COMPOSITE SINGER ARRAYS WITH HOLE-FREE COARRAYS AND ENHANCED ROBUSTNESS

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ABSTRACT

In array processing, minimum redundancy arrays (MRA) can identify up to $\mathcal{O}(N^2)$ uncorrelated sources (the $\mathcal{O}(N^2)$ property) with N physical sensors, but this property is susceptible to sensor failures. On the other hand, uniform linear arrays (ULA) are robust, but they resolve only $\mathcal{O}(N)$ sources. Recently, the robust MRA (RMRA) was shown to possess the $\mathcal{O}(N^2)$ property and to be as robust as ULA. But finding RMRA is computationally difficult for large N. This paper proposes a novel array geometry called the composite Singer array, which is related to a classic paper by Singer in 1938, and to other results in number theory. For large N, composite Singer arrays could own the $\mathcal{O}(N^2)$ property and are as robust as ULA. Furthermore, the sensor locations for the composite Singer array can be readily computed by the proposed recursive procedure. These properties will also be demonstrated by using numerical examples.

Index Terms**—** Sparse arrays, difference coarrays, robustness, Singer arrays, composite arrays.

1. INTRODUCTION

Sparse arrays have drawn attention in array signal processing due to their capability of identifying $\mathcal{O}(N^2)$ uncorrelated sources with N physical sensors, in particular, more sources than sensors [1–4]. This $\mathcal{O}(N^2)$ property is in contrast to uniform linear arrays (ULA), which distinguish at most $N-1$ uncorrelated sources [5]. The $\mathcal{O}(N^2)$ property is because the difference coarray of sparse arrays, defined as the set of differences between sensor locations, has a central ULA segment of size $\mathcal{O}(N^2)$ [3, 4]. In particular, sparse arrays with the $\mathcal{O}(N^2)$ property include minimum redundancy arrays (MRA) [6], nested arrays [3], and coprime arrays [4], to name a few [7]. However, it was empirically observed that the structure of the difference coarray of MRA is not robust to sensor failures [8]. This phenomenon could lead to degradation of the estimation performance.

In early work, array robustness was studied based on array manifold [9], or in terms of peak sidelobe level [10]. In this paper, we focus on the theory of the essentialness property and the fragility, since they are closely related to difference coarrays [8]. A sensor at n in an array is *essential* if the deletion of that sensor (with all other sensors intact) modifies the difference coarray. The fragility is defined as the ratio of the number of essential sensors to the number of all sensors. It can be shown that the fragility is bounded between $2/N$ (the most robust) and 1 (the least robust). In particular, the ULA achieves the fragility $2/N$ while the MRA reaches the fragility 1 [8]. With these tools, the robust MRA (RMRA) was defined through an integer program such that RMRA enjoys a hole-free difference coarray with size $\mathcal{O}(N^2)$ and minimum fragility $2/N$ at the same time [11]. Nevertheless, solving for the sensor locations of RMRA is computationally expensive when N is large.

This paper continues to investigate novel array geometries other than RMRA, that own a hole-free difference coarray with size $\mathcal{O}(N^2)$ as well as minimum fragility $2/N$. This is done by the following steps. First, the composite array is constructed according to two array configurations $\mathbb{S}_1$ and $\mathbb{S}_2$ such that the difference coarray of the composite array has a larger size than the coarray of each of $\mathbb{S}_1$ and $\mathbb{S}_2$. Similar techniques can be found in earlier works [12–16]. We show that the composite array reduces the fragility, but it does not necessary achieve minimum fragility. Next, by combining Singer arrays [17], the newly proposed *supplementary array*, and the composite array, we obtain the *composite Singer array*, which owns a hole-free difference coarray, minimum fragility $2/N$, and a simple and recursive formulation for large N. The above design steps will be elaborated in detail.

Paper outline: Section 2 reviews the difference coarray and the Singer array, and introduces the composite array. Section 3 first studies the fragility of the composite array and then presents the supplementary array. It will also be shown that the composite Singer array enjoys a hole-free difference coarray with size $\mathcal{O}(N^2)$ and fragility $2/N$. Section 4 demonstrates numerical examples of the composite Singer array while Section 5 concludes this paper.

