Graphs and Combinatorics (2018) 34:1049–1075
https://doi.org/10.1007/s00373-018-1929-1

ORIGINAL PAPER

Tiling Tripartite Graphs with 3-Colorable Graphs: The Extreme Case

Kirsten Hogenson[1] · **Ryan R. Martin**[2] · **Yi Zhao**[3]

Received: 17 May 2016 / Revised: 20 July 2018 / Published online: 6 August 2018

Abstract
There is a sufficiently large $N \in h\mathbb{N}$ such that the following holds. If G is a tripartite graph with N vertices in each vertex class such that every vertex is adjacent to at least $2N/3 + 2h - 1$ vertices in each of the other classes, then G can be tiled perfectly by copies of $K_{h,h,h}$. This extends work by Martin and Zhao (Electron J Combin 16(1):109, 2009) and also gives a sufficient condition for tiling by any fixed 3-colorable graph. Furthermore, we show that $2N/3 + 2h - 1$ in our result can not be replaced by $2N/3 + h - 2$ and that if N is divisible by $6h$, then we can replace it with the value $2N/3 + h - 1$ and this is tight.

Keywords Tiling · Hajnal–Szemerédi · Multipartite · Regularity

Mathematics Subject Classification 05C35 · 05C70

Martin was supported in part by NSA Grants H98230-05-1-0257 and H98230-08-1-0015 and by a Grant from the Simons Foundation (#353292, Ryan Martin). Zhao was supported in part by NSA Grants H98230-05-1-0079, H98230-07-1-0019 and NSF Grant DMS-1400073. Part of this research was done while Zhao was working at University of Illinois at Chicago.

✉ Kirsten Hogenson
kirsten.hogenson@coloradocollege.edu

Ryan R. Martin
rymartin@iastate.edu

Yi Zhao
yzhao6@gsu.edu

[1] Colorado College, Colorado Springs, CO 80903, USA

[2] Iowa State University, Ames, IA 50010, USA

[3] Georgia State University, Atlanta, GA 30303, USA

1 Introduction

Let H be a graph on h vertices, and let G be a graph on n vertices. An H-tiling of G is a subgraph of G which consists of vertex-disjoint copies of H and a *perfect H-tiling*, or *H-factor*, of G is an H-tiling consisting of $\lfloor n/h \rfloor$ copies of H. The celebrated Hajnal–Szemerédi Theorem [11] says that each n-vertex graph G with $\delta(G) \geq (r-1)n/r$ contains a K_r-factor. (Corrádi and Hajnal [6] proved the case $r = 3$.) Using Szemerédi's regularity lemma [26], Alon and Yuster [1,2] obtained results on H-tiling for arbitrary H. Their results were improved substantially [16,17,19,25], in particular, Kühn and Osthus [19] determined the minimum degree threshold for H-factors for arbitrary H up to an additive constant, see the survey [18] for details.

In this paper, we consider multipartite tiling, which restricts G to be an r-partite graph. For $r = 2$, this is an immediate consequence of the König–Hall Theorem (e.g. see [3]). Wang [27] considered $K_{s,s}$-factors in bipartite graphs for all $s > 1$; Zhao [28] gave the best possible minimum degree condition for this problem. With the exception of one case, Hladký and Schacht [13] found best possible minimum degree conditions for $K_{s,t}$-factors in bipartite graphs with $s < t$; the last case was settled by Czygrinow and DeBiasio [8]. Later, Bush and Zhao [4] considered tiling bipartite graphs with an arbitrary graph H.

For a tripartite graph $G = (A, B, C; E)$, the graphs induced by (A, B), (A, C) and (B, C) are called the *natural bipartite subgraphs* of G. Let $\mathcal{G}_r(N)$ be the family of r-partite graphs with N vertices in each partition set. Such a graph is called *balanced* because the number of vertices in each partition set is the same. In an r-partite graph G, $\delta^*(G)$ stands[1] for the minimum degree over all natural bipartite subgraphs of G.

There are two classes of multipartite graphs that we will reference in this paper. One is Γ_k, which is in $\mathcal{G}_k(k)$. The vertices of Γ_k are $h_i^{(j)}$, $i = 1, \ldots, k$ and $j = 1, \ldots, k$, and the adjacency rules are as follows: $h_i^{(j)} \sim h_{i'}^{(j')}$ iff $i \neq i'$, $j \neq j'$, and either j or j' is in $\{1, \ldots, k-2\}$. Also, $h_i^{(k-1)} \sim h_{i'}^{(k-1)}$ and $h_i^{(k)} \sim h_{i'}^{(k)}$ for $i \neq i'$. The other graph is $\Theta_{r\times n}$, which is in $\mathcal{G}_r(n)$. The vertices of $\Theta_{r\times n}$ are $a_i^{(j)}$, $i = 1, \ldots, r$ and $j = 1, \ldots, n$ such that $a_i^{(j)} \sim a_{i'}^{(j')}$ if and only if $i \neq i'$ and $j \neq j'$. We will also discuss the so-called blow-ups of these graphs. The *blow-up graph*, $G(N)$, for a graph G is obtained by replacing each edge of G with a copy of $K_{N,N}$ and replacing each non-edge by an $N \times N$ bipartite graph with no edges.

In addition to the bipartite results discussed above, there have also been a number of results on multipartite graphs with $r \geq 3$, many of which were inspired by a conjecture of Fischer. Fischer [10] conjectured that if $G \in \mathcal{G}_r(N)$ satisfies $\delta^*(G) \geq \frac{r-1}{r}N$, then G contains a K_r-factor. However, if r and N/r are odd integers, then Catlin [5] had earlier given an example of a graph without a K_r-factor where $\delta^*(G) = \frac{r-1}{r}N$. In [21], Magyar and Martin proved that, for large N, this graph is a unique counterexample to Fisher's conjecture for $r = 3$ by showing that if N is a sufficiently large odd multiple of 3, the blow-up graph $\Gamma_3(N/3) \in \mathcal{G}_3(N)$ (see Fig. 1) is the unique graph with $\delta^*(G) \geq 2N/3$ and no K_3-factor. The conjecture of Fischer can be modified to exclude this case. This gives the following Corrádi–Hajnal-type result.

[1] In [24], $\bar{\delta}$ was used in place of δ^*.

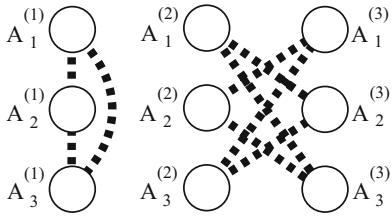

Fig. 1 The diagram for the blow-up of Γ_3, with the vertex classes in rows and the dotted lines representing *nonedges*. Note that $\delta^*(\Gamma_3) = 2$ but Γ_3 has no K_3-factor

Theorem 1 ([21]) *Let $G \in \mathcal{G}_3(N)$ have $\delta^*(G) \geq (2/3)N$. If $N \geq N_0$ for some absolute constant N_0, then G contains a K_3-factor or $G = \Gamma_3(N/3)$ for $N/3$ an odd integer.*

Martin and Szemerédi [23] proved a quadripartite version of the Hajnal–Szemerédi Theorem. Han and Zhao [12] reproved the results of [21,23] by using the absorbing method. An approximate version of the multipartite Hajnal–Szemerédi Theorem was given by Csaba and Mydlarz [7]. Keevash and Mycroft [14] and independently Lo and Markstrom [20] confirmed Fischer's conjecture asymptotically, and finally Keevash and Mycroft [15] proved the modified Fischer conjecture exactly for any sufficiently large graph. More recently, an asymptotic multipartite version of the the Alon-Yuster Theorem was proved by Martin and Skokan [22].

It was shown [24, Theorem 1.2] that in the tripartite case, $2/3$ is the correct coefficient of N required to have a $K_{h,h,h}$-factor.

Theorem 2 ([24]) *For any positive real number γ and any positive integer h, there is N_0 such that the following holds. Given an integer $N \geq N_0$ such that N is divisible by h, if G is a tripartite graph with N vertices in each vertex class such that every vertex is adjacent to at least $(2/3 + \gamma)N$ vertices in each of the other classes, then G contains a $K_{h,h,h}$-factor.*

Let $f(N, h)$ be the smallest integer f such that every balanced tripartite graph G on $3N$ vertices with $\delta^*(G) \geq f$ contains a $K_{h,h,h}$-factor. Our main result is the following more precise theorem.

Theorem 3 *Fix a positive integer h and let N be sufficiently large. If $h \geq 2$ and $N = (6q + r)h$ with $0 \leq r < 6$, then*

$$\begin{aligned} f(N,h) &= \frac{2N}{3} + h - 1, && \text{if } r = 0; \\ h\left\lceil \frac{2N}{3h} \right\rceil + h - 2 \leq f(N,h) &\leq h\left\lceil \frac{2N}{3h} \right\rceil + h - 1, && \text{if } r = 1, 2, 4, 5; \\ \frac{2N}{3} + h - 1 \leq f(N,h) &\leq \frac{2N}{3} + 2h - 1, && \text{if } r = 3. \end{aligned}$$

So, the result is tight when $6h$ divides N, almost tight unless N/h is an odd multiple of 3 and, in the worst case, the upper and lower bounds differ by h. We are not sure whether the upper or lower bounds of Theorem 3 are correct in the cases when they are not equal.

Clearly the complete tripartite graph $K_{h,h,h}$ can itself be perfectly tiled by any 3-colorable graph on h vertices. Since $f(N,h) \leq \frac{2N}{3} + 2h - 1$ whenever N is divisible by h, we have the following corollary.

Corollary 1 *Let H be a 3-colorable graph of order h. There exists a positive integer N_0 such that if $N \geq N_0$ and N divisible by h, then every $G \in \mathcal{G}_3(N)$ with $\delta^*(G) \geq \frac{2N}{3} + 2h - 1$ contains an H-factor.*

The lower bound for $f(N,h)$ in Theorem 3 is due to two constructions, one which is from [24] and another which is similar. They are stated in Proposition 1, and proven in Sect. 2.

Proposition 1 *Fix a positive integer $h \geq 2$. There exists an N_0 such that*

(1) *if $N \geq N_0$, $h \mid N$ and N/h is divisible by 3, then there is a graph $G_2 \in \mathcal{G}_3(N)$ with no $K_{h,h,h}$-factor and $\delta^*(G_2) \geq 2N/3 + h - 2$; and*
(2) *if $N \geq N_0$, $h \mid N$ and N/h is not divisible by 3, then there is a graph $G_3 \in \mathcal{G}_3(N)$ with no $K_{h,h,h}$-factor and $\delta^*(G_3) \geq h\left\lceil \frac{2N}{3h} \right\rceil + h - 3$.*

As to the upper bound, we use Theorem 4 [24, Theorem 1.4] to take care of the main case. For vertex sets A and B, let $d(A,B) := \frac{e(A,B)}{|A||B|}$ denote the *density* of A and B. Before we can prove the main case, we need the following definition.

Definition 1 Given $\alpha > 0$, we say that $G = (V_1, V_2, V_3; E) \in \mathcal{G}_3(N)$ is α-**extremal** when there are three sets A_1, A_2, A_3 such that $A_i \subseteq V_i$, $|A_i| = \lfloor N/3 \rfloor$ for all i and $d(A_i, A_j) \leq \alpha$ for $i \neq j$.

If G is α-extremal and $\delta^*(G) \geq 2N/3$, then for $i \neq j$, the pair $(A_i, V_j - A_j)$ is a very dense bipartite graph. Thus, we expect most members of our $K_{h,h,h}$-factor with vertices in A_i to have h vertices in A_i and the remaining $2h$ vertices in $\big(V_j - A_j\big) \cup (V_k - A_k)$, where $\{j,k\} = [3] - \{i\}$.

Theorem 4 ([24]) *Given any positive integer h and any $\alpha > 0$, there exists an $\varepsilon > 0$ and an integer N_0 such that whenever $N \geq N_0$, and h divides N, the following occurs: If $G \in \mathcal{G}_3(N)$ satisfies $\delta^*(G) \geq (2/3 - \varepsilon)N$, then either G contains a $K_{h,h,h}$-factor or G is α-extremal.*

Hence, for the upper bound, it suffices to assume that $G \in \mathcal{G}_3(N)$ is α-extremal. The proof, given in Sect. 3, is detailed and involves a case analysis. Moreover, it requires the definition of a particular structure we call *the very extreme case*, which we deal with in Sect. 3.5. This definition is given below, but roughly, it means that the graph looks like $\Gamma_3(N)$.

Definition 2 A balanced tripartite graph G on $3N$ vertices is in **the very extreme case** if the following occurs: First, there are integers h, q such that $N = (6q+3)h$. Second, there are sets $U_i^{(j)} \subseteq V_i$ for $i, j \in \{1,2,3\}$, each with size at least $2qh + 1$, such that if $v \in U_i^{(j)}$ then v is nonadjacent to at most $3h-3$ vertices in $U_{i'}^{(j')}$ whenever $(h_i^{(j)}, h_{i'}^{(j')})$ is an edge in the graph Γ_3.

Note that we use different language for α-extremal and the very extreme case because the definition of α-extremal requires a parameter, whereas the very extreme case does not.

Now that we have defined the very extreme case, we can formally state the upper bound theorem as follows:

Theorem 5 *Fix $h \geq 2$. Let $N \in h\mathbb{N}$ be sufficiently large and assume $G \in \mathcal{G}_3(N)$. If $\delta^*(G) \geq h\left\lceil \frac{2N}{3h} \right\rceil + h - 1$, then G has a $K_{h,h,h}$-factor or G is in the very extreme case. If G is in the very extreme case and $\delta^*(G) \geq h\left\lceil \frac{2N}{3h} \right\rceil + 2h - 1$, then G has a $K_{h,h,h}$-factor.*

2 Lower Bound

First, we need a lemma (Lemma 2.1 in [24]) which permits sparse tripartite graphs with no triangles and with no quadrilaterals in its natural bipartite subgraphs:

Lemma 1 *For each integer $d \geq 0$, there exists an n_0 such that, if $n \geq n_0$, there exists a balanced tripartite graph, $Q(n, d)$ on $3n$ vertices such that each of the 3 natural bipartite subgraphs are d-regular with no C_4 and $Q(n, d)$ has no K_3.*

Finally, we prove the lower bound given in Proposition 1. Note that Proposition 1(1) is proved by Proposition 1.5 in [24], so here we only address Proposition 1(2).

