Game Dynamics as the Meaning of a Game

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Learning dynamics have traditionally taken a secondary role to Nash equilibria in game theory. We propose a new approach that places the understanding of game dynamics over mixed strategy profiles as the central object of inquiry. We focus on the stable recurrent points of the dynamics, *i.e.* states which are likely to be revisited infinitely often; obviously, pure Nash equilibria are a special case of such behavior. We propose a new solution concept, the Markov-Conley Chain (MCC), which has several favorable properties: It is a simple randomized generalization of the pure Nash equilibrium, just like the mixed Nash equilibrium; every game has at least one MCC; an MCC is invariant under additive constants and positive multipliers of the players' utilities; there is a polynomial number of MCCs in any game, and they can be all computed in polynomial time; the MCCs can be shown to be, in a well defined sense, surrogates or traces of an important but elusive topological object called the sink chain component of the dynamics; finally, it can be shown that a natural game dynamics surely ends up at one of the MCCs of the game.

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1. INTRODUCTION: NASH EQUILIBRIUM AND GAME DYNAMICS

Games are mathematical thought experiments. The reason we study them is because we think they will help us understand and predict the behavior of rational agents in interesting situations. Games first appeared in the work of Borel and von Neumann in the 1920s, but the field of Game Theory begun with Nash's 1950 paper [Nash et al. 1950]. Nash proposed to focus the study of games on the stationary points now called Nash equilibria, the mixed strategy profiles from which no player would deviate. Significantly, he employed Brouwer's fixed point theorem to prove that all games have a Nash equilibrium. As Roger Myerson argued in 1999, the solution concept of Nash equilibrium defined Game Theory and is a key ingredient of modern economic thought [Myerson 1999].

When computer scientists took up in earnest the study of games two decades ago, they had to come to grips with two aspects of the Nash equilibrium which, from their point of view, seemed remarkably awkward: first, the Nash equilibrium is less than an algorithmic concept in that there are often *many Nash equilibria* in a game, and resolving this ambiguity leads to the quagmire known as *equilibrium selection* [Harsanyi and Selten 1988]. Second, computing the Nash equilibrium seemed to be a combinatorial problem of exponential complexity, like the ones they had learned

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to avoid — a suspicion that eventually became a theorem.

The Nash equilibrium is a *dynamic* concept, because it draws its strength from the possibility that players *may move away* from a mixed strategy in the pursuit of utility. Game dynamics — processes whereby the players move, in discrete or continuous time, in the space of mixed strategy profiles according to precise rules, each in response to what others are currently doing — became mainstream during the 1980s, often, but not exclusively, in the context of evolutionary game theory [Smith 1982; Weibull 1995; Hofbauer and Sigmund 1998; Sandholm 2010]. Computer scientists recognized in game dynamics the key computational phenomenon of *learning* — change in behavior caused by experience — and made it an important part of algorithmic game theory's mathematical repertoire. The most commonly studied game dynamics is the *multiplicative weight updates* (MWU) algorithm [Arora et al. 2012; Littlestone and Warmuth 1994] (multiply the weight of each strategy by a multiplier reflecting the strategy's expected utility in the present mixed strategy environment created by the other players); its continuous time limit called *replicator dynamics* [Schuster and Sigmund 1983; Taylor and Jonker 1978] had been studied for some time.

An often-heard informal justification of the Nash equilibrium is the claim that "Players will eventually end up there." *But will they?* There is a long tradition of work in game dynamics whose goal is to establish that specific dynamics will eventually lead to a Nash equilibrium, or to circumscribe the conditions under which they would. The outcome is always somewhat disappointing. In fact, we now have an interesting proof [Benaïm et al. 2012] establishing that, in a precise sense, *there can be no possible dynamics that leads to a Nash equilibrium in all games*[1]. Even for the benchmark case of zero-sum games recent work has shown that any dynamics in the broad class Follow-the-Regularized-Leader always lead to cycling, recurrent behavior [Mertikopoulos et al. 2018]. It seems that *game dynamics are profoundly incompatible with the main solution concept in Game Theory.* Tacitly, this had been considered by many a drawback of game dynamics: Interesting as they may be, if game dynamics fail to reach the standard prediction of Game Theory then they have to be largely irrelevant.

