Transfer operator approach to 1d random band matrices

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Abstract

We discuss an application of the transfer operator approach to the analysis of the different spectral characteristics of 1d random band matrices (correlation functions of characteristic polynomials, density of states, spectral correlation functions). We show that when the bandwidth W crosses the threshold $W = N^{1/2}$, the model has a kind of phase transition (crossover), whose nature can be explained by the spectral properties of the transfer operator.

1 Introduction

Random band matrices (RBM) represent quantum systems on a large box in $\mathbb{Z}^d$ with random transition amplitudes effective up to distances of order W, which is called a bandwidth. They are natural intermediate models to study eigenvalue statistics and quantum propagation in disordered systems as they interpolate between Wigner matrices and random Schrödinger operators: Wigner matrix ensembles represent mean-field models without spatial structure, where the quantum transition rates between any two sites are i.i.d. random variables; in contrast, random Schrödinger operator has only a random diagonal potential in addition to the deterministic Laplacian on a box in $\mathbb{Z}^d$.

In the simplest 1d case RBM H is a Hermitian or real symmetric $N \times N$ matrix with independent (up to the symmetry condition) entries H_{ij} such that

$$\mathbb{E}\{H_{ij}\} = 0, \quad \mathbb{E}\{\,|H_{ij}|^2\} = (2W)^{-1}\mathbf{1}_{|i-j|\le W},$$

i.e. H is a Hermitian matrix with which has inly non $2W+1$ zero diagonals whose entries are i.i.d. random variables (up to the symmetry) and the sum of the variances of entries in each line is 1.

In a more general case H is a Hermitian random $N \times N$ matrix, whose entries H_{jk} are independent (up to the symmetry) complex random variables with mean zero and variances scaled as

$$\mathbb{E}\{|H_{jk}|^2\} = \frac{1}{W^d} J\Big(\frac{|j-k|}{W}\Big). \tag{1.1}$$

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Here Λ is a box in $\mathbb{Z}^d$, $|\Lambda| = N$, and $J : \mathbb{R}^d \to \mathbb{R}_+$ is a function having the compact support or decaying sufficiently fast at infinity and normalized in such a way that

$$\sum_{k \in \Lambda} J(|k|/W) = 1,$$

and the bandwidth $W \gg 1$ is a large parameter.

The density of states ρ of a general class of RBM with $W \gg 1$ is given by the well-known Wigner semicircle law (see [3, 18]):

$$\rho(E) = (2\pi)^{-1}\sqrt{4 - E^2}, \quad E \in [-2, 2]. \tag{1.2}$$

As it was mentioned above, a substantial interest to random band matrices is caused by the fact that they have a non-trivial spatial structure like random Schrödinger matrices (in contrast to classical random matrix ensembles), and furthermore RBM and random Schrödinger matrices are expected to have some similar qualitative properties (for more details on these conjectures see [32]). For instance, RBM can be used to model the celebrated Anderson metal-insulator phase transition in $d \geq 3$. Moreover, the crossover for RBM can be investigated even in $d = 1$ by varying the bandwidth W.

The key physical parameter of RBM is the localization length ℓ_ψ, which describes the length scale of the eigenvector $\psi(E)$ corresponding to the energy $E \in (-2, 2)$. The system is called delocalized if for all E in the bulk of spectrum ℓ_ψ is comparable with the system size, $\ell_\psi \sim N$, and it is called localized otherwise. Delocalized systems correspond to electric conductors, and localized systems are insulators.

In the case of 1d RBM there is a fundamental conjecture stating that for every eigenfunction $\psi(E)$ in the bulk of the spectrum ℓ_ψ is of order W^2 (see [7, 15]). In $d = 2$, the localization length is expected to be exponentially large in W, in $d \geq 3$ it is expected to be macroscopic, $\ell_\psi \sim N$, i.e. system is delocalized.

Notice that the global eigenvalue statistics for 1d RBM such as density of states does not feel any difference between the regime $W \gg \sqrt{N}$ and $W \ll \sqrt{N}$ (see (1.2)). Same situation with the central limit theorem for the linear eigenvalue statistics which was proved in [21] for any $W \gg 1$ (see also [17] for CLT in the regime $W \gg \sqrt{N}$). However, the questions of the localization length are closely related to the universality conjecture of the bulk *local* regime of the random matrix theory. The bulk local regime deals with the behaviour of eigenvalues of $N \times N$ random matrices on the intervals whose length is of the order $O(N^{-1})$. According to the Wigner – Dyson universality conjecture, this local behaviour does not depend on the matrix probability law (ensemble) and is determined only by the symmetry type of matrices (real symmetric, Hermitian, or quaternion real in the case of real eigenvalues and orthogonal, unitary or symplectic in the case of eigenvalues on the unit circle). In terms of eigenvalue statistics the conjecture about the localization length of RBM in $d = 1$ means that 1d RBM in the bulk of the spectrum changes the spectral local behaviour of random operator type with Poisson local eigenvalue statistics (for $W \ll \sqrt{N}$) to the local spectral behaviour of the GUE/GOE type (for $W \gg \sqrt{N}$). In particular, it means that if we consider the second correlation function R_2

defined by the equality

$$\mathbb{E}\Big\{\sum_{j_1\neq j_2}\varphi(\lambda_{j_1},\lambda_{j_2})\Big\}=\int_{\mathbb{R}^2}\varphi(\lambda_1,\lambda_2)R_2(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2, \tag{1.3}$$

where $\{\lambda_j\}$ are eigenvalues of a random matrix, the function $\varphi:\mathbb{R}^2\to\mathbb{C}$ is bounded, continuous and symmetric in its arguments, and the summation is over all pairs of distinct integers $j_1,j_2\in\Lambda$, then in the delocalization region $W\gg\sqrt{N}$

$$(N\rho(E))^{-2}R_2\left(E+\frac{\xi_1}{\rho(E)\,N},E+\frac{\xi_2}{\rho(E)\,N}\right)\longrightarrow 1-\frac{\sin^2(\pi(\xi_1-\xi_2))}{\pi^2(\xi_1-\xi_2)^2}, \tag{1.4}$$

while in the localization region

$$(N\rho(E))^{-2}R_2\left(E+\frac{\xi_1}{\rho(E)\,N},E+\frac{\xi_2}{\rho(E)\,N}\right)\longrightarrow 1. \tag{1.5}$$

The conjecture on the crossover in RBM with $W\sim\sqrt{N}$ is supported by physical derivation due to Fyodorov and Mirlin (see [15]) based on supersymmetric formalism, and also by the so-called Thouless scaling. However, there are only a few partial results on the mathematical level of rigour. At the present time only some upper and lower bounds for ℓ_ψ for the general class of 1d RBM are proved rigorously. It is known from the paper [20] that $\ell_\psi\le W^8$. Recently this bound was improved in [19] to W^7. On the other side, for the general Wigner matrices (i.e. $W=N$) the bulk universality has been proved in [14, 33], which gives $\ell_\psi\ge W$. By the developing the Erdős-Yau approach, there were also obtained some other results, where the localization length is controlled: $\ell_\psi\ge W^{7/6}$ in [12] and $\ell_\psi\ge W^{5/4}$ in [13]. GUE/GOE gap distributions for $W\sim N$ was proved recently in [4].

The study of the eigenfunctions decay is closely related to properties of the Green function $(H-E-i\varepsilon)^{-1}$ with a small ε. For instance, if $|(H-E-i\varepsilon)^{-1}_{ii}|^2$ (without expectation) is bounded for all i and all $E\in(-2,2)$, then each normalized eigenvector ψ of H is delocalized on the scale ε^{-1} in a sense that

$$\max_i|\psi_i|^2\le C\varepsilon^{-1},$$

and so ψ is supported on at least ε^{-1} sites. In particular, if $|(H-E-i\varepsilon)^{-1}_{ii}|^2$ can be controlled down to the scale $\varepsilon\sim 1/N$, then the system is in the complete delocalized regime. Moreover, in view of the bound

$$\mathbb{E}\{|(H-E-i\varepsilon)^{-1}_{jk}|^2\}\sim C\varepsilon^{-1}\,e^{-\|j-k\|/\ell}$$

which holds for the localized regime, the problem of localization/delocalization reduces to controlling

$$\mathbb{E}\{|(H-E-i\varepsilon)^{-1}_{jk}|^2\}$$

for $\varepsilon\sim 1/N$. As it will be shown below, similar estimates of $\mathbb{E}\{|\mathrm{Tr}\,(H-E-i\varepsilon)^{-1}|^2\}$ for $\varepsilon\sim 1/N$ are required to work with the correlation functions of RBM.