Proof of Proposition 1(2) Let $h \geq 3$ and $N = (3q + r)h$ so that, in this case, $r \in \{1, 2\}$. Let G_3 be defined such that $V_i = A_i^{(1)} \cup A_i^{(2)} \cup A_i^{(3)}$ (the notation $\cup$ emphasizes that it is a disjoint union of sets) in which *column j* is defined to be the triple of the form $(A_1^{(j)}, A_2^{(j)}, A_3^{(j)})$. Let the graph in column 1 be $Q(qh + rh - 1, rh + h - 4)$ where $rh + h - 4 \geq 2$, the graph in column 2 be $Q(qh, h - 3)$ and the graph in column 3 be $Q(qh + 1, h - 2)$. If two vertices are in different columns and different vertex-classes, then they are adjacent. It is easy to verify that $\delta^*(G_3) = 2qh + rh + (h - 3) = h\lceil (2N)/(3h) \rceil + h - 3$. Suppose, by way of contradiction, that G_3 has a $K_{h,h,h}$-factor.

If a copy of $K_{h,h,h}$ has vertex classes U_1, U_2, U_3, then $U_i \subseteq V_j$ for some j. Since there are no triangles in any column and no C_4's in the natural bipartite subgraphs of a column, the intersection of a copy of $K_{h,h,h}$ with a column is either a star with all leaves in the same vertex-class, or a set of vertices in the same vertex-class. So each copy of $K_{h,h,h}$ has at most $h + 1$ vertices in column 1 and at most h vertices in each of column 2 and column 3.

There are three cases for a copy of $K_{h,h,h}$. Case 1 has h vertices in each column. Case 2 has $h + 1$ vertices in column 1, $h - 1$ vertices in column 2 and h vertices in column 3. Case 3 has $h + 1$ vertices in column 1, h vertices in column 2 and $h - 1$ vertices in column 3.

In Cases 1 and 2, since G_3 contains no $K_{1,h-1}$ in column 3, having h vertices of a $K_{h,h,h}$ in column 3 implies that all of them are in the same vertex class. In Case 3, since G_3 has no $K_{1,h-1}$ in column 2, having h vertices in column 2 means that all are in the same vertex-class. Since $h + 1$ vertices in column 1 means that they form a star,

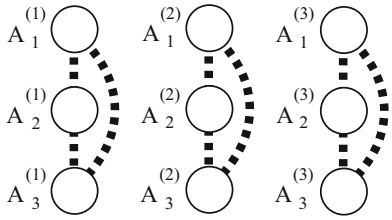

Fig. 2 The diagram for $\Theta_{3\times 3}$, with the vertex classes in rows and the dotted lines representing *nonedges*

the remaining $h-1$ vertices in column 3 must be in the same vertex-class (the same vertex-class as the center of the star). Hence, the intersection of any copy of $K_{h,h,h}$ with column 3 is contained within a single vertex-class. Therefore, the number of copies of $K_{h,h,h}$ in the $K_{h,h,h}$-factor of G_3 is at least $3\left\lceil \frac{qh+1}{h} \right\rceil = 3q+3$, a contradiction because the factor has exactly $3q+r \leq 3q+2$ copies of $K_{h,h,h}$.

Next consider the case when $h=2$ and $N=2(3q+r)$ with $r \in \{1,2\}$. Let G_3 be defined such that the graph in column 1 is $Q(2q+1,0)$, but all other possible edges in G_3 are present. It is easy to verify that $\delta^*(G_3) = 4q+2r-1 = 2\lceil N/3 \rceil - 1$. Suppose, by way of contradiction, that G_3 has a $K_{2,2,2}$-factor. The intersection of one copy of $K_{2,2,2}$ with column 1 must be contained within a single vertex class and can contain at most 2 vertices. So at least $3\left\lceil \frac{2q+1}{2} \right\rceil = 3q+3$ copies of $K_{2,2,2}$ are needed to cover all of column 1. This is a contradiction, because the factor has exactly $3q+r \leq 3q+2$ copies of $K_{2,2,2}$. □

3 The Extreme Case

Throughout Sect. 3, assume that G is minimal, *i.e.*, no edge of G can be deleted so that the minimum degree condition still holds. As we complete the proof of Theorem 3 by proving Theorem 5, we will develop the usual hierarchy of constants:

$$\alpha \ll \alpha_1 \ll \alpha_2 \ll \alpha_3 \ll \alpha_4 \ll \alpha_5 \ll 1-\theta \ll h^{-1}.$$

Brief outline of the proof. There are 4 parts to the proof. Part 1 begins with G being α-extremal and seeks a $K_{h,h,h}$-factor. If such a tiling is not found in G, we deduce that G looks like the graph in Fig. 3 and move to Part 2. We again seek a $K_{h,h,h}$-factor in G, and if it is not found, then we move on to Part 3 which addresses the two potential structures G must have. In Part 3a, G is approximately $\Theta_{3\times 3}(N/3)$ (see Fig. 2). In Part 3b, G is approximately $\Gamma_3(N/3)$ (see Fig. 1.) Proofs for the lemmas and propositions stated in this section are deferred until Sect. 3.6.

The following definition will come into play as we describe the structure of G.

Definition 3 For δ, $0 < \delta < 1$, a graph H and positive integer m, we say a graph G is δ-*approximately* $H(m)$ if $V(G)$ can be partitioned into $|V(H)|$ nearly-equally sized pieces, each of size m or $m+1$, corresponding to a vertex of H so that for vertices $v, w \in V(H)$ with $v \nsim_H w$, the parts of $V(G)$ corresponding to v and w have pairwise density less than δ.

Note that if $v \sim_H w$, we do not require that the parts of $V(G)$ corresponding to v and w have pairwise density close to 1.

We will assume for Parts 1, 2, 3a and 3b (Sects. 3.1, 3.2, 3.3 and 3.4, respectively) that $\delta^*(G) \geq h\left\lceil \frac{2N}{3h} \right\rceil + h - 1$. This takes care of everything except for the very extreme case, which we will consider in Sect. 3.5. For this last part, we will require $\delta^*(G) \geq h\left\lceil \frac{2N}{3h} \right\rceil + 2h - 1$ to complete the proof.

3.1 Part 1: The Basic Extreme Case

For Part 1, we will prove that either a $K_{h,h,h}$-factor exists in G, or G is in Part 2.

Let $A_i \subset V_i$ for $i = 1, 2, 3$ be the three pairwise sparse sets given by the definition of α-extremal and $B_i = V_i - A_i$ for $i = 1, 2, 3$. Recall that $|A_i| = \lfloor N/3 \rfloor$, so $|B_i| = \lceil 2N/3 \rceil$. We then define $\widetilde{A}_i$ to be the set of *typical* vertices with respect to A_i, $\widetilde{B}_i$ to be the set of *typical* vertices with respect to B_i, and $\widetilde{C}_i$ are what remain. Formally, for $i = 1, 2, 3$,

$$\begin{aligned}
\widetilde{A}_i &= \left\{ x \in V_i : \forall j \neq i,\ \deg_{A_j}(x) \leq \alpha_1 |A_j| \right\} \\
\widetilde{B}_i &= \left\{ y \in V_i : \forall j \neq i,\ \deg_{A_j}(y) \geq (1-\alpha_1)|A_j| \right\} \\
\widetilde{C}_i &= V_i - \left(\widetilde{A}_i \cup \widetilde{B}_i \right).
\end{aligned}$$

Let $\{i, j, k\} = \{1, 2, 3\}$. Using these definitions, the fact that G is α-extremal and the bound on δ^*, and the fact that every member of $A_i - \widetilde{A}_i$ is adjacent to at least an α_1 proportion of either A_j or A_k, we obtain the following:

$$\begin{aligned}
|A_i - \widetilde{A}_i| \cdot \alpha_1 \lfloor N/3 \rfloor &< e(A_i - \widetilde{A}_i, A_j) + e(A_i - \widetilde{A}_i, A_k) \leq e(A_i, A_j) + e(A_i, A_k) \\
&\leq 2\alpha \lfloor N/3 \rfloor^2,
\end{aligned}$$

and

$$\begin{aligned}
\left\lceil \frac{2N}{3} \right\rceil |A_j| &\leq e(V_i, A_j) \\
&\leq \alpha |A_i||A_j| + |B_i - \widetilde{B}_i|(1-\alpha_1)|A_j| + |B_i \cap \widetilde{B}_i||A_j| \\
&= \alpha |A_i||A_j| + |B_i||A_j| - \alpha_1 |B_i - \widetilde{B}_i||A_j|.
\end{aligned}$$

As a result, we have that $|A_i - \widetilde{A}_i| \leq 2(\alpha/\alpha_1)\lfloor N/3 \rfloor$ and $|B_i - \widetilde{B}_i| \leq 2(\alpha/\alpha_1)\lfloor N/3 \rfloor$. So, with $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = 4\alpha_1^2$, we get the following bounds for $|\widetilde{A}_i|$ and $|\widetilde{B}_i|$:

$$(1-\alpha_2)\lfloor N/3 \rfloor \leq |\widetilde{A}_i| \leq (1+\alpha_2)\lfloor N/3 \rfloor$$

and

$$(1-\alpha_2)\lceil 2N/3 \rceil \leq |\widetilde{B}_i| \leq (1+\alpha_2)\lceil 2N/3 \rceil.$$

Step 1 Adjusting the sizes of the $\widetilde{A}_i$ sets. Let $N = (3q + r)h$ with $r \in \{0, 1, 2\}$ and $T = h\lfloor N/(3h)\rfloor$.

Without loss of generality, assume that $|\widetilde{A}_1| \geq |\widetilde{A}_2| \geq |\widetilde{A}_3|$. For $i = 1, 2, 3$, define $a_i = T + h$ if $i \leq r$; otherwise, $a_i = T$. If $|\widetilde{A}_i| > a_i$, then we will move $|\widetilde{A}_i| - a_i$ vertices of $\widetilde{A}_i$ to $\widetilde{B}_i$ by applying Lemma 2 below, which is proved in Sect. 3.6. It is applied several times throughout this paper to different sets.

Lemma 2 *Let us be given $\epsilon_2 > 0$ and a positive integer M.*

(1) *Let $(A_1, A_2; E)$ be a bipartite graph such that every vertex in A_2 is adjacent to at least d_1 vertices in A_1. Suppose further that $||A_i| - M| < \epsilon_2 M$ and $d_i < \epsilon_2 M$ for $i = 1, 2$.*
Provided $\epsilon_2 < ((h + 1)h)^{-1}$, there is a family of $\max\{0, d_1 - h + 1\}$ vertex-disjoint copies of $K_{1,h}$ all of whose centers lie in A_1.

(2) *Let $(A_1, A_2, A_3; E)$ be a tripartite graph such that every vertex not in A_i is adjacent to at least d_i vertices in A_i, for $i = 1, 2, 3$. Suppose further that $||A_i| - M| < \epsilon_2 M$ and $d_i < \epsilon_2 M$ for $i = 1, 2, 3$.*
Provided $\epsilon_2 < (2(h + 2)(h + 1)h)^{-1}$, there is a family of $\max\{0, d_i - h + 1\}$ vertex-disjoint copies of $K_{1,h}$ all of whose centers lie in A_i and leaves lie in A_{i+1} (index arithmetic is modulo 3).

Since $\lceil \frac{2N}{3h} \rceil + \lfloor \frac{N}{3h} \rfloor = \frac{N}{h}$, we have $h\lceil \frac{2N}{3h} \rceil + T = N$. As $\delta^*(G) \geq h\lceil \frac{2N}{3h} \rceil + h - 1 \geq N - T + h - 1$, we can guarantee that each vertex not in V_i is adjacent to at least $|\widetilde{A}_i| - T + h - 1$ vertices in $\widetilde{A}_i$. So we apply Lemma 2(2) to the graph induced by $(\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3)$, with $d_i = |\widetilde{A}_i| - T + h - 1$, $\epsilon_2 = \alpha_2$, and $M = N/3$. This will construct stars with the property that there are exactly enough centers in $\widetilde{A}_i$ such that, when removed, the resulting set has its size bounded above by a_i, which is either T or $T + h$, depending on the case. Let Z_i denote the set of these centers and move the desired number of vertices of Z_i from $\widetilde{A}_i$ into $\widetilde{B}_i$.

If $|\widetilde{A}_i| < a_i$, then we will move $a_i - |\widetilde{A}_i|$ vertices of $\widetilde{B}_i \cup \widetilde{C}_i$ to $\widetilde{A}_i$, as follows.

For a subgraph $K_{1,h,h}$, with $h \geq 2$, define the *center* to be the vertex that is adjacent to all others. We will refer to the remaining vertices as *leaves*, although their degree is $h + 1$.

In $B := \bigcup_{i=1}^{3} \left(\widetilde{B}_i \cup \widetilde{C}_i\right)$, we will find vertex-disjoint copies of $K_{1,h,h}$ such that each of $\max\{a_i - |\widetilde{A}_i|, 0\}$ copies has its center vertex in $\widetilde{B}_i \cup \widetilde{C}_i$ for $i \leq r$ and such that each of $a_i - |\widetilde{A}_i|$ copies has its center vertex in $\widetilde{B}_i \cup \widetilde{C}_i$ otherwise. This will be accomplished with Lemma 3, which is proved in Sect. 3.6. It is applied several times throughout this paper to slight variations of the sets $\widetilde{B}_i$.

