

Landau-Ginzburg models for certain fiber products with curves

Zhuo Chen¹, Tony Pantev², Eric Sharpe³

^{1,3} Dep't of Physics
Virginia Tech
850 West Campus Dr.
Blacksburg, VA 24061
zhuo2012@vt.edu, tpantev@math.upenn.edu, ersharpe@vt.edu

² Dep't of Mathematics
University of Pennsylvania
David Rittenhouse Lab.
209 South 33rd Street
Philadelphia, PA 19104-6395

In this paper we describe a physical realization of a family of non-compact Kähler threefolds with trivial canonical bundle in hybrid Landau-Ginzburg models, motivated by some recent non-Kähler solutions of Strominger systems, and utilizing some recent ideas from GLSMs. We consider threefolds given as fiber products of compact genus g Riemann surfaces and noncompact threefolds. Each genus g Riemann surface is constructed using recent GLSM tricks, as a double cover of \mathbb{P}^1 branched over a degree $2g+2$ locus, realized via nonperturbative effects rather than as the critical locus of a superpotential. We focus in particular on special cases corresponding to a set of Kähler twistor spaces of certain hyperKähler four-manifolds, specifically the twistor spaces of \mathbb{R}^4 , $\mathbb{C}^2/\mathbb{Z}_k$, and $S^1 \times \mathbb{R}^3$. We check in all cases that the condition for trivial canonical bundle arising physically matches the mathematical constraint.

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1 Introduction

Over the years, there has been much work on non-Kähler solutions of heterotic compactifications with H flux, known as the Strominger system [1]. This paper is inspired by the recent work [2] in which a new family of compact non-Kähler analogues of Calabi-Yau threefolds, a new set of potential solutions to the Strominger system, was constructed. In this paper, we do not construct physical theories for non-Kähler targets, but instead apply recent tricks in GLSMs to build physical theories for Kähler analogues of the fiber products discussed in [2].

Specifically, suppose M is a compact hyperKähler manifold with real dimension four and Σ is a compact Riemann surface of genus $g \geq 3$. Then, a manifold X with holomorphically-trivial canonical bundle can be constructed

as the pullback

$$\begin{array}{ccc}
 X = \varphi^* Z = Z \times_{\mathbb{P}^1} \Sigma & \longrightarrow & Z = M \times \mathbb{P}^1 \\
 \downarrow & & \downarrow \pi \\
 \Sigma & \xrightarrow{\varphi} & \mathbb{P}^1
 \end{array}$$

where Z is the twistor space of M together with the natural holomorphic projection π and φ is a nonconstant holomorphic map. It was argued in [2] that the threefold X has trivial canonical bundle as long as

$$\varphi^* \mathcal{O}(2) \cong K_\Sigma.$$

In this paper, the curve Σ will be constructed as a branched double cover of \mathbb{P}^1 , for which case the condition above for the fiber product to have trivial canonical bundle reduces to simply $g = 3$, independent of the details of M .

Motivated by the mathematical construction above, we will give a physical realization of threefolds of trivial canonical bundle constructed as fiber products of genus g curves with noncompact Kähler threefolds, including as special cases certain noncompact Kähler¹ twistor spaces. We will take the Riemann surface of genus g and the holomorphic map φ to be a branched double cover over \mathbb{P}^1 , realized physically via nonperturbative tricks as in [3]. Because we describe fiber products with Kähler threefolds, including Kähler twistor spaces of certain hyperKähler four-manifolds, the Calabi-Yau threefolds we realize are non-compact and Kähler, as opposed to non-Kähler spaces of trivial canonical bundle which were the focus of [2].

We will construct higher-energy theories that realize these geometries as (2,2) supersymmetric hybrid Landau-Ginzburg models. These hybrid models do not seem to have a UV description as GLSMs, though some GLSMs do come close, as we shall explain later.

We begin in section 2 with a review of GLSMs for genus g curves, constructed via nonperturbative methods as branched double covers. In section 3 we construct (2,2) supersymmetric hybrid Landau-Ginzburg models for the fiber products above, of curves with a few noncompact Kähler threefolds. In section 4 we specialize to fiber products of curves and twistor spaces, which arise as special cases. In appendix A we review some pertinent mathematics.

¹ Most twistor spaces are not Kähler.

Although we are not able to give physical realizations of any non-Kähler geometries in this paper, it is our hope that the ideas we present here will later be extended to non-Kähler fiber product constructions.