Despite many attempts, such control so far has not been achieved. The standard approaches of [14] and [13] do not seem to work for $\varepsilon\le W^{-1}$, and so cannot give an information about the strong form of delocalization (i.e. for *all* eigenfunctions). Classical moment methods, even with a delicate renormalization approach [31], could not break the barrier $\varepsilon \sim W^{-1}$ either.

Another method which allows to work with random operators with non-trivial spatial structures and breaks that barrier, is supersymmetry techniques (SUSY). It is based on the representation of the determinant as an integral (formal) over the Grassmann variables. Combining this representation with the representation of the inverse determinant as an integral over the Gaussian complex field, SUSY allows to obtain the integral representation for the main spectral characteristics such as averaged density of states and correlation functions, as well as for $\mathbb{E}\{G_{jk}(E+i\varepsilon)\}$, $\mathbb{E}\{|G_{jk}(E+i\varepsilon)|^2\}$, etc. For instance, according to the properties of the Stieljes transform, the second correlation function can be rewritten in the form

$$\begin{aligned} R_2(\lambda_1,\lambda_2) =&(\pi N)^{-2}\lim_{\varepsilon\to 0}\mathbb{E}\{\Im\,\mathrm{Tr}\,(H-\lambda_1-i\varepsilon)^{-1}\Im\,\mathrm{Tr}\,(H-\lambda_2-i\varepsilon)^{-1}\} \\ =&(2i\pi N)^{-2}\lim_{\varepsilon\to 0}\mathbb{E}\Big\{\Big(\mathrm{Tr}\,(H-\lambda_1-i\varepsilon)^{-1}-\mathrm{Tr}\,(H-\lambda_1+i\varepsilon)^{-1}\Big) \\ &\times\Big(\mathrm{Tr}\,(H-\lambda_2-i\varepsilon)^{-1}-\mathrm{Tr}\,(H-\lambda_2+i\varepsilon)^{-1}\Big)\Big\}, \end{aligned} \tag{1.6}$$

and since

$$\mathbb{E}\{\mathrm{Tr}\,(H-z_1)^{-1}\mathrm{Tr}\,(H-z_2)^{-1}\}=\frac{d^2}{dz_1'dz_2'}\mathbb{E}\Big\{\frac{\det(H-z_1)\det(H-z_2))}{\det(H-z_1')\det(H-z_2'))}\Big\}\Big|_{z'=z}, \tag{1.7}$$

R_2 can be represented as a sum of derivatives of the expectation of ratio of four determinants. Besides, it is expected that if we set

$$\begin{aligned} z_1&=E+i\varepsilon/N+\xi_1/N\rho(E), \quad z_2=E+i\varepsilon/N+\xi_2/N\rho(E),\\ z_1'&=E+i\varepsilon/N+\xi_1'/N\rho(E), \quad z_2'=E+i\varepsilon/N+\xi_2'/N\rho(E), \end{aligned} \tag{1.8}$$

then the r.h.s. of (1.7) before taking derivatives is an analytic function in $\xi_1,\xi_2,\xi_1',\xi_2'$. Thus, to study the second correlation function, it suffices to study the ratio of four determinants, which we call the second "generalized" correlation functions

$$\begin{aligned} \mathcal{R}_2^{+-}(z_1,z_1';z_2,z_2')&=\mathbb{E}\Big\{\frac{\det(H-z_1)\det(H-\overline{z}_2)}{\det(H-z_1')\det(H-\overline{z}_2')}\Big\},\\ \mathcal{R}_2^{++}(z_1,z_1';z_2,z_2')&=\mathbb{E}\Big\{\frac{\det(H-z_1)\det(H-z_2)}{\det(H-z_1')\det(H-z_2')}\Big\}. \end{aligned} \tag{1.9}$$

Similarly the derivative of the first "generalized" correlation function

$$\mathcal{R}_1(z_1,z_1'):=\mathbb{E}\Big\{\frac{\det(H-z_1')}{\det(H-z_1)}\Big\}$$

gives the Stieltjes transform of of the density of states (the first correlation function).

Instead of eigenvalue correlation functions one can consider more simple objects which are the correlation functions of characteristic polynomials:

$$\mathcal{R}_0(\lambda_1,\lambda_2)=\mathbb{E}\Big\{\det(H-\lambda_1)\det(H-\lambda_2)\Big\}, \quad \lambda_{1,2}=E\pm\xi/N\rho(E). \tag{1.10}$$

Characteristic polynomials are the objects of independent interest because of their connections to the number theory, quantum chaos, integrable systems, combinatorics, representation theory and others. But in our context the main point is that from the SUSY point of view correlation functions of characteristic polynomials correspond to the so-called fermion-fermion (Grassmann) sector of the supersymmetric full model describing the usual correlation functions (since they represent two determinants in the numerator of (1.9)). They are especially convenient for the SUSY approach and were successfully studied by the techniques for many ensembles (see [5], [6], [26], [27], etc.). In addition, although $\mathcal{R}_0(\lambda_1, \lambda_2)$ is not a local object, it is also expected to be universal in some sense. Moreover, correlation functions of characteristic polynomials are expected to exhibit a crossover which is similar to that of local eigenvalue statistics. In particular, for 1d RBM they are expected to have the same local behaviour as for GUE for $W \gg \sqrt{N}$, and the different behaviour for $W \ll \sqrt{N}$. Besides, the analysis of $\mathcal{R}_0(\lambda_1, \lambda_2)$ is much less involved than that for $\mathcal{R}_2^{+-}(z_1, z_1'; z_2, z_2')$, but on the other hand, this analysis allows to understand the nature of the crossover in RBM when W crosses the threshold $W \sim \sqrt{N}$.

The derivation of SUSY integral representation is basically an algebraic step, and usually it can be done by the standard algebraic manipulations. SUSY is widely used in the physics literature, but the rigour analysis of the obtained integral representation is a real mathematical challenge. Usually it is quite difficult, and it requires a powerful analytic and statistical mechanics techniques, such as a saddle point analysis, transfer operators, cluster expansions, renormalization group methods, etc. However, it can be done rigorously for some special class of RBM.

There exist especially convenient classes of RBM, where the control of SUSY integral representation becomes more accessible. One of them was introduced in [9]: it is (1.1) with Gaussian elements with variance

$$\mathbb{E}\{|H_{jk}|^2\} = \left(-W^2\Delta + 1\right)^{-1}_{jk}, \qquad (1.11)$$

where $\triangle$ is the discrete Laplacian on Λ with Neumann boundary conditions: for the case $d = 1$,

$$(-\Delta f)_j = \begin{cases} -f_{j-1} + 2f_j - f_{j+1}, & j \neq 1, n, \\ -f_{j-1} + f_j - f_{j+1}, & j = 1, n \end{cases} \qquad (1.12)$$

with $f_0 = f_{n+1} = 0$. It is easy to see that in 1d case $J_{jk} \approx C_1 W^{-1} \exp\{-C_2|j-k|/W\}$, and so the variance of matrix elements is exponentially small when $|j-k| \gg W$.

Another class of convenient models are the Gaussian block RBM which are the special class of Wegner's orbital models (see [35]). Gaussian block RBM are $N \times N$ Hermitian block matrices composed from n^2 blocks of the size $W \times W$ ($N = nW$). Only 3 block diagonals are non zero:

$$H = \begin{pmatrix} A_1 & B_1 & 0 & 0 & 0 & \dots & 0 \\ B_1^* & A_2 & B_2 & 0 & 0 & \dots & 0 \\ 0 & B_2^* & A_3 & B_3 & 0 & \dots & 0 \\ . & . & B_3^* & . & . & . & . \\ . & . & . & . & . & A_{n-1} & B_{n-1} \\ 0 & . & . & . & 0 & B_{n-1}^* & A_n \end{pmatrix}.$$

Here $A_1, \dots A_n$ are independent $W \times W$ GUE-matrices with i.i.d. (up to the symmetry) Gaussian entries with variance $(1-2\alpha)/W$, $\quad \alpha < \frac{1}{4}$, and $B_1, \dots B_{n-1}$ are independent $W \times W$ Ginibre matrices with i.i.d. Gaussian entries with variance α/W.

