(0,2) versions of exotic (2,2) GLSMs

Hadi Parsian, Eric Sharpe, Hao Zou

Dep't of Physics
Virginia Tech
850 West Campus Dr.
Blacksburg, VA 24061
varzi61@vt.edu, ersharpe@vt.edu, hzou@vt.edu

In this paper we extend work on exotic two-dimensional (2,2) supersymmetric gauged linear sigma models (GLSMs) in which, for example, geometries arise via nonperturbative effects, to (0,2) theories, and in so doing find some novel (0,2) GLSM phenomena. For one example, we describe examples in which bundles are constructed physically as cohomologies of short complexes involving torsion sheaves, a novel effect not previously seen in (0,2) GLSMs. We also describe examples related by RG flow in which the physical realizations of the bundles are related by quasi-isomorphism, analogous to the physical realization of quasi-isomorphisms in D-branes and derived categories, but novel in (0,2) GLSMs. Finally, we also discuss (0,2) deformations in various duality frames of other examples.

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1 Introduction

Over the last decade there have been numerous advances in understanding two-dimensional (2,2) supersymmetric gauged linear sigma models (GLSMs). These have included nonperturbative realizations of geometry in nonabelian (see *e.g.* [1, 2]) and abelian (see *e.g.* [3, 4]) GLSMs, and perturbative realizations of non-complete-intersections such as Pfaffians [5], as well numerous advances in other areas. Two-dimensional (0,2) GLSMs have also seen a number of advances over the last decade, but so far there has not been any work applying nonperturbative geometric realizations to (0,2) theories.

In this paper we begin to fill this gap, by describing some novel properties of (0,2) GLSMs that result from considering nonperturbatively realized geometries and other non-complete-intersections in (0,2) rather than (2,2) settings.

For one example, we find examples of bundles constructed physically in (0,2) GLSMs that involve short complexes of both bundles and skyscraper sheaves, whereas previously all such physically-realized monad constructions involved short complexes of bundles only. For another example, we find that bundles related by RG flow and dualities are sometimes constructed physically by quasi-isomorphic complexes, yielding a physical use for quasi-isomorphism outside of derived categories.

For another example, we find a physical realization of quasi-isomorphisms, outside of physical realizations of derived categories [6, 7], relating monad constructions for (0,2) theories related by RG flow.

We begin in section 2 by describing the physical realization of tangent bundles of branched double cover constructions first described in [3, 8]. These GLSM constructions are nonperturbative, in the sense that geometry is not realized perturbatively as the critical locus of a superpotential. These furnish the examples of bundles realized by extensions of torsion sheaves. We discuss both tangent bundles as well as (0,2) deformations of the theories. In broad brushstrokes, the rest of this paper concerns (0,2) versions of the theories discussed in [4]. In section 3 we discuss physical realizations of the tangent bundle of a Veronese embedding, and show how quasi-isomorphism plays a role in relating presentations of tangent bundles in theories related by RG flow. In section 4 we discuss (0,2) deformations of the Segre embeddings discussed in [4], and again see that physical realizations of tangent bundles in theories related by RG flow, are related mathematically by quasi-isomorphisms. In section 5 we describe (0,2) moduli of intersections $G(2, N) \cap G(2, N)$ in various duality frames, and for completeness we conclude in section 6 with a few concrete examples of anomaly-free (0,2) models on the Calabi-Yau $G(2, 5) \cap G(2, 5)$.

There are nonperturbatively-constructed geometries in both nonabelian [1] as well as abelian [3] GLSMs. Unfortunately, we do not have a simple realization of the tangent bundles for the nonabelian cases, and so we do not discuss (0,2) deformations or tangent bundles in phases of $G(2,N) \cap G(2,N)$ realized ala [1].

Other work on two-dimensional (0,2) theories from just the past few months includes [9–19].

2 Tangent bundles of branched double covers

Ordinarily in (0,2) GLSMs [20, 21], bundles are described as the cohomology of a monad, a three-term complex of vector bundles on the ambient space, in which each vector bundle corresponds to a set of massless worldsheet fermions.

In this section we will discuss examples in which the tangent bundle is realized physically in a different form, as an extension of a set of skyscraper sheaves. To our knowledge, the only previous cases in which anything analogous was described were in [22]; however, there the sheaves arose because the E or J maps failed to be injective or surjective, respectively, whereas by contrast here one is getting torsion sheaves as part of the original three-term complex.

We will analyze two examples from [3]. This paper described examples of abelian GLSMs with exotic phases, in which geometry was realized via nonperturbative effects, and geometries of different phases were not birational to one another. (See also [1] for nonabelian examples with analogous properties.) In broad brushstrokes, the examples in [3] describe, in one phase, complete intersections of quadrics, and in another phase, either branched double covers or noncommutative resolutions of branched double covers. We will restrict ourselves in this paper to cases describing ordinary branched double covers and not noncommutative resolutions.

2.1 First example: branched covers of $\mathbb{P}^1$

2.1.1 (2,2) locus

Our first example [3][section 4.1] is the GLSM for $\mathbb{P}^{2g+1}[2,2]$. We first recall the (2,2) theory, and then will describe (0,2) deformations. This is a $U(1)$ gauge theory with matter

- $2g+2$ chiral superfields ϕ_i of charge $+1$,
- 2 chiral superfields p_a of charge -2,

and superpotential

$$W \;=\; \sum_a p_a G_a(\phi) \;=\; \sum_{ij} \phi_i \phi_j A^{ij}(p),$$

where $G_a(\phi)$ are a pair of quadric polynomials and $A^{ij}(p)$ is a symmetric $(2g+2) \times (2g+2)$ matrix with entries that are linear in the ps.

For large FI parameter $r \gg 0$, the analysis of this GLSM is standard, and it describes a complete intersection of the two quadrics $\{G_a = 0\}$ in $\mathbb{P}^{2g+2}$. Note that for $g = 1$, this is $\mathbb{P}^3[2,2]$, an elliptic curve.