2. PRELIMINARIES

2.1. The Difference Coarray

Consider a linear array of sensors located at $n\lambda/2$, where n belongs to an integer-valued set $\mathbb{S}$ and λ is the wavelength of the incoming far-field and monochromatic sources. The direction-of-arrivial (DOA) of the ith source is denoted by $\theta_i \in [-\pi/2, \pi/2]$. Under the assumption of uncorrelateness of the source amplitudes and the additive noise term and other mild assumptions, the estimation of the DOA based on the physical array $\mathbb{S}$ can be converted into the estimation of the DOA based on the difference coarray $\mathbb{D}$ [1–4]. The difference coarray $\mathbb{D}$ is defined as

Definition 1. The difference coarray of the sensor array $\mathbb{S}$ is defined as $\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$.

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Array processing on $\mathbb{D}$ enjoys several advantages over that on $\mathbb{S}$. First, it was shown in [1, 3, 4] that exsiting DOA estimators on $\mathbb{S}$, such as the MUSIC algorithm, can be readily adapted to the autocorrelation vector on $\mathbb{D}$. Second, for some arrays with N physical sensors, their difference coarrays own $\mathcal{O}(N^2)$ degrees of freedom, which is much larger than N. This property admits to resolve more uncorrelated sources than sensors based on $\mathbb{D}$ [1–4]. Finally, it was demonstrated that processing on $\mathbb{D}$ typically has higher spatial resolution than that on the ULA with the same N [1, 3, 4].

In what follows, we will review some quantities related to the difference coarray. First, the central ULA segment is defined as $\mathbb{U} \triangleq \{0, \pm 1, \ldots, \pm m\}$, where m is the largest number such that $\{0, \pm 1, \ldots, \pm m\} \subseteq \mathbb{D}$. A hole $h \in \mathbb{Z}$ in the difference coarray is defined such that $\min(\mathbb{D}) \leq h \leq \max(\mathbb{D})$ but $h \notin \mathbb{D}$. An array has a hole-free difference coarray if there are no holes in $\mathbb{D}$, or equivalently $\mathbb{D} = \mathbb{U}$.

Definition 2. The weight function $w(m)$ of an array $\mathbb{S}$ is defined as the number of sensor pairs with coarray location m. That is, $w(m) = \left|\{(n_1, n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\}\right|$.

Next we will introduce the essentialness of sensors, which is fundamental to the robustness of arrays [8, 18].

Definition 3. Let $\mathbb{S}$ be a sensor array and $\mathbb{D}$ be its difference coarray. A sensor at $n \in \mathbb{S}$ is *essential* (or n is essential) if its removal from $\mathbb{S}$ changes the difference coarray. Namely, n is essential if $\overline{\mathbb{D}} \neq \mathbb{D}$, where $\overline{\mathbb{D}}$ is the difference coarray of $\overline{\mathbb{S}} \triangleq \mathbb{S} \backslash \{n\}$.

Based on Definition 3, we say that $n \in \mathbb{S}$ is *inessential* if n is not essential [8]. An array $\mathbb{S}$ is *maximally economic* if all the sensors in $\mathbb{S}$ are essential [18]. The fragility of an array is defined based on the essentialness of sensors [8]:

Definition 4. The fragility F of an array $\mathbb{S}$ is defined as the ratio of the number of essential sensors in $\mathbb{S}$ to the total number of sensors in $\mathbb{S}$.

Conceptually, the fragility quantifies the robustness of $\mathbb{S}$. For a fixed number of sensors, large F implies that there are more essential sensors so the difference coarray tends to change in the presence of sensor failures. Therefore the difference coarray is less robust to sensors failures. In particular, if the number of sensors $N \geq 4$, then $2/N \leq F \leq 1$ [8]. The ULA achieves $F = 2/N$ for $N \geq 4$, while maximally economic sparse arrays, such as minimum redundancy arrays (MRA) [6], minimum hole arrays (MHA) [19], and nested arrays [3], own $F = 1$.

