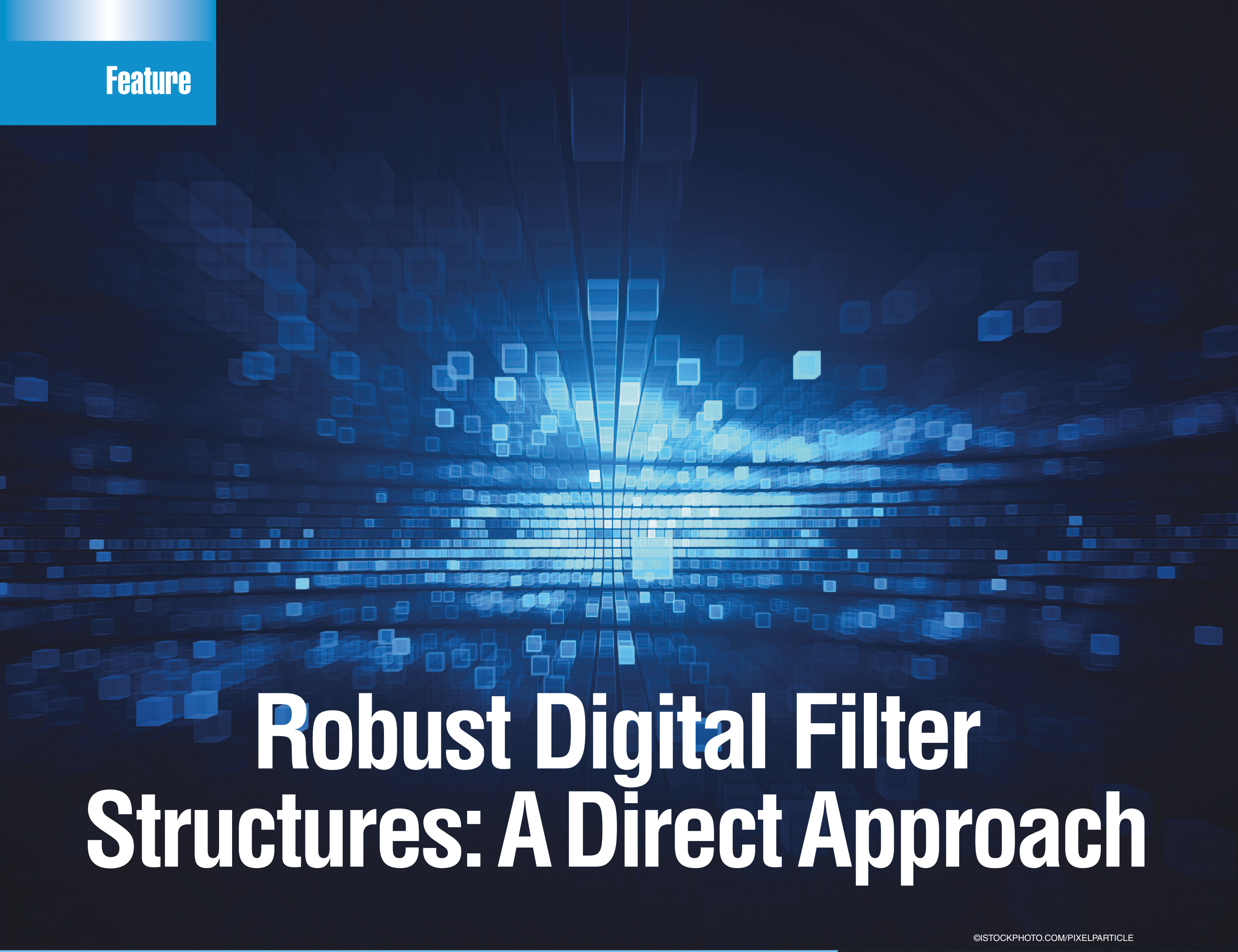



# Robust Digital Filter Structures: A Direct Approach

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## Abstract

One of the many contributions of Prof. Fettweis was the invention of wave digital filters. These filters are obtained from classical RLC filters, in particular doubly terminated lossless two-ports, by using some transformations. Namely, the voltages and currents in the circuit elements are transformed into wave-variables and then a bilinear transformation is performed. If the wave transformation is performed appropriately it results in a realizable digital filter structure which furthermore enjoys a number of robustness properties such as low passband sensitivity, low roundoff noise, and freedom from limit cycle oscillations. Prof. Fettweis and his colleagues also showed that these properties are due to the inheritence of passivity properties from the continuous-time domain into the digital filter domain. Subsequent to this landmark work, a number of researchers worked on the problem of obtaining robust digital filter structures without starting from continuous-time circuits. One of these is the structurally bounded or structurally passive class of digital filters. These structures are based on inducing structural passivity directly into the implementation and are therefore simpler, both conceptually and from a practical viewpoint. They are also more general and lead to new structures which have no natural connection to electrical circuits. This paper gives an overview of some of these developments.


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## I. Introduction

Prof. Fettweis was a giant in many different areas of circuit theory, signal processing, physics and related mathematics. One of his contributions was the invention in 1971 of wave digital filters [18], [19]. These filters are obtained from classical *RLC* filters [35], in particular doubly terminated lossless two-ports, by using some transformations. Namely, the voltages $V_i$ and currents $I_i$ in the circuit elements are transformed into wave-variables using formulas of the form

$$A_i = V_i + R_i I_i \quad B_i = V_i - R_i I_i \tag{1}$$

where $A_i$ and $B_i$ are called the "incident" and "reflected" wave variables, and $R_i > 0$ are free parameters to be chosen by the designer. In addition to *RLC* elements, transformers and gyrators are also sometimes included. If the wave transformation is performed appropriately, then the bilinearly transformed version of the circuit results in a realizable digital filter structure (i.e., a structure without delay-free loops), which furthermore enjoys a number of robustness properties such as low passband sensitivity, low roundoff noise, and freedom from limit cycle oscillations [26]. Prof. Fettweis and his colleagues also showed that these properties are related to the passivity properties of the underlying continuous time electrical circuit [20].

Subsequent to this landmark work, a number of researchers explored the possibility of obtaining robust digital filter structures without starting from continuous-time circuits. This includes the work of Bruton and Vaughan-Pope [10], Constantinides [13], Mitra and Sherwood [49], Deprettere and Dewilde [15], Rao and Kailath [59], and Vaidyanathan and Mitra [71], [74], [77]. Some (but not all) of these structures exhibited low sensitivity and other robustness properties. These include the structurally bounded class of digital filters [71], [73], [77], and orthogonal digital filter structures [15], [59].

The structurally bounded or structurally passive class of digital filters [71] are based on inducing passivity directly into the structure and are therefore simpler, both conceptually and implementation wise. They explain in a unified way the robustness properties of wave digital filters, wave lattice filters [21], orgthogonal digital filters [15], [59], and the Gray-Markel lattice structures [32]–[34]. The structurally bounded class is also more general and leads to new structures which have no natural connection to electrical circuits such as the cascaded FIR power complementary lattice [78], the FIRBR structure [74], and the single-input two-output IIR power complementary lattice [59].

This paper gives an overview of some of these developments. We first review the fundamental reason for the low sensitivity of doubly terminated lossless networks based on an argument advanced by Orchard in 1966 [55], [56]. After a brief review of wave digital filters we explain structural boundedness in detail. The approach of structural boundedness [71] is based on the premise that if boundedness can be directly realized in the digital domain by constraining the structure, then there is no need to copy electrical circuits into the digital domain using painstaking and detailed formulas. We also discuss a number of robust digital filter structures derived using structural boundedness. It turns out that the two-port extraction method proposed by Mitra, Kamat, and Huey [50] long before the introduction of structural boundedness, can in fact be used to develop cascaded networks with structurally bounded properties. The beauty is that this procedure gives rise to wave filters, Gray-Markel lattices, and orthogonal digital filters as special cases [75]. But that is not all. Many new low sensitivity structures, not based on the two-port cascade, emerge from the theory of structural boundedness. Some of these are reviewed here as well, such as the parallel allpass structure [77], FIR power complementary lattices [78], and FIRBR structures [74].

## II. Low Sensitivity and Structural Boundedness

When a digital filter is implemented with finite precision for the multiplier coefficients, the response $H(e^{j\omega})$ changes, and may fail to satisfy the original specifications. In fact the filter may even become unstable if the number of bits used for the multipliers is too small. Figure 1 shows the response of a fifth order digital elliptic

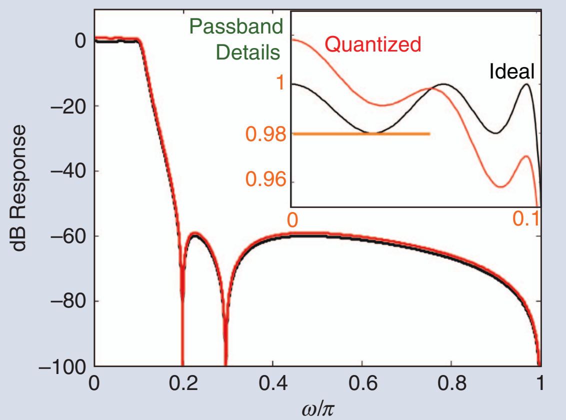


**Figure 1.** The effect of coefficient quantization. The magnitude response $|H(e^{j\omega})|$ of a 5th order elliptic lowpass filter is shown with 18 bit quantization (each multiplier is quantized to 18 bits). The direct-form structure [52], [54] is used. The red plot shows the response of the quantized structure whereas the black plot is the unquantized response. The passband details are shown in the inset. While the stopband response is nearly perfect even after quantization, it is clear that the passband response after quantization deviates significantly from the ideal.

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filter implemented in direct-form [16], [52], [54]. The ideal response is shown in black and the response with multiplier coefficients quantized to 18 bits is shown in red. Notice that the passband response (shown separately in the inset) deviates considerably from the ideal, even with 18 bits of precision for each multiplier coefficient. For higher order filters which have sharp cutoff and very small passband ripples, this effect is even more severe. The good news is that if the direct-form structure is replaced with a properly chosen structure, then these effects of quantization can be reduced to a considerable extent.

### A. A Lesson Learned From Passive Electrical Filters

Historically, even before the advent of digital filters, it was well known that continuous time electrical filter circuits exhibited very low passband sensitivity (with respect to circuit element variations) if they are implemented as lossless (i.e., LC) circuits terminated at both ends appropriately with resistors. Such a doubly terminated lossless two-port is shown in Fig. 2(a). Define the filter transfer function as the voltage ratio $H(s) = 2Y(s)/X(s)$. With element values appropriately chosen, this can be designed to be a lowpass filter with response as in Fig. 2(b). It is found that such a filter exhibits low passband sensitivity with respect to element variations.