We disagree with this point of view. We feel that the profound incompatibility between the Nash equilibrium and game dynamics exposes and highlights another serious flaw of the Nash equilibrium concept. More importantly, we believe that this incompatibility is a pristine opportunity and challenge to *define a new solution concept of an algorithmic nature* that aims to address the ambiguities and selection problems of Nash equilibria and is based on game dynamics.

2. RECURRENT POINTS

If games are experiments, shouldn't we be interested in their *outcome?*. The incumbent concept, the Nash equilibrium, declares that the outcome of a game is any point from which no player wants to deviate. But let us examine closely the

[1] The example game in which no dynamics can converge to the Nash equilibria presented in [Benaïm et al. 2012] is highly degenerate; we conjecture that there are *complexity* reasons why dynamics cannot converge to the Nash equilibria of even a non-degenerate two-player game, see the open problems section and [Papadimitriou and Piliouras 2018] .

two simplest games in the world: Matching Pennies (MP) and Coordination (CO) and their dynamics. The two games are shown in Figure 1, together with a plot of the trajectories of the replicator dynamics. In each game, the replicator dynamics (the continuous version of MWU) has five stationary points, the fully mixed Nash equilibrium and the four pure strategy outcomes. Interestingly, the actual behavior is totally different in the two games. In the MP game the dynamics revolve around the interior equilibrium without ever converging to it (it turns out that this is true of any (network) zero-sum game with fully mixed equilibria [Piliouras and Shamma 2014]). In CO out of the five stationary points three, including the mixed equilibrium, are *unstable*. The other two are stable and correspond to the two pure Nash equilibria of the coordination game. The space of mixed strategies (in this case the unit square) is divided into two *regions of attraction:* if the dynamics starts inside one such region, then the corresponding stable point will be eventually reached (at the limit). Such stability analysis and even computations of the geometry, volume of the regions of attraction of stable equilibria can be achieved even for network coordination games [Panageas and Piliouras 2016].

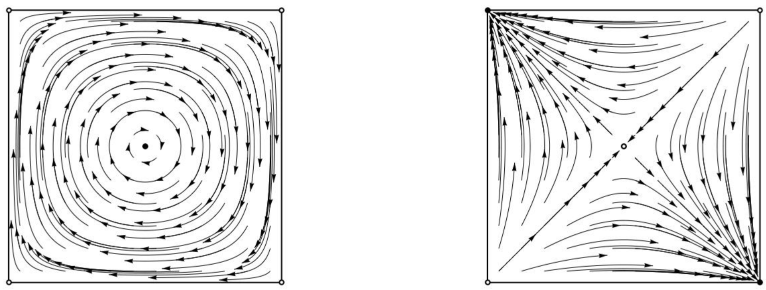


Fig. 1: Replicator trajectories in the Matching Pennies game (left) and the simple coordination game with common payoff matrix equal to $-I$ (right). The coordinates of each point are the probabilities assigned by the players to their first strategy.

Looking at this picture, one starts doubting if the mixed Nash equilibrium of CO deserves to be called "an outcome of this game:" to end up at this play you essentially have to start there — and in MP, the unique Nash equilibrium is otherwise never reached! What are then the outcomes of these two games? If we think of the games as behavioral experiments that evolve according to these rules, how should we describe the behavior in these pictures[2]. We want to propose that a point is an outcome of the game if it is *recurrent*, that is, if it appears again and again in the dynamics, and does so *stably*, i.e. even after it has been "jolted" a little. In this sense, only the two stable equilibria are outcomes in CO, while in MP all points are outcomes!

[2]It is well known that the replicator, as well as all no-regret dynamics, converges *in a time average sense* to a coarse correlated equilibrium. But of course, in an experiment we should be interested in *actual outcomes*, not time averages of the behavior.

Recurrence[3], informally, is the property of a point that is approached arbitrarily closely infinitely often by its own trajectory — e.g., lies on a cycle — is a key concept in dynamics. It is obviously a generalization of stationarity (a stationary point *is* its own trajectory). In these two simple games recurrence leads to a natural notion of "outcome," which greatly generalizes the Nash equilibrium — but also restricts it a bit: *The outcomes of the game are its recurrent points, with the exception of unstable stationary points.* Now all we have to do is generalize this simple picture to larger games!