Finally, before starting, we should add a caution. We discuss non-compact Kähler manifolds with trivial canonical bundle. However, Yau’s theorem does not apply to non-compact cases, so it is possible² that some might not have Ricci-flat metrics. (Nevertheless, we will sometimes call these noncompact Kähler spaces of trivial canonical bundle, “Calabi-Yau’s,” though this terminology is inaccurate.) This is an issue for both the spaces themselves as well as for Landau-Ginzburg models on such spaces that do not have known UV completions as GLSMs. In the case of Landau-Ginzburg models, if the metric is not Ricci-flat, not even asymptotically, then, RG flow would be more complicated, and our analysis likely too naive. Our proposed hybrid Landau-Ginzburg models are constructed on the assumption that they have Ricci-flat metrics, at least asymptotically, so that the renormalization group flow works as expected.

It is not entirely out of the realm of possibility that complications in RG flow, alluded to above, might actually generate nonzero H flux in a low-energy theory, especially in $(0,2)$ supersymmetric versions of this construction where one has less control over RG flow. We will leave this possibility to future work.

2 GLSM for $\mathbb{P}^{2g+1}[2,2]$ and curves of genus g

One essential piece of our construction will be a trick from [3], in which GLSMs describe geometries nonperturbatively, rather than as the critical locus of a superpotential. As it plays a critical role in this paper, we review the highlights in this section.

Section 4.1 of [3] discusses a gauged linear sigma model for $\mathbb{P}^{2g+1}[2,2]$

² One set of examples is discussed in [4]. A second set of examples arises by removing an anticanonical divisor with deep singularities from a projective manifold, then the complement will not have a complete Ricci-flat metric, though it may still have a non-complete Ricci-flat metric. A third set of examples arises from the cotangent bundle of a compact Kähler manifold X . On a tubular neighborhood of the zero section, one can find a unique Ricci-flat metric which restricts to the given Kähler metric on the zero section and is circle invariant. This metric extends to a Ricci-flat metric on the whole cotangent bundle if and only if X is a homogeneous Fano or semi-abelian variety. For example, if X is a Riemann surface of genus at least two, there is a Ricci-flat metric on a disk bundle of fixed radius but it cannot be extended beyond that bundle to the entire cotangent bundle.

(with $g \geq 1$) which realizes a genus g Riemann surface, via nonperturbative tricks, in its $r \ll 0$ phase. That model will play an essential role in this paper, so we shall quickly review it here.

The GLSM in question is a $U(1)$ gauge theory with $(2,2)$ supersymmetry and $2g + 2$ chiral superfields ϕ_i of charge 1 and two chiral superfields p_1, p_2 of charge -2 , with superpotential

$$W = p_1 Q_1(\phi) + p_2 Q_2(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j, \quad (1)$$

where Q_i are quadratic functions of ϕ 's, and $A^{ij}(p)$ is a $(2g + 2) \times (2g + 2)$ symmetric matrix whose entries are linear in the p_a .

For $r \gg 0$ (geometric phase), one can do the usual analysis of the critical loci to argue that the GLSM flows to a sigma model on a complete intersection of two quadrics in \mathbb{P}^{2g+1} .

The $r \ll 0$ phase is more interesting. D terms imply that p_1 and p_2 can not simultaneously vanish, and the superpotential generically gives a mass to the ϕ 's. On that open set where all the ϕ 's are massive, since the p 's have nonminimal charges, physics sees a double cover of the \mathbb{P}^1 mapped out by p 's [3, 5, 6]. On the locus where any ϕ becomes massless, specifically the degree $2g + 2$ locus $\{\det A = 0\}$ where the mass matrix develops at least one zero eigenvalue, the double cover collapses to a single cover.

Put simply, $\{\det A^{ij} = 0\}$ defines the branch locus on the double cover of \mathbb{P}^1 . (Monodromies about the branch locus correspond to Berry phases and are described in [3].) The resulting geometry, a double cover of \mathbb{P}^1 branched over a degree $2g + 2$ loci, is a compact Riemann surface of genus g .

3 Hybrid Landau-Ginzburg models for fiber products

3.1 Fiber products with vector bundles on \mathbb{P}^1

In this section we will consider a general set of fiber products, between curves and vector bundles on \mathbb{P}^1 . Specifically, let V be the total space of the rank-two vector bundle $\mathcal{O}(a) + \mathcal{O}(b) \rightarrow \mathbb{P}^1$.

