ENERGY COMPACTION FILTERS ON GRAPHS

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ABSTRACT

In classical signal processing spectral concentration is an important problem that was first formulated and analyzed by Slepian. The solution to this problem gives the optimal FIR filter that can confine the largest amount of energy in a specific bandwidth for a given filter order. The solution is also known as the prolate sequence. This study investigates the same problem for polynomial graph filters. The problem is formulated in both graph-free and graph-dependent fashions. The graph-free formulation assumes a continuous graph spectrum, in which case it becomes the polynomial concentration problem. This formulation has a universal approach that provides a theoretical reference point. However, in reality graphs have discrete spectrum. The graph-dependent formulation assumes that the eigenvalues of the graph are known and formulates the energy compaction problem accordingly. When the eigenvalues of the graph have a uniform distribution, the graph-dependent formulation is shown to be asymptotically equivalent to the graph-free formulation. However, in reality eigenvalues of a graph tend to have different densities across the spectrum. Thus, the optimal filter depends on the underlying graph operator, and a filter cannot be universally optimal for every graph.

Index Terms— Graph signal processing, polynomial filters, concentrated polynomials, Hilbert matrix.

1. INTRODUCTION

Recent years have observed an increased interest in network structured data in which the underlying dependency structure is modeled by a graph that allows irregular signal domains unlike in classical signal processing [1, 2]. In the area of graph signal processing, the analysis of signals is based on the graph operator, whose eigenvectors serve as the graph Fourier basis (GFB) and eigenvalues indicate the frequency. With the use of GFB, sampling, reconstruction, multirate processing of graph signals and some uncertainty results have been extended to the case of graphs in [3–13].

In the study of graph signals polynomial filters play an important role. Their significance follows from their localization property: when implemented on a graph, a polynomial filter of order L requires a node to communicate only with its L-hop neighbors. Moreover, polynomial graph filters are analogous to finite impulse response (FIR) filters of classical signal processing. Elements of the graph Fourier basis can be amplified or suppressed according to the behavior of the filter. Thus, the design of such polynomials in the context of graphs is an important problem.

The spectral concentration problem in classical signal processing searches for the optimal FIR filter (of fixed order) that confines the largest amount of energy into a specific bandwidth. The problem was first formulated and analyzed by Slepian in his seminal works [14, 15]. The solution to the problem is known as the prolate sequence, and it provides the optimal (in the least squares sense) window for the filter design problem [16].

In this study, we consider the spectral concentration problem for polynomial graph filters. Given a filter order L and a bandwidth σ, we consider the optimal selection of the coefficients such that the energy confined in the band (of the graph) is maximized. This problem is analogous to the classical spectral concentration problem [14–16]. The difference lies in the definition of the spectrum: in the classical case the spectrum is defined with respect to the unit circle, whereas in the case of graphs the spectrum is an interval on the real line. In spite of their conceptual similarity, the analysis in the graph case differs from the classical one and requires additional attention.

For the energy compaction problem on graphs, we take two approaches. In the first one, we assume that the spectrum of the graph is continuous. In this case the problem reduces to the polynomial concentration problem studied more generally by Slepian in [17]. We re-visit the problem, compare it with the classical case and present its asymptotic behavior in the case of narrow bandwidth. Although the continuous approach provides a theoretical and graph-free reference point, it is not applicable to graphs directly as graphs have discrete spectrum of finite size. In the second approach, we define the problem with respect to the spectrum of the graph. Thus, the optimal filter becomes specific to the underlying graph. We consider different examples of graphs, and compare the behavior of the maximum energy compaction as well as the optimal filter.

We would like to note that the studies in [11–13] focus on the concentration and localization properties of graph signals. In particular, [12,13] extend the classical time-frequency concentration problem to the case of graphs. Different from [11–13], this study focuses on the energy concentration properties of polynomial graph filters. Thus, results here do not involve vertex domain properties.

In Section 2 we provide a quick overview of the classical spectral concentration problem. In Section 3 we define the energy compaction problem for the continuous case and study the behavior of the solution. In Section 4 we consider the discrete graph-dependent counter-part of the problem and investigate the effect of the spectrum of the graph.

