Opportunities to Engage Secondary Students in Proof Generated by Pre-service Teachers

Orly Buchbinder, University of New Hampshire
Sharon McCrone, University of New Hampshire

For reasoning and proving to become a reality in mathematics classrooms, pre-service teachers (PSTs) must develop knowledge and skills for creating lessons that engage students in proof-related activities. Supporting PSTs in this process was among the goals of a capstone course: Mathematical Reasoning and Proving for Secondary Teachers. During the course, the PSTs designed and implemented in local schools four lessons that integrated within the regular secondary curriculum one of the four proof themes discussed in the course: quantification and the role of examples in proving, conditional statements, direct proof and argument evaluation, and indirect reasoning. In this paper we report on the analysis of 60 PSTs' lesson plans in terms of opportunities for students to learn about the proof themes, pedagogical features of the lessons and cognitive demand of the proof-related tasks.

Keywords: Reasoning and Proving, Preservice Secondary Teachers, Lesson Plans

Despite persistent calls to make reasoning and proof an integral part of everyday teaching and learning mathematics, the reality in secondary schools is far from what mathematics educators and policy leaders had in mind (NCTM, 2009, 2014, 2018; CCSSI, 2010). Studies consistently show that proof and proving are "notoriously difficult for students to learn and for teachers to teach," and that making proof a reality in mathematics classrooms requires systemic change in classroom culture (Nardi & Knuth, 2017, p. 267). Since teachers are instrumental to any instructional change (NCTM, 2014), pre-service teachers (PSTs) need to develop knowledge and skills to successfully implement reasoning and proving in their future classrooms.

To address this goal, we developed a capstone course *Mathematical Reasoning and Proving for Secondary Teachers,* which is part of an NSF-funded, 3-year design-based research project. The course comprised four modules, each focused on one proof theme: quantified statements and the role of examples in proving, conditional statements, direct proof / argument evaluation, and indirect reasoning. The themes were chosen because they are known in the literature as central to proof production and comprehension, but challenging for students and teachers alike (e.g., Antonini & Mariotti, 2008; Durand-Guerrier, 2003; Weber, 2010). Our primary goal was to support future teachers in integrating reasoning and proving in their classroom instruction, regardless of the content or grade level, with the specific focus on these proof themes. Thus, the course activities aimed (a) to increase PSTs' awareness of the importance of the logical aspects of proof, and student difficulties with proving, (b) to teach PSTs to identify *within regular school curricula* opportunities to integrate proof-related tasks, and (c) to equip PSTs with pedagogical tools and ideas on how to create or modify mathematical tasks to integrate proof within them.

In Buchbinder and McCrone (in press) we describe the theoretical foundations of the course design and detail its structure and activities. Here, we focus on one critical component of the course: having the PSTs design and implement, in local schools, lessons that integrate mathematical topics with one of the four proof themes. In this paper, we analyze the PSTs' lesson plans, focusing on the opportunities that PSTs engineered for secondary students to learn about the four proof themes. Our analysis addressed two overarching questions:

1. What opportunities to learn about reasoning and proving, specifically about the four proof-themes, did PSTs integrate in their lesson plans?

2. How were these learning opportunities realized in the lesson plans?

While the most intriguing question might be: "what did the secondary students learn from such lessons?", our ability to answer this question is limited. First, the focus of the study was on the PSTs' ability to engage students in proving. Second, most PSTs taught multiple different groups of students throughout the semester, and although all lessons were video-taped, we can at most assess student engagement with the lesson rather than their learning from a single lesson.

Background and Theoretical Perspectives

For the purpose of our study, we adopt a definition of proof that is appropriate for the secondary school context: "a mathematical argument for or against a mathematical claim that is both mathematically sound and conceptually accessible to the members of the local community where the argument is offered" (Stylianides & Stylianides, 2017, p. 212). By reasoning and proving, we refer to a wide range of processes such as conjecturing, generalizing and making valid arguments on the basis of mathematical deductions rather than authority or empirical evidence (Ellis, Bieda & Knuth, 2012; Stylianides, 2008). This definition and these processes were used in the analysis of the PSTs' lesson plans.

