Minimal Non-Uniform Sampling For Multi-Dimensional Period Identification

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***Abstract*—This paper addresses a fundamental question in the context of multi-dimensional periodicity. Namely, to distinguish between two N-dimensional periodic patterns, what is the least number of (possibly non-contiguous) samples that need to be observed? This question was only recently addressed for one-dimensional signals. This paper generalizes those results to N-dimensional signals. It will be shown that the optimal sampling pattern takes the form of sparse and uniformly separated bunches. Apart from new theoretical insights, this paper's results may provide the foundation for fast N-dimensional period recognition algorithms that use minimal number of samples[1].**

Keywords—Multidimensional periodicity, period estimation, sparse sampling, non-uniform sampling.

I. Introduction

Periodicity in multi-dimensions is a beautiful phenomenon. While most DSP applications of periodicity involve one dimensional signals, such as in speech, ECG, EEG, machine vibrations, DNA microsatellites, protein repeats, etc., there are applications in art [3], [5], crystallography [4], [8], texture analysis, image denoising [7] etc. that involve multi-dimensional periodic signals. In terms of practical techniques for estimating periodicity, there is today a rich variety of algorithms available in the literature. However, one fundamental aspect of this field has received surprisingly little mathematical attention in the past. Namely, is there a precise bound on the minimum number of samples required to identify the period unambiguously? In this paper, we investigate this question for periodic sequences, which have integer valued periods (shortest repeat length) in 1 dimension (1D) or integer valued matrix periods in higher dimensions (although the sequence itself could be real or complex valued).

The earliest results related to minimum datalength can be dated back to Carathéodary and Fejér [1]. They derived bounds on the minimum contiguous datalength needed to estimate the frequencies in a 1-dimensional (1D) sum of sinusoids. If we express 1D periodic signals as sums of sinusoids using Fourier series, these results tell us that the number of samples must be at-least twice the largest expected period. Carathéodary and Fejér's results were extended to N-dimensional signals in [9]. However, notice that periodic signals are not just arbitrary sums of sinusoids. There is a nice harmonic relationship between the frequencies of a periodic signal, which is not taken into account when using Carathéodary's results. So these classical bounds can only yield sufficiency results for periodicity.

In a recent work [12], the following result was proved for 1D signals: To identify the true period from a set of possible integer periods $\{P_1, P_2, \ldots, P_K\}$ using L consecutive samples, it is both necessary and sufficient that:

$$L \geq L_{min} = \max_{P_i, P_j} P_i + P_j - (P_i, P_j) \tag{1}$$

This result was also generalized to mixtures of 1D signals in [12]. But what if we are free to choose our samples in a non-contiguous fashion? Can we estimate the period using fewer samples than (1)? If so, what is the optimal way to choose those samples so that we have the smallest number of samples?

This question is quite difficult to answer in general. As a first result in this regard, [13] showed that, given a 1D periodic signal $x(n)$ whose period P lies in the set $\mathbb{P} = \{P_1, P_2\}$, where $P_1 < P_2$, the following number of samples is necessary and sufficient to identify P:

$$M_{min} = \begin{cases} P_2 & \text{if } P_1 \text{ divides } P_2 \\ P_1 & \text{otherwise} \end{cases} \tag{2}$$

Although the scope of this result is limited to resolving between two periods, its proof was still quite involved [13]. M_{min} in (2) can be significantly smaller than L_{min} in (1). For example, suppose we were to distinguish between periods 8 and 50. While L_{min} is 56 samples, M_{min} is only 8 samples. Directly applying Caratheodary's bounds [1] for complex exponentials in this case tells us that $2 \times 50 = 100$ samples are sufficient to estimate the correct period, which is far more than M_{min}.

In this paper, we will generalize the above non-contiguous samples result to N-dimensional signals. The proof in N-D is even more involved than 1D, and gives rise to interesting bunched sampling patterns in N-dimensions as shown in Fig. 4. The statement of our main result for N-D is as follows:

Theorem 1. *Let $x(\mathbf{n})$, $\mathbf{n} \in \mathbb{Z}^N$ be an N-dimensional periodic signal whose period $\mathbf{P}$ is one of the two integer matrices in the set $\mathbb{P} = \{\mathbf{P}_1, \mathbf{P}_2\}$, where $|det(\mathbf{P}_1)| \leq |det(\mathbf{P}_2)|$. Then, the following number of samples is both necessary and sufficient to identify $\mathbf{P}$:*

$$N_{min} = \begin{cases} |det(\mathbf{P}_2)| & \text{if } \mathbf{P}_1 \text{ is a left divisor of } \mathbf{P}_2 \\ |det(\mathbf{P}_1)| & \text{otherwise.} \end{cases} \tag{3}$$

◊

The notation used in Theorem 1 will be discussed shortly. Before doing so, we note that necessity in Theorem 1 implies that there exist signals with periods $\mathbf{P}_1$ and $\mathbf{P}_2$ for which there is absolutely no way of identifying the true period if the number of samples is smaller than N_{min} in (3). The sufficiency part is shown using a new period estimation method that can provably estimate the period using N_{min} number of samples. A new non-uniform bunched sampling pattern emerges as the minimal set of samples needed to identify the true N-D period.