Mathematically, we are considering the fiber product³

$$\begin{array}{ccc} X = \varphi^* V = V \times_{\mathbb{P}^1} \Sigma & \longrightarrow & V \\ \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\varphi} & \mathbb{P}^1 \end{array}$$

The fiber product X will have trivial canonical bundle if

$$K_\Sigma = \varphi^*(\det V). \quad (2)$$

We will consider the special case of genus g curves Σ constructed as branched double covers, for which the condition above reduces to

$$a + b = g - 1. \quad (3)$$

Now, in general, we will want to describe cases in which a or b are positive, and the total space of such V is challenging to describe with a GLSM. Recall that the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ can be described by a GLSM with a single $U(1)$ gauge field and four chiral superfields:

- two chiral superfields p_a of charge $+1$ corresponding to homogeneous coordinates on the base \mathbb{P}^1 ,
- two chiral superfields y_a of charge -1 corresponding to the two line bundles $\mathcal{O}(-1)$.

Naively, one could try a similar GLSM with the charges of the chiral superfields y_a flipped to $+1$ describing two $\mathcal{O}(+1)$ line bundles. However, D terms in the resulting GLSM make it clear that that GLSM will describe the space \mathbb{P}^3 . To describe the total space of the bundle above, one would need to remove a different exceptional locus than the one canonically dictated by the D terms for a quotient of flat space.

Setting aside the issue above, the fiber product would formally be described by the GLSM with gauge group $U(1)$ and matter

- $2g + 2$ chiral superfields ϕ_i of charge -1 ,
- 2 chiral superfields p_a of charge $+2$,

³ As an aside, if $\pi : \Sigma \rightarrow \mathbb{P}^1$ is the projection from the genus g curve to \mathbb{P}^1 , then the fiber product X is the total space of $L_a \oplus L_B \rightarrow \Sigma$, where $L_n = \pi^* \mathcal{O}(n)$.

- 2 chiral superfields y_a, y_b of charges $2a, 2b$,

and superpotential

$$W = \sum_{ij} \phi_i \phi_j A^{ij}(p),$$

where A^{ij} is a symmetric $(2g+2) \times (2g+2)$ matrix with entries linear in the p 's.

We have taken the matrix A^{ij} to be independent of the y 's, to preserve translation invariance along the fibers as well as a global $SU(2)$ rotation symmetry between the y 's. In models discussed later we will make similar restrictions so as to reproduce the desired geometries.

In passing, if we were to add terms to the superpotential to realize the most general case compatible with gauge invariance, *i.e.* adding terms involving y_a, y_b , and sufficient ϕ factors, we would get the GLSM for $\mathbb{P}^{2g+1}[2, 2, 2a, 2b]$ (with an overall sign flip on the charges, inverting the $r \gg 0$ and $r \ll 0$ phases). (This includes the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$, studied in [3] because of the geometric realization of its $r \ll 0$ phase.) Note that the Calabi-Yau condition for that complete intersection also reduces to (3).

Assuming that p_1 and p_2 are homogeneous coordinates on \mathbb{P}^1 , one can easily see, modulo the issue with D terms, that the mass matrix in the F term imply the fiber product geometry in the phase $r \gg 0$. First, following the same argument in section (2), the fields ϕ_i and p_a describe the genus g Riemann surface as the branched double cover of \mathbb{P}^1 . On the other hand, the fields y_a and y_b correspond to the fiber coordinates on $\mathcal{O}(a) \oplus \mathcal{O}(b)$ since we assumed that p_1 and p_2 are homogeneous coordinates on \mathbb{P}^1 . The fiber product structure is achieved by identifying two different \mathbb{P}^1 in the holomorphic map and the total space of V . The identification is manifest in our theory.

Finally, let us consider the Calabi-Yau condition. The sum of the charges in this theory is precisely

$$-2g + 2 + 2a + 2b,$$

and so vanishes precisely when the mathematical condition (3) for trivial canonical bundle is satisfied.