More precisely, H is Hermitian matrices with complex zero-mean random Gaussian entries $H_{jk,\alpha\beta}$, where $j,k \in \Lambda \subset \mathbb{Z}^d$ (they parameterize the lattice sites) and $\alpha, \gamma = 1, \dots, W$ (they parametrize the orbitals on each site), such that

$$\langle H_{j_1k_1,\alpha_1\gamma_1} H_{j_2k_2,\alpha_2\gamma_2} \rangle = \delta_{j_1k_2}\delta_{j_2k_1}\delta_{\alpha_1\gamma_2}\delta_{\gamma_1\alpha_2} J_{j_1k_1} \tag{1.13}$$

with

$$J = 1/W + \alpha\Delta/W, \tag{1.14}$$

where $W \gg 1$ and Δ is the discrete Laplacian on Λ (as in (1.11)). The probability law of H can be written in the form

$$P_N(dH) = \exp\Big\{ -\frac{1}{2} \sum_{j,k\in\Lambda} \sum_{\alpha,\gamma=1}^{W} \frac{|H_{jk,\alpha\gamma}|^2}{J_{jk}} \Big\} dH. \tag{1.15}$$

This model is one of the possible realizations of the Gaussian RBM, for example for $d = 1$ they correspond to the band matrices with the bandwidth $2W+1$. Let us remark that for this model $N = nW$, hence the crossover is expected for $n \sim W$.

The main advantage of both models (1.11) and (1.13) – (1.14) is that the main spectral characteristics such as density of states, R_2, $\mathbb{E}\{|G_{jk}(E+i\varepsilon)|^2\}$ for these models can be expressed via SUSY as the averages of certain observables of *nearest-neighbour* statistical mechanics models on Λ, which makes the model easier. For instance, the detailed information about the averaged density of states Gaussian RBM (1.11) in dimension 3 including local semicircle low at arbitrary short scales and smoothness in energy (in the limit of infinite volume and fixed large band width W) was obtained in [9]. The techniques of this paper was used in [8] to obtain the same result in 2d. The rigorous application of SUSY to the Gaussian block RBM (1.13) – (1.14) was developed in [29], where the universality of the bulk local regime for $n = const$ was proved. Combining this approach with Green's function comparison strategy it has been proved in [1] that $\ell \geq W^{7/6}$ (in a strong sense) for the block band matrices with rather general element's distribution.

The nearest-neighbour structure of the model also allows to combine the SUSY techniques with a transfer matrix approach.

2 Idea of the transfer operator approach

The supersymmetric transfer matrix formalism was first suggested by Efetov (see [11]) and on a heuristic level it was adapted specifically for RBM in [16] (see also references therein). The rigorous application of this method to the density of states and correlation function of characteristic polynomials was done in [22], [23], [24], [30]. The approach is based on the fact that many nearest-neighbour statistical mechanics problems in 1d can be formulated in terms of properties of some integral operator K that is called a transfer operator. More precisely, the discussion above yields that for 1d RBM of the form (1.11) or (1.13) – (1.14) the SUSY techniques helps to find a scalar kernel $\mathcal{K}_0(X_1, X_2)$ and matrix kernels $\mathcal{K}_1(X_1, X_2)$, $\mathcal{K}_2(X_1, X_2)$ (containing $z_{1,2}, z'_{1,2}$ as parameters) such that

$$\mathcal{R}_0(\lambda_1, \lambda_2) = C_N \int g_0(X_1)\mathcal{K}_0(X_1, X_2) \dots \mathcal{K}_0(X_{n-1}, X_n) f_0(X_n) \prod dX_i, \tag{2.1}$$
$$\mathcal{R}_1(z_1, z'_1) = W^2 \int g_1(X_1)\mathcal{K}_1(X_1, X_2) \dots \mathcal{K}_1(X_{n-1}, X_n) f_1(X_n) \prod dX_i,$$
$$\mathcal{R}_2(z_1, z'_1; z_2, z'_2) = W^4 \int g_2(X_1)\mathcal{K}_2(X_1, X_2) \dots \mathcal{K}_2(X_{n-1}, X_n) f_2(X_n) \prod dX_i,$$

where $\{X_j\}$ are Hermitian 2×2 matrices for the cases of $\mathcal{R}_0$, 2×2 matrices whose entries depend on 2 spacial variables $x_{1j}, y_{1j} \in \mathbb{R}$ for the cases $\mathcal{R}_1$, and for the case of $\mathcal{R}_2$ $\{X_j\}$ are 70×70 matrices whose entries depend on 4 spacial variables $x_{1j}, x_{2j}, y_{1j}, y_{2j} \in \mathbb{R}$, unitary 2×2 matrix U_i, and hyperbolic 2×2 matrix S_j, dX_j means the standard measure on Herm(2) for $\mathcal{R}_0$, $dX_j = dx_{j1}dy_{j1}$ for $\mathcal{R}_1$, and for $\mathcal{R}_2$ dX_j means the integration over $dx_{1j}dx_{2j}dy_{1j}dy_{2j}dU_jdS_j$ with dU, dS being the corresponding Haar measures.

Remark, that for the model (1.11) $n = N$, while for the block band matrix (1.13) – (1.15) n is a number of blocks on the main diagonal.

The idea of the transfer operator approach is very simple and natural. Let $\mathcal{K}(X, Y)$ be the matrix kernel of the compact integral operator in $\oplus_{i=1}^p L_2[X, d\mu(X)]$. Then

$$\int g(X_1)\mathcal{K}(X_1, X_2) \dots \mathcal{K}(X_{n-1}, X_n) f(X_n) \prod d\mu(X_i) = (\mathcal{K}^{n-1} f, \bar{g})$$
$$= \sum_{j=0}^{\infty} \lambda_j^{n-1}(\mathcal{K}) c_j, \quad \textit{with} \quad c_j = (f, \psi_j)(g, \tilde{\psi}_j). \tag{2.2}$$

Here $\{\lambda_j(\mathcal{K})\}_{j=0}^{\infty}$ are the eigenvalues of $\mathcal{K}$ ($|\lambda_0| \ge |\lambda_1| \ge \dots$), ψ_j are corresponding eigenvectors, and $\tilde{\psi}_j$ are the eigenvectors of $\mathcal{K}^*$. Hence, to study the correlation function, it suffices to study the eigenvalues and eigenfunctions of the integral operator with a kernel $\mathcal{K}(X, Y)$.

The main difficulties here are the complicated structure and non self-adjointness of the corresponding transfer operators.

In fact, since the analysis of eigenvectors of non self-adjoint operators is rather involved, it is simpler to work with the resolvent analog of (2.2)

$$\mathcal{R}_\alpha = (\mathcal{K}_\alpha^{n-1} f, \bar{g}) = -\frac{1}{2\pi i} \oint_{\mathcal{L}} z^{n-1} (\mathcal{G}_\alpha(z) f, \bar{g}) dz, \quad \mathcal{G}_\alpha(z) = (\mathcal{K}_\alpha - z)^{-1}, \quad \alpha = 0, 1, 2, \tag{2.3}$$

where $\mathcal{L}$ is any closed contour which contains all eigenvalues of $\mathcal{K}_\alpha$. For any α if we set

$$\lambda_* = \lambda_0(\mathcal{K}_\alpha), \quad (\lambda_* \sim 1),$$

then it suffices to choose $\mathcal{L}$ as $\mathcal{L}_0 = \{z : |z| = |\lambda_*|(1 + O(n^{-1}))\}$. However, it is more convenient to choose $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_2 = \{z : |z| = |\lambda_*|(1 - \log^2 n/n)\}$, and $\mathcal{L}_1$ is some contour in the domain between $\mathcal{L}_0$ and $\mathcal{L}_2$ which contains all eigenvalues of $\mathcal{K}_\alpha$ outside of $\mathcal{L}_2$. Then

$$(\mathcal{K}_\alpha^{n-1} f, \bar{g}) = -\frac{1}{2\pi i} \oint_{\mathcal{L}_1} z^{n-1} (\mathcal{G}_\alpha(z) f, \bar{g}) dz - \frac{1}{2\pi i} \oint_{\mathcal{L}_2} z^{n-1} (\mathcal{G}_\alpha(z) f, \bar{g}) dz$$

and if we have a reasonable bound for $\|\mathcal{G}_\alpha(z)\|$ ($z \in \mathcal{L}_2$), then the second integral is small comparing with $|\lambda_*|^{n-1}$, since

$$|z|^{n-1} \le |\lambda_*|^{n-1} e^{-\log^2 n}.$$

Hence, it is natural to expect that the integral over $\mathcal{L}_1$ gives the main contribution to $\mathcal{R}_\alpha$.