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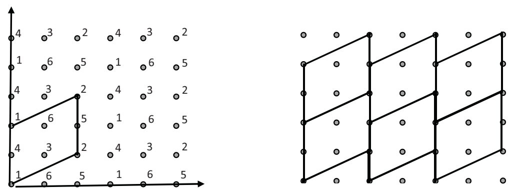

(a) (b)

Fig. 1. Part (a) - A two dimensional periodic signal according to the definition in (4), whose period is represented by the matrix in (5). The grid of dots is the set $\mathbb{Z}^2$. The numbers shown indicate the value of the signal at those integer points. The parallelogram shown is the FPD. Part (b) - A convenient way to show the shape of the periodicity for such periodic signals. Note that the FPD, when tiled along its edges, generates the whole signal.

Paper Outline: Important concepts summarizing multi-dimensional periodicity on a discrete grid are discussed in Sec. II. Sec. III presents a proof-by-construction for the sufficiency aspect of Theorem 1. The necessity aspect of Theorem 1 is proved in Sec. IV.

Notations: Vectors and matrices are indicated using bold lower case and bold upper case fonts respectively (e.g., vector $\mathbf{n}$, matrix $\mathbf{A}$). The determinant of a matrix $\mathbf{A}$ is denoted as $\det(\mathbf{A})$. Sets are indicated using blackboard font (e.g., set of all integers is $\mathbb{Z}$).

II. Periodicity in N-Dimensions: Overview

Periodicity on a multi-dimensional discrete grid can be defined in the following way [2], [11], [14]. A signal $x(\mathbf{n})$, $\mathbf{n} \in \mathbb{Z}^N$ is said to be periodic if there exists a non-singular integer matrix $\mathbf{P} \in \mathbb{Z}^{N\times N}$ such that:

$$x(\mathbf{n}+\mathbf{P}\mathbf{r}) = x(\mathbf{n}) \; \forall \; \mathbf{n},\mathbf{r} \in \mathbb{Z}^N \tag{4}$$

Such a $\mathbf{P}$ is called a repetition matrix of $x(\mathbf{n})$. The parallelepiped whose edges are represented by the column vectors of $\mathbf{P}$ is known as a repetition region, since this parallelepiped when tiled periodically along the directions represented by the columns of $\mathbf{P}$, generates $x(\mathbf{n})$. Such a parallelepiped is called the Fundamental Parallelepiped of $\mathbf{P}$, denoted as $\mathrm{FPD}(\mathbf{P})$. The number of integer vectors inside $\mathrm{FPD}(\mathbf{P})$ equals $|\det(\mathbf{P})|$. If $\mathbf{P}$ is a repetition matrix with a determinant that has the smallest absolute value among all possible repetition matrices for $x(\mathbf{n})$, then such a $\mathbf{P}$ is known as a period of $x(\mathbf{n})$. Fig. 1(a) shows an example of a two dimensional periodic signal with the following period:

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \tag{5}$$

For simplicity, we will indicate periodic signals such as the one in Fig. 1(a) by plots similar to Fig. 1(b). (The horizontal direction in Fig. 1(a) and (b) represents the first coordinate.)

It is worth noting that unlike in 1D, the period of an N-D periodic signal is not unique. For any $\mathbf{P}$ in (4), $\mathbf{PU}$ also satisfies (4) for any integer matrix $\mathbf{U}$.

$$x(\mathbf{n}+\mathbf{PU}\mathbf{r}) = x(\mathbf{n}) \; \forall \; \mathbf{n},\mathbf{r} \in \mathbb{Z}^N \tag{6}$$

In particular, if $\mathbf{U}$ is a unimodular[2] integer matrix, then $|\det(\mathbf{P})| = |\det(\mathbf{PU}|)$, so that if $\mathbf{P}$ is a period, so is $\mathbf{PU}$. We will refer to such periods as "equivalent periods of $\mathbf{P}$" in this paper[3].

Before we proceed further, it is important to define the following notions of divisibility among integer matrices [14]:

1) $\mathbf{D}$ is a left divisor of $\mathbf{P}$ if $\mathbf{P} = \mathbf{DK}$ for some integer matrix $\mathbf{K}$. $\mathbf{D}$ being a left divisor of $\mathbf{P}$ is denoted as $\mathbf{D}|\mathbf{P}$. If $\mathbf{D}$ is not a left divisor of $\mathbf{P}$, we denote it as $\mathbf{D} \nmid \mathbf{P}$
2) $\mathbf{D}$ is a left common divisor (LCD) of $\mathbf{P}_1$ and $\mathbf{P}_2$ if $\mathbf{P}_1 = \mathbf{DK}_1$ and $\mathbf{P}_2 = \mathbf{DK}_2$ for some integer matrices $\mathbf{K}_1$ and $\mathbf{K}_2$.
3) $\mathbf{G}$ is a greatest left common divisor (GLCD) of $\mathbf{P}_1$ and $\mathbf{P}_2$ if, for every LCD $\mathbf{D}$ of $\mathbf{P}_1$ and $\mathbf{P}_2$, $\mathbf{G} = \mathbf{DK}$ for some integer matrix $\mathbf{K}$. We will denote GLCD of $\mathbf{P_1}$ and $\mathbf{P_2}$ as $(\mathbf{P_1}, \mathbf{P_2})$.

We will now proceed to proving Theorem 1.

III. Sufficiency Proof

Let $\mathbf{G} = (\mathbf{P_1}, \mathbf{P_2})$. We will address the sufficiency proof of Theorem 1 in three cases: (A) when $\mathbf{P}_1|\mathbf{P}_2$; (B) when $\mathbf{P}_1 \nmid \mathbf{P}_2$, but $\mathbf{G} = \mathbf{I}_{N\times N}$ (Identity matrix), i.e, $\mathbf{P_1}$ and $\mathbf{P_2}$ are coprime; (C) and more generally when $\mathbf{P}_1 \nmid \mathbf{P}_2$, but $\mathbf{G}$ is not necessarily the identity matrix.