1.1. Notation & Preliminaries

We will assume that $\boldsymbol{A} \in \mathcal{C}^{N\times N}$ is an operator on the graph with N nodes. We consider $\boldsymbol{A}$ to be a *local* operator, that is, $A_{i,j} = 0$ when the nodes i and j are not neighbors. We allow $A_{i,i}$ to be non-zero. The operator $\boldsymbol{A}$ can be the adjacency matrix, the Laplacian, the normalized Laplacian, and so on. Assuming that $\boldsymbol{A}$ is diagonalizable, the eigenvalue decomposition of $\boldsymbol{A}$ will be denoted as:

$$\boldsymbol{A} = \boldsymbol{V}\ \boldsymbol{\Lambda}\ \boldsymbol{V}^{-1}, \tag{1}$$

where $\boldsymbol{V}$ is a matrix consisting of eigenvectors of $\boldsymbol{A}$, and $\boldsymbol{\Lambda}$ is the diagonal matrix with the eigenvalues, which are assumed to be real and ordered such that $\lambda_1 \leqslant \cdots \leqslant \lambda_N$.

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2. THE ENERGY COMPACTION PROBLEM

Let $H(e^{j\omega})$ denote the frequency response of a causal FIR filter of order L that is defined as follows:

$$H(e^{j\omega}) = \sum_{k=0}^{L} h_k \, e^{-j\omega k}, \tag{2}$$

where $h_k \in \mathcal{R}$ denote the coefficients of the filter. The problem of energy compaction (or, spectral concentration) searches for the filter whose energy is maximized in the specified passband. This can be described precisely with the following optimization problem:

$$\phi(\sigma) = \max_{h_k} \int_0^{\pi\sigma} \left|H(e^{j\omega})\right|^2 \frac{\mathrm{d}\omega}{\pi} \quad \text{s.t.} \quad \int_0^{\pi} \left|H(e^{j\omega})\right|^2 \frac{\mathrm{d}\omega}{\pi} = 1, \tag{3}$$

where $0 < \sigma < 1$ denotes the (normalized) bandwidth of the passband, and $\phi(\sigma)$ denotes the maximum amount of energy that can be confined in the band $[0 \;\; \sigma]$. As described clearly in Chapter 3.2.2 of [16], when the coefficients of an FIR filter are represented as a vector $\boldsymbol{h} = [h_0 \; \cdots \; h_L]^T$, the problem in (3) can be reformulated as the following Rayleigh quotient:

$$\widehat{\boldsymbol{h}} \;=\; \arg\max_{\boldsymbol{h}} \quad \boldsymbol{h}^* \, \boldsymbol{P} \, \boldsymbol{h} \qquad \text{s.t.} \qquad \|\boldsymbol{h}\|_2^2 = 1, \tag{4}$$

where the kernel matrix $\boldsymbol{P} \in \mathcal{R}^{(L+1)\times(L+1)}$ is given as follows:

$$(\boldsymbol{P})_{m,n} = \sigma \;\, \mathrm{sinc}\left(\, \sigma \,(m-n)\,\right), \qquad 1 \leqslant m, n \leqslant L{+}1, \tag{5}$$

where $\mathrm{sinc}(x) = \sin(\pi\, x)/(\pi\, x)$. Then, the optimal filter, $\widehat{\boldsymbol{h}}$, and its energy compaction, $\phi(\sigma)$, can be found as the dominant eigenvector-eigenvalue pair of the positive-definite and Toeplitz matrix $\boldsymbol{P}$. The solution, $\widehat{\boldsymbol{h}}$, is also known as the prolate sequence. Many other properties of the eigenvectors and the eigenvalues of the matrix $\boldsymbol{P}$ were studied by Slepian in [15].

3. GRAPH INDEPENDENT CONTINUOUS SPECTRUM

Given a graph operator $\boldsymbol{A}$, a polynomial graph filter of order L (or FIR graph filter) is defined as follows:

$$H(\boldsymbol{A}) = \sum_{k=0}^{L} h_k \, \boldsymbol{A}^k. \tag{6}$$

Since $\boldsymbol{A}$ and $H(\boldsymbol{A})$ are simultaneously diagonalizable, the filter scales a graph Fourier component corresponding to an eigenvalue λ with $H(\lambda)$. Thus, the frequency response of a polynomial filter can be written as follows:

$$H(\lambda) = \sum_{k=0}^{L} h_k \, \lambda^k. \tag{7}$$

At the core of most practical applications lie low-pass filters, which can be described conceptually as follows:

$$\left|H(\lambda)\right| \approx \begin{cases} 1, & \text{if } \lambda \leqslant \sigma, \\ 0, & \text{if } \lambda > \sigma, \end{cases} \tag{8}$$

where $0 < \sigma < 1$, and σ denotes the cut-off frequency of the filter. Depending on the design criteria one can construct different filters to achieve the behavior in (8). Motivated by the results in [14–16], we consider here the energy compaction filter similar to the one in (3). More specifically we consider the following problem, which was first addressed in [17]:

$$\gamma(\sigma) = \max_{h_k} \int_0^{\sigma} |H(\lambda)|^2 \, \mathrm{d}\lambda \quad \text{s.t.} \quad \int_0^{1} |H(\lambda)|^2 \, \mathrm{d}\lambda = 1. \tag{9}$$

It should be noted that we treat $\lambda \in \mathcal{R}$ (spectrum of the graph) as a continuous parameter in (9), which is contrary to the fact that a spectrum of a graph is discrete and has at most N eigenvalues (N being the size of graph). More importantly, eigenvalues of a graph are not spaced uniformly, they may be concentrated (or, clustered) around some specific intervals. (See Figure 4 later.) Nevertheless, the problem in (9) has two theoretical advantages: 1) The formulation is graph-free. Therefore, it considers a unified approach to the filter design problem. 2) It provides a theoretical reference point and allows us to answer the following question: can we ignore the underlying graph structure and design filters universally? As we shall discuss in Secion 4, *the answer is no: the graph spectrum matters.*

In order to convert the problem (9) into matrix-vector equations we first define the following vector variables:

$$\boldsymbol{\lambda}^T = [1 \quad \lambda \quad \cdots \quad \lambda^L], \qquad \boldsymbol{h}^T = [h_0 \quad \cdots \quad h_L]. \tag{10}$$

Since $\boldsymbol{\lambda} \in \mathcal{R}^{L+1}$ we have

$$H(\lambda) = \boldsymbol{\lambda}^* \, \boldsymbol{h}, \qquad\qquad |H(\lambda)|^2 = \boldsymbol{h}^* \, \boldsymbol{\lambda} \, \boldsymbol{\lambda}^* \, \boldsymbol{h}. \tag{11}$$

Then, the objective function (as well as the constraint) in (9) can be written as follows:

$$\int_0^{\sigma} |H(\lambda)|^2 \, \mathrm{d}\lambda = \int_0^{\sigma} \boldsymbol{h}^* \, \boldsymbol{\lambda} \boldsymbol{\lambda}^* \, \boldsymbol{h} \, \mathrm{d}\lambda = \boldsymbol{h}^* \, \boldsymbol{Q}(\sigma) \, \boldsymbol{h}, \tag{12}$$

where the matrix $\boldsymbol{Q}(\sigma)$ consists of the following terms:

$$\Big(\boldsymbol{Q}(\sigma)\Big)_{m,n} = \int_0^{\sigma} \lambda^{m+n-2} \, \mathrm{d}\lambda \;=\; \frac{\sigma^{m+n-1}}{m+n-1}. \tag{13}$$

Using (12), the problem in (9) can be written as follows:

$$\gamma(\sigma) = \max_{\boldsymbol{h}} \quad \boldsymbol{h}^* \, \boldsymbol{Q}(\sigma) \, \boldsymbol{h} \qquad \text{s.t.} \qquad \boldsymbol{h}^* \, \boldsymbol{Q}(1) \, \boldsymbol{h} = 1. \tag{14}$$

Thus, the optimal energy compaction problem on graphs can be formulated as a *generalized Rayleigh quotient* problem. Before elaborating on the solution of (14), we first present the following lemma whose proof is omitted due to the space limitations:

Lemma 1. *$\boldsymbol{Q}(\sigma)$ is a symmetric matrix with the Hankel structure. Moreover, it satisfies the following ordering for $0 < \sigma_1 < \sigma_2 \leqslant 1$:*

$$\boldsymbol{0} \prec \boldsymbol{Q}(\sigma_1) \prec \boldsymbol{Q}(\sigma_2) \prec \sigma_2 \, \pi \, \boldsymbol{I}. \tag{15}$$