Definition 2.1. *We shall say that the operator $\mathcal{A}_{n,W}$ is equivalent to $\mathcal{B}_{n,W}$ ($\mathcal{A}_{n,W} \sim \mathcal{B}_{n,W}$), if for some certain contour $\mathcal{L}_1$ (the choice of $\mathcal{L}_1$ depends on the problem)*

$$((\mathcal{A}_{n,W} - z)^{-1} f, \bar{g}) = ((\mathcal{B}_{n,W} - z)^{-1} f, \bar{g})(1 + o(1)), \quad n, W \to \infty,$$

with f, g of (2.2).

The idea is to find some $\mathcal{K}_{*\alpha} \sim \mathcal{K}_\alpha$ whose spectral analysis we are ready to perform.

3 Mechanism of the crossover for $\mathcal{R}_0$

As it was mentioned in Section 1, the simplest object which allows to understand the crossover's mechanism for the 1d RBM (1.11) is the correlation function of characteristic polynomials $\mathcal{R}_0$. Using SUSY and the idea of the transfer operator approach, one can write $\mathcal{R}_0$ (see [23]) as

$$\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = C_n \cdot W^{-4n} \det{}^{-2} J \cdot (K_{0\xi}^{n-1} \mathcal{F}_\xi, \bar{\mathcal{F}}_\xi), \tag{3.1}$$

where $(\cdot,\cdot)$ is a standard inner product in $L_2(\mathrm{Herm}(2), dX)$ (i.e., 2×2 Hermitian matrices), with respect to the measure

$$dX_j = d(X_j)_{11} d(X_j)_{22} d\Re(X_j)_{12} d\Im(X_j)_{12},$$

C_n is some ξ-independent constant, $K_{0\xi} : \mathcal{H} \to \mathcal{H}$ be the operators with the kernels

$$K_\xi(X, Y) = \frac{W^4}{2\pi^2} \mathcal{F}_\xi(X) \exp\Big\{ -\frac{W^2}{2} \mathrm{Tr}\,(X - Y)^2 \Big\} \mathcal{F}_\xi(Y). \tag{3.2}$$

where $\hat{\xi} = \mathrm{diag}\,\{\xi, -\xi\}$, $\Lambda_0 = E \cdot I_2$, and $\mathcal{F}_\xi(X)$ is the operator of multiplication by

$$\mathcal{F}_\xi(X) = \mathcal{F}(X) \cdot \exp\Big\{ -\frac{i}{2n\rho(E)}\, \mathrm{Tr}\, X\hat{\xi} \Big\} \tag{3.3}$$

with

$$\mathcal{F}(X) = \exp\Big\{ -\frac{1}{4}\,\mathrm{Tr}\,\Big(X + \frac{i\Lambda_0}{2}\Big)^2 + \frac{1}{2}\,\mathrm{Tr}\,\log\big(X - i\Lambda_0/2\big) - C_+ \Big\}$$

and some specific C_+. Notice that the stationary points of $\mathcal{F}$ are

$$a_+ = -a_- = \sqrt{1 - E^2/4} = \pi\rho(E). \tag{3.4}$$

The first step is to show that if we introduce the projection $P_\pm$ onto the $W^{-1/2}\log W$-neighbourhood of the"surface" $X_*(U)=UDU^*$ with $D=\mathrm{diag}\,\{a_+,a_-\}$ and $U\in \mathring{U}(2):=U(2)/U(1)\times U(1)$, then in the sense of Definition 2.1

$$K_{0\xi}\sim P_\pm K_{0\xi}P_\pm. \tag{3.5}$$

To study the operators $P_\pm K_{0\xi}P_\pm$ we use the "polar coordinates". Namely, introduce

$$t=(x_1-y_1)(x_2-y_2),\quad p(x,y)=\frac{\pi}{2}(x-y)^2, \tag{3.6}$$

and denote by dU the integration with respect to the Haar measure on the group $\mathring{U}(2)$. Consider the space $L_2[\mathbb{R}^2,p]\times L_2[\mathring{U}(2),dU]$. The inner product and the action of an integral operator in this space are

$$(f,g)_p=\int f(x,y)\bar{g}(x,y)p(x,y)\,dx\,dy; \tag{3.7}$$

$$(Mf)(x_1,y_1,U_1)=\int M(x_1,y_1,U_1;x_2,y_2,U_2)\,f(x_2,y_2,U_2)\,p(x_2,y_2)dx_2dy_2dU_2.$$

Changing the variables

$$X=U^*\Lambda U,\quad \Lambda=\mathrm{diag}\{x_1,x_2\},\quad x_1>x_2,\quad U\in\mathring{U}(2),$$

we obtain that $K_{0\xi}$ can be represented as an integral operator in $L_2[\mathbb{R}^2,p]\times L_2[\mathring{U}(2),dU]$ defined by the kernel

$$\mathcal{K}_{0\xi}(X,Y)\to\mathcal{K}_{0\xi}(x_1,y_1,U_1;x_2,y_2,U_2) \tag{3.8}$$

where

$$\mathcal{K}_{0\xi}(x_1,y_1,U_1;x_2,y_2,U_2)=t^{-1}A_1(x_1,x_2)A_2(y_1,y_2)K_{*0\xi}(t,U_1,U_2)(1+O(n^{-1}W^{-1/2}));$$

$$A_{1,2}(x_1,x_2)=(2\pi)^{-1/2}e^{-W^2(x_1-x_2)^2/2}e^{f_{1,2}(x_1)+f_{1,2}(x_2)}; \tag{3.9}$$

$$K_{*0\xi}(t,U_1,U_2):=W^2t\cdot e^{tW^2\mathrm{Tr}\,U_1U_2^*L(U_1U_2^*)^*L/4-tW^2/2}e^{-i\xi\pi(\nu(U_1)+\nu(U_2))/n}; \tag{3.10}$$

$$\nu(U)=\mathrm{Tr}\,U^*LUL/2,\quad L=\mathrm{diag}\{1,-1\},$$

and t is defined in (3.6). The concrete form of $f_{1,2}$ in (3.9) is not important for us now. It is important that they are analytic functions with stationary points $a_\pm$ (see (3.4)). The analysis of the resolvent of A_1 and A_2 allows us to show that only eigenfunctions localized in the $W^{-1/2}\log W$ neighbourhood of a_+ and a_- give essential contribution in (2.2). More precisely, the resolvent analysis of $A_{1,2}$ allows to prove (3.5). Further resolvent analysis gives

$$P_\pm\mathcal{K}_{0\xi}P_\pm\sim\mathcal{K}_{*\xi}\otimes\mathcal{A}, \tag{3.11}$$

$$\mathcal{K}_{*\xi}(U_1,U_2):=K_{*0\xi}(t^*,U_1,U_2)\quad \textit{with}\quad t^*=(a_+-a_-)^2=4\pi^2\rho(E)^2,$$

$$\mathcal{A}(x_1,x_2,y_1,y_2)=A_1(x_1,x_2)A_2(y_1,y_2).$$

Then from (2.3) and Definition 2.1 it is easy to obtain

$$\mathcal{R}_\xi = C_n(\mathcal{K}_{*\xi}^{n-1} \otimes \mathcal{A}^{n-1} f, \bar{g})(1+o(1)) = (\mathcal{K}_{*\xi}^{n-1} f_0, f_0)(\mathcal{A}^{n-1} f_1, \bar{g}_1)(1+o(1)),$$

where we used that both f, g asymptotically can be replaced by $f_0(U) \otimes f_1(x,y)$ with

$$f_0 \equiv 1. \tag{3.12}$$

If we introduce

$$D_2 = \mathcal{R}_0(E,E), \tag{3.13}$$

then the above consideration yields

$$D_2^{-1}\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{(\mathcal{K}_{*\xi}^{n-1} f_0, f_0)}{(\mathcal{K}_{*0}^{n-1} f_0, f_0)}(1+o(1)). \tag{3.14}$$