Lemma 3 *Given $\delta > 0$, there exists an $\epsilon_3 = \epsilon_3(\delta) > 0$ such that the following occurs:*
Let $(B_1, B_2, B_3; E)$ be a tripartite graph on $6M$ vertices such that for all $i \neq j$, each vertex in B_i is adjacent to at least $(1 - \epsilon_3)M$ vertices in B_j. Furthermore, $||B_i| - 2M| < \epsilon_3 M$.
If $(B_1, B_2, B_3; E)$ contains no copy of $K_{1,h,h}$ with 1 vertex in B_1, and h vertices in each of B_2 and B_3, then the graph $(B_1, B_2, B_3; E)$ is δ-approximately $\Theta_{3\times 2}(M)$.

Lemma 3 can be repeatedly applied to B at most $\lceil \alpha_2(N/3) \rceil$ times with $\delta = \alpha_3$, $\alpha_2 \ll \epsilon_3$ and $M = T$. Each time, either a $K_{1,h,h}$ is found and removed, or the

current incarnation of B is α_3-approximately $\Theta_{3\times 2}(T)$ and we stop applying the lemma. When we are finished applying Lemma 3, add the center vertices of the $K_{1,h,h}$ subgraphs to the appropriate sets $\widetilde{A}_i$. Put the leaves back into B and denote the result as $B = (B_1, B_2, B_3; E)$.

If necessary, place vertices from $\widetilde{C}_i$ into the set $\widetilde{A}_i$, for $i = 1, 2, 3$, so that the resulting set, relabeled as $A_i^{(1)}$, is of size a_i and $\sum_{i=1}^{3} |A_i^{(1)}| = N$.

Step 2 Finding a $K_{h,h}$*-factor in* B. Now we try to find a $K_{h,h}$-factor among the remaining vertices in B with the goal of extending each $K_{h,h}$ into a $K_{h,h,h}$ using vertices in $A_1^{(1)} \cup A_2^{(1)} \cup A_3^{(1)}$. Before we do so, however, we need to address the following concerns:

- Vertices in copies of $K_{1,h,h}$ where the center vertex is in some $A_i^{(1)}$ must be in a specified copy of $K_{h,h}$ in B.
- Recall that Z_i is the set of centers of h-stars which were found in Step 1. If $v \in Z_i$ is the center of a $K_{1,h}$ with leaves in $A_k^{(1)}$, then v will be *assigned to* B_j, where $\{j\} = \{1, 2, 3\} - \{i, k\}$. This means that v will be adjacent to vertices in B_j in a $K_{h,h}$ in B.
- For $\{i, j, k\} = \{1, 2, 3\}$, vertices $v \in \widetilde{C}_i$ will be *assigned to* B_j or B_k, respectively. This means that v will be adjacent to either h vertices in B_j or h vertices in B_k in a $K_{h,h}$ to be formed in B together with $h - 1$ vertices in B_i. We know this can be accomplished because if $v \in \widetilde{C}_i$, then we may assume, without loss of generality, that v is adjacent to at least $\alpha_1|A_j|$ vertices in A_j.

Moreover, because all but a α_2-proportion of the sets A_i and B_i are typical, we have that $|\widetilde{C}_i| \le \alpha_2|A_i| + \alpha_2|B_i| \le 3\alpha_2 T$. Recall that we applied Lemma 2 with $d_i = |\widetilde{A}_i| - T + h - 1$. Thus $|Z_i| \le \alpha_2|A_i| + h - 1 \le 2\alpha_2 T$ and there are at most $\alpha_2|A_i| + h \le 2\alpha_2 T$ copies of $K_{1,h,h}$ with the center vertex in a given $A_i^{(1)}$.

Lemma 4 is proved in Sect. 3.6. We will apply it to an adjusted B where we know from Step 1 there are copies of $K_{h,h}$ which must belong to any $K_{h,h}$-factor.

Lemma 4 *Given $\delta > 0$, there exists $\epsilon_4 = \epsilon_4(\delta) > 0$ and a positive integer $T_0 = T_0(\delta)$ such that the following occurs. Let T_1, T_2, T_3 be three positive integers which are divisible by h and with $|T_i - T_j| \in \{0, h\}$, for all $i, j \in \{1, 2, 3\}$ and $T_1 > T_0$. Let $(B_1, B_2, B_3; E)$ be a tripartite graph such that for $\{i, j, k\} = \{1, 2, 3\}$ $|B_i| = T_j + T_k$, and for $i \ne j$, each vertex in B_i is adjacent to at least $(1 - \epsilon_4)T_1$ vertices in B_j. Then one of the following holds.*

(1) *There is a $K_{h,h}$-factor in the graph induced by $(B_1, B_2, B_3; E)$ with the following properties. Each copy is a subgraph of (B_i, B_j) for some $i \ne j$. If we fix a set of at most $\epsilon_4 T_1$ vertex-disjoint copies of $K_{h,h}$ and at most $\epsilon_4 T_1$ vertex-disjoint copies of $K_{1,h}$, then the $K_{h,h}$-factor contains them as subgraphs.*
(2) *The graph induced by $(B_1, B_2, B_3; E)$ can be partitioned such that $B_i = A_i^{(2)} \cup A_i^{(3)}$, $|A_i^{(2)}| = T_1$ for $i = 1, 2, 3$ and $d(A_j^{(2)}, A_2^{(2)}) \le \delta$ and $d(A_j^{(3)}, A_2^{(3)}) \le \delta$ for $j = 1, 3$.*

Now to find our $K_{h,h,h}$-factor, we first match vertices in $\widetilde{C}_i$ that are assigned to B_j with h typical neighbors in B_j and those h vertices with $h - 1$ typical neighbors in

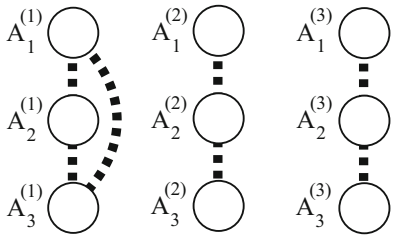

Fig. 3 The diagram that defines Part 2. A dotted line represents a sparse pair

B_i. As the name implies, a typical neighbor is a neighbor which is a typical vertex. This forms a copy of $K_{h,h,h}$. Then, place the vertices that were moved in prior steps into copies of $K_{h,h,h}$ by matching the $K_{h,h}$ with vertices in the appropriate "$\widetilde{A}$" set. Remove all of these from B, and apply Lemma 4 to the remaining adjusted graph with $\delta = \alpha_2$ and $\epsilon_4 = \alpha_2$. If the appropriate $K_{h,h}$-factor cannot be found, then we are in the case of Part 2, and G has the form shown in Fig. 3. A more rigorous definition of this case is provided in Sect. 3.2.

Step 3 Completing the *$K_{h,h,h}$-factor*. If the $K_{h,h}$-factor above is found, then we will recycle notation to define $A_i^{(1)}$ to be the vertices that remain from $\widetilde{A}_i$ after removing copies of $K_{h,h,h}$ as above. It is easy to see that each $A_i^{(1)}$ will have size close to T and divisible by h. Further define $A_i^{(j)}$, $i = 1, 2, 3$ and $j = 2, 3$ so that each member of the $K_{h,h}$-factor of B lies in a pair $(A_2^{(2)}, A_3^{(3)})$, $(A_3^{(2)}, A_1^{(3)})$ or $(A_1^{(2)}, A_2^{(3)})$, and so that each of the triples $(A_1^{(1)}, A_2^{(2)}, A_3^{(3)})$, $(A_2^{(1)}, A_3^{(2)}, A_1^{(3)})$ and $(A_3^{(1)}, A_1^{(2)}, A_2^{(3)})$ consist of sets of the same size. Note that this can be done arbitrarily.

We use Proposition 2, which allows us to complete a $K_{h,h}$-factor into a $K_{h,h,h}$-factor. The proof follows easily from König–Hall.

Proposition 2 *Let $h \geq 1$.*

(1) *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = M$, h divides M, and each vertex is adjacent to at least $(1 - 1/(2h^2))M$ vertices in the other part. Then, we can find a $K_{h,h}$-factor in G.*

(2) *Let $G = (V_1, V_2, V_3; E)$ be a tripartite graph with $|V_1| = |V_2| = |V_3| = M$, h divides M, and each vertex is adjacent to at least $(1 - 1/(4h^2))M$ vertices in each of the other parts. Furthermore, let there be a $K_{h,h}$-factor in (V_2, V_3). Then, we can extend it into a $K_{h,h,h}$-factor in G.*

Proposition 2(2) allows us to find $K_{h,h,h}$-factors in each of $(A_1^{(1)}, A_2^{(2)}, A_3^{(3)})$, $(A_2^{(1)}, A_3^{(2)}, A_1^{(3)})$ and $(A_3^{(1)}, A_1^{(2)}, A_2^{(3)})$ which completes the $K_{h,h,h}$-factor in G.

3.2 Part 2: *G* is Approximately the Graph Represented by Fig. 3

Let H_3 be the graph on vertices $v_i^{(j)}$ for $i, j \in \{1, 2, 3\}$ with the following non-adjacencies: $v_1^{(j)} \nsim v_2^{(j)}$ for $j \in \{1, 2, 3\}$, $v_2^{(j)} \nsim v_3^{(j)}$ for $j \in \{1, 2, 3\}$ and $v_1^{(1)} \nsim v_3^{(1)}$.

In this part, our graph G is α_2-approximately H_3. It therefore corresponds to the diagram in Fig. 3 in which partite sets are represented as rows, and each row is split

into three columns. Note that the first column of G consists of the pairwise sparse sets from the definition of α-extremal, and the second and third columns are defined by the exceptional case of Lemma 4. We will group the vertices into sets $A_i^{(j)}$ of size between $(1-3\alpha_2^{2/3})T$ and $(1+3\alpha_2^{2/3})T$ so that each vertex in $A_i^{(j)}$ is adjacent to at least θT vertices in each set $A_{i'}^{(j')}$ when $v_i^{(j)} \sim_{H_3} v_{i'}^{(j')}$. In other words, the vertices in $A_i^{(j)}$ are typical according to the rules established by Fig. 3. The non-typical vertices in row i will be collected in the set $\widetilde{C}_i$. From this point forward we have issues related to divisibility that we did not have before. Namely, we may need to modify $A_2^{(2)}$ and $A_2^{(3)}$ so that their sizes are divisible by h.

Step 1 Ensuring small $A_i^{(j)}$ *sets of proper size.* Each $A_i^{(j)}$ set has a target size that we will denote $s_{i,j}$. If $N = 3T$, then $s_{i,j} = T$ for all i, j. If $N = 3T+h$, let $s_{i,j} = T+h$ for all i, j such that 3 divides $i+j$ and $s_{i,j} = T$ otherwise. If $N = 3T+2h$, let $s_{i,j} = T$ for all i, j such that 3 divides $i+j$ and $s_{i,j} = T+h$ otherwise. Note that if $N = 3T+h$, we can remove one copy of $K_{h,h,h}$ from the triple $(A_1^{(2)}, A_2^{(1)}, A_3^{(3)})$, and if $N = 3T+2h$, we can remove two copies of $K_{h,h,h}$ from triples where $i+j$ is not divisible by 3.

Apply Lemma 2 to obtain $\max\{0, |A_i^{(j)}| - s_{i,j}\}$ disjoint stars with centers in $A_i^{(j)}$ and leaves in $A_i^{(j+1)}$. Then move these star centers to $A_i^{(j')}$ where $j' \neq j$ so that $|A_i^{(j)}| = s_{i,j}$ holds for all i, j.

Step 2 Partitioning the sets. Before we partition the sets, we must examine the behavior of $(A_1^{(2)} \cup A_1^{(3)}, A_3^{(2)} \cup A_3^{(3)})$. If this is α_5- approximately $\Theta_{2\times 2}(T)$, then call the *dense* pairs (E_1, E_3) and (F_1, F_3). Note that the sets E_i and F_i need not be uniquely-defined as long as they satisfy the given condition. If $(A_1^{(2)} \cup A_1^{(3)}, A_3^{(2)} \cup A_3^{(3)})$ is not α_5-approximately $\Theta_{2\times 2}(T)$, do nothing.

For $i \in \{1,3\}, \{j, j'\} = \{2,3\}$, we say that E_i and $A_i^{(j)}$ *coincide* if the intersection of their typical vertices is large and therefore the intersection of the typical vertices of E_i and $A_i^{(j')}$ is small. We will determine the quantities that constitute "large" and "small" later. If (E_1, E_3) and (F_1, F_3) coincide with $(A_1^{(2)}, A_3^{(3)})$ and $(A_1^{(3)}, A_3^{(2)})$, respectively, then G is approximately $\Theta_{3\times 3}(N/3)$. This case will be handled in Sect. 3.3. If (E_1, E_3) and (F_1, F_3) coincide with $(A_1^{(2)}, A_3^{(2)})$ and $(A_1^{(3)}, A_3^{(3)})$, respectively, then G is approximately $\Gamma_3(N/3)$. This case will be handled in Sect. 3.4. Otherwise, there may be no coincidence, or coincidence may occur in exactly one of V_1 and V_3. Without loss of generality, we will assume that if there is coincidence in only one part, then it occurs in V_1. More specifically, we will assume that E_1 coincides with $A_1^{(2)}$, F_1 coincides with $A_1^{(3)}$, and neither E_3 nor F_3 coincides with $A_3^{(j)}$, $j = 2, 3$.

In addition, note that if, say $A_1^{(2)}$ coincides with E_1, then every vertex in $A_1^{(2)}$ is adjacent to at least θT vertices in E_3 and vice versa. If there is no coincidence, then let E_1 and E_3 be redefined so that every vertex in E_1 is adjacent to at least θT vertices in E_3 and vice versa. Similarly for (F_1, F_3).

