L_z : z = 0$, in coordinates y and z on $\mathbb{P}^2$.

Let $\mathfrak{o}_\nu$ be the valuation ring of ν in $K(Q)$ and consider the center of ν in $\mathbb{P}^2$. If $\mathfrak{o}_\nu \supset K$ then the $\partial_\nu(\alpha') = 0$; hence there are two cases to consider:

- The center is the generic point of a curve $C_\nu \subset \mathbb{P}^2$; we denote the corresponding residue map $\partial_{C_\nu} : H^2(K) \to H^1(\kappa(C_\nu))$.
- The center is a closed point $\mathfrak{p}_\nu \in \mathbb{P}^2$.

Codimension 1. The inclusion of discrete valuation rings $\mathfrak{o}_{\mathbb{P}^2, C_\nu} \subset \mathfrak{o}_\nu$ induces a commutative diagram [CTO89, p. 143]:

(4.1)
$$
\begin{array}{ccc}
H^2(K(Q)) & \xrightarrow{\ \partial_\nu\ } & H^1(\kappa(\nu)) \\
\uparrow & & \uparrow \\
H^2(K) & \xrightarrow{\ \partial_{C_\nu}\ } & H^1(\kappa(C_\nu))
\end{array}
$$

The right vertical arrow need not be the functorial homomorphism induced by inclusion of the residue fields when there is ramification.

Hence we have the following cases:

(1) C_ν is different from L_x, L_y, or L_z. Then $\partial_{C_\nu}(\alpha) = 0$, so that $\partial_\nu(\alpha')$ is zero from the diagram above.
(2) C_ν is one of the lines L_x, L_y or L_z. Note that modulo the equation of C_ν, the element $d := F(x, y, z)$ is a nonzero square, so that [Pir16, Cor. 3.12] gives $\partial_\nu(\alpha') = 0$.

We deduce that for any valuation ν of $K(Q)$ with center a codimension 1 point in $\mathbb{P}^2_{\mathbb{C}}$ the residue $\partial_\nu(\alpha')$ vanishes.

Codimension 2. Let $\mathfrak{p}_\nu$ be the center of ν on $\mathbb{P}^2$. We have an inclusion of local rings $\mathfrak{o}_{\mathbb{P}^2, \mathfrak{p}_\nu} \subset \mathfrak{o}_\nu$ inducing the inclusion of corresponding completions $\widehat{\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}_\nu}} \subset \widehat{\mathfrak{o}_\nu}$ with quotient fields $K_{\mathfrak{p}_\nu} \subset K(Q)_\nu$ respectively. We have three possibilities:

(1) If $\mathfrak{p}_\nu \notin L_x \cup L_y \cup L_z$, then α is a cup product of units in $\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}_\nu}$, hence units in $\mathfrak{o}_v$, so that $\partial_\nu(\alpha') = 0$.
(2) If $\mathfrak{p}_\nu$ lies on one curve, e.g., $\mathfrak{p}_\nu \in L_x \setminus (\mathfrak{p}_y \sqcup \mathfrak{p}_z)$, where $\mathfrak{p}_y = (0, 1, 0)$ and $\mathfrak{p}_z = (0, 0, 1)$, then the image of y in $\kappa(\mathfrak{p}_\nu)$ is a nonzero complex number, hence a square in $\widehat{\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}_\nu}}$, and y is also a square in $\widehat{\mathfrak{o}_\nu}$. (We are using Hensel's Lemma.) Thus $\alpha' = (x, y) = 0$ in $H^2(K(Q)_\nu, \mathbb{Z}/2)$, and $\partial_\nu(\alpha') = 0$.
(3) If $\mathfrak{p}_\nu$ lies on two curves, e.g., $\mathfrak{p}_\nu = L_x \cap L_y$, then the image of $F(x, y, 1)$ in $\kappa(\mathfrak{p}_\nu)$ is a nonzero complex number, hence a square. By [Pir16, Corollary 3.12], we have $\partial_\nu(\alpha') = 0$.

$\square$

5. Singularities of the special fiber

In this section we analyze the singularities of the fourfold introduced in Example 8 and studied in Section 4. Our main result is:

Proposition 11. *The fourfold $X \subset \mathbb{P}^2 \times \mathbb{P}^3$, with coordinates (x, y, z) and (s, t, u, v), respectively, given by*

$$(5.1) \qquad yzs^2 + xzt^2 + xyu^2 + F(x, y, z)v^2 = 0,$$

with

$$(5.2) \qquad F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz),$$

admits a universally CH_0-trivial resolution of singularities.

We proceed as follows:

- identify the singular locus of X;
- construct a resolution of singularities $\beta : \tilde{X} \to X$;
- verify universal CH_0-triviality of β.

5.1. **The singular locus.** Here we describe the singularities explicitly using affine charts on $\mathbb{P}^2 \times \mathbb{P}^3$. The equations (5.1) and (5.2) are symmetric with respect to compatible permutations of $\{x, y, z\}$ and $\{s, t, u\}$. In addition, there is the symmetry

$$(s, t, u, v) \to (\pm s, \pm t, \pm u, v)$$

so altogether we have an action by the semidirect product $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_3$.