A good news here is that the operator $\mathcal{K}_{*0}$ is self-adjoint and his kernel depends only on $|(U_1U_2^*)_{12}|^2$. By [34], his eigenfunctions are associated Legendre polynomials P_k^j. Moreover since $\mathcal{K}_{*0}$ is reduced by the space $\mathcal{E}_0 \subset L_2(U(2))$ of the functions which depends only on $|U_{12}|^2$, and $f_0 \in \mathcal{E}_0$, we can restrict our spectral analysis to $\mathcal{E}_0$. In this space eigenfunctions of $\mathcal{K}_{*0}$ are Legendre polynomials P_j and it is easy to check that correspondent eigenvalues have the form

$$\lambda_j = 1 - j(j+1)/t^*W^2 + O((j(j+1)/W^2)^2), \quad j = 0, 1 \ldots \tag{3.15}$$

with t^* of (3.11). Moreover, it follows from (3.10) that

$$\mathcal{K}_{*\xi} = \mathcal{K}_{*0} - 2n^{-1}\pi i \xi \hat{\nu} + o(n^{-1}),$$

where $\hat{\nu}$ is the operator of multiplication by ν of (3.10). Thus the eigenvalues of $\mathcal{K}_{*\xi}$ are in the n^{-1}-neighbourhood of λ_j. This implies that for $W^{-2} \gg n^{-1} = N^{-1}$

$$|\lambda_1(\mathcal{K}_{*\xi})| \le 1 - O(W^{-2}), \quad \lambda_0 = 1 - 2n^{-1}\pi i \xi(\nu f_0, f_0) + o(n^{-1})$$

Since

$$(\nu f_0, f_0) = 0,$$

we obtain that the numerator and the denominator of (3.14) tends to 1 in this regime.

To study the regime $W^{-2} = Cn^{-1} = CN^{-1}$, observe that the Laplace operator Δ_U on $U(2)$ is also reduced by $\mathcal{E}_0$ and has the same eigenfunctions as $\mathcal{K}_{*0}$ with eigenvalues

$$\lambda_j^* = j(j+1)$$

Hence, we can write $\mathcal{K}_{*\xi}$ as

$$\mathcal{K}_{*\xi} \sim 1 - n^{-1}(C\Delta_U - 2i\xi\pi\nu) \Rightarrow (\mathcal{K}_{*\xi}^{n-1} f_0, f_0) \to (e^{-C\Delta_U + 2i\xi\pi\hat{\nu}} f_0, f_0),$$

where

$$\Delta_U = -\frac{d}{dx}x(1-x)\frac{d}{dx}, \quad x = |U_{12}|^2. \tag{3.16}$$

And in the regime $W^{-2} \ll n^{-1}$ we have $\mathcal{K}_{*0}^{n-1} \to I$ in the strong vector topology, hence

$$\mathcal{K}_{*\xi} \sim 1 - n^{-1} 2i\xi\pi\nu \Rightarrow (\mathcal{K}_{*\xi}^{n-1} f_0, f_0) \to (e^{-2i\xi\pi\hat{\nu}} f_0, f_0)$$

and the numerator of (3.14) is given by the multiplication of f_0 by $e^{-2i\xi\pi\hat{\nu}}$, which gives the same form as for the correlation function of the Wigner model.

The last result was proved in [28] with a different method:

Theorem 3.1 ([**28**]). *For the 1d RBM of (1.11) with $W^2 = N^{1+\theta}$, where $0 < \theta \le 1$, we have*

$$\lim_{n\to\infty} D_2^{-1} \mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{\sin(2\pi\xi)}{2\pi\xi}, \tag{3.17}$$

i.e. the limit coincides with that for GUE. The limit is uniform in ξ varying in any compact set $C \subset \mathbb{R}$. Here $\rho(x)$ and $\mathcal{R}_0$ are defined in (1.2) and (1.10), $E \in (-2, 2)$.

The regime $W^{-2} \gg N^{-1}$ was studied in [23]:

Theorem 3.2. *For the 1d RBM of (1.11) with $1 \ll W \le \sqrt{N/C_* \log N}$ for sufficiently big C_*, we have*

$$\lim_{n\to\infty} D_2^{-1} \mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = 1,$$

where the limit is uniform in ξ varying in any compact set $C \subset \mathbb{R}$. Here $E \in (-2, 2)$, and $\rho(x)$, $\mathcal{R}_0$, and D_2 are defined in (1.2), (1.10), and (3.13).

Remark 3.1. *Although the result is formulated for $\xi_1 = -\xi_2 = \xi$ in (1.8), one can prove Theorem 3.2 for $\xi_1, \xi_2 \in [-C, C] \subset \mathbb{R}$ by the same arguments with minor revisions. The only difference is a little bit more complicated expressions for D_2 and K_ξ.*

The regime $W^{-2} = C_* N^{-1}$ is studied in [30]:

Theorem 3.3. *For the 1d RBM of (1.11) with $N = C_* W^2$, we have*

$$\lim_{n\to\infty} D_2^{-1} \mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = (e^{-C\Delta_U - 2\pi i\xi\hat{\nu}} f_0, f_0),$$

where $C = 1/t^* C_*$ with t^* of (3.11), and the limit is uniform in ξ varying in any compact subset of $\mathbb{R}$. Here $E \in (-2, 2)$.

4 Analysis of $\mathcal{R}_1$

In the case of $\mathcal{R}_1$ the transfer operator $\mathcal{K}_1$ of (2.2) has the form

$$\mathcal{K}_1 = A_1(x_1, x_2) A_2(y_1, y_2) \hat{Q}, \quad \hat{Q} := \begin{pmatrix} 1 + L(\bar{x}, \bar{y})/W^2 & -1/W^2 \\ -L(\bar{x}, \bar{y}) & 1 \end{pmatrix} \tag{4.1}$$

with some explicit function L whose form is not important for us now. Operators $A_{1,2}$ (the same as for $\mathcal{R}_0$) contain a large parameter W in the exponent, hence only $W^{-1/2}$- neighbourhood of the stationary point gives the main contribution. The spectral analysis of A_1 gives us that

$$\begin{aligned} &A_1 \sim e^{\xi g_+(E)/N} A_+, \quad A_2 \sim A_+, \\ &A_+(x,y) = (2\pi)^{-1/2} W^2 e^{-W^2(x-y)/2 + c_+(x^2+y^2)/2}, \quad c_+ = 1 + a_+^{-2}, \\ &g_+(E) = (-E + i\sqrt{4-E^2})/2. \end{aligned} \tag{4.2}$$

Then since

$$\lambda_j(A_+) = \Big(1 + \frac{2\alpha_+}{W} + \frac{c_+}{W^2}\Big)^{-1/2-j}, \tag{4.3}$$

$$\alpha_+ = \sqrt{\frac{c_+}{2}}\Big(1 + \frac{c_+}{2W^2}\Big)^{1/2}, \tag{4.4}$$

we obtain that the spectral gap for $A_{1,2}$ is of the order $W^{-1} \gg N^{-1}$, hence one could expect that A_1^{N-1} converges in the strong vector topology to the projection

$$A_{1,2}^{N-1} \to \lambda_0^{N-1}(A_1)\psi_0 \otimes \psi_0^*$$

where

$$A_1\psi_0 = \lambda_0(A_1)\psi_0, \quad A_1^*\psi_0^* = \overline{\lambda_0(A_1)}\psi_0^*.$$

The entry Q_{12} here is small hence the main order of our operator contains the Jordan cell. A simple computation shows that if we just replace in (4.1) $A_{1,2}$ by A_+ and Q_{12} by 0, then the answer will be wrong. Hence one should apply more refine analysis. An important point of such analysis is an application of the "gauge" transformation of $\mathcal{K}_1$ with matrix T

$$\begin{aligned} &\mathcal{K}_1 \to \mathcal{K}_{1T} = T\mathcal{K}_1 T^{-1} = A_1 A_2 \hat{S}, \quad \hat{S} = T\hat{Q}T^{-1}; \\ &T = \begin{pmatrix} 0 & W^{-1/2} \\ W^{1/2} & 0 \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} 1 & -L/W \\ -1/W & 1 + L/W^2 \end{pmatrix}. \end{aligned} \tag{4.5}$$

