WEAK SUBCONVEXITY WITHOUT A RAMANUJAN HYPOTHESIS

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Abstract

We describe a new method to obtain weak subconvexity bounds for L-functions with mild hypotheses on the size of the Dirichlet coefficients. We verify these hypotheses for all automorphic L-functions and (with mild restrictions) the Rankin–Selberg L-functions attached to two automorphic representations. The proof relies on a new unconditional log-free zero density estimate for Rankin–Selberg L-functions.

1. Statement of results

In [29], the first author obtained a weak subconvexity result bounding central values of a large class of L-functions, assuming a weak Ramanujan hypothesis on the size of Dirichlet series coefficients of the L-function. If C denotes the analytic conductor of the L-function in question, then $C^{1/4}$ is the size of the convexity bound, and the weak subconvexity bound established there was of the form $C^{1/4}/(\log C)^{1-\epsilon}$. In this paper, we establish a weak subconvexity bound of the shape $C^{1/4}/(\log C)^{\delta}$ for some small $\delta > 0$, but with a much milder hypothesis on the size of the Dirichlet series coefficients. In particular, our results will apply to all automorphic L-functions and (with mild restrictions) to the Rankin–Selberg L-functions attached to two automorphic representations.

In order to make clear the scope and limitations of our results, we axiomatize the properties of L-functions that we need. In Section 2, we shall discuss how automorphic L-functions and Rankin–Selberg L-functions fit into this framework. Let $m \geq 1$ be a natural number. We now describe axiomatically a class of L-functions, which we shall denote by $\mathcal{S}(m)$.

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1.1. Dirichlet series and Euler product

The functions $L(s,\pi)$ appearing in the class $\mathcal{S}(m)$ will be given by a Dirichlet series and Euler product

$$\begin{aligned} L(s,\pi) &= \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} = \prod_p L_p(s,\pi), \\ L_p(s,\pi) &= \prod_{j=1}^{m} \Big(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\Big)^{-1} = \sum_{j=0}^{\infty} \frac{a_\pi(p^j)}{p^{js}}, \end{aligned} \tag{1.1}$$

with both the series and the product converging absolutely for $\mathrm{Re}(s) > 1$. It will also be convenient for us to write

$$\log L_p(s,\pi) = \sum_{k=1}^{\infty} \frac{\lambda_\pi(p^k)}{kp^{ks}}, \quad \text{where } \lambda_\pi(p^k) = \sum_{j=1}^{m} \alpha_{j,\pi}(p)^k. \tag{1.2}$$

Setting $\lambda_\pi(n) = 0$ if n is not a prime power, we have

$$-\frac{L'}{L}(s,\pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)\Lambda(n)}{n^s}, \qquad \text{and} \qquad \log L(s,\pi) = \sum_{n=2}^{\infty} \frac{\lambda_\pi(n)\Lambda(n)}{n^s \log n}. \tag{1.3}$$

1.2. Functional equation

Write

$$L_\infty(s,\pi) = N_\pi^{s/2} \pi^{-ms/2} \prod_{j=1}^{m} \Gamma\Big(\frac{s + \mu_\pi(j)}{2}\Big), \tag{1.4}$$

where $N_\pi \geq 1$ is known as the "conductor" of the L-function and the $\mu_\pi(j)$ are complex numbers. We suppose that there is an integer $0 \leq r = r_\pi \leq m$ such that the completed L-function $s^r(1-s)^r L(s,\pi) L_\infty(s,\pi)$ extends to an entire function of order 1, and satisfies the functional equation

$$s^r(1-s)^r L(s,\pi) L_\infty(s,\pi) = \kappa_\pi s^r (1-s)^r L(1-s,\widetilde{\pi}) L_\infty(1-s,\widetilde{\pi}). \tag{1.5}$$

Here κ_π is a complex number with $|\kappa_\pi| = 1$, and

$$L(s,\widetilde{\pi}) = \sum_{n=1}^{\infty} \frac{\overline{a_\pi(n)}}{n^s}, \qquad L_\infty(s,\widetilde{\pi}) = N_\pi^{s/2} \pi^{-ms/2} \prod_{j=1}^{m} \Gamma\Big(\frac{s + \overline{\mu_\pi(j)}}{2}\Big). \tag{1.6}$$

We suppose that r has been chosen such that the completed L-function does not vanish at $s = 1$ and $s = 0$. Thus, if $L(s,\pi)$ has a pole at $s = 1$, then we are assuming that the order of this pole is at most m, and r is taken to be the order of the pole. If

$L(s,\pi)$ has no pole at $s=1$, then we take $r=0$ and are making the assumption that the $L(1,\pi)\neq 0$. In our work, a key measure of the "complexity" of the L-function $L(s,\pi)$ is the "analytic conductor," which is defined to be

$$C(\pi)=N_\pi\prod_{j=1}^{m}\bigl(1+|\mu_\pi(j)|\bigr). \tag{1.7}$$

1.3. Bounds toward the generalized Ramanujan and Selberg conjectures

The absolute convergence of the Euler product in (1.1) implicitly includes the assumption that $|\alpha_{j,\pi}(p)|<p$ for all p and j. Further, the Euler product shows that $L(s,\pi)$ is nonzero in $\mathrm{Re}(s)>1$, which implies that $\mathrm{Re}(\mu_\pi(j))>-1$ for all j (otherwise, there would be a trivial zero of $L(s,\pi)$ in $\mathrm{Re}(s)>1$ to compensate for a pole of $\Gamma((s+\mu_\pi(j))/2)$). We impose a modest strengthening of these estimates. Namely, we assume that for all $1\le j\le m$,

$$|\alpha_{j,\pi}(p)|\le p^{1-1/m},\qquad \mathrm{Re}\bigl(\mu_\pi(j)\bigr)\ge -(1-1/m). \tag{1.8}$$

The widely believed generalized Ramanujan and Selberg conjectures for automorphic L-functions state that the bounds in (1.8) hold with $1-1/m$ replaced by 0. While these conjectures are still open, the weak bounds in (1.8) are known both for the L-functions associated to automorphic representations and their Rankin–Selberg convolutions. We could also weaken (1.8) further by replacing $1-1/m$ with $1-\delta$ for some $\delta>0$, but the present formulation is convenient and includes all L-functions of interest to us.