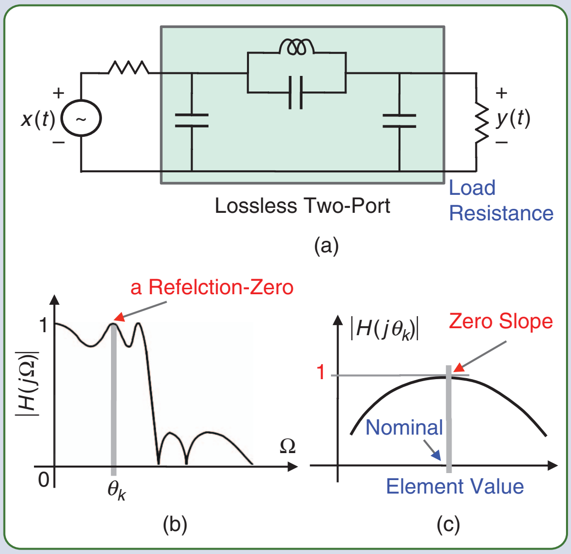



**Figure 2.** Fundamentals of low passband-sensitivity in *LCR* filters. (a) A lossless (LC) circuit, terminated at both ends with resistances. The resistances are such that maximum power is transferred from the source to the load at certain frequencies in the passband of the filter. (b) A typical lowpass filter response, realized as the ratio $H(s) = 2Y(s)/X(s)$. The passband maxima occur at the "reflection zeros" $\theta_k$, where maximum power is transferred from the voltage source to the load resistance. (c) Variation of the response at $\Omega = \theta_k$ with respect to variation in a circuit element. The response can only decrease as the element value departs from nominal. This behavior was used by Orchard [55] to explain the low passband sensitivity of doubly terminated lossless two-ports.

An explanation for the low passband sensitivity was given by Orchard [55]: the two port is usually designed such that there is maximum transfer of power from the source $x(t)$ to the load $y(t)$ at the passband maxima. Thus, at a frequency $\omega = \theta_k$ in the passband (Fig. 2(b)), where the filter has maximum gain, there is maximum transfer of power. When a circuit element is perturbed, the transfer of power, hence the gain $|H(j\theta_k)|$, can only get smaller as demonstrated in Fig. 2(c). Thus the passband maxima exhibit low sensitivity with respect to element values. If there is a number of such maxima in the passband then the entire passband response has low sensitivity. A more quantitative explanation was given later in [56]. Incidentally, the frequencies $\theta_k$ are called **reflection zeros** because there is no power reflected back from the load resistance at these frequencies. Note that networks designed as above do not guarantee low *stop band* sensitivity.

The term low sensitivity can have multiple meanings. In this paper it is used to indicate the small sensitivity of the *magnitude response* in the *passband*. This is often quantified by the derivative of the passband magnitude response with respect to element values (as in the left hand side of Eq. (8)). The sensitivity of the phase response, or that of the response in the stop band will not be the focus here.

### B. Fettweis's Vision

Fettweis recognized that this low sensitivity property of a doubly terminated lossless network can be inherited by a digital filter structure, if the structure is derived from the electrical network by an appropriate transformation. In his pioneering work in 1971, he achieved this [18], [19] by obtaining a digital equivalent for every circuit element (inductor, capacitor, resistor, open circuit, short circuit, voltage source, and so forth) by using the wave variable transformation (1) followed by the bilinear transformation [54]. In this process the quantities $R_i$, called the port resistances are chosen carefully so that, when the digital equivalents were interconnected, there were no *delay-free loops*. Fettweis developed the so-called series wave adaptor and parallel wave adaptor for the purpose of interconnecting digital equivalents of circuit elements [27].

To be specific, let us consider the case of an inductor $L$. When this is appropriately transformed, its digital equivalent is $-z^{-1}$ where $z^{-1}$ represents a unit delay. To see this recall that the inductor is characterized by the relation $V(s) = sLI(s)$. With the wave variables defined as $A(s) = V(s) + RI(s)$ and $B(s) = V(s) - RI(s)$, we have $B(s) = (sL/R - 1)/(sL/R + 1)A(s)$, With the free port-resistance chosen as $R = L$ this reduces to

$$B(s) = \frac{s-1}{s+1}A(s) \tag{2}$$

If we now use the bilinear transform $s = (1-z^{-1})/(1+z^{-1})$, then $(s-1)/(s+1)$ reduces to $-z^{-1}$ so that the digital equivalent of (2) becomes

$$B_d(z) = -z^{-1}A_d(z). \tag{3}$$

Thus an inductor transforms into $-z^{-1}$. Similarly a capacitor can be transformed into $z^{-1}$. When the doubly terminated lossless network of Fig. 2(a) is transformed using such digital equivalent building blocks, it results in the wave digital filter shown in Fig. 3. Notice the use of series and parallel adaptors for interconnecting the elements. The figure also shows the internal details of one of the adaptors. The main complexity and computational load of wave digital filters come from these adaptors.

As wave digital filters have been widely written about, we do not go into further details of the construction here. The interested reader will enjoy reading the original articles [18], [19], [27], or the excellent presentation in Antoniou's text book [3]. A short section on wave digital filters can also be found in Sec. XIII of [83] (chapter in an edited handbook), and will serve as an introduction for new readers.

As envisioned by Fettweis, wave digital filters indeed exhibited very low passband sensitivity. In addition, they also enjoyed freedom from parasitic oscillations or limit cycles, as shown in later papers by Fettweis and Meerkotter [26]. Wave digital filters were soon also extended to wave lattice filters [21], [30] and other variations. Wave filters for multirate applications have also been developed by Fettweis and Nossek [29]. A detailed overview article on wave filters, written by Fettweis himself, can be found in [23].

Wave digital filters have also been extended to the case of *multiple dimensions* [22], [28] but we shall not discuss those in our limited overview here. Many subtle aspects regarding passivity and stability of multidimensional filters are discussed in papers by Basu and Fettweis [6], [24], [25]. For papers focussing on multidimensional stability the reader is referred to [8], [9], [39], [67] and [60].

### *C. Wave Filters From Two-Port Viewpoint*

Subsequent to the invention of wave digital filters, some researchers made efforts to simplify the procedure. In particular, considerable simplification could be obtained if the wave adaptors can be avoided, or implicitly incorporated without having to worry about cumbersome rules for interconnecting them. Swamy and Thyagarajan [68] came up with an ingenious way to do this. Their wisdom was to regard each circuit element itself as a two-port rather than a one-port as demonstrated in Fig. 4(a). Using this idea an $LC$ ladder network can then be transformed directly into a "cascaded structure" of the form shown in Fig. 4(b). Here each rectangular box represents a $2 \times 2$ digital transfer matrix, also known variously as the **digital two-port** or **digital two-pair** [49]. It represents the digital equivalent of electrical elements such as inductors, capacitors, and even series or parallel $LC$ circuits. Notice that the cascade in Fig. 4 is not a traditional cascade because the arrows are running in different directions, creating feedback loops. This sort of cascade is often called a **chain-cascade**. The reason for this name is that, in such a cascade, the so-called chain matrices (rather than transfer matrices) of the systems in cascade are multiplied. Please see Box 1 for details.

Each of these two-pairs in Fig. 4(b) is first-order (i.e., has one $z^{-1}$ element) if it represents an $L$ or a $C$ element, and is second-order if it represents a series or parallel $LC$ circuit. It was shown by Swamy and Thyagarajan that

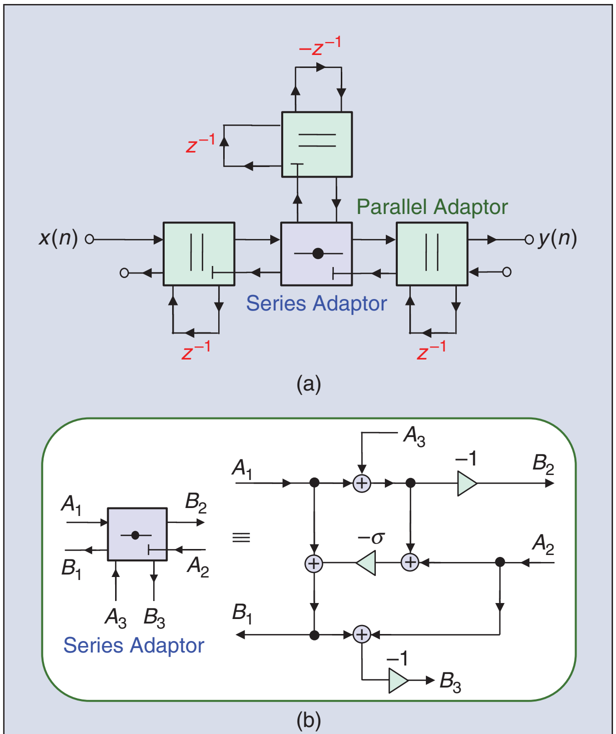



**Figure 3.** The wave digital filter obtained from the doubly terminated $LC$ filter of Fig. 2. Details of a typical series adaptor used in wave digital filters are shown in the bottom. Parallel adaptors have a similar structure. For more details please see [18], [19], [27], or the text book [3], or Sec. XIII of [83].

LC ladder networks and more generally LCR circuits can be transformed into such a cascade of two-pairs. Furthermore, certain free parameters in the transformation can be selected such that delay free loops are avoided in the back to back interconnections. With this new type of wave digital filters we do not have to worry about the design of adaptors, as they are implicitly and automatically included in the two-ports of Fig. 4(b).

## Box 1: Two Types of Cascaded Systems

A 2-input 2-output system has also been referred to as a *two-port* or a *two-pair*. With $\mathbf{x}(n)=[x_1(n)\quad x_2(n)]^T$ and $\mathbf{y}(n)=[y_1(n)\quad y_2(n)]^T$ denoting the input and output, there are two popular ways to describe an LTI two port. The *transfer matrix* and the *chain matrix* descriptions:

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \underbrace{\begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix}}_{\text{transfer matrix } \mathbf{T}(z)} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix},$$

$$\begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix}}_{\text{chain matrix } \mathbf{\Pi}(z)} \begin{bmatrix} Y_2(z) \\ X_2(z) \end{bmatrix}$$