As mentioned above, the putative GLSM above does not quite work, because the D terms will not describe the correct exceptional locus in general. To evade the issue of positive-degree line bundles on \mathbb{P}^1 in GLSMs, we construct a hybrid Landau-Ginzburg model, an ungauged sigma model on the

total space of

$$\mathcal{O}(-1/2)^{2g+2} \oplus \mathcal{O}(a) \oplus \mathcal{O}(b) \rightarrow \mathbb{P}^1, \quad (4)$$

with superpotential

$$W = \sum_{ij} \phi_i \phi_j A^{ij}(p), \quad (5)$$

where the mass matrix $A^{ij}(p)$ should now be interpreted as a generic symmetric $(2g+2) \times (2g+2)$ matrix of sections of $\mathcal{O}(+1) \rightarrow \mathbb{P}^1$.

Notice that the \mathbb{P}^1 in the target space (4) is actually a \mathbb{Z}_2 gerbe on \mathbb{P}^1 indicated by the line bundle denoted $\mathcal{O}(-1/2)$. The bundle $\mathcal{O}(-1/2)$ is a special kind of line bundle that only exist for gerbes. On the other hand, it is also a fiber bundle of \mathbb{P}^1 whose fibers are the orbifolds $[\mathbb{C}/\mathbb{Z}_2]$. More details of such line bundles on gerbes over projective spaces are discussed in appendix B of [7].

Generically on the \mathbb{P}^1 , the ϕ_i are massive, away from the locus $\{\det A = 0\}$, and so the \mathbb{Z}_2 gerbe implies a branched double cover, as usual, and hence the fiber product of the genus g Riemann surface and the vector bundle V over \mathbb{P}^1 .

The condition for the canonical class to be trivial in this hybrid model is that the first Chern class of the vector bundle (4) match the first Chern class of the canonical bundle, meaning specifically that

$$(2g+2)(-1/2) + a + b = -2,$$

which again reduces to the mathematical condition (3).

There is a possible technical issue with this construction, due to the fact that there is no non-compact version of Yau's theorem, as described in the introduction. We describe above a hybrid Landau-Ginzburg model over a Kähler space with holomorphically trivial canonical bundle (in the case $g = 3$); however, that does not guarantee that a Ricci-flat metric exists in the noncompact case. If the metric is not Ricci-flat, at least asymptotically, then the RG flow may be more complicated than we have naively supposed. If the model arose from a GLSM, we could appeal to RG flow from the GLSM, but as we have not been able to write down a UV GLSM, we cannot guarantee that an asymptotically Ricci-flat metric exists. Analogous potential issues arise in every hybrid Landau-Ginzburg model described in this paper.

3.2 Fiber products with hypersurfaces in vector bundles

Let V denote the rank-three vector bundle

$$\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \rightarrow \mathbb{P}^1,$$

and consider the hypersurface $f(x, y, z) = 0$, where x, y, z are coordinates along the fibers of V , and f is of degree⁴ d . Mathematically, for the fiber product of this hypersurface with the curve Σ of genus g to have trivial canonical bundle, we must require that the degree d match the degree of φ^*V , for $\varphi : \Sigma \rightarrow \mathbb{P}^1$, which means, for Σ realized as a branched double cover of \mathbb{P}^1 ,

$$d = a + b + c + 1 - g. \quad (6)$$

Modulo the same issue with D terms and positive-degree line bundles discussed in the last section, we can construct a ‘fake’ GLSM for the fiber product with the hypersurface in V as a $U(1)$ gauge theory with matter

- $2g + 2$ chiral superfields ϕ_i of charge -1 ,
- 2 chiral superfields p_a of charge $+2$,
- 3 chiral superfields x, y, z of charges $2a, 2b, 2c$, respectively,
- 1 chiral superfield q of charge $-2d$,

and superpotential

$$W = \sum_{ij} \phi_i \phi_j A^{ij}(p) + qf(x, y, z),$$

where A^{ij} is a symmetric $(2g + 2) \times (2g + 2)$ matrix with entries linear in the p ’s. (As before, we do not consider more general possible terms, in order to preserve pertinent symmetries.) The sum of the charges in this theory vanishes when

$$g + d = a + b + c + 1, \quad (7)$$

matching the mathematical condition given above for the canonical bundle of the fiber product to be trivial.

⁴ In the sense of weighted projective spaces, so that the monomials x, y, z have weights a, b, c , respectively.