We randomly partition each set $A_i^{(j)}$ into two pieces of size divisible by h and as equal as possible. By the Chernoff bound, with high probability each vertex in $A_i^{(j)}$

has at least $(1-2\alpha_3-6\alpha_2^{2/3})(T/2)$ neighbors in each piece of the partition of $A_{i'}^{(j')}$, $i' \neq i$, $j' \neq j$. Moreover, if a vertex has degree at least $\alpha_3 T$ in an $A_i^{(j)}$ set, it has degree at least $(\alpha_3/3)(T/2)$ in each of the two partitions.

Let Σ_3 denote the symmetric group that permutes the elements of $\{1,2,3\}$. For all $i, j \in \{1,2,3\}$, we assign to each part of $A_i^{(j)}$ a permutation $\sigma \in \Sigma_3$ such that $\sigma(i) = j$ (there are exactly two such permutations) and denote it by $A_{i,\sigma}$. Furthermore, it is possible to arrange the assignment such that $|A_{1,\sigma}| = |A_{2,\sigma}| = |A_{3,\sigma}|$ for all $\sigma \in \Sigma_3$. After some adjustments, these permutations will identify which sets the copies of $K_{h,h,h}$ in our covering will span. For example, a $K_{h,h,h}$ which spans $A_1^{(2)}$, $A_2^{(1)}$ and $A_3^{(3)}$ will be contained in the parts of those sets corresponding to $\sigma = 213$, and a $K_{h,h,h}$ which spans $A_1^{(2)}$, $A_2^{(3)}$ and $A_3^{(1)}$ will be contained in the parts of those sets corresponding to $\sigma = 231$. Note that the permutations 213 and 231 are expressed using the notation $\sigma(1)\sigma(2)\sigma(3)$.

Step 3 Assigning vertices. Each vertex $c \in \widetilde{C}_2$ has the property that, for all $j \in \{1,2,3\}$ and distinct $i', i'' \in \{1,3\}$, if c is adjacent to fewer than $\alpha_3 T$ vertices in $A_{i'}^{(j)}$, then c is adjacent to at least $\alpha_3 T$ vertices in $A_{i''}^{(j)}$.

For $i \in \{1,3\}$, each vertex $c \in \widetilde{C}_i$ has the property that, for all $j \in \{1,2,3\}$, c cannot be adjacent to fewer than $\alpha_3 T$ vertices in either $A_2^{(2)}$ or $A_2^{(3)}$. Also, c cannot be adjacent to fewer than $\alpha_3 T$ vertices in both $A_2^{(1)}$ and $A_{4-i}^{(1)}$ or both $A_2^{(2)}$ and F_{4-i} (if it exists) or both $A_2^{(3)}$ and E_{4-i} (if it exists). Note that when F_{4-i} and E_{4-i} do not exist, it is because $(A_1^{(2)} \cup A_1^{(3)}, A_3^{(2)} \cup A_3^{(3)})$ is not approximately $\Theta_{2\times 2}(T)$.

Trivially, each vertex in V_i is adjacent to at least $(1/2-\alpha_3)T$ vertices in at least two of $\{A_{i'}^{(1)}, A_{i'}^{(2)}, A_{i'}^{(3)}\}$ and in at least two of $\{A_{i''}^{(1)}, A_{i''}^{(2)}, A_{i''}^{(3)}\}$, where i', i'' are distinct members of $\{1,2,3\} - \{i\}$. This is particularly important for vertices in $\widetilde{C}_i$.

The $\widetilde{C}_i$ vertices, as well as star-leaves and star-centers, may only be able to form a $K_{h,h,h}$ with respect to one particular permutation.

For example, consider a vertex c which had been in $\widetilde{C}_1$ but was put into $A_1^{(1)}$ in Step 2. Then, for either the pair $(A_2^{(2)}, A_3^{(3)})$ or the pair $(A_2^{(3)}, A_3^{(2)})$, the vertex c is adjacent to at least $(1/2-\alpha_3)T$ vertices in one set and at least $\alpha_3 T$ vertices in the other; otherwise, it would have been a typical vertex in $A_1^{(1)}$, $A_1^{(2)}$ or $A_1^{(3)}$.

Assume that c is adjacent to at least $\alpha_3 T$ vertices in $A_2^{(3)}$ and at least $(1/2-\alpha_3)T$ vertices in $A_3^{(2)}$. In this case, if c were placed into the partition corresponding to the identity permutation in Step 3, then exchange c with a vertex in $A_{1,132}$.

In a similar fashion, if there is a star with center in, say $A_1^{(2)}$, and leaves in, say $A_2^{(1)}$, then we will use it to form a $K_{h,h,h}$ with respect to the permutation $213 \in \Sigma_3$. Again, if any such leaf or center was placed in the wrong partition, exchange it with a typical vertex in the other partition.

The number of leaves in any set is at most $2h(6\alpha_2^{2/3}T+h)$ and the number of centers is at most $2(6\alpha_2^{2/3}T+h)$; the number of $\widetilde{C}_i$ vertices is at most $9\alpha_2^{2/3}T$. So, if N is large enough, the total number of typical vertices in any $A_i^{(j)}$ which were exchanged is at most $2(12h+21)\alpha_2^{2/3}T + 4h^2 + 4h$.

With the partition established and the $\widetilde{C}_i$, star center and leaf vertices in the proper parts, we consider the triple formed by three sets:

- $A_2^{(1)}$, which will also be denoted $\widetilde{S}_2$
- the union of the piece of $A_1^{(2)}$ corresponding to 213 and the piece of $A_1^{(3)}$ corresponding to 312, denoted $\widetilde{S}_1$, and
- the union of the piece of $A_3^{(2)}$ corresponding to 312 and the piece of $A_3^{(3)}$ corresponding to 213, denoted $\widetilde{S}_3$.

Let the graph induced by the triple $(\widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3)$ be denoted $\widetilde{\mathcal{S}}$.

Step 4 Finding a $K_{h,h,h}$ *cover in* $\widetilde{\mathcal{S}}$. We will first find a $K_{h,h,h}$-factor in $\widetilde{\mathcal{S}}$. This task is complicated because the parts of $\widetilde{\mathcal{S}}$ correspond to the permutations 213 and 312, meaning the $K_{h,h,h}$'s in our covering either will span $A_1^{(2)}$, $A_2^{(1)}$ and $A_3^{(3)}$ or will span $A_1^{(3)}$, $A_2^{(1)}$ and $A_3^{(2)}$. If $(A_1^{(2)} \cup A_1^{(3)}, A_3^{(2)} \cup A_3^{(3)})$ is approximately $\Theta_{2\times 2}(T)$, then for $i = 1, 3$, we will need to exchange vertices in $\widetilde{S}_i$ with typical vertices in E_i and F_i. Doing this in the right way will ensure that a $K_{h,h}$-factor of $(\widetilde{S}_1, \widetilde{S}_3)$ can be found, and we will extend that $K_{h,h}$-factor to a $K_{h,h,h}$-factor of $\widetilde{\mathcal{S}}$. Note that this complication does not arise when finding $K_{h,h,h}$'s with respect to permutations in $\Sigma_3 - \{213, 312\}$.

To begin, let $T_0 = |A_2^{(1)}|$. First, take each existing copy of $K_{1,h}$ in $\widetilde{\mathcal{S}}$ and complete it to form disjoint copies of $K_{h,h,h}$, using unexchanged typical vertices. This can be done because α_4 is small enough and the centers are typical vertices. Remove all the copies of $K_{h,h,h}$ that contain stars.

Second, take each vertex c from $\widetilde{C}_i$ and use it to complete a $K_{h,h,h}$. We can guarantee, because of the random partitioning, that c is adjacent to at least $(\alpha_3/3)T_0$ vertices in one partition set and $(1/3 - 2\alpha_3)T_0$ vertices in the other. Without loss of generality, assume that $c \in \widetilde{S}_1$ has degree at least $(\alpha_3/3)T_0$ in $\widetilde{S}_2$ and at least $(1/3 - 2\alpha_3)T_0$ in $\widetilde{S}_3$. Since $\alpha_3 \gg \alpha_2$, we can guarantee h neighbors of c in $\widetilde{S}_2$ among unexchanged typical vertices and, if $\alpha_3 \ll \alpha_4 \ll 1$, then h common neighbors of those among unexchanged typical vertices in $N(c) \cap \widetilde{S}_3$. Finally, $\alpha_4 \ll h^{-1}$ implies this $K_{h,h}$ has at least $h - 1$ more common neighbors in $\widetilde{S}_1$. This is our $K_{h,h,h}$ and we can remove it. Repeat this process for all former members of a $\widetilde{C}_i$.

Third, take each exchanged typical vertex and put it into a $K_{h,h,h}$ and remove it. Throughout this process, we have removed at most $C_h\sqrt{\alpha_2} \times T_0$ vertices where C_h is a constant depending only on h. What remains are three sets of the same size, $T' \geq (1 - C_h\sqrt{\alpha_2})T_0$, with each vertex in $\widetilde{S}_1$ adjacent to at least, say $(1/2 - 2\alpha_4)T'$, vertices in $\widetilde{S}_3$ and vice versa. Each vertex in $\widetilde{S}_1$ and in $\widetilde{S}_3$ is adjacent to at least $(1/2 - 2\alpha_4)T'$ vertices in $\widetilde{S}_2$ and each vertex in $\widetilde{S}_2$ is adjacent to at least $(1/2 - 2\alpha_4)T'$ vertices in $\widetilde{S}_1$ and in $\widetilde{S}_3$.

Lemma 5 [28, Theorem 9] shows that we can find a factor of $(\widetilde{S}_1, \widetilde{S}_3)$ with vertex-disjoint copies of $K_{h,h}$ unless $(\widetilde{S}_1, \widetilde{S}_3)$ is approximately $\Theta_{2\times 2}(T/2)$.

Lemma 5 (Zhao [28]) *For every $\epsilon > 0$ and integer $h \geq 1$, there exists an $\alpha > 0$ and an N_0 such that the following holds. Suppose that $N > N_0$ is divisible by h. Then every bipartite graph $G = (A, B; E)$ with $|A| = |B| = N$ and $\delta(G) \geq (1/2 - \alpha)N$ either contains a $K_{h,h}$-factor, or contains $A' \subseteq A$, $B' \subseteq B$ such that $|A'| = |B'| = N/2$ and $d(A, B) \leq \epsilon$.*

If we can find the factor, apply König-Hall to form a factor of $\widetilde{\mathcal{S}}$ of vertex-disjoint copies of $K_{h,h,h}$. If not, apply Lemma 6. Lemma 6 states, in particular, that if a random partition results in $(\widetilde{S}_1, \widetilde{S}_3)$ being approximately $\Theta_{2\times 2}(T/2)$ with high probability, then $(A_1^{(2)} \cup A_1^{(3)}, A_3^{(2)} \cup A_3^{(3)})$ is approximately $\Theta_{2\times 2}(T)$. The proof of Lemma 6 follows from similar arguments to those in the proof of Lemma 3.3 of [21] and in Sect. 3.3.1 of [23] so we omit it.

Lemma 6 *For every $\epsilon > 0$ and integer $h \geq 1$, there exists a $\beta > 0$ and positive integer T_0 such that if $T \geq T_0$ the following holds. Let (A, B) be a bipartite graph such that $|A|, |B| \in \{2T - h, 2T, 2T + h\}$ with minimum degree at least $(1 - \epsilon)T$ and is minimal with respect to this condition. Let $A' \subset A$, $B' \subset B$, $|A'| = |B'| = T$ be chosen uniformly at random. If*

$$\Pr\{(A', B') \text{ contains a subpair with density at most } \epsilon\} \geq 1/4$$

then (A, B) is β-approximately $\Theta_{2\times 2}(T)$.

We can, therefore, assume the existence of (E_1, E_3) and (F_1, F_3). Further, we can assume that coincidence occurs only in V_1 or not at all; otherwise, we would be in Part 3.

As a result, recall that we let the typical vertices in the dense pairs in $(A_1^{(2)} \cup A_1^{(3)}, A_3^{(2)} \cup A_3^{(3)})$ be denoted (E_1, E_3) and (F_1, F_3). If the dense pairs do not coincide, then we will work to ensure that $|E_1 \cap \widetilde{S}_1| = |E_3 \cap \widetilde{S}_3|$ and $|F_1 \cap \widetilde{S}_1| = |F_3 \cap \widetilde{S}_3|$ and both are divisible by h. Do this by moving typical vertices from $(A_1^{(2)} \cap E_1) - \widetilde{S}_1$ into $(A_1^{(2)} \cap E_1) \cap \widetilde{S}_1$ and move the same number from $(A_1^{(2)} \cap F_1) \cap \widetilde{S}_1$ into $(A_1^{(2)} \cap F_1) - \widetilde{S}_1$. In addition, move vertices from $(A_3^{(2)} \cap E_3) - \widetilde{S}_3$ into $(A_3^{(2)} \cap E_3) \cap \widetilde{S}_3$ and move the same number from $(A_3^{(2)} \cap F_3) \cap \widetilde{S}_3$ into $(A_3^{(2)} \cap F_3) - \widetilde{S}_3$.

This can be done unless one of the intersections $A_i^{(j)} \cap E_i$ or $A_i^{(j)} \cap F_i$ is too small. This implies the coincidence that we discussed at the beginning of this part. But then, we have guaranteed that the remaining vertices of $A_1^{(2)}$ are not only typical in that set but also typical in E_1. The same is true of $A_1^{(3)}$ and F_1.