Analysis in local charts. Let $L_x, L_y, L_z \subset \mathbb{P}^2$ be the coordinate lines given by

$$x = 0, \ y = 0, \ z = 0,$$

respectively, and

$$\mathfrak{p}_x := (1, 0, 0), \ \mathfrak{p}_y := (0, 1, 0), \ \mathfrak{p}_z := (0, 0, 1)$$

their intersections.

The quadrics in the family (5.1) drop rank over coordinate lines L_x, L_y, L_z and over the conic $C \subset \mathbb{P}^2$, with equation (5.2)

$$F(x, y, z) = 0.$$

This conic is tangent to the coordinate lines in the points

$$\mathfrak{r}_x := (0, 1, 1), \ \mathfrak{r}_y := (1, 0, 1), \ \mathfrak{r}_z := (1, 1, 0),$$

respectively.

By symmetry, and since no singular point satisfies $s = t = u = 0$, it suffices to consider just two affine charts:

Chart 1: $z = u = 1$. Equation (5.1) takes the form

$$(5.3) \qquad ys^2 + xt^2 + xy + F(x, y, 1)v^2 = 0.$$

Derivatives with respect to s, t, v give

$$(5.4) \qquad ys = 0, \ xt = 0, \ vF(x, y, 1) = 0.$$

Hence $xy = 0$, from (5.3). Derivatives with respect to y, x give

$$(5.5) \qquad s^2 + x + (2y - 2x - 2)v^2 = 0, \ t^2 + y + (2x - 2y - 2)v^2 = 0.$$

Since $xy = 0$, we have two cases, modulo symmetries:
Case 1: $y = 0$;
Case 2: $x = 0, y \neq 0$.
We analyze each of these cases:
Case 1: $y = 0$. Then $vF(x, y, 1) = 0$ (from (5.4)) implies

$$v(x - 1)^2 = 0.$$

So either $v = 0$ or $x = 1$. If $v = 0$, from (5.5) we obtain $s^2 + x = t = 0$. Hence we obtain the following equations for the singular locus:

$$(5.6) \qquad y = v = t = s^2 + x = 0.$$

If $x = 1$ then (5.4) implies $t = 0$, and the remaining equation from (5.5) gives $s^2 + 1 - 4v^2 = 0$. Hence we obtain the following equations:

$$(5.7) \qquad x - 1 = y = t = s^2 + 1 - 4v^2 = 0.$$

Case 2: $x = 0, y \neq 0$. From (5.4) the condition $ys = 0$ implies $s = 0$. There are two more cases: $v = 0$ or $v \neq 0$. If $v = 0$ the remaining equation (5.5) gives $t^2 + y = 0$. Hence we obtain equations for the singularity:

$$(5.8) \qquad x = v = s = t^2 + y = 0.$$

If $v \neq 0$, then $F(0, y, 1) = (y - 1)^2 = 0$ from (5.4), hence $y = 1$. The remaining equation from (5.5) gives

$$t^2 + y + (2x - 2y - 2)v^2 = t^2 + 1 - 4v^2 = 0.$$

So we obtain equations for the singular locus:

$$(5.9) \qquad x = y - 1 = s = t^2 + 1 - 4v^2 = 0.$$

Chart 2: $z = s = 1$. The equation of the quadric bundle is

$$y + xt^2 + xyu^2 + F(x, y, 1)v^2 = 0.$$

As above, derivatives with respect to t, u, v give

(5.10) $$xt = 0, \ xyu = 0, \ vF(x, y, 1) = 0.$$

Thus $y = 0$ from the equation. The conditions above and derivatives with respect to y and x yield

(5.11) $$xt = v(x - 1)^2 = 1 + xu^2 + (-2x - 2)v^2 = t^2 + (2x - 2)v^2 = 0.$$

The second equation implies that either $x = 1$ or $v = 0$.

If $x = 1$, we obtain:

(5.12) $$x - 1 = y = t = 1 + u^2 - 4v^2 = 0.$$

If $v = 0$, we obtain:

(5.13) $$y = t = v = 1 + xu^2 = 0.$$

Collecting these computations, we obtain the following singularities:

(1) In Chart 1:

$$y = v = t = s^2 + x = 0$$
$$x - 1 = y = t = s^2 + 1 - 4v^2 = 0$$
$$x = v = s = t^2 + y = 0$$
$$x = y - 1 = s = t^2 + 1 - 4v^2 = 0$$

(2) In Chart 2:

$$x - 1 = y = t = 1 + u^2 - 4v^2 = 0$$
$$y = t = v = 1 + xu^2 = 0$$

Enumeration of strata. Using the symmetries, we deduce that the singular locus of X is a union of 6 conics. We distinguish between