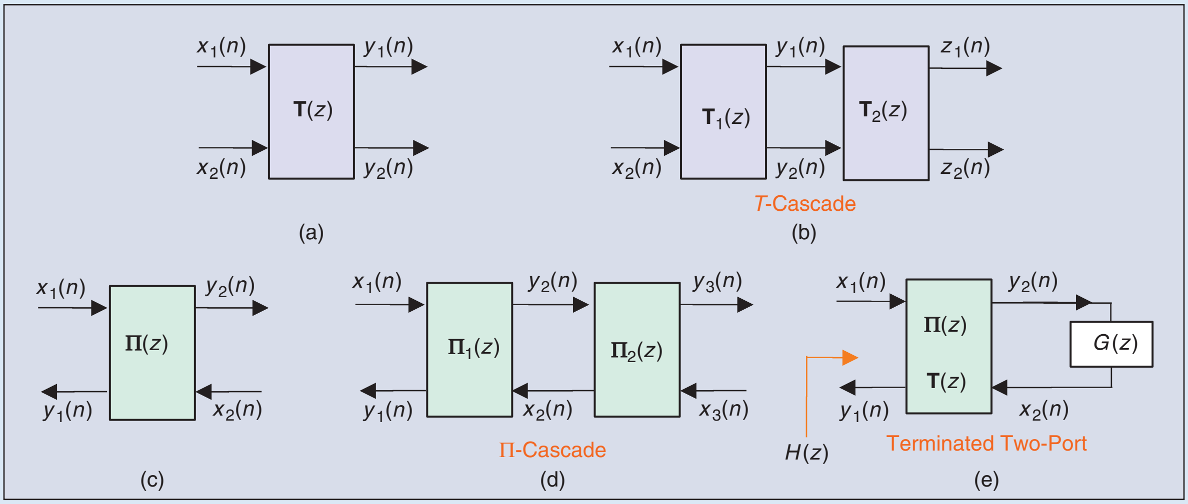
Depending on the description chosen it is convenient to show the inputs and outputs either as in (a) or as in (c) in the figure. The transfer matrix $\mathbf{T}(z)$ is convenient when two-ports are connected in a so-called ***T*-cascade** as shown in part (b). In this case the transfer matrix of the cascade is the product $\mathbf{T}(z)=\mathbf{T}_2(z)\mathbf{T}_1(z)$. The chain-matrix description is convenient when two-ports are connected in a so-called **$\Pi$-cascade** as shown in part (d). This interconnection generates new feedback loops and the description of the cascaded system in terms of transfer matrices becomes cumbersome. But the chain-matrix description becomes extremely convenient: with $\mathbf{\Pi}_k(z)$ denoting the chain matrices of the systems, the chain matrix of the $\Pi$-cascade is just the product $\mathbf{\Pi}(z)=\mathbf{\Pi}_1(z)\mathbf{\Pi}_2(z)$. The two descriptions are interrelated as follows: $T_{11}=C/A$, $T_{12}=\det\mathbf{\Pi}/A$, $T_{21}=1/A$, $T_{22}=-B/A$ and similarly, $A=1/T_{21}$, $B=-T_{22}/T_{21}$, $C=T_{11}/T_{21}$, $D=-\det T/T_{21}$ where the argument $(z)$ has been omitted for simplicity.

Now consider part (e) in the figure where a two-port is "terminated" at one end by a transfer function $G(z)$. In this case, the transfer function $H(z)=Y_1(z)/X_1(z)$ can be expressed either in terms of the transfer parameters $T_{km}(z)$ or chain parameters $A(z), B(z), C(z), D(z)$ as follows:

$$H(z)=T_{11}(z)+\frac{T_{12}(z)\,T_{21}(z)\,G(z)}{1-T_{22}(z)\,G(z)}, \quad \text{or equivalently}$$

$$H(z)=\frac{C(z)+D(z)\,G(z)}{A(z)+B(z)\,G(z)}$$

The chain matrix description has its origin in electrical circuit theory. *LCR* circuits in a ladder configuration can be conveniently expressed as a $\Pi$-cascade of two-ports where each two-port represents a series or parallel branch in the ladder. The chain matrix is therefore inherited into wave digital filters as seen explicitly from the work of Swamy and Thyagarajan (1975). It has also been used in direct synthesis of digital ladder filter structures by Mitra, Kamat, and Huey (1977). Later on it was also used extensively by Vaidyanathan and Mitra (1984) for the synthesis of structurally passive digital filters.



## III. Direct Digital Synthesis

In the early to middle seventies, many other researchers besides Fettweis got interested in synthesis of digital filter structures inspired by *LC* network synthesis [10], [13], [49]. Many of these structures had qualitative similarities to *LC* ladder networks, but they were not necessarily designed to inherit specific properties such as low sensitivity or passivity. In 1977, Mitra, Kamat and Huey [50] proposed a way to synthesize digital transfer functions directly in the *z*-domain by extracting digital two-pairs (as in Fig. 4) in such a way that there is a degree reduction at each step in the extraction. (We will return to this in Fig. 7 again.) This procedure resulted in a number of new realizations for digital filters, but again, the two-pairs were not designed with any specific properties that would induce low sensitivity or passivity. However this basic idea of digital two-pair extraction, which realizes digital filters by successive order reduction, laid the foundation for future work which incorporated such robustness properties systematically into digital filter synthesis. More specifically, the approach introduced in [71] showed how to develop two-pair extraction methods to obtain digital filter structures with low sensitivity and other passivity properties, without recourse to continuous-time electrical circuits. This is based on a concept called structural passivity. This property is crucial to low sensitivity, and it can be incorporated directly into digital filter structures as explained next.

### *A. Structural Boundedness or Structural Passivity*

Any digital filter structure is essentially an interconnection of delay elements, scalar multipliers, and two-input adders, as shown schematically in Fig. 5(a). Imagine now that we have a structure with the following special property: no matter what the values of the multipliers $m_i$ are, the frequency response is always bounded by unity, that is, $|H(e^{j\omega})| \le 1$ for all $\omega$. We say that such an implementation is **structurally bounded**, that is, the structural interconnection itself ensures that the frequency response never exceeds unity. The term **structurally passive** is also used for reasons described below.

In practice we constrain the multipliers $m_i$ to belong to some reasonable range (such as, for example $|m_i| < 1$) as we shall indicate explicitly in the context of specific examples. In practice one also likes to make sure the transfer function remains stable. In order to be precise with these ideas, we introduce a number of important definitions here:

**Definition 1.**

*Bounded transfer functions.* A digital filter transfer function $H(z)$ is said to be *bounded* if it is stable (i.e., all poles are in $|z| < 1$) and $|H(e^{j\omega})| \le 1$ for all $\omega$. Notice also the following definitions, properties, and remarks:

1) It can be shown that a stable rational transfer function $H(z)$ is bounded if and only if

$$\sum_n |y(n)|^2 \le \sum_n |x(n)|^2 \tag{4}$$

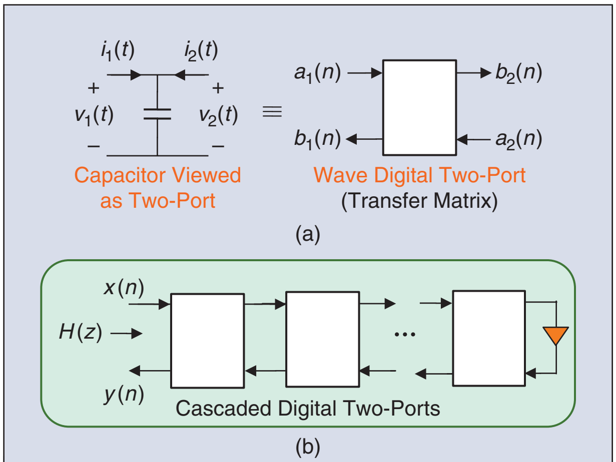



**Figure 4.** The wave digital filter synthesis method of Swamy and Thyagarajan [68]. (a) Each electrical element is regarded as a two-port as demonstrated, rather than as a one port. (b) The entire electrical circuit is transformed into a digital chain-cascaded network of $2\times 2$ digital transfer matrices (two-ports). This avoids the need for explicit use of wave adaptors.

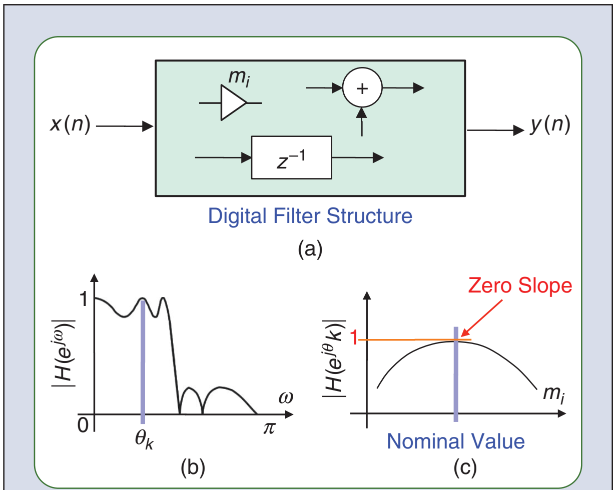



**Figure 5.** Structural boundedness and low passband sensitivity. (a) Any digital filter structure is an interconnection of delay elements, multipliers, and adders. (b) A typical lowpass response, with peak frequencies $\theta_k$ in the passband. (c) Variation of the response $|H(e^{j\theta_k})|$ with respect to a multiplier $m_i$. In a structurally bounded (or passive) system, the quantity $|H(e^{j\theta_k})|$ can only decrease as $m_i$ deviates from its nominal value. This induces low passband sensitivity.

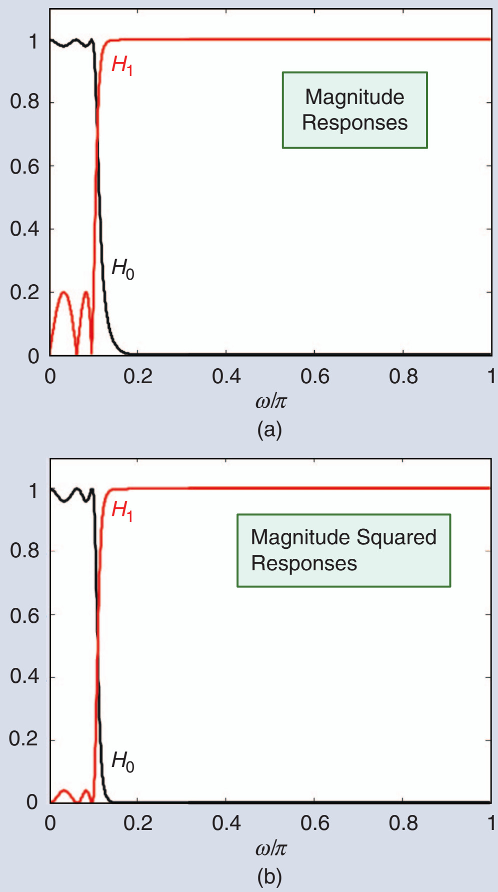



**Figure 6.** Responses of fifth order power complementary elliptic digital filters. (a) Magnitude responses, and (b) magnitude squared responses. Notice in (b) that the shapes of the ripples of the two filters are exactly complementary so that they add up to unity everywhere.

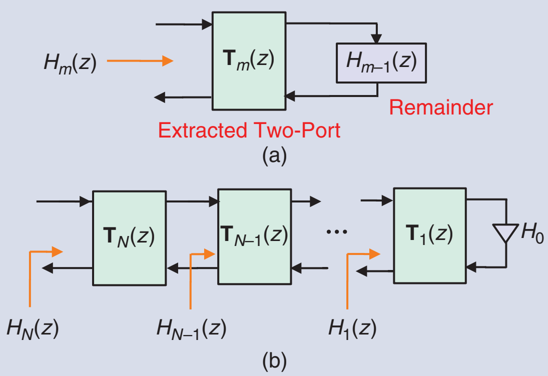



**Figure 7.** (a) The two-pair or two-port extraction step. Here the mth order transfer function $H_m(z)$ is implemented as a two-pair $\mathbf{T}_m(z)$ terminated at one end with a transfer function $H_{m-1}(z)$ of smaller order $m-1$. (b) Repeated extraction of two-pairs results in the realization of a scalar transfer function as a cascade of two-pairs terminated in a multiplier $H_0$.

where $y(n)$ is the output of $H(z)$ in response to $x(n)$. That is, the signal energy cannot be increased by the system; so a bounded system is also said to be *passive*.