Now, we want to move vertices in V_3 to ensure that $|E_3 \cap \widetilde{S}_3| = |A_1^{(2)} \cap \widetilde{S}_1|$ and $|F_3 \cap \widetilde{S}_3| = |A_1^{(3)} \cap \widetilde{S}_1|$. Note that we have ensured that both $|A_1^{(2)} \cap \widetilde{S}_1|$ and $|A_1^{(3)} \cap \widetilde{S}_1|$ are divisible by h and approximately $T/2$.

We can do this as follows: Move vertices from $E_3 \cap A_3^{(2)} - \widetilde{S}_3$ to $(E_3 \cap A_3^{(2)}) \cap \widetilde{S}_3$ and move the same amount from $(F_3 \cap A_3^{(2)}) - \widetilde{S}_3$ to $(F_3 \cap A_3^{(2)}) \cap \widetilde{S}_3$. Also move vertices from $(E_3 \cap A_3^{(3)}) - \widetilde{S}_3$ to $(E_3 \cap A_3^{(3)}) \cap \widetilde{S}_3$ and move the same amount from $(F_3 \cap A_3^{(3)}) - \widetilde{S}_3$ to $(F_3 \cap A_3^{(3)}) \cap \widetilde{S}_3$. Since none of the intersections are small, this is possible. Moving around these vertices will let us find a $K_{h,h}$-factor of $(\widetilde{S}_1, \widetilde{S}_3)$ which we can complete to a $K_{h,h,h}$-factor of $\widetilde{\mathcal{S}}$ by applying Proposition 2(2).

Step 5 Completing the *$K_{h,h,h}$-factor in G*. Now that we have found a $K_{h,h,h}$-factor that corresponds to permutations 213 and 312, we consider the other permutations in Σ_3. For a $\sigma \in \Sigma_3 - \{213, 312\}$, let $\mathcal{S}(\sigma) \stackrel{\text{def}}{=} (A_{1,\sigma}, A_{2,\sigma}, A_{3,\sigma})$ be a triple of parts formed by the random partitioning after the exchange of vertices has taken place. The set $A_{i,\sigma}$

is a subset of $A_i^{(\sigma(i))}$. We have also ensured that $s_\sigma \stackrel{\text{def}}{=} |A_{1,\sigma}| = |A_{2,\sigma}| = |A_{3,\sigma}|$ and s_σ is divisible by h. It is now easy to ensure that this triple contains a $K_{h,h,h}$-factor:

First, take each star in $\mathcal{S}(\sigma)$ and complete it to form disjoint copies of $K_{h,h,h}$, using unexchanged typical vertices. This can be done if α_4 is small enough. Remove all such $K_{h,h,h}$'s containing stars.

Second, take each c which had been a member of some $\widetilde{C}_i$ and use it to complete a $K_{h,h,h}$. We can guarantee, because of the random partitioning, that c is adjacent to at least $(\alpha_3/3)s_\sigma$ vertices in one set and $(2/3 - 2\alpha_3)s_\sigma$ vertices in the other. Without loss of generality, let $c \in A_{1,\sigma}$ with degree at least $(\alpha_3/3)s_\sigma$ in $A_{2,\sigma}$ and at least $(1/2 - 2\alpha_3)s_\sigma$ in $A_{3,\sigma}$. Since $\alpha_3 \gg \alpha_2$, we can guarantee h neighbors of c in $A_{2,\sigma}$ among unexchanged typical vertices and, if $\alpha_3 \ll \alpha_4 \ll 1$, then h common neighbors of those among unexchanged typical vertices in $N(c) \cap A_{3,\sigma}$. Finally, $\alpha_4 \ll h^{-1}$ implies this $K_{h,h}$ has at least $h - 1$ more common neighbors in $A_{1,\sigma}$. This is our $K_{h,h,h}$ and we can remove it. Do this for all former members of a $\widetilde{C}_i$.

Finally, take each exchanged typical vertex and put it into a $K_{h,h,h}$ and remove it. Throughout this process, we have removed at most $C_h\sqrt{\alpha_2} \times s_\sigma$ vertices where C_h is a constant depending only on h. What remains are three sets of the same size, $s' \geq (1 - C_h\sqrt{\alpha_2})s_\sigma$, with each vertex adjacent to at least, say $(1 - 2\alpha_4)s'$, vertices in each of the other parts. If N is large enough, then we can use the Blow-up Lemma or Proposition 2(2) to complete the factor of $\mathcal{S}(\sigma)$ by copies of $K_{h,h,h}$.

3.3 Part 3a: G is Approximately $\Theta_{3\times 3}(\lfloor N/3 \rfloor)$

Figure 2 shows $\Theta_{3\times 3}$ and we are in the case where G is α_2-approximately $\Theta_{3\times 3}(\lfloor N/3 \rfloor)$, so $A_i^{(j)}$ and $A_{i'}^{(j')}$ being connected with a dotted line means that the pair $(A_i^{(j)}, A_{i'}^{(j')})$ is sparse.

We will assume for this part that each vertex is adjacent to at least $h\left\lceil \frac{2N}{3h} \right\rceil + h - 1$ vertices in each of the other pieces of the partition. Again, let $T = h\lfloor N/(3h) \rfloor$.

We will group the vertices of G into sets $A_i^{(j)}$ of size between $(1 - \sqrt{\alpha_2})T$ and $(1 + \sqrt{\alpha_2})T$ so that each vertex in $A_i^{(j)}$ is adjacent to at least θT vertices in each set $A_{i'}^{(j')}$ where $i' \neq i$ and $j' \neq j$. In other words, the vertices in $A_i^{(j)}$ are typical according to the rules established by Fig. 2. The non-typical vertices in row i will be collected in the set $\widetilde{C}_i$. Note that each vertex $c \in \widetilde{C}_i$ has the property that, for all $j \in \{1, 2, 3\}$ and distinct $i', i'' \in \{1, 2, 3\} - \{i\}$, if c is adjacent to fewer than $\alpha_3 T$ vertices in $A_{i'}^{(j)}$, then c is adjacent to at least $\alpha_3 T$ vertices in $A_{i''}^{(j)}$; otherwise c is in some set $A_i^{(j)}$. Furthermore, c is adjacent to at least $(1/2 - \alpha_3)T$ vertices in at least two of $\{A_{i'}^{(1)}, A_{i'}^{(2)}, A_{i'}^{(3)}\}$ and in at least two of $\{A_{i''}^{(1)}, A_{i''}^{(2)}, A_{i''}^{(3)}\}$.

Step 1 Ensuring small $A_i^{(j)}$ *sets of proper size.* As in Sect. 3.2, each $A_i^{(j)}$ set has a target size $s_{i,j}$. If $N = 3T$, then $s_{i,j} = T$ for all i, j. If $N = 3T + h$, let $s_{i,j} = T + h$ when $i = j$ and $s_{i,j} = T$ otherwise. If $N = 3T + 2h$, let $s_{i,j} = T$ when $i = j$ and $s_{i,j} = T + h$ otherwise.

Take each triple $(A_1^{(j)}, A_2^{(j)}, A_3^{(j)})$, $j = 1, 2, 3$, and construct disjoint copies of stars so that there are at most T non-center vertices in each set $A_i^{(j)}$. We use the fact that every vertex is adjacent to at least $h\left\lceil \frac{2N}{3h} \right\rceil + h - 1$ vertices in each of the other parts as well as Lemma 2. Move these star centers to $A_i^{(j')}$ where $j' \neq j$ so that $|A_i^{(j)}| = s_{i,j}$ holds for all i, j.

Step 2 Partitioning the sets. We will randomly partition each set $A_i^{(j)}$ into two pieces, as close as possible to equal size but which have size divisible by h, and assign them to a permutation, $\sigma \in \Sigma_3$, which assigns $\sigma(i) = j$. Each part assigned to σ will be the same size, and these permutations will identify which sets the copies of $K_{h,h,h}$ in our covering will span.

When N is large, this random partition of $A_i^{(j)}$ will have the following properties with high probability. A typical vertex in $A_i^{(j)}$ has at least $(1 - 2\alpha_4 - 2\sqrt{\alpha_2})(T/2)$ neighbors in each piece of the partition of $A_{i'}^{(j')}$, $i' \neq i$, $j' \neq j$. Moreover, if a vertex has degree at least $\alpha_3 T$ in a set, it has degree at least $(\alpha_3/3)(T/2)$ in each of the two partitions.

Step 3 Assigning vertices. The $\widetilde{C}_i$ vertices, as well as star centers together with their star-leaves, may only be able to form a $K_{h,h,h}$ with respect to one particular permutation.

For example, consider a vertex c which had been in $\widetilde{C}_1$ but is now in $A_1^{(1)}$. Then, for either the pair $(A_2^{(2)}, A_3^{(3)})$ or the pair $(A_2^{(3)}, A_3^{(2)})$, the vertex c is adjacent to at least $(1/2 - \alpha_3)T$ in one set and at least $\alpha_3 T$ vertices in the other. It is easy to see, since $\alpha_2 \ll \alpha_3$, that if this were not true, then c would have been typical with respect to one of the sets $A_1^{(1)}$, $A_1^{(2)}$ or $A_1^{(3)}$, which is a contradiction to the definition of c.

Assume that c is adjacent to at least $\alpha_3 T$ vertices in $A_2^{(3)}$ and at least $(1/2 - \alpha_3)T$ vertices in $A_3^{(2)}$. In this case, if c were placed into the partition corresponding to the identity permutation, then exchange c with a typical vertex in the partition assigned to 132.

In a similar fashion, if there is a star with center in, say $A_1^{(2)}$, and leaves in, say $A_2^{(1)}$, then we will form a $K_{h,h,h}$ with respect to the permutation $213 \in \Sigma_3$. Again, if any such leaf or center was in the wrong partition, exchange it with a typical vertex in the other partition.

The number of leaves in any set is at most $2h(\sqrt{\alpha_2}\, T + h)$ and the number of centers is at most $2(\sqrt{\alpha_2}\, T + h)$, the number of $\widetilde{C}_i$ vertices is at most $3\sqrt{\alpha_2}\, T$. So, if N is large enough, the total number of typical vertices in any $A_i^{(j)}$ which were exchanged is at most $(2h + 6)\sqrt{\alpha_2}\, T$.

Step 4 Completing the cover. For some $\sigma \in \Sigma_3$, let $\mathcal{S}(\sigma) \stackrel{\text{def}}{=} \left(S_1^{(\sigma(1))}, S_2^{(\sigma(2))}, S_3^{(\sigma(3))}\right)$ be a triple of parts formed by the random partitioning after the exchange has taken place. The set $S_i^{(\sigma(i))}$ is a subset of $A_i^{(\sigma(i))}$. We have also ensured in Step 3 that $s_\sigma \stackrel{\text{def}}{=} \left|S_1^{(\sigma(1))}\right| = \left|S_2^{(\sigma(2))}\right| = \left|S_3^{(\sigma(3))}\right|$ and s_σ is divisible by h. It is now easy to ensure that this triple contains a $K_{h,h,h}$-factor:

First, take each star in $\mathcal{S}(\sigma)$ and complete it to form disjoint copies of $K_{h,h,h}$, using unexchanged typical vertices. This can be done if α_4 is small enough. Remove all such $K_{h,h,h}$'s containing stars.

Second, take each c which had been a member of some $\widetilde{C}_i$ and use it to complete a $K_{h,h,h}$. We can guarantee, because of the random partitioning, that c is adjacent to at least $(\alpha_3/3)s_\sigma$ vertices in one set and $(2/3-2\alpha_3)s_\sigma$ vertices in the other. Without loss of generality, let $c \in S_1^{(\sigma(1))}$ have degree at least $(\alpha_3/3)s_\sigma$ in $S_2^{(\sigma(2))}$ and at least $(1/2-2\alpha_3)s_\sigma$ in $S_3^{(\sigma(3))}$. Since $\alpha_3 \gg \alpha_2$, we can guarantee h neighbors of c in $S_2^{(\sigma(2))}$ among unexchanged typical vertices and, since $\alpha_3 \ll \alpha_4 \ll 1$, h common neighbors of those among unexchanged typical vertices in $N(c) \cap S_3^{(\sigma(3))}$. Finally, $\alpha_4 \ll h^{-1}$ implies this $K_{h,h}$ has at least $h-1$ more common neighbors in $S_1^{(\sigma(1))}$. This is our $K_{h,h,h}$ and we can remove it. Do this for all former members of a $\widetilde{C}_i$.

Finally, take each exchanged typical vertex and put it into a $K_{h,h,h}$ and remove it. Throughout this process, we have removed at most $\alpha_2^{1/3}s_\sigma$ vertices if α_2 is small enough. What remains are three sets of the same size, $s' \geq (1-\alpha_2^{1/3})s_\sigma$, with each vertex adjacent to at least, say $(1-2\alpha_4)s'$, vertices in each of the other parts. If N is large enough, then we can use Proposition 2(2) to complete the factor of $\mathcal{S}(\sigma)$ by copies of $K_{h,h,h}$.

3.4 Part 3b: G is Approximately $\Gamma_3(\lfloor N/3 \rfloor)$

Figure 1 shows Γ_3 and we are in the case where G is α_2-approximately $\Gamma_3(\lfloor N/3 \rfloor)$, where $A_i^{(j)}$ and $A_{i'}^{(j')}$ being connected with a dotted line means that the pair $(A_i^{(j)}, A_{i'}^{(j')})$ is sparse.

We will assume for this part that each vertex is adjacent to at least $h\left\lceil \frac{2N}{3h} \right\rceil + h - 1$ vertices in each of the other pieces of the partition. We also assume that G is not in the very extreme case (see Definition 2). We must deal with the very extreme case separately.