With this transformation it can be shown that for any W

$$\lambda_0(\mathcal{K}_{1T}) = e^{\xi g_+(E)/N}(1 + O(n^{-2})), \quad |\lambda_1(\mathcal{K}_{1T})| \le 1 - c/W, \quad c > 0.$$

Hence for any $1 \ll W \ll N$ we get that $(\mathcal{K}_{1T})^{N-1}$ converges in the strong vector topology to the projection (non-orthogonal) on the eigenvector, corresponding to $\lambda_0(\mathcal{K}_{1T})$. This gives

Theorem 4.1. *Let H be 1d Gaussian RBM defined in (1.11) with $N \ge C_0 W \log W$, and let $|E| \le 4\sqrt{2}/3 \approx 1.88$.*

$$\mathcal{R}_1(E + \xi/N, E) \to e^{\xi g(E)}, \quad \Big|\frac{\partial}{\partial \xi}\mathcal{R}_1(E + \xi/N, E)\Big|_{\xi=0} - g_+(E)\Big| \le C/W.$$

The second relation implies that

$$|\bar{\rho}_N(E) - \rho(E)| \le C/W, \tag{4.6}$$

where $\bar{\rho}_N(E) = R_1(E)$ is the first correlation function, and $\rho(E)$ is defined in (1.2).

Remark 4.1. *The statement is expected to be true for all* $|E| < 2$*. The condition* $|E| \le 4\sqrt{2}/3 \approx 1.88$ *is technical, and it can be removed by the proper deformation of the integration contour in the integral representation.*

Theorem 4.1 yields, in particular, that for $g_N(E + i\varepsilon)$ (the Stieltjes transform of the first correlation function $\bar{\rho}_N(E)$) and $g(E + i\varepsilon)$ (the Stieltjes transform of $\rho(E)$) we have

$$|\bar{g}_N(E + i\varepsilon) - g(E + i\varepsilon)| \le C/W \tag{4.7}$$

uniformly in any arbitrary small $\varepsilon \ge 0$. As it was mentioned above, similar asymptotics (with correction C/W^2) for RBM of (1.11) in 3d was obtained in [9] and in 2d was obtained in [8] (by the same techniques), however their method cannot be directly applied to 1d case since it essentially uses the Fourier analysis which is different in 1d. All other previous results about the density of states for RBM deal with $\varepsilon \gg W^{-1}$ or bigger (for fixed $\varepsilon > 0$ the asymptotics (4.7) follows from the results of [3]; [12] gives (4.7) with $\varepsilon \gg W^{-1/3}$; [31] yields (4.7) for 1d RBM with Bernoulli elements distribution for $\varepsilon \ge W^{-0.99}$, and [14] proves similar to (4.7) asymptotics with correction $1/(W\varepsilon)^{1/2}$ for $\varepsilon \gg 1/W$). On the other hand, the methods of [12], [14] allow to control $N^{-1}\mathrm{Tr}\,(E + i\varepsilon - H_N)^{-1}$ and $(E + i\varepsilon - H_N)^{-1}_{xy}$ for $\varepsilon \gg W^{-1}$ without expectation, which gives some information about the localization length. This cannot be obtained from Theorem 4.1, since it requires estimates on $\mathbb{E}\{|(E + i\varepsilon - H_N)^{-1}_{xy}|^2\}$.

5 Analysis of $\mathcal{R}_2$ for the block RBM

5.1 Sigma-model approximation for $\mathcal{R}_2$ for the block RBM

We start from the analysis of so-called sigma-model approximation for the model (1.13) – (1.14). Sigma-model approximation is often used by physicists to study a complicated statistical mechanics systems. In such approximation spins take values in some symmetric space (± 1 for Ising model, S^1 for the rotator, S^2 for the classical Heisenberg model, etc.). It is expected that sigma-models have all the qualitative physics of more complicated models with the same symmetry (for more details see, e.g., [32]). The sigma-model approximation for RBM was introduced by Efetov (see [11]), and the spins there are 4×4 matrices with both complex and Grassmann entries (this approximation was studied in [15], [16]). Let us mention also the paper [10], where the average conductance for 1d Efetov's sigma-model for RBM was computed.

In the subsection we present rigorous results on the derivation of the sigma-model approximation for 1d RBM and the analysis of the model in the delocalization regime. The results are published in [24].

To derive a sigma-model approximation for the model (1.13) – (1.14), we take α in (1.14) $\alpha = \beta/W$, i.e. put

$$J = 1/W + \beta\Delta/W^2, \quad \beta > 0, \tag{5.1}$$

fix β and n, and consider the limit $W \to \infty$, for the generalized correlation functions

$$\mathcal{R}^{+-}_{Wn\beta}(E,\varepsilon,\xi)=\mathbf{E}\Big\{\frac{\det(H-z_1)\det(H-\overline{z}_2)}{\det(H-z_1')\det(H-\overline{z}_2')}\Big\}, \tag{5.2}$$
$$\mathcal{R}^{++}_{Wn\beta}(E,\varepsilon,\xi)=\mathbf{E}\Big\{\frac{\det(H-z_1)\det(H-z_2)}{\det(H-z_1')\det(H-z_2')}\Big\}$$

for $\xi=(\xi_1,\xi_2,\xi_1',\xi_2')$.

Theorem 5.1. *Given $\mathcal{R}^{+-}_{Wn\beta}$ of (5.2) ,(1.13) and (5.1), with any dimension d, any fixed β, $|\Lambda|$, $\varepsilon>0$, and $\xi=(\xi_1,\bar{\xi}_2,\xi_1',\bar{\xi}_2')\in\mathbb{C}^4$ ($|\Im\xi_j|<\varepsilon\cdot\rho(E)/2$) we have, as $W\to\infty$:*

$$\mathcal{R}^{+-}_{Wn\beta}(E,\varepsilon,\xi)\to\mathcal{R}^{+-}_{n\beta}(E,\varepsilon,\xi),\quad \frac{\partial^2\mathcal{R}^{+-}_{Wn\beta}}{\partial\xi_1'\partial\xi_2'}(E,\varepsilon,\xi)\to\frac{\partial^2\mathcal{R}^{+-}_{n\beta}}{\partial\xi_1'\partial\xi_2'}(E,\varepsilon,\xi), \tag{5.3}$$

$$\textit{where}\quad \mathcal{R}^{+-}_{n\beta}(E,\varepsilon,\xi)=C_{E,\xi}\int\exp\Big\{\frac{\tilde\beta}{4}\sum\operatorname{Str}Q_jQ_{j-1}-\frac{c_0}{2|\Lambda|}\sum\operatorname{Str}Q_j\Lambda_{\xi,\varepsilon}\Big\}dQ,$$

$\tilde\beta=(2\pi\rho(E))^2\beta$, $U_j\in\mathring{U}(2)$, $S_j\in\mathring{U}(1,1)=U(1,1)/U(1)\times U(1)$,

$$C_{E,\xi}=e^{E(\xi_1+\xi_2-\xi_1'-\xi_2')/2\rho(E)},\quad \rho(E)=(2\pi)^{-1}\sqrt{4-E^2},$$

and Q_j are 4×4 supermatrices with commuting diagonal and anticommuting off-diagonal 2×2 blocks

$$Q_j=\left(\begin{array}{cc}U_j^* & 0\\ 0 & S_j^{-1}\end{array}\right)\left(\begin{array}{cc}(I+2\hat\rho_j\hat\tau_j)L & 2\hat\tau_j\\ 2\hat\rho_j & -(I-2\hat\rho_j\hat\tau_j)L\end{array}\right)\left(\begin{array}{cc}U_j & 0\\ 0 & S_j\end{array}\right), \tag{5.4}$$

$$dQ=\prod dQ_j,\quad dQ_j=(1-2n_{j,1}n_{j,2})\,d\rho_{j,1}d\tau_{j,1}\,d\rho_{j,2}d\tau_{j,2}\,dU_j\,dS_j$$

with

$$n_{j,1}=\rho_{j,1}\tau_{j,1},\quad n_{j,2}=\rho_{j,2}\tau_{j,2},$$
$$\hat\rho_j=\operatorname{diag}\{\rho_{j1},\rho_{j2}\},\quad \hat\tau_j=\operatorname{diag}\{\tau_{j1},\rho_{j2}\},\quad L=\operatorname{diag}\{1,-1\}$$