Since $\boldsymbol{Q}(\sigma)$ does not have a null-space and is bounded, the problem in (14) is well-defined. Moreover, it can be converted into a standard Rayleigh quotient problem. For this purpose, consider the Cholesky decomposition of $\boldsymbol{Q}(1)$:

$$\boldsymbol{Q}(1) = \boldsymbol{C} \, \boldsymbol{C}^*, \tag{16}$$

where we assume that $\boldsymbol{C}$ is a lower triangular matrix with strictly positive diagonal entries, hence $\boldsymbol{C}$ is unique and invertible. Then, the problem in (14) can be equivalently written as follows:

$$\begin{aligned} \gamma(\sigma) &= \max_{\boldsymbol{v}} \quad \boldsymbol{v}^* \, \boldsymbol{C}^{-1} \, \boldsymbol{Q}(\sigma) \, \boldsymbol{C}^{-*} \, \boldsymbol{v} \qquad \text{s.t.} \qquad \boldsymbol{v}^* \boldsymbol{v} = 1 && (17) \\ &= \|\boldsymbol{C}^{-1} \, \boldsymbol{Q}(\sigma) \, \boldsymbol{C}^{-*}\|_2. && (18) \end{aligned}$$

Furthermore, the optimal filter that achieves the maximum energy compaction can be found as follows:

$$\widehat{\boldsymbol{h}} = \boldsymbol{C}^{-*} \, \boldsymbol{v}, \tag{19}$$

where $\boldsymbol{v}$ is the dominant eigenvector of the symmetric matrix in (18).

It should be noted that the matrix $\boldsymbol{Q}(1)$ corresponds to *a Hilbert matrix* of size L+1 [18], which has been used extensively in the study of polynomial approximations. A Hilbert matrix has many interesting properties and challenges, among which lies the condition number. A Hilbert matrix is positive definite for any size as shown by Lemma 1. However, the condition number grows like $(1+\sqrt{2})^{4n}/\sqrt{n}$ for the size n Hilbert matrix [19] making the matrix so ill-conditioned that MATLAB *fails* to compute the Cholesky decomposition in (16) for $L \geqslant 13$. Nevertheless, researchers have obtained closed form expressions for the matrices that are related to a Hilbert matrix. For example, the study in [20] shows that the inverse of the Cholesky factor in (16) has the following entries:

$$\left(\boldsymbol{C}^{-1}\right)_{m,n} = (\text{-}1)^{m+n}\sqrt{2m\text{-}1}\binom{m{+}n\text{-}2}{n\text{-}1}\binom{m\text{-}1}{n\text{-}1}, \quad m \geqslant n, \tag{20}$$

which allows the direct computation of the matrix in (18). It is important to note that entries in $\boldsymbol{C}^{-1}$ grow exponentially with its size L. Thus, direct computation of the matrix in (18) is still prone to the numerical problems for large values of L.

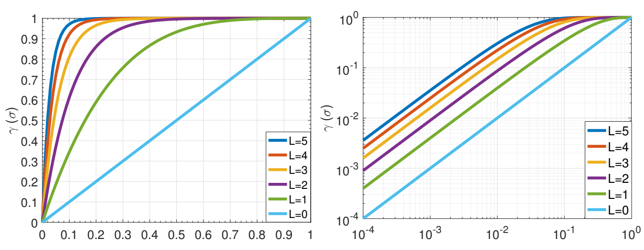
It is well-known that (14) can be converted into the following generalized eigenvalue problem with the use of Lagrange multiplier:

$$\boldsymbol{Q}(\sigma)\ \boldsymbol{h} = \gamma\ \boldsymbol{Q}(1)\ \boldsymbol{h}, \tag{21}$$

whose dominant eigenvalue-eigenvector pair provides the maximum amount of energy compaction and the corresponding filter that achieves it. Although the formulation in (21) is easier to implement in numerical environments, it still suffers from numerical precision even for moderate values of L.

3.1. The Optimal Filter and the Maximum Energy Compaction

Although closed form solution for the dominant eigenpair of (21) is not available, a numerical solution is possible to obtain for small values of L. In Figure 1 we present the maximum energy compaction, $\gamma(\sigma)$, as a function of the bandwidth σ for different values of L. For a fixed order L, notice that $\gamma(\sigma)$ is an increasing function of σ, that is, larger amount of energy can be confined in a larger bandwidth. Moreover, for a fixed bandwidth σ, the amount of energy compaction increases as the filter order L gets larger. This shows a trade-off between the locality of the graph filter and better (close to the ideal) low-pass characteristics.



