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Unimodular graded Poisson Hopf algebras

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Abstract

Let A be a Poisson Hopf algebra over an algebraically closed field of characteristic zero. If A is finitely generated and connected graded as an algebra and its Poisson bracket is homogeneous of degree $d \geqslant 0$, then A is unimodular; that is, the modular derivation of A is zero. This is a Poisson analogue of a recent result concerning Hopf algebras which are connected graded as algebras.

Introduction

Poisson algebras have lately been playing an important role in algebra, geometry, mathematical physics and other subjects. For example, Poisson structures have been used in the study of discriminants, in the work of Nguyen–Trampel–Yakimov [**23**] and Levitt–Yakimov [**16**], and in the representation theory of Sklyanin algebras of global dimension 3, in the work of Walton–Wang–Yakimov [**29**]. Restricted Poisson Hopf algebras were introduced in [**4**], and further investigated in [**2**].

Throughout, $\Bbbk$ will denote an algebraically closed field of characteristic zero; all algebras considered in what follows are $\Bbbk$-algebras, and all unadorned tensor products are over $\Bbbk$.

We are concerned here with the unimodularity of a Poisson Hopf algebra A which is an affine connected graded algebra. Here, in the light of the smoothness which holds in our characteristic zero setting, A is a polynomial algebra in finitely many variables. Since Poisson Hopf algebras were introduced by Drinfeld [**10**] in 1985, (see Definition 1.1), they have been intensively studied in connection with homological algebra and deformation quantization; see, for example, recent work in [**17**–**20**]. A Poisson algebra is said to be *unimodular* if the class of its modular derivation in a certain factor of the first Poisson cohomology group is trivial; further details and references are given in Subsection 3.1. The purpose of this paper is to provide further evidence that unimodularity for commutative Poisson algebras is an analogue of the Calabi–Yau property of a noncommutative algebra, reinforcing results in this direction in [**9**, **19**, **20**]. Thus, just as is the case for the Calabi–Yau property in a noncommutative setting, unimodularity can be viewed as a homological property of Poisson algebras.

A result of [**6**] states that a Hopf $\Bbbk$-algebra of finite Gelfand–Kirillov dimension which is connected graded as an algebra is Calabi–Yau. One might suspect that there should be a Poisson version of this result, and indeed our main result is the following theorem, whose proof uses this noncommutative result from [**6**], applied to the Poisson enveloping algebra of a graded Poisson Hopf algebra.

Theorem 1. *Let A be a Poisson Hopf $\Bbbk-$algebra. Suppose that*

(i) *as an algebra, A is finitely generated and connected graded; and that*
(ii) *the Poisson bracket $\{-,-\}$ is homogeneous of degree $d \geqslant 0$.*

Then A is unimodular; that is, the modular derivation of A is zero.

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In parallel with duality results for Hochschild homology and cohomology in the setting of noncommutative Calabi–Yau algebras, the above result yields consequences for duality of Poisson homology and cohomology. Thus, as an immediate consequence of the theorem and [**20**, Theorem 3.5 and Remark 3.6], we have

Corollary 2. *Let A be as in the above theorem and let $n = \mathrm{GKdim}\, A$, the rank of the polynomial algebra A. Then there is a Poincaré duality between the Poisson homology and cohomology: for every Poisson A-module M, there is a functorial isomorphism*

$$HP_i(M) \cong HP^{n-i}(M)$$

for all $i = 0, \ldots, n$.

We refer to [**19**, **20**] for other terminology in Corollary 2. Section 1 contains background on Poisson Hopf algebras, algebra, coalgebra and Hopf gradings. In Section 2 we recall the definition of the enveloping algebra $\mathcal{U}(A)$ of a Poisson algebra A, and collect some of the properties of $\mathcal{U}(A)$ needed for the proof of Theorem 1. In particular, we show that if A is a connected Poisson Hopf algebra, then $\mathcal{U}(A)$ is a connected Hopf algebra. Background on the modular derivation is contained in Section 3, which also includes the proofs of Theorem 1 and Corollary 2. An example (which is not an enveloping algebra of a Lie algebra) to which the main theorem applies is given in Example 1.9, and further examples to illustrate the necessity of the hypotheses of the theorem appear in Section 4. Given a Hopf algebra, its coproduct, antipode and counit will be respectively denoted by Δ, S and ϵ. The definition of the Gelfand–Kirillov dimension, denoted GKdim, can be found in [**14**].

1. *Definitions and preliminaries*

1.1. *Poisson algebras*

Recall (from [**13**, Definition 1.1], for example) that a commutative associative $\Bbbk$-algebra is a *Poisson algebra* if there is a $\Bbbk$-bilinear map, the *Poisson bracket* of A,

$$\{-,-\} : A \otimes A \longrightarrow A,$$

such that $(A, \{-,-\})$ is a Lie algebra, and the Leibniz rule holds, namely

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

for all $a, b, c \in A$. Given Poisson algebras A and B, an algebra homomorphism $\alpha : A \longrightarrow B$ is a *Poisson morphism* if $\alpha(\{a, c\}) = \{\alpha(a), \alpha(c)\}$ for all $a, c \in A$. If A and B are Poisson algebras, then by [**13**, Proposition 1.2.10] so is $A \otimes B$, with bracket

$$\{a \otimes b, c \otimes d\} = \{a, c\} \otimes bd + ac \otimes \{b, d\}.$$

Definition 1.1 ([**11**, p. 801 and 802; **13**, Definitions 3.1.3 and 3.1.6]).