Here $\rho_{j,l}$, $\tau_{j,l}$, $l=1,2$ are anticommuting Grassmann variables,

$$\operatorname{Str}\left(\begin{array}{cc}A & \sigma\\ \eta & B\end{array}\right)=\operatorname{Tr}A-\operatorname{Tr}B,$$

and

$$\Lambda_{\xi,\varepsilon}=\operatorname{diag}\{\varepsilon-i\xi_1/\rho(E),-\varepsilon-i\xi_2/\rho(E),\varepsilon-i\xi_1'/\rho(E),-\varepsilon-i\xi_2'/\rho(E)\}.$$

Theorem 5.2. *Given $\mathcal{R}^{++}_{Wn\beta}$ of (5.2) ,(1.13) and (5.1), with any dimension d, any fixed β, $|\Lambda|$, $\varepsilon>0$, and $\xi=(\xi_1,\xi_2,\xi_1',\xi_2')\in\mathbb{C}^4$ ($|\Im\xi_j|<\varepsilon\cdot\rho(E)/2$) we have, as $W\to\infty$:*

$$\mathcal{R}^{++}_{Wn\beta}(E,\varepsilon,\xi)\to e^{ia_+(\xi_1'+\xi_2'-\xi_1-\xi_2)/\rho(E)}, \tag{5.5}$$
$$\frac{\partial^2\mathcal{R}^{++}_{Wn\beta}}{\partial\xi_1'\partial\xi_2'}(E,\varepsilon,\xi)\to -a_+^2/\rho^2(E)\cdot e^{ia_+(\xi_1'+\xi_2'-\xi_1-\xi_2)/\rho(E)},\qquad a_+=(iE+\sqrt{4-E^2})/2.$$

Note that $Q_j^2 = I$ for Q_j of (5.4) and so the integral in the r.h.s of (5.3) is a sigma-model approximation similar to Efetov's one (see [11]).

The kernel of the transfer operator for $\mathcal{R}_2^{(\sigma)}$ has a form

$$\mathcal{K}_2^{(\sigma)} = \hat{F}\hat{Q}\hat{F}$$

where $\hat{F}$ and $\hat{Q}$ are 6×6 matrix kernels, such that $\hat{F}_{\mu\nu}$ are the operators of the multiplication by some function of U, S and $\hat{Q}_{\mu\nu} = \hat{Q}_{\mu\nu}(U_1 U_2^*, S_1 S_2^{-1})$ are the "difference" operators.
After some asymptotic analysis $\mathcal{K}_2^{(\sigma)}$ and some "gauge" transformation similar to (4.5) we obtain that $T\mathcal{K}_2^{(\sigma)}T$ can be replaced by the 4×4 "effective" matrix kernel

$$T\mathcal{K}_2^{(\sigma)}T \sim \tilde{F}\hat{K}_0\tilde{F}, \tag{5.6}$$

$$\hat{K}_0 = \begin{pmatrix} K & \tilde{K}_1 & \tilde{K}_2 & \tilde{K}_3 \\ 0 & K & 0 & \tilde{K}_2 \\ 0 & 0 & K & \tilde{K}_1 \\ 0 & 0 & 0 & K \end{pmatrix}, \quad \tilde{F} = F \begin{pmatrix} 1 & \tilde{F}_1 & \tilde{F}_2 & \tilde{F}_1\tilde{F}_2 \\ 0 & 1 & 0 & \tilde{F}_2 \\ 0 & 0 & 1 & \tilde{F}_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $K = K_U \otimes K_S$

$$K_U(U_1, U_2) \sim \beta e^{-\beta|(U_1 U_2^*)_{12}|^2}, \;\; K_S(S_1, S_2) \sim \beta e^{-\beta|(S_1 S_2^{-1})_{12}|^2},$$

$\tilde{K}_i = \tilde{K}_i(U_1 U_2^*; S_1 S_2^{-1})$, F is an operator of multiplication by $e^{\varphi(U,S)/2n}$, and $\tilde{F}_{1,2}$ are operators of multiplication by $n^{-1}\varphi_{1,2}(U, S)$ with some specific φ, φ_1 and φ_2. An important feature of $\tilde{K}_i$ that they satisfy the operator bound

$$|\tilde{K}_i| \le C\beta^{-1}(\Delta_U + \Delta_S)$$

where Δ_U, Δ_S are the Laplace operator on the correspondent groups (see e.g. (3.16) for the definition of Δ_U). The bounds imply that for sufficiently smooth function f $\tilde{K}_i f \sim \beta^{-1}$.

Similarly to Section 3 the idea is to show that in the regime $\beta \gg n$

$$\tilde{F}\hat{K}_0\tilde{F} \sim \tilde{F}^2$$

Then we get

$$\begin{aligned} \mathcal{R}^{+-}_{n\tilde{\beta}}(E, \varepsilon, \xi) &= \frac{C_E^*}{2\pi i} \oint_{\omega_A} z^{n-1}(\widehat{G}_0(z)\widehat{f}, \widehat{g})dz + o(1) = C_E^*(\widehat{F}^{2n-2}\widehat{f}, \widehat{g}) + o(1) \\ &= C_E^* \int \big(4n^2 F_1 F_2 - 2\big)F^{2n} dU dS + o(1), \end{aligned}$$

where

$$C_E^* = e^{-g_+(E)(\xi_1 + \xi_1' - \xi_2 - \xi_2')/\rho(E)}, \quad g_+(E) = (-E + i\sqrt{4 - E^2})/2. \tag{5.7}$$

This relation allows us to prove

Theorem 5.3. *If $n, \beta \to \infty$ in such a way that $\beta > Cn \log^2 n$, then for any fixed $\varepsilon > 0$ and $\xi = (\xi_1, \xi_2, \xi_1', \xi_2') \in \mathbb{C}^4$ $(|\Im \xi_j| < \varepsilon \cdot \rho(E)/2)$ we have*

$$\mathcal{R}_{n\beta}^{+-} \to C_E^* \Big(\frac{\delta_1 \delta_2}{\alpha_1 \alpha_2} (e^{2c_0\alpha_1} - 1) - \frac{\delta_1 + \delta_2}{\alpha_2} e^{2c_0\alpha_1} + e^{2c_0\alpha_1} \frac{\alpha_1}{\alpha_2} \Big), \tag{5.8}$$

$$\text{where} \quad \alpha_1 = \varepsilon - i(\xi_1 - \xi_2)/2\rho(E), \quad \alpha_2 = \varepsilon - i(\xi_1' - \xi_2')/2\rho(E), \tag{5.9}$$
$$\delta_1 = i(\xi_1' - \xi_1)/2\rho(E), \quad \delta_2 = i(\xi_2 - \xi_2')/2\rho(E),$$

and C_E^* is defined in (5.7).

Theorem 5.3 combined with Theorem 5.2 gives the GUE type behaviour for the spectral correlation function:

Theorem 5.4. *In the dimension $d = 1$ the behaviour of the sigma-model approximation of the second order correlation function (5.2) of (1.13), (5.1), as $\beta \gg n$, in the bulk of the spectrum coincides with those for the GUE. More precisely, if $\Lambda = [1, n] \cap \mathbb{Z}$ and H_N, $N = Wn$ are matrices (1.13) with J of (5.1), then for any $|E| < \sqrt{2}$ (1.4) holds in the limit first $W \to \infty$, and then $\beta, n \to \infty$, $\beta \geq Cn \log^2 n$.*

5.2 Analysis of $\mathcal{R}_2$ for block RBM of (1.13)-(1.14)

As it was mentioned in Section 2 in the case of $\mathcal{R}_2$ the transfer operator $\mathcal{K}_2$ is a 70×70 matrices whose entries depend on 8 spacial variables $x_1, x_2, y_1, y_2; x_1', x_2', y_1', y_2' \in \mathbb{R}$, two unitary 2×2 matrix U, U', and two hyperbolic 2×2 matrix S, S', which acts in the direct sum of 70 Hilbert spaces $L_2(\mathbb{R}^4) \otimes L_2(\mathring{U}(2), dU) \otimes L_2(\mathring{U}(1,1), dS)$, where dU, dS are integrations with respect to the corresponding Haar measures. In general the analysis of such operator is a very involved problem, unless there is a possibility to take into account some special features of the matrix kernel and to reduce it (in the sense of Definition 2.1) by some matrix kernel of smaller dimensionality.