1.4. Rankin–Selberg and Brun–Titchmarsh bounds on $\lambda_\pi(n)$

Our final hypothesis prescribes two mild average bounds on $|\lambda_\pi(n)|$, which can be verified by Rankin–Selberg theory for the class of L-functions associated to automorphic representations and their Rankin–Selberg convolutions. First, we assume that for all $\eta>0$,

$$\sum_{n=1}^{\infty}\frac{|\lambda_\pi(n)|\Lambda(n)}{n^{1+\eta}}\le\frac{m}{\eta}+m\log C(\pi)+O(m^2). \tag{1.9}$$

Second, we assume that, for all $T\ge 1$,

$$\sum_{x<n\le xe^{1/T}}|\lambda_\pi(n)|\Lambda(n)\ll_m\frac{x}{T},\quad \text{provided } x\gg_m\bigl(C(\pi)T\bigr)^{144m^3}. \tag{1.10}$$

There is considerable latitude in formulating the conditions (1.9) and (1.10); for example, we could have chosen the range for x in (1.10) differently. The specific choice made here is based on the applicability of these conditions to automorphic

L-functions. When T is of constant size, the criterion (1.10) may be viewed as a Chebyshev-type estimate for $|\lambda(n)|$ (generalizing $\sum_{n\le x}\Lambda(n)\ll x$), while for larger T the criterion (1.10) is an analogue of the classical Brun–Titchmarsh inequality

$$\sum_{x<n\le x+h}\Lambda(n)\ll_\epsilon h,\quad \text{for all } x\ge h\ge x^\epsilon. \tag{1.11}$$

We denote by $\mathcal{S}(m)$ the class of L-functions satisfying the properties laid out in articles 1.1 to 1.4 above; see (1.1)–(1.10). Before stating our results, we introduce the quantity

$$N_\pi(\sigma,T):=\#\big\{\rho=\beta+i\gamma\colon L(\rho,\pi)=0,\beta>\sigma,|\gamma|\le T\big\}, \tag{1.12}$$

which arises in the study of "zero density estimates."

THEOREM 1.1
If $L(s,\pi)$ is an L-function in the class $\mathcal{S}(m)$ and $0\le\delta<1/2$, then

$$\log\big|L(1/2,\pi)\big|\le\Big(\frac{1}{4}-10^{-9}\delta\Big)\log C(\pi)+10^{-7}\delta N_\pi(1-\delta,6)+2\log\big|L(3/2,\pi)\big|+O(m^2).$$

Theorem 1.1 adds to a long line of investigations relating the size of L-functions to the distribution of their zeros. For example, it is well known that the generalized Riemann hypothesis (GRH) implies the generalized Lindelöf hypothesis. One could weaken the assumption of the GRH, and establish (as Backlund did originally for $\zeta(s)$) that if almost all the zeros of the L-function up to height 1 are in the region $\mathrm{Re}(s)<1/2+\epsilon$, then the Lindelöf bound $L(1/2,\pi)\ll C(\pi)^\epsilon$ would follow. In contrast, Theorem 1.1 states that the more modest assumption that not too many of the zeros of $L(s,\pi)$ are very close to the line $\mathrm{Re}(s)=1$ leads to a subconvex bound for $L(1/2,\pi)$ (which is a modest form of the Lindelöf bound). For recent related work in the context of character sums and zeros of Dirichlet L-functions, see [12]. The proof of Theorem 1.1 is a refinement of an argument of Heath-Brown [13] to prove sharp convexity bounds for L-values.

To obtain from Theorem 1.1 a genuine subconvexity bound of the form $L(1/2,\pi)\ll C(\pi)^{1/4-\delta}$ for some $\delta>0$, we would need a zero density estimate of the form $N_\pi(1-\delta,6)\le 10^{-4}\log C(\pi)$, which we are unable to establish for any fixed $\delta>0$. However, one can establish a "log-free" zero density estimate which will permit values of δ of size $(\log\log C(\pi))/\log C(\pi)$. This will then lead to the weak subconvexity bound where a power of $\log C(\pi)$ is saved over the convexity bound.

THEOREM 1.2
Let $L(s,\pi) \in \mathcal{S}(m)$ and $T \geq 1$. For all $1/2 \leq \sigma \leq 1$,

$$N_\pi(\sigma, T) \ll_m \big(C(\pi)T\big)^{10^7 m^3(1-\sigma)}.$$

Log-free zero density estimates have a long history, going back to Linnik's pioneering work on the least prime in arithmetic progressions. Our proof of Theorem 1.2 follows an argument of Gallagher, based on Turán's power sum method. A key feature is the formulation of hypotheses (1.9) and (1.10), which are L^1-bounds that can be verified for L-functions associated to automorphic representations and their Rankin–Selberg convolutions. Thus, Theorem 1.2 applies to a larger class of L-functions than the earlier log-free zero density estimates established by (for example) Kowalski and Michel [18], Motohashi [24], Akbary and Trudgian [1], and Lemke Oliver and Thorner [19]. We have not made any attempt to optimize the exponent $10^7 m^3$, but our argument does not seem to yield an exponent independent of m.