- Horizontal conics $C_x, C_y, C_z \subset X$: these project onto the coordinate lines $L_x, L_y, L_z \subset \mathbb{P}^2$. We express them using our standard coordinates on $\mathbb{P}^2 \times \mathbb{P}^3$:

$$C_y = \{y = t = v = 0, \ zs^2 + xu^2 = 0\}$$
$$C_x = \{x = s = v = 0, \ zt^2 + yu^2 = 0\}$$
$$C_z = \{z = u = v = 0, \ xt^2 + ys^2 = 0\}$$

The conics intersect transversally over $\mathfrak{p}_z, \mathfrak{p}_x, \mathfrak{p}_y \in \mathbb{P}^2$, respectively:

$$
\begin{aligned}
C_x \cap C_y &= \mathfrak{q}_z := (0,0,1) \times (0,0,1,0), & \pi(\mathfrak{q}_z) &= \mathfrak{p}_z \\
C_y \cap C_z &= \mathfrak{q}_x := (1,0,0) \times (1,0,0,0), & \pi(\mathfrak{q}_x) &= \mathfrak{p}_x \\
C_x \cap C_z &= \mathfrak{q}_y := (0,1,0) \times (0,1,0,0), & \pi(\mathfrak{q}_y) &= \mathfrak{p}_y
\end{aligned}
$$

- Vertical conics $R_y, R_x, R_z \subset X$: these project to the points $\mathfrak{r}_y, \mathfrak{r}_x, \mathfrak{r}_z \in \mathbb{P}^2$:

$$
\begin{aligned}
R_y &= \{x - z = y = t = 0, \ s^2 + u^2 - 4v^2 = 0\} \\
R_x &= \{y - z = x = s = 0, \ t^2 + u^2 - 4v^2 = 0\} \\
R_z &= \{x - y = z = u = 0, \ s^2 + t^2 - 4v^2 = 0\}
\end{aligned}
$$

Vertical conics intersect the corresponding horizontal conics transversally in two points:

$$
\begin{aligned}
R_y \cap C_y &= \{\mathfrak{r}_{y+}, \mathfrak{r}_{y-}\} = (1,0,1) \times (\pm i, 0, 1, 0) \\
R_x \cap C_x &= \{\mathfrak{r}_{x+}, \mathfrak{r}_{x-}\} = (0,1,1) \times (0, 1, \pm i, 0) \\
R_z \cap C_z &= \{\mathfrak{r}_{z+}, \mathfrak{r}_{z-}\} = (1,1,0) \times (1, \pm i, 0, 0)
\end{aligned}
$$

Local étale description of the singularities. The structural properties of the resolution become clearer after identifying étale normal forms for the singularities.

We now provide a local-étale description of the neighborhood of $\mathfrak{q}_z$. Equation (5.3) takes the form

$$
ys^2 + xt^2 + xy + F(x, y, 1)v^2 = 0.
$$

At $\mathfrak{q}_z$ we have $F(x, y, 1) \neq 0$, so we can set

$$
v_0 = \sqrt{F(x, y, 1)}\, v
$$

to obtain

$$
ys^2 + xt^2 + xy + v_0^2 = 0.
$$

Set $x = m - n$ and $y = m + n$ to get

$$
(m + n)s^2 + (m - n)t^2 + m^2 - n^2 + v_0^2 = 0
$$

or

$$
m(s^2 + t^2) + n(s^2 - t^2) + m^2 - n^2 + v_0^2 = 0.
$$

Then let

$$
m = m_0 - (s^2 + t^2)/2 \quad \text{and} \quad n = n_0 + (s^2 - t^2)/2
$$

to obtain

$$(5.14) \qquad m_0^2 - n_0^2 + v_0^2 = ((s^2 + t^2)^2 - (s^2 - t^2)^2)/4 = s^2 t^2.$$

We do a similar analysis in an étale-local neighborhood of either of the points $\mathfrak{r}_{y\pm}$. The singular strata for C_y and R_y are given in (5.6) and (5.7):

$$\{y = t = v = s^2 + x = 0\}, \{y = t = x - 1 = s^2 + 1 - 4v^2 = 0\}.$$

We first introduce a new coordinate $w = x - 1$. Thus the singular stratum is the intersection of the monomial equations $y = t = vw = 0$ and the hypersurface

$$s^2 + w + 1 - 4v^2.$$

We regard this as a local coordinate near $\mathfrak{r}_{y\pm}$. Equation (5.3) transforms to

$$ys^2 + wt^2 + t^2 + wy + y + v^2(-4y + (w - y)^2) = 0.$$

Regroup terms to obtain

$$y(s^2 + w + 1 - 4v^2) + t^2(1 + w) = -v^2(w - y)^2.$$

Note that $w \neq -1$ because $x \neq 0$ near $\mathfrak{r}_{y\pm}$. Let $t_0 = t\sqrt{1 + w}$, $s_0 = s^2 + w + 1 - 4v^2$, and $w_0 = w - y$ we obtain

$$(5.15) \qquad\qquad\qquad ys_0 + t_0^2 = -v^2 w_0^2.$$

The normal forms (5.14) and (5.15) are both equivalent to

$$a_1^2 + a_2^2 + a_3^2 = (b_1 b_2)^2,$$

with ordinary threefold double points along the lines

$$\ell_1 = \{a_1 = a_2 = a_3 = b_1 = 0\}, \quad \ell_2 = \{a_1 = a_2 = a_3 = b_2 = 0\}.$$