We cannot. The mixed strategy space of two-by-two games is two-dimensional, and it is well known that two-dimensional dynamical systems are especially well behaved: the Poincaré – Bendixson theorem [Hirsch et al. 2012] guarantees essentially that all trajectories end in cycles (or stationary points, as a special case). In three or more dimensions, or in games with more than two players and/or strategies, the trajectories can be very complicated. The dynamics of such games can be *chaotic* [Sato et al. 2002], a severe setback in our ambition to use game dynamics in order to predict the outcome of games.

3. THE FUNDAMENTAL THEOREM OF DYNAMICAL SYSTEMS

A deep theorem in the topology of dynamical systems comes to the rescue of our project. Ever since the Poincaré – Bendixson theorem in the beginning of the 20th century illuminated the stark difference between two- and three-dimensional dynamical systems, mathematicians have been working hard on proving a similar result for the much more challenging environment of three or more dimensions. In particular, they strived for decades to define a relaxed concept of a "cycle" so that all trajectories end up in one of them, just as they do in two dimensions. This effort culminated in the late 1970s with the proof of the Fundamental Theorem of Dynamical Systems by Charles Conley [Conley 1978], through the definition of the concept of *chain recurrence.* Like many important results in Dynamical Systems, this theorem holds for both continuous and discrete-time dynamical systems; we shall only introduce it here the discrete time version.

Consider a discrete time dynamical system, that is, a continuous function f mapping a compact space (such as the space of all mixed strategy profiles of a game) to itself. A finite sequence of points $x_0, x_1, x_2, \ldots, x_n$ is called an ϵ-chain if for all $i < n, |x_{i+1} - f(x_i)| < \epsilon$. That is, an ϵ-chain is a sequence of f-steps each followed by an ϵ-jump.

Now comes an important definition: We say that two points x and y are *chain equivalent*, written $x \sim y$, if *for every* $\epsilon > 0$ there is an ϵ-chain from x to y and from y to x. Notice the severe requirement that such chains must exist for arbitrarily small jumps. Relation $\sim$ is an equivalence relation, and its non-singleton equivalence classes are called *chain components* of the dynamical system. Together, they comprise the *chain recurrent* part of the dynamical system: the points which eventually, *and after arbitrarily small jumps*, can and will return to themselves. In a familiar CS interpretation, point x is chain recurrent if, whenever Alice starts at x, Bob can convince her that she is on a cycle by manipulating the round off error of her computer — no matter how much precision Alice brings to bear.

[3] A recurrent point for a function f is a point that is in its own limit set by f.

Conley's Fundamental Theorem of Dynamical Systems states that *any dynamical system will eventually converge to its recurrent part* — that is to say, to one of its chain components — and in fact will do so in the strongest way possible: There is a *Lyapunov function,* that is, a real valued function defined on the domain that is *strictly decreasing*[4] on any point outside the chain recurrent set of the system, and constant on any chain component. In other words, if we adopt the chain recurrent points of the game dynamics as the outcome of the game (a reasonable proposal, which however we will further refine in the next section), then *every game is a potential game!* To summarize:

THEOREM 3.1. *(The Fundamental Theorem of Dynamical Systems, [Conley 1978]) The domain of any dynamical system can be decomposed into its transient part and its chain recurrent part, which is the union of the chain connected components. Furthermore, there is a Lyapunov function leading the dynamics to the chain recurrent part. Specifically, the Lyapunov function is strictly decreasing in the transient part and is constant along any chain component.*

Although this theorem is known within evolutionary game theory [Sandholm 2010] and more generally the notion of chain recurrence has been used to provide reductions between stochastically perturbed and continuous time dynamical systems, some of which are inspired by game theory [Benaim 1996; Benaım and Hirsch 1999; Hofbauer and Sandholm 2002; Benaïm and Faure 2012], the general structure and properties of chain recurrence sets for game theoretic dynamics is still not well understood. A stepping stone in this direction is to explore approximations of these notions that have a discrete, combinatorial structure.

The Sink Chain Components

Computer scientists know well that any directed graph can be decomposed into strongly connected components (SCCs), and these components can be partially ordered into a DAG. It is also well known that any random walk on the nodes and edges of this directed graph will end up almost certainly at one of the *sink* SCCs (assuming the directed graph is finite). In this sense, the sink SCCs are the stable components of the directed graph; it is quite natural to think of them as the "outcomes" of the process modeled by the graph. These familiar intuitions will be very helpful for understanding how chain components of game dynamics are outcomes of the game.

