Optimizing Minimum Redundancy Arrays for Robustness

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***Abstract*—Sparse arrays have received considerable attention due to their capability of resolving $\mathcal{O}(N^2)$ uncorrelated sources with N physical sensors, unlike the uniform linear array (ULA) which identifies at most $N-1$ sources. This is because sparse arrays have an $\mathcal{O}(N^2)$-long ULA segment in the difference coarray, defined as the set of differences between sensor locations. Among the existing array configurations, minimum redundancy arrays (MRA) have the largest ULA segment in the difference coarray with no holes. However, in practice, ULA is robust, in the sense of coarray invariance to sensor failure, but MRA is not. This paper proposes a novel array geometry, named as the robust MRA (RMRA), that maximizes the size of the hole-free difference coarray subject to the same level of robustness as ULA. The RMRA can be found by solving an integer program, which is computationally expensive. Even so, it will be shown that the RMRA still owns $\mathcal{O}(N^2)$ elements in the hole-free difference coarray. In particular, for sufficiently large N, the aperture for RMRA, which is approximately half of the size of the difference coarray, is bounded between $0.0625N^2$ and $0.2174N^2$.[1]**

***Index Terms*—Sparse arrays, minimum redundancy arrays, difference coarray, robustness, fragility.**

I. INTRODUCTION

Sparse arrays are capable of resolving $\mathcal{O}(N^2)$ uncorrelated source directions, using N physical sensors, in contrast to the uniform linear array (ULA), which identifies at most $N-1$ sources [1]. This $\mathcal{O}(N^2)$ property is because the difference coarray, defined as the set of differences between sensor locations, possesses a central ULA segment of size $\mathcal{O}(N^2)$. These sparse arrays include minimum redundancy arrays (MRA) [2], nested arrays [3], and coprime arrays [4], to name a few [5]. For a fixed number of sensors, MRA has the largest hole-free difference coarray among the above-mentioned arrays. Due to this property, MRA typically has the best estimation performance.

The robustness of arrays to sensor failure is also a significant and practical issue in array processing, since sensor failures typically lead to degradation of the array performance [6], [7]. It was empirically known that the difference coarray of the MRA is susceptible to sensor failures [8]. In particular, any faulty sensor in the MRA shrinks its difference coarray [8], which affects the applicability of array processing algorithms, such as the spatial smoothing MUSIC [3]. This observation was explained quantitatively in [8] by using the concept of fragility of arrays. It was also demonstrated that the ULA is one of the most robust array geometries.

This paper proposes a new array geometry, named as the robust minimum redundancy array (RMRA), that is robust to sensor failures (as robust as ULA) while at the same time enjoying a large central ULA segment in the difference coarray (similar to MRA). The RMRA is defined as the solution to the integer program $(\mathrm{P1})$ given in Section III. Even though solving $(\mathrm{P1})$ is computationally expensive, it will be shown that the feasible region of $(\mathrm{P1})$ is not empty, the solutions are not unique, and the aperture of a N-sensor RMRA is bounded between $0.0625N^2$ and $0.2174N^2$. These bounds indicate that the size of the difference coarray of the RMRA is $\mathcal{O}(N^2)$, which is as good as the MRA. These new results will also be proved rigorously in this paper.

This paper is organized as follows. Section II reviews the MRA and the robustness of the difference coarray to sensor failures. Section III presents the RMRA and characterizes the size of the difference coarray of RMRA (Theorem 1). This theorem is proved rigorously in Section IV. Numerical examples are presented in Section V while Section VI concludes this paper.

II. PRELIMINARIES

Assume that the sensors in an array are located at $n\lambda/2$, where λ is the wavelength of the incoming monochromatic sources and n belongs to an integer-valued set $\mathbb{S}$. For uncorrelated source amplitudes and noise, the estimation of direction-of-arrival (DOA) of the sources, based on the sensor array $\mathbb{S}$, can be converted into the estimation of DOA on the difference coarray [3], [4]. The difference coarray of $\mathbb{S}$ is defined as

$$\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}. \tag{1}$$

In some DOA estimators such as spatial smoothing MUSIC [3], the estimation performance depends on the central ULA segment of the difference coarray, which is defined as $\mathbb{U} \triangleq \{0, \pm 1, \ldots, \pm m\}$. Here m is the largest integer such that $\{0, \pm 1, \ldots, \pm m\} \subseteq \mathbb{D}$.