For $r \ll 0$, the analysis of this example is more exotic. The D terms imply that not all the p's can vanish, in which case the superpotential acts as a mass matrix for the ϕ fields. Naively, this phase then appears to describe a sigma model on $\mathbb{P}^1$; however, since we know that for $g = 1$ the GLSM describes a Calabi-Yau, the $r \ll 0$ phase cannot describe a non-Calabi-Yau, and so this cannot be the answer.

To understand this phase, we must take into account the fact that generically on the space of p's, the only massless fields have charge 2 rather than one. Theories with nonminimal charges – equivalently, theories in which nonperturbative sectors are restricted – were analyzed in [23–25]. In particular, [8] argued that in two-dimensional gauge theories with nonminimal charges, the theory 'decomposes' into a disjoint union of theories. In the present case, this means that generically on the space of p's, the theory describes a double cover of $\mathbb{P}^1$. Further analysis [3] shows that this is a branched double cover of $\mathbb{P}^1$, branched away from the locus $\{\det A = 0\}$, which has degree $2g + 2$. This is precisely a genus g curve. In particular, for $g = 1$, both the $r \ll 0$ and $r \gg 0$ phases describe an elliptic curve, exactly as expected.

As a consistency check, let us compare Witten indices. To do this, we need to take into account the discrete Coulomb vacua which exist in the $r \ll 0$ phase. These arise as the solutions to

$$\sigma^{2g+2}(-2\sigma)^{-2}(-2\sigma)^{-2} \;=\; q,$$

which has $2g - 2$ solutions. A genus g Riemann surface has $\chi = 2 - 2g$, so between the Higgs and Coulomb branches, we see that altogether the Witten index of the $r \ll 0$ phase is

$$(2g - 2) + (2 - 2g) \;=\; 0.$$

It is straightforward to show that for the $r \gg 0$ phase, $\chi(\mathbb{P}^{2g+1}[2,2]) = 0$. Hence both phases have the same (vanishing) Witten index. (See also [26] for a more detailed analysis of Witten indices in this and related examples.)

Now, let us turn to the physical realization of the tangent bundle of the genus g curve appearing in the $r \ll 0$ phase. Locally over the space of p vevs, the left-moving fermions include a left-moving gaugino λ_-, the superpartners ψ_{pa} of the p fields, and the superpartners

$\psi_{\phi i}$ of the ϕ fields. However, the latter are only massless at special points on the moduli space, specifically the points where $\{\det A = 0\}$. This suggests that the tangent bundle should be given as the cohomology of the complex

$$0 \longrightarrow \mathcal{O} \xrightarrow{E_a} \mathcal{O}(2)^2 \xrightarrow{*} \oplus \mathcal{O}_p \longrightarrow 0, \tag{1}$$

where each $\mathcal{O}_p$ is a skyscraper sheaf. The map $E_a \propto p_a$ arises from the usual analysis of (0,2) theories [20, 21]. We take the second map to be

$$* = \frac{\partial}{\partial p_a} \det A(p).$$

This is determined by the need for this sequence to be a complex: from homogeneity of the matrix A,

$$* \circ E_a = p_a \frac{\partial}{\partial p_a} \det A(p) \propto \det A(p),$$

which vanishes over the skyscraper sheaves above.

Mathematically, we can understand this[1] as a special case of the Hurwitz formula, which can be described as follows. Let $\pi : X \to S$ be a finite cover of smooth varieties, and suppose that π has simple ramification, meaning that the branch divisor $B \subset S$ is smooth, and that over a neighborhood of each point of B, the cover π looks like a ramified cover plus non-intersecting sheets. Let $D \subset X$ denote the ramification divisor; in the case of simple ramification, π is an isomorphism between D and B. In this case, there is a short exact sequence

$$0 \longrightarrow TX \longrightarrow \pi^* TS \longrightarrow i_* N_{D/X} \longrightarrow 0, \tag{2}$$

where $N_{D/X} = \mathcal{O}_X(D)|_D$ is the normal bundle of D in X, and $i : D \hookrightarrow X$ is inclusion.

In the present case, for Σ the genus g curve realized as a branched double cover of $\mathbb{P}^1$, branched over the divisor D consisting of $2g + 2$ points,

$$0 \longrightarrow T\Sigma \longrightarrow \pi^* T\mathbb{P}^1 \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

It is straightforward to compare this to our GLSM result above (after normalizing the charges of the p_a to be 1 in mathematics conventions, rather than 2). There, note that the cokernel of the map $\mathcal{O} \to \mathcal{O}(2)^2$ is $\pi^* T\mathbb{P}^1$, so the we see that the GLSM sequence is equivalent to

$$\pi^* T\mathbb{P}^1 \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

which by virtue of the Hurwitz result above, has cohomology given by $T\Sigma$, as desired.

In passing, we should mention there is an analogousr construction of vector bundles described in [27][section 6.2.7], as an extension of an ideal sheaf $\mathcal{I}$ rather than a torsion sheaf:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I} \longrightarrow 0.$$

[1] We would like to thank T. Pantev for explaining this to us.

We do not claim to have a physical realization of this construction in GLSMs, but thought it useful to mention the analogy.

2.1.2 (0,2) deformations

Before going on to our next example, let us pause to discuss (0,2) deformations of the (2,2) theory above. To review, so far we have discussed a (2,2) theory with superpotential

$$W \;=\; \sum_a p_a G_a(\phi) \;=\; \sum_{ij} \phi_i \phi_j A^{ij}(p),$$

for G_a a set of quadrics in the ϕ's, and A^{ij} a symmetric $(2g+2)\times(2g+2)$ matrix, with entries linear in the p's. In (0,2) language, this would be described by potential functions

$$E_i \;=\; -\sigma\phi_i, \quad E_a \;=\; 2\sigma p_a,$$
$$J_i \;=\; \sum_a p_a \frac{\partial G_a}{\partial \phi_i}, \quad J_a \;=\; G_a,$$

where on the (2,2) locus, each J is a derivative of W.