(1) A *Poisson Hopf algebra* is a Poisson algebra A which is also a Hopf algebra, such that $\Delta : A \longrightarrow A \otimes A$ is a Poisson morphism.
(2) A *Poisson algebraic group G over* $\Bbbk$ is an affine algebraic $\Bbbk$-group whose coordinate algebra $\Bbbk(G)$ is a Poisson Hopf algebra; equivalently, the multiplication $m : G \times G \longrightarrow G$ is a Poisson map.

It is well known that the category of affine commutative Hopf algebras is dual to the category of affine algebraic groups. When restricted to the Poisson setting, the category of affine Poisson Hopf algebras is dual to the category of Poisson algebraic groups.

In fact, when Definition 1.1(1) applies, it is not hard to show that ϵ is a Poisson morphism and S is a Poisson anti-morphism; see for example, [**18**, p. 96].

1.2. *Gradings on algebras and coalgebras*

We fix some terminology to discuss gradings on three operations — the multiplication, the Poisson bracket and the comultiplication. The graded algebras A under consideration in this paper will typically be $\mathbb{N}$-graded; that is, $A = \bigoplus_{i\in\mathbb{N}} A(i)$ for $\Bbbk$-subspaces $A(i)$, with $A(i)A(j) \subseteq A(i+j)$. We say that A is *connected as an algebra* if $A(0) = \Bbbk$.

Suppose that an $\mathbb{N}$- or $\mathbb{Z}$-graded algebra A is commutative and Poisson. If there exists an integer d with

$$\{A(i), A(j)\} \subseteq A(i+j+d)$$

for all i and j, we say that the bracket of A has *degreed*. In this case, A is called a *graded Poisson algebra*, or — equivalently — that $\{-,-\}$ is *homogeneous*.

Recall that the *coradical filtration* of a coalgebra H is the ascending chain of subcoalgebras $\{H_n : n \geqslant 0\}$ of H, with H_0 the coradical of H and H_n defined inductively by

$$H_n \quad := \quad \Delta^{-1}(H \otimes H_0 + H_{n-1} \otimes H)$$

for $n > 0$, [**22**, 5.2]. Then $H = \bigcup_{n\geqslant 0} H_n$; and if H is a pointed Hopf algebra, then $\{H_n\}$ is an algebra filtration, [**22**, Lemma 5.2.8]. We shall say that the Hopf algebra H is *connected as a Hopf algebra* if it is connected as a coalgebra — that is, if $H_0 = \Bbbk$.

The following lemma is easily proved by induction on n.

Lemma 1.2 [**18**, Lemma 7.1]. *Let A be a pointed Poisson Hopf algebra with coradical filtration $\{A_n\}$. Suppose that $\{g,h\} = 0$ for all group-like elements $g,h \in A$. Then, for all $i,j \geqslant 0$,*

$$\{A_i, A_j\} \subseteq A_{i+j}.$$

Turning now to gradings on Hopf algebras, the relevant definitions are as follows.

Definition 1.3. Let H be a Hopf algebra.

(1) H is a *graded Hopf algebra* if it is simultaneously $\mathbb{N}$-graded as an algebra and as a coalgebra — that is, $H = \bigoplus_{i\geqslant 0} H(i)$ for some vector subspaces $H(i)$, with $H(i)H(j) \subseteq H(i+j)$, $S(H(i)) \subseteq H(i)$ and

$$\Delta(H(i)) \subseteq \bigoplus_{\ell\geqslant 0} H(\ell) \otimes H(i-\ell)$$

for all $i,j \geqslant 0$.

(2) H is a *coradically graded Hopf algebra* if it is a graded Hopf algebra with $H_i = \bigoplus_{j\leqslant i} H(j)$ for all $i \geqslant 0$.

When working with graded Hopf algebras it is frequently useful to make use of the adjustment permitted by the following lemma, a slight improvement of [**6**, Lemma 2.1(2)].

Lemma 1.4. *Let H be a Hopf algebra which is a connected graded algebra, $H = \bigoplus_{i\geqslant 0} H(i)$.*

(i) *There is an algebra grading of H, $H = \bigoplus_{i\geqslant 0} B(i)$, such that $\ker \epsilon = \bigoplus_{i\geqslant 1} B(i)$.*

(ii) *Suppose that H is a graded Hopf algebra through the given grading $H = \bigoplus_{i\geqslant 0} H(i)$. Then $\ker \epsilon = \bigoplus_{i\geqslant 1} H(i)$.*

Proof. (i) This is [6, Lemma 2.1(2)].

(ii) Clearly it is enough to show that $H(i) \subseteq \ker \epsilon$ for all $i \geqslant 1$. So, let $i \geqslant 1$ and let $x \in H(i)$. We show that $\epsilon(x) = 0$ by induction on i. Since H is a graded coalgebra and $H(0) = k$,

$$\Delta(x) = x_1 \otimes 1 + 1 \otimes x_2 + y, \tag{E1.4.1}$$

where $y \in \bigoplus_{j=1}^{i-1} H(j) \otimes H(i-j)$. By induction, $y = \sum y_1 \otimes y_2$ with all terms $y_1, y_2 \in \ker \epsilon$. Apply $\mu \circ (\mathrm{Id} \otimes \epsilon)$ to (E1.4.1), where $\mu : H \otimes H \to H$ is multiplication, yielding

$$x = x_1 + \epsilon(x_2).$$

Since x and x_1 are in $H(i)$ while $\epsilon(x_2) \in H(0)$, we deduce that $\epsilon(x_2) = 0$ and $x_1 = x$. Similarly, using instead $\mu \circ (\epsilon \otimes \mathrm{Id})$, it follows that $\epsilon(x_1) = 0$. Thus $x \in \ker \epsilon$ as required. □

Coradically graded Hopf algebras occur very naturally in the study of pointed Hopf algebras. The first part of following result is recorded for example in [1, Definition 1.13]; the second part is proved in [33, Theorem 6.9].