One can interpret Conley's theorem in a similar graph theoretic fashion. The long term system behavior is captured by the chain recurrent set, which itself decomposes into its chain components (CC) — the dynamical systems analogue of strongly connected components. Chain recurrent points and the chain components on which they reside can be partially ordered in the following way: We say that CC A *precedes* CC B if there exist (chain recurrent) points $x \in A$ and $y \in B$ such that there is an ϵ-chain from x to y for all $\epsilon > 0$.

Furthermore, the random walk insight form directed graphs carries over as well: Suppose that we run a *noisy* version of the discrete-time dynamical system, where,

[4] This means that given any initial condition that is not a chain recurrent point the function will decrease when we ones moves from this point to the next.

after each step, we make a random jump bounded by a small $\epsilon > 0$; it can be easily shown that, sooner or later, any CC *that is not a sink CC in the partial order* will be left behind. At this point it is tempting to adopt robustness under random walk as our notion of stability, and proclaim that the outcomes of the game are the sink CCs of the game dynamics.

But there is a catch: in the topology of dynamical systems, the DAG can be infinite and *sink CCs may not exist!*

We conjecture that *sink CC's always exist in the special case of game dynamics.* We know that, if they do exist, they are finitely many (in fact, polynomially many, see the next theorem). In the next section we introduce a simple combinatorial object — a probabilistic generalization of pure Nash equilibria, just like the mixed Nash equilibrium — which is a compelling surrogate of the sink CC, and the solution concept, the alternative to the mixed Nash equilibrium, that we are proposing.

4. THE RESPONSE GRAPH AND THE CONLEY-MARKOV CHAINS OF A GAME

There is a very natural — almost familiar — finite directed graph associated with every game: The nodes are all pure strategy profiles of the game. There is a directed edge from node x to node y if and only if (a) the two profiles differ only in the strategy used by one player, say player p; and (b) the utility of player p is at y is *strictly* greater, by some difference $\Delta > 0$, than the utility of the same player at x. Note that no "simultaneous moves," or moves with zero utility differential, are included. This graph is called the *response graph* of the game; an example is shown in Figure 2.

The sink SCCs of the response graph are important. Their importance stems from the fact that they are "combinatorial traces" of the sink chain components of the game dynamics, as the following result states.

THEOREM 4.1. *Any sink CC of the game dynamics contains at least one sink SCC of the response graph, and no two sink CCs can contain the same sink SCC. Therefore, the sink CCs are finitely many.*

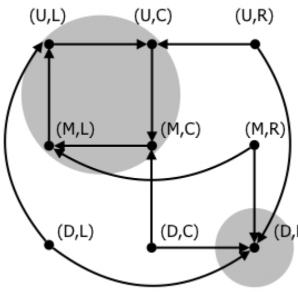
Theorem 4.1 has a remarkable liberating effect for our project of defining the "outcomes" of a game: It enables us to shift our attention from the sink CCs of the game dynamics — chain connected components of mixed strategy profiles which could be in general huge, complicated and unwieldy, let alone nonexistent... — to the sink SCCs of the response graph — tangible, simple, intuitive, present in all games.

Markov-Conley Chains (MCC) and Noisy Dynamics

The SCCs of the response graph are already of great interest to our quest for a solution concept. But we want to go a little further: We want to define in a natural way transition probabilities on the edges of the sink SCCs so they become Markov chains. Second, we shall prove that a natural dynamics converges to them almost surely.

We start by defining a natural randomized game dynamics: In any game, we define the *noisy MWU dynamics* as follows:

> We start with the familiar MWU dynamics, enhanced by noise: the frequency x_i of each strategy becomes $\max\{x_i \cdot e^{\alpha \cdot u_i} + \eta_i, 0\}/Z$, where η_i is a Gaussian



	L	C	R
U	(2, 0)	(0, 2)	(0, 0)
M	(0, 2)	(2, 0)	(0, 0)
D	(0, 0)	(0, 0)	(1, 1)

Fig. 2: Response graph (left) of a 3x3 game (right). The sink connected components are identified by the two grey circles. One is a pure Nash whereas the other is a 4-cycle of best response moves.

noise with mean zero and variance $\epsilon > 0$ and Z is a normalizing denominator keeping the frequencies of unit sum.[5] Notice that, by the way it is defined so far, the dynamics are non-explorative, that is, zero frequencies stay zero.