In principle, we can define a (0,2) deformation by replacing the J_i above with

$$J_i \;=\; \sum_a p_a \left(\frac{\partial G_a}{\partial \phi_i} \;+\; G_{ai}(\phi) \right),$$

where the G_{ai} are a set of (linear) functions of ϕ obeying

$$\sum_{a,i} \phi_i p_a G_{ai} \;=\; 0$$

(so that $EJ = 0$ is obeyed).

Now, for convenience, define

$$B^{ij}(p) \;=\; \frac{1}{2} \sum_a p_a \frac{\partial}{\partial \phi_j} G_{ai}(\phi),$$

so that (since G_{ai} is linear in ϕs)

$$J_i \;=\; 2 \sum_j \phi_j \left(A^{ij}(p) + B^{ij}(p) \right).$$

Note that the potential term derived from J_a is quartic in ϕs, whereas the potential term derived from J_i is quadratic in ϕs, and so the quantity $A^{ij}(p) + B^{ij}(p)$ acts as a mass matrix for the ϕs.

Now, let us consider the phases of this GLSM. For $r \gg 0$, we have a (0,2) deformation of a complete intersection of quadrics, here $\mathbb{P}^{2g+1}[2,2]$. The (0,2) deformation in this phase acts as a modification of the left-moving gauge bundle.

For $r \ll 0$, the analysis is also very similar to the (2,2) locus, except that because the mass matrix is $A + B$ instead of just A, the branched double cover of the space of p's is branched over the locus

$$\det(A + B) = 0,$$

rather than the locus $\{\det A = 0\}$. Thus we see that the (0,2) deformation of the complete intersection has, as its $r \ll 0$ phase, a slightly different geometry than one would have obtained on the (2,2) locus.

Such a result is not unusual in (0,2) theories, where the phases are determined by the gauge bundle rather than the complete intersection *per se* [20, 21].

2.2 Second example: branched covers of $\mathbb{P}^2$

Our second example, from [3][section 2.8], involves the GLSM for $\mathbb{P}^5[2,2,2]$. This is a $U(1)$ gauge theory with

- 6 chiral superfields ϕ_i of charge 1,
- 3 chiral superfields p_a of charge -2,

and a superpotential

$$W \;=\; \sum_a p_a G_a(\phi) \;=\; \sum_{ij} \phi_i \phi_j A^{ij}(p),$$

where the G_a are quadric polynomials and A^{ij} is a symmetric 6×6 matrix with entries linear in the p's.

For large FI parameter $r \gg 0$, the analysis is standard and the GLSM describes the complete intersection $\mathbb{P}^5[2,2,2]$, which is a K3 surface.

The analysis of the other phase, $r \ll 0$, proceeds as above. From the D terms, the p's are not all zero, hence the superpotential defines a mass matrix for the ϕ_i over the space of p's, a $\mathbb{P}^2$. Because at generic points the p's are nonminimally charged and the only massless fields, physics sees a branched double cover of $\mathbb{P}^2$, branched over the degree six locus $\{\det A = 0\}$. Such a branched double cover is another K3 surface, and so we see that both phases in this model correspond to K3 surfaces.

Proceeding as before, the left-moving fermions describe the tangent bundle as the cohomology of the short complex

$$0 \longrightarrow \mathcal{O} \stackrel{E_a}{\longrightarrow} \mathcal{O}(2)^3 \stackrel{*}{\longrightarrow} \mathcal{O}(12) \otimes \mathcal{O}_D \longrightarrow 0, \tag{3}$$

where

$$E_a \,=\, p_a, \quad * \,=\, \frac{\partial}{\partial p_a} \det A(p),$$

The left-most $\mathcal{O}$ corresponds to the left-moving gaugino, the middle $\mathcal{O}(2)^3$ from the superpartners of the p_a, and the right-most term from the superpartners of the ϕ_i, massless only along the locus $D \equiv \{\det A = 0\}$. This is a complex due to homogeneity of the matrix $A^{ij}(p)$:

$$* \circ E_a \,=\, p_a \frac{\partial}{\partial p_a} \det A(p) \,\propto\, \det A(p),$$

which vanishes along D. As before, the superpartners of the ϕ_i are not themselves charge 12 objects, but correspond to a term coupling to the line bundle $\mathcal{O}(12)$ ultimately because they are only supported along the locus D.

Now, let us compare to the mathematics prediction. In this case, the Hurwitz formula (2) says

$$0 \longrightarrow T(\mathrm{K3}) \longrightarrow \pi^* T\mathbb{P}^2 \longrightarrow (\pi^* \mathcal{O}(6)) \otimes \mathcal{O}_D \longrightarrow 0.$$

Normalizing the charge of p_a to be 1 instead of 2, we see that the sequence (3) above matches the Hurwitz prediction for this case.

3 Quasi-isomorphism and the tangent bundle of Veronese embeddings

In this section we will see examples of theories related by RG flow in which the physical realizations of the tangent bundles are related mathematically by quasi-isomorphisms, a trick previously only seen in discussions of D-branes and derived categories.

Consider a Veronese embedding of degree d, mapping $\mathbb{P}^n$ to a projective space of dimension

$$N \,=\, \left(\begin{array}{c} n+d \\ d \end{array} \right) - 1.$$

The corresponding GLSM [4] is a $U(1)$ gauge theory with matter:

- $n+1$ chiral superfields x_i of charge 1,
- $N+1$ chiral superfields $y_{i_1 \cdots i_d}$ (symmetric in their indices) of charge d,

- $N+1$ chiral superfields $p_{i_1\cdots i_d}$ (symmetric in their indices) of charge d,

with superpotential

$$W \;=\; p_{i_1\cdots i_d}\left(y_{i_1\cdots i_d} - x_{i_1}\cdots x_{i_d}\right).$$

Now, the geometry described by this GLSM is technically the graph of the Veronese embedding, which is isomorphic to the original $\mathbb{P}^n$. This projective space by itself does not have any tangent bundle deformations; however, the physical realization of its tangent bundle is related to that of $\mathbb{P}^n$ by quasi-isomorphism, a relationship ordinarily only encountered in derived categories [6, 7].