(a) (b)

Fig. 1. Dominant eigenvalue of (21) in (a) linear-scale (b) log-scale.

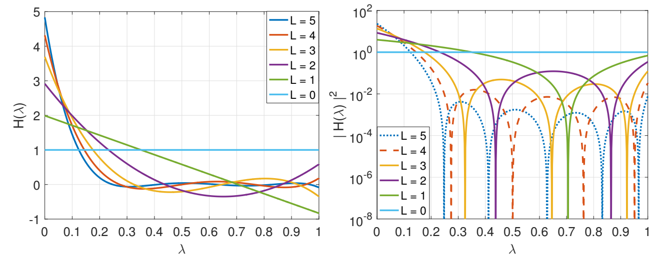
The optimal filters that achieve the maximum energy compaction are presented in Figure 2 in which the bandwidth is selected as $\sigma = 0.2$, and filters with different orders are considered. As seen in Figure 2(b) the filters have zeros in the interval $[0.2\ \ 1]$. In fact, numerical observations suggests that the optimal filter of order L for the bandwidth σ has exactly L zeros in the interval $(\sigma\ \ 1]$. This is an expected result since the problem in (9) minimizes the energy confined in $(\sigma\ \ 1]$. Thus, all the zeros are located in this interval.

3.2. Narrow Bandwidth Behavior

Although a closed form solution for the dominant eigenpair of (21) does not exist, for small values of σ, Figure 1(b) suggests that the amount of energy compaction depends *linearly* on the bandwidth. The following theorem, whose proof is omitted due to space limitations, shows that this is in fact the case:

Theorem 1. *For small σ, the maximum amount of energy concentration of an order L filter is approximated as follows:*

$$\gamma(\sigma) \approx \sigma\ (L+1)^2. \tag{22}$$



(a) (b)

Fig. 2. The optimal filter responses for $\sigma = 0.2$ and different values of L. Filter responses in (a) linear-scale, (b) log-scale.

Moreover, the coefficients of the optimal filter can be approximated as:

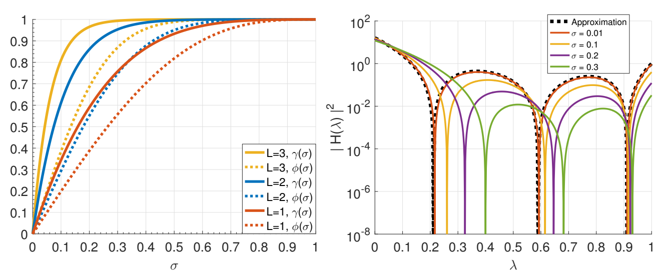
$$h_k = (-1)^k\ \frac{(L{+}k{+}1)!}{(L\text{-}k)!\ \ k!\ \ (k{+}1)!}, \qquad 0 \leqslant k \leqslant L. \tag{23}$$

The asymptotic behavior of the energy compaction of polynomial filters resembles that of the classical FIR filters. Eq. (64) of [15] approximated the solution of the energy compaction problem in (3) as $\phi(\sigma) \approx \sigma\,(L+1)$ for small values of σ. Although both $\phi(\sigma)$ and $\gamma(\sigma)$ depend linearly on the bandwidth, $\phi(\sigma)$ depends on the order linearly, whereas $\gamma(\sigma)$ has a quadratic dependence resulting in $\gamma(\sigma) \geqslant \phi(\sigma)$ in the case of narrow bandwidth. In fact, as observed in Figure 3(a), $\gamma(\sigma) \geqslant \phi(\sigma)$ for all values of σ. Thus, polynomial filters (graph filters) can confine more energy.

It is also interesting to see that the approximation of the optimum filter given in (23) has *integer valued coefficients* with alternating signs. In the case of $L = 3$, which is illustrated in Figure 3(b), these coefficients can be found as follows:

$$h_0 = 4, \qquad h_1 = -30, \qquad h_2 = 60, \qquad h_3 = -35. \tag{24}$$

As Figure 3(b) shows, (23) approximates the optimal filter very well, and the approximation gets better as σ decreases.



(a) (b)

Fig. 3. (a) Comparison of the maximum energy compaction achieved in classical and graph filters. (b) Magnitude response of the optimal filter for $L = 3$ for different values of σ. Response of the approximation (24) is also shown.