Combining Theorem 1.1 and Theorem 1.2, we deduce the following bound for $L(1/2,\pi)$.

COROLLARY 1.3
Let $L(s,\pi) \in \mathcal{S}(m)$. Then

$$\big|L(1/2,\pi)\big| \ll_m \big|L(3/2,\pi)\big|^2 \frac{C(\pi)^{1/4}}{(\log C(\pi))^{1/(10^{17}m^3)}}.$$

In the above corollary, one should expect the term $|L(3/2,\pi)|$ (which is evaluated in the region of absolute convergence) to be bounded, in which case the corollary furnishes a weak subconvexity bound. The boundedness of $|L(3/2,\pi)|$ would follow, for example, from a stronger version of assumption (1.8), and we shall check that this holds for automorphic L-functions. For Rankin–Selberg convolutions of automorphic representations, we cannot give a satisfactory bound for the L-value at $3/2$ in complete generality. Compared to the work in [29], Corollary 1.3 extends considerably the class of L-functions for which a weak subconvexity bound may be established, but the power of $\log C(\pi)$ saved is smaller than in [29].

2. Applications to automorphic L-functions

In this section, we describe how the framework and results described in Section 1 apply to automorphic L-functions. We restrict attention to automorphic representations over $\mathbb{Q}$, and we let $\mathcal{A}(m)$ denote the set of all cuspidal automorphic representations of GL_m over $\mathbb{Q}$ with unitary central character. Here we give a brief description of the analytic properties of the standard L-functions associated to such

automorphic representations. Our goal is twofold: we wish to show that elements of $\mathcal{A}(m)$ give rise to L-functions in the class $\mathcal{S}(m)$, and also that if $\pi_1 \in \mathcal{A}(m_1)$ and $\pi_2 \in \mathcal{A}(m_2)$ then the Rankin–Selberg L-function $L(s, \pi_1 \times \pi_2)$ fits into the framework of $\mathcal{S}(m_1 m_2)$. For proofs and further discussion of the properties that we need, we refer to [11], [15], [23], or the surveys in Michel [22, Lecture 1] or Brumley [5, Section 1].

Properties 1.1 to 1.3 listed in Section 1 follow from the standard theory of automorphic forms, while Property 1.4 will require further discussion. Thus, given $\pi \in \mathcal{A}(m)$, its standard L-function $L(s, \pi)$ has a Dirichlet series, Euler product, and satisfies a functional equation, exactly as described in (1.1) to (1.6). Note also that here $\widetilde{\pi}$ denotes the representation which is contragredient to π. Concerning Property 1.3, for $\pi \in \mathcal{A}(m)$, it is known that

$$\big|\alpha_{j,\pi}(p)\big| \le p^{\theta_m}, \qquad \big|\mathrm{Re}\big(\mu_\pi(j)\big)\big| \le \theta_m, \tag{2.1}$$

where

$$\theta_m = \begin{cases} 0 & \text{if } m = 1, \\ 7/64 & \text{if } m = 2, \\ 1/2 - 1/(m^2+1) & \text{if } m \ge 3. \end{cases} \tag{2.2}$$

The bounds follow from the work of Luo, Rudnick, and Sarnak [21] for $m \ge 3$ and Kim and Sarnak [16, Appendix 2] for $m = 2$ in the unramified cases. The ramified cases are handled by Müller and Speh [25, Proposition 3.3] for $m \ge 3$ and Brumley and Blomer [3] for $m = 2$. The generalized Ramanujan and Selberg conjectures assert that θ_m may be taken as 0 in (2.1).

Now we turn to Rankin–Selberg L-functions. If $\pi_1 \in \mathcal{A}(m_1)$ and $\pi_2 \in \mathcal{A}(m_2)$ are two automorphic representations, then the Euler product and Dirichlet series expansions of the Rankin–Selberg L-function $L(s, \pi_1 \times \pi_2)$ are given by

$$L(s, \pi_1 \times \pi_2) = \sum_{n=1}^{\infty} \frac{a_{\pi_1 \times \pi_2}(n)}{n^s} = \prod_p \prod_{j_1=1}^{m_1} \prod_{j_2=1}^{m_2} \Big(1 - \frac{\alpha_{j_1,j_2,\pi_1\times\pi_2}(p)}{p^s}\Big)^{-1}.$$

Here we may index the parameters $\alpha_{j_1,j_2,\pi_1\times\pi_2}(p)$ in such a way that, for all $p \nmid N_{\pi_1} N_{\pi_2}$, one has

$$\alpha_{j_1,j_2,\pi_1\times\pi_2}(p) = \alpha_{j_1,\pi_1}(p)\alpha_{j_2,\pi_2}(p). \tag{2.3}$$

At the archimedean place, we write

$$L_\infty(s, \pi_1 \times \pi_2) = N_{\pi_1\times\pi_2}^{s/2} \pi^{-m_1 m_2 s/2} \prod_{j_1=1}^{m_1} \prod_{j_2=1}^{m_2} \Gamma\Big(\frac{s + \mu_{\pi_1\times\pi_2}(j_1, j_2)}{2}\Big). \tag{2.4}$$

If both π_1 and π_2 are unramified at infinity, one may write

$$\mu_{\pi_1\times\pi_2}(j_1,j_2)=\mu_{\pi_1}(j_1)+\mu_{\pi_2}(j_2). \tag{2.5}$$

See Lemma 2.1 below for a complete description of $\mu_{\pi_1\times\pi_2}(j_1,j_2)$ in the general case. As part of the Langlands functoriality conjectures, one expects that $\pi_1\times\pi_2$ corresponds to an automorphic representation of $GL(m_1m_2)$ (not necessarily cuspidal), but this remains unknown, apart from the work of Ramakrishnan [26] in the case $m_1=m_2=2$ and the work of Kim and Shahidi [17] in the case $m_1=2$ and $m_2=3$.