PROPOSITION 1.5. *Let H be a pointed Hopf algebra with coradical filtration $\mathcal{C} = \{H_n\}$. Set $H(0) := H_0$ and $H(i) := H_i/H_{i-1}$ for $i \geqslant 1$, and let*

$$\mathrm{gr}_{\mathcal{C}} H = \bigoplus_{i \geqslant 0} H(i)$$

be the associated graded algebra, which is a Hopf algebra with the operations induced from H.

(i) *$\mathrm{gr}_{\mathcal{C}} H$ is a coradically graded Hopf algebra.*

(ii) *If H is a connected Hopf algebra of finite GK-dimension n, then $\mathrm{gr}_{\mathcal{C}} H$ is a commutative polynomial algebra in n variables.*

Let H be a graded Hopf algebra. Then it is straightforward to show that, keeping the assigned grading,

$$H \text{ a connected graded algebra} \Longrightarrow H \text{ a connected Hopf algebra.} \tag{E1.5.1}$$

To prove (E1.5.1), note first that the coradical H_0 is graded, and then prove that $H_0 \cap H(i) = \{0\}$ for $i \geqslant 1$ by induction on i. However, the reverse implication is false even when H is commutative, as explained in Theorem 1.6.

1.3. *Gradings of commutative Hopf algebras*

Our focus in this paper is on polynomial algebras and their deformations. The possible Hopf structures in this case are as described in the following theorem. Details concerning part (i) can be found at [6, Theorem 1]. Its nontrivial content is due to Serre for (1) $\Longrightarrow$ (3), (see for example, [3, Theorem 6.2.2]), and to Lazard [15] for (3) $\Longrightarrow$ (4). Note that a unipotent algebraic $\Bbbk$-group U is *Carnot* if its Lie algebra $\mathfrak{u}$ is $\mathbb{N}$-graded, $\mathfrak{u} = \bigoplus_{i=1}^{t} \mathfrak{u}_i$, with $\mathfrak{u} = \langle \mathfrak{u}_1 \rangle$. See for example [7] for further details and references.

THEOREM 1.6. *Let H be an affine commutative Hopf $\Bbbk$-algebra.*

(i) *Then the following are equivalent.*
  (1) *H is connected graded as an algebra.*
  (2) *H is connected as a Hopf algebra.*
  (3) *H is a polynomial algebra of finite rank.*
  (4) *H is the coordinate algebra of a unipotent algebraic $\Bbbk$-group U.*

(ii) *Let H satisfy the equivalent hypotheses of* (i). *Then the following are equivalent.*
(5) *H is coradically graded.*
(6) *U is a Carnot group.*

Proof. (ii)(5)$\Longrightarrow$(6): Suppose that H is coradically graded, with, in view of (i),

$$H = k(U) = k[X_1, \dots, X_n] = k \oplus \bigoplus_{i \geqslant 1} H(i),$$

for some $n \geqslant 1$. Moreover, setting $\mathfrak{u}$ to be the Lie algebra of U we have

$$\mathcal{U}(\mathfrak{u}) \cong \bigoplus_{i \geqslant 0} H(i)^*,$$

the graded dual of H, by [**5**, Proposition 5.5(1) and (3)], and $\mathcal{U}(\mathfrak{u})$ is generated by $\mathfrak{u}_1 = H(1)^*$, by [**1**, Lemma 5.5], see also [**5**, Proposition 5.5(4)]. Thus $\mathfrak{u}$ is a graded Lie algebra generated in degree 1. Hence $\mathfrak{u}$ and, equivalently, U, are Carnot.

(6)$\Longrightarrow$(5): Suppose $H = k(U)$ with U a Carnot group. Consider the associated graded algebra $\mathrm{gr}_{\mathcal{C}} H$ with respect to the coradical filtration $\mathcal{C} = \{H_n\}$ of H. By Lemma 1.5 this is a connected and coradically graded Hopf algebra, and it is easy to see — for example, it follows by considering the graded dual, as in the proof of (5)$\Longrightarrow$(6) — that $\mathrm{gr}_{\mathcal{C}} H$ is the coordinate algebra of the associated graded (Carnot) unipotent group $\mathrm{gr}(U)$ of U. But U is Carnot, so $U = \mathrm{gr}(U)$, and hence H is coradically graded. □

It is natural to synthesise the concepts defined in Subsection 1.2 for the Poisson and Hopf categories, thus arriving at graded versions of the concepts introduced by Drinfeld, see Definition 1.1, as follows.

Definition 1.7. Let $n \geqslant 1$, let $d_1, \dots, d_n$ be nonnegative integers and let $A = k[x_1, \dots, x_n]$, graded so that x_i is homogeneous of degree d_i for $i = 1, \dots, n$. Suppose that $\{-,-\}$ is a Poisson bracket on A.

(1) If A is a Poisson Hopf algebra, simultaneously graded both as a Hopf algebra and a Poisson algebra, then A is called a *graded Poisson Hopf algebra*.

(2) If in (1) the Poisson bracket has degree d, as defined in Subsection 1.2, then A is a *graded Poisson Hopf algebra of degree d*.