Another quantity associated with the difference coarray is the weight function $w(m)$, defined as the number of sensor pairs with separation m. That is,

$$w(m) \triangleq \left|\left\{(n_1, n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\right\}\right|, \tag{2}$$

where $m \in \mathbb{D}$.

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The redundancy R of an array measures the size of the central ULA segment $\mathbb{U}$ with respect to the the number of sensors [2]:

$$R \triangleq \frac{\binom{|\mathbb{S}|}{2}}{(|\mathbb{U}|-1)/2}. \tag{3}$$

For a fixed number of sensors, the smaller R is, the larger the central ULA segment is, so more uncorrelated sources are resolvable [2]–[4]. Based on this concept, the MRA with N sensors is defined as[2]

$$(\text{P0}): \quad \mathbb{S}_{\text{MRA}} \triangleq \arg\min_{\mathbb{S}} R \qquad \text{subject to} \tag{4}$$

$$|\mathbb{S}| = N, \qquad \mathbb{D} = \mathbb{U}. \tag{5}$$

Here the constraint $\mathbb{D} = \mathbb{U}$ denotes that the difference coarray consists of consecutive integers. Namely, the difference coarray is *hole-free*, which is crucial for the applicability of DOA estimators such as spatial smoothing MUSIC [3], [4]. Note that in (P0), minimizing the redundancy is equivalent to maximizing the size of the difference coarray due to the constraints in (5).

If $\mathbb{S}_{\text{MRA}}$ is a solution to (P0), then its translated version $\{n + n_0 : n \in \mathbb{S}_{\text{MRA}}\}$ and its reversed version $\{-n : n \in \mathbb{S}_{\text{MRA}}\}$ are both solutions to (P0). Such ambiguity arises because the number of sensors $|\mathbb{S}|$ and the difference coarray are invariant under these operations. In this paper, unless specified, the leftmost element in an array is at the location 0 and for simplicity, the reversed version of an array will not be considered.

A notable property of MRA is the $\mathcal{O}(N^2)$ property. For sufficiently large N, the size of the difference coarray of MRA satisfies $0.5974N^2 \leq |\mathbb{D}_{\text{MRA}}| \leq 0.8217N^2$, or equivalently $|\mathbb{D}_{\text{MRA}}| = \mathcal{O}(N^2)$ [2], [9]. This $\mathcal{O}(N^2)$ property makes it possible to resolve more source directions than sensors using MRA [2], [10].

Next we will review the essentialness property [8]. This property characterizes the influence of faulty sensors on the difference coarray. It is defined as follows:

Definition 1. Assume that $n \in \mathbb{S}$. Then the sensor at n is essential with respect to $\mathbb{S}$ if the removal of n from $\mathbb{S}$ changes the difference coarray. Namely, $\overline{\mathbb{D}} \neq \mathbb{D}$, where $\overline{\mathbb{D}}$ and $\mathbb{D}$ are the difference coarrays of $\overline{\mathbb{S}} \triangleq \mathbb{S}\backslash\{n\}$ and $\mathbb{S}$, respectively.

We say that $n \in \mathbb{S}$ is *inessential* if n is not essential. An array $\mathbb{S}$ is *maximally economic* if all sensors in $\mathbb{S}$ are essential [11]. Based on Definition 1, the fragility F is defined as [8]:

$$F \triangleq \frac{\text{The number of essential sensors in } \mathbb{S}}{\text{The number of sensors in } \mathbb{S}}. \tag{6}$$

The fragility F can be used to quantify the robustness of the difference coarray to sensor failures [8]. As F increases, any sensor failure tends to modify the difference coarray, so we say that the array is not robust. In particular, for any array with $N \geq 4$, their fragility satisfies $2/N \leq F \leq 1$. The minimum fragility $F = 2/N$ is achieved for ULA while the fragility $F = 1$ corresponds to maximally economic sparse arrays [12], such as MRA [2], MHA [13], [14], and nested arrays [3].