Example 1. Consider the array $\mathbb{S} = \{0, 1, 3, 4, 5, 6\}$. According to Definition 1, the difference coarray is $\mathbb{D} = \{0, \pm 1, \ldots, \pm 6\}$. It is clear that $\mathbb{D} = \mathbb{U}$ so the difference coarray is hole-free. The weight function $w(4)$ is 2 since there exist two sensor pairs $(4, 0)$ and $(5, 1)$ with separation 4. It can be shown that the sensors at 0 and 6 are essential and the remaining sensors are inessential. Therefore, the fragility $F = 2/6 = 1/3$.

It was demonstrated in [20] that for the same number of sensors N, larger difference coarrays *usually* make them less robust to sensor failures, but there are exceptions. Given two arrays with the same N and the same difference coarray, one could be much more robust than the other. It is possible to design novel sparse arrays that strike a balance between the size and the robustness of the difference coarray. These arrays would be useful in applications where the estimation performance is controlled both by the size of the difference coarrays and by robustness properties.

The robust minimum redundancy array (RMRA) was recently proposed to have the largest hole-free difference coarray (like MRA) and minimum fragility $F = 2/N$ (like ULA), where N is the number of sensors [11]. At the same time the RMRA was proved to own $\mathcal{O}(N^2)$ degrees of freedom in the difference coarray, like MRA.

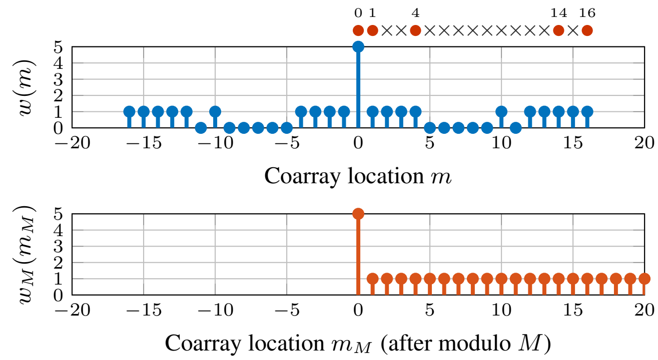

Fig. 1. (Top) The weight function $w(m)$ of the Singer array with $q = 2^2$ and $M = q^2 + q + 1 = 21$ and (bottom) the weight function $w_M(m_M)$ after the modulo-M operation.

However, for large N, it is computationally expensive to find the sensor locations of RMRA, because a combinatorial search is needed.

2.2. The Singer Array

The Singer array is a family of array geometries that are built on the following theorem in number theory [17, 21, 22]:

Theorem 1. Let q be a power of a prime number. Assume that $M = q^2 + q + 1$. Then there exist $q + 1$ integers $d_1, d_2, \ldots, d_{q+1}$ such that

1. $0 = d_1 < d_2 < \cdots < d_{q+1} < M$ and
2. The set of pairwise differences between d_i and d_j for $i \neq j$ is equivalent to $\{1, 2, \ldots, M - 1\}$, under the modulo-M operation. That is,

$$\begin{aligned} &\{((d_i - d_j))_M : i \neq j, \quad i, j = 1, 2, \ldots, q + 1\} \\ &= \{1,\ 2,\ \ldots,\ M - 1\}, \end{aligned} \tag{1}$$

where $((a))_b$ denotes the remainder of a divided by b.

The property in (1) is analogous to the hole-free property of the difference coarray [3, 6], but (1) is with respect to the modulo-M operation. Using Theorem 1, the Singer array is defined as [17, 22]

Definition 5. A Singer array with the parameter q (a power of a prime) is defined as the set $\{d_1,\ d_2,\ \ldots,\ d_{q+1}\}$, where the numbers $d_1, d_2, \ldots, d_{q+1}$ are given in Theorem 1.