(3) If in (1) [respectively, (2)] A is coradically graded, then A is a *coradically graded Poisson Hopf algebra* [respectively, *of degree d*].

Here are important mechanisms for the construction of coradically graded Poisson Hopf algebras. The first is an immediate corollary of Lemma 1.2 and Proposition 1.5(i), and the second and third follow from Proposition 1.5(ii), using the commutator bracket on H to induce a Poisson bracket on $\mathrm{gr}_{\mathcal{C}} H$. Note that (ii) is the more familiar case $t = 1$ of (iii). The proof of the general case is straightforward, and is left to the reader.

Corollary 1.8. (i) *Let A be a pointed Poisson Hopf algebra with coradical filtration $\mathcal{C} = \{A_n\}$, and suppose $\{g, h\} = 0$ for all group-like elements $g, h, \in A$. Then $\mathrm{gr}_{\mathcal{C}} A$ carries a Poisson bracket induced from the bracket on A, and as such it is coradically graded Poisson Hopf algebra of degree* 0.

(ii) *Let H be a connected Hopf algebra of finite GK-dimension n and coradical filtration $\mathcal{C}$. Then $\mathrm{gr}_{\mathcal{C}} H = \Bbbk[x_1, \dots, x_n]$ is a coradically graded Poisson Hopf algebra of degree -1.*

(iii) *In the setting of part* (ii), *let $x_i \in \mathrm{gr}_{\mathcal{C}} H(m_i)$ where $m_i \in \mathbb{N}$, and choose $y_1, \dots, y_n \in H$ be such that $\mathrm{gr}_{\mathcal{C}} y_i = x_i$, for $i = 1, \dots, n$. Let $t \in \mathbb{Z}$ be maximal such that $[y_i, y_j] \in H_{m_i + m_j - t}$*

for all i and j, (so that $t \geqslant 1$ by Proposition 1.5(ii)). Taking images of the brackets $[y_i, y_j]$ in the space $H_{m_i+m_j-t}/H_{m_i+m_j-t-1} \subseteq \mathrm{gr}_{\mathcal{C}} H$ yields a coradically graded Poisson Hopf algebra structure of degree $d = -t$ on $\Bbbk[x_1, \ldots, x_n]$. This Poisson bracket is trivial if and only if H is commutative.

Similar terminology to that in Definition 1.7 can be introduced in an obvious way for a commutative $\mathbb{N}$-graded bialgebra A.

We conclude this section by giving an example of a Poisson Hopf algebra to which Theorem 1 applies. It should be observed that this example illustrates the fact that a Poisson Hopf polynomial algebra may be a graded Poisson Hopf algebra with respect to more than one grading, and that the hypotheses of the main theorem may apply in a proper subset of the possible cases.

EXAMPLE 1.9. Let H be the connected Hopf algebra of GK-dimension 5 constructed in [**6**, Theorem 5.6]. That is, $H = k\langle \hat{a}, \hat{b}, \hat{c}, \hat{z}, \hat{w}\rangle$ with relations given by setting all commutators of the generators equal to 0, except for $[\hat{a}, \hat{b}] = \hat{c}$ and $[\hat{z}, \hat{w}] = \frac{1}{3}\hat{c}^3$. Here, $\hat{a}, \hat{b}, \hat{c}$ are primitive,

$$\Delta(\hat{z}) = 1 \otimes \hat{z} + \hat{z} \otimes 1 + \hat{a} \otimes \hat{c} + \hat{c} \otimes \hat{a}.$$

and

$$\Delta(\hat{w}) = 1 \otimes \hat{w} + \hat{w} \otimes 1 + \hat{b} \otimes \hat{c} + \hat{c} \otimes \hat{b}.$$

It is shown in [**6**] that H is not isomorphic as an algebra to $U(L)$ for any Lie algebra L. It is easy to confirm that

$$H_1 = k + k\hat{a} + k\hat{b} + k\hat{c} \textit{ and } H_2 = H_1 + H_1^2 + k\hat{z} + k\hat{w},$$

so that $A := \mathrm{gr}_{\mathcal{C}} H = k[a, b, c, z, w]$, with a, b and c having degree 1, z and w degree 2. Thus, by Corollary 1.8, A is a coradically graded Poisson Hopf algebra of degree -1, with

$$\{a, b\} = c, \qquad \{z, w\} = \frac{1}{3}c^3,$$

and all other Poisson brackets of the generators equal to 0.

Another way of understanding A is the following. Let T be the subgroup of $SL(4, k)$ with 1s on the diagonal and 0s below the diagonal, and let $U = T/Z$, where $Z = Z(T)$, which is the subgroup of T with all off-diagonal entries equal to 0 except for the (1,4)-entry. Then it is easy to check that $A \cong \mathcal{O}(U)$; indeed, with the obvious notation, one takes $a = X_{12}, b = X_{34}$, $c = X_{23}, z = X_{13} - S(X_{13}) = 2X_{13} - X_{12}X_{23}, w = X_{24} - S(X_{24}) = 2X_{24} - X_{34}X_{23}$.

Now we set

$$\deg a = \deg b = 1; \qquad \deg c = 2; \qquad \deg z = \deg w = 3.$$

Then one checks that A is a graded Poisson Hopf algebra of Poisson degree 0; but of course A is no longer coradically graded. Nevertheless, the main theorem, Theorem 1, still applies, and we conclude that A is unimodular. Another way of checking the unimodularity of A is to use (E3.0.1).