A. Sufficiency when $\mathbf{P}_1|\mathbf{P}_2$

The fact that $|\det(\mathbf{P_2})|$ samples are sufficient is relatively easy to derive in this case. We first note that when $\mathbf{P}_1|\mathbf{P}_2$, there exist equivalent periods of $\mathbf{P_1}$ and $\mathbf{P_2}$, $\mathbf{P_1'} = \mathbf{P_1 U}$ and $\mathbf{P_2'} = \mathbf{P_2 U'}$ for unimodular matrices $\mathbf{U}, \mathbf{U'}$, such that $\mathbf{P_2'} = \mathbf{P_1'}\mathbf{\Lambda}$ for some integer diagonal matrix $\mathbf{\Lambda}$.

To see this, let $\mathbf{P_2} = \mathbf{P_1 R}$ for some integer matrix $\mathbf{R}$. We can write $\mathbf{R}$ as $\mathbf{R} = \mathbf{U\Lambda V}$ using its Smith-form decomposition [10], where $\mathbf{U}$ and $\mathbf{V}$ are unimodular matrices and $\mathbf{\Lambda}$ is a diagonal integer matrix. So we can write:

$$\mathbf{P_2} = \mathbf{P_1 R} = \mathbf{P_1 U \Lambda V} \tag{7}$$

which leads to

$$\mathbf{P_2 V^{-1}} = \mathbf{P_1 U \Lambda} \tag{8}$$

So $\mathbf{P_2'} = \mathbf{P_2 V^{-1}}$ and $\mathbf{P_1'} = \mathbf{P_1 U}$ are the equivalent periods of $\mathbf{P_2}$ and $\mathbf{P_1}$ respectively such that $\mathbf{P_2'} = \mathbf{P_1'}\mathbf{\Lambda}$.

Notice that since $\mathbf{\Lambda}$ is integer valued and diagonal, by periodically tiling the parallelepiped $\mathrm{FPD}(\mathbf{P_1'})$, we can obtain the parallelepiped $\mathrm{FPD}(\mathbf{P_2'})$. So to check if the period of $x(\mathbf{n})$ is $\mathbf{P_1}$ or $\mathbf{P_2}$, all we need to do is check if the values of $x(\mathbf{n})$ on the tiles of $\mathrm{FPD}(\mathbf{P_1'})$ inside $\mathrm{FPD}(\mathbf{P_2'})$ are identical to each other (in which case the period of $x(\mathbf{n})$ is $\mathbf{P_1}$). Alternatively, if there is at least one tile of $\mathrm{FPD}(\mathbf{P_1'})$ that is different from the rest, the period of $x(\mathbf{n})$ will be $\mathbf{P_2}$. The samples in $\mathrm{FPD}(\mathbf{P_2'})$ are sufficient to check this, which are $|\det(\mathbf{P}_2)|$ in number.

B. Sufficiency when $\mathbf{P}_1$ and $\mathbf{P}_2$ are coprime

We will first prove the following result:

Theorem 2. *Suppose the period of $x(\mathbf{n})$ is either $\mathbf{P_1}$ or $\mathbf{P_2}$, $\mathbf{P_1} \nmid \mathbf{P_2}$, and $(\mathbf{P_1}, \mathbf{P_2}) = \mathbf{I}$. Consider a downsampled signal $y(\mathbf{n}) = x(\mathbf{P}_2\mathbf{n})$. Then, $y(\mathbf{n})$ will be a constant valued signal if and only if the period of $x(\mathbf{n})$ is $\mathbf{P}_2$.* ◇

[2]A unimodular matrix is a matrix with determinant ± 1. So its inverse will also be an integer matrix.

[3]One may wonder if defining the period in N-D as a parallelepiped is general enough to capture all possible periodicities in N-D. We refer the interested reader to [11] for an interesting analysis in this regard.

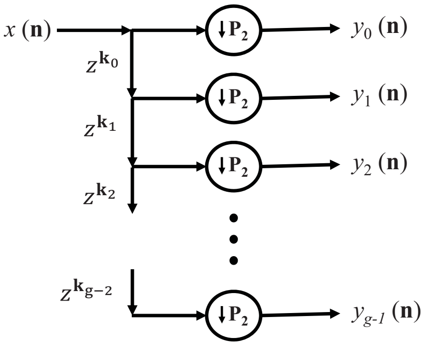

Fig. 2. Finding the period of $x(\mathbf{n})$ when $\mathbf{P_1} \nmid \mathbf{P_2}$, and $(\mathbf{P_1}, \mathbf{P_2}) = \mathbf{G}$. $|\det(\mathbf{G})|$ is denoted as g. See Sec. III-C for details.

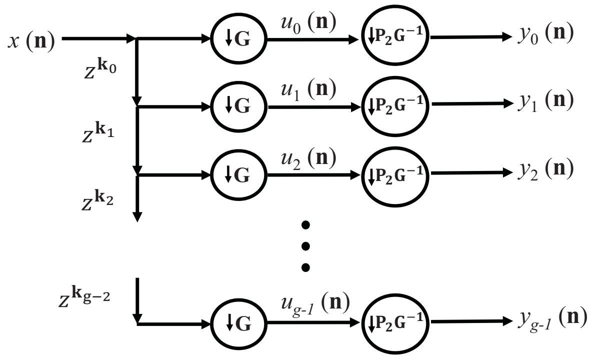

Fig. 3. Re-drawing Fig. 2 for analysis. See Sec. III-C for details.