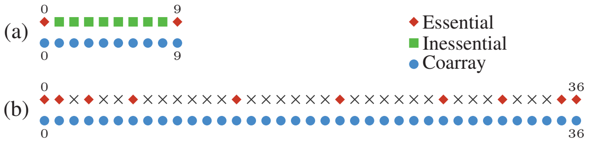

Fig. 1. The array geometries and the nonnegative parts of the difference coarrays for (a) ULA and (b) MRA. These arrays have 10 physical sensors.

Example 1. Fig. 1 shows the array geometries of ULA and MRA. These arrays have 10 physical sensors. The essential sensors and the inessential sensors are depicted by red diamonds and green squares, respectively. The nonnegative part of the difference coarray is illustrated in blue circles. First, both arrays have hole-free difference coarrays and the largest element in the difference coarray is 9 for ULA and 36 for MRA. This is since $|\mathbb{D}| = \mathcal{O}(N)$ for ULA while $|\mathbb{D}| = \mathcal{O}(N^2)$ for MRA. Second, the fragility F is $2/10$ for ULA and 1 for MRA, implying that MRA is much less robust than ULA, in the sense of coarray invariance to sensor failure.

III. MRA OPTIMIZED FOR ROBUSTNESS

As defined in Section II, the MRA has the largest hole-free difference coarray among all arrays with N sensors. However, the MRA with N sensors is not robust ($F = 1$) [8]. This attribute hinders the applicability of MRAs in the presence of sensor failures. On the contrary, the ULA is one of the most robust arrays, since it has minimum fragility $2/N$. However, the difference coarray of ULA has only $\mathcal{O}(N)$ elements (in particular, $|\mathbb{D}| = 2N - 1$). This property prevents ULA from identifying more sources than sensors [1] (assuming we use autocorrelation based methods).

Presented with these issues, in this part, we will propose a new array geometry that strikes a balance between the redundancy R (or equivalently the size of the central ULA segment of the difference coarray), and the fragility F, as defined in (6). This array is named as the *robust minimum redundancy arrays* (RMRA), formally defined as

$$(\text{P1}): \quad \mathbb{S}_{\text{RMRA}} \triangleq \arg\min_{\mathbb{S}} R \qquad \text{subject to} \tag{7}$$

$$|\mathbb{S}| = N, \qquad \mathbb{D} = \mathbb{U}, \tag{8}$$

$$F = \frac{2}{N}, \qquad N \geq 4. \tag{9}$$

The physical meaning of the problem (P1) is as follows. Eqs. (7) and (8) show that the redundancy is minimized subject to the hole-free difference coarray. The new array $\mathbb{S}_{\text{RMRA}}$ is as robust as ULA ($F = 2/N$), as in (9). Furthermore, in (P1), minimizing the redundancy is equivalent to maximizing the size of the difference coarray.

The constraint $N \geq 4$ is to guarantee a nonempty feasible region of (P1). For any arrays with $N = 1$ or 3, it can be shown that $F = 1 \neq 2/N$ [12]. For $N \geq 4$, the ULA ($\mathbb{S}_{\text{ULA}} = \{0, 1, \ldots, N-1\}$) satisfies $|\mathbb{S}_{\text{ULA}}| = N$, $\mathbb{D}_{\text{ULA}} = \mathbb{U}_{\text{ULA}}$, and $F = 2/N$, so the feasible region of (P1) is not empty.

The integer program (P1) requires a combinatorial search over the feasible region. This task becomes computationally

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[2] In [2], this array is called the restricted MRA.

TABLE I
ARRAY CONFIGURATIONS OF RMRA

N	Array Configuration
4	#1: ◆ 🟩 🟩 ◆ ◆ Essential 🟩 Inessential 0 3
5	#1: ◆ 🟩 🟩 🟩 ◆ 0 4
6	#1: ◆ 🟩 🟩 🟩 × 🟩 ◆ #2: ◆ 🟩 🟩 × 🟩 🟩 ◆ 0 2 4 6
7	#1: ◆ 🟩 🟩 × 🟩 × 🟩 × 🟩 ◆ #2: ◆ 🟩 🟩 × × 🟩 🟩 × 🟩 ◆ 0 2 5 9
8	#1: ◆ 🟩 🟩 🟩 × 🟩 × × 🟩 × × 🟩 ◆ #2: ◆ 🟩 🟩 🟩 × × × 🟩 🟩 × × 🟩 ◆ 0 3 5 8 12
9	#1: ◆ 🟩 🟩 🟩 🟩 × × × × 🟩 🟩 × × × 🟩 ◆ 0 4 9 15
10	#1: ◆ 🟩 🟩 × × × 🟩 🟩 × × × 🟩 × × × 🟩 🟩 × 🟩 ◆ #2: ◆ 🟩 🟩 × × × 🟩 🟩 🟩 × × × × × × 🟩 🟩 × 🟩 ◆ 0 2 6 11 15 19