2. *The enveloping algebra of a Poisson algebra*

2.1. *Definition of $\mathcal{U}(A)$*

The proof of Theorem 1 is carried out by passing to the enveloping algebra $\mathcal{U}(A)$ of the Poisson algebra A, whose definition we recall from [**24**]. Let A be a Poisson algebra, and let $m_A = \{m_a \mid a \in A\}$ and $h_A = \{h_a \mid a \in A\}$ be two copies of the vector space A endowed with two $\Bbbk$-linear isomorphisms $m : A \to m_A : a \mapsto m_a$ and $h : A \to h_A : a \mapsto h_a$. Then the universal enveloping algebra $\mathcal{U}(A)$ is an associative algebra over $\Bbbk$, with an identity 1, generated by m_A and h_A with relations

$$m_{xy} = m_x m_y, \tag{E2.0.1}$$

$$h_{\{x,y\}} = h_x h_y - h_y h_x, \tag{E2.0.2}$$

$$h_{xy} = m_y h_x + m_x h_y, \tag{E2.0.3}$$

$$m_{\{x,y\}} = h_x m_y - m_y h_x = [h_x, m_y], \tag{E2.0.4}$$

$$m_1 = 1. \tag{E2.0.5}$$

Given a commutative $\Bbbk$-algebra A, an A-module and Lie algebra L, and an A-module and Lie algebra map ρ from L to $\mathrm{Der}_{\Bbbk} A$ satisfying a compatibility condition, Rinehart [**28**] defined in 1963 a certain associative algebra which he denoted $V(A, L)$ and which is nowadays called the *the enveloping algebra of the Lie–Rinehart algebra* of A, L and the *anchor map* ρ. Huebschmann showed in 1990 [**12**, Theorem 3.8] that one can construct such an enveloping algebra starting from any Poisson algebra $(A, \{-,-\})$, taking L to be the A-module Ω_A of Kähler differentials of A, with $\rho(da) = \{a, -\}$ for $a \in A$. In fact it follows from earlier work of Weinstein *et al* [**8**, **31**], that there is an algebra isomorphism

$$\mathcal{U}(A) \;\cong\; V(A, \Omega_A),$$

which is the identity on A; a detailed account of this isomorphism can be found as [**18**, Proposition 5.7].

2.2. *Properties of* $\mathcal{U}(A)$

Most of the following facts about $\mathcal{U}(A)$ which we need are already in the literature, or are easy consequences of known results.

PROPOSITION 2.1. *Let A be an affine Poisson $\Bbbk$-algebra, $A = \Bbbk\langle x_1, \ldots, x_n\rangle$. Let $\mathcal{U}(A)$ be the Poisson enveloping algebra of A.*

(i) *$\mathcal{U}(A)$ is an affine $\Bbbk$-algebra, generated by $\{m_{x_i}, h_{x_i} : 1 \leqslant i, j \leqslant n\}$.*
(ii) *(Here we abuse notation slightly by simply writing a for the image m_a of $a \in A$ in $\mathcal{U}(A)$.) When $\mathcal{U}(A)$ is $\mathbb{Z}_{\geqslant 0}$-filtered by the filtration $\mathcal{F}$ obtained by assigning* $\deg a = 0$ *for $a \in A$ and* $\deg h_{x_i} = 1$ *for $i = 1, \ldots, n$, $\mathrm{gr}_{\mathcal{F}}(A)$ is a commutative affine $\Bbbk$-algebra with a generating set of cardinality at most $2n$.*
(iii) *Suppose that A is regular, with module of Kähler differentials $\Omega(A)$. Then*

$$\mathrm{gr}_{\mathcal{F}}(A) \;\cong\; \mathrm{Sym}_A(\Omega_A),$$

the symmetric algebra of Ω_A over A.
(iv) *Suppose that A is regular of global dimension t. Then* $\mathrm{GKdim}(\mathcal{U}(A)) = 2t$.

Proof. (i), (ii): These are easily deduced from the defining relations for $\mathcal{U}(A)$, (E2.0.1)–(E2.0.5).

(iii) This is the Rinehart's PBW theorem [**28**, Theorem 3.1], since Ω_A is A-projective when A is regular.

(iv) It follows from [**21**, Section 1.4, Corollary] that

$$\mathrm{GKdim}(A) \;=\; \mathrm{GKdim}(\mathrm{gr}_{\mathcal{F}}(A)),$$

since $\mathrm{gr}_{\mathcal{F}}(A)$ is an affine commutative $\Bbbk$-algebra by (3). However the GK-dimension of $\mathrm{gr}_{\mathcal{F}}(A)$ equals its Krull dimension. Let $\mathfrak{m}$ be a maximal ideal of $\mathrm{gr}_{\mathcal{F}}(A)$, so that $\mathfrak{m}' := \mathfrak{m} \cap A$ is a maximal ideal of A. The localisation of $\mathrm{gr}_{\mathcal{F}}(A)$ at $A \setminus \mathfrak{m}'$ is $\mathrm{Sym}_{A_{\mathfrak{m}'}}(\Omega_{A_{\mathfrak{m}'}})$. This algebra is thus a polynomial algebra in t' variables over the $t'-$dimensional regular local ring $A_{\mathfrak{m}'}$, where $t' = \mathrm{height}(\mathfrak{m}') \leqslant t$. Thus all of its maximal ideals, and in particular $\mathfrak{m}\mathrm{Sym}_{A_{\mathfrak{m}'}}(\Omega_{A_{\mathfrak{m}'}})$, have height $2t'$. Since the maximum value of t' is t, $\mathrm{Sym}_A(\Omega_A)$ has Krull dimension $2t$ as required. □

2.3. *Gradings on $\mathcal{U}(A)$*

When A is graded there are straightforward ways to extend the grading to $\mathcal{U}(A)$. The following result summarises what we will need.