Specifically, the tangent bundle is realized in the GLSM above as the cohomology of the following monad over $\mathbb{P}^n$:

$$0 \longrightarrow \stackrel{E}{\longrightarrow} \mathcal{O}(1)^{n+1} \oplus \mathcal{O}(d)^{N+1} \stackrel{J}{\longrightarrow} \mathcal{O}(d)^{N+1} \longrightarrow 0, \tag{4}$$

where the left-most $\mathcal{O}$ corresponds to the gaugino λ_-, the middle bundle corresponds to the superpartners of x_i, $y_{i_1\cdots i_d}$, and the right-most bundle $\mathcal{O}(d)^{N+1}$ corresponds to the superpartners of the $p_{i_1\cdots i_d}$, and

$$E \;=\; (x_i, y_{i_1\cdots, i_d}), \quad J \;=\; (-x_{i_1}\cdots x_{i_{d-1}}, 1).$$

($EJ = 0$ along the critical locus of the superpotential, namely the $\mathbb{P}^n$.) Because of the presence of the identity maps in the J's, arising from

$$J_{y_{j_1\cdots j_d}} \;=\; \frac{\partial}{\partial y_{j_1\cdots j_d}}\left(y_{i_1\cdots i_d} - x_{i_1}\cdots x_{i_d}\right),$$

the same tangent bundle is obtained from the cohomology of the complex

$$0 \longrightarrow \mathcal{O} \stackrel{x_i}{\longrightarrow} \mathcal{O}(1)^{n+1}. \tag{5}$$

Mathematically, the two complexes (4) and (5) are said to be quasi-isomorphic, as claimed. In the next section we will see further examples of quasi-isomorphisms relating physical realizations of bundles in theories related by RG flow.

4 Deformations of tangent bundles of Segre embeddings

The Segre embedding is an embedding of a product $\mathbb{P}^n \times \mathbb{P}^m$ in a higher-dimensional projective space. Mathematically, it is the map

$$s:\; \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^{(n+1)(m+1)-1}$$

defined by

$$[x_0,\ldots,x_n]\times[y_0,\ldots,y_m]\mapsto[x_0y_0,x_0y_1,\ldots,x_ny_m].$$

A (2,2) GLSM realizing the Segre embedding was described in [4], and is given as follows. It is a $U(1)\times U(1)$ gauge theory with matter

- $n+1$ chiral superfields x_i of charge $(1,0)$,
- $m+1$ chiral superfields y_i of charge $(0,1)$,
- $(n+1)(m+1)$ chiral superfields z_{ij} of charge $(1,1)$,
- $(n+1)(m+1)$ chiral superfields p_{ij} of charge $(-1,-1)$,

and with superpotential

$$W=\sum_{i,j}p_{ij}G_{ij}(x,y,z)=\sum_{i,j}p_{ij}(z_{ij}-x_iy_j).$$

We will see momentarily that physics realizes the RG flow from this model to that for $\mathbb{P}^n\times\mathbb{P}^m$, and its (0,2) deformations, via a quasi-isomorphism, just as in the last section.

Now, the tangent bundle of $\mathbb{P}^n\times\mathbb{P}^m$ admits deformations, and this is in fact used as a canonical example in discussions of quantum sheaf cohomology in (0,2) theories, see for example [28–33]. Mathematically, these deformations of the tangent bundle are given as a cokernel $\mathcal{E}$, where

$$0\longrightarrow\mathcal{O}^2\stackrel{*}{\longrightarrow}\mathcal{O}(1,0)^{n+1}\oplus\mathcal{O}(0,1)^{m+1}\longrightarrow\mathcal{E}\longrightarrow 0,$$

with

$$*=\left[\begin{array}{cc}Ax & Bx\\ Cy & Dy\end{array}\right],$$

with A,B $(n+1)\times(n+1)$ matrices and C,D $(m+1)\times(m+1)$ matrices. In effect, this is a deformation of two copies of the Euler sequences for the tangent bundles of the two separate projective spaces, reducing to the tangent bundle in the special case that A, D are the identity and $B=0=C$. Physically, in a (0,2) GLSM, the map $*$ is realized in the E potentials associated to the Fermi superfields associated with $\mathcal{O}(1,0)^{n+1}\oplus\mathcal{O}(0,1)^{m+1}$.

The (0,2) deformations above also exist in the Segre embedding, as expected. For completeness, we list them here: Define

$$E_{xi}=\sigma_1(Ax)_i+\sigma_2(Bx)_i,\quad E_{yj}=\sigma_1(Cy)_j+\sigma_2(Dy)_j,$$

$$J_{xi}=-p^{ij}y_j,\quad J_{yj}=-p^{ij}x_i,$$

$$E_{zij} = \sigma_1(A_{ik}z_{kj} + C_{jk}z_{ik}) + \sigma_2(B_{ik}z_{kj} + D_{jk}z_{ik}), \quad J_{zij} = p^{ij},$$
$$E_{pij} = -\sigma_1(A_{ki}p^{kj} + C_{kj}p^{ik}) - \sigma_2(B_{ki}p^{kj} + D_{kj}p^{ik}), \quad J_{pij} = z_{ij} - x_i y_j.$$

One can show that $EJ = 0$.