2) A bounded transfer function with $|H(e^{j\omega})|=1$ for all $\omega$ is called **lossless** and is nothing but a stable **allpass** filter. In this case Eq. (4) holds with equality for all inputs $x(n)$.
3) A bounded transfer function is said to be *bounded real* or **BR** if all the filter coefficients are real i.e., the impulse response $h(n)$ is real. In this case $H(z)$ is real for real $z$. A lossless function with real filter coefficients is called a *lossless bounded real* or **LBR** function. It is nothing but a stable allpass filter with real coefficients.
4) An $M \times K$ transfer matrix $\mathbf{T}(z)$ is said to be lossless if it is stable (i.e., all entries $T_{km}(z)$ are stable), and furthermore $\mathbf{T}(e^{j\omega})$ is unitary for all frequencies:

$$\mathbf{T}^H(e^{j\omega})\mathbf{T}(e^{j\omega}) = \mathbf{I}_K \quad \forall \omega \tag{5}$$

where the superscript $H$ denotes transpose conjugation. If the lossless matrix $\mathbf{T}(z)$ also has real coefficients, then we say it is a *LBR* transfer matrix. Note that we require $M \geq K$ for (5) to hold. For the special case where $K=1$, $\mathbf{T}(z)$ becomes a column vector $\mathbf{T}(z) = [H_0(z)\, H_1(z) \ldots H_{M-1}(z)]^T$ and (5) implies

$$\sum_{k=0}^{M-1} |H_k(e^{j\omega})|^2 = 1 \tag{6}$$

which is also referred to as the **power complementary** property. ◊

Figure 6 demonstrates the meaning of the power complementary property for the case where $M=2$. If the elements of $\mathbf{T}(z)$ are rational functions of $z$, then the property (5) implies

$$\widetilde{\mathbf{T}}(z)\mathbf{T}(z) = \mathbf{I} \quad \forall z \tag{7}$$

where $\widetilde{\mathbf{T}}(z) = \mathbf{T}^H(1/z^*)$. The property (7) is called the **paraunitary** property. In short a rational lossless matrix is a stable and paraunitary matrix. The mathematical origin of this property can be traced back to **scattering matrices** in classical network theory in the context of lossless multiports [2], [7], [53]. Multidimensional extensions are also well-known, please see [5] and references therein.

### *B. Low Sensitivity Induced by Structural Boundedness*

Now consider a digital lowpass filter with response as shown in Fig. 5(b). This is a bounded filter, with maximum response of $|H(e^{j\omega})|=1$ at some frequencies $\theta_k$ in the passband. Assume this filter is realized using a structurally bounded implementation. In this implementation, the multipliers are such that $|H(e^{j\omega})|=1$ at $\omega=\theta_k$. Now, if a particular multiplier $m_i$ is disturbed from its ideal value $m_{i0}$ (say due to quantization), then the response $|H(e^{j\theta_k})|$ can only decrease from the maximum of unity. We therefore have the behavior shown in Fig. 5(c). This shows that

$$\left.\frac{\partial |H(e^{j\theta_k})|}{\partial m_i}\right|_{m_i=m_{i0}}=0 \tag{8}$$

That is, the sensitivity with respect to $m_i$ is zero at the maxima $\omega=\theta_k$. Thus, the structure exhibits low passband sensitivity with respect to multipliers, especially if there are a number of maxima $\theta_k$ in the passband. Thus the behavior of a structurally bounded system is similar to that of a doubly terminated lossless two-port with maximum power transfer at the maximal points (reflection zeros) of the passband. The main point however is that structural boundedness can be directly achieved in the $z$-domain without recourse to electrical filters as we explain next. Like doubly terminated lossless electrical circuits, structural passivity does not guarantee low stop band sensitivity.

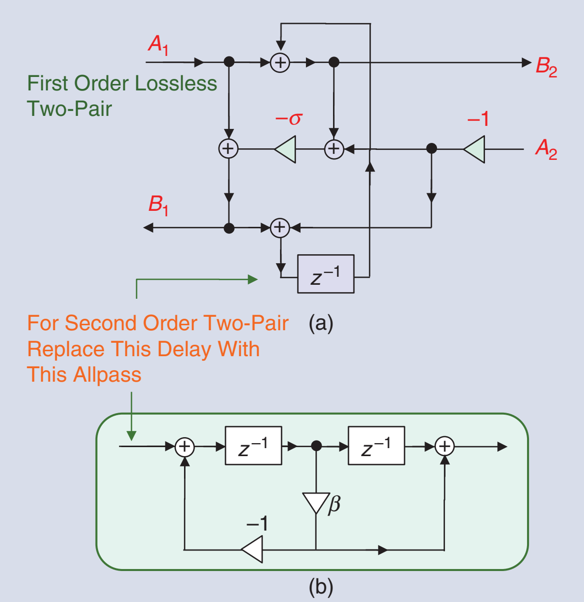



**Figure 8.** (a) Example of a first order lossless two-port or two-pair developed in [71]. Notice similarity to series wave adaptor (Fig. 3), with third port terminated by a delay. (b) One way to obtain a second order lossless two-port, is to replace the delay element $z^{-1}$ with an allpass filter, as indicated.

### *C. A Synthesis That Achieves Structural Boundedness*

The basic step in the structurally bounded realization of a BR transfer function, as described in [71], is as follows: given an $m$th order BR function $H_m(z)$, we "extract" an LBR two-pair $\mathbf{T}_m(z)$ and a BR "remainder" $H_{m-1}(z)$ with smaller order $m-1$, such that $H_m(z)$ can be implemented as in Fig. 7(a). The conditions on $H_m(z)$ under which this is possible, as well as the details of the specific two-pairs $\mathbf{T}_m(z)$ to be used are described in [71]. Since the remainder $H_{m-1}(z)$ remains BR, we can repeat this extraction process until the final remainder $H_0$ is a constant BR function (i.e., $-1\le H_0\le 1$). Thus starting from an $N$th order BR function $H_N(z)$, we can obtain the chain-cascaded structure shown in Fig. 7(b). While it is not obvious, it can be shown that such a synthesis is always possible for classical transfer functions (elliptic, Cheybyshev, and Butterworth filters). Broadly speaking, two types of LBR two-pair building blocks are necessary for this: first-order and second-order building blocks. Each of these comes with some minor variations depending on the details of the transfer function to be synthesized as elaborated in Tables 2, 3, and 4 of [71]. A typical first order LBR two-pair involved in the synthesis takes the form

$$\mathbf{T}(z)=\frac{1}{1+\sigma z^{-1}}\begin{bmatrix}1-\sigma & \sqrt{\sigma}(1+z^{-1})\\ \sqrt{\sigma}(1+z^{-1}) & (\sigma-1)z^{-1}\end{bmatrix} \tag{9}$$

where $0\le\sigma<1$ so that the pole is inside the unit circle and furthermore $\sqrt{\sigma}$ is real. It is readily verified that $\widetilde{\mathbf{T}}(z)\mathbf{T}(z)=\mathbf{I}$ so that $\mathbf{T}(z)$ is LBR. Figure 8(a) shows an implementation of the first order two-pair (9). The multipliers $\sqrt{\sigma}$ do not appear because they can be removed by a denormalization process which does not change the transfer functions $H_m(z)$. To be more specific, if a two-pair $\mathbf{T}_m(z)$ has the general form

$$\begin{bmatrix}T_{11}(z) & T_{12}(z)\\ T_{21}(z) & T_{22}(z)\end{bmatrix} \tag{10}$$

then only the product $T_{12}(z)T_{21}(z)$ matters in determining the transfer functions $H_m(z)$. So replacing $T_{12}(z)$ with $\alpha T_{12}(z)$ and $T_{21}(z)$ with $T_{21}(z)/\alpha$ where $\alpha=\sqrt{\sigma}$ or $\alpha=1/\sqrt{\sigma}$ gets rid of $\sqrt{\sigma}$. Fig. 8(a) is one such denormalized structure.

A typical second order LBR two pair arising in the synthesis of BR functions is obtained simply by replacing $z^{-1}$ in Eq. (9) with the allpass function

$$z^{-1}\frac{\beta+z^{-1}}{1+\beta z^{-1}} \tag{11}$$

where $-1<\beta<1$. Please see Fig. 8(b). Some other minor variations of these first and second order LBR two-pairs

**Historically the mathematics of the lattice structure can be traced back to the mathematical works of Schur and Szego in the early 1900s, and the work of Levinson.**

are tabulated in [71] and are sufficient to realize a large class of BR transfer functions in the form of Fig. 7(b).

How does this cascade achieve structural boundedness? If the multipliers $\sigma_i$ and $\beta_i$ are restricted to their specific ranges in spite of quantization (i.e., $0 \le \sigma_i < 1$ and $-1 < \beta_i < 1$) then each two-port remains LBR. If the rightmost multiplier $H_0$ is quantized such that the property $-1 \le H_0 \le 1$ continues to be respected, then all the transfer functions $H_m(z)$ will remain BR in spite of multiplier quantization. Thus boundedness of $H_N(z)$ can be structurally enforced.

## IV. Generality of the Structurally Passive Approach

The synthesis of a BR transfer function by extraction of LBR building blocks gives rise to a number of well known low sensitivity structures as special cases. In fact, notice the similarity between the LBR two-pair shown in Fig. 8(a) and the wave adaptor shown Fig. 3. This similarity is not coincidental. The direct digital synthesis described in Sec. 3.3 does give rise to the type of wave digital filters developed by Swamy and Thyagarajan (Sec. 2.3) as special cases; please see [71].

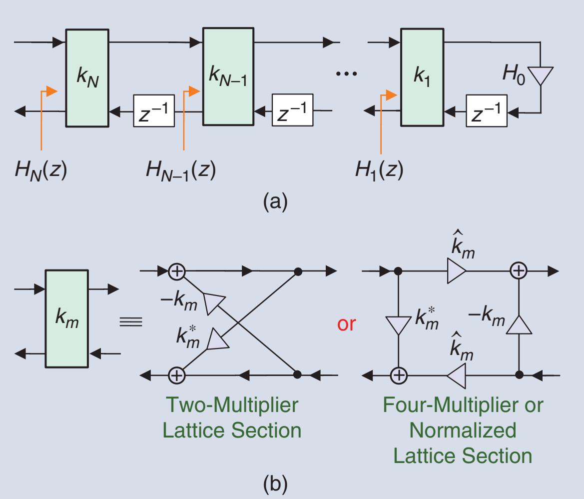



**Figure 9.** (a) The Gray-and-Markel recursive lattice structure for allpass filters. (b) Two possible choices for the lattice sections. The transfer functions $H_k(z)$ are the same regardless of which of these sections is used. This lattice is also known as the linear-prediction lattice because it has its origin in the theory of linear prediction, and was extensively studied in the context of speech coding.