Proof: **Case (I)**: When the period of $x(\mathbf{n})$ is $\mathbf{P}_2$,

$$y(\mathbf{n}) = x(\mathbf{P_2 n}) = x(\mathbf{P_2 n} + \mathbf{0}) = x(\mathbf{0}) \tag{9}$$

The second equality follows from (4). Hence, $y(\mathbf{n})$ has a constant value across all $\mathbf{n}$.

Case (II): When the period of $x(\mathbf{n})$ is $\mathbf{P}_1$, downsampling by a coprime $\mathbf{P}_2$ is in fact a lossless operation. For this, we will show that for every $\mathbf{n} \in \mathbb{Z}^N$, there exists an $\mathbf{n}' \in \mathbb{Z}^N$ such that $x(\mathbf{n}) = y(\mathbf{n}')$.

Since $(\mathbf{P}_1, \mathbf{P}_2) = \mathbf{I}_{N\times N}$, Bezout's identity [14] tells us that there exist matrices $\mathbf{A}, \mathbf{B}$ such that:

$$\mathbf{P_1 A} + \mathbf{P_2 B} = \mathbf{I} \tag{10}$$

multiplying both sides by $\mathbf{n}$, we have:

$$\mathbf{P_1}(\mathbf{An}) + \mathbf{P_2}(\mathbf{Bn}) = \mathbf{n} \tag{11}$$

Let us now denote $\mathbf{Bn}$ as $\mathbf{n}'$. So we have:

$$x(\mathbf{n}) = x(\mathbf{P_1}(\mathbf{An}) + \mathbf{P_2 n}') = x(\mathbf{P_2 n}') = y(\mathbf{n}') \tag{12}$$

That is, downsampling by $\mathbf{P}_2$ is a lossless operation, and only periodically rearranges the samples of $x(\mathbf{n})$. In this subsection, we assumed that $\mathbf{P}_1 \nmid \mathbf{P}_2$, which in particular means that $\mathbf{P}_1 \neq \mathbf{I}$. Hence, when $x(\mathbf{n})$ has period $\mathbf{P}_1$, in particular it cannot be a constant signal, and so $y(\mathbf{n})$ cannot be a constant signal either. This completes the proof of the theorem. ∎

In the proof of Theorem 2, how many samples do we need to check if $y(\mathbf{n})$ is a constant? The answer is, at most $|\det(\mathbf{P_1})|$, as it suffices to check the value of y on the following set:

$$\{\mathbf{Bn} \;:\; \mathbf{n} \in \text{FPD}(\mathbf{P_1})\} \tag{13}$$

This is so, since if the period of $x(\mathbf{n})$ is $\mathbf{P}_1$, then $x(\mathbf{n})$ cannot be a constant on $\text{FPD}(\mathbf{P_1})$. And hence y cannot be a constant on the above set because of (12) where $\mathbf{n}' = \mathbf{Bn}$. The size of the above set is atmost $|\det(\mathbf{P_1})|$, which completes the sufficiency proof of Theorem 1 for the case of $\mathbf{P}_1$ and $\mathbf{P}_2$ being coprime.

C. When $\mathbf{P}_1 \nmid \mathbf{P}_2$ and $(\mathbf{P}_1, \mathbf{P}_2) = \mathbf{G}$

We propose a generalization of the single downsampling operation of Sec. III-B to the structure shown in Fig. 2. There are $|\det(\mathbf{G})|$ downsamplers, and each channel shifts the input by a vector $\mathbf{k_i}$ in $\text{FPD}(\mathbf{G})$ before downsampling by $\mathbf{P_2}$. That is, $y_i(\mathbf{n}) = x(\mathbf{P_2 n} + \mathbf{k_i})$ for each channel i. The following theorem can now be proved:

Theorem 3. *Suppose the period of $x(\mathbf{n})$ is either $\mathbf{P_1}$ or $\mathbf{P_2}$, $\mathbf{P_1} \nmid \mathbf{P_2}$, and $(\mathbf{P_1}, \mathbf{P_2}) = \mathbf{G}$. Then, for every channel i, $y_i(\mathbf{n})$ in Fig. 2 is a constant $\forall \mathbf{n}$ if and only if the period of $x(\mathbf{n})$ is $\mathbf{P}_2$. If $x(\mathbf{n})$'s period is $\mathbf{P}_1$, at least one of the $y_i(\mathbf{n})$'s will not be a constant signal.*

Proof: **Case (I):** When the period of $x(\mathbf{n})$ is $\mathbf{P_2}$, all shifted versions of $x(\mathbf{n})$ will also have period $\mathbf{P_2}$. Hence, similar to (9) in Theorem 2, each $y_i(\mathbf{n})$ is a constant signal.

Case (II): When the period of $x(\mathbf{n})$ is $\mathbf{P_1}$, it is useful to look at an equivalent representation of Fig. 2 as shown in Fig. 3. We will first show that at least one of the $u_i(\mathbf{n})$'s is not a constant.