As before, there is a problem involving D terms in the putative GLSM above. To evade this issue, we can construct a hybrid Landau-Ginzburg model which describes the geometry. Specifically, this will be an ungauged sigma model on the total space of

$$\mathcal{O}(-1/2)^{2g+2} \oplus \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)\mathcal{O}(-d) \rightarrow \mathbb{P}^1, \quad (8)$$

where we interpret the bundle in terms of a \mathbb{Z}_2 gerbe on the \mathbb{P}^1 , and with superpotential

$$W = \sum_{ij} \phi_i \phi_j A^{ij}(p) + qf(x, y, z),$$

where the mass matrix $A^{ij}(p)$ is a generic symmetric $(2g+2) \times (2g+2)$ matrix of sections $\mathcal{O}(+1) \rightarrow \mathbb{P}^1$.

This hybrid Landau-Ginzburg model realizes the same fiber product structure as the fake GLSM above. The superpotential contains a mass matrix for the ϕ_i , $i = 1, \dots, 2g+2$, that gives them a mass away from the locus $\{\det A = 0\}$. As a result, at generic points on the \mathbb{P}^1 , the remaining massless fields are invariant under the gerbe \mathbb{Z}_2 , which physics sees [3, 5] as a double cover of \mathbb{P}^1 , branched over the locus $\{\det A = 0\}$. Consequently, one obtains a fiber product of the genus g curve and the hypersurface.

The Calabi-Yau condition for the hybrid Landau-Ginzburg model is the condition that c_1 of the bundle (8) match c_1 of the canonical bundle of \mathbb{P}^1 , which in this case implies

$$(2g+2)(-1/2) + a + b + c - d = -2.$$

It is straightforward to check that this matches the condition (7) given earlier.

4 Fiber products with twistor spaces

In this section, we will construct (2,2) supersymmetric hybrid Landau-Ginzburg theories which should RG flow to sigma models on the non-compact Kähler Calabi-Yau threefolds constructed as fiber products of genus three curves and twistor spaces, as explained in the introduction. The (Kähler) twistor spaces we consider here are the twistor spaces⁵ of \mathbb{R}^4 , $\mathbb{C}^2/\mathbb{Z}_k$ and $S^1 \times \mathbb{R}^3$. These will all correspond to special cases of the constructions in section 3, so we

⁵ Sometimes, blowdowns of the twistor spaces.

will strive to be brief. In each case, since the curve is realized as a branched double cover of \mathbb{P}^1 , the Calabi-Yau condition is that the curve be of genus 3 – details of the hyperKähler manifold are otherwise irrelevant. In each of our models, we will recover the genus three condition as a consistency check.

4.1 Fiber product with twistor space of \mathbb{R}^4

From [8], the twistor space of \mathbb{R}^4 can be described as the total space of $\mathcal{O}(+1) \oplus \mathcal{O}(+1) \rightarrow \mathbb{P}^1$. Our construction for this case is a special case of the construction in section 3.1.

As discussed in section 3.1, there is a technical question of how to realize positive-degree line bundles in GLSMs, so we instead construct a lower energy theory, a hybrid Landau-Ginzburg model. Specifically, this will be an ungauged sigma model on the total space of

$$\mathcal{O}(-1/2)^{2g+2} \oplus \mathcal{O}(+1) \oplus \mathcal{O}(+1) \rightarrow \mathbb{P}^1, \quad (9)$$

with superpotential

$$W = \sum_{ij} \phi_i \phi_j A^{ij}(p),$$

where the mass matrix $A^{ij}(p)$ should now be interpreted as a generic symmetric $(2g+2) \times (2g+2)$ matrix of sections of $\mathcal{O}(+1) \rightarrow \mathbb{P}^1$.

The superpotential contains a mass matrix for the $\phi_i, i = 1, \dots, 2g+2$, that gives them a mass away from the locus $\{\det A = 0\}$. Therefore, at generic points on the \mathbb{P}^1 , the remaining massless fields are all non-minimally charged. The Riemann surface of genus g is given by a double cover of \mathbb{P}^1 branched over a degree $2g+2$ locus as before. Also, the fields y_1 and y_2 are the coordinates on the fibers of $\mathcal{O}(+1) \oplus \mathcal{O}(+1)$ of the same \mathbb{P}^1 . Consequently, one obtains a fiber product of the genus g Riemann surface and twistor space of \mathbb{R}^4 over \mathbb{P}^1 .

The Calabi-Yau condition for the total space of a vector bundle over \mathbb{P}^1 is that the first Chern class of the vector bundle should be the same as the first Chern class of the canonical bundle of \mathbb{P}^1 . In this case, one gets

$$(2g+2)(-1/2) + 1 + 1 = -2.$$

It implies that the genus of the Riemann surface is three, as expected.