A direct computation – which will be presented in Section 5.2 – shows this is resolved by blowing up ℓ_1 and ℓ_2 in *either* order. The exceptional fibers over the generic points of ℓ_1 and ℓ_2 are smooth quadric surfaces, isomorphic to the Hirzebruch surface $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Over the origin, we obtain a copy

$$\mathbb{F}_0 \cup_\Sigma \mathbb{F}_2$$

where $\Sigma \simeq \mathbb{P}^1$ is the (-2)-curve on $\mathbb{F}_2$ and has self-intersection 2 on $\mathbb{F}_0$.

By symmetry, this analysis is valid at *all* nine special points

$$\mathfrak{q}_x, \mathfrak{q}_y, \mathfrak{q}_z, \mathfrak{r}_{x\pm}, \mathfrak{r}_{y\pm}, \mathfrak{r}_{z\pm}$$

where components of the singular locus (the horizontal and vertical conics) intersect. This explains why we can blow these conics up in any order.

5.2. **Resolution of singularities.**

What we need to compute. We propose blowing up as follows:
(1) blow up C_y;
(2) blow up the proper transform of C_x;
(3) blow up the proper transform of C_z;
(4) blow up the union of the proper transforms of R_x, R_y, and R_z, which are disjoint.

Taking into account the symmetry, after the first step we must understand:

- What are the singularities along the proper transform of C_x?
- What are the singularities along the proper transform of R_y?

Of course, answering the first questions clarifies the behavior along the proper transform of C_z. And R_x and R_z behave the exactly the same as R_y.

Let X_1 denote the blow up of C_y and $E_{1,y}$ the resulting exceptional divisor. We shall see that

- $E_{1,y}$ is smooth, except where it meets the proper transforms of C_x, C_z, R_y.
- Since $E_{1,y} \subset X_1$ is Cartier, X_1 is also smooth at any point of $E_{1,y}$, except where $E_{1,y}$ meets the proper transforms of C_x, C_z, R_y.
- The fibers of $E_{1,y} \to C_y$ are smooth quadric surfaces away from $\mathfrak{q}_x, \mathfrak{q}_z, \mathfrak{r}_{y\pm}$, over which the fibers are quadric cones.

Since the quadric bundle $E_{1,y} \to C_y$ admits sections, $E_{1,y}$ is rational over the function field of C_y and all fibers of $E_{1,y} \to C_y$ are rational as well.

First blow up—local charts. We describe the blow up of C_y in charts. We start in Chart 1, where $z = u = 1$. Local equations for the center are given in (5.6) and we have a local chart for each defining equation.

- **Chart associated with y:** Equations for the blow up of the ambient space take the form

$$v = yv_1, \ t = yt_1, \ s^2 + x = yw_1.$$

The equation of the proper transform of the quadric bundle is

$$w_1 + xt_1^2 + F(x, y, 1)v_1^2 = 0, \ s^2 + x = yw_1.$$

The exceptional divisor $E_{1,y}$ is given by $y = 0$, i.e.,

$$w_1 + xt_1^2 + (x-1)^2 v_1^2 = 0, \ s^2 + x = 0.$$

The blow up is smooth at any point of the exceptional divisor in this chart, as the derivative of the first equation with respect to w_1 is 1 and the derivative of the second equation with respect to w_1 (resp. x) is 0 (resp. 1). (The proper transforms of R_y and C_x do not appear in this chart.) We analyze $E_{1,y} \to C_y$: for any field $\kappa/\mathbb{C}$ and $a \in \kappa$, the fiber above $s = a$, $x = -a^2$, $y = v = t = 0$, is given by

$$(5.16) \qquad w_1 - a^2 t_1^2 + (1+a^2)^2 v_1^2 = 0,$$

which is smooth in this chart. Equation (5.16) makes clear that the exceptional divisor is rational and admits a section over the center.

  • **Chart associated with $s^2 + x$:** Equations for the blow up of the ambient space take the form

$$y = (s^2 + x)y_1, \ v = (s^2 + x)v_1, \ t = (s^2 + x)t_1.$$

The proper transform of the quadric bundle has equation

$$y_1 + xt_1^2 + F(x, (s^2+x)y_1, 1)v_1^2 = 0.$$

The exceptional divisor $E_{1,y}$ satisfies

$$y_1 + xt_1^2 + (x-1)^2 v_1^2 = 0, \ s^2 + x = 0.$$

The blow up is smooth at any point of the exceptional divisor in this chart. (Again, the proper transforms of R_y and C_x do not appear in this chart.) The fiber above $s = a$, $x = -a^2$, $y = v = t = 0$, is given by

$$(5.17) \qquad y_1 - a^2 t_1^2 + (1+a^2)^2 v_1^2 = 0,$$

which is smooth and rational in this chart.