difficult as N increases. To the best of our knowledge, closed-form expressions for $\mathbb{S}_{\text{RMRA}}$ are not available. Even so, it is still manageable to enumerate RMRA for small N. The results are summarized in the following example.

Example 2. Table I tabulates some of the solutions to (P1) for a given N, where the essential sensors and the inessential sensors are marked by red diamonds and green squares, respectively. The leftmost element in the array is shifted to the location 0 and the reversed version of these solutions are omitted. For N being 4 or 5, RMRA are the same as ULA. Among those in Table I, the solutions to (P1) are not unique, such as those for $N = 6, 7, 8, 10$.

Next we will compare MRA with RMRA. For instance, the MRA with $N = 10$, as depicted in Fig. 1(b), own the aperture $A = 36$ and $F = 1$. On the other hand, the RMRA with $N = 10$, as in Table I, has a smaller aperture $A = 19$ and minimum fragility $F = 2/10$. That is, in this example, the RMRA approximately halves the aperture to decrease the fragility, compared with the MRA.

Based on this empirical observation, next we will show that the RMRA also owns the $\mathcal{O}(N^2)$ property, i.e., $|\mathbb{D}_{\text{RMRA}}| = \mathcal{O}(N^2)$. Before presenting this result, we first define a quantity r for the relation between the number of sensors N and the aperture A. It is defined as

$$r \triangleq \frac{N^2}{A}. \tag{10}$$

For any array with sufficiently large N, the redundancy can be approximated by $R \approx r/2$. Then, the following theorem states the lower and the upper bounds of r for the RMRA, where the details of the proof will be elaborated in Section IV.

Theorem 1. Let $\mathbb{S}_{\text{RMRA}}$ be a solution to (P1) with $N \geq 4$ physical sensors. The aperture of $\mathbb{S}_{\text{RMRA}}$ is denoted by A_{RMRA}. Define the ratio $r_{\text{RMRA}} \triangleq N^2/A_{\text{RMRA}}$. Then

$$4 + \frac{4\sqrt{2}}{3\pi} \leq r_{\text{RMRA}} < 16. \tag{11}$$

Theorem 1 makes it possible to show the $\mathcal{O}(N^2)$ property of RMRA. First (11) can be rearranged as $0.0625N^2 < A_{\text{RMRA}} \leq 0.2174N^2$. Then the size of the hole-free difference coarray is $|\mathbb{D}_{\text{RMRA}}| = 2A_{\text{RMRA}} + 1 = \mathcal{O}(N^2)$.

Finally we will compare the aperture of MRA [2] with that of RMRA. Due to [2] and Theorem 1, for sufficiently large N, the apertures of these arrays satisfy

$$0.2987N^2 \leq A_{\text{MRA}} \leq 0.4108N^2, \tag{12}$$

$$0.0625N^2 < A_{\text{RMRA}} \leq 0.2174N^2. \tag{13}$$

Eqs. (12) and (13) indicate that $A_{\text{RMRA}}/A_{\text{MRA}} \leq 0.7278$. That is, for sufficiently large N, A_{RMRA} is at most 72.78% of A_{MRA}. In particular, if $N = 10$, then $A_{\text{RMRA}}/A_{\text{MRA}} = 19/36 \approx 52.8\% < 72.78\%$, due to Fig. 1 and Table I.

IV. PROOF OF THEOREM 1

A. The Main Proof

This section aims to derive the relation in (11). Before presenting the details, we will invoke Lemma 1 to relate the constraint $F = 2/N$ to another constraint on the weight function $w(m)$.