To see this, let us assume the contrary. That is,

$$u_i(\mathbf{n}) = x(\mathbf{Gn} + \mathbf{k_i}) = \lambda_i \;\; \forall \mathbf{n} \tag{14}$$

for each channel i. Now, given any $\mathbf{n} \in \mathbb{Z}^N$, there exist unique $\mathbf{n}' \in \mathbb{Z}^N$ and $\mathbf{k_j} \in \text{FPD}(\mathbf{G})$ such that:

$$\mathbf{n} = \mathbf{Gn}' + \mathbf{k_j} \tag{15}$$

This follows from a generalization of the division theorem of integers to $N-$D (see Sec. 12.4.2 in [14]). Combining (14) and (15), we can write:

$$x(\mathbf{n}) = x(\mathbf{Gn}' + \mathbf{k_j}) = \lambda_j \tag{16}$$

and for any $\mathbf{r} \in \mathbb{Z}^N$:

$$x(\mathbf{n} + \mathbf{Gr}) = x(\mathbf{G}(\mathbf{n}' + \mathbf{r}) + \mathbf{k_j}) = \lambda_j \tag{17}$$

Hence, we have:

$$x(\mathbf{n} + \mathbf{Gr}) = x(\mathbf{n}) \tag{18}$$

Since $\mathbf{n}$ and $\mathbf{r}$ were arbitrarily chosen, the above equation holds $\forall \mathbf{n}, \mathbf{r} \in \mathbb{Z}^N$. And so $\mathbf{G}$ is also a repetition matrix of $x_{\mathbf{P_1}}(\mathbf{n})$. However, note that:

$$\mathbf{P_1} = \mathbf{GR_1}, \quad \mathbf{P_2} = \mathbf{GR_2} \tag{19}$$

for some integer matrices $\mathbf{R_1}, \mathbf{R_2}$ since $\mathbf{G} = (\mathbf{P_1}, \mathbf{P_2})$. So in particular, $|\det(\mathbf{G})| \leq |\det(\mathbf{P_1})|$. But since $\mathbf{P_1}$ is the period of the input here, it becomes necessary that $|\det(\mathbf{R_1})| = 1$. That is:

$$\mathbf{PU} = \mathbf{G} \tag{20}$$

where $\mathbf{U}$ is the (integer valued) inverse of the unimodular $\mathbf{R_1}$. Substituting (20) in (19) shows that:

$$\mathbf{P_2} = \mathbf{G}\mathbf{R_2} = \mathbf{P_1}(\mathbf{U}\mathbf{R_2}) \tag{21}$$

That is, $\mathbf{P_1}|\mathbf{P_2}$, which is contrary to our main assumption in this subsection. Hence, (14) cannot be true for all channels i simultaneously.

In the following two paragraphs, we will show that if $u_i(\mathbf{n})$ is not a constant signal, then the corresponding $y_i(\mathbf{n})$ is also not a constant signal. To see this, let us assume that the period of $u_i(\mathbf{n})$ is $\mathbf{D}$. We will first show that $\mathbf{D}|\mathbf{G^{-1}P_1}$. Note that:

$$u_i(\mathbf{n} + \mathbf{G^{-1}P_1 r}) = x(\mathbf{Gn} + \mathbf{P_1 r} + \mathbf{k_i}) = x(\mathbf{Gn} + \mathbf{k_i}) = u_i(\mathbf{n})$$

Hence, $\mathbf{G^{-1}P_1}$ is a repetition matrix of u_i. So its period $\mathbf{D}$ must necessarily be a left divisor of $\mathbf{G^{-1}P_1}$, i.e, $\mathbf{G^{-1}P_1} = \mathbf{DH}$ for some integer matrix $\mathbf{H}$ (see Theorem 6 in the Appendix).

Now, using the extension of Euclid's theorem to N-D (see Lemma 13.5.1 in [14]), we have:

$$\mathbf{P_1 A} + \mathbf{P_2 B} = \mathbf{G} \tag{22}$$

for some integer matrices $\mathbf{A}$ and $\mathbf{B}$. We can re-write this as:

$$\mathbf{G^{-1}P_1 A} + \mathbf{G^{-1}P_2 B} = \mathbf{I} \tag{23}$$

Substituting $\mathbf{G^{-1}P_1} = \mathbf{DH}$, we have:

$$\mathbf{D}(\mathbf{HA}) + \mathbf{G^{-1}P_2 B} = \mathbf{I} \tag{24}$$

Using Bezout's identity [14], we can conclude that $\mathbf{D}$ and $\mathbf{G^{-1}P_2}$ must be coprime. In other words, the period of u_i is coprime to the dowsampling index $\mathbf{G^{-1}P_2}$ in Fig. 3. Using Theorem 2, we can then conclude that y_i cannot be a constant signal. This completes the proof of the current theorem. ■

The above result shows that the period of $x(\mathbf{n})$ can be estimated by checking if the outputs y_i are constant signals. Using the same arguments as in Sec. III-B (Eq. 13), we need at most $|\det(\mathbf{D})| \leq |\det(\mathbf{G^{-1}P_1})|$ samples of y_i for checking if each y_i is a constant signal. These are given by the points:

$$\{\mathbf{Bn} \;:\; \mathbf{n} \in \mathrm{FPD}(\mathbf{G^{-1}P_1})\} \tag{25}$$

Since there are $|\det(\mathbf{G})|$ channels, we need at most $|\det(\mathbf{P_1})|$ samples in total. The corresponding samples of $x(\mathbf{n})$ that are required to be checked occur in an interesting bunched pattern as shown in Fig. 4. Please see Fig. 4 for more details. This completes the sufficiency proof of Theorem 1.