Stein, Remilard and Smith (2007) distinguish between *written curriculum*, which is written artifacts that teachers and students use, *intended curriculum*, which is the teacher's lesson plan, and *enacted curriculum* that is the lesson as it unfolds in the classroom. A lesson plan contains information on the mathematical content of the lesson, the types of tasks, how students will be engaged in them and the goals the teacher seeks to achieve. All these aspects shape the quality of students' mathematical experiences. For example, mathematical tasks of high vs. low cognitive demand determine whether students will be engaged in meaningful mathematical processes such as exploring and justifying, or simply applying standard procedures and recalling facts (Smith et al., 2004). Pedagogical features of the lessons provide information on how it will be enacted and on the organizational aspects of the lesson that "have potential to generate opportunities for students to develop or display mathematical understanding" (Silver et al., 2009, p. 511). In this paper, we analyze PSTs' lesson plans and focus on the *opportunities* to learn about reasoning and proof embedded in them and *how* the PSTs intended to enact these opportunities.

Methods

Fifteen PSTs participating in the capstone course *Mathematical Reasoning and Proving for Secondary Teachers* took part in this study. The PSTs (4 middle-school, and 11 high-school track; 6 males and 9 females) were in their senior year, meaning that they have completed most of their content courses and two courses on methods of teaching mathematics.

During the course, the PSTs designed four lesson plans integrating a particular proof theme with a mathematical topic from the secondary curriculum, based on information from cooperating teachers from the local schools. Due to the course structure, the PSTs were required to address particular proof themes while the current classroom mathematical topic might have been more conducive to a different proof theme. The PSTs were encouraged to include in their lessons high cognitive demand tasks, and to use pedagogical tools that were illustrated and discussed in the course, among them proof task models, such as, *Who is right?, True-or-False?, Always-Sometimes-Never?* and *Is it a coincidence?.* These task models have been shown to elicit rich student engagement with the logical aspects of proof and can be modified for various mathematical topics, while maintaining their original structure and goals (Buchbinder & Zaslavsky, 2013). However, turning these pedagogical devices into a lesson plan was up to the

PSTs; we did not offer lesson plan templates that were specific to the proof themes, and PSTs were not directly told how to integrate these themes into the content of their lessons.

The lessons were 50 minutes long, designed for small groups of 4-8 students. The PSTs then taught each lesson and videotaped their teaching. The lesson plans followed a particular format that included: (1) general information on grade level, subject area, topic of the lesson, student prior knowledge, content and process objectives; (2) outline of the lesson explaining what the teacher and the students will be doing, description of anticipated student difficulties and ways to address them; and (3) student worksheets with solutions. These lesson plans, 60 in total, comprise the main corpus of data for this paper. Supplementary data sources supporting our analysis were the PSTs' reflections on each lesson and on the course overall, and video records of the course sessions and of the PSTs' classroom teaching.

In our analysis we relied on the frameworks developed by Silver et al., (2009) who analyzed lessons submitted by teachers seeking national board certification. The analysis proceeded in several stages. First, we mapped out the grade level, mathematical content and pedagogical features of each lesson plan. Second, since each lesson intended to integrate some proof theme, we assigned each lesson plan, as a whole, a rating (high, medium or low) reflecting the prevalence of the proof theme in it. We illustrate this coding and its outcomes in the results section below. Next, we examined the level of cognitive demand of the tasks designed by the PSTs. In each lesson we identified *proof-related tasks*, that is, tasks in which students had to develop/evaluate an argument, justify, explain, or compare their own mathematical work with that of others. Regardless of whether or not the tasks were focused on the proof theme, we coded them as high or low-demand using Silver's et al (2009) framework. Note that the attributes of proof-related tasks are often associated with high-cognitive demand (Stein, et al., 2000), however our analysis showed that these two characteristics are not identical.

The coding procedures were carried out as follows: the two researchers coded each lesson plan independently, and then compared and discussed their coding until agreement was reached.

Results

Mathematical Topics and Pedagogical Features of the Lesson Plans

Table 1 summarizes the mathematical content and pedagogical features of the lesson plans.