  • **Chart associated with t:** Equations for the blow up of the ambient space are

$$y = ty_1, \ v = tv_1, \ s^2 + x = tw_1$$

and the proper transform of the quadric bundle satisfies

$$y_1 w_1 + x + F(x, ty_1, 1)v_1^2 = 0, \ s^2 + x = tw_1.$$

The exceptional divisor is given by $t = 0$, i.e.

$$y_1 w_1 + x + (x-1)^2 v_1^2 = 0, \ s^2 + x = 0.$$

The blow up is smooth along the exceptional divisor, except at the point

$$t = v_1 = y_1 = s = w_1 = x = 0,$$

which lies over the point $\mathfrak{q}_z$. Thus the only singularity is along the proper transform of C_x. The fiber above $s = a$, $x = -a^2$, $y = v = t = 0$, is given by

(5.18) $$y_1 w_1 - a^2 + (1 + a^2)^2 v^2 = 0,$$

which is smooth in this chart unless $a = 0$.
- **Chart associated with** v: The equations are

$$y = vy_1, \quad t = vt_1, \quad s^2 + x = vw_1$$

and

$$y_1 w_1 + xt_1^2 + F(x, vy_1, 1) = 0, \quad s^2 + x = vw_1.$$

The exceptional divisor is given by $v = 0$, i.e.

$$y_1 w_1 + xt_1^2 + (x - 1)^2 = 0, \quad s^2 + x = 0.$$

The blow up is smooth at any point of the exceptional divisor except for

$$y_1 = v = w_1 = t_1 = 0, \quad x = 1, \quad s = \pm i.$$

Thus the only singularities are along the proper transform of $\mathfrak{r}_y$. The fiber above $s = a$, $x = -a^2$, $y = v = t = 0$, is given by

(5.19) $$y_1 w_1 - a^2 t_1^2 + (1 + a^2)^2 = 0,$$

which is smooth in this chart unless $a = \pm i$.

What is missed on restricting to Chart 1? For C_y, we omit only

$$(1, 0, 0) \times (1, 0, 0, 0) = \mathfrak{q}_x = C_y \cap C_z$$

but the symmetry exchanging x and z (and s and u) takes this to $\mathfrak{q}_z$, which lies over Chart 1. For R_y, we omit the locus

$$x - z = y = t = u = s^2 - 4v^2 = 0$$

which equals $(1, 0, 1) \times (\pm 2, 0, 0, 1)$. However, the same symmetry takes these to $(1, 0, 1) \times (0, 0, \pm 2, 1)$, which is over Chart 1. Thus modulo symmetries computations over Chart 1 cover these points as well.

Singularities above $\mathfrak{p}_z$. Our goal is to show explicitly that the singularity of the blow up in the exceptional divisor $E_{1,y}$ over $(x, y, z) = (0, 0, 1) = \mathfrak{p}_z$ is resolved on blowing up the proper transform of C_x. It suffices to examine the chart associated with t, where we have equations

$$y_1 w_1 + x + F(x, t y_1, 1) v_1^2 = 0, \ \ s^2 + x = t w_1,$$

i.e.,

$$(5.20) \quad (y_1 + t) w_1 - s^2 + F(-s^2 + t w_1, t y_1, 1) v_1^2 = 0, \ \ s^2 + x = t w_1,$$

and the proper transform of C_x satisfies

$$y_1 + t = 0, \ \ w_1 = s = v_1 = 0.$$

If we compute the singular locus for the equation (5.20) above, at the points of the exceptional divisor $t = 0$ and above $x = 0$, we recover the equations for the proper transform of C_x in this chart.

We analyze X_2, the blowup along the proper transform of C_x. In any chart above $y_1 = t = 0$ we have $F = 1$ so étale locally we can introduce a new variable $v_2 = \sqrt{F} v_1$ to obtain

$$(y_1 + t) w_1 - s^2 + v_2^2 = 0.$$

After the change of variables $y_2 = y_1 + t$:

$$y_2 w_1 - s^2 + v_1^2 = 0,$$

the singular locus is $y_2 = s = w_1 = v_2 = 0$. Here t is a free variable corresponding to an $\mathbb{A}^1$-factor. This is the product of an ordinary threefold double point with a curve, thus is resolved on blowing up the singular locus. Note the exceptional divisor is a smooth quadric surface bundle over the proper transform of C_x, over this chart. (There is a singular fiber over the point where it meets the proper transform of C_z.)