In the case of $\mathcal{K}_2$ the first observation is that it can be factorised as

$$\mathcal{K}_2 = \hat{F} \hat{Q} \hat{A} \hat{F}$$

where $\hat{F}$, $\hat{Q}$ and $\hat{A}$ are 70×70 matrix kernels, such that $\hat{F}_{\mu\nu}$ are the operators of multiplication by some function of U, S,

$$\hat{Q}_{\mu\nu} = K_U K_S Q_{\mu\nu}(U(U')^*; S(S')^{-1}),$$
$$K_U = \alpha t W e^{-\alpha W t |(U(U')^*)_{12}|^2}, \quad K_S = \alpha \tilde{t} W e^{-\alpha W \tilde{t} |(S(S')^{-1})_{12}|^2},$$

with t, $\tilde{t}$ defined similarly to (3.6) and functions $Q_{\mu\nu}$ which do not depend on W, and

$$\hat{A}_{\mu\nu} = A_1(x_1, x_1') A_2(y_1, y_1') A_3(x_2, x_2') A_4(y_2, y_2') \mathcal{A}_{\mu,\nu}(\bar{x}, \bar{x}', \bar{y}, \bar{y}')$$

with $A_{1,2,3,4}$ being a scalar kernels similar to that for $\mathcal{R}_0$ (see (3.9)) and functions $A_{\mu\nu}$ which do not depend on W. It is straightforward to prove that only $W^{-1/2} \log W$ -neighbourhoods of some stationary points in $\mathbb{R}^8$ give essential contributions. Further analysis shows that after some "gauge" transformation similar to (4.5) $T\mathcal{K}_2T^{-1}$ can be replaced (in the sense of Definition 2.1) by 4×4 effective kernel of the form similar to (5.6).

Remark that the analysis justifies the physics conjecture that the behaviour of the "generalized" correlation function $\mathcal{R}_2$ for the model (1.13) – (1.14) and of its sigma-model approximation $\mathcal{R}_2^{\sigma}$ of are very similar.

As a result we obtain (cf with Theorem 5.4)

Theorem 5.5. *In the dimension $d=1$ the behaviour of the second order correlation function (1.6) of the model (1.13) – (1.14), as $W\gg n$, in the bulk of the spectrum coincides with those for the GUE. More precisely, for any $|E|<\sqrt{2}$ (1.4) holds in the limit $W,n\to\infty$ with $W/\log^2 W>Cn$.*

The theorem is the main result of the paper [25].

References

[1] Bao, J., Erdős, L.: Delocalization for a class of random block band matrices. Probab. Theory Relat. Fields, **167**, pp. 673 – 776 (2017)

[2] Berezin, F.A.: Introduction to the algebra and analysis of anticommuting variables. Moscow State University Publ., Moscow (1983) (Russian)

[3] Bogachev, L. V., Molchanov, S. A., and Pastur, L. A.: On the level density of random band matrices. Mat. Zametki, **50:6**, 31 – 42(1991)

[4] Bourgade, P., Erdős, L., Yau, H.-T., Yin, J. Universality for a class of random band matrices, arXiv:1602.02312

[5] Brézin, E., Hikami, S.: Characteristic polynomials of random matrices. Commun. Math. Phys. **214**, 111 – 135 (2000)

[6] Brézin, E., Hikami, S.: Characteristic polynomials of real symmetric random matrices. Commun. Math. Phys. **223**, 363 – 382 (2001)

[7] Casati, G., Molinari, L., Israilev, F.: Scaling properties of band random matrices, Phys. Rev. Lett. **64** (1990), 1851–1854.

[8] Disertori, M., Lager, M.: Density of states for random band matrices in two dimensions, Ann. Henri Poincare, **18:7**, pp. 2367 – 2413 (2017)

[9] Disertori, M., Pinson, H., and Spencer, T.: Density of states for random band matrices. Comm. Math. Phys., vol. 232, p. 83 – 124 (2002)

[10] Disertori, M., Spencer, T., Zirnbauer, M. R. Supersymmetric 1d sigma model, unpublished

[11] Efetov, K.: Supersymmetry in disorder and chaos. Cambridge university press, New York (1997)

[12] Erdös, L., Knowles, A.: Quantum diffusion and eigenfunction delocalization in a random band matrix model. Comm. Math. Phys. **303**, 509 – 554 (2011).

[13] Erdös, L., Knowles, A., Yau, H.-T., Yin, J.: Delocalization and diffusion profile for random band matrices. arXiv:1205.5669v1

[14] Erdös, L., Yau, H.-T., Yin, J.: Bulk universality for generalized Wigner matrices, Preprint arXiv:1001.3453.

[15] Fyodorov, Y.V., Mirlin, A.D.: Scaling properties of localization in random band matrices: a σ-model approach, Phys. Rev. Lett. **67**, 2405 – 2409 (1991).

[16] Fyodorov, Y.V., Mirlin, A.D.: Statistical properties of eigenfunctions of random quasi 1d one-particle Hamiltonians, Int. J. Mod. Phys. B **8**, 3795 – 3842 (1994)

[17] Li, L., Soshnikov, A.: Central Limit Theorem for Linear Statistics of Eigenvalues of Band Random Matrices. Random Matrices: Theory and Applications, 2, 04 (2013)

[18] Molchanov, S. A., Pastur, L. A., Khorunzhii, A. M.: Distribution of the eigenvalues of random band matrices in the limit of their infinite order, Theor. Math. Phys. **90**, 108 – 118 (1992)

[19] Peled, R., Schenker, J. , Shamis, M., Sodin, A.: On the Wegner orbital model, International Mathematics Research Notices, rnx145, https://doi.org/10.1093/imrn/rnx145

[20] Schenker, J.: Eigenvector localization for random band matrices with power law band width, Comm. Math. Phys. **290**, 1065 – 1097 (2009)

[21] Shcherbina, M.: On fluctuations of eigenvalues of random band matrices, J.Stat.Phys., **161**, Issue 1, pp 73-90 (2015)

[22] Shcherbina, M., Shcherbina, T.: Transfer matrix approach to 1d random band matrices: density of states, J.Stat.Phys., vol. 164, p. 1233 – 1260 (2016)

[23] Shcherbina, M., Shcherbina, T.: Characteristic polynomials for 1d random band matrices from the localization side, Commun. Math. Phys. **351**, p. 1009 – 1044 (2017)

[24] Shcherbina, M., Shcherbina, T.: Universality for 1d random band matrices: sigma-model approximation. Submitted to JSP

[25] Shcherbina, M., Shcherbina, T.: Transfer matrix approach to 1d random band matrices: universality regime, in preparation

[26] Shcherbina, T.: On the correlation function of the characteristic polynomials of the Hermitian Wigner ensemble. Commun. Math. Phys. **308**, p. 1 – 21 (2011)

[27] Shcherbina, T.: On the correlation functions of the characteristic polynomials of the Hermitian sample covariance ensemble, Probab. Theory Relat. Fields **156**, p. 449 – 482 (2013)

[28] Shcherbina, T. : On the second mixed moment of the characteristic polynomials of the 1D band matrices. Commun. Math. Phys., vol. 328, p. 45 – 82 (2014)

[29] Shcherbina, T.: Universality of the local regime for the block band matrices with a finite number of blocks. J.Stat.Phys., vol. 155(3), p. 466 – 499 (2014)

[30] Shcherbina, T.: Characteristic polynomials for random band matrices near the threshold, in preparation

[31] Sodin, S.: An estimate for the average spectral measure of random band matrices. J. Stat. Phys., vol. 144, p. 46 – 59 (2011)

[32] Spencer, T.: SUSY statistical mechanics and random band matrices. Quantum many body system, Cetraro, Italy 2010, Lecture notes in mathematics 2051 (CIME Foundation subseries) (2012)

[33] Tao, T., Vu, V.: Random matrices: Universality of the local eigenvalue statistics. Acta Math. **206**, 127 – 204 (2011).

[34] Vilenkin, N. Ja.: Special Functions and the Theory of Group Representations. Translations of Mathematical Monographs, AMS 1968; 613 pp;

[35] Wegner, F.J.: Disordered system with n orbitals per site: $n \to \infty$ limit, Phys. Rev. B **19**, 783-792 (1979)