Now, let $T \stackrel{\text{def}}{=} h\lfloor N/(3h) \rfloor$. We may group the vertices of G into sets $A_i^{(j)}$ of size between $(1-\sqrt{\alpha_2})T$ and $(1+\sqrt{\alpha_2})T$ so that each vertex in $A_i^{(1)}$ is adjacent to at least $(1-\alpha_3)T$ vertices in each set $A_{i'}^{(j')}$ where $i' \neq i$ and $j' \in \{2,3\}$. For $j = 2, 3$, each vertex in $A_i^{(j)}$ is adjacent to at least $(1-\alpha_3)T$ vertices in each set $A_{i'}^{(1)}$ and $A_{i'}^{(j)}$, where $i' \neq i$. In other words, the vertices in $A_i^{(j)}$ are typical according to the rules established by Fig. 1. The non-typical vertices in row i will be collected in the set $\widetilde{C}_i$. Note that each vertex $c \in \widetilde{C}_i$ has the following property: for all $j \in \{1,2,3\}$ and distinct $i', i'' \in \{1,2,3\} - \{i\}$, if c is adjacent to fewer than $\alpha_3 T$ vertices in $A_{i'}^{(j)}$, then c is adjacent to at least $\alpha_3 T$ vertices in $A_{i''}^{(j)}$. Furthermore, c is adjacent to at least $(1/2-\alpha_4)T$ vertices in at least two of $\left\{A_{i'}^{(1)}, A_{i'}^{(2)}, A_{i'}^{(3)}\right\}$ and $\left\{A_{i''}^{(1)}, A_{i''}^{(2)}, A_{i''}^{(3)}\right\}$.

Step 1 Ensuring small $A_i^{(j)}$ *sets of proper size.* As in the previous two sections, each $A_i^{(j)}$ has a target size $s_{i,j}$. There are several cases for $s_{i,j}$ according to the divisibility of N/h. Let $N/h = 6q + r$ where $0 \leq r < 6$.

- $r = 0, 3$: $s_{i,j} = T$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.
- $r = 1$: $s_{i,j} = T$ for $i = 1, 2, 3$ and $j = 1, 3$; and $s_{i,2} = T + h$ for $i = 1, 2, 3$.
- $r = 2, 5$: $s_{i,1} = T$ for $i = 1, 2, 3$; and $s_{i,j} = T + h$ for $i = 1, 2, 3$ and $j = 2, 3$.
- $r = 4$: $s_{i,1} = T$ for $i = 1, 2, 3$; and $s_{1,3} = s_{2,3} = s_{3,2} = T$; and $s_{1,2} = s_{2,2} = s_{3,3} = T + h$.

Without loss of generality, we will assume that both $|A_1^{(2)}| \geq |A_1^{(3)}|$ and $|A_2^{(2)}| \geq |A_2^{(3)}|$.

If $|A_3^{(2)}| \geq |A_3^{(3)}|$, then $A_i^{(2)}$ is larger than $A_i^{(3)}$ for $i = 1, 2, 3$. Use Lemma 2(1) to construct $\max\left\{|A_i^{(2)}| - T, 0\right\}$ disjoint copies of $K_{1,h}$ in the pair[2] $(A_i^{(2)}, A_{i+1}^{(3)})$ with centers in $A_i^{(2)}$. Move these star-centers into $A_i^{(3)}$.

If $|A_3^{(2)}| < |A_3^{(3)}|$, we do something similar except that first we use Lemma 2(1) to create the appropriate number of stars in $(A_1^{(2)}, A_2^{(3)})$ and $(A_2^{(2)}, A_1^{(3)})$ with the centers in $A_1^{(2)}$ and $A_2^{(2)}$, respectively. Move these star-centers into $A_1^{(3)}$ and $A_2^{(3)}$, respectively. Then, after the star-centers have been removed from $A_2^{(2)}$, we apply Lemma 2(1) to the pair $(A_3^{(3)}, A_2^{(2)})$, and move the star-centers into $A_3^{(2)}$.

By the conditions on Lemma 2(1), we see that each remaining set $A_i^{(j)}$ is of size at most T. Now, apply Lemma 2(2) to the triple $(A_1^{(1)}, A_2^{(1)}, A_3^{(1)})$. For star-centers in $A_i^{(1)}$, move $T - |A_i^{(2)}|$ into $A_i^{(2)}$ and $T - |A_i^{(3)}|$ into $A_i^{(3)}$.

If necessary, place vertices from $\widetilde{C}_i$ into $A_i^{(j)}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$, while ensuring that we still have $|A_i^{(j)}| \leq s_{i,j}$.

For $j = 2, 3$, let $\mathcal{A}^{(j)} = (A_1^{(j)}, A_2^{(j)}, A_3^{(j)})$. We remove some copies of $K_{h,h,h}$ from among typical vertices of these sets as follows:

- $r = 1$: One from $\mathcal{A}^{(2)}$.
- $r = 2$: One from each of $\mathcal{A}^{(2)}$ and $\mathcal{A}^{(3)}$.
- $r = 4$: One from $\mathcal{A}^{(2)}$.
- $r = 5$: Two from $\mathcal{A}^{(2)}$.

Recalling $N = (6q + r)h$, each $A_i^{(j)}$ is now of size $2qh$, $2qh + h$ or $2qh + 2h$.

Step 2a Partitioning the sets ($r \neq 3$). Let $r \in \{0, 1, 2, 4, 5\}$, $\tau_1 = qh$ and $\tau_2 = qh+h$. Partition each $A_i^{(j)}$ set into parts of nearly equal size. Each part of the partition will receive a label $\sigma \in \{1, 2, 3\} \times \{2, 3\}$, where $\sigma = (i, j)$ corresponds to row i and column j. The part with label (i, j) will be denoted $S_{i,j}$. A $K_{h,h,h}$ which is associated with the label (i, j) will span the triple with one part in $A_i^{(1)}$ and two parts in column j. Now, partition each $A_i^{(j)}$ as follows:

Each $A_i^{(1)}$ will be split into two pieces. For $r = 0, 1, 2$ and $i = 1, 2, 3$, both pieces will have size τ_1 and we will arbitrarily assign the two pieces with the labels $S_{i,2}$ and $S_{i,3}$. For $r = 4$ and $i = 3$, assign the piece of size τ_1 with label $S_{3,3}$ and the one of size τ_2 with $S_{3,2}$. For $r = 4$ and $i = 1, 2$ and for $r = 5$ and $i = 1, 2, 3$, assign the smaller piece with label $S_{i,2}$ and the larger with label $S_{i,3}$.

[2] Arithmetic in the indices is always done modulo 3.

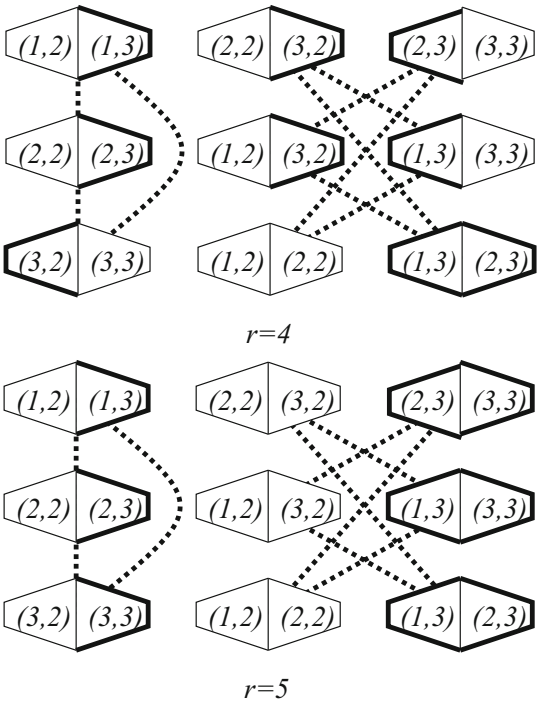

Fig. 4 Partitioning the sets. The light outlined half of a set is the piece of size τ_1, the bold outlined half of a set is the piece of size τ_2

Each $A_i^{(2)}$ will be split into two pieces. Unless both $r = 4$ and $i \in \{1, 2\}$, both pieces will be of size τ_1 and will be assigned $S_{i',2}$ and $S_{i'',2}$ arbitrarily, where $\{i, i', i''\} = \{1, 2, 3\}$. If $r = 4$ and $i \in \{1, 2\}$, the one of size τ_1 is labeled $S_{3-i,2}$ and the one of size τ_2, is labeled $S_{3,2}$.

Each $A_i^{(3)}$ will be split into two pieces. If $r \in \{0, 1, 2\}$, both pieces will be of size τ_1, and if $r = 5$ or if $r = 4$ and $i = 3$, both pieces will be of size τ_2. In these cases, arbitrarily assign the pieces with labels $S_{i',3}$ and $S_{i'',3}$ where $\{i, i', i''\} = \{1, 2, 3\}$. If $r = 4$ and $i \in \{1, 2\}$, the one of size τ_1 is labeled $S_{3,3}$ and one of size τ_2 is labeled $S_{3-i,3}$.

Figure 4 diagrams the partitioning for $r = 4$ and $r = 5$. Note that when $r \in \{0, 1, 2\}$, the partition labeling is identical to the case when $r = 5$, but all parts have size τ_1.

Partitioning the sets at random again ensures that the above can be accomplished so that all of the vertices' neighborhoods maintain roughly the same proportion, as in Part 3a, Step 3.

Step 2b Partitioning the vertices ($r = 3$, *not The very extreme case)*. Let $r = 3$ (recall $N = (6q + r)h$) and let G not be in the very extreme case. We will use Lemma 2(1) to find additional stars between sparse pairs. Without loss of generality, we seek stars with centers in either $A_1^{(1)}$ or $A_1^{(3)}$. If we can find at least h centers in one of these sets, then we can make that $A_1^{(j)}$ set of size $2qh$. If we are not able to do this, every vertex $v \in A_i^{(j)}$ must be adjacent to at most $2h - 2$ vertices in $A_{i'}^{(j')}$ where $(A_i^{(j)}, A_{i'}^{(j')})$ is a sparse pair. In turn, we have that every vertex $v \in A_i^{(j)}$ is nonadjacent to at most

$3h-3$ vertices in $A_{i'}^{(j')}$ where $(A_i^{(j)}, A_{i'}^{(j')})$ is a dense pair. Since G is approximately $\Gamma_3(\lfloor N/3 \rfloor)$, this means G is in the very extreme case.

Suppose star-centers are removed to make either $|A_1^{(1)}| = 2qh$ or $|A_1^{(3)}| = 2qh$. We will make the set $A_1^{(2)}$ of size $(2q+2)h$ by adding star-centers and vertices from the set $\widetilde{C}_1$.

In each case, if the star-centers that were placed into $A_1^{(2)}$ were themselves originally in $A_1^{(2)}$, then we just treat them as typical vertices again, ignoring the star that was formed. Note that all sets are of size $(2q+1)h$, except $|A_1^{(2)}| = (2q+2)h$ and either $A_1^{(1)}$ or $A_1^{(3)}$, which has size $2qh$. If $A_1^{(1)}$ is the small set, then remove one copy of $K_{h,h,h}$ in the triple $(A_1^{(3)}, A_2^{(3)}, A_3^{(3)})$.

Now we partition each set as follows: Each $A_i^{(1)}$ will have one piece of size qh with label $(i,3)$. The other set will have label $(i,2)$ and will be of size $(q+1)h$ in the case of $A_2^{(1)}$ and $A_3^{(1)}$ and of size either qh or $(q+1)h$ in the case of $A_1^{(1)}$. The set $A_1^{(2)}$ is partitioned into two pieces of size $(q+1)h$, one labeled $(2,2)$, the other labeled $(3,2)$. For $A_i^{(2)}$, $i=2,3$, we have one piece of size qh and labeled $(1,2)$ and the other of size $(q+1)h$, labeled $(5-i,2)$. For $A_1^{(3)}$, it will have two pieces of size qh, one labeled $(2,3)$, the other $(3,3)$. Finally, for $A_i^{(3)}$, $i=2,3$, we have one piece of size qh with label $(5-i,3)$ and the other will have size either qh or $(q+1)h$ and label $(1,3)$.

Partitioning the sets at random again ensures that the above can be accomplished so that all of the vertices' neighborhoods maintain roughly the same proportion, as in Part 3a, Step 3.

Step 3 Assigning vertices. For any $\sigma \in \{1,2,3\} \times \{2,3\}$, we will show that the star-centers and $\widetilde{C}_i$ vertices, in any $A_i^{(j)}$ can be assigned to one of the two parts of the partition.

For example, consider a vertex c which had been in $\widetilde{C}_1$ but is now in $A_1^{(1)}$. Then, for either the pair $(A_2^{(2)}, A_3^{(2)})$ or the pair $(A_2^{(3)}, A_3^{(3)})$, the vertex c is adjacent to at least $(1/2-\delta)T$ in one set and at least $\alpha_3 T$ vertices in the other. If such a pair is $(A_2^{(2)}, A_3^{(2)})$ then if c were labeled $(1,2)$ exchange it with a typical vertex with label $(1,3)$.

Now, for example, consider a vertex c which had been in $\widetilde{C}_1$ but is now in $A_1^{(2)}$. It is easy to check that for either the pair $(A_2^{(1)}, A_3^{(2)})$ or the pair $(A_3^{(1)}, A_2^{(2)})$, the vertex c is adjacent to at least $(1/2-\alpha_3)T$ in one set and at least $\alpha_3 T$ vertices in the other. If such a pair is, say, $(A_2^{(1)}, A_3^{(2)})$, and c is not labeled $(2,2)$, then exchange it for a typical vertex of that label.

A similar analysis can be applied to any $c \in \widetilde{C}_i$ for $i=1,2,3$.

Now we consider stars. All star-centers are in sets $A_i^{(2)}$ or $A_i^{(3)}$. Without loss of generality, assume z is such a center in $A_1^{(2)}$ and the leaves are in V_2. If the leaves are in $A_2^{(1)}$, then z must have been a member of $A_1^{(1)}$ originally. So, z and its leaves must have label $(2,2)$. If the leaves are in $A_2^{(2)}$, then z must have been a member of $A_1^{(3)}$ originally. So, z and its leaves must have label $(3,2)$. Exchange z with typical vertices to ensure this.