Example 2. Let $q = 2^2 = 4$. Then $M = q^2 + q + 1 = 21$ and the Singer array is given by $\mathbb{S} = \{0, 1, 4, 14, 16\}$, as listed in [17, 22]. The geometry of the Singer array is depicted on the top of Fig. 1, where dots and crosses denote physical sensors and empty space. According to Definition 2, the weight function $w(m)$ is illustrated on the top of Fig. 1. It can be observed that the difference coarray of the Singer array has holes at $\pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 11$. However, under the modulo-M operation, we define

$$\begin{aligned} m_M &\triangleq ((m))_M, \\ w_M(m_M) &\triangleq \left|\{(n_1, n_2) \in \mathbb{S}^2 : ((n_1 - n_2))_M = m_M\}\right|. \end{aligned}$$

Here m_M denotes the differences folded within the range from 0 to $M - 1$ and $w_M(m_M)$ represents the number of sensor pairs with separation m_M modulo-M. The bottom of Fig. 1 shows the function $w_M(m_M)$ for the Singer array. It is deduced that each $m_M > 0$ corresponds to exactly one sensor pair, which is in accordance with Theorem 1.

The Singer array was applied to the proof of the $\mathcal{O}(N^2)$ property of MRA [13, 14] and has been used in active sensing as well [22].

Furthermore, the Singer arrays can be constructed using the methods in [17]. For other details and extensions of the Singer array, the interested readers are referred to [21].

2.3. The Composite Array

In this paper we will adapt the technique of duplicating a certain array geometry based on the geometry of another array, to form a composite array. This technique has in the past been used in proving the $\mathcal{O}(N^2)$ property of MRA [12, 14], and in constructing large MRAs [15] and generalized nested subarrays (GNSA) [16]. In this paper, we will reformulate this concept into a unified definition, called the *composite array*.

Definition 6. Suppose α is a positive integer. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ denote two sensor arrays with $\min(\mathbb{S}_1) = \min(\mathbb{S}_2) = 0$. Then the $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array $\mathbb{S}_{\mathrm{c}}$ is defined as

$$\mathbb{S}_{\mathrm{c}} \triangleq \alpha\mathbb{S}_1 + \mathbb{S}_2 = \{\alpha n_1 + n_2 : n_1 \in \mathbb{S}_1, n_2 \in \mathbb{S}_2\}, \quad (2)$$

where $\alpha > \max(\mathbb{S}_2)$.

Several existing array geometries are special cases of (2). For instance, in the proof of the $\mathcal{O}(N^2)$ property of MRA, the authors selected $\mathbb{S}_1$ to be the MRA, $\mathbb{S}_2$ to be the Singer array with parameter q, and $\alpha = q^2 + q + 1$ [13, 14]. As another example, the GNSA considers all the combinations of ULA, MRA, and nested arrays for $\mathbb{S}_1$ and $\mathbb{S}_2$ and the parameter α is chosen according to $\mathbb{S}_2$.

Using Definition 6, the difference coarray of $\mathbb{S}_{\mathrm{c}}$ is characterized by the following corollary [16, Proposition 1]:

Corollary 1. Let $\mathbb{S}_{\mathrm{c}}$ be a $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array. Suppose the difference coarrays of $\mathbb{S}_{\mathrm{c}}$, $\mathbb{S}_1$, and $\mathbb{S}_2$ are denoted by $\mathbb{D}_c$, $\mathbb{D}_1$, and $\mathbb{D}_2$, respectively. Then $\mathbb{D}_c = \alpha\mathbb{D}_1 + \mathbb{D}_2$.

Example 3. Let us consider a numerical example of the composite array. If $\alpha = 9$, $\mathbb{S}_1 = \{0, 1\}$ and $\mathbb{S}_2 = \{0, 1, 3, 4\}$, then according to Definition 6, the $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array $\mathbb{S}_{\mathrm{c}} = \{0, 1, 3, 4, 9, 10, 12, 13\}$. The difference coarray of $\mathbb{S}_{\mathrm{c}}$ is given by $\{0, \pm 1, \ldots, \pm 13\}$, which is hole-free and satisfies Corollary 1.

3. COMPOSITE SINGER ARRAYS FOR ROBUSTNESS

The earlier work related to the composite array, such as [13–16], focused on generating difference coarrays with large central ULA segments. In this paper, we will show that this technique can also make the array more robust to sensor failures, but minimum fragility is not necessary achieved. To overcome this issue, the *supplementary array* will be proposed in addition to the composite array so that their union could lead to a hole-free difference coarray *with* minimum fragility, as we shall elaborate later.