4.2 Fiber product with twistor space of $\mathbb{C}^2/\mathbb{Z}_k$

Next, we will give a physical theory describing the fiber product with (the blowdown of) a different twistor space, namely the the twistor space of $\mathbb{C}^2/\mathbb{Z}_k$ [8,9]. The group $\Gamma = \mathbb{Z}_k$ acts on \mathbb{C}^2 as follows:

$$(z_1, z_2) \rightarrow (e^{2\pi in/k} z_1, e^{-2\pi in/k} z_2).$$

Notice that the monomials $x = z_1^k$, $y = z_2^k$, $z = z_1 z_2$ are invariant under the group Γ . Therefore the singular surface \mathbb{C}^2/Γ can be described by a hypersurface embedding in \mathbb{C}^3 ,

$$\{xy = z^k\} \subset \mathbb{C}^3 = \text{Spec } \mathbb{C}[x, y, z].$$

One can turn on a universal family of complex structure deformations which is given by adding lower order terms in z . As a result, the hypersurface defining equation becomes

$$xy = z^k + a_1 z^{k-1} + \cdots + a_k = \prod_{i=1}^k (z - f_i),$$

where a_i and f_i are constant parameters. The twistor space is a resolution of the hypersurface

$$\{xy = \prod_{i=1}^k (z - f_i(p))\} \subset \text{Tot}(\mathcal{O}(+k) \oplus \mathcal{O}(+k) \oplus \mathcal{O}(+2) \rightarrow \mathbb{P}^1),$$

where x, y are fiber coordinates on the bundle $\mathcal{O}(+k)$, z on $\mathcal{O}(+2)$, and f_i are sections of $\mathcal{O}(+2) \rightarrow \mathbb{P}^1$. In particular, for each point of \mathbb{P}^1 , the fiber is a deformation of $\mathbb{C}^2/\mathbb{Z}_k$. In the special case $k = 2$, the fiber space is also known as an Eguchi-Hanson space.

We can realize the hypersurface above, a blowdown of the twistor space, using the same ideas as in section 3.2. Specifically, we propose a hybrid Landau-Ginzburg model, an ungauged sigma model whose target space is the total space of

$$\mathcal{O}(-1/2)^{2g+2} \oplus \mathcal{O}(+k)^2 \oplus \mathcal{O}(+2) \oplus \mathcal{O}(-2k) \rightarrow \mathbb{P}^1,$$

with fiber coordinates ϕ_i on $\mathcal{O}(-1/2)^{2g+2}$, x, y on $\mathcal{O}(+k)^2$, z on $\mathcal{O}(+2)$ and q on $\mathcal{O}(-2k)$. The superpotential is

$$W = \sum_{ij} \phi_i \phi_j A^{ij}(p) + q(xy - \prod_{i=1}^k (z - f_i(p))), \quad (10)$$

where f_i are sections of $\mathcal{O}(2) \rightarrow \mathbb{P}^1$ and $A^{ij}(p)$ is symmetric $(2g+2) \times (2g+2)$ matrix with elements which are sections of $\mathcal{O}(1) \rightarrow \mathbb{P}^1$.

Going through the same analysis as in section 3.2, one obtains the desired fiber product geometry. Note that the Calabi-Yau condition in this case is

$$(-1/2)(2g+2) + k + k + 2 + (-2k) = -2$$

hence $g = 3$, as expected.

4.3 Fiber product with twistor space of $S^1 \times \mathbb{R}^3$

The last case we will discuss here is the fiber product with (the blowdown of) the twistor space of $S^1 \times \mathbb{R}^3$ [8]. Since the analyses is similar to previous sections, we will present our proposition briefly here. The space $S^1 \times \mathbb{R}^3$ can be defined as \mathbb{C}^2/Γ where Γ is given by

$$(z_1, z_2) \rightarrow (z_1 + 1, z_2).$$

Following the same process as in section 4.2, the twistor space is defined by a resolution of the hypersurface

$$\begin{aligned} \{y^2 + zx^2 = z + (f_1(p)^2 + f_2(p)^2) + 2f_1(p)^2 f_2(p)^2 x\} \\ \subset \text{Tot}(\mathcal{O} \oplus \mathcal{O}(+2) \oplus \mathcal{O}(+4) \rightarrow \mathbb{P}^1), \end{aligned}$$

where x is a fiber coordinate on the line bundle \mathcal{O} , y on $\mathcal{O}(+2)$, z on $\mathcal{O}(+4)$ and f_1, f_2 are two sections of $\mathcal{O}(+2) \rightarrow \mathbb{P}^1$.