Finally, we consider typical vertices moved from $A_i^{(2)} \cup A_i^{(3)}$ to $A_i^{(1)}$. Without loss of generality, suppose z is such a vertex in $A_1^{(1)}$. If z were originally from $A_1^{(2)}$, then it is a typical vertex with respect to $A_2^{(2)}$ and $A_3^{(2)}$ and z should receive label $(1, 2)$. Otherwise, it is typical with respect to $A_2^{(3)}$ and $A_3^{(3)}$ and z should receive label $(1, 3)$.

This completes the verification that all moved vertices can receive at least one label of the $A_i^{(j)}$ set in which it is placed.

Step 4 Completing the cover. For any $\sigma \in \{1, 2, 3\} \times \{2, 3\}$, let $\mathcal{S}(\sigma)$ be the triple of parts with label σ. Note that the label (i, j) corresponds to a triple with one part in $A_i^{(1)}$ and two parts in column j. We can finish the $K_{h,h,h}$-factor as in Part 3a, Step 5.

3.5 The very extreme case

Recall the very extreme case:

> There are integers N, q such that $N = (6q + 3)h$. There are sets $A_i^{(j)}$ for $i, j \in \{1, 2, 3\}$, with sizes at least $2qh + 1$, such that if $v \in A_i^{(j)}$ then v is nonadjacent to at most $3h - 3$ vertices in $A_{i'}^{(j')}$ whenever the pair $(A_i^{(j)}, A_{i'}^{(j')})$ corresponds to an edge in the graph Γ_3 with respect to the usual correspondence.

In this case, we must raise the minimum degree condition to $2N/3 + 2h - 1$. Recalling Part 4, Step 3b, we were able to proceed if we were able to make one of the sets $A_i^{(j)}$ small by means of creating stars. Each vertex in $A_2^{(2)}$ is adjacent to at least $|A_1^{(3)}| - N/3 + 2h - 1$ vertices in $A_1^{(3)}$. Using Lemma 2(1), we have that there is a family of $|A_1^{(3)}| - N/3 + h$ vertex-disjoint stars with centers in $A_1^{(3)}$. We move the centers to $A_1^{(2)}$. Then we can proceed from Part 3b, Step 4.

3.6 Proofs of Lemmas

Lemma 2 is used to find vertex-disjoint h-stars in a graph G. Part (1) deals with the case where $G = (A_1, A_2; E)$, and part (2) deals with the case where $G = (A_1, A_2, A_3; E)$.

Proof of Lemma 2 (1) Let $\delta_1 = d_1 - h + 1$. If the stars cannot be created greedily, then there is a set $S \subset A_1$ and a set $T \subset A_2$ such that $|S| \leq \delta_1 - 1$ and $|T| = |S|h$ and each vertex in $A_1 - S$ is adjacent to at most $h - 1$ vertices in $A_2 - T$. In this case,

$$(d_1 - |S|)|A_2 - T| \leq e(A_1 - S, A_2 - T) \leq (h - 1)|A_1 - S|.$$

This gives

$$\begin{aligned}|S| &\geq \delta_1 - (h-1)\frac{|A_1 - S| - |A_2 - T|}{|A_2 - T|}\\ &= \delta_1 - (h-1)\frac{|A_1| - |A_2| + (h-1)|S|}{|A_2| - h|S|}\\ &\geq \delta_1 - (h-1)\frac{(h+1)\epsilon M}{(1-(h+1)\epsilon)M}.\end{aligned}$$

If $\epsilon < (h^2+h)^{-1}$, then this gives $|S| > \delta_1 - 1$. Since $|S|$ is an integer, $|S| \geq \delta_1$, contradicting the condition we put on $|S|$.

(2) Let $\delta_i = \max\{0, d_i - h + 1\}$ for $i = 1, 2, 3$. If, say, $\delta_3 = 0$, then apply part (1) to the pair (A_2, A_3) to create δ_2 vertex-disjoint stars with centers in A_2. Let Z_2 be the set of the centers. Apply part (1) to $(A_1, A_2 - Z_2)$ and we can find δ_1 vertex-disjoint stars with centers in A_1 if $2\epsilon < (h^2+h)^{-1}$.
So, we may assume that $\delta_i > 0$ for $i = 1, 2, 3$. Note that if it is possible to construct $\delta_1 + \delta_2$ disjoint copies of $K_{1,h}$ in (A_1, A_2) with centers, $Z_1 \subset A_1$, then we can finish by applying part (1). To see this, apply part (1) to $(A_3, A_1 - Z_1)$, with $3\epsilon < (h^2+h)^{-1}$, creating δ_3 stars with centers $Z_3 \subset A_3$. Then apply part (1) to $(A_2, A_3 - Z_3)$. (Here, we need $2\epsilon < (h^2+h)^{-1}$.) There will be δ_1 stars remaining in (A_1, A_2) which are vertex-disjoint from the rest.
So, we will assume that it is not possible to create $\delta_1 + \delta_2$ vertex-disjoint copies of $K_{1,h}$ in (A_1, A_2) with centers in A_1. That means there is an $S \subset A_1$ and a $T \subset A_2$ such that $|S| < \delta_1 + \delta_2$, $|T| = h|S|$ and every vertex in $A_1 - S$ is adjacent to at most $h-1$ vertices in $A_2 - T$.
Now apply part (1) to $(A_3, A_1 - S)$ to obtain δ_3 vertex-disjoint copies of $K_{1,h}$ with centers $Z_3 \subset A_3$. (Here, we need $3\epsilon < (h^2+h)^{-1}$.) Next, apply part (1) to $(A_2, A_3 - Z_3)$ to obtain δ_2 vertex-disjoint copies of $K_{1,h}$ with centers $Z_2 \subset A_2$. (Here, we need $2\epsilon < (h^2+h)^{-1}$.) Finally, apply part (1) to $(A_1, A_2 - (Z_2 \cup T))$ to obtain δ_1 vertex-disjoint copies of $K_{1,h}$ with centers $Z_1 \subset A_1$. (Here, we need $(2h+2)\epsilon < (h^2+h)^{-1}$.) But, because no vertex in $A_1 - S$ is adjacent to h vertices in $A_2 - (Z_2 \cup T)$, it must be the case that $Z_1 \subset S$ and our $\delta_1 + \delta_2 + \delta_3$ copies of $K_{1,h}$ are, indeed, vertex-disjoint.

□

Lemma 3 is used to find a copy of $K_{1,h,h}$ in a tripartite graph $(B_1, B_2, B_2; E)$. If a $K_{1,h,h}$ cannot be found, then the graph must be approximately $\Theta_{3\times 2}(M)$.

Proof of Lemma 3 We can first apply the following theorem of Erdős, Frankl and Rödl [9]:

Theorem 6 *For every $\epsilon' > 0$ and graph F, there is a constant n_0 such that for any graph G of order $n \geq n_0$, if G does not contain F as a subgraph, then G contains a set E' of at most $\epsilon' n^2$ edges such that $G - E'$ contains no K_r with $r = \chi(F)$.*

Here, $F = K_{1,h,h}$ and $r = 3$.

Let us remove at most $\epsilon'(6M)^2$ edges from G so that it becomes triangle-free. In doing so, some vertices might be nonadjacent to many more vertices than before. We want to remove such vertices so that we can apply Proposition 3, which appeared in [21] and is rephrased below:

Proposition 3 *For a Δ small enough, there exists $\epsilon'' > 0$ such that if H is a tripartite graph with at least $2\left(1-\epsilon''\right)t$ vertices in each vertex class and each vertex is nonadjacent to at most $\left(1+\epsilon''\right)t$ vertices in each of the other classes. Furthermore, let H contain no triangles. Then, each vertex class is of size at most $2\left(1+\epsilon''\right)t$ and H is Δ-approximately $\Theta_{3\times 2}(t)$.*

For $\epsilon'' \gg \epsilon'$, at least $2(1-\epsilon'')M$ vertices are nonadjacent to at most $(1+\epsilon'')M$ vertices in each of the other classes. Otherwise, we would have had to delete a total of at least $\Omega(\epsilon'')M$ edges incident to each of these vertices, of which there would be at least $\Omega(\epsilon'')M$. But this means deleting $\Omega((\epsilon'')^2)M^2$ edges, which is more than $\epsilon'(6M)^2$.

So we apply Proposition 3. Thus, G is approximately $\Theta_{3\times 2}(M)$, and so the lemma follows. □

Lemma 4 is used to find a $K_{h,h}$-factor in a tripartite graph $(B_1, B_2, B_2; E)$. If the factor cannot be found, then the graph has a structure like columns 2 and 3 of the diagram in Fig. 3.

Proof of Lemma 4 Let ϵ' be chosen such that $\epsilon' \ll \delta$.

For this lemma, we partition the possibilities according to whether the pairs (B_i, B_j) are approximately $\Theta_{2\times 2}(T_1)$. That is, there are two pairs of sets of size T_1 which have density less than ϵ'. Minimality gives the rest.

In addition, we say that graphs $\Theta_{2\times 2}(T_1)$ *coincide* if (B_i, B_j) and (B_j, B_k) are approximately $\overline{B}_i \subseteq B_i$, $\overline{B}_j \subseteq B_j$, $\overline{B}_k \subseteq B_k$, all of size T_1, such that both $(\overline{B}_i, \overline{B}_j)$ and $(\overline{B}_j, \overline{B}_k)$ have density less than ϵ'. Note that this means that $(B_i - \overline{B}_i, B_j - \overline{B}_j)$ and $(B_i - \overline{B}_i, B_j - \overline{B}_j)$

Case 1 No pair is $\Theta_{2\times 2}(T_1)$. For each distinct $i, j, k \in \{1, 2, 3\}$, partition B_i into two pieces, $B_i[j]$ and $B_i[k]$ with $|B_i[j]| = T_j$ and $|B_i[k]| = T_k$. If this partition is done uniformly at random, then with probability approaching 1, each vertex in $B_i[k]$ is adjacent to at least $(1/2 - \epsilon^{1/2})T_k$ vertices in $B_j[k]$. So there exists a partition such that each vertex in B_i is adjacent to at least $(1/2 - \epsilon^{1/2})T_1$ vertices in each of the pieces $B_j[k]$, $j, k \neq i$ and such that the pair $(B_2[1], B_3[1])$ fails to contain a subpair with $\lfloor T_1/2 \rfloor$ vertices in each part and density at most $\epsilon^{1/3}$.

The vertices that are reserved will have to be placed in the proper set. For example, if a reserved $K_{h,h}$ is in the pair (B_i, B_j), then those vertices will need to be in the pair $(B_i[k], B_j[k])$. So, we exchange vertices in $B_i[k]$ for vertices in $B_i[j]$ so that reserved vertices are in the proper place. At most $4(\epsilon + \epsilon)T_1$ vertices are either reserved or moved in each set $B_i[j]$. After such exchanges occur, place the moved vertices into vertex-disjoint copies of $K_{h,h}$ that lie entirely within the given pairs. This can be done because each vertex not in B_i is adjacent to almost half of the vertices in both $B_i[j]$ and $B_i[k]$.

Consider what remains of these sets. The number of vertices is still divisible by h and at most $8h(\epsilon)T_1$ have been placed into these copies of $K_{h,h}$. We look for a perfect

$K_{h,h}$-factor in each of the pairs $(B_1[3], B_2[3])$, $(B_1[2], B_3[2])$ and $(B_2[1], B_3[1])$. Recall that each of these pairs has minimum degree at least $(1/2-\epsilon^{1/2})T_1$. Utilizing a lemma in [28]—stated as Lemma 5 in Sect. 3.2 above—we are able to find such a factor unless at least one of those pairs is $\alpha(\epsilon^{1/2})$-approximately $\Theta_{2\times 2}(T_1/2)$. (Minimality gives the other sparse pair.)

Lemma 6 says that if random selections give a graph that is approximately $\Theta_{2\times 2}(T_1/2)$, then the original graph was, too. So, along with Lemma 5, it establishes that if, after moving our vertices, we are unable to complete our $K_{h,h}$-cover in $(B_i[k], B_j[k])$ with nontrivial probability, then the pair (B_i, B_j) is ϵ'-approximately $\Theta_{2\times 2}(T_1)$, where $\epsilon' = \beta(\alpha(\epsilon^{1/2}))$.

Since none of the pairs is ϵ'-approximately $\Theta_{2\times 2}(T_1)$, we can find the required factor of (B_1, B_2, B_3) by copies of $K_{h,h}$.

Case 2 Exactly one pair is $\Theta_{2\times 2}(T_1)$. Here, we will assume that $B_1 = \overline{B}_1 \cup \widehat{B}_1$ and $B_2 = \overline{B}_2 \cup \widehat{B}_2$, where $|\overline{B}_1| = |\widehat{B}_2| = T_1$ and $d(\overline{B}_1, \widehat{B}_2), d(\widehat{B}_1, \overline{B}_2) \le \epsilon'$. A random partition of B_1 into pieces, with probability approaching 1 as T_1 approaches infinity, will partition $\overline{B}_1$ into two approximately equal pieces. In particular, let the *typical vertices* in $\overline{B}_1$ be those that are nonadjacent to at most $(\epsilon')^{1/2}T_1$ in $\widehat{B}_2$. There are at most $(\epsilon')^{1/2}T_1$ such vertices. A similar conclusion can be drawn from $\overline{B}_2$, $\widehat{B}_1$ and $\widehat{B}_2$.