First, we have the following proposition for the fragility of a $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array.

Proposition 1. Assume that $\mathbb{S}_{\mathrm{c}}$ is a $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array. Then $F(\mathbb{S}_{\mathrm{c}}) \leq F(\mathbb{S}_1)$, where $F(\mathbb{S})$ denotes the fragility of $\mathbb{S}$.

Proof. First we will invoke the following property:

1. If $n_1 \in \mathbb{S}_1$ is inessential with respect to $\mathbb{S}_1$, then $\alpha n_1 + n_2$ is inessential with respect to $\mathbb{S}_{\mathrm{c}}$ for all $n_2 \in \mathbb{S}_2$.

The proof of Property 1 is as follows. Since n_1 is inessential with respect to $\mathbb{S}_1$, the difference coarray $\overline{\mathbb{D}}_1$ of $\overline{\mathbb{S}}_1 \triangleq \mathbb{S}_1 \backslash \{n\}$ is identical to $\mathbb{D}_1$. Let us consider the $(\alpha, \overline{\mathbb{S}}_1, \mathbb{S}_2)$-composite array $\overline{\mathbb{S}}_c$ with difference coarray $\overline{\mathbb{D}}_c$. Due to Corollary 1, we have $\overline{\mathbb{D}}_c = \alpha\overline{\mathbb{D}}_1 + \mathbb{D}_2 = \alpha\mathbb{D}_1 + \mathbb{D}_2 = \mathbb{D}_c$. Therefore removing $\alpha n_1 + n_2$ from $\mathbb{S}_{\mathrm{c}}$ for $n_2 \in \mathbb{S}_2$ does not modify the difference coarray, which proves this property.

Suppose the number of essential sensors and inessential sensors in $\mathbb{S}$ are denoted by $E(\mathbb{S})$ and $I(\mathbb{S})$, respectively. We have $E(\mathbb{S}) + I(\mathbb{S}) = |\mathbb{S}|$. Therefore, the fragility of $\mathbb{S}_{\mathrm{c}}$ becomes

$$F(\mathbb{S}_{\mathrm{c}}) = \frac{E(\mathbb{S}_{\mathrm{c}})}{|\mathbb{S}_{\mathrm{c}}|} = 1 - \frac{I(\mathbb{S}_{\mathrm{c}})}{|\mathbb{S}_{\mathrm{c}}|} \leq 1 - \frac{I(\mathbb{S}_1)|\mathbb{S}_2|}{|\mathbb{S}_{\mathrm{c}}|}, \quad (3)$$

where the inequality is because each inessential sensor in $\mathbb{S}_1$ corresponds to $|\mathbb{S}_2|$ sensors in $\mathbb{S}_{\mathrm{c}}$ (Property 1), and all these sensor locations do not overlap ($\alpha > \max(\mathbb{S}_2)$). Next using the relation $|\mathbb{S}_{\mathrm{c}}| = |\mathbb{S}_1||\mathbb{S}_2|$ leads to $F(\mathbb{S}_{\mathrm{c}}) \leq 1 - I(\mathbb{S}_1)/|\mathbb{S}_1| = F(\mathbb{S}_1)$. ☐

Proposition 1 indicates that the $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array $\mathbb{S}_{\mathrm{c}}$ tends to be more robust than $\mathbb{S}_1$. However the fragility of $\mathbb{S}_{\mathrm{c}}$ might not achieve the minimum $2/|\mathbb{S}_{\mathrm{c}}|$. In order to make the array geometry as robust as ULA, we introduce the *supplementary array* as follows:

Definition 7. Let P and Q be positive integers satisfying $Q > 2P(P-1)$. The supplementary array $\mathbb{S}_{\mathrm{supp}}$ with parameters P and Q is defined as

$$\mathbb{S}_{\mathrm{supp}} = \{u,\ uP,\ Q-u,\ Q-uP : \\ u = 0, 1, \ldots, P-1\}. \quad (4)$$

According to (4), the supplementary array $\mathbb{S}_{\mathrm{supp}}$ is the union of four ULAs with separations 1, P, 1, and P, respectively. $\mathbb{S}_{\mathrm{supp}}$ can also be divided into two parts $\{u, Q - uP\}$ and $\{uP, Q - u\}$. The latter is a reversed version of the former, where the reversed version of $\mathbb{S}$ is defined as $\{\max(\mathbb{S}) + \min(\mathbb{S}) - n : n \in \mathbb{S}\}$ [18]. Furthermore, the part of $\{u, Q - uP\}$ was utilized in [12, 14] to fill missing numbers in the differences. Even though the supplementary array is related to these earlier works, the novelty here is that we would like not only to fill the holes but also to achieve minimum fragility.