Properties 1.1 and 1.2 may thus be verified for Rankin–Selberg L-functions. As for Property 1.3, using (2.1) and (2.2), and proceeding as in [27, Appendix] (see also [4, Section 3] and [5, Section 1]), we obtain for all primes p,

$$\bigl|\alpha_{j_1,j_2,\pi_1\times\pi_2}(p)\bigr|\le p^{\theta_{m_1}+\theta_{m_2}},\qquad \operatorname{Re}\bigl(\mu_{\pi_1\times\pi_2}(j_1,j_2)\bigr)\ge-\theta_{m_1}-\theta_{m_2}. \tag{2.6}$$

The reader may also consult the explicit description of L_∞ given in Lemma 2.1 below, and the explicit description of L_p given by (A.6) in the Appendix.

So far we have discussed how automorphic L-functions and Rankin–Selberg L-functions satisfy Properties 1.1 and 1.3 of Section 1. To facilitate our discussion of Property 1.4, we require two lemmas.

LEMMA 2.1
If $\pi_1\in\mathcal{A}(m_1)$ and $\pi_2\in\mathcal{A}(m_2)$, then

$$C(\pi_1\times\pi_2)\le e^{O(m_1m_2)}C(\pi_1)^{m_2}C(\pi_2)^{m_1},$$

and

$$C(\pi_1\times\widetilde{\pi}_1)^{m_2^2}C(\pi_2\times\widetilde{\pi}_2)^{m_1^2}\le e^{O((m_1m_2)^2)}C(\pi_1\times\pi_2)^{4m_1m_2}.$$

Proof
We write $\pi\in\mathcal{A}(m)$ and $\pi'\in\mathcal{A}(m')$ instead of $\pi_1\in\mathcal{A}(m_1)$ and $\pi_2\in\mathcal{A}(m_2)$ to avoid having too many subscripts. Let

$$K_\pi=\prod_{j=1}^{m}\bigl(1+\bigl|\mu_\pi(j)\bigr|\bigr) \tag{2.7}$$

so that $C(\pi)=N_\pi K_\pi$. For a prime p, let $\operatorname{ord}_p(N_\pi)$ be the exponent of p in the prime factorization of N_π; in particular, $\operatorname{ord}_p(N_\pi)=0$ if and only if $p\nmid N_\pi$. Bushnell and Henniart proved that (see [6, Theorem 1] or [7, Corollary C])

$$\operatorname{ord}_p(N_{\pi\times\pi'})\le m'\cdot\operatorname{ord}_p(N_\pi)+m\cdot\operatorname{ord}_p(N_{\pi'})-\min\bigl\{\operatorname{ord}_p(N_\pi),\operatorname{ord}_p(N_{\pi'})\bigr\}, \tag{2.8}$$

and also that (see [7, Corollary B])

$$\frac{\operatorname{ord}_p(N_{\pi\times\pi'})}{m'm} \geq \frac{1}{2}\Big(\frac{\operatorname{ord}_p(N_{\pi\times\widetilde{\pi}})}{m^2} + \frac{\operatorname{ord}_p(N_{\pi'\times\widetilde{\pi}'})}{(m')^2}\Big).$$

These bounds imply

$$N_{\pi\times\pi'} \leq \frac{N_{\pi}^{m'} N_{\pi'}^{m}}{\gcd(N_\pi, N_{\pi'})} \qquad \text{and} \qquad N_{\pi\times\pi'} \geq N_{\pi\times\widetilde{\pi}}^{\frac{m'}{2m}} N_{\pi'\times\widetilde{\pi}'}^{\frac{m}{2m'}}.$$

We require corresponding bounds for $K_{\pi\times\pi'}$. Brumley [14, Appendix] proved that

$$K_{\pi\times\pi'} \leq e^{O(m'm)} K_{\pi}^{m'} K_{\pi'}^{m}.$$

It remains to establish the bound

$$K_{\pi\times\widetilde{\pi}} K_{\pi'\times\widetilde{\pi}'} \leq e^{O((m'm)^2)} K_{\pi\times\pi'}^{4m'm}. \tag{2.9}$$

In order to prove (2.9) regardless of the ramification at infinity, we use the archimedean case of the local Langlands correspondence as described by Müller and Speh [25, Proof of Lemma 3.1, $F = \mathbb{R}$]. We give a brief account of the archimedean factors. Langlands proved that there exist collections of irreducible representations $\{\varphi_i\}_{i\in\mathcal{I}}$ and $\{\varphi'_j\}_{j\in\mathcal{J}}$ of the Weil group $W_{\mathbb{R}}$ such that π_∞ and π'_∞ correspond to the direct sums $\bigoplus_{i\in\mathcal{I}}\varphi_i$ and $\bigoplus_{j\in\mathcal{J}}\varphi'_j$, respectively. Each irreducible representation φ of $W_{\mathbb{R}}$ is of dimension 1 or 2; furthermore, one has the factorizations

$$L_\infty(s,\pi) = \prod_{i\in\mathcal{I}} L(s,\varphi_i), \qquad L_\infty(s,\pi') = \prod_{j\in\mathcal{J}} L(s,\varphi'_j),$$

$$L_\infty(s,\pi\times\pi') = \prod_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}} L(s,\varphi_i\otimes\varphi'_j).$$

To describe further the L-functions above, it is convenient to define $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$.