### *A. The Gray-Markel Allpass Lattice*

Another special case is the well-known Gray and Markel lattice structure for allpass filters [32]. Please see Box 2 for a review of allpass filters; these filters have many applications [61], including low sensitivity implementations [77] and frequency transformations [12].

The allpass lattice structure is shown in Fig. 9. In this structure the lattice coefficients $k_m$ satisfy $|k_m| < 1$, and

$$\hat{k}_m = \sqrt{1 - |k_m|^2} \qquad (12)$$

It can be shown that the transfer function $H_N(z)$ is stable and **allpass**, that is, $|H_N(e^{j\omega})| = 1$ for all $\omega$. In fact any stable rational allpass filter can be implemented this way. The coefficients $k_m$ are real for real-coefficient allpass filters.

This structure was derived independently in 1973 without reference to either the wave filter approach or structural passivity [48], [57], [90]. In fact, historically, the mathematics of the structure can be traced back to the mathematical works of Schur and Szegö in early 1900s [62], [63], [70] and the work of Levinson [44]. It was developed further in the signal processing literature in the 1970s, in the context of linear prediction theory [4], [38], [43], [48], [57], [90]. Please also see the classic tutorial articles by Kailath [40] and Makhoul [45] in this context.

Now, since $H_N(z)$ is allpass, it is in particular a bounded function, and it can be synthesized in the form of a chain-cascade by extracting lossless two-ports as described in Sec. 3.3. There are many choices of lossless two ports that make this synthesis possible. One specific synthesis, described in Sec. 3.4.3 of [88], yields the specific structure of Fig. 9. More details can be found in [81]. Thus the Gray-Markel lattice can be regarded as a special case of the lossless two-port extraction method. The transfer matrix of each lossless two-port in this example takes the form

$$\mathbf{T}_m = \begin{bmatrix} k_m^* & \hat{k}_m \\ \hat{k}_m & -k_m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \qquad (13)$$

The constant matrix above is the four-multiplier or normalized building block [33] shown in Fig. 9(b). There are many denormalized versions of the lattice, as explained in detail in [88].

In addition to structural passivity, this implementation also involves an **internal passivity** which leads to

## Box 2: Allpass Functions

Allpass filters are fundamental building blocks in signal processing. A digital filter $H(z)$ is said to be allpass if $|H(e^{j\omega})| = 1$ for all $\omega$, that is $H(e^{j\omega}) = e^{j\phi(\omega)}$. A rational allpass filter has the form

$$H(z) = \frac{b_N^* + b_{N-1}^* z^{-1} + \cdots + b_1^* z^{-(N-1)} + b_0^* z^{-N}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}} = \frac{z^{-N}\tilde{B}(z)}{B(z)}$$

where $\tilde{B}(z) = B^*(1/z^*)$. This notation is equivalent to replacing the coefficients $b_k$ by their complex conjugates, and replacing $z$ with $1/z$. For a rational transfer function the allpass property can be rewritten as $\tilde{H}(z)H(z) = 1$ for all $z$. Allpass filters are used in phase equalization, and in the implementation of certain filters. For example, as reviewed in this article, classical Butterworth, Chebyshev, and elliptic filters can be expressed as a *sum of two allpass filters*, leading to a structurally passive implementation with low passband sensitivity. Such structures also have very few multipliers compared to direct-form and other structures. The nonzero poles $p_k$ and zeros $z_k$ of a rational allpass filter have a reciprocal symmetry: $p_k = 1/z_k^*$.

In fact, for a causal stable rational allpass function of order $\geq 1$, a curious symmetry with respect to the unit circle, called the *modulus property* holds:

$$|H(z)| \begin{cases} <1 & \text{for } |z| > 1 \\ >1 & \text{for } |z| < 1 \\ =1 & \text{for } |z| = 1 \end{cases}$$

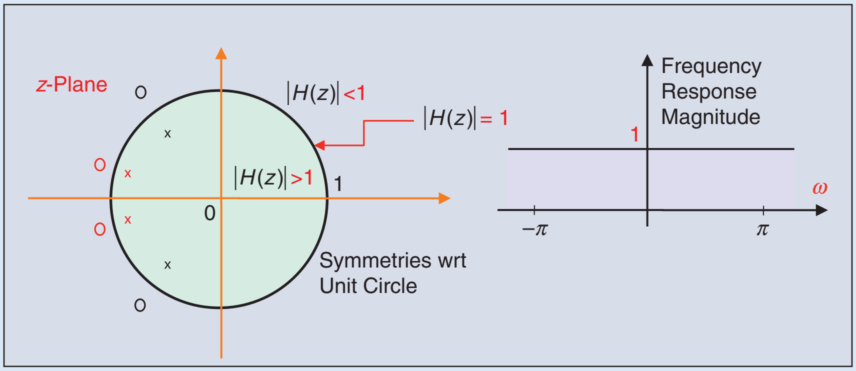
This is at the heart of the derivation of lattice structures and stability test procedures based on allpass systems as elaborated by Vaidyanathan and Mitra (1987). These symmetries with respect to the unit circle are summarized in the figure. Allpass filters are also very effective in the implementation of notch and antinotch filters. Many efficient structures exist for allpass filters such as the Gray and Markel lattice (1973), and the Mitra and Hirano class of structures (1974). A detailed review of allpass functions can be found in Regalia, Mitra, and Vaidyanathan (1988). An early application of allpass filters for frequency transformations was developed by Constantinides (1970). The allpass property can be generalized to MIMO systems as follows: an $M \times N$ transfer matrix $\mathbf{T}(z)$ is allpass if it is unitary on the unit circle, that is,

$$\mathbf{T}^H(e^{j\omega})\mathbf{T}(e^{j\omega}) = \mathbf{I}_N$$

for all $\omega$. This requires $M \geq N$. For rational transfer matrices this implies the *paraunitary* property: $\tilde{\mathbf{T}}(z)\mathbf{T}(z) = \mathbf{I}_N$ for all $z$. Here $\tilde{\mathbf{T}}(z) = \mathbf{T}^H(1/z^*)$. If $\mathbf{y}(n)$ is the output of a stable paraunitary system in response to input $\mathrm{x}(n)$, then

$$\sum_n y^H(n)y(n) = \sum_n x^H(n)x(n)$$

That is, the output energy is equal to the input energy. So, stable paraunitary matrices and allpass filters are called *lossless* systems. Historically, lossless systems had a fundamental role in circuit and system theory as elaborated by Belevitch (1968) and by Anderson and Vongpanitlerd (1973). Paraunitary matrices arise in the cascaded synthesis of digital filters with structural passivity. For the special case where $N = 1$, $\mathbf{T}(z)$ is a column vector with components $H_k(z)$, and the allpass property becomes $\Sigma_{k=0}^{M-1}|H_k(e^{j\omega})|^2 = 1$, which is the *power complementary* property. There is a systematic way to factorize FIR paraunitary matrices in terms of planar rotations and delay elements. These are summarized in the review paper by Vaidyanathan and Doğanata (1989). Paraunitary matrices also arise in the design of orthonormal digital filter banks. Please see Vaidyanathan (1993) for details.



**Thus, wave filters, orthogonal filters, and cascaded lattice structures are nicely unified by the structurally passive synthesis methods which use LBR building blocks.**

many useful properties, including *suppression of limit cycle oscillations* [34], [76]. The importance of this internal passivity in suppression of limit cycles has also been established independently in the context of wave digital filters [20], [26]. Further generalizations as well as simplifications can be found in [76], [79].

### *B. Orthogonal Digital Filters and Rotation Operators*

Before we discuss further examples it is useful to introduce the planar rotation operator which turns out to be an important building block for many types of digital filter structures. Thus, consider the matrix

$$\mathbf{\Theta}_m = \begin{bmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{bmatrix} \tag{14}$$

and the operation $\mathbf{y} = \mathbf{\Theta}_m \mathbf{x}$. It is readily shown that $\mathbf{y}$ is the clockwise rotated version of $\mathbf{x}$, by the angle $\theta_m$ (page 290, [88]). This operator is therefore called the **planar rotation** operator, and is schematically denoted as shown in Fig. 10(a). It is also known as the *Givens rotation* operator or the *cordic* processor [31], [37], [42]. As an example of how this operator arises, consider the allpass lattice structure with the four-multiplier or normalized building block (Fig. 9). If the filter has real coefficients, then $k_m$ are real and we can write $k_m = \cos\theta_m$ and $\hat{k}_m = \sqrt{1-k_m^2} = \sin\theta_m$ for some real $\theta_m$ so that the computational blocks become

$$\begin{bmatrix} k_m & \hat{k}_m \\ \hat{k}_m & -k_m \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\mathcal{R}} \underbrace{\begin{bmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{bmatrix}}_{\mathbf{\Theta}_m} \tag{15}$$

Here $\mathcal{R}$ is just a reflection operator, as it merely reverses the sign of the $y$-component. Thus, the allpass filter can be implemented entirely in terms of planar rotations as the computational units.

Similarly it has been shown [75] that the first order LBR two-pair of Fig. 8(a) can be rearranged in terms of two planar rotations as shown in Fig. 10(b). In fact, any BR transfer function, synthesized in terms of the first and second order LBR two-pairs of Fig. 8 can be expressed in terms of planar rotations as the only computational units [75]. Digital filter structures which can be expressed entirely in terms of rotation operators were first noticed by Deprettere and Dewilde in the context of a family of structures called **orthogonal filter** structures [15], [58], which enjoy several robustness properties under quantization. It should also be mentioned here that paraunitary matrices can be neatly factorized into planar rotations (or other fundamental unitary blocks such as Householder matrices) and delay elements [17], [84]–[86], [88].

The preceding discussions show that the Gray-Markel lattice structures (Fig. 9) and the structurally passive cascaded structures (Fig. 7(b) with building blocks as in Fig. 8) can be expressed as orthogonal digital filter structures. Thus, wave filters, orgthogonal filters, and cascaded lattice structures are nicely unified by the structurally passive synthesis methods which use LBR building blocks.