Table 1. Mathematical Content and Pedagogical Features

	8th Grade Mathematics Pre-Algebra (HS) 22 lessons	Algebra 1 College-Prep Alg. 1 18 lessons	Geometry College-Prep Geometry 20 lessons
Mathematical Content	• Rules of exponents • Scientific notation • Order of operations • Problem solving • Variable expressions • Distributive property	• Proportions • Order of operations • Combining "like" terms • Solving equations • Linear functions/graphs	• Quadrilaterals • Parallel lines • Vertical angles • Line and angle proofs • Pythagorean theorem • Simplifying square roots
Pedagogical Features of Lessons	• Manipulatives (e.g., dice and playing cards) • Matching activities • Logic riddles	• Manipulatives (e.g., algebra tiles)	• Card sorting tasks • Exploration and conjecturing
	• Real-world context; • Assessing sample student work; • Using task models (e.g., Is this a Coincidence?); • Games (e.g., Jeopardy, Math Baseball)		

We were encouraged to see the PSTs' efforts to creatively incorporate multiple pedagogical techniques for addressing a range of mathematical content at various grade levels. Other common features of the lesson plans, not reflected in Table 1, were due to the special nature of this teaching experience. One such feature is the use of PST-developed worksheets to reduce reliance on students' textbooks to which the PSTs often had no access. Second, since the lessons were designed for small groups of students, all plans embedded opportunities for students to work with their peers and share ideas. In the next section we describe how the PSTs used these and other features to focus on the proof themes.

Focus on the Proof Themes

There was substantial variation in how focused the lesson plans were on the proof themes for the four modules of the capstone course. Since a proof theme could appear in multiple parts of the lesson e.g., exposition, warm-up, some or all student tasks, we took the whole lesson plan as a unit of analysis. Based on how prevalent a proof theme was in the lesson plan, we broadly categorized each plan as having high, medium or low focus on a given proof theme. For example, for the lesson in which PSTs were asked to integrate the proof theme of quantification and the role of examples in proving, Ellen's (all names are pseudonyms) Geometry lesson contained several two-column proofs about parallel lines and vertical angles, but nothing related to the proof theme; thus, it was coded as having *low* proof theme focus.

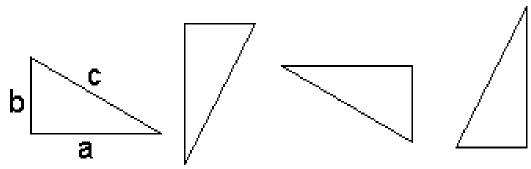
Nate's lesson plan aimed to integrate this proof theme with the topic of proportions and unit conversion in Algebra 1. Nate used a real-world context to create a problem about two investors buying land in the United States and Europe; the solution required area and money conversion to decide who got a better deal. The lesson also contained four sample arguments, each claiming that another investor got a better deal. The task for students was to evaluate these arguments, decide whether they were correct or not and justify their decisions. Nate wrote that he intended to use these explanations as counterexamples to the claims made by the imaginary students in the problem. That is, if an imaginary student made a claim that one investor got a better deal, but the students in class could refute this argument by showing that the second investor got a better deal, this would illustrate that a counterexample disproves a claim. Although it might be possible to use Nate's problem in this way, we felt unconvinced that the lesson plan was sufficiently explicit in positioning the problem in this light, hence we coded it as having *medium* focus on the proof theme.

On the contrary, Rebecca's lesson plan was categorized as *highly* focused on the proof theme. It started with exposition on what a universal statement is, and used examples outside mathematics, such as "A man who is wearing a suit and tie is attending a funeral," to explain that one needs a general proof to prove a universal statement, and a counterexample to disprove it. Next, Rebecca had students explore and develop a conjecture about types of quadrilaterals created by connecting the midpoints of the sides of another quadrilateral. The students were not required to prove their conjectures, but only to consider what information may be needed to prove or disprove it. This lesson constitutes creative and high integration of the proof theme.