We can construct a hybrid Landau-Ginzburg model realizing this geometry as a special case of the construction in section 3.2. This hybrid Landau-Ginzburg model is defined on the total space of

$$\mathcal{O}(-1/2)^{2g+2} \oplus \mathcal{O}(0) \oplus \mathcal{O}(+2) \oplus \mathcal{O}(+4) \oplus \mathcal{O}(-4) \rightarrow \mathbb{P}^1,$$

with superpotential

$$W = \sum_{ij} \phi_i \phi_j A^{ij}(p) + q(y^2 + zx^2 - z - (f_1(p)^2 + f_2(p)^2) - 2f_1(p)^2 f_2(p)^2 x), \quad (11)$$

where A^{ij} is a symmetric $(2g+2) \times (2g+2)$ matrix with entries that are sections of $\mathcal{O}(1) \rightarrow \mathbb{P}^1$.

The Calabi-Yau condition

$$(-1/2)(2g+2) + 3(+2) + (-4) = -2$$

implies that $g = 3$, as expected.

5 Conclusions

In this paper, we have constructed hybrid Landau-Ginzburg models that RG flow to a new family of non-compact Calabi-Yau threefolds, constructed as fiber products of genus g curves and noncompact Kähler threefolds. We only consider curves given as branched double covers of \mathbb{P}^1 . Our construction utilizes ‘nonperturbative’ constructions of the genus g curves given in [3], and so provides a new set of exotic UV theories that should RG flow to sigma models on Calabi-Yau manifolds, in which the Calabi-Yau is not realized simply as the critical locus of a superpotential.

As important special cases, we applied these constructions to describe fiber products with certain Kähler twistor spaces of noncompact hyperKähler four-manifolds, specifically \mathbb{R}^4 , $\mathbb{C}^2/\mathbb{Z}_k$ and $S^1 \times \mathbb{R}^3$. We check that the Calabi-Yau condition one sees in physics matches that from mathematics, namely that the curve have genus three, independent of details of the four-manifold.

We see this work as a first step to realizing GLSMs for compact non-Kähler analogues of Calabi-Yau threefolds constructed in [2], to which we hope to return in the future.

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A Review of pertinent mathematics

According to proposition 2.2 of [2], the fiber product of a twistor space X and a genus g curve Σ with *some* map $\varphi : \Sigma \rightarrow \mathbb{P}^1$ has trivial canonical bundle if and only if

$$\varphi^* \mathcal{O}(2) \cong K_\Sigma. \quad (12)$$

We can see this as follows. Let $\pi : X \rightarrow \mathbb{P}^1$ be the twistor space for

any hyperKähler surface, then the relative symplectic form is a nowhere-zero section of $K_{X/\mathbb{P}^1} \otimes \pi^*\mathcal{O}(2)$, hence $K_{X/\mathbb{P}^1} \cong \pi^*\mathcal{O}(-2)$. Furthermore, by definition,

$$\begin{aligned} K_{X/\mathbb{P}^1} &= K_X \otimes \pi^*K_{\mathbb{P}^1}^{-1}, \\ &= K_X \otimes \pi^*\mathcal{O}(2). \end{aligned}$$

This gives

$$K_X = \pi^*\mathcal{O}(-4).$$

Next, for the fiber product $Z = X \times_{\mathbb{P}^1} \Sigma$,

$$\begin{aligned} K_{Z/\Sigma} &= p_X^*K_{X/\mathbb{P}^1}, \\ &= p_X^*\pi^*\mathcal{O}(-2) = \pi_Z^*\mathcal{O}(-2), \end{aligned}$$

using $\pi_Z = \pi \circ p_X = \varphi \circ p_\Sigma$, and where $p_\Sigma : Z \rightarrow \Sigma$, $\pi_Z : Z \rightarrow \mathbb{P}^1$ are projections. Hence,

$$\begin{aligned} K_Z &= p_\Sigma^*K_\Sigma \otimes K_{Z/\Sigma}, \\ &= p_\Sigma^*K_\Sigma \otimes \pi_Z^*\mathcal{O}(-2). \end{aligned}$$