PROPOSITION 2.2. *Let A be an affine Poisson $\Bbbk$-algebra, $A = \Bbbk\langle x_1, \dots, x_n\rangle$. Let $\mathcal{U}(A)$ be the Poisson enveloping algebra of A.*

(i) *Suppose that A is $\mathbb{Z}$-graded and $\{-,-\}$ is homogeneous of degree d. Then $\mathcal{U}(A)$ is $\mathbb{Z}$-graded with $\deg m_x = \deg x$ and $\deg h_x = \deg x + d$ for all homogeneous elements x in A.*

(ii) *If, in* (i), *A is connected $\mathbb{N}$-graded with $d \geqslant 0$, then $\mathcal{U}(A)$ is also connected $\mathbb{N}$-graded.*

(iii) *Suppose that A is a connected $\mathbb{N}$-graded algebra, generated in degree 1, and that $\{-,-\}$ is homogeneous of degree $d \geqslant 0$. Then $\mathcal{U}(A)$ is a connected $\mathbb{N}$-graded algebra, and is minimally generated by m_{A_1} and h_{A_1}.*

Proof. (i),(ii) First note that $m : A \to m_A$ and $h : A[d] \to h_A$ are graded $\Bbbk$-linear maps. It is clear that $\mathcal{U}(A)$ is generated by homogeneous elements in m_A and h_A since both m and h are $\Bbbk$-linear. The relations of $\mathcal{U}(A)$ given in (E2.0.1)–(E2.0.5) are homogeneous. Therefore $\mathcal{U}(A)$ is naturally $\mathbb{Z}$-graded. Finally, (ii) is an immediate consequence of (i).

(iii) In this case we can assume that the generators x_i are linearly independent and homogeneous of degree 1, so $\deg m_{x_i} = 1$ and $\deg h_{x_i} = 1 + d \geqslant 1$ for all $i = 1, \dots, n$. Therefore $\mathcal{U}(A)$ is connected graded, by (ii). Since $\deg m_{x_i} = 1$ for all i, $\{m_{x_i} : 1 \leqslant i \leqslant n\}$ is a linearly independent subset of a minimal homogeneous generating set. Let $\mathfrak{m}$ be the maximal graded ideal of $\mathcal{U}(A)$. It is clear that $\mathfrak{m}/\mathfrak{m}^2$ is spanned by $\{m_{x_i} : 1 \leqslant i \leqslant n\} \cup \{h_{x_i} : 1 \leqslant i \leqslant n\}$. Let $x \in A(1)$. Since $\deg h_x = 1 + d$, if h_x is in $\mathfrak{m}^2$, then it is generated by elements of small degrees, namely by $m_{A(1)}$. However, any homogeneous relation involving h_x (see (E2.0.2)–(E2.0.4)) has degree at least $d + 2$. Therefore $h_x \in \mathfrak{m}^2$ does not follow from any relations of $\mathcal{U}(A)$, a contradiction. Thus $\{h_{x_i} : 1 \leqslant i \leqslant n\}$ maps to a linearly independent subset of $\mathfrak{m}/\mathfrak{m}^2$. The same argument shows that $(m_{A(1)} + \mathfrak{m}^2/\mathfrak{m}^2) \cap (h_{A(1)} + \mathfrak{m}^2/\mathfrak{m}^2) = 0$. Hence $\mathcal{U}(A)$ is minimally generated by $m_{A(1)} \cup h_{A(1)}$. □

2.4. *Bialgebra structure on $\mathcal{U}(A)$*

We shall need the following results of [**18**, **25**]. The proof of the claim in [**25**] that a Poisson Hopf structure on an algebra A always induces a structure of Hopf algebra on $\mathcal{U}(A)$ appears not to be completely clear as regards the extension of the antipode from A to $\mathcal{U}(A)$, so we address that aspect separately, for the cases of concern to us, in what follows.

THEOREM 2.3. *Let A be a Poisson Hopf algebra.*

(i) *$\mathcal{U}(A)$ admits a bialgebra structure with A as a sub-bialgebra.*

(ii) *The coradical of $\mathcal{U}(A)$ is the coradical of A.*

(iii) *If A is pointed, then $\mathcal{U}(A)$ is a Hopf algebra with sub-Hopf algebra A, with $G(\mathcal{U}(A)) = G(A)$.*

(iv) *If A is a connected Hopf algebra, then so is $\mathcal{U}(A)$.*

Proof. (i) See [**25**, Theorem 10].

(ii) This is [**18**, Proposition 6.6].

(iii) Suppose that A is pointed. Then (ii) shows that $\mathcal{U}(A)$ is also pointed, with $G(\mathcal{U}(A)) = G(A)$, so that every grouplike element of $\mathcal{U}(A)$ is invertible. It follows from [**26**, Theorem 1] that the bialgebra $\mathcal{U}(A)$ is a Hopf algebra.