The dynamics so far has one serious disadvantage for our project: *Any pure strategy profile is a stationary point of the dynamics.* We surely do not want our theory to predict any pure strategy profile as outcome! For this reason we add the possibility of exploring new strategies: At a pure strategy profile, one player is chosen uniformly at random, then one strategy of this player is chosen also uniformly at random, and a component of ϵ is subtracted from the current pure strategy and added to the chosen one.

This concludes the definition of the noisy MWU dynamics. The following result is now not too difficult to show:

THEOREM 4.2. *The noisy MWU dynamics almost certainly ends up at a sink SCC of the response graph.*

This result — which we believe to hold for a broad spectrum of different dynamics replacing MWU, see the open problems section — points to the sink SCCs of the response graph as a very natural conception of the "fate," or "meaning," of a game — and a fine finish line for our quest.

But, as we have mentioned, we can go a little further: We can render each sink SCC of the response graph into an ergodic (strongly connected) Markov chain, that we will call *Markov-Conley Chains (MCCs)*, by defining in a natural way transition probabilities. But which transition probabilities, exactly?

There are several reasonable and principled ways of defining these transition probabilities. A natural starting point is to assign *equal probabilities to all edges* leaving a pure strategy profile of the SCC. Other more sophisticated approaches which are tailored to the game and dynamic being applied can also be explored. For example, since MWU dynamics are known to be regret minimizing, one possible

[5]Strictly speaking, to make sure that $Z > 0$ we should restrict the values of η_i to be bounded above by the inverse of the number of strategies; that is, the noise distribution is a truncated Gaussian.

desideratum for these Markov chains is that they converge to the set of coarse correlated equilibria, i.e., to distributions where agents do not experience regret. We leave this more detailed discussion for future work.

We propose that *a game's MCCs constitute a reasonable notion of outcome, or meaning, of the game*, a useful prediction and an attractive solution concept. It follows from the discussion above that the MCC concept has several strong advantages: (a) Every game has at least one MCC; (b) in fact, it has polynomially many in the description of the game; (c) they can all be computed in polynomial time; (d) pure Nash equilibria are MCCs, as an MCC is a simple randomized generalization of the concept of pure Nash equilibrium, an alternative to the mixed Nash equilibrium; (e) if the utilities of the players are multiplied by any positive constants, or increased by additive constants, specific to the players, the MCCs of the game do not change; (f) there is a class of natural dynamics that converges to the MCC's with probability one; (g) finally, and importantly, by Theorem 4.1 the MCCs are tangible surrogates of the game's sink chain components, a key topological concept underlying the Fundamental Theorem of Dynamical Systems, which however is quite elusive. In other words, MCCs can be thought as rough discretizations of chain components and as we will see in the next section, give rise to new important research questions.

5. DISCUSSION

To summarize our main points:

1. The Nash equilibrium is paramount in Game Theory, its standard solution concept: intuitive, compelling, universal, productive. But it has its problems: Computational complexity, multiplicity.
2. Furthermore, and contrary to popular belief, game dynamics *do not* (and *cannot*) converge to Nash equilibria.
3. The topology of game dynamics can be complicated and chaotic, but it becomes quite simple "if you squint a little:" Conley's Theorem establishes that, with respect to chain connectivity (connectivity through arbitrarily small jumps) the game dynamics always converges to the chain recurrent components (CCs) of the game via a potential/Lyapunov function argument.
4. Hence, *all games are potential games!*
5. If one adds to the model arbitrarily small noise, only the *sink CCs* are reached.
6. The Markov-Conley chain is a fundamental and simple combinatorial object associated with a game. It is a useful surrogate of the sink CCs of a game. It is a randomized generalization of the pure Nash equilibrium — just like the mixed Nash equilibrium.
7. All games have at least one Markov-Conley chain, and at most polynomially many, in the description of the game. They can be computed efficiently. A natural dynamics is guaranteed to reach one of them almost certainly. The Markov-Conley chain is our proposed new solution concept of a game.
8. In view of the above, any game can be seen as map from the set of mixed strategy profiles to the set of Markov-Conley chains. *Games are algorithms!*