Overall, 28 lessons were coded as having high focus on the proof theme, 13 as medium and 19 as low (see Table 2 below). Table 2 shows that the highest focus on proof themes occurred in lessons on conditional statements (11 out of 28) and on direct proof/argument evaluation (10 out of 28). Most lesson plans on these proof themes contained tasks engaging students in evaluating the mathematical work or arguments of imagined students, providing justification for why these arguments are true or finding and correcting mistakes in them. Another frequently used feature was the task model *Is this a coincidence?* In this type of task students are given a description of a mathematical exploration along with one or two related examples generated by an imaginary student, and an observation that he/she made based on these examples. A set of prompts, including: “Is this a coincidence?”, invite students to formulate a conjecture, explore it and then prove or disprove it. Figure 2 shows Angela’s task of this type.

A student said: I took four congruent triangles, with side lengths 3in, 4in, and 5in, and found that I could rearrange them in a square. I tried to do the same thing with four triangles of side lengths 6in, 7in, and 8in and I couldn’t make a square.

Is this a coincidence?



[redacted]

Figure 2: Angela’s task using the model of “Is this a coincidence?”

The two proof themes in which the majority of lessons were coded as having low focus on the intended theme were: (a) quantification and the role of examples in proving and (b) indirect reasoning (Table 2). Despite the attempt to integrate the proof theme with the ongoing mathematical topic, in reality these lesson plans were only tangentially related to the proof themes. However, some PSTs found creative ways to incorporate proof themes in their lessons, cf. Rebecca’s lesson on quantification. Another strong example is Logan’s lesson on indirect reasoning. Logan created six problems on applications of Pythagorean theorem, each asking students to explain why certain measures of triangle sides cannot be true (see Figure 3 for one problem). Indirect reasoning would come into play by assuming that the ramp is 9 feet long, and using the Pythagorean theorem to calculate the length of the ramp to arrive at a contradiction.

You’re working as an independent contractor and your latest client needs a ramp built at one of their properties. The client knows that the ramp must come to an elevation of three feet and that they only have enough room for the ramp to come out six feet from the wall. The client mentions that the length of the ramp’s surface will be 9 feet. *Explain to the client why the length of the ramp cannot be nine feet.* Also include what the correct third measurement is for the ramp.

Figure 3: Logan’s task on indirect reasoning. Emphasis added.

Although conditional statements and direct proofs are relatively common in the high school geometry curriculum, it was reassuring to see the PSTs implementing such lessons within algebra and prealgebra. We turn now to describing a cognitive demand of the proof-related tasks.

Cognitive Demand

In each lesson plan, we identified proof-related tasks and examined how cognitively demanding they were, using the framework of Silver et al. (2009, p. 511). The tasks coded as high-demand asked students to: (a) explain, describe, justify, compare or assess; (b) make decisions or choices, formulate questions or problems, (c) work with multiple representations; (d) read, comprehend or complete proofs. Tasks coded as low-demand: (a) required application of routine procedures, (b) lowered expectations or provided too much guidance making a potentially high-demand task into a routine one, (c) targeted non-challenging issues (e.g., required explanation of standard procedures). If a plan contained more than one proof-related task, the lesson plan was assigned the score of the task with the highest demand. Table 2 shows, for each proof theme, the number of the lesson plans with high, medium and low focus on that proof theme, and the cognitive demand of proof-related tasks.

Table 2. Focus on proof themes vs. cognitive demand of proof-related tasks.

Proof theme	Focus of the lesson of a proof theme			Cognitive demand of proof-related tasks	
	High	Medium	Low	High-demand	Low demand
Quantification and the role of examples	3	4	8	5	10
Conditional statements	11	1	3	5	10
Direct proof, argument evaluation	10	5	0	14	1
Indirect reasoning	4	3	8	7	8

Although we only coded *proof-related* tasks, i.e., tasks that require developing / evaluating arguments, justifying, explaining, or comparing one's mathematical work to that of others, not all tasks were highly demanding. In fact, in all proof themes, except for direct proof and argument evaluation, the number of low-demand tasks exceeded the number of high-demand. This often happened when PSTs lowered the cognitive demand of a proof-related task. For example, Audrey created a worksheet with several problems that called for identifying and correcting student mistakes. One item was: "Carly thinks that $(x^2)^4 = x^6$. Is she correct? Explain why or why not". The answer key showed that Audrey expected students to respond: "Carly is not correct because you do not add the exponents", an answer that relies on rule memorization characteristic of low-demand tasks, rather than mathematical reasoning.