(1) If φ is one-dimensional, then there exist $\nu\in\mathbb{C}$ and $\varepsilon\in\{0,1\}$ such that

$$L(s,\varphi) = \Gamma_{\mathbb{R}}(s+\nu+\varepsilon).$$

We define $K(\varphi) = 1 + |\nu+\varepsilon|$.

(2) If φ is two-dimensional, then there exist $k\in\mathbb{Z}$ and $\nu\in\mathbb{C}$ such that

$$L(s,\varphi) = \Gamma_{\mathbb{C}}\big(s+\nu+|k|/2\big).$$

We define $K(\varphi) = (1+|\nu+|k|/2|)(1+|\nu+|k|/2+1|)$.

In both of the above cases, Rudnick and Sarnak [27, Appendix A.3] proved that

$$\big|\mathrm{Re}(\nu)\big| < 1/2. \tag{2.10}$$

Müller and Speh also describe the L-functions associated to the tensor products $\varphi \otimes \varphi'$.

(1) If both φ and φ' are one-dimensional, then $\varphi \otimes \varphi'$ is one-dimensional and

$$L(s, \varphi \otimes \varphi') = \Gamma_{\mathbb{R}}(s + \nu + \nu' + \varepsilon_{\varphi\otimes\varphi'}),$$

where $\varepsilon_{\varphi\otimes\varphi'} \in \{0, 1\}$ and $\varepsilon_{\varphi\otimes\varphi'} \equiv \varepsilon + \varepsilon' (\mathrm{mod}\ 2)$. In this case, we define $K(\varphi \otimes \varphi') = 1 + |\nu + \nu' + \varepsilon_{\varphi\otimes\varphi'}|$.

(2) If φ is one-dimensional and φ' is two-dimensional, then $\varphi \otimes \varphi'$ is two-dimensional and

$$L(s, \varphi \otimes \varphi') = \Gamma_{\mathbb{C}}\big(s + \nu + \nu' + |k'|/2\big).$$

In this case, we define

$$K(\varphi \otimes \varphi') = \big(1 + |\nu + \nu' + |k'|/2|\big)\big(1 + |\nu + \nu' + |k'|/2 + 1|\big).$$

(3) Suppose that φ and φ' are two-dimensional. Then $\varphi \otimes \varphi'$ is the direct sum of two two-dimensional representations and

$$L(s, \varphi \otimes \varphi') = \Gamma_{\mathbb{C}}\big(s + \nu + \nu' + |k + k'|/2\big)\Gamma_{\mathbb{C}}\big(s + \nu + \nu' + |k - k'|/2\big).$$

In this case, we define

$$\begin{aligned} K(\varphi \otimes \varphi') = {} & \big(1 + |\nu + \nu' + |k + k'|/2|\big)\big(1 + |\nu + \nu' + |k + k'|/2 + 1|\big) \\ & \times \big(1 + |\nu + \nu' + |k - k'|/2|\big)\big(1 + |\nu + \nu' + |k - k'|/2 + 1|\big). \end{aligned}$$

These definitions give us a complete description of

$$K_{\pi\times\pi'} = \prod_{\substack{i\in\mathcal{J} \\ j\in\mathcal{J}}} K(\varphi_i \otimes \varphi'_j). \tag{2.11}$$

We now address (2.9). First, assume that both π and π' are unramified at infinity, in which case (2.5) holds. Suppose z_1, z_2, w_1, and w_2 are complex numbers all having real part $\geq -1/2$. We claim that

$$\frac{(1 + |z_1 + \overline{w_1}|)(1 + |z_2 + \overline{w_2}|)}{(1 + |z_1 + z_2|)(1 + |z_1 + w_2|)(1 + |w_1 + z_2|)(1 + |w_1 + w_2|)} \leq C \tag{2.12}$$

for some absolute constant C. The triangle inequality gives

$$1+|z_1+\overline{w_1}| = \left|1+z_1+z_2-(z_2-\overline{w_1})\right| \le 1+|z_1+z_2|+|z_2-\overline{w_1}|$$
$$\ll 1+|z_1+z_2|+|z_2+w_1|,$$

where the last estimate follows because the real parts of z_1 and w_1 are both bounded below by $-1/2$ so that $|z_2-\overline{w_1}| \le O(1)+|z_2+w_1|$. In the same way one sees that $1+|z_1+\overline{w_1}| \ll 1+|w_1+w_2|+|z_1+w_2|$, and two similar inequalities for $1+|z_2+\overline{w_2}|$ hold. Multiplying these four estimates together and taking square roots yields (2.12).

Apply (2.12) with $z_1=\mu_\pi(i_1)$, $w_1=\mu_\pi(i_2)$ and $z_2=\mu_{\pi'}(j_1)$, $w_2=\mu_{\pi'}(j_2)$, where $1\le i_1, i_2 \le m$ and $1\le j_1, j_2 \le m'$. Taking the product over all the inequalities so obtained, we arrive at (2.9) in the case when both π and π' are unramified at infinity.