9. This creates a new framework for the price of anarchy, and raises many open questions, explored next.

Precursors

Over the past three decades, there have been several interesting works that anticipated our point of view and some of the ingredients of our proposal. Peyton Young in his influential 1993 paper "The evolution of social conventions" [Young 1993] introduces a randomized game dynamics which, even though in detail quite different from our noisy MWU, can be shown to converge to the MCCs of the game *in the special case in which the MCCs are singletons*, that is to say, pure Nash equilibria; Young proposes the induced distribution on these as the outcome of the game. Another influential work in economics refining the Nash equilibrium in a way that is a bit reminiscent of the MCCs are the CURB sets [Basu and Weibull 1991]. In AGT, Goemans, Mirokni, and Vetta, in their pursuit of an alternative definition of the price of anarchy [Goemans et al. 2005], study and analyze something akin to the MCCs (see the open problems section for new opportunities for such analysis opened by the present work). Finally, [Candogan et al. 2011] is an elegant algebraic study of the response graph of games, establishing that it can be the basis of an intriguing taxonomy and decomposition of games.

Open Problems

These ideas release a torrent of open problems; here we mention the ones that are most central to our point of view.

We conjecture that a very broad class of game dynamics, certainly including the MWU and the replicator, have sink chain components for all games, and thus the role of the MCCs as surrogates of the sink chain components is no longer conditional on existence. This seems a plausible and reasonably demanding conjecture in the theory of dynamical systems. We also conjecture that Theorem 4.2 holds not just for the modification of MWU by Gaussian noise, but also for (a) a very large class of dynamics that will essentially include a large variety of rational reactions of the players to the current mix; and (b) under less liberal noise than Gaussian; for example, uniform in the ϵ ball. Both kinds of extensions would be important improvements of this work. Extension (b) would diminish the reliance of our result on rare events, and make it more realistic. Extension (a) is important for a more fundamental reason: One objection to dynamics as the basis of rationality is that the behavior of others can be predicted, and this leads to incentives to deviate from the behavior implied by the dynamics. But such deviation would be itself a new kind of dynamics, and ideally it will also be covered by a similar convergence result.

For the traditional theory, a game is the specification of an equilibrium selection problem; we now know that it is very hard to make progress in that direction. When one turns to the dynamics as the meaning of the game, the selection problem does not exactly vanish, but it is a well defined mathematical exercise: The meaning of the game is a finite distribution on the MCCs of the game — each one has a basin of attraction, and we must calculate the probability mass of this basin, presumably given some prior on mixed strategy profiles. This mathematical exercise becomes even more meaningful, and often analytically demanding, when one wants to calculate quantities such as *the price of anarchy* of the game; see [Panageas and Piliouras 2016] for an early example of such analysis in potential games where MCCs correspond to Nash equilibria. Extending such average case analysis to other classes of games with non-equilibrating dynamics is an important future direction.

An MCC is an interesting mathematical object associated with the game. It would be insightful to discover connections with other types of equilibria. For example, is there any connection between the steady state distribution of an MCC and mixed Nash equilibria? (There is obviously a strong connection with pure Nash equilibria.) How about the correlated equilibria? The latter question is interesting because there are well known connections between the steady state distributions of Markov chains and correlated equilibria [Hart and Schmeidler 1989; Myerson 1997]. Also, an MCC implies intuitively a certain circularity of payoffs, when is such circularity enough to guarantee the existence of a mixed Nash equilibrium?

Regarding convergence of dynamics to the Nash equilibria: As we mentioned in the introduction, there is an impossibility result for a *highly degenerate* two-player game [Benaïm et al. 2012]. For non-degenerate two-player games, we can show that Nash converging dynamics exist, albeit of exponential complexity. We conjecture that, unless P = NP, impossibility (now for complexity reasons) prevails as well: There are no possible tractable dynamics, i.e. dynamics for which each update step can be computed in polynomial time, such that they always converge (even arbitrarily slowly) to Nash equilibria. Specifically, in contrast to [Hart and Mas-Colell 2003] we conjecture that even if agents dynamics are allowed to depend on the whole payoff structure of the game, there do not exist efficiently computable update rules that always converge to Nash equilibria.

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REFERENCES

Arora, S., Hazan, E., and Kale, S. 2012. The multiplicative weights update method: a meta-algorithm and applications. *Theory of Computing 8,* 1, 121–164.

Basu, K. and Weibull, J. W. 1991. Strategy subsets closed under rational behavior. *Economics Letters 36,* 2, 141–146.

Benaim, M. 1996. A dynamical system approach to stochastic approximations. *SIAM Journal on Control and Optimization 34,* 2, 437–472.

Benaïm, M. and Faure, M. 2012. Stochastic approximation, cooperative dynamics and supermodular games. *The Annals of Applied Probability 22,* 5, 2133–2164.