In this case, we randomly partition B_1, B_2 and B_3 into the sets $B_i[k]$ as prescribed. Exchange the vertices as we have done above and complete both the reserved and exchanged vertices to form copies of $K_{h,h}$. This encompasses at most $8h\epsilon T_1$ vertices. Exchange vertices in $B_1[3]$ with vertices in $B_1[2]$ and vertices in $B_2[3]$ with vertices in $B_2[1]$ so that there are exactly $h\lfloor T_1/(2h)\rfloor$ typical vertices of $\overline{B}_1$ in $B_1[3]$ and $h\lfloor T_1/(2h)\rfloor$ typical vertices of $\widehat{B}_2$ in $B_2[3]$. Let the rest of the vertices, not matched into a $K_{h,h}$, in $B_1[3]$ be typical vertices in $\widehat{B}_1$ and the rest of the vertices in $B_2[3]$ be typical in $\overline{B}_2$. Using Proposition 2(1) on each pair of sets of typical vertices in $(B_1[3], B_2[3])$ will easily have a $K_{h,h}$-factor. With ϵ' small enough, we can guarantee that at most $(\epsilon')^{1/3}T_1$ vertices in $(B_1[2], B_3[2])$ and $(B_2[1], B_3[1])$ were moved. Applying Lemmas 5 and 6, and the fact that no pair other than (B_1, B_2) can be ϵ'-approximately $\Theta_{2\times 2}(T_1)$, we conclude that the pairs $(B_1[2], B_3[2])$ and $(B_2[1], B_3[1])$ can be completed to $K_{h,h}$-factors.

Case 3 Exactly two pairs are $\Theta_{2\times 2}(T_1)$, *which do not coincide.* Let the pairs in question be (B_1, B_2) and (B_2, B_3). Let the dense pairs in the subgraph induced by (B_1, B_2) be $(\overline{B}_1, \overline{B}_2)$ and $(\widehat{B}_1, \widehat{B}_2)$. Let the dense pairs in (B_2, B_3) be $(\mathring{B}_2, \mathring{B}_3)$ and $(\ddot{B}_2, \ddot{B}_3)$. Moreover, since the pairs fail to coincide, we can conclude that the intersection of the typical vertices of $\overline{B}_2$ with the typical vertices of each of $\mathring{B}_2$ and $\ddot{B}_2$ is at least $(\epsilon')^{1/4}T_1$ and similarly for $\widehat{B}_2$.

Once again, we randomly partition the vertices in B_1, B_2 and B_3 and move vertices so as to ensure that the reserved vertices and the vertices exchanged for them are placed into vertex-disjoint copies of $K_{h,h}$. Our concern at this point is the vertices in B_2.

Consider the vertices in $(B_1[3], B_2[3])$. Approximately half are typical vertices of $\overline{B}_2$ and approximately half are typical vertices of $\widehat{B}_2$. Take each non-typical vertex in $B_1[3]$ and in $B_2[3]$, match them with a copy of $K_{h,h}$ in the pair $(B_1[3], B_2[3])$ and remove them. Do the same for vertices in $B_2[1]$ that are not typical in $\mathring{B}_2$ or $\ddot{B}_2$ and in $B_3[1]$ that are not typical in $\mathring{B}_3$ or $\ddot{B}_3$. Remove those copies of $K_{h,h}$ also.

Observe that there are at least $\epsilon^{1/4}t_1/4$ vertices in each intersection of $\overline{B}_2$ or $\widehat{B}_2$ with $\mathring{B}_2$ or $\ddot{B}_2$ and with $B_2[3]$ or $B_2[1]$.

First, move a vertices from $\overline{B}_2 \cap \mathring{B}_2 \cap B_2[3]$ to $\overline{B}_2 \cap \mathring{B}_2 \cap B_2[1]$ to make $|\overline{B}_2 \cap B_2[3]|$ divisible by h. Second, move $a+b$ vertices from $\widehat{B}_2 \cap \mathring{B}_2 \cap B_2[1]$ to $\widehat{B}_2 \cap \mathring{B}_2 \cap B_2[3]$ to make $|\mathring{B}_2 \cap B_2[1]|$ divisible by h. Third, move $a+b+c$ vertices from $\widehat{B}_2 \cap \ddot{B}_2 \cap B_2[3]$ to $\widehat{B}_2 \cap \ddot{B}_2 \cap B_2[1]$. This will make both $|\widehat{B}_2 \cap B_2[3]|$ and $|\ddot{B}_2 \cap B_2[1]|$ divisible by h.

Here a, b and c are the remainders of $|\overline{B}_2 \cap B_2[3]|$, $|\mathring{B}_2 \cap B_2[1]|$ and $|\widehat{B}_2 \cap B_2[3]|$, respectively, when each is divided by h. Observe that both $|\overline{B}_2 \cap B_2[3]| + |\widehat{B}_2 \cap B_2[3]|$ and $|\mathring{B}_2 \cap B_2[1]| + |\ddot{B}_2 \cap B_2[1]|$ are divisible by h.

Finally, we exchange vertices in $\overline{B}_1 \cap B_1[3]$ with those in $\overline{B}_1 \cap B_1[2]$ so that $|\overline{B}_1 \cap B_1[3]| = |\overline{B}_2 \cap B_2[3]|$ and similarly for $\widehat{B}_2$. Also, exchange vertices in $\mathring{B}_3 \cap B_3[1]$ with those in $\mathring{B}_3 \cap B_3[2]$ so that $|\mathring{B}_3 \cap B_3[1]| = |\mathring{B}_2 \cap B_2[1]|$ and similarly for $\ddot{B}_2$.

Then, in $(\overline{B}_1 \cap B_1[3], \overline{B}_2 \cap B_2[3])$, first greedily place each moved vertex into copies of $K_{h,h}$ and then finish the factor via Proposition 2(1). Do the same for $\left(\widehat{B}_1 \cap B_1[3], \widehat{B}_2 \cap B_2[3]\right)$, $\left(\mathring{B}_2 \cap B_2[1], \mathring{B}_3 \cap B_3[1]\right)$ and $\left(\mathring{B}_2 \cap B_2[1], \mathring{B}_3 \cap B_3[1]\right)$.

Finally, we can complete the factor of $(B_1[2], B_3[2])$ because if it is not possible, Lemmas 5 and 6 would require (B_1, B_3) to be approximately $\Theta_{2\times 2}(T_1)$, excluded by this case.

Case 4 Three pairs are $\Theta_{2\times 2}(T_1)$, *none of which coincide.* Let the dense pairs in (B_1, B_2) be $(\overline{B}_1, \overline{B}_2)$ and $(\widehat{B}_1, \widehat{B}_2)$. Let the dense pairs in (B_2, B_3) be $(\mathring{B}_2, \mathring{B}_3)$ and $(\ddot{B}_2, \ddot{B}_3)$. Let the dense pairs in (B_1, B_3) be $(B_1^\sharp, B_3^\sharp)$ and $(B_1^\flat, B_3^\flat)$. Moreover, since the pairs fail to coincide, we can conclude that the intersection of the typical vertices of one set of sparse pairs with the typical vertices of another is at least $(\epsilon')^{1/4}T_1$.

Partition B_1, B_2 and B_3 into appropriately-sized sets as before, uniformly at random. The degree conditions hold with high probability as before. Take non-typical vertices and complete them greedily to place them in vertex-disjoint copies of $K_{h,h}$ within each of the pairs $(B_1[3], B_2[3])$, $(B_2[1], B_3[1])$ and $(B_1[2], B_3[2])$. Remove these copies of $K_{h,h}$ from the graph.

Let M be the largest multiple of h less than or equal to the size of the intersection of what remains of any sparse set (*i.e.*, $\overline{B}_i$, $\widehat{B}_i$, $\mathring{B}_i$, $\ddot{B}_i$, $B_i^\sharp$, $B_i^\flat$) with a set of the form $B_i[k]$.

We can move vertices as in Case 3 by letting $a = |\overline{B}_2 \cap B_2[3]| - M$, $b = |\mathring{B}_2 \cap B_2[1]| - M$ and $c = |\widehat{B}_2 \cap B_2[3]| + M - T_3$, which is also equal to $T_1 - M - a - b - |\ddot{B}_2 \cap B_2[1]|$. We can perform similar operations to guarantee that, among the vertices that remain in the graph, that

$$\begin{aligned} M &= \left|\overline{B}_1 \cap B_1[3]\right| = \left|\overline{B}_2 \cap B_2[3]\right| = \left|\mathring{B}_2 \cap B_2[1]\right| = \left|\mathring{B}_3 \cap B_3[1]\right| \\ &= \left|B_1^\sharp \cap B_1[2]\right| = \left|B_3^\sharp \cap B_3[2]\right| \end{aligned}$$

The fact that the pairs do not coincide ensures that there are enough vertices to make these moves.

Place the moved vertices into vertex-disjoint copies of $K_{h,h}$ and finish the factor via Proposition 2(1).

Case 5 There are at least two pairs which are $\Theta_{2\times 2}(T_1)$ *and which coincide.* This is exactly the exceptional case stated in the lemma and without loss of generality the pairs $(A_1^{(2)}, A_2^{(2)})$ and $(A_2^{(2)}, A_3^{(2)})$ are those that witness the coincidence of the copies of $\Theta_{2\times 2}(T_1)$. □

Acknowledgements The authors would like to acknowledge and thank the Department of Mathematics, Statistics, and Computer Science at the University of Illinois at Chicago for their supporting Martin via a visitor fund. The authors also wish to acknowledge the support of National Security Agency Grant H98230-08-1-0015 for Hogenson for a summer research assistantship.

References

1. Alon, N., Yuster, R.: Almost H-factors in dense graphs. Graphs Combin. **8**(2), 95–102 (1992)
2. Alon, N., Yuster, R.: H-factors in dense graphs. J. Combin. Theory Ser. B **66**(2), 269–282 (1996)
3. Bollobás, B.: Extremal graph theory. Dover Publications, Inc., Mineola, NY (2004). Reprint of the 1978 original
4. Bush, A., Zhao, Y.: Minimum degree thresholds for bipartite graph tiling. J. Graph Theory **70**(1), 92–120 (2012)
5. Catlin, P.: On the Hajnal–Szemerédi theorem on disjoint cliques. Utilitas Math. **17**, 163–177 (1980)
6. Corrádi, K., Hajnal, A.: On the maximal number of independent circuits in a graph. Acta Math. Acad. Sci. Hungar. **14**, 423–439 (1963)
7. Csaba, B., Mydlarz, M.: Approximate multipartite version of the Hajnal–Szemerédi theorem. J. Combin. Theory Ser. B **102**(2), 395–410 (2012)
8. Czygrinow, A., DeBiasio, L.: A note on bipartite graph tiling. SIAM J. Discrete Math. **25**(4), 1477–1489 (2011)
9. Erdős, P., Frankl, P., Rödl, V.: The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent. Graphs Combin. **2**(2), 113–121 (1986)
10. Fischer, E.: Variants of the Hajnal–Szemerédi theorem. J. Graph Theory **31**(4), 275–282 (1999)
11. Hajnal, A., Szemerédi, E.: Proof of a conjecture of P. Erdős. In: Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), pp. 601–623. North-Holland, Amsterdam (1970)
12. Han, J., Zhao, Y.: On multipartite Hajnal–Szemerédi theorems. Discrete Math. **313**(10), 1119–1129 (2013)
13. Hladký, J., Schacht, M.: Note on bipartite graph tilings. SIAM J. Discrete Math. **24**(2), 357–362 (2010)
14. Keevash, P., Mycroft, R.: A geometric theory for hypergraph matching. Mem. Amer. Math. Soc. **233**(1098), vi+95 (2015)
15. Keevash, P., Mycroft, R.: A multipartite Hajnal–Szemerédi theorem. J. Combin. Theory Ser. B **114**, 187–236 (2015)
16. Komlós, J.: Tiling Turán theorems. Combinatorica **20**(2), 203–218 (2000)
17. Komlós, J., Sárközy, G., Szemerédi, E.: Proof of the Alon–Yuster conjecture. Discrete Math. **235**(1–3), 255–269 (2001) **(Combinatorics (Prague, 1998))**
18. Kühn, D., Osthus, D.: Embedding large subgraphs into dense graphs. In: Surveys in combinatorics 2009, London Math. Soc. Lecture Note Ser., vol. 365, pp. 137–167. Cambridge Univ. Press, Cambridge (2009)
19. Kühn, D., Osthus, D.: The minimum degree threshold for perfect graph packings. Combinatorica **29**(1), 65–107 (2009)
20. Lo, A., Markström, K.: A multipartite version of the Hajnal–Szemerédi theorem for graphs and hypergraphs. Combin. Probab. Comput. **22**(1), 97–111 (2013)
21. Magyar, C., Martin, R.: Tripartite version of the Corrádi–Hajnal theorem. Discrete Math. **254**(1–3), 289–308 (2002)
22. Martin, R.R., Skokan, J.: Asymptotic multipartite version of the Alon–Yuster theorem. J. Combin. Theory Ser. B. **127**, 32–52 (2017)
23. Martin, R., Szemerédi, E.: Quadripartite version of the Hajnal–Szemerédi theorem. Discrete Math. **308**(19), 4337–4360 (2008)

24. Martin, R., Zhao, Y.: Tiling tripartite graphs with 3-colorable graphs. Electron. J. Combin. **16**(1), 109 (2009) **(Research Paper)**
25. Shokoufandeh, A., Zhao, Y.: Proof of a tiling conjecture of Komlós. Random Struct. Algorithms **23**(2), 180–205 (2003)
26. Szemerédi, E.: Regular partitions of graphs. In: Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, pp. 399–401. CNRS, Paris (1978)
27. Wang, H.: Vertex-disjoint hexagons with chords in a bipartite graph. Discrete Math. **187**(1–3), 221–231 (1998)
28. Zhao, Y.: Bipartite graph tiling. SIAM J. Discrete Math. **23**(2), 888–900 (2009)