IV. Necessity Proof

Motivated by the approach for the one dimensional setting in [13], we will first show that $|\det(\mathbf{P_1})|$ samples are necessary to find the period from the set $\{\mathbf{P_1}, \mathbf{P_2}\}$, irrespective of whether $\mathbf{P_1}|\mathbf{P_2}$ or $\mathbf{P_1} \nmid \mathbf{P_2}$. Later, we will show that when $\mathbf{P_1}|\mathbf{P_2}$, $|\det(\mathbf{P_2})|$ samples are necessary.

Theorem 4. *Given any set of $L < |\det(\mathbf{P_1})|$ integer vectors $\mathbb{N}_T = \{\mathbf{n_1}, \mathbf{n_2}, \ldots, \mathbf{n_L}\} \subset \mathbb{Z}^N$, and $|det(\mathbf{P}_1)| \leq |det(\mathbf{P}_2)|$, there exist periodic signals $x_{\mathbf{P_1}}(\mathbf{n})$ and $x_{\mathbf{P_2}}(\mathbf{n})$ with periods $\mathbf{P_1}$ and $\mathbf{P_2}$ respectively such that*

$$x_{\mathbf{P_1}}(\mathbf{n}) = x_{\mathbf{P_2}}(\mathbf{n}) \;\forall\; \mathbf{n} \in \mathbb{N}_T \tag{26}$$

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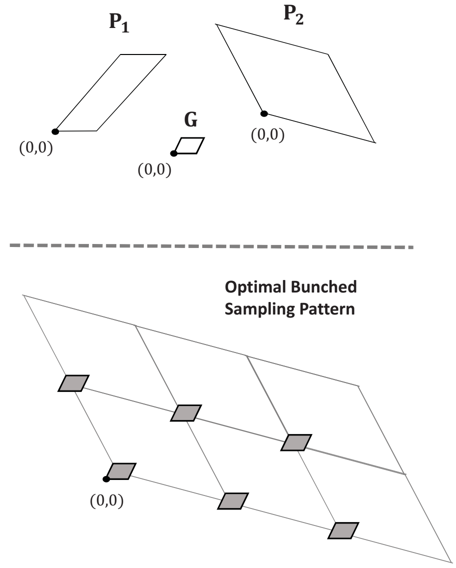

Fig. 4. A pictorial illustration of Theorem 3 in two dimensions: (Top) Let the parallelograms corresponding to the potential periods $\mathbf{P}_1$ and $\mathbf{P}_2$ in Theorem 3 be as shown on the top part of the figure. While we consider discrete time periodic signals in this work, for convenience, we have omitted the 2-D integer grid shown in Fig 1 in these figures. The origin of the 2-D plane is shown for reference for each parallelogram. Let the parallelogram corresponding to $\mathbf{G}$ also be as shown. (Bottom) The 2-D integer points inside the dark gray regions in the image on the bottom are the locations of the samples that are used in the proof of Theorem 3. Each gray region has shape same as $\mathrm{FPD}(\mathbf{G})$. There are $\det(\mathbf{P}_1)/\det(\mathbf{G})$ such gray regions, spaced apart on a grid generated by the columns of $\mathbf{P}_2$. This sampling pattern provably yields the absolute minimum number of samples required for estimating the period.

Proof: For every $\mathbf{n} \in \mathbb{Z}^N$, there exist unique $\mathbf{n'} \in \mathbb{Z}^N$ and $\mathbf{k} \in \mathrm{FPD}(\mathbf{P_1})$ such that:

$$\mathbf{n} = \mathbf{P_1 n'} + \mathbf{k} \tag{27}$$

(See Sec. 12.4.2 in [14].) Notice that the value of any signal with period $\mathbf{P_1}$ will be equal on both $\mathbf{n}$ and $\mathbf{k}$. Now, let us map each $\mathbf{n_i} \in \mathbb{N}_T$ to its corresponding point $\mathbf{k_i}$ in $\mathrm{FPD}(\mathbf{P_1})$ according to (27). Let us denote the set of such $\mathbf{k_i}$ as $\mathbb{K}_{\mathbf{P_1}}$, i.e,

$$\mathbb{K}_{\mathbf{P_1}} = \{\mathbf{k} \in \mathrm{FPD}(\mathbf{P_1}) \;:\; \exists \mathbf{n} \in \mathbb{N}_T \; s.t. \; \mathbf{n} = \mathbf{P_1 n'} + \mathbf{k}\}$$

Since $L < |\det(\mathbf{P_1})|$, there exists at least one integer vector in $\mathrm{FPD}(\mathbf{P}_1)$ that does not belong to the set $\mathbb{K}_{\mathbf{P_1}}$. Let $\mathbf{m}$ be such a point. We will now define a period $\mathbf{P}_1$ signal $x_{\mathbf{P_1}}(\mathbf{n})$ by specifying its values on $\mathrm{FPD}(\mathbf{P_1})$ as follows:

$$x_{\mathbf{P_1}}(\mathbf{n}) = \begin{cases} 0 & \text{if } \mathbf{n} = \mathbf{m} \\ 1 & \text{otherwise} \end{cases} \tag{28}$$

The values of $x_{\mathbf{P_1}}(\mathbf{n})$ at other points in space are generated by periodically tiling its values on $\mathrm{FPD}(\mathbf{P_1})$. Notice that $x_{\mathbf{P_1}}(\mathbf{n}) = 1 \;\forall\; \mathbf{n} \in \mathbb{N}_T$. In the same way, we can construct a period $\mathbf{P_2}$ signal $x_{\mathbf{P_2}}(\mathbf{n})$ that satisfies $x_{\mathbf{P_2}}(\mathbf{n}) = 1 \;\forall\; \mathbf{n} \in \mathbb{N}_T$. Clearly, for these $x_{\mathbf{P_1}}(\mathbf{n})$ and $x_{\mathbf{P_2}}(\mathbf{n})$, (26) is satisfied. ■

Theorem 4 shows that given any such set of $L < |\det(\mathbf{P_1})|$ integer vectors, there exist signals with periods $\mathbf{P_1}, \mathbf{P_2}$, whose true period cannot be identified. We will now argue that when $\mathbf{P_1}|\mathbf{P_2}$, one needs at least $|\det(\mathbf{P_2})|$ samples to estimate the period.