(iv) This is a special case of (iii). □

3. *Proof of theorem* 1

3.1. *The modular derivation.*

We refer to [**9**, **20**] for further details about several important invariants of a Poisson algebra. There is a geometric notion of unimodular Poisson manifold introduced by Weinstein [**32**]. In algebraic terms, let A be a smooth Poisson algebra with trivial canonical bundle — for instance in our case A will be a polynomial $\Bbbk$-algebra in n variables. A *Poisson derivationd* of A is a derivation in the usual sense with the additional property that

$$d(\{a, b\}) = \{d(a), b\} + \{a, d(b)\}$$

for all $a, b \in A$. Then the *modular derivation* of A (with respect to a given volume form) is a certain Poisson derivation δ, defined in [**20**, Definition 2.3]. Now the *modular class* of A is the class of δ in the space of Poisson derivations modulo the subgroup of *log-Hamiltonian* derivations, see both [**9**, p. 208; **20**, Section 2.2]. If the modular class of A is trivial, then A is called *unimodular*. If A is the polynomial ring (or more generally, if the only invertible elements in A are scalars), then there are no nonzero log-Hamiltonian derivations. In this case, A is unimodular precisely when the modular derivation is zero.

When A is the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ (and the volume form can be taken to be $\eta := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$), the modular derivation δ of A is given in [**20**, Lemma 2.4]: namely,

$$\delta(f)(= \delta_\eta(f)) = \sum_{j=1}^{n} \frac{\partial\{f, x_j\}}{\partial x_j} \tag{E3.0.1}$$

for all $f \in A$.

We refer to [**27**] for some basic definitions concerning Calabi–Yau algebras (or CY algebras for short), skew Calabi–Yau algebras and the Nakayama automorphism. There is a close connection between modular derivations and Nakayama automorphisms via deformation theory, see [**9**, Theorem 2]. A further such result, in this case connecting the modular derivation of a Poisson algebra with the Nakayama automorphism of its Poisson enveloping algebra, is the following. We shall apply the result below in the case where A is a polynomial algebra. In its statement, one has to interpret the expression 'the ... automorphism ... of $\mathcal{U}(A)$ ... is ... 2δ' in the following manner. Recall that $\mathcal{U}(A)$ is generated by $\{m_a, h_a : a \in A\}$. So, if τ is a Poisson derivation of A, we define the corresponding map on $\mathcal{U}(A)$ by $\tau(m_a) = m_a$ and $\tau(h_a) = h_a + m_{\tau(a)}$. For details, see [**19**, Lemma 2.2].

Proposition 3.1 [**19**, Corollary 5.6, Proposition 1.12, Theorem 5.8 and Remark 5.9]. *Let A be a CY Poisson algebra; that is, by definition, A is an affine smooth Poisson algebra of finite global dimension n, with trivial canonical bundle. Then $\mathcal{U}(A)$ is skew CY of dimension $2n$. Moreover, the Nakayama automorphism of $\mathcal{U}(A)$ is given by 2δ, where δ is the modular derivation of A. As a consequence, if A is a polynomial ring, then A is unimodular if and only if the modular derivation is zero, and if and only if $\mathcal{U}(A)$ is CY.*

3.2. *Proof of Theorem* 1

Theorem 1 follows easily from the above proposition and [**6**, Theorem 0.2].

Proof. Since A is a commutative affine connected graded algebra of finite global dimension, it is a polynomial algebra, by [**3**, Theorem 6.2]. Its coalgebra structure is thus also connected, by Theorem 1.6. By the hypothesis on the grading of the Poisson bracket and by Proposition 2.2(ii), $\mathcal{U}(A)$ is a connected $\mathbb{N}$-graded algebra, and has finite GKdimension by Proposition 2.1(iv). By Theorem 2.3(iv), $\mathcal{U}(A)$ is a connected Hopf algebra which is moreover connected graded as an algebra. Applying [**6**, Theorem 0.2], $\mathcal{U}(A)$ is Calabi–Yau. It follows from Lemma 3.1 that the modular derivation δ of A is zero — that is, A is unimodular. □

Corollary 2 is an immediate consequence of Theorem 1 and [**20**, Theorem 3.5 and Remark 3.6] (or [**19**, Theorem 4.3]).

4. *Examples and counterexamples*

Theorem 1 does not remain true for Poisson bialgebras as the following example demonstrates.

Example 4.1. Let A be the polynomial algebra $\mathbb{C}[x,y]$, let i be a nonnegative integer, and consider the bialgebra structure on A given by

$$\Delta(y) = y \otimes y, \quad \Delta(x) = x \otimes 1 + y^i \otimes x,$$

with $\epsilon(x) = 0$ and $\epsilon(y) = 1$.Then, setting $\{x, y\} = xy$, it is straightforward to check that A is a Poisson bialgebra. Moreover, A is a connected graded algebra, with the Poisson bracket homogeneous of degree 0. However, A is *not* unimodular — one calculates from (E3.0.1) that the modular derivation δ of A is given by $\delta(x) = x$ and $\delta(y) = -y$.

Example 4.2 [**20**, Example 2.8(2)]. *The Kostant – Souriau Poisson bracket.* The familiar Poisson bracket induced by the bracket of a Lie algebra $\mathfrak{g} = \sum_{i=1}^{n} \Bbbk x_i$ on its symmetric algebra $S(\mathfrak{g})$ gives rise to a Poisson Hopf structure on $S(\mathfrak{g})$.

For, with its polynomial generators x_i all assigned degree 1, $S(\mathfrak{g})$ is a coradically graded Poisson Hopf algebra of degree $d = -1$. Using (E3.0.1) one easily calculates that the modular derivation is given by

$$\delta(x_i) = \mathrm{tr}(\mathrm{ad}(x_i)) \quad 1 \leqslant i \leqslant n,$$

where ad denotes the adjoint representation of $\mathfrak{g}$.