Our analysis suggests that the relationship between the lesson's focus on proof themes and cognitive demand of proof-related tasks was not straightforward. While 19 of highly-demanding tasks occurred in lessons with high focus on a proof theme, and 17 of low-demanding lessons appeared in the lessons with a low proof theme focus, other combinations were also present in the data. For example, Nate's problem on unit conversion was proof-related and highly demanding, but it had only medium focus on the proof theme for which it was designed, namely, quantification and the role of examples in proving.

Discussion and Implications for Education

Our study focused on two overarching research questions:

1. What opportunities to learn about reasoning and proving, specifically about the four proof-themes, did PSTs integrate in their lesson plans?
2. How were these learning opportunities realized in the lesson plans?

We operationalized the first question by examining the ways PSTs integrated the four proof themes in their lesson plans and noting the prevalence of these proof themes in the plans. We addressed the second question by examining the pedagogical features of the lessons as a whole and the cognitive demand of the proof-related tasks.

The lesson plans encompassed a variety of mathematical topics and embedded multiple pedagogical features demonstrating that a wide range of topics can provide opportunities for introducing reasoning and proof across the grades, and that PSTs were capable of identifying and capitalizing on these opportunities in their lesson plans. The variation in the level of focus on the proof themes stems from several factors, some beyond the PSTs' control (e.g., responding to a cooperating teacher's request to devote time to exam review). Data from other sources, such as course classroom video and the PSTs' course reflections, suggest that two main reasons for low or moderate focus on proof themes were: (a) the PSTs' own doubts about feasibility of integrating proof themes in secondary mathematics, and (b) lack of experience with proof-related tasks at the secondary level. The quotes by Ethan and Laura illustrate these points, with Ethan sharing what he saw as challenging and Laura explaining how she addressed the challenge:

> It was definitely easier to implement certain proof topics compared to others. I found implementing two themes the role of examples in proving and evaluating arguments to be rather easy/less challenging and beneficial to the students. On the other hand, I found conditional statements and proof by contradiction to be challenging to teach middle school students, even if it was at the most basic level. These topics can be very difficult to grasp so finding a way to relate them to exponents or linear equations I found to be challenging. (Ethan)

> At the start of this class, I believed that proofs were only appropriate in geometry classrooms, or in proving Calculus theorems. However, I was tasked with teaching a geometry class, a pre-algebra class, and two Algebra 1 classes. I found that if you focus on the kinds of reasoning involved in different proof-themes, and if you don't overwhelm students by attempting formal proof right away, the four proof-themes could easily be applied to any mathematics topic. (Laura)

As instructors, we invested a considerable amount of course time and efforts to get the PSTs on board with the idea that all students are capable of doing proof-related tasks and can benefit from them. Some of this included providing examples of pedagogical features, such as assessing sample student work or proof task models to inspire PSTs' creativity. Our data suggest that PSTs could benefit from greater exposure to examples of successful integration of proof themes with mathematics instruction. We plan to use the current sample of lesson plans as a pool of examples on how this can be achieved. Another critical point that came up in our data is the cognitive demand of proof-related tasks. We found it somewhat surprising that inclusion of a proof theme in a lesson plan did not automatically translate to highly demanding proof-related task. We intend to address this in the next iteration of the course by having the PSTs assess cognitive demand of their own tasks and the tasks of their peers, to increase their awareness of different learning opportunities in tasks with high vs. low cognitive demand.

We conclude this paper by noting that there is a long way between creating lesson plans that integrate reasoning and proving in secondary mathematics as a course assignment and being able to identify opportunities to integrate proving in mathematics instruction as a part of one's regular teaching practice. We hope that our course helped the PSTs to make an important step in this direction, as the following quote from Angela's reflection suggests:

> So while the task of incorporating the proof themes into our lessons was challenging, it was also very eye-opening into the multitude of ways that higher-level mathematics topics can be brought into lower level subjects and it is something I want to continue to try and do in my own practice.

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