Theorem 5. *Let $\mathbf{P_1}|\mathbf{P_2}$. Given any set of $L < |\det(\mathbf{P_2})|$ integer vectors $\mathbb{N}_T = \{\mathbf{n_1}, \mathbf{n_2}, \ldots, \mathbf{n_L}\} \subset \mathbb{Z}^N$, and any period $\mathbf{P_1}$ signal $x_{\mathbf{P_1}}(\mathbf{n})$, there exists a period $\mathbf{P_2}$ signal $x_{\mathbf{P_2}}(\mathbf{n})$ such that:*

$$x_{\mathbf{P_1}}(\mathbf{n}) = x_{\mathbf{P_2}}(\mathbf{n}) \; \forall \; \mathbf{n} \in \mathbb{N}_T \tag{29}$$

◊

Proof: We will essentially construct an $x_{\mathbf{P_2}}(\mathbf{n})$ satisfying the conditions of the theorem. Let us first specify the values of $x_{\mathbf{P_2}}(\mathbf{n})$ on $\text{FPD}(\mathbf{P_2})$. To do so, we first define the set $\mathbb{K}_{\mathbf{P_2}}$ in a similar fashion as in the proof of Theorem 4 as follows:

$$\mathbb{K}_{\mathbf{P_2}} = \{\mathbf{k} \in \text{FPD}(\mathbf{P_2}) \; : \; \exists \mathbf{n} \in \mathbb{N}_T \; s.t. \; \mathbf{n} = \mathbf{P_2}\mathbf{n}' + \mathbf{k}\}$$

Since $L < |\det(\mathbf{P_2})|$, there exists at least one integer vector $\mathbf{m} \in \text{FPD}(\mathbf{P_2})$ that does not belong to $\mathbb{K}_{\mathbf{P_2}}$. Let u and v be real numbers such that $u > \max_{\mathbf{n}} x_{\mathbf{P_1}}(\mathbf{n})$ and $v < \min_{\mathbf{n}} x_{\mathbf{P_1}}(\mathbf{n})$. We can now define $x_{\mathbf{P_2}}(\mathbf{n})$ by specifying its values on $\text{FPD}(\mathbf{P_2})$ as follows:

$$x_{\mathbf{P_2}}(\mathbf{n}) = \begin{cases} x_{\mathbf{P_1}}(\mathbf{n}) & \text{if } \mathbf{n} \in \mathbb{K}_{\mathbf{P_2}} \\ u & \mathbf{n} = \mathbf{m} \\ v & \text{otherwise} \end{cases} \tag{30}$$

It is easy to see that $x_{\mathbf{P_2}}(\mathbf{n})$, when generated by periodically tiling the above values along the edges of $\text{FPD}(\mathbf{P_2})$, is a period $\mathbf{P_2}$ signal (u occurs only once every tile of $\text{FPD}(\mathbf{P_2})$).

It remains to be shown that $x_{\mathbf{P_2}}(\mathbf{n}) = x_{\mathbf{P_1}}(\mathbf{n})$ for all points in $\mathbb{N}_T$. Let $\mathbf{n} \in \mathbb{N}_T$. We can decompose $\mathbf{n}$ as:

$$\mathbf{n} = \mathbf{P_2}\mathbf{n}' + \mathbf{k} \tag{31}$$

for some $\mathbf{k} \in \mathbb{K}_{\mathbf{P_2}}$. From (30), we have:

$$x_{\mathbf{P_2}}(\mathbf{k}) = x_{\mathbf{P_1}}(\mathbf{k}) \tag{32}$$

Substituting (31), we get:

$$x_{\mathbf{P_2}}(\mathbf{n} - \mathbf{P_2}\mathbf{n}') = x_{\mathbf{P_1}}(\mathbf{n} - \mathbf{P_2}\mathbf{n}') \tag{33}$$

The left hand side is just $x_{\mathbf{P_2}}(\mathbf{n})$ since $x_{\mathbf{P_2}}$ has period $\mathbf{P_2}$. Substituting $\mathbf{P_2} = \mathbf{P_1}\mathbf{R}$ into the right hand side, since $\mathbf{P_1}|\mathbf{P_2}$, the right hand side reduces to $x_{\mathbf{P_1}}(\mathbf{n})$. Hence, we have:

$$x_{\mathbf{P_2}}(\mathbf{n}) = x_{\mathbf{P_1}}(\mathbf{n}) \; \forall \mathbf{n} \in \mathbb{N}_T \tag{34}$$

This completes the proof. ■

Theorem 5 shows that when $\mathbf{P_1}|\mathbf{P_2}$, given any set of $L < |\det(\mathbf{P_2})|$ integer vectors, there exist signals with periods $\mathbf{P_1}, \mathbf{P_2}$, whose true period cannot be identified. So $|\det(\mathbf{P_2})|$ samples are necessary in this case. This concludes the proof of Theorem 1.