Using Definition 7, the supplementary array can be shown to have these properties, where the proof technique is analogous to those in [11, 14] and the details can be found in [23].

Proposition 2. Let $\mathbb{S}_{\mathrm{supp}}$ be the supplementary array with parameters P and Q. Let $\mathbb{D}_{\mathrm{supp}}$ be the difference coarray of $\mathbb{S}_{\mathrm{supp}}$. Define the new set

$$\mathbb{L} \triangleq \{1,\ 2,\ 3,\ \ldots,\ P^2 - P, \quad (5)$$

$$Q-1,\ Q-2,\ Q-3,\ \ldots,\ Q-(P^2-1)\}. \quad (6)$$

Then the following properties hold

1. $|\mathbb{S}_{\mathrm{supp}}| = 4P - 2$.
2. $\mathbb{L} \subseteq \mathbb{D}_{\mathrm{supp}}$.
3. Let $n \in \mathbb{S}_{\mathrm{supp}} \backslash \{0, Q\}$. Denote the difference coarray of $\overline{\mathbb{S}}_{\mathrm{supp}} \triangleq \mathbb{S}_{\mathrm{supp}} \backslash \{n\}$ by $\overline{\mathbb{D}}_{\mathrm{supp}}$. Then $\mathbb{L} \subseteq \overline{\mathbb{D}}_{\mathrm{supp}}$.

The third property in Proposition 2 indicates that the part of $\mathbb{L}$ in $\mathbb{D}_{\mathrm{supp}}$ is invariant to one sensor failure (except for those on the boundaries) in $\mathbb{S}_{\mathrm{supp}}$, which can be utilized to reduce the fragility of the composite array. This property motivates us to define the *composite Singer array* based on a composite array, a Singer array, and a supplementary array.

Definition 8. Let $\mathbb{S}_{\mathrm{c}}$ and $\mathbb{S}_{\mathrm{supp}}$ be defined in Definitions 6 and 7, respectively. We define *the composite Singer array* as $\mathbb{S}_{\mathrm{cs}} \triangleq \mathbb{S}_{\mathrm{c}} \cup \mathbb{S}_{\mathrm{supp}}$. The important items involved in the design are

1. $\mathbb{S}_1$, which is a sensor array with $F(\mathbb{S}_1) = 2/|\mathbb{S}_1|$ and owning a hole-free difference coarray. The number of sensors in $\mathbb{S}_1$ is at least 4.
2. $\mathbb{S}_2 = \{d_1, d_2, \ldots, d_{q+1}\}$, which is is a Singer array with the parameter q, as in Definition 5.