Next, let us compare complexes. Recall the analogue of the Euler complex for the deformation $\mathcal{E}$ of the tangent bundle of $\mathbb{P}^n \times \mathbb{P}^m$ has the form

$$0 \longrightarrow \mathcal{O}^2 \stackrel{E}{\longrightarrow} \mathcal{O}(1,0)^{n+1} \oplus \mathcal{O}(0,1)^{m+1} \longrightarrow \mathcal{E} \longrightarrow 0.$$

The analogous complex for the tangent bundle deformation of the Segre embedding is

$$0 \longrightarrow \mathcal{O}^2 \stackrel{E'}{\longrightarrow} \mathcal{O}(1,0)^{n+1} \oplus \mathcal{O}(0,1)^{m+1} \oplus \mathcal{O}(1,1)^{(n+1)(m+1)} \stackrel{J}{\longrightarrow} \mathcal{O}(1,1)^{(n+1)(m+1)} \longrightarrow 0.$$

In this case, the tangent bundle deformation $\mathcal{E}$ is the cohomology of this complex. It is straightforward to check that the complex above is quasi-isomorphic to the complex

$$0 \longrightarrow \mathcal{O}^2 \stackrel{E}{\longrightarrow} \mathcal{O}(1,0)^{n+1} \oplus \mathcal{O}(0,1)^{m+1},$$

and so we see again that quasi-isomorphism is the mathematical realization of RG flow in (0,2) theories, just as it is in the physical realization of derived categories.

5 (0,2) deformations of $G(2,N) \cap G(2,N)$

In [4], a (2,2) GLSM was given for the Calabi-Yau constructed as the self-intersection of the Grassmannian $G(2,5)$, as well as several dual descriptions of that GLSM. In this section, we will describe the deformations of that GLSM in its various duality frames, and compare the results.

5.1 First description

The first (2,2) GLSM for $G(2,N) \cap G(2,N)$, presented in [4][section 4.1], was as a

$$\frac{U(1) \times SU(2) \times SU(2)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

gauge theory with matter

- N chiral multiplets ϕ_a^i in the $(\mathbf{2},\mathbf{1})_1$ representation,
- N chiral multiplets $\tilde{\phi}_{a'}^j$ in the $(\mathbf{1},\mathbf{2})_1$ representation,

- $(1/2)N(N-1)$ chiral multiplets $p_{ij} = -p_{ji}$ in the $(\mathbf{1},\mathbf{1})_{-2}$ representation,

with superpotential
$$W = \sum_{i<j} p_{ij} \left(f^{ij}(B) - \tilde{B}^{ij} \right),$$
where
$$B^{ij} = \epsilon^{ab} \phi^i_a \phi^j_b, \quad \tilde{B}^{ij} = \epsilon^{a'b'} \tilde{\phi}^i_{a'} \tilde{\phi}^j_{b'}$$
are the baryons in each $SU(2)$ factor, and $f^{ij}(x)$ define a linear isomorphism on the homogeneous coordinates of $\mathbb{P}^{(1/2)N(N-1)-1}$, defining the deformation of one of the copies of the Plücker embedding. Put another way,
$$f^{ij}(B) = f^{ij}_{k\ell} B^{k\ell}$$
for a constant invertible matrix $f^{ij}_{k\ell}$. Each $\mathbb{Z}_2$ factor in the gauge group linked the center of one of the two $SU(2)$'s with a $\mathbb{Z}_2$ subgroup of $U(1)$, and it is straightforward to check that the matter is invariant.

In this section, we shall describe (0,2) deformations of this theory.

The tangent bundle defined implicitly by this GLSM in its $r \gg 0$ phase is given[2] by the cohomology of the sequence
$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O}(\mathbf{3},\mathbf{1})_0 \oplus \mathcal{O}(\mathbf{1},\mathbf{3})_0 \stackrel{E}{\longrightarrow} \mathcal{O}(\mathbf{2},\mathbf{1})_1^{\oplus N} \oplus \mathcal{O}(\mathbf{1},\mathbf{2})_1^{\oplus N} \stackrel{J}{\longrightarrow} \mathcal{O}(\mathbf{1},\mathbf{1})_2^{\oplus (1/2)N(N-1)} \longrightarrow 0.$$
In our notation, $\mathcal{O}(\mathbf{n},\mathbf{m})_p$ is the bundle defined by representation $\mathbf{n}$ of the first $SU(2)$, $\mathbf{m}$ of the second $SU(2)$, and charge p of the $U(1)$ factor. The leftmost factor, $\mathcal{O} \oplus \mathcal{O}(\mathbf{3},\mathbf{1})_0 \oplus \mathcal{O}(\mathbf{1},\mathbf{3})_0$, is defined by the gauginos in the theory. The middle factor is defined by the chiral multiplets ϕ^i_a, $\tilde{\phi}^i_{a'}$. The rightmost factor is defined by the p_{ij}. As a consistency check, note that the rank of the resulting bundle is given by
$$2N + 2N - 7 - (1/2)N(N-1)$$
coinciding with the expected dimension given in [4][section 3.2.1].

The $r \ll 0$ phase is realized nonperturbatively in the form of [1], so as mentioned in the introduction, we shall not try to write down a purely mathematical description of the tangent bundle.