V. Concluding Remarks

Precise bounds on the least number of possibly non-contiguous samples required to distinguish between two multi-dimensional periodic patters were derived in this paper. Although the scope Theorem 1 is restricted to resolving between two periodic patterns, it could pave the way for a generalization to larger sets of periods. This is motivated by the fact that the 1-D result in Eq. (1) for contiguous samples was also derived in [12] starting from distinguishing between pairs of periods. Such a generalization of Theorem 1 will be a part of our future research efforts. It would also be interesting to see if the sparse and minimal sampling patterns resulting from Theorem 1 can lead to fast algorithms for period estimation in N-dimesnions. This will also be a promising direction to pursue in the future.

References

[1] C. Caratheodory, L. Fejer, "Uber den Zusammenghang der Extemen von harmonischen Funktionen mit ihren Koeffizienten und uber den Picard-Landauschen Satz," Rendiconti del Circolo Matematico di Palermo, vol. 32, pp. 218-239, 1911.

[2] D. E. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing*, Englewood Cliffs, NJ, USA: Prentice Hall, 1984.

[3] M. C. Escher, *Escher on Escher: Exploring the Infinite*, New York, NY, USA: Harry N. Abrams, 1989.

[4] C. Hammond, *The Basics of Crystallography and Diffraction*, New York, NY, USA: Oxford Univ. Press, 2009.

[5] D. H. Johnson, C. R. Johnson, A. G. Klein, W. A. Sethares, H. Lee and E. Hendriks, "A thread counting algorithm for art forensics," 2009 IEEE 13th Digital Signal Processing Workshop and 5th IEEE Signal Processing Education Workshop, Marco Island, FL, 2009, pp. 679-684.

[6] C. C. MacDuffee, *The Theory of Matrices*, New York, NY, USA: Chelsea, 1946.

[7] S-C. Pei and K-W. Chan, "Two-Dimensional Period Estimation by Ramanujan's Sum", IEEE Trans. on Sig. Proc., vol. 65, no. 19, pp. 5108 - 5120, 2017.

[8] B. Rupp, *Biomolecular Crystallography: Principles, Practice, and Application to Structural Biology*, New York, NY, USA: Garland Sci ence, 2010.

[9] N. D. Sidiropoulos, "Generalizing Caratheodory's uniqueness of harmonic parameterization to N dimensions," in IEEE Transactions on Information Theory, vol. 47, No. 4, pp. 1687-1690, May 2001.

[10] H. J. S. Smith, "On systems of linear indeterminate equations and congruences," Philos. Trans. Royal Soc. London, vol. 151, pp. 293326, 1861.

[11] S. V. Tenneti and P. P. Vaidyanathan, "Arbitrarily Shaped Periods in Multidimensional Discrete Time Periodicity", IEEE Sig. Proc. Letters, vol. 22, no. 10, pp. 1748 - 1751, 2015.

[12] S. V. Tenneti and P. P. Vaidyanathan, "Minimum Data Length for Integer Period Estimation," IEEE Trans. on Sig. Proc., vol. 66, no. 10, pp. 2733-2745, 2018.

[13] S. V. Tenneti and P. P. Vaidyanthan, "Minimum Number of Possibly Non-Contiguous Samples to Distinguish Two Periods", Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing, New Orleans, March 2017.

[14] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, Englewood Cliffs, NJ, USA, 1993.

Appendix

Theorem 6. *Let $\mathbf{P}$ be a period of $x(\mathbf{n})$. For any repetition matrix $\mathbf{Q}$ of $x(\mathbf{n})$, $\mathbf{P}$ must be a left divisor of $\mathbf{Q}$.* ◊

Proof: Let $\mathbf{G} = (\mathbf{P}, \mathbf{Q})$. Using the N-D extension of Euclid's theorem (see Lemma 13.5.1 in [14]), there exist matrices $\mathbf{A}$ and $\mathbf{B}$ such that:

$$\mathbf{PA} + \mathbf{QB} = \mathbf{G} \tag{35}$$

Since $\mathbf{Q}$ is a repetition matrix of x, so will $\mathbf{QB}$ be,

$$x(\mathbf{n} + \mathbf{QBr}) = x(\mathbf{n}) \; \forall \; \mathbf{n}, \mathbf{r} \in \mathbb{Z}^N \tag{36}$$

we can substitute (35) into (36) to get:

$$x(\mathbf{n} + \mathbf{Gr} - \mathbf{PAr}) = x(\mathbf{n}) \; \forall \; \mathbf{n}, \mathbf{r} \in \mathbb{Z}^N \tag{37}$$

But since $\mathbf{P}$ is a period, $\mathbf{PA}$ will also be a repetition matrix so that (38) is equivalent to:

$$x(\mathbf{n} + \mathbf{Gr}) = x(\mathbf{n}) \; \forall \; \mathbf{n}, \mathbf{r} \in \mathbb{Z}^N \tag{38}$$

i.e., $\mathbf{G}$ is also a repetition matrix. Using arguments identical to Eq. (19) to Eq. (21), with $\mathbf{P_1}, \mathbf{P_2}$ replaced by $\mathbf{P}, \mathbf{Q}$ respectively, we can then prove that $\mathbf{P}$ is a left divisor of $\mathbf{Q}$.

■