To double-check, we can also compute

$$\begin{aligned} K_Z &= p_X^*K_X \otimes K_{Z/X}, \\ &= p_X^*K_X \otimes p_\Sigma^*K_{\Sigma/\mathbb{P}^1}, \\ &= p_X^*\pi^*\mathcal{O}(-4) \otimes p_\Sigma^*(K_\Sigma \otimes \varphi^*\mathcal{O}(2)), \\ &= \pi_Z^*\mathcal{O}(-4) \otimes p_\Sigma^*K_\Sigma \otimes \pi_Z^*\mathcal{O}(2), \\ &= p_\Sigma^*K_\Sigma \otimes \pi_Z^*\mathcal{O}(-2), \end{aligned}$$

matching the result above. In any event, using the fact that $\pi_Z = \pi \circ p_X = \varphi \circ p_\Sigma$, we see that K_Z is trivial if and only if

$$K_\Sigma = \varphi^*\mathcal{O}(2).$$

The fact that this condition does not depend upon X follows from the fact that K_X is always a pullback of $\mathcal{O}(-4)$.

Now, under what circumstances is (12) satisfied?

Let us consider the case that Σ is a spectral cover of \mathbb{P}^1 , the case of relevance for this paper. Let d be the degree of the projection map $\varphi :$

$\Sigma \rightarrow \mathbb{P}^1$, then $2d = 2g - 2$ for g the genus of Σ , hence $g = d + 1$. From Hurwitz, $K_\Sigma = \varphi^* \mathcal{O}(-2) \otimes \mathcal{O}(R)$, where $R \subset \Sigma$ is the ramification divisor of $\varphi : \Sigma \rightarrow \mathbb{P}^1$. Thus, to satisfy (12), we must satisfy $\mathcal{O}(R) = \varphi^* \mathcal{O}(4)$.

For a hyperelliptic double cover, the degree of the ramification divisor is $2g + 2$ and $\mathcal{O}(R) = f^* \mathcal{O}(g + 1)$. Thus, in such a case, one must require $g = 3$.

More generally spectral covers have the property that their ramification divisor is a pullback. If $\varphi : \Sigma \rightarrow \mathbb{P}^1$ is a spectral cover of degree d embedded in the total space of $\mathcal{O}(k)$, then $\mathcal{O}(R) = \varphi^* \mathcal{O}((d - 1)k)$. To satisfy (12), one must require $(d - 1)k = 4$, which gives three options (for the case that Σ is a spectral cover of \mathbb{P}^1):

- $d - 1 = 1, k = 4$, which gives a hyperelliptic curve of genus 3 (the case that arises in this paper),
- $d - 1 = 2, k = 2$, which gives a 3-sheeted spectral cover of \mathbb{P}^1 of genus 4,
- $d - 1 = 4, k = 1$, which gives a 5-sheeted spectral cover of \mathbb{P}^1 of genus 6.

Note this condition is not satisfied for Σ a genus one curve.

If we drop the spectral curve constraint on Σ , then there are solutions in any genus ≥ 3 . We can see this as follows. First, the condition (12) for the fiber product to have trivial canonical class can be rephrased as the statement that φ is given by a basepoint-free pencil of sections in some spin structure L on Σ . From a theorem of Harris [10], if $g \geq 3$, the moduli space of pairs (Σ, L) such that Σ is smooth of genus g and L is a spin structure which has a pencil of sections has dimension $3g - 4$.

For another example, there is a theorem of Farkas [11] which says that for any $g \geq 10$, the moduli space of pairs (Σ, L) such that Σ is a smooth curve of genus g and L is a spin structure which gives an embedding of Σ in \mathbb{P}^3 is of dimension $3g - 9$. Given such a pair and compose the embedding $\phi_L : \Sigma \rightarrow \mathbb{P}^3$ with a generic projection from some point not in the image, one will get a morphism $\Sigma \rightarrow \mathbb{P}^1$ which has the desired property. Thus, for $g \geq 10$, there is a $3g - 9$ -dimensional space of pairs with the desired property.

So far we have discussed conditions for the fiber product Z to have holomorphically trivial canonical bundle. Next, let us turn to the question of when Z is Kähler. Since Z is a finite cover of X , Z is Kähler if and only if X is Kähler.

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