Thus, when $\mathfrak{g}$ is, for example, the two-dimensional solvable nonabelian Lie algebra, $S(\mathfrak{g})$ is not unimodular. This shows that the hypothesis in Theorem 1 that the Poisson bracket is homogeneous of degree $d \geqslant 0$ is necessary.

Example 4.3 (*Connected Hopf algebras of small GK – dimension*). The connected Hopf $\Bbbk$-algebras of GK-dimensions 3 and 4 are determined in [**30**, **33**] respectively. Applied to these algebras, the recipe of Corollary 1.8(ii) and (iii) yields infinite families of coradically graded Poisson Hopf algebra structures on the polynomial $\Bbbk$-algebras in 3 and 4 variables. Two of these families, manufactured respectively from [**30**, Example 4.4; **33**, Section 7, Example 2], are given as [**18**, Examples 3.2 and 3.3]. Here are brief details of the three-dimensional examples.

(a) Let $\lambda, \mu \in k$ and $\alpha \in \{0, 1\}$, and let $A := A(\lambda, \mu, \alpha)$ be the family of Hopf $\Bbbk$-algebras of GK-dimension 3 defined at [**33**, Section 7, Example 2]. So $A = k\langle X, Y, Z\rangle$, with

$$[X, Y] = 0, \quad [Z, X] = \lambda X + \alpha Y, \quad [Z, Y] = \mu Y.$$

The space of primitive elements is $kX \oplus kY$, and $\Delta(Z) = Z \otimes 1 + 1 \otimes Z + X \otimes Y$, so that $Z \in A_2$. Thus, applying Corollary 1.8(iii) and using the obvious notation,

$$\mathrm{gr}_{\mathcal{C}} A(\lambda, \mu, \alpha) = k[x, y, z]; \quad \mathrm{deg} x = \mathrm{deg} y = 1; \quad \mathrm{deg} z = 2.$$

Therefore $\mathrm{gr}_{\mathcal{C}} A(\lambda, \mu, \alpha) = \mathcal{O}(U)$ is a coradically graded Poisson Hopf algebra of Poisson degree -2, with U the three-dimensional Heisenberg group and

$$\{x, y\} = 0, \quad \{z, x\} = \lambda x + \alpha y, \quad \{z, y\} = \mu y.$$

Using (E3.0.1), the modular derivation of the Poisson algebra $\mathcal{O}(U)$ is determined by

$$\delta(x) = 0,\ \delta(y) = 0,\ \delta(z) = \lambda + \mu.$$

By [**33**, Proposition 7.9], a complete set of isomorphism classes of the Hopf algebras $A(\lambda, \mu, \alpha)$ is given by

$$\{(1, 0, 0), (0, 0, 0), (0, 0, 1), (1, 1, 1), (1, \mu^{\pm 1}, 0) : \mu \in k^*\}. \tag{E4.3.1}$$

It is not hard to deduce from this result that the same list of parameter values (E4.3.1) gives the complete list of isomorphism classes of the Poisson Hopf algebras $\mathrm{gr}_{\mathcal{C}} A(\lambda, \mu, \alpha)$. We sketch an argument. It is enough to show that distinct parameter triples from (E4.3.1) yield distinct Poisson Hopf algebras. Suppose that θ is an isomorphism of Poisson Hopf algebras between two such algebras. Since θ preserves the counits, that is $\theta(\mathfrak{m}) = \mathfrak{n}$ say, it induces an isomorphism of Lie algebras $(\mathfrak{m}/\mathfrak{m}^2, \{-, -\}) \cong (\mathfrak{n}/\mathfrak{n}^2, \{-, -\})$. Moreover, θ preserves the coalgebra structure, and hence preserves the structures of Lie bialgebras (see [**18**, Section 6]) on $(\mathfrak{m}/\mathfrak{m}^2, \{-, -\})$ and $(\mathfrak{n}/\mathfrak{n}^2, \{-, -\})$. From this the desired conclusion easily follows.

(b) Let $B(\lambda)$ be the family of Hopf k-algebras of GK-dimension 3 defined at [**33**, Section 7, Example 3]. So $B(\lambda) = k\langle X, Y, Z\rangle$, with $\lambda \in k$ and

$$[X, Y] = Y; \quad [Z, X] = -Z + \lambda Y; \quad [Z, Y] = \frac{1}{2} Y^2.$$

The space of primitive elements is $kX \oplus kY$ and $\Delta(Z) = Z \otimes 1 + 1 \otimes Z + X \otimes Y$. Thus, exactly as with (a) we get a coradically graded Poisson Hopf algebra

$$\mathrm{gr}_{\mathcal{C}} B(\lambda) = k[x, y, z]; \quad \deg x = \deg y = 1; \quad \deg z = 2.$$

This time, however, the Poisson degree d is -1. We see that $\mathrm{gr}_{\mathcal{C}} B(\lambda)$ is a Poisson Hopf algebra with the underlying Poisson unipotent group again being the three-dimensional Heisenberg group, with

$$\{x, y\} = y, \quad \{z, x\} = -z, \quad \{z, y\} = \frac{1}{2} y^2.$$

Using (E3.0.1), the modular derivation of this Poisson algebra is determined by

$$\delta(x) = 2,\ \delta(y) = 0,\ \delta(z) = y.$$

From [**33**, Proposition 7.10], $B(\lambda) \cong B(\mu)$ as Hopf algebras if and only if $\lambda = \mu$, whereas it is evident that one obtains the *same* Poisson Hopf algebra $\mathrm{gr}_{\mathcal{C}} B(\lambda)$ for every value of $\lambda \in k$.

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