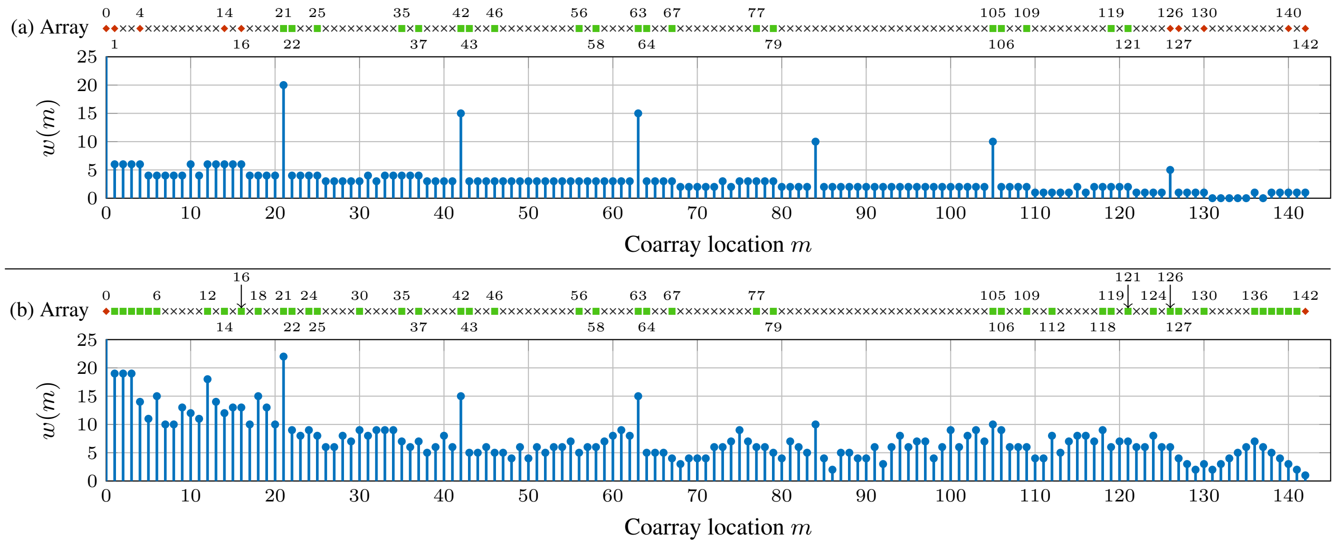

Fig. 2. The array configurations and the weight functions of (a) the $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array $\mathbb{S}_{\text{c}}$ and (b) the composite Singer array $\mathbb{S}_{\text{cs}}$. In the array configurations, red diamonds, green squares, and crosses represent essential sensors, inessential sensors, and empty space.

3. $\alpha = q^2 + q + 1$.

4. $P = \left\lceil \sqrt{2d_{q+1} + 1} \right\rceil$.

5. $Q = \alpha A_1 + d_{q+1}$, where A_1 is the aperture of $\mathbb{S}_1$.

In addition, P and Q satisfy $Q > 2P(P-1)$.

Using Definition 8, it can be shown that the composite Singer array has the following property [23].

Theorem 2. Suppose $\mathbb{S}_{\text{cs}}$ is a composite Singer array, as in Definition 8, whose difference coarray is denoted by $\mathbb{D}_{\text{cs}}$. Then $\mathbb{D}_{\text{cs}}$ is hole-free and $F(\mathbb{S}_{\text{cs}}) = 2/|\mathbb{S}_{\text{cs}}|$.

The proof of Theorem 2 is based on Theorem 1, Corollary 1, and Proposition 2.

Remark 1: Definition 8 can be viewed as a recursive procedure that starts with an array $\mathbb{S}_1$ having minimum fragility and a hole-free difference coarray, to generate another array $\mathbb{S}_{\text{cs}}$ with similar attributes. This concept can be applied repeatedly by substituting $\mathbb{S}_{\text{cs}}$ for $\mathbb{S}_1$ in Definition 8 to obtain another array $\mathbb{S}'_{\text{cs}}$.

Remark 2: Let N_1 be the number of sensors in $\mathbb{S}_1$. Suppose that $\mathbb{S}_1$ has the $\mathcal{O}(N^2)$ property. Then for sufficiently large q and N_1, the $\mathcal{O}(N^2)$ property holds for the composite Singer array. In particular, if 1) qN_1 is much larger than N_1 and q and 2) $|\mathbb{D}_1| = \mathcal{O}(N_1^2)$, then $|\mathbb{D}_{\text{cs}}| = \mathcal{O}(|\mathbb{S}_{\text{cs}}|^2)$. This is because according to Definition 8, $|\mathbb{S}_{\text{cs}}|$ is bounded between $|\mathbb{S}_{\text{c}}|$ and $|\mathbb{S}_{\text{c}}| + |\mathbb{S}_{\text{supp}}|$, which can be simplified as

$$|\mathbb{S}_{\text{c}}| = |\mathbb{S}_1||\mathbb{S}_2| = (q+1)N_1 = \mathcal{O}(qN_1). \quad (7)$$

$$|\mathbb{S}_{\text{c}}| + |\mathbb{S}_{\text{supp}}| = (q+1)N_1 + (4P-2) = \mathcal{O}(qN_1). \quad (8)$$