Lemma 1. Let $\mathbb{S}$ be a sensor array with N physical sensors. Let $A \triangleq \max(\mathbb{S}) - \min(\mathbb{S})$ be the aperture of $\mathbb{S}$. Assume that $N \geq 4$. If $F = 2/N$, then $w(m) \geq 2$ for all $m \in \mathbb{D}\backslash\{\pm A\}$.

Proof: Assume that there exists $\widehat{m} \in \mathbb{D}\backslash\{\pm A\}$ such that $w(\widehat{m}) < 2$. Since $\widehat{m} \in \mathbb{D}$, we have $w(\widehat{m}) = 1$, implying that there exists a unique sensor pair $(n_1, n_2) \in \mathbb{S}^2$ such that 1) $n_1 - n_2 = \widehat{m}$ and 2) n_1 and n_2 are both essential [11, Lemma 1]. Next let us match the essential sensors n_1 and n_2 with $\min(\mathbb{S})$ and $\max(\mathbb{S})$, which are known to be essential for any arrays [12]. If $\{n_1, n_2\} = \{\min(\mathbb{S}), \max(\mathbb{S})\}$, then $|\widehat{m}| = |n_1 - n_2| = |\min(\mathbb{S}) - \min(\mathbb{S})| = A$, which contradicts with the assumption that $\widehat{m} \in \mathbb{D}\backslash\{\pm A\}$. If $\{n_1, n_2\} \neq \{\min(\mathbb{S}), \max(\mathbb{S})\}$, then there are at least three essential sensors in $\mathbb{S}$, so $F \geq 3/N$. This statement disagrees with the assumption that $F = 2/N$. ■

Next let us move on to the main results. In what follows, the lower bound and the upper bound in (11) will be derived separately.

The lower bound in (11)*:* Let $\mathbb{S}$ be a sensor array belonging to the feasible region of (P1), i.e. satisfying the constraints (8) and (9). The beampattern of the array $\mathbb{S}$ is defined as

$$B(\omega; \mathbb{S}) = \sum_{n \in \mathbb{S}} e^{\jmath \omega n}, \tag{14}$$

where $\jmath = \sqrt{-1}$ is the imaginary unit and the parameter ω is real-valued. Based on (14), we have

$$|B(\omega; \mathbb{S})|^2 = \sum_{n_1, n_2 \in \mathbb{S}} e^{\jmath \omega (n_1 - n_2)} = \sum_{m \in \mathbb{D}} w(m) e^{\jmath \omega m}, \tag{15}$$

where $\mathbb{D} = \{0, \pm 1, \pm 2, \ldots, \pm A\}$. Due to Lemma 1 and the property that $w(\pm A) = 1$ and $w(\pm(A-1)) = 2$, we can divide Eq. (15) into three terms $T_1(\omega)$, $T_2(\omega)$, and $T_3(\omega)$ as follows

$$|B(\omega;\mathbb{S})|^2 = \sum_{m=-A}^{A} w(m)e^{\jmath\omega m} = \underbrace{\sum_{m=-A}^{A} e^{\jmath\omega m}}_{T_1(\omega)} + \underbrace{\sum_{m=-(A-1)}^{A-1} e^{\jmath\omega m}}_{T_2(\omega)} + \underbrace{\sum_{m=-(A-2)}^{A-2} (w(m)-2)e^{\jmath\omega m}}_{T_3(\omega)}. \tag{16}$$

The term $T_1(\omega) + T_2(\omega)$ can be simplified as

$$\begin{aligned} T_1(\omega) + T_2(\omega) &= \frac{\sin((A+1/2)\omega) + \sin((A-1/2)\omega)}{\sin(\omega/2)} \\ &= \frac{2\sin(A\omega)\cos(\omega/2)}{\sin(\omega/2)}. \end{aligned} \tag{17}$$