If at least one of π and π' is ramified at infinity, then by (2.11), the bound (2.9) is equivalent to the bound

$$\prod_{\substack{i_1\in\mathcal{I}\\ j_1\in\mathcal{J}}}\prod_{\substack{i_2\in\mathcal{I}\\ j_2\in\mathcal{J}}}\frac{K(\varphi_{i_1}\otimes\widetilde{\varphi}_{i_2})K(\varphi'_{j_1}\otimes\widetilde{\varphi}'_{j_2})}{K(\varphi_{i_1}\otimes\varphi'_{j_1})K(\varphi_{i_1}\otimes\varphi'_{j_2})K(\varphi_{i_2}\otimes\varphi'_{j_1})K(\varphi_{i_2}\otimes\varphi'_{j_2})} \ll e^{O((m'm)^2)}. \tag{2.13}$$

If each of φ_{i_1}, φ_{i_2}, φ'_{j_1}, and φ'_{j_2} is one-dimensional, then we are led to the quotient

$$\frac{(1+|\nu_{i_1}+\overline{\nu_{i_2}}+\varepsilon_{\varphi_{i_1}\otimes\widetilde{\varphi}_{i_2}}|)}{(1+|\nu_{i_1}+\nu'_{j_1}+\varepsilon_{\varphi_{i_1}\otimes\varphi'_{j_1}}|)(1+|\nu_{i_1}+\nu'_{j_2}+\varepsilon_{\varphi_{i_1}\otimes\varphi'_{j_2}}|)}$$
$$\times\frac{(1+|\nu'_{j_1}+\overline{\nu'_{j_2}}+\varepsilon_{\varphi'_{j_1}\otimes\widetilde{\varphi}'_{j_2}}|)}{(1+|\nu_{i_2}+\nu'_{j_1}+\varepsilon_{\varphi_{i_2}\otimes\varphi'_{j_1}}|)(1+|\nu_{i_2}+\nu'_{j_2}+\varepsilon_{\varphi_{i_2}\otimes\varphi'_{j_2}}|))}.$$

Recall that $\varepsilon_{\varphi\otimes\varphi'}\in\{0,1\}$. In light of (2.10), this quotient is a mild perturbation of (2.12), and we conclude that it is absolutely bounded. Proceeding similarly for the other cases, we observe that the product in (2.13) is a product of mild perturbations of (2.12), each of which is absolutely bounded. This proves (2.13). □

LEMMA 2.2

Let $\pi_1\in\mathcal{A}(m_1)$ and $\pi_2\in\mathcal{A}(m_2)$. With the notation

$$\log L(s,\pi_1\times\pi_2)=\sum_{n=2}^{\infty}\frac{\lambda_{\pi_1\times\pi_2}(n)\Lambda(n)}{n^s\log n},$$

for all prime powers n, we have

$$\big|\lambda_{\pi_1\times\pi_2}(n)\big| \le \sqrt{\lambda_{\pi_1\times\widetilde{\pi}_1}(n)\lambda_{\pi_2\times\widetilde{\pi}_2}(n)} \le \frac{1}{2}\big(\lambda_{\pi_1\times\widetilde{\pi}_1}(n) + \lambda_{\pi_2\times\widetilde{\pi}_2}(n)\big).$$

Further, for any $\pi \in \mathcal{A}(m)$, we have

$$\big|\lambda_{\pi}(n)\big| \le \sqrt{\lambda_{\pi\times\widetilde{\pi}}(n)} \le \frac{1}{2}\big(1 + \lambda_{\pi\times\widetilde{\pi}}(n)\big).$$

If n is the power of an unramified prime p, then from (2.3) one may see that $\lambda_{\pi_1\times\pi_2}(n) = \lambda_{\pi_1}(n)\lambda_{\pi_2}(n)$, and that $\lambda_{\pi_1\times\widetilde{\pi}_1}(n) = |\lambda_{\pi_1}(n)|^2$ and $\lambda_{\pi_2\times\widetilde{\pi}_2}(n) = |\lambda_{\pi_2}(n)|^2$. In this situation, the bound of Lemma 2.2 follows readily by Cauchy–Schwarz. The point of the lemma is that the same bound applies in the ramified case also. We thank Farrell Brumley for supplying a proof of this fact in the Appendix.

We now discuss Property 1.4 with relation to automorphic L-functions, starting with the estimate (1.9). In the next section, we shall establish the following lemma, from which we can deduce (1.9).

LEMMA 2.3
If $\pi \in \mathcal{A}(m)$ is a cuspidal automorphic representation then for any $\eta > 0$

$$\sum_{n=1}^{\infty} \frac{\lambda_{\pi\times\widetilde{\pi}}(n)\Lambda(n)}{n^{1+\eta}} \le \frac{1}{\eta} + \frac{1}{2}\log C(\pi\times\widetilde{\pi}) + O(m^2). \tag{2.14}$$

Verifying (1.9) for $\pi \in \mathcal{A}(m)$
Applying Lemma 2.2, we find that

$$\sum_{n=1}^{\infty} \frac{|\lambda_{\pi}(n)|\Lambda(n)}{n^{1+\eta}} \le \frac{1}{2}\Big(\sum_{n=1}^{\infty}\big(1 + \lambda_{\pi\times\widetilde{\pi}}(n)\big)\frac{\Lambda(n)}{n^{1+\eta}}\Big) \le \frac{1}{\eta} + \frac{1}{4}\log C(\pi\times\widetilde{\pi}) + O(m^2),$$

by Lemma 2.3. Now applying Lemma 2.1, we see that $\log C(\pi\times\widetilde{\pi}) \le 2m\log C(\pi)$, and therefore

$$\sum_{n=1}^{\infty} \frac{|\lambda_{\pi}(n)|\Lambda(n)}{n^{1+\eta}} \le \frac{1}{\eta} + m\log C(\pi) + O(m^2).$$