Eq. (8) is because the parameter $P \approx \sqrt{2d_{q+1}+1} = \mathcal{O}(q)$ [13,14]. Therefore $|\mathbb{S}_{\text{cs}}| = \mathcal{O}(qN_1)$. Next, due to Theorem 2, the difference coarray $\mathbb{D}_{\text{cs}}$ is given by $\{0, \pm 1, \ldots, \pm(\alpha \max(\mathbb{D}_1) + d_{q+1})\}$. Since the parameters $\alpha = \mathcal{O}(q^2)$, $\max(\mathbb{D}_1) = \mathcal{O}(N_1^2)$, and $d_{q+1} = \mathcal{O}(q^2)$, we have

$$|\mathbb{D}_{\text{cs}}| = 2(\alpha \max(\mathbb{D}_1) + d_{q+1}) + 1 = \mathcal{O}(q^2 N_1^2) = \mathcal{O}(|\mathbb{S}_{\text{cs}}|^2).$$

Therefore, the composite Singer array has the $\mathcal{O}(N^2)$ property.

4. NUMERICAL EXAMPLES

In this example, we will first construct a composite array $\mathbb{S}_{\text{c}}$ based on the Singer array. We select $\mathbb{S}_1 = \{0, 1, 2, 3, 5, 6\}$ (the RMRA with 6 sensors [11]), $\mathbb{S}_2$ as in Fig. 1, and $\alpha = 21$. The array geometry and the weight function are illustrated in Fig. 2(a), where red diamonds and green squares denote essential sensors and inessential sensors, respectively. The number of sensors in $\mathbb{S}_{\text{c}}$ is 30 and the aperture of $\mathbb{S}_{\text{c}}$ is 142. It can be deduced from Fig. 2(a) that the fragility of $\mathbb{S}_{\text{c}}$ is $10/30$ and the fragility of $\mathbb{S}_1$ is $2/6$, which confirm Proposition 1. The difference coarray $\mathbb{D}_c$ of $\mathbb{S}_{\text{c}}$ contains a central ULA segment up to 130 and there are holes at $131, 132, 133, 134, 135, 137$. As a result, $\mathbb{D}_c$ has a large central ULA segment, but $\mathbb{D}_c$ has holes and the fragility $F(\mathbb{S}_{\text{c}})$ is not minimum.

Next let us consider the composite Singer array $\mathbb{S}_{\text{cs}}$. Definition 8 leads to $P = 6$ and $Q = 142$, which satisfy the constraint that $Q > 2P(P-1)$ in Definitions 7 and 8. Fig. 2(b) illustrates the array geometry and the weight function of the composite Singer array $\mathbb{S}_{\text{cs}}$. It is observed that the number of sensors in $\mathbb{S}_{\text{cs}}$ becomes 46 but the aperture remains 142. It can be deduced from Fig. 2(b) that the difference coarray of $\mathbb{S}_{\text{cs}}$ is now hole-free and the fragility $F(\mathbb{S}_{\text{cs}}) = 2/|\mathbb{S}_{\text{cs}}|$, achieving the minimum. These results confirm Theorem 2.

5. CONCLUDING REMARKS

This paper proposed the composite Singer array, which owns a hole-free difference coarray (like MRA) and minimum fragility (like ULA). Unlike the RMRA, this array can be readily defined in terms of three arrays: (a) one array with a hole-free difference coarray and minimum fragility, (b) a Singer array, and (c) a supplementary array. Furthermore, the composite Singer array enjoys the $\mathcal{O}(N^2)$ property for large number of sensors.

In the future, it is of interest to study other array configurations with minimum fragility and hole-free difference coarrays. For instance, designing the parameters α, $\mathbb{S}_1$, and $\mathbb{S}_2$ properly could lead to a composite array $\mathbb{S}_{\text{c}}$ with fewer sensors than the composite Singer array, and satisfying the above-mentioned design criteria.

Another future direction is to extend this topic to other scenarios such as correlated sources or two-dimensional arrays.

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