Next, we consider $(0,2)$ deformations. The E-terms are
$$\begin{aligned} E_{p_{i_1 i_2}} &= -\sigma \left(\tilde{N}^j_{i_1} p_{j i_2} - \tilde{N}^j_{i_2} p_{j i_1} \right), \\ E^i_a &= \sigma^b_a \phi^i_b + N^i_j \sigma \phi^j_a, \\ \tilde{E}^i_{a'} &= \tilde{\sigma}^{b'}_{a'} \tilde{\phi}^i_{b'} + \tilde{N}^i_j \sigma \tilde{\phi}^j_{a'}, \end{aligned}$$

[2] See *e.g.* [34] for a discussion of physical realizations of tangent bundles of PAX and PAXY models, which form the prototype for this observation.

where N^i_j, $\tilde{N}^i_j$ are related by the constraint

$$N^k_{[j_1} f^{i_1 i_2}_{j_2]k} \; = \; \tilde{N}^{[i_1}_k f^{i_2]k}_{j_1 j_2}, \tag{6}$$

with J terms

$$\begin{aligned} J_{p_{i_1 i_2}} &= f^{i_1 i_2}_{j_1 j_2} B^{j_1 j_2} - \tilde{B}^{i_1 i_2}, \\ J_{\phi^k_a} &= p_{i_1 i_2} f^{i_1 i_2}_{j_1 j_2} \frac{\partial B^{j_1 j_2}}{\partial \phi^k_a}, \\ J_{\tilde{\phi}^k_{a'}} &= -p_{i_1 i_2} \frac{\partial \tilde{B}^{i_1 i_2}}{\partial \tilde{\phi}^k_{a'}}, \end{aligned}$$

where σ^a_b is traceless. It is straightforward to check that $EJ = 0$, as required by supersymmetry.

The (2,2) locus is given by taking

$$N^i_j \; = \; \delta^i_j \; = \; \tilde{N}^i_j,$$

which is easily checked to satisfy condition (6).

As a demonstration that other solutions to constraint (6) exist, the reader can verify that in the case

$$f^{i_1 i_2}_{j_1 j_2} \; = \; \frac{1}{2} \left(\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \; - \; \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right),$$

constraint (6) is satisfied for

$$N^i_j \; = \; \alpha \delta^i_1 \delta^2_j \; = \; \tilde{N}^i_j,$$

where α is a constant.

5.2 Double dual description

Next, we turn to the 'double dual' of this GLSM described in [4][section 4.2], obtained by dualizing both of the $SU(2)$ factors in the GLSM for $G(2,N) \cap G(2,N)$ using the duality described in [2]. The result is a

$$\frac{U(1) \times Sp(N-3) \times Sp(N-3)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

gauge theory with

- N fields φ^a_i in the $(\mathbf{N-3}, \mathbf{1})_{-1}$ representation,
- N fields $\tilde{\varphi}^{a'}_i$ in the $(\mathbf{1}, \mathbf{N-3})_{-1}$ representation,

- $(1/2)N(N-1)$ fields $b^{ij} = -b^{ji}$ in the $(\mathbf{1},\mathbf{1})_2$ representation,
- $(1/2)N(N-1)$ fields $\tilde{b}^{ij} = -\tilde{b}^{ji}$ in the $(\mathbf{1},\mathbf{1})_2$ representation,
- $(1/2)N(N-1)$ fields $p_{ij} = -p_{ji}$ in the $(\mathbf{1},\mathbf{1})_{-2}$ representation,

with superpotential

$$\begin{aligned} W &= \sum_{i<j} p_{ij} \left(f^{ij}(b) - \tilde{b}^{ij} \right) \, + \, \varphi_i^a \varphi_j^b J_{ab} b^{ij} \, + \, \tilde{\varphi}_i^{a'} \tilde{\varphi}_j^{b'} J_{a'b'} \tilde{b}^{ij}, \\ &= \left(A(p)_{ij} + \varphi_i^a \varphi_j^b J_{ab} \right) b^{ij} \, + \, \left(C(p)_{ij} + \tilde{\varphi}_i^{a'} \tilde{\varphi}_j^{b'} J_{a'b'} \right) \tilde{b}^{ij}, \end{aligned}$$

where J is the antisymmetric symplectic form, and $A(p)$, $C(p)$ are matrices that can be derived from the first line of the expression for the superpotential.

The $r \gg 0$ phase realizes geometry nonperturbatively in the sense of [1], so as described in the introduction, we shall not try to write down a purely mathematical description of the tangent bundle. The $r \ll 0$ phase, on the other hand, can be described perturbatively.

In the $r \ll 0$ phase, D terms imply that not all of the φ_i^a, $\tilde{\varphi}_i^{a'}$, and p_{ij} can vanish. The tangent bundle is built physically[3] as the cohomology of the complex

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O}((\mathrm{adj},\mathbf{1})_0) \oplus \mathcal{O}(\mathbf{1},\mathrm{adj})_0) \stackrel{E}{\longrightarrow} A \stackrel{J}{\longrightarrow} \mathcal{O}((\mathbf{1},\mathbf{1})_{-2})^{\oplus (2)(1/2)N(N-1)} \longrightarrow 0,$$

where

$$A \, = \, \mathcal{O}((\mathbf{N-3},\mathbf{1})_{-1})^{\oplus N} \oplus \mathcal{O}((\mathbf{1},\mathbf{N-3})_{-1})^{\oplus N} \oplus \mathcal{O}((\mathbf{1},\mathbf{1})_{-2})^{\oplus (1/2)N(N-1)}.$$

The leftmost terms are from the gauginos, the middle terms (A) are from φ_i^a, $\tilde{\varphi}_i^{a'}$, and p_{ij}, and the rightmost terms are from b^{ij}, $\tilde{b}^{ij}$. As a consistency check, note that for $N=5$ the rank of this bundle is

$$(2)(5) + (2)(5) + 10 - 7 - 20 \, = \, 3,$$

as expected for a threefold.