Singularities above $\mathfrak{r}_y = (1, 0, 1) \in \mathbb{P}^2$. By the analysis above, we have only to consider the chart of the first blowup associated with v. Recall that it is obtained by setting

$$y = v y_1, \ \ t = v t_1, \ \ s^2 + x = v w_1$$

with equation

$$y_1 w_1 + x t_1^2 + F(x, v y_1, 1) = 0.$$

The exceptional divisor is given by $v = 0$. The proper transform R'_y of the conic

$$R_y : x - 1 = y = t = 0, \ \ s^2 + 1 - 4v^2 = 0$$

is then

(5.21) $\qquad x - 1 = y_1 = t_1 = 0, \ w_1 - 4v = 0, \ s^2 + 1 - 4v^2 = 0.$

We obtain these equations by inverting the local equation for the exceptional divisor. Eliminating x from the equation of X_1 yields an equation that can be put in the form

$$y_1(w_1 - 4v) + (-s^2 + vw_1)t_1^2 + (s^2 - vw_1 + vy_1 + 1)^2 = 0.$$

Writing $w_2 = w_1 - 4v$ we obtain

$$y_1 w_2 + (-s^2 + vw_2 + 4v^2)t_1^2 + (s^2 - vw_2 - 4v^2 + vy_1 + 1)^2 = 0.$$

The curve R'_y may be expressed as a complete intersection

$$y_1 = w_2 = t_1 = 0, \ \sigma := (s^2 - 4v^2 + 1) + v(y_1 - w_2) = 0;$$

the coefficient

$$c := -s^2 + vw_2 + 4v^2$$

is non-vanishing along R'_y in this chart so we may introduce an étale local coordinate $t_2 = \sqrt{c}\,t_1$. Then our equation takes the form

$$y_1 w_2 + t_2^2 + \sigma^2 = 0.$$

In other words, we have ordinary threefold double points along each point of R'_y. Blowing up R'_y resolves the singularity, and the exceptional divisor over R'_y is fibered in smooth quadric surfaces.

5.3. CH_0-triviality of the resolution.

Let $E_{1,y}$ denote the exceptional divisor after blowing up C_y. We've seen that the projection $E_{1,y} \to C_y$ is a quadric surface bundle. The fibers are smooth away from $\mathfrak{q}_x, \mathfrak{q}_z$, and $\mathfrak{r}_{y\pm}$; over these points the fibers are quadric cones.

Let $E_{1,x}$ denote the exceptional divisor after blowing up the proper transform C'_x of C_x. The fibration $E_{1,x} \to C'_x$ is also a quadric surface bundle. The fibers are smooth away from $\mathfrak{q}_y$ and $\mathfrak{r}_{x\pm}$, where the fibers are quadric cones.

Let $E_{1,z}$ denote the exceptional divisor on blowing up the proper transform C'_z of C_z, after the first two blow ups. Again $E_{1,z} \to C'_z$ is a quadric surface bundle, smooth away from $\mathfrak{r}_{z\pm}$; the fibers over these points are quadric cones.

Finally, we blow up the proper transforms R'_x, R'_y, R'_z of the disjoint vertical conics. The local computations above show that the resulting fourfold $\tilde{X}$ is smooth and the exceptional divisors

$$E_{2,x} \to R'_x, \ E_{2,y} \to R'_y, \ E_{2,z} \to R'_z,$$

are smooth quadric surface bundles with sections.

To summarize, fibers of $\beta : \tilde{X} \to X$ are one of the following:

- if x is not contained in any of the conics, $\beta^{-1}(x)$ is a point;
- if x is contained in exactly one of the conics, $\beta^{-1}(x)$ is a smooth quadric surface isomorphic to $\mathbb{F}_0$; when x is a generic point of one of the conics, then $\beta^{-1}(x)$ is rational over the residue field of x;
- if x is contained in two of the conics, $\beta^{-1}(x) = \mathbb{F}_0 \cup_\Sigma \mathbb{F}_2$, where $\mathbb{F}_2$ is the proper transform of a quadric cone appearing as a degenerate fiber, $\Sigma \subset \mathbb{F}_2$ is the (-2) curve, and $\Sigma \subset \mathbb{F}_0$ has self-intersection 2.

By Proposition 3 and Example 2, we conclude that β is universally CH_0-trivial.

6. Analysis of Hodge classes

Our approach follows Section 2 of [Voi15a]. As explained in Proposition 6, a quadric surface bundle over a rational surface $\pi : X \to S$ is rational provided X admits an integral class of type $(2,2)$ meeting the fibers of π in odd degree. Here we analyze how these classes occur.

We start by reviewing the Hodge-theoretic inputs. Let $\mathcal{Y} \to B$ be the family of all smooth hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2,2)$, i.e., B is the complement of the discriminant in $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(2,2)))$. For each $b \in B$, let Y_b denote the fiber over b. The Lefschetz hyperplane theorem gives Betti/Hodge numbers

- $b_{2i+1}(Y_b) = 0$
- $b_2(Y_b) = h^{1,1}(Y_b) = 2, \ b_6(Y_b) = h^{3,3}(Y_b) = 2.$

We compute $b_4(Y_b)$ by analyzing $Y_b \to \mathbb{P}^2$; its degeneracy divisor is an octic plane curve D_b, of genus 21. Indeed, the fibers away from D_b are smooth quadric surfaces and the fibers over D_b are quadric cones, so we have

$$
\begin{aligned}
\chi(Y_b) &= \chi(\mathbb{P}^1 \times \mathbb{P}^1)\chi(\mathbb{P}^2 \setminus D_b) + \chi(\text{quadric cone})\chi(D_b) \\
&= 4 \cdot (3 - (-40)) + 3 \cdot (-40) = 52.
\end{aligned}
$$

It follows that $b_4(Y_b) = 46$.