We conclude by mentioning that the lossless two-port extraction approach of Sec. 3.3 has also been extended to transfer functions with multiple inputs and multiple outputs (MIMO). Figure 11(a) shows an example of a single-input multi-output transfer matrix with transfer function $\mathbf{H}_N(z)$. Here the transfer matrices $\mathbf{T}_m(z)$ are MIMO lossless transfer matrices, that is, they are stable and satisfy $\tilde{\mathbf{T}}_m(z)\mathbf{T}_m(z) = \mathbf{I}$. It can be shown that if $\mathbf{H}_N(z)$ is SIMO lossless, it can be synthesized in this form by using the lossless multiport extraction approach [75]. A special case of this is the beautiful family of single-input

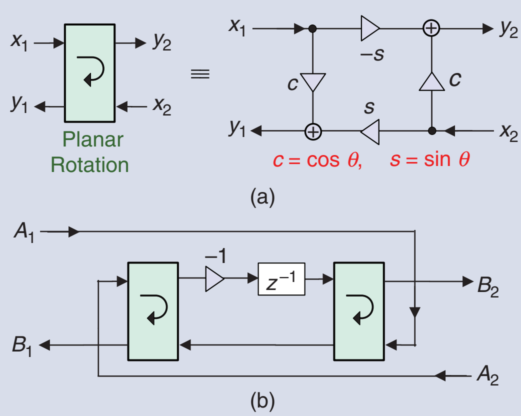



**Figure 10.** (a) The planar rotation operator. This rotates the vector $\mathbf{x} = [x_1 \ \ x_2]^T$ clockwise by an angle $\theta$ to produce $\mathbf{y} = [y_1 \ \ y_2]^T$. This is also known as the Givens rotation operator [31], or the cordic processor [37], [42]. (b) Implementation of the first order LBR two-port of Fig. 8(a) using planar rotation operators.

two-output lattice structures developed first by Rao and Kailath in 1984 [59] shown in Fig. 11(b). In this structure the matrices $\mathbf{R}_k$ are constant unitary matrices, and the transfer matrix $[H_N(z) \quad G_N(z)]^T$ is lossless. That is, $H_N(z)$ and $G_N(z)$ are stable and satisfy

$$\tilde{H}_N(z)H_N(z)+\tilde{G}_N(z)G_N(z)=1 \tag{16}$$

The above property implies the power complementary property $|H_N(e^{j\omega})|^2+|G_N(e^{j\omega})|^2=1$. Thus, given any BR transfer function $H_N(z)$ we can always find its power complementary partner $G_N(z)$, and realize the pair as in Fig. 11(b). It can be shown in this specific case [75] that the $3\times3$ unitary matrices $\mathbf{R}_k$ can be implemented using two planar rotations each, as shown in Fig. 11(c). This is a structurally passive implementation of the BR function $H_N(z)$ in the sense that, regardless of the angular values of the rotations the transfer functions remain BR. In particular, the structure exhibits low passband sensitivity as explained in Sec. 3.2.

## V. Further Examples of Structurally Passive Implementations

In this section we review a number of structurally passive implementations and demonstrate their low sensitivity properties. These methods are quite simple and can be understood independently in the $z$-domain without any background on circuit theory or electrical filters.

### *A. Parallel-Allpass Implementations*

Consider Fig. 12 where $A_0(z)$ and $A_1(z)$ are stable rational allpass filters and $H_0(z)$ and $H_1(z)$ are obtained by adding and subtracting the outputs of the allpass filters as shown. It turns out that a large class of IIR digital filters, including Butterworth, Chebyshev, and elliptic filters, can be implemented in this way. This result is known in classical continuous-time filter theory and it is implicit in the design of wave lattice digital filters pioneered by Fettweis [21], [30]. However, the result is more general, and it can be proved quite easily and directly without reference to continuous-time circuit theory [77]. Thus, Theorem 3.6.1 in [88] establishes some sufficient conditions on $H_0(z)$ and $H_1(z)$ which allow their implementation as in Fig. 12(a). There are many special cases of filters which satisfy these sufficient conditions. For example if $H_0(z)$ is an odd order lowpass Butterworth, Chebyshev, or elliptic filter, it satisfies the conditions of the above theorem, and it can be expressed as

$$H_0(z)=\frac{A_0(z)+A_1(z)}{2} \tag{17}$$

where $A_0(z)$ and $A_1(z)$ are real-coefficient, stable, allpass filters. That is, they have the form [88]

$$A_k(z)=\frac{a_{k,n_k}+a_{k,n_k-1}z^{-1}+\cdots+z^{-n_k}}{1+a_{k,1}z^{-1}+\cdots+a_{k,n_k}z^{-n_k}} \tag{18}$$

(Please see Box 2 for a review of allpass filters.) There are systematic ways to identify the coefficients of the

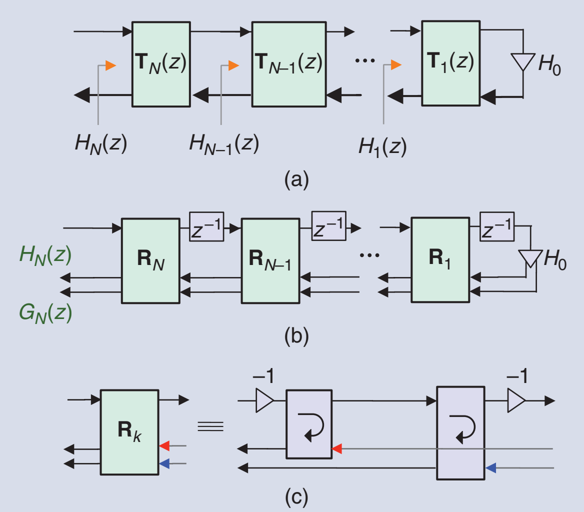



**Figure 11.** (a) The two-port extraction approach extended to single-input multi-output systems. (b) Example of a single-input two-output lattice, generating an IIR allpass vector, that is, an IIR power complementary pair $[H_N(z) \quad G_N(z)]^T$. (c) Implementation of each building block using two planar rotations. The example in (b), (c) was first developed by Rao and Kailath in a pioneering work in the 1980s [59].

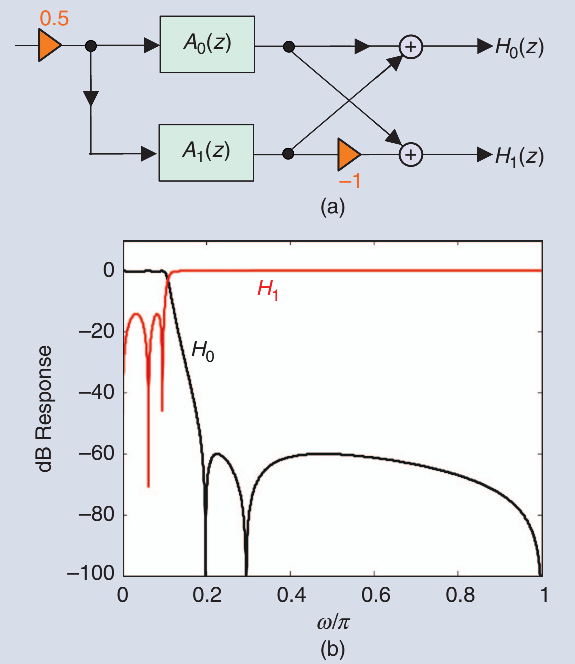



**Figure 12.** (a) The parallel allpass structure to implement a power complementary pair of bounded real transfer functions. (b) Example of filter responses.

allpass filters starting from the coefficients of $H_0(z)$ [77], [88]. With $A_k(z)$ identified, the filter

$$H_1(z) = \frac{A_0(z) - A_1(z)}{2} \tag{19}$$

turns out to be a highpass filter of the same kind (Butterworth, Chebyshev, or elliptic). From (17) and (19) it is easy to verify that the two filters are power complementary:

$$\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1 \tag{20}$$

This is demonstrated in Fig. 12(b) where the filters are fifth order elliptic filters. The implementation of Fig. 12 is called the **parallel-allpass** implementation or sum-of-allpass implementation.

A similar implementation is possible for even-order Butterworth, Chebyshev, and elliptic lowpass filters, but $A_0(z)$ has complex coefficients $a_{k,i}$, and the coefficients of $A_1(z)$ are the conjugates of those of $A_0(z)$; see [80] for details. In this case $H_0(z)$ can be realized by taking the real part of the output of $A_0(z)$, and $H_1(z)$ realized by taking the imaginary part. In short, a single **complex allpass filter** can be used to implement the pair $H_0(z), H_1(z)$.

We now argue that Fig. 12(a) gives rise to a structurally passive implementation. While there exist many structures for implementation of allpass filters [32], [51], [61], [69], the Gray-Markel lattice is especially attractive in this context. If $A_i(z)$ are implemented using the Gray-Markel allpass lattice of Fig. 9, then as long as the quantized multipliers $k_m$ continue to satisfy $|k_m| < 1$, the filters remain allpass as well as stable [81]. Thus, even when the multipliers are quantized $H_0(z)$ and $H_1(z)$ continue to be a sum and difference of two stable allpass filters as in Eqs. (17), (19). Since $|A_i(e^{j\omega})| = 1$ it is obvious that $|H_k(e^{j\omega})| \le 1$ which proves structural boundedness of $H_0(z)$ and $H_1(z)$. As explained in Sec. 3.2 these structures therefore enjoy low passband sensitivity.

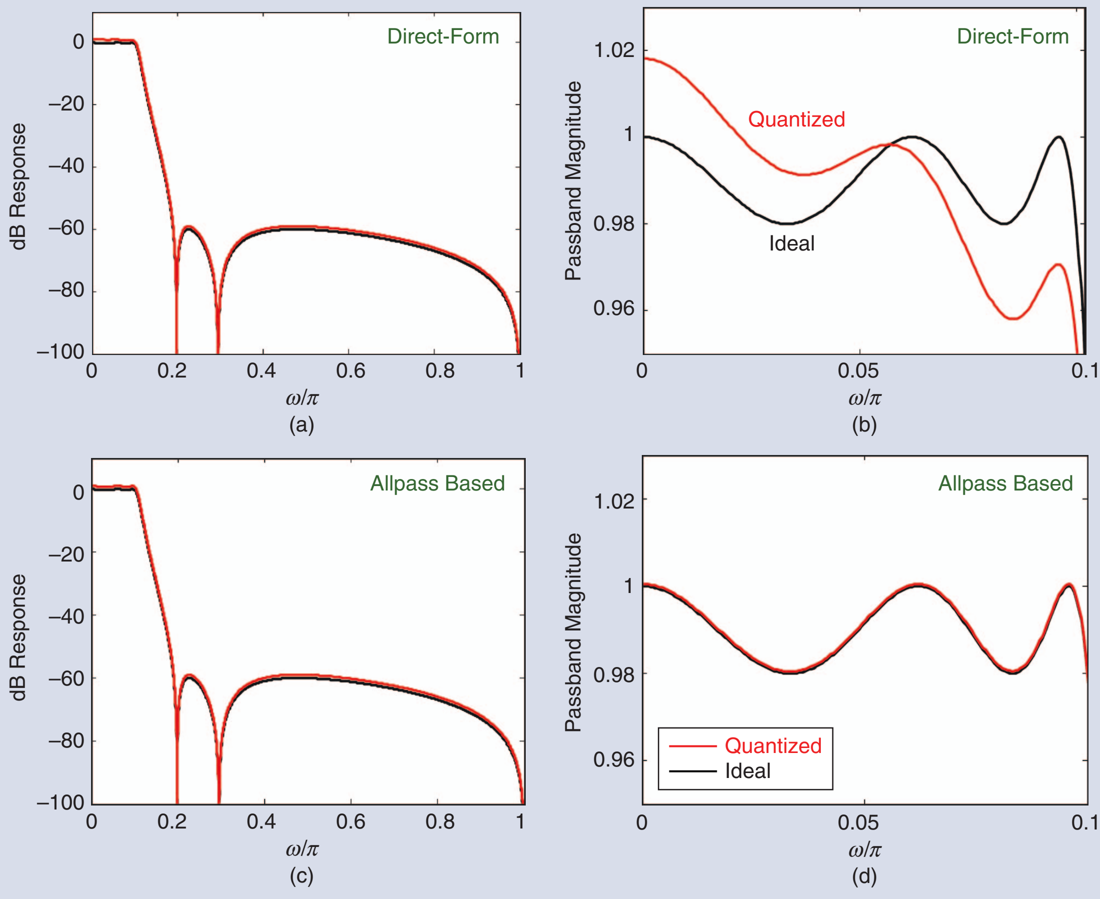



**Figure 13.** Responses of the direct-form structure and the parallel-allpass based structure under quantization. A fifth order IIR elliptic filter is simulated, and coefficients are quantized to 18 bits. The responses of quantized structures are shown in red, and responses of unquantized structures are in black. Notice that the parallel-allpass based system (structurally passive system) has very low passband sensitivity: the quantized response is indistinguishable from ideal.