We can describe (0,2) deformations of this theory as follows. We take

$$\begin{aligned} E_{p_{ij}} &= -\sigma \left(\tilde{N}_i^{\ell} p_{\ell j} - \tilde{N}_j^{\ell} p_{\ell i} \right), \\ E_{b^{ij}} &= \sigma \left(N_k^i b^{kj} - N_k^j b^{ki} \right), \\ E_{\tilde{b}^{ij}} &= \sigma \left(\tilde{N}_k^i \tilde{b}^{kj} - \tilde{N}_k^j \tilde{b}^{ki} \right), \\ E_{\varphi_i^a} &= \sigma_b^a \varphi_i^b - \sigma N_i^j \varphi_j^a, \\ E_{\tilde{\varphi}_i^{a'}} &= \tilde{\sigma}_{b'}^{a'} \tilde{\varphi}_i^{b'} - \sigma \tilde{N}_i^j \tilde{\varphi}_j^{a'}, \end{aligned}$$

[3] See *e.g.* [34] for a discussion of physical realizations of tangent bundles of PAX and PAXY models, which form the prototype for this observation.

where N^i_j, $\tilde{N}^i_j$ are related by the constraint (6), namely

$$N^k_{[j_1} f^{i_1 i_2}_{j_2]k} \;=\; \tilde{N}^{[i_1}_k f^{i_2]k}_{j_1 j_2},$$

and for J's:

$$\begin{aligned}
J_{p_{ij}} &= f^{ij}_{i'j'} b^{i'j'} - \tilde{b}^{ij}, \\
J_{b^{ij}} &= p_{i_1 i_2} f^{i_1 i_2}_{ij} + \varphi^a_i \varphi^b_j J_{ab}, \\
J_{\tilde{b}^{ij}} &= -p_{ij} + \tilde{\varphi}^a_i \tilde{\varphi}^b_j J_{ab}, \\
J_{\varphi^a_i} &= 2\varphi^b_k J_{ab} b^{ik}, \\
J_{\tilde{\varphi}^{a'}_i} &= 2\tilde{\varphi}^{b'}_k J_{a'b'} \tilde{b}^{ik}.
\end{aligned}$$

It can be shown that for the deformations above, $E \cdot J = 0$, using the relation

$$\sigma^a_c J_{ab} \;=\; \sigma^a_b J_{ac}, \tag{7}$$

following from properties of the Sp Lie algebra.

On the (2,2) locus,

$$N^i_j \;=\; \tilde{N}^i_j,$$

just as in the original description.

5.3 Single dual description

In this section we dualize only one of the $SU(2)$ gauge factors of the first model discussed to $Sp(N-3)$, giving gauge group

$$\frac{U(1) \times Sp(N-3) \times SU(2)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

with

- N fields φ^a_i in the $(\mathbf{N-3}, \mathbf{1})_{-1}$ representation,
- $(1/2)N(N-1)$ fields $b^{ij} = -b^{ji}$ in the $(\mathbf{1}, \mathbf{1})_2$ representation,
- N fields $\tilde{\phi}^j_{a'}$ in the $(\mathbf{1}, \mathbf{2})_1$ representation,
- $(1/2)N(N-1)$ fields $p_{ij} = -p_{ji}$ in the $(\mathbf{1}, \mathbf{1})_{-2}$ representation,

with superpotential

$$W = \sum p_{ij}(f^{ij}(b) - \tilde{B}^{ij}) + \varphi^a_i \varphi^b_j J_{ab} b^{ij}$$

where $\tilde{B}^{ij} = \epsilon^{a'b'} \tilde{\phi}^i_{a'} \tilde{\phi}^j_{b'}$ and p_{ij} have charge -2, b^{ij} have charge 2, φ^a_i have charge -1 and $\tilde{\phi}^i_{a'}$ have charge 1 under the $U(1)$ factor in the gauge group.

In this duality frame, both the $r \gg 0$ and $r \ll 0$ phases have geometry determined in part by nonperturbative effects as in [1], so as mentioned in the introduction, at this time we cannot provide a simple monad description of either.

In (0,2) language, E deformations are

$$\begin{aligned}
E_{p_{ij}} &= -\sigma \left(\tilde{N}^k_i p_{kj} - \tilde{N}^k_j p_{ki} \right), \\
E_{b^{ij}} &= \sigma \left(N^i_k b^{kj} - N^j_k b^{ki} \right), \\
E_{\varphi^a_i} &= \sigma^a_b \varphi^b_i - \sigma N^j_i \varphi^a_j, \\
E_{\tilde{\phi}^i_{a'}} &= \tilde{\sigma}^{b'}_{a'} \tilde{\phi}^i_{b'} + \sigma \tilde{N}^i_j \tilde{\phi}^j_{a'},
\end{aligned}$$

where N^i_j, $\tilde{N}^i_j$ are related by the same constraint (6) as in the last two duality frames, namely

$$N^k_{[j_1} f^{i_1 i_2}_{j_2]k} \; = \; \tilde{N}^{[i_1}_k f^{i_2]k}_{j_1 j_2},$$

and J deformations are

$$\begin{aligned}
J_{p_{ij}} &= f^{ij}(b) - \tilde{B}^{ij}, \\
J_{b^{ij}} &= p_{mn} f^{mn}_{ij} + \varphi^a_i \varphi^b_j J_{ab}, \\
J_{\varphi^a_i} &= 2 \varphi^b_j J_{ab} b^{ij}, \\
J_{\tilde{\phi}^i_{a'}} &= -2 p_{ij} \epsilon^{a'b'} \tilde{\phi}^j_{b'}.
\end{aligned}$$

It is straightforward to show that $EJ = 0$, using the symplectic property (7) of σ^a_b and the tracelessness of $\tilde{\sigma}^{a'}_{b'}$.