Next let us consider the term $T_3(\omega)$. Since the weight function $w(m)$ is real-valued and evenly-symmetric, the term $T_3(\omega)$ is real-valued for any real ω. Applying the triangular inequality and the constraint that $w(m) \geq 2$ to $T_3(\omega)$ yields

$$\begin{aligned} T_3(\omega) \leq |T_3(\omega)| &\leq \sum_{m=-(A-2)}^{A-2} |(w(m)-2)e^{\jmath\omega m}| \\ &= \left(\sum_{m=-(A-2)}^{A-2} w(m)\right) - 2(2(A-2)+1) \\ &= (N^2-6) - (4A-6) = N^2 - 4A, \end{aligned} \tag{18}$$

where we use the fact that $\sum_{m\in\mathbb{D}} w(m) = N^2$ for any array with N sensors [3]. Substituting (17) and (18) into the property that $|B(\omega;\mathbb{S})|^2 \geq 0$ leads to

$$\frac{N^2}{A} \geq 4 - \frac{2\sin(A\omega)\cos(\omega/2)}{A\sin(\omega/2)}. \tag{19}$$

Note that (19) holds for any real ω. Substituting $\omega = 3\pi/(2A)$ into (19) gives

$$\frac{N^2}{A} \geq 4 + \frac{2\cos\frac{3\pi}{4A}}{A\sin\frac{3\pi}{4A}}. \tag{20}$$

Since $N \geq 4$, we have $A \geq 3$ and $0 \leq \frac{3\pi}{4A} \leq \frac{\pi}{4}$. Thus, the sine and the cosine terms in (20) satisfy

$$0 \leq \sin\frac{3\pi}{4A} \leq \frac{3\pi}{4A}, \qquad \frac{1}{\sqrt{2}} \leq \cos\frac{3\pi}{4A} \leq 1. \tag{21}$$

Finally, combining (20) and (21) shows that $N^2/A \geq 4 + 4\sqrt{2}/(3\pi)$. Since RMRA belongs to the feasible region of (P1), we have $N^2/A_{\text{RMRA}} \geq 4 + 4\sqrt{2}/(3\pi)$.

The upper bound of (11)*:* It suffices to find a solution $\mathbb{S}$ in the feasible region of (P1), namely, a solution that satisfies (8) and (9), that gives $N^2/A < 16$, where A is the aperture of $\mathbb{S}$. Based on N, we have the following cases:

1) $4 \leq N \leq 14$*:* In this case, $\mathbb{S}$ is chosen as the ULA with N sensors: $\{0, 1, \ldots, N-1\}$, which satisfies $F = 2/N$ and $N^2/A = N^2/(N-1) \leq 196/13 < 16$.

2) $N \geq 15$ *and* N *is even:* We consider the symmetric nested array with N sensors [11], defined as

Definition 2. Assume that N is a positive even number. The symmetric nested array with N sensors is defined as the union of two arrays $\mathbb{S}_1$ and $\mathbb{S}_2$, i.e., $\mathbb{S}_{\text{sym}} \triangleq \mathbb{S}_1 \cup \mathbb{S}_2$, where

$$\begin{aligned} \mathbb{S}_1 \triangleq \{&1,\ 2,\ \ldots,\ N_1, \\ &(N_1+1),\ 2(N_1+1),\ \ldots,\ N_2(N_1+1)\}, \end{aligned} \tag{22}$$

$$\mathbb{S}_2 \triangleq \{N_2(N_1+1) + 1 - n : n \in \mathbb{S}_1\}. \tag{23}$$

Here the parameters N_1 and N_2 are given by

$$N_1 = \lfloor (N+2)/4 \rfloor, \qquad N_2 = \lceil (N+2)/4 \rceil. \tag{24}$$

The ceiling and the floor functions are denoted by $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$, respectively.

The array $\mathbb{S}_1$ is the nested array with parameters N_1 and N_2 [3] while the array $\mathbb{S}_2$ is the reversed version of $\mathbb{S}_1$. Note that the definition of the symmetric nested array from (22) to (24) is applicable for any positive even N.

Properties of the symmetric nested array are listed below:

Lemma 2. Let $\mathbb{S}_{\text{sym}}$ denote a symmetric nested array with a positive even N. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be given by (22) and (23), respectively. Then we have

1) $\mathbb{S}_1 \cap \mathbb{S}_2 = \{1, N_2(N_1+1)\}$.
2) Let $\mathbb{D}_{\text{sym}}$ be the difference coarray of $\mathbb{S}_{\text{sym}}$. Then $\mathbb{D}_{\text{sym}} = \{0, \pm 1, \ldots, \pm(N_2(N_1+1)-1)\}$.
3) The fragility F for $\mathbb{S}_{\text{sym}}$ is $2/N$.