4. GRAPH DEPENDENT DISCRETE SPECTRUM

In the previous section analysis of the energy compaction is based on the continuous spectrum. Although such an analysis is theoretically important, its practical importance is limited since graphs have finite number of eigenvalues. In this section, we take the eigenvalues of the graph into account and formulate the discrete counterpart of the problem in (9) as follows:

$$\rho(K) = \max_{h_k}\ \frac{1}{N}\sum_{i=1}^{K}|H(\lambda_i)|^2 \quad \text{s.t.} \quad \frac{1}{N}\sum_{i=1}^{N}|H(\lambda_i)|^2 = 1. \tag{25}$$

where $K < N$ determines the pass band "width" of the filter.

Following the formulation in Section 3, (25) can be reformulated as the following generalized Rayleigh quotient problem:

$$\max_{\boldsymbol{h}} \quad \boldsymbol{h}^* \, \boldsymbol{S}(K) \, \boldsymbol{h} \qquad \text{s.t.} \qquad \boldsymbol{h}^* \, \boldsymbol{S}(N) \, \boldsymbol{h} = 1, \tag{26}$$

where

$$\big(\boldsymbol{S}(K)\big)_{m,n} = \frac{1}{N} \sum_{i=1}^{K} \lambda_i^{m+n-2}. \tag{27}$$

Then, the optimum filter and the maximum amount of energy compaction can be found as the dominant eigenpair of the following generalized eigenvalue problem:

$$\boldsymbol{S}(K) \, \boldsymbol{h} = \rho \, \boldsymbol{S}(N) \, \boldsymbol{h}. \tag{28}$$

Although the problems in (14) and (26) have the same form, their characteristics differ from each other in two respects. Firstly, K in (26) is a discrete parameter as opposed to σ in (14) being a continuous. Nevertheless, they can be conceptually related as $\sigma = K/N$, which denotes the fraction of eigenvalues in the baseband of the graph spectrum. Secondly, and more importantly, the spectrum of a graph has a finite number of possibly repeated eigenvalues. Thus, the matrix $\boldsymbol{S}(N)$ in (26) may have a null-space unlike $\boldsymbol{Q}(1)$. More precisely, we have the following lemma:

Lemma 2. *Let $\bar{N}$ denote the number of distinct eigenvalues of the graph operator. If $L < \bar{N}$, then $\boldsymbol{S}(N) \succ \boldsymbol{0}$; if otherwise, $\boldsymbol{S}(N)$ has a null-space, hence positive semi-definite.*

When the matrix $\boldsymbol{S}(N)$ has a null-space it can be shown that the problem in (26) does not have a unique maximizer. Thus, the optimal energy compaction filter is not unique. This means that the order of the polynomial filter is larger than what is necessary, and a lower order filter can obtain the same amount of energy compaction. In most applications low orders are preferred in order to have filters that are localized on the graph. So, the condition in Lemma 2 is almost always satisfied in practice yielding a positive definite $\boldsymbol{S}(N)$. Moreover, when the order of the filter satisfies $L \geqslant \bar{N}$-1 *any* frequency response can be realized with a polynomial [21]. Thus, the maximum energy compaction becomes $\rho(K) = 1$ for all values of K.

Since $\boldsymbol{S}(N)$ depends on the eigenvalues of the underlying graph operator, a closed form expression for it does not exist in general. Nevertheless, it is possible to obtain closed forms in some specific cases. For example, if the graph is an undirected cycle of size N and its graph Laplacian is used as the operator, Lemma 2 of [22] reveals that $\boldsymbol{S}(N)$ has the following closed form

$$\big(\boldsymbol{S}(N)\big)_{m,n} = \binom{2m+2n-4}{m+n-2} \tag{29}$$

as long as the order of the filter satisfies $L < N/2$.