This verifies (1.9) for cuspidal automorphic representations. □

Verifying (1.9) for $\pi_1 \times \pi_2$
If $\pi_1 \in \mathcal{A}(m_1)$ and $\pi_2 \in \mathcal{A}(m_2)$ are two cuspidal automorphic representations, then from Lemma 2.2 and Lemma 2.3, we see that

$$\sum_{n=1}^{\infty} \frac{|\lambda_{\pi_1\times\pi_2}(n)|\Lambda(n)}{n^{1+\eta}}$$

$$\leq \frac{1}{2}\sum_{n=1}^{\infty}\big(\lambda_{\pi_1\times\widetilde{\pi}_1}(n)+\lambda_{\pi_2\times\widetilde{\pi}_2}(n)\big)\frac{\Lambda(n)}{n^{1+\eta}}$$

$$\leq \frac{1}{\eta}+\frac{1}{4}\log C(\pi_1\times\widetilde{\pi}_1)+\frac{1}{4}\log C(\pi_2\times\widetilde{\pi}_2)+O(m_1^2+m_2^2).$$

Appealing now to Lemma 2.1, we conclude that, for any $\eta>0$,

$$\sum_{n=1}^{\infty} \frac{|\lambda_{\pi_1\times\pi_2}(n)|\Lambda(n)}{n^{1+\eta}} \leq \frac{1}{\eta}+m_1m_2\log C(\pi_1\times\pi_2)+O\big((m_1m_2)^2\big).$$

This completes our verification of (1.9) for the Rankin–Selberg convolution $\pi_1\times\pi_2$. □

In Section 6, we will prove the following theorem, from which we will deduce (1.10) for $L(s,\pi_1)$ and $L(s,\pi_1\times\pi_2)$.

THEOREM 2.4
Let $\pi\in\mathcal{A}(m)$ be a cuspidal automorphic representation. If $x\gg_m C(\pi\times\widetilde{\pi})^{36m^2}$ and $1\leq T\leq x^{1/9m^2}$, then

$$\sum_{x<n\leq xe^{1/T}} \lambda_{\pi\times\widetilde{\pi}}(n)\Lambda(n) \ll_m \frac{x}{T}.$$

Deducing (1.10) for $L(s,\pi)$
By Lemma 2.2

$$\sum_{x<n\leq xe^{1/T}} \big|\lambda_{\pi}(n)\big|\Lambda(n) \leq \frac{1}{2}\sum_{x<n\leq xe^{1/T}} \big(1+\lambda_{\pi\times\widetilde{\pi}}(n)\big)\Lambda(n). \tag{2.15}$$

By Theorem 2.4, the second term in the right-hand side above contributes $\ll x/T$, provided $1\leq T\leq x^{1/9m^2}$ and $x\geq C(\pi\times\widetilde{\pi})^{36m^2}$. In view of Lemma 2.1, it suffices to assume that $x\geq (C(\pi)T)^{72m^3}$. For the same range of x and T, the Brun–Titchmarsh inequality (1.11) bounds the first term in the right side of (2.15) by $\ll x/T$, which completes our deduction. □

Deducing (1.10) for $L(s,\pi_1\times\pi_2)$
This follows similarly, appealing to Lemma 2.1, Lemma 2.2, and Theorem 2.4. □

Gathering together the observations made so far, we arrive at the following proposition.

PROPOSITION 2.5
If $\pi \in \mathcal{A}(m)$ is a cuspidal automorphic representation, then $L(s,\pi)$ is in the class $\mathcal{S}(m)$. If $\pi_1 \in \mathcal{A}(m_1)$ and $\pi_2 \in \mathcal{A}(m_2)$ are two cuspidal automorphic representations, then $L(s,\pi_1 \times \pi_2)$ is in the class $\mathcal{S}(m_1 m_2)$.

Therefore the results given in Section 1 apply in the context of automorphic L-functions and yield the following corollaries.

COROLLARY 2.6
If $\pi \in \mathcal{A}(m)$ is a cuspidal automorphic representation, then for all $T \geq 1$ and $1/2 \leq \sigma \leq 1$ we have

$$N_\pi(\sigma, T) \ll_m \big(C(\pi)T\big)^{10^7 m^3(1-\sigma)}.$$

Further, if $\pi_1 \in \mathcal{A}(m_1)$ and $\pi_2 \in \mathcal{A}(m_2)$ are two cuspidal automorphic representations, then for all $T \geq 1$ and $1/2 \leq \sigma \leq 1$ we have

$$N_{\pi_1 \times \pi_2}(\sigma, T) \ll_{m_1, m_2} \big(C(\pi_1 \times \pi_2)T\big)^{10^7 m_1^3 m_2^3 (1-\sigma)}.$$

Apart from the exponent, this corollary gives a general result which in special situations (or with additional hypotheses) was given by a number of authors (see Kowalski and Michel [18], Motohashi [24], Akbary and Trudgian [1], and Lemke Oliver and Thorner [19]).

As a consequence of Corollary 1.3 we obtain the following weak subconvexity results for automorphic L-functions.