We extract the remaining Hodge numbers using techniques of Griffiths for hypersurfaces in projective space, extended to the toric case by Batyrev and Cox. Let F be the defining equation of bidegree $(2,2)$ and consider the bigraded *Jacobian ring*:

$$
\mathrm{Jac}(F) = \mathbb{C}[x,y,z;s,t,u,v]/\mathcal{I}(F),
$$

where $\mathcal{I}(F)$ is the ideal of partials of F. Note the partials satisfy the Euler relations:

$$(6.1) \qquad x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = 2F = s\frac{\partial F}{\partial s} + t\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v}.$$

Consider the *vanishing cohomology*

$$H^4(Y_b)_{van} := H^4(Y_b)/H^4(\mathbb{P}^2 \times \mathbb{P}^3),$$

i.e., we quotient by $\langle h_1^2, h_1 h_2, h_2^2 \rangle$ where h_1 and h_2 are pull-backs of the hyperplane classes of $\mathbb{P}^2$ and $\mathbb{P}^3$ respectively. Then we have [BC94, Theorem 10.13]:

- $H^{4,0}(Y_b) = H^{4,0}(Y_b)_{van} = \mathrm{Jac}(F)_{(-1,-2)} = 0$
- $H^{3,1}(Y_b) = H^{3,1}(Y_b)_{van} \simeq \mathrm{Jac}(F)_{(1,0)} = \mathbb{C}[x,y,z]_1 \simeq \mathbb{C}^3$
- $H^{2,2}(Y_b)_{van} \simeq \mathrm{Jac}(F)_{(3,2)} \simeq \mathbb{C}^{37}$
- $H^{1,3}(Y_b) = H_{1,3}(Y_b)_{van} \simeq \mathrm{Jac}(F)_{(5,4)} \simeq \mathbb{C}^3$.

The first two dimension computations imply the others by the formula

$$b_4(Y_b) = \sum_{p+q=4} h^{p,q}(Y_b);$$

or one may compute the Hilbert function of an ideal generated by three forms of degree $(1,2)$ and four forms of degree $(2,1)$, subject to the relations (6.1) but otherwise generic.

We recall the technique of Green [CHM88, Sect. 5] and Voisin [Voi07, proof of 5.3.4], which applies as our variation of Hodge structures is effective of weight two after a suitable Tate twist.

Proposition 12. *Suppose there exists a $b_0 \in B$ and $\gamma \in H^{2,2}(Y_{b_0})_{van}$ such that the infinitesimal period map evaluated at γ*

$$\bar{\nabla}(\gamma) : T_{B,b_0} \to H^{1,3}(Y_{b_0})$$

is surjective. Then there exists a Zariski-dense set of $b \in B$ such that for any simply connected Euclidean neighborhood B' of b, the image of the natural map (composition of inclusion with local trivialization):

$$\tau_b : \mathcal{H}^{2,2}_{\mathbb{R}} \to H^4(Y_b, \mathbb{R})_{van}$$

contains an open subset $V_b \subset H^4(Y_b, \mathbb{R})_{van}$. Here $\mathcal{H}^{2,2}_{\mathbb{R}}$ is a vector bundle over B' with fiber

$$\mathcal{H}^{2,2}_{\mathbb{R},u} = H^4(Y_u, \mathbb{R})_{van} \cap F^2 H^4(Y_u, \mathbb{C})_{van}$$

over $u \in B'$.

Note that the image is the set of real degree four vanishing classes that are of type $(2,2)$ for some $b' \in B'$.

The infinitesimal condition is easy to check here. Since

$$B \subset \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(2,2)))$$

we may identify

$$T_{B,b_0} = (\mathbb{C}[x,y,z;s,t,u,v]/\langle F_0 \rangle)_{(2,2)},$$

where F_0 is the defining equation of Y_{b_0}. The infinitesimal period map

$$T_{B,b_0} \to \mathrm{Hom}(H^{2,2}(Y_b), H^{1,3}(Y_b))$$

was interpreted by Carlson and Griffiths as a multiplication map

$$(\mathbb{C}[x,y,z;s,t,u,v]/\langle F_0 \rangle)_{(2,2)} \times \mathrm{Jac}(F_0)_{(3,2)} \to \mathrm{Jac}(F_0)_{(5,4)}.$$

For fixed $\gamma \in \mathrm{Jac}(F_0)_{(3,2)}$, the differential in Voisin's hypothesis is computed by multiplying γ with the elements of bidegree $(2,2)$ [Voi07, Theorem 6.13].