Proof: The proof of Lemma 2 is sketched below. The first item follows from (22) and (23) directly.

For the second item, let $\mathbb{D}_1$ and $\mathbb{D}_2$ be the difference coarray of $\mathbb{S}_1$ and $\mathbb{S}_2$, respectively. We have $\mathbb{D}_1 = \mathbb{D}_2 = \{0, \pm 1, \ldots, \pm(N_2(N_1+1)-1)\}$ [3]. Since 1) $\mathbb{S}_1$, $\mathbb{S}_2$, and $\mathbb{S}_{\text{sym}}$ share the same aperture and 2) $\mathbb{D}_1 = \mathbb{D}_2$ are both hole-free, we have $\mathbb{D}_{\text{sym}} = \mathbb{D}_1 = \mathbb{D}_2$.

The third item is due to the following chain of arguments. If $N = 2$, then it is obviously true [8]. Next we consider $N \geq 4$. Let $n \in \mathbb{S}_{\text{sym}} \backslash \{1, N_2(N_1+1)\}$. By definition, n belongs to either $\mathbb{S}_1$ or $\mathbb{S}_2$. If $n \in \mathbb{S}_1$, then removing n from $\mathbb{S}_{\text{sym}}$ does not change the difference coarray. This is because the difference coarray of $\mathbb{S}_{\text{sym}} \backslash \{n\}$ is a superset of $\mathbb{D}_2 = \mathbb{D}_{\text{sym}}$. Thus n is inessential with respect to $\mathbb{S}_{\text{sym}}$. Similar arguments apply to the case when $n \in \mathbb{S}_2$, which completes the proof. ■

Lemma 2 shows that the symmetric nested array is a feasible solution to (P1). Now let us move on to the ratio N^2/A for the symmetric nested array. Since $\lfloor x \rfloor > x - 1$ and $\lceil x \rceil \geq x$ for any real x, the aperture of $\mathbb{S}_{\text{sym}}$ satisfies

$$A_{\text{sym}} = N_2(N_1+1) - 1 > \frac{1}{16}(N+6)(N-2). \tag{25}$$

For $N \geq 15$ and N is even, it can be shown that $N^2/A_{\text{sym}} < 16/((1+6/N)(1-2/N)) \leq 16$, which proves this case.

3) $N \geq 15$ *and* N *is odd:* In this case, the symmetric nested array with $N' \triangleq N-1$ sensors is considered, which is denoted by $\mathbb{S}'_{\text{sym}}$. Definition 2 gives $N'_1 = \lfloor (N+1)/4 \rfloor$, $N'_2 = \lceil (N+1)/4 \rceil$, and the aperture

$$A'_{\text{sym}} = N'_2(N'_1+1) - 1 > \frac{1}{16}(N+5)(N-3). \tag{26}$$

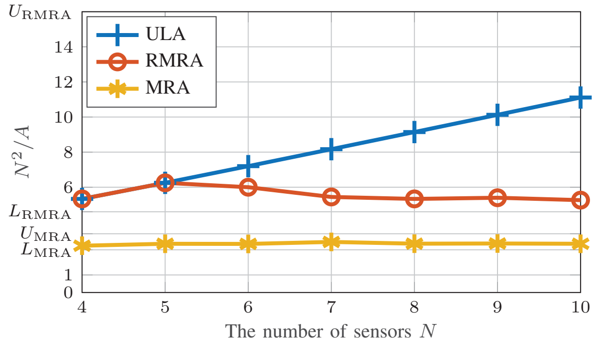

Fig. 2. The dependence of the ratio $r = N^2/A$ on the number of sensors N for (a) the ULA, (b) the RMRA, and (c) the MRA. Here $L_{\text{MRA}} = 2.434$, $U_{\text{MRA}} = 3.348$, $L_{\text{RMRA}} = 4 + 4\sqrt{2}/(3\pi) \approx 4.6.$, and $U_{\text{RMRA}} = 16$.