**In short, *we get two Nth order filters $H_0(z)$ and $H_1(z)$ at the total cost of only N multipliers!***

In addition to low sensitivity, the implementation Fig. 12 is also amazingly economic in terms of computational complexity. For example, assume $H_0(z)$ is an $N$th order lowpass elliptic filter with odd $N$. Then the allpass filters have orders $n_0$ and $n_1$ where $n_0 + n_1 = N$. Each allpass filter can be implemented using a lattice structure as in Fig. 9. Now, instead of using the two-multiplier or four-multiplier lattice sections in Fig. 9(b), it is always possible to use one-multiplier sections (see Fig. 3.4-11 of [88]). If we do this then $A_k(z)$ requires only $n_k$ multipliers, so that the entire implementation of Fig. 12 requires only $N$ multipliers where $N$ is the order of each filter $H_k(z)$. In short, *we get two Nth order filters $H_0(z)$ and $H_1(z)$ at the total cost of only $N$ multipliers!* Indeed, this is *one of the most efficient ways* to implement Butterworth, Chebyshev and elliptic filters.

To demonstrate low sensitivity, consider a 5th order elliptic lowpass filter $H_0(z)$. In Fig. 13 we show the magnitude response of $H_0(z)$ with the quantized direct-form structure [54], and the quantized parallel allpass structure of Fig. 12(a). The responses of quantized structures are shown in red, and responses of unquantized structures are in black. For both structures, the dB plot of the entire response is shown, and the details of the passband region is shown separately. We have used 18 bits per multiplier coefficient in both structures. While both structures perform satisfactorily in the stopband, the passband response of the quantized direct-form deviates significantly from the ideal. In contrast, the quantized response of the structurally passive implementation (Fig. 12(a)) is nearly perfect in the passband as well, demonstrating very low passband sensitivity.

It is well known that if the Gray-Markel lattice structure is used to implement the allpass filters, then limit cycle oscillations can be suppressed [34], and furthermore, the roundoff noise gain is small [33]. Thus, the structurally passive implementation of Fig. 12(a) enjoys a number of robustness properties in addition to its computational economy. A special case of (17) where $A_0(z) = 1$ is very useful for the design of notch and anti-notch filters [89].

### *B. The FIR Power Complementary Lattice*

Consider Fig. 14(a) which is a cascaded lattice structure. Unlike in the earlier cascaded structures (e.g., Figs. 7 and 9), there are no feedback loops here. So, this is a **nonrecursive** or FIR lattice. The internal details of the lattice sections are shown in Fig. 14(b). In this figure we use the notations

$$c_m = \cos\theta_m, \quad s_m = \sin\theta_m \tag{21}$$

So the building blocks are planar rotations or denormalized versions of such rotations. This structure was introduced in [78] and has a number of interesting properties. First, the two transfer functions are $H_0(z)$ and $H_1(z)$ are guaranteed to be power complementary that is, $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$ regardless of the choice of the angles $\theta_m$. (If denormalized lattice sections are used, then $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = c$ for some constant $c > 0$). Secondly, given any pair of power complementary FIR filters $H_0(z)$ and $H_1(z)$ (with real coefficients), they can always be implemented using this lattice structure, by choosing the planar rotation angles $\theta_m$ appropriately. Now assume that we are given some FIR filter $H_0(z)$ with real coefficients and normalized such that $|H_0(e^{j\omega})| \le 1$, i.e., we are given an FIR BR function $H_0(z)$. Then we can always find an FIR BR $H_1(z)$ such that $\{H_0(z), H_1(z)\}$ is power complementary. For this we simply take $H_1(z)$ to be any spectral factor of $|H_1(e^{j\omega})|^2 \triangleq 1 - |H_0(e^{j\omega})|^2$. Then we can implement the pair as in Fig. 14. *This shows that we can obtain this cascaded lattice implementation for any FIR BR filter $H_0(z)$.*

Now, given such an implementation, if the angles $\theta_m$ in the rotations are perturbed, the power complementary property is not affected, and therefore the property $|H_0(e^{j\omega})| \le 1$ continues to hold. In this sense the

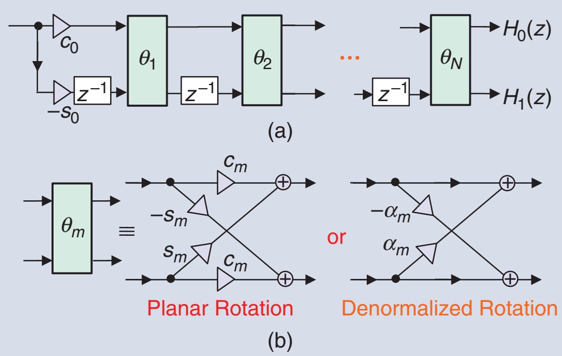



**Figure 14.** (a) The lattice structure for an FIR power complementary pair $[H_0(z), H_1(z)]$ [78]. (b) Details of the building blocks. Each block is a planar rotation operator. It can also be replaced by a denormalized section with two multipliers as shown.

implementation is structurally passive and therefore enjoys low passband sensitivity. In practice even the lattice with the denormalized sections shown in Fig. 14(b) exhibits low passband sensitivity. To demonstrate this, we consider a 60th order linear phase lowpass equiripple filter $H_0(z)$ designed using the McClellan-Parks algorithm [54], and implement it using the lattice structure (using the algorithm presented in [78]. We use the denormalized (two-multiplier) lattice sections of Fig. 14(b) and quantize the lattice coefficients $\alpha_m$ to 8 bits. We compare the resulting frequency responses with those of the direct form structure with multipliers quantized to 8 bits as well. Fig. 15 shows the filter magnitude responses for these implementations. The responses of quantized structures are shown in red, and responses of unquantized structures are in black. For both structures, the dB plot of the entire response is shown, and the details of the passband region is shown separately. While both structures perform satisfactorily in the stopband, the passband response of the quantized direct-form deviates significantly from the ideal. In contrast, the quantized response of the structurally passive implementation (Fig. 14(a)) is nearly perfect in the passband as well, demonstrating very low passband sensitivity.

The FIR lattice structure Fig. 14 is called the FIR power complementary lattice or the **FIR structurally passive lattice**. Readers familiar with the FIR linear prediction lattice or **LPC lattice** might wonder what the difference is. The linear-prediction lattice also has an appearance similar to Fig. 14 with two-multiplier lattice sections, but the minus sign on the $\alpha_m$ is not there, and furthermore, $|\alpha_m| < 1$. In terms of theoretical properties, this is a major difference. The FIR lattice structure of Fig. 14 can realize arbitrary BR $H_0(z)$. But the LPC lattice cannot be used to realize arbitrary FIR filters. It is typically used to realize prediction filters $H_0(z)$ with all zeros strictly inside the unit circle (minimum-phase filters). The filter $H_1(z)$ in an LPC lattice is not power complementary to $H_0(z)$, but rather, it is the mirror image polynomial $H_1(z) = z^{-N} H_0^*(1/z^*)$. So, in the LPC lattice, $H_1(z)$ has all its zeros *outside* the unit circle.

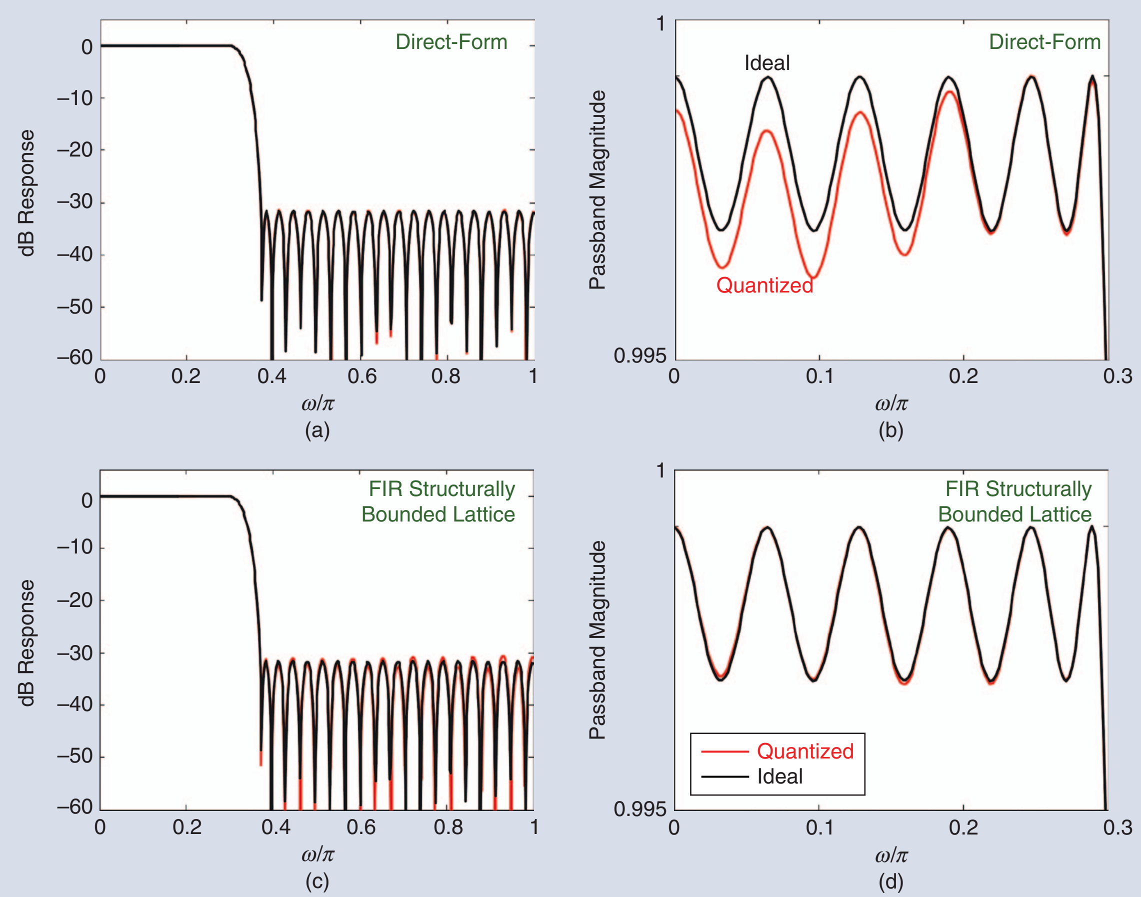



**Figure 15.** Responses of the direct-form structure and the FIR structurally passive lattice [78] under quantization. A 60th order FIR filter $H_0(z)$ is simulated, and coefficients are quantized to 8 bits. The responses of quantized structures are shown in red, and responses of unquantized structures are in black. Notice the low passband sensitivity of the structurally passive FIR lattice.