Example 13. Consider the hypersurface $Y_{b_0} \subset \mathbb{P}^2 \times \mathbb{P}^3$ with equation

$$
\begin{aligned}
F_0 \;=\; & (u^2 + uv + ts)x^2 + (-t^2 + u^2 - v^2 - s^2)xy + (t^2 + uv + ts)y^2 \\
& + (-t^2 + u^2 - v^2 - s^2)xz + (t^2 - 16tu - u^2 + v^2 + s^2)yz \\
& + (-3uv - 3ts + s^2)z^2.
\end{aligned}
$$

We computed the Jacobian ring using Macaulay2 [GS]. In particular, we verified that

- $\mathrm{Jac}(F_0)_{(m_1,m_2)} = 0$ for

$$(m_1, m_2) \geq (13,2), (7,3), (3,5),$$

  so in particular Y_{b_0} is smooth;
- the monomials $\{xz^4v^4, yz^4v^4, z^5v^4\}$ form a basis for $\mathrm{Jac}(F_0)_{(5,4)}$.

Setting $\gamma = z^3v^2$, the multiples of γ generate $\mathrm{Jac}(F_0)_{(5,4)}$. Hence this example satisfies Voisin's hypothesis on the differential of the period map.

Proposition 14. *Consider the Noether-Lefschetz loci*

$$\{b \in B : Y_b \text{ admits an integral class of type } (2,2) \text{ meeting}$$
$$\text{the fibers of } Y_b \to \mathbb{P}^2 \text{ in odd degree}\}.$$

These are dense in the Euclidean topology on B.

Proof. We retain the set-up of Proposition 12. The intersection of the Noether-Lefschetz loci with B' may be expressed as

$$\{u \in B' : \mathcal{H}^{2,2}_{\mathbb{R},u} \cap \tau_b^{-1} H^4(Y_b, \mathbb{Q})_{van} \neq 0\}.$$

The density of the Noether-Lefschetz loci reflects the fact that

$$H^4(Y_b, \mathbb{Q})_{van} \subset H^4(Y_b, \mathbb{R})_{van}$$

is dense.

However, we are interested in vectors of $H^4(Y_b, \mathbb{Q})_{van}$ that are rational multiples of those associated with *odd* degree multisections M of $Y_b \to \mathbb{P}^2$. Such multisections exist because we can write a bidegree $(2,2)$ hypersurface containing a constant section of $\mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}^2$. The parity condition corresponds to a congruence on the image of M in $H^4(Y_b, \mathbb{Z})_{van}$: Indeed, write $\Lambda = H^4(Y_b, \mathbb{Z})$ and consider the natural inclusions and homomorphisms

$$\Lambda \supset \langle h_1^2, h_1 h_2, h_2^2 \rangle^{\perp} \hookrightarrow \Lambda / \langle h_1^2, h_1 h_2, h_2^2 \rangle = H^4(Y_b, \mathbb{Z})_{van};$$

the cokernel of the middle arrow is the *discriminant group* of the lattice $\langle h_1^2, h_1 h_2, h_2^2 \rangle^{\perp}$, a finite abelian group. The class M yields an element of this group and the parity condition translates into

$$M \cdot h_1^2 \equiv 1 \bmod 2.$$

Rational multiples of the elements satisfying this condition remain dense in $H^4(Y_b, \mathbb{R})_{van}$, so Proposition 12 gives the desired result. $\square$

The Noether-Lefschetz loci produced by this argument have codimension at most three in moduli; each is an algebraic subvariety of $B \subset \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(2,2))) \simeq \mathbb{P}^{59}$ [CDK95]. Any projective threefold in $\mathbb{P}^{59}$ will meet the closures of infinitely many of these loci.

7. Proof of Theorem 1

We assemble the various ingredients developed above:

(1) Theorem 4 guarantees that a very general hypersurface of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ fails to be stably rational, provided we can find a special X satisfying its hypotheses.
(2) The candidate example is introduced in Example 8.
(3) In Section 4, we show that X has non-trivial unramified second cohomology. This verifies the first hypothesis of Theorem 4.
(4) In Section 5, we analyze the singularities of X, checking that it admits a resolution with universally CH_0-trivial fibers.

(5) Proposition 6 gives a cohomological sufficient condition for rationality of $(2, 2)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$; Proposition 14 shows this condition is satisfied over a dense subset of the moduli space.

Consider a family $\phi : \mathcal{X} \to B$ of smooth $(2, 2)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ over a connected base B. If the base meets both the locus parametrizing non-stably rational varieties and the Noether-Lefschetz loci then ϕ has both rational and irrational fibers.

Remark 15. Concrete examples of rational hypersurfaces $X_{rat} \subset \mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(2, 2)$ are easy to produce, e.g., those containing a constant section of the first projection. Any very general pencil containing X_{rat} will have both rational and irrational fibers.

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