Since $N \geq 15$, we have $A'_{\text{sym}} > N'$. That is, there exists at least one empty space in $\mathbb{S}'_{\text{sym}}$. Therefore we can construct another array $\overline{\mathbb{S}}$ by adding a new sensor to $\mathbb{S}'_{\text{sym}}$. Namely, $\overline{\mathbb{S}} \triangleq \mathbb{S}'_{\text{sym}} \cup \{n\}$, where $\min(\mathbb{S}'_{\text{sym}}) < n < \max(\mathbb{S}'_{\text{sym}})$ and $n \notin \mathbb{S}'_{\text{sym}}$. We can show that $|\overline{\mathbb{S}}| = N$, the difference coarray of $\overline{\mathbb{S}}$ is hole-free, and the fragility of $\overline{\mathbb{S}}$ is $2/N$, implying $\overline{\mathbb{S}}$ resides in the feasible region of (P1). Based on these results and (26), we obtain $N^2/A'_{\text{sym}} < 16/((1+5/N)(1-3/N)) \leq 16$. Since RMRA has the largest hole-free difference coarray in the feasible region of (P1), we have $A_{\text{RMRA}} \geq A'_{\text{sym}}$ so $N^2/A_{\text{RMRA}} < 16$.

B. Remarks on the Proof

The lower bound $4 + 4\sqrt{2}/(3\pi) \approx 4.6002$ in (11) holds for any feasible solution to (P1). This bound can be tightened to be 4.6436 if the parameter ω is set to be $4.3514/A$ in (19).

The proof technique of (14) and (15) is inspired by [9], [15], [16]. The novelty of this paper in deriving the lower bound is as follows. First, we presented Lemma 1 to convert the constraint on the fragility (9) into the constraints on the weight function. Second, these constraints were utilized in (16) to (18), leading to our new result.

In the presented proof, the upper bound of $r_{\text{RMRA}} = N^2/A_{\text{RMRA}}$, is obtained by analyzing the quantity r of the symmetric nested arrays. However this upper bound of r_{RMRA} is empirically found to be loose, as we will demonstrated later in Fig. 2. It remains a future research direction to tighten the upper bound of r_{RMRA}.

V. NUMERICAL EXAMPLES

In this section, we consider three array configurations: the ULA, the RMRA, as proposed in (P1), and the MRA, as in (P0). Fig. 2 plots the dependence of the ratio $r = N^2/A$ on the number of sensors N among these array configurations. The notations r_{ULA}, r_{RMRA}, and r_{MRA} denotes the ratio for ULA, RMRA, and MRA, respectively. The lower and upper bounds for MRA are given by $L_{\text{MRA}} = 2.434$ and $U_{\text{MRA}} = 3.348$, respectively [9]. Furthermore, the lower and upper bounds for RMRA are $L_{\text{RMRA}} = 4 + 4\sqrt{2}/(3\pi)$ and $U_{\text{RMRA}} = 16$, as in Theorem 1. It is verified through Fig. 2 that the bounds for r_{MRA} and r_{RMRA} are valid. Empirically, r_{RMRA} becomes close to L_{RMRA} as N increases. By contrast, r_{ULA} grows linearly for sufficiently large N, as observed in Fig. 2. These results confirm the property that the MRA and the RMRA own $\mathcal{O}(N^2)$ elements in the hole-free difference coarrays while the ULA only has $\mathcal{O}(N)$ elements in its difference coarray.

VI. CONCLUDING REMARKS

This paper proposed a novel array configuration, called the robust minimum redundancy array (RMRA), that maximizes the hole-free difference coarray, subject to the highest level of robustness to sensor failures. This concept was cast as an integer program, which is computationally expensive to solve in general. Using Theorem 1, it was proved that the size of the difference coarray for the RMRA with N sensors achieves $\mathcal{O}(N^2)$, which is as good as that of MRAs. This property was also verified through numerical examples for the number of sensors up to 10.

Future directions can be towards finding suboptimal solutions with low computational complexity in the feasible region of (P1), while at the same time preserving the $\mathcal{O}(N^2)$ property. These suboptimal solutions may also lead to a tighter upper bound of r_{RMRA} than that in Theorem 1. These solutions would be very appealing to applications where it is desirable to have large difference coarray and high level of robustness.

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