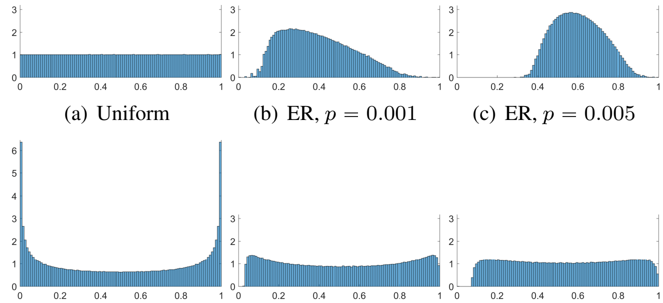
It is also important to note that $\boldsymbol{Q}(\sigma)$ and $\boldsymbol{S}(K)$ are asymptotically identical when the underlying eigenvalues are uniformly separated. That is, if $\lambda_i = i/N$ for $1 \leqslant i \leqslant N$, then

$$\lim_{N \to \infty} \boldsymbol{S}(\sigma N) = \boldsymbol{Q}(\sigma). \tag{30}$$

Thus, the problems in (14) and (26) are also asymptotically equivalent when the eigenvalues are separated uniformly. However, the spectrum of a graph is almost never distributed uniformly [23]. So, the solution to the energy compaction problem in (26) depends heavily on the underlying graph operator as we shall demonstrate next.

Figure 4 shows the histogram of the eigenvalues of the Laplacian of different examples of graphs including Erdos-Renyi (ER), random regular (RR) and the undirected cycle graph. We also consider the case of uniform eigenvalue separation (which does not correspond to a graph) as a reference. The size of the graphs is set to be $N = 10^4$, and the eigenvalues are scaled such that $0 \leqslant \lambda_i \leqslant 1$ since the scaling does not affect the generality of the results.

Figure 5 visualizes the numerical solution of (26) for the graphs considered above for filter order $L = 2$. In Figure 5(a) we consider the maximum amount of energy compaction with respect to

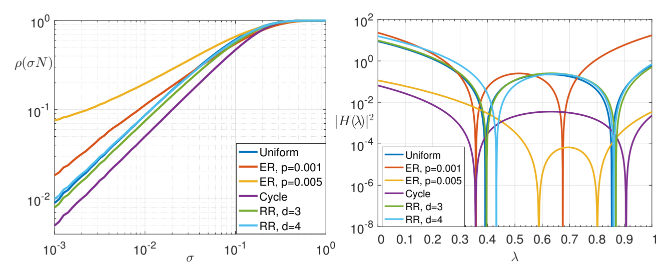


(a) Uniform (b) ER, $p = 0.001$ (c) ER, $p = 0.005$

(d) Undirected cycle (e) RR, $d = 3$ (f) RR, $d = 4$

Fig. 4. Histogram of the eigenvalues of (a) the uniform case, (b) Erdos-Renyi (ER) graph with $p = 0.001$, (c) ER graph with $p = 0.005$, (d) undirected cycle graph, (e) random regular (RR) graph with degree $d = 3$, and (f) RR graph with degree $d = 4$.

$\sigma = K/N$. In Figure 5(b) we show the response of the optimum filters for $\sigma = 0.1$. As seen clearly from the figures, both the maximum energy compaction and the optimal filter are affected by the distribution of the eigenvalues. Among all considered examples, the compaction filter for the ER graph with $p = 0.005$ has the most "concentrated" spectrum, and Figure 5(a) shows that the optimum filter for the ER graph can confine more energy in a band compared to the other graphs. Similarly, Figure 5(b) shows that the zeros of the optimum polynomial are located where the eigenvalues are denser. On the other hand, the undirected cycle graph has the most "spread-out" spectrum with two different peaks. Figure 5(a) shows that the optimal filter on the undirected cycle graph has the least amount of energy confinement. Correspondingly, zeros of the optimum polynomial are also spread-out from each other in order to accommodate the spread-out in the spectrum.



(a) (b)

Fig. 5. (a) Comparison of energy compaction on different graphs, (b) the optimal filters on different graphs for $\sigma = 0.1$.

5. CONCLUDING REMARKS & FUTURE DIRECTIONS

In this study we investigated the spectral concentration problem for polynomial graph filters and considered two approaches. In the first one, we assumed that the graph spectrum is continuous, in which case it reduced to the polynomial concentration problem studied by Slepian. We re-visited the problem and compared its solution with the classical spectral concentration problem. In the second approach, we took the discrete graph spectrum into consideration and formulated the problem accordingly. We showed that the maximum amount of energy compaction as well as the optimum filter depend on the spectrum of the underlying graph.

In future we will rigorously analyze properties of the optimum filter such as the behavior of the filter coefficients, and the distribution of the filter zeros. We will extend these results to band-pass filters as well. We will also study the relationship between the structure of the graph and the corresponding optimum filters and maximum possible compaction.

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