Just as in the last two duality frames, on the (2,2) locus,

$$N^i_j \; = \; \delta^i_j \; = \; \tilde{N}^i_j.$$

5.4 Comparison of deformations

It is tempting to identify the N in any one duality frame with the N in any other, and similarly the $\tilde{N}$ in any one duality frame with the $\tilde{N}$ in any other. As a cautionary note, however, we observe that this is potentially too simplistic. For example, (0,2) deformations of

rank	A	B
6	$(\mathbf{2},\mathbf{1})_{-1}\oplus(\mathbf{1},\mathbf{2})_{-1}\oplus(\mathbf{1},\mathbf{1})_{2}\oplus(\mathbf{1},\mathbf{1})_{6}^{\oplus 2}$	$(\mathbf{1},\mathbf{1})_{10}$
7	$(\mathbf{2},\mathbf{1})_{-1}\oplus(\mathbf{1},\mathbf{2})_{-1}\oplus(\mathbf{1},\mathbf{1})_{2}^{\oplus 2}\oplus(\mathbf{1},\mathbf{1})_{4}^{\oplus 2}$	$(\mathbf{1},\mathbf{1})_{8}$
7	$(\mathbf{2},\mathbf{1})_{-1}\oplus(\mathbf{1},\mathbf{2})_{-3}\oplus(\mathbf{1},\mathbf{1})_{4}^{\oplus 2}\oplus(\mathbf{1},\mathbf{1})_{6}^{\oplus 2}$	$(\mathbf{1},\mathbf{1})_{12}$
7	$(\mathbf{2},\mathbf{1})_{-3}\oplus(\mathbf{1},\mathbf{2})_{7}\oplus(\mathbf{1},\mathbf{1})_{4}\oplus(\mathbf{1},\mathbf{1})_{6}\oplus(\mathbf{1},\mathbf{1})_{-2}^{\oplus 2}$	$(\mathbf{1},\mathbf{1})_{14}$

Table 1: A few anomaly-free bundles on $G(2,5)\cap G(2,5)$.

a Grassmannian $G(k,n)$ are parametrized by [35,36] an $n\times n$ matrix B^i_j, and one might guess that the transpose would give the corresponding (0,2) deformations of the dual Grassmannian $G(n-k,n)$. However, as observed in [35], merely taking the transpose of B^i_j does not generate the quantum sheaf cohomology ring of the deformed dual Grassmannian, hence the correct parameter match is more complicated than merely taking the transpose. In the present case, it is possible that the correct parameter match is more complicated than merely identifying all instances of N^i_j and $\tilde{N}^i_j$ in different duality frames.

6 Examples of anomaly-free bundles on $G(2,5)\cap G(2,5)$

In table 1, we summarize several anomaly-free bundles built as kernels, in the form

$$0\longrightarrow\mathcal{E}\longrightarrow\mathcal{O}(A)\longrightarrow\mathcal{O}(B)\longrightarrow 0,$$

where A and B are representations of the gauge group.

Each of these bundles is represented by a (0,2) GLSM with gauge group

$$\frac{U(1)\times SU(2)\times SU(2)}{\mathbb{Z}_2\times\mathbb{Z}_2}$$

with matter

- 5 chiral superfields ϕ^i_a in the $(\mathbf{2},\mathbf{1})_1$ representation,
- 5 chiral superfields $\tilde{\phi}^i_{a'}$ in the $(\mathbf{1},\mathbf{2})_1$ representation,
- 10 Fermi superfields $\Lambda_{ij}=-\Lambda_{ji}$ in the $(\mathbf{1},\mathbf{1})_{-2}$ representation,
- Fermi superfields Λ^α in representation A,
- chiral superfields P_β in the dual of representation B,

with (0,2) superpotential

$$\Lambda_{ij}\left(f^{ij}(B)-\tilde{B}^{ij}\right)\;+\;P_\beta J^\beta_\alpha \Lambda^\alpha.$$

It is straighforward to check that each representation in table 1 is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$, where each $\mathbb{Z}_2$ factor relates the center of one $SU(2)$ to a subgroup of $U(1)$.

In addition, there is also an anomaly-free bundle defined similarly by the data

rank	A	B
4	$(\mathbf{2},\mathbf{2})_4 \oplus (\mathbf{1},\mathbf{1})_2^{\oplus 3} \oplus (\mathbf{1},\mathbf{1})_{-2}$	$(\mathbf{2},\mathbf{1})_5 \oplus (\mathbf{1},\mathbf{2})_5$

in the GLSM with gauge group

$$U(1)\times SU(2)\times SU(2)$$

and the same matter and superpotential as above. (Here, representation A is not invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

In each case, Green-Schwarz anomaly cancellation requires, schematically,

$$\sum_{R_{\rm left}} {\rm tr}\left(T^a T^b\right)\;=\;\sum_{R_{\rm right}} {\rm tr}\left(T^a T^b\right).$$

Anomaly cancellation for the $U(1)$ factor can be understood in the standard form, as a sum of squares of charges. For the nonabelian factors, anomaly cancellation in each factor can be written more explicitly as [34][section 3.1]

$$\sum_{R_{\rm left}} (\dim R_{\rm left})\,{\rm Cas}_2\left(R_{\rm left}\right)\;+\;(\dim {\rm adj})\,{\rm Cas}_2\left({\rm adj}\right)\;=\;\sum_{R_{\rm right}} (\dim R_{\rm right})\,{\rm Cas}_2\left(R_{\rm right}\right),$$

where we have explicitly incorporated the left-moving gauginos into the expression above. As $SU(2)$ generators are traceless, there are $SU(2)-SU(2)$ and $SU(2)'-SU(2)'$ anomalies, but no $U(1)-SU(2)$ or $SU(2)-SU(2)'$ anomalies to check. In checking such anomalies, it is handy to note that for an n-dimensional representation of $SU(2)$, ${\rm Cas}_2$ is given by [37][equ'n (7.27)]

$$\frac{1}{2}\left(n^2-1\right)$$

(using the fact that $\lambda_1 = n-1$ in that reference's conventions).

7 Conclusions

In this paper we have examined (0,2) deformations and properties of some exotic GLSMs. We have seen that in GLSMs realizing branched double covers nonperturbatively, the tangent bundle is realized as the cohomology of three-term sequence involving both bundles and torsion sheaves, a novel effect in (0,2) theories. We have also seen several examples in which quasi-isomorphisms arise physically, relating monads in theories at different points along RG flow, their first occurrence outside of applications to B-branes and derived categories. Finally, we have examined (0,2) deformations in the various duality phases of the intersection $G(2,N) \cap G(2,N)$, and have listed a few examples of anomaly-free (0,2) theories on $G(2,5) \cap G(2,5)$.

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