Benaim, M. and Hirsch, M. W. 1999. Mixed equilibria and dynamical systems arising from fictitious play in perturbed games. *Games and Economic Behavior 29,* 1-2, 36–72.

Benaïm, M., Hofbauer, J., and Sorin, S. 2012. Perturbations of set-valued dynamical systems, with applications to game theory. *Dynamic Games and Applications 2,* 2, 195–205.

Candogan, O., Menache, I., Ozdaglar, A., and Parrilo, P. A. 2011. Flows and decompositions of games: Harmonic and potential games. *Mathematics of Operations Research 36,* 3, 474–503.

Conley, C. 1978. Isolated invariant sets and the morse index. cbms regional conference series in mathematics, 38. *American Mathematical Society, Providence, RI 16.*

GOEMANS, M., MIRROKNI, V., AND VETTA, A. 2005. Sink equilibria and convergence. In *Foundations of Computer Science, 2005. FOCS 2005. 46th Annual IEEE Symposium on*. IEEE, 142–151.

HARSANYI, J. C. AND SELTEN, R. 1988. A general theory of equilibrium selection in games. *MIT Press Books 1*.

HART, S. AND MAS-COLELL, A. 2003. Uncoupled dynamics do not lead to nash equilibrium. *American Economic Review 93,* 5, 1830–1836.

HART, S. AND SCHMEIDLER, D. 1989. Existence of correlated equilibria. *Mathematics of Operations Research 14,* 1, 18–25.

HIRSCH, M. W., SMALE, S., AND DEVANEY, R. L. 2012. *Differential equations, dynamical systems, and an introduction to chaos*. Academic press.

HOFBAUER, J. AND SANDHOLM, W. H. 2002. On the global convergence of stochastic fictitious play. *Econometrica 70,* 6, 2265–2294.

HOFBAUER, J. AND SIGMUND, K. 1998. *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge.

LITTLESTONE, N. AND WARMUTH, M. K. 1994. The weighted majority algorithm. *Inf. Comput. 108,* 2 (Feb.), 212–261.

MERTIKOPOULOS, P., PAPADIMITRIOU, C., AND PILIOURAS, G. 2018. Cycles in adversarial regularized learning. In *ACM-SIAM Symposium on Discrete Algorithms (SODA)*.

MYERSON, R. B. 1997. Dual reduction and elementary games. *Games and Economic Behavior 21,* 1-2, 183–202.

MYERSON, R. B. 1999. Nash equilibrium and the history of economic theory. *Journal of Economic Literature 37,* 3, 1067–1082.

NASH, J. F. ET AL. 1950. Equilibrium points in n-person games. *Proceedings of the national academy of sciences 36,* 1, 48–49.

PANAGEAS, I. AND PILIOURAS, G. 2016. Average case performance of replicator dynamics in potential games via computing regions of attraction. In *Proceedings of the 2016 ACM Conference on Economics and Computation*. ACM, 703–720.

PAPADIMITRIOU, C. AND PILIOURAS, G. 2018. Games are algorithms: Game dynamics, chain components and markov-conley chains. *Arxiv Preprints*.

PILIOURAS, G. AND SHAMMA, J. S. 2014. Optimization despite chaos: Convex relaxations to complex limit sets via poincaré recurrence. In *Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms*. SIAM, 861–873.

SANDHOLM, W. H. 2010. *Population Games and Evolutionary Dynamics*. MIT Press.

SATO, Y., AKIYAMA, E., AND FARMER, J. D. 2002. Chaos in learning a simple two-person game. *Proceedings of the National Academy of Sciences 99,* 7, 4748–4751.

SCHUSTER, P. AND SIGMUND, K. 1983. Replicator dynamics. *Journal of Theoretical Biology 100,* 3, 533 – 538.

SMITH, J. M. 1982. *Evolution and the Theory of Games*. Cambridge university press.

TAYLOR, P. D. AND JONKER, L. B. 1978. Evolutionary stable strategies and game dynamics. *Mathematical Biosciences 40,* 12, 145 – 156.

WEIBULL, J. W. 1995. *Evolutionary Game Theory*. MIT Press; Cambridge, MA: Cambridge University Press.

YOUNG, H. P. 1993. The evolution of conventions. *Econometrica: Journal of the Econometric Society*, 57–84.