**In fact, these power complementary FIR lattices have also inspired the theory of multirate filter banks with perfect reconstruction, leading to a whole generation of filter bank structures with orthogonality properties.**

We conclude by mentioning some generalizations. If $\{H_0(z), H_1(z), \ldots, H_{M-1}(z)\}$ are causal real coefficient FIR filters with power complementary property, that is, $\sum_k |H_k(e^{j\omega})|^2 = 1$, then there exists a cascaded lattice structure similar in principle to Fig. 14(a), and can be implemented with planar rotations as the only computational units. For details please see [78]. Such structures are useful in the implementation of power complementary filter banks. In fact, these power complementary FIR lattices have also inspired the theory of multirate filter banks with perfect reconstruction, leading to a whole generation of filter bank structures with orthogonality properties. Details can be found in [1], [41], [46], [47], [65], [82], [84]–[88]. Such filter banks retain the perfect-reconstruction property in spite of coefficient quantization, and this can be exploited in the design of the filter responses under quantized conditions [36]. In addition to their applications in signal compression and digital communications, orthonormal filter banks have a role in the construction of orthonormal wavelets [11], [14], [66], [88], [91].

### *C. The FIRBR Structure*

We now present an example of a structurally passive FIR system called the **FIRBR** structure [74]. It is one of the simplest ways to achieve structural passivity – the only background required is a first course in digital signal processing. The method only works for Type 1 linear phase FIR filters [54] with equiripple passbands. Such a filter has transfer function of the form $H(z) = \sum_{n=0}^{N} h(n) z^{-n}$ and satisfies the following properties:

1) $N$ is even,
2) $h(n)$ is real, and
3) $h(n)$ is symmetric, that is $h(n) = h(N-n)$.

So the frequency response has the form

$$H(e^{j\omega}) = e^{-j\omega N/2} H_R(e^{j\omega}) \tag{22}$$

where $H_R(e^{j\omega})$ is called the zero-phase part. It is real, with $H_R(e^{j\omega}) \geq 0$ in the passband. Figure 16(a) shows a plot of the zero-phase part $H_R(e^{j\omega})$ for a lowpass filter. The equiripple property ensures in particular that all the peaks in the passband are equal to unity.

We now show how to implement such a filter in a structurally passive manner. Define the companion filter

$$G(z) = z^{-N/2} - H(z) \tag{23}$$

so that

$$G(e^{j\omega}) = e^{-j\omega N/2} \underbrace{(1 - H_R(e^{j\omega}))}_{G_R(e^{j\omega})} \tag{24}$$

Clearly this is a Type 1 linear phase highpass filter with zero-phase response $G_R(e^{j\omega})$ as shown in the figure. Furthermore, its response satisfies $G_R(e^{j\omega}) \geq 0$ for all $\omega$. In fact at the passband peak frequencies of $H(z)$ where $H_R(e^{j\omega}) = 1$ and its derivative is zero, we have $G_R(e^{j\omega}) = 0$ and these are guaranteed to be *double zeros* of $G(z)$. Therefore $G(z)$ can be factorized into the form

$$G(z) = \underbrace{\prod_{k=1}^{M} (1 - 2z^{-1}\cos\omega_k + z^{-2})^2}_{G_1(z)} G_2(z) \tag{25}$$

where $G_1(z)$ represents all the double zeros on the unit circle and $G_2(z)$ represents all zeros which are not on the unit circle. So we can implement the original lowpass filter $H(z) = z^{-N/2} - G(z)$ using the structure shown in Fig. 16(b) where $G(z)$ is implemented in the factored form (25).

Now consider the effect of quantization. $G_1(z)$ is implemented in the factored form (25) where the multipliers are

$$m_k = 2\cos\omega_k \tag{26}$$

So if these multipliers are perturbed slightly, the zeros of $G_1(z)$ remain on the unit circle and they continue to be double zeros (as long as the quantized $m_k$ satisfies $|m_k| \leq 2$). So $G_R(e^{j\omega}) \geq 0$ which shows that $H_R(e^{j\omega}) \leq 1$.

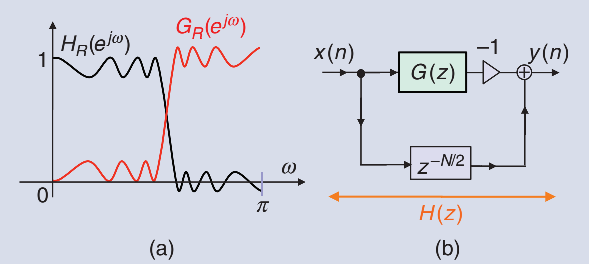



**Figure 16.** (a) The relation between the responses of $H(z)$ and $G(z)$ where $H(z)$ is Type-1 linear phase FIR, and $G(z) = z^{-N/2} - H(z)$. Here $H_R(e^{j\omega})$ and $G_R(e^{j\omega})$ are the zero-phase responses of $H(z)$ and $G(z)$ as defined in Eqs. (22) and (24). (b) An implementation of $H(z)$. If $G(z)$ is implemented in the factored form (25), this implementation of $H(z)$ is structurally passive. This is called the FIRBR implementation, and enjoys low passband sensitivity.

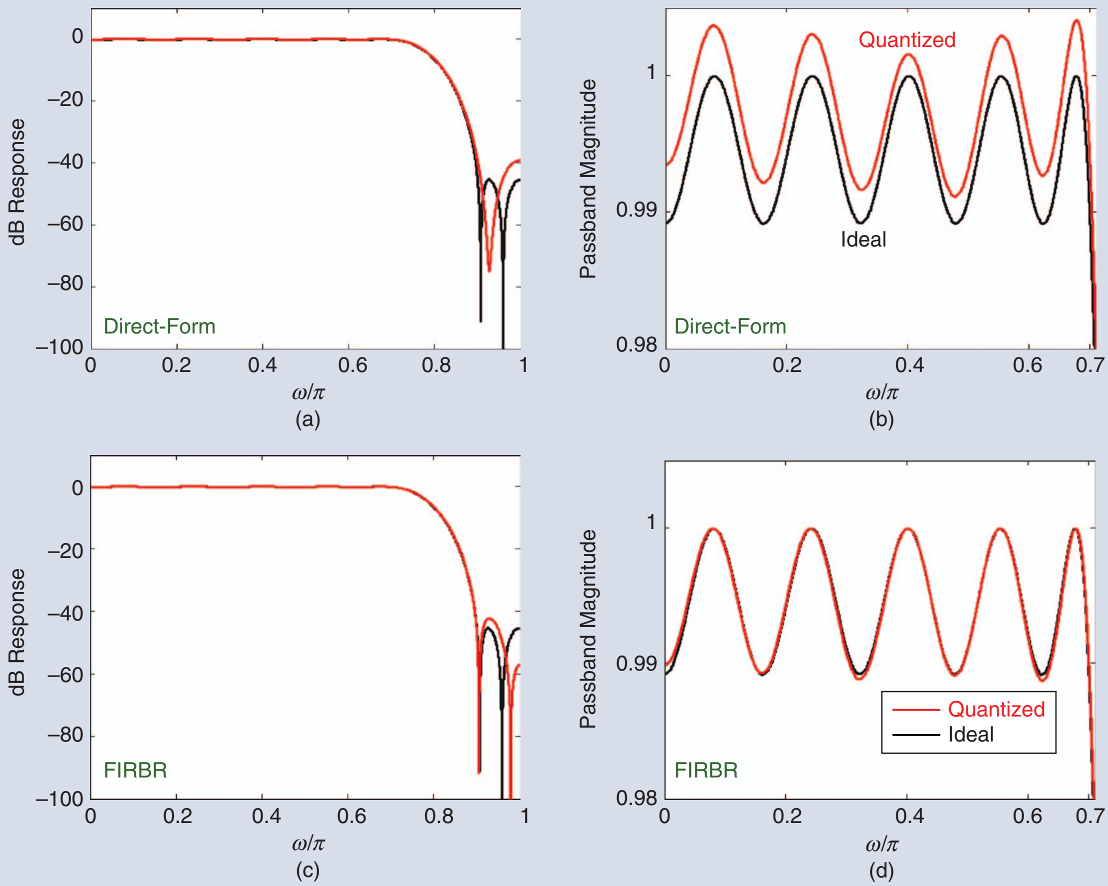



**Figure 17.** Responses of the direct-form structure and the FIRBR structure [74] under quantization. A 24th order linear phase FIR filter $H(z)$ is simulated, and coefficients are quantized to 7 bits. The responses of quantized structures are shown in red, and responses of unquantized structures are in black. Notice the low passband sensitivity of the FIRBR structure, induced by structural passivity.

Thus the passband response of $H(z)$ *remains bounded by unity*.[1] Fig. 16(b) is therefore a structurally bounded implementation as long as $G(z)$ is implemented in factored form (25). The structure therefore enjoys low passband sensitivity.

To demonstrate the low sensitivity, we consider a 24th order linear phase lowpass equiripple filter $H(z)$ designed using the McClellan-Parks algorithm [54]. We implement this using Fig. 16(b), with $G(z)$ implemented in factored form (25). We compare the resulting frequency response with those of the direct-form structure, with multipliers quantized to 7 bits in both structures. Fig. 17 shows the filter magnitude responses for these implementations. The responses of quantized structures are shown in red, and responses of unquantized structures are in black. For both structures, the dB plot of the entire response is shown, and the details of the passband region is shown separately. While the structures have comparable performances in the stopband, the passband response of the quantized direct-form deviates significantly from the ideal. In contrast, the quantized response of the structurally passive implementation (FIRBR structure of Fig. 16(b)) is nearly perfect in the passband as well, demonstrating very low passband sensitivity.

[1]Notice that $G_2(z)$ is itself a Type 1 linear phase filter, and has zeros in reciprocal conjugate pairs $\{z_k, 1/z_k^*\}$. When the coefficients of $G_2(z)$ are quantized, these zeros continue to remain inside and outside the unit circle, with reciprocal conjugate symmetry.

## VI. Concluding Remarks

The world of circuit theory has been home to many legends in the last hundred years who gave a rock solid foundation to the field. Prof. Fettweis was one such giant who was a legend even during his life time. His contributions spanned a much wider area than we have discussed in this limited space. In this article we focussed only on digital filter structures with low passband sensitivity. Even in the area of wave digital filters, low sensitivity is only one of the many aspects addressed by Prof. Fettweis. His contributions to other aspects of robustness such as freedom from limit cycles are addressed in other articles in this issue.

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