COROLLARY 2.7
If $\pi \in \mathcal{A}(m)$ is a cuspidal automorphic representation, then

$$\big|L(1/2, \pi)\big| \ll_m \frac{C(\pi)^{1/4}}{(\log C(\pi))^{1/(10^{17} m^3)}}.$$

If $\pi_1 \in \mathcal{A}(m_1)$ and $\pi_2 \in \mathcal{A}(m_2)$ are two cuspidal automorphic representations then

$$\big|L(1/2, \pi_1 \times \pi_2)\big| \ll_{m_1, m_2} \big|L(3/2, \pi_1 \times \pi_2)\big|^2 \frac{C(\pi_1 \times \pi_2)^{1/4}}{(\log C(\pi_1 \times \pi_2))^{1/(10^{17} m_1^3 m_2^3)}}.$$

In the first part of Corollary 2.7, we dropped the term $|L(3/2,\pi)|^2$. This is permissible because (2.1) and (2.2) give $|\lambda_\pi(n)| \ll n^{\theta_m}$, so the bound $|L(3/2,\pi)| \ll_m 1$

follows from (1.3). For the general Rankin–Selberg L-function $L(s, \pi_1 \times \pi_2)$, we are not able to obtain the bound $|L(3/2, \pi_1 \times \pi_2)| \ll 1$—without additional hypotheses, the best known bound for $|L(3/2, \pi_1 \times \pi_2)|$ follows from Theorem 2 of [20], and this is larger than any power of $\log C(\pi_1 \times \pi_2)$.

Nevertheless, in a number of special situations the term $|L(3/2, \pi_1 \times \pi_2)|^2$ may be dropped, and we give a few such examples.

Example 1

If either π_1 or π_2 satisfies the Ramanujan conjecture, then using (2.1) and (2.2), we obtain $|\lambda_{\pi_1 \times \pi_2}(n)| \ll n^{1/2-\delta}$ for some $\delta = \delta(m_1, m_2) > 0$. Therefore,

$$|L(3/2, \pi_1 \times \pi_2)| \ll_{m_1, m_2} 1$$

by (1.3).

Example 2

Since θ_2 may be taken as $7/64$ (see (2.2)), if π_1 and π_2 are both cuspidal automorphic forms on $GL(2)$, then $|L(3/2, \pi_1 \times \pi_2)| \ll 1$ and

$$\big|L(1/2, \pi_1 \times \pi_2)\big| \ll \frac{C(\pi_1 \times \pi_2)^{1/4}}{(\log C(\pi_1 \times \pi_2))^{1/10^{19}}}.$$

Alternatively, here we could use the work of Ramakrishnan [26] which shows that $\pi_1 \times \pi_2$ is an isobaric sum of cuspidal automorphic representations of dimension at most 4, and then use our bound for each constituent.

Example 3

If π_1 and π_2 are cuspidal automorphic representations in $\mathcal{A}(2)$, then $\mathrm{Sym}^2 \pi_1$ is an automorphic representation on $GL(3)$ (by the work of Gelbart and Jacquet [10]). Since $\theta_2 = 7/64$, we find that $|\lambda_{\mathrm{Sym}^2 \pi_1 \times \pi_2}(n)| \ll n^{21/64}$, and so

$$|L(3/2, \mathrm{Sym}^2 \pi_1 \times \pi_2)| \ll 1.$$

Therefore, if $\mathrm{Sym}^2 \pi_1$ is cuspidal, then

$$\big|L(1/2, \mathrm{Sym}^2 \pi_1 \times \pi_2)\big| \ll \frac{C(\mathrm{Sym}^2 \pi_1 \times \pi_2)^{1/4}}{(\log C(\mathrm{Sym}^2 \pi_1 \times \pi_2))^{1/10^{20}}}.$$

The bound also applies when $\mathrm{Sym}^2 \pi_1$ is not cuspidal, upon decomposing this and using our result for each component. Similarly, one can obtain

$$\big|L(1/2, \mathrm{Sym}^2 \pi_1 \times \mathrm{Sym}^2 \pi_2)\big| \ll \frac{C(\mathrm{Sym}^2 \pi_1 \times \mathrm{Sym}^2 \pi_2)^{1/4}}{(\log C(\mathrm{Sym}^2 \pi_1 \times \mathrm{Sym}^2 \pi_2))^{1/10^{20}}}.$$

Example 4
If π_1 and π_2 are in $\mathcal{A}(2)$, then $\mathrm{Sym}^3\pi_1$ is an automorphic form on $GL(4)$ by the work of Kim and Shahidi [17]. As in Example 3, we can obtain a weak subconvexity bound for $L(1/2, \mathrm{Sym}^3\pi_1 \times \pi_2)$.

Example 5
While we have formulated our results for the L-values at the central point $1/2$, with trivial modifications the results apply equally to any point $1/2 + it$ on the critical line. If π_1 in $\mathcal{A}(m_1)$ and π_2 in $\mathcal{A}(m_2)$ are considered fixed, then in the t-aspect our work gives the weak subconvexity bound

$$\big|L(1/2 + it, \pi_1 \times \pi_2)\big| \ll_{\pi_1,\pi_2} \frac{(2 + |t|)^{m_1 m_2/4}}{(\log(2 + |t|))^{1/(10^{17} m_1^3 m_2^3)}}.$$

Here we have used the absolute convergence of $L(s, \pi_1 \times \pi_2)$ for $\mathrm{Re}(s) > 1$ (due to Jacquet, Piatetski-Shapiro, and Shalika [15]) to bound $|L(3/2 + it, \pi_1 \times \pi_2)|$ by $\ll_{\pi_1,\pi_2} 1$.

3. Preliminary lemmas

Let $L(s, \pi) \in \mathcal{S}(m)$. Since the Euler product expansion of $L(s, \pi)$ converges absolutely and $L_\infty(s, \pi) \neq 0$ for $\mathrm{Re}(s) > 1$, there are no zeros of $L(s, \pi)L_\infty(s, \pi)$ in this region. By the functional equation, the same must be true in the region $\mathrm{Re}(s) < 0$. Thus all of the zeros of $L(s, \pi)L_\infty(s, \pi)$ lie in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$; we call these zeros the nontrivial zeros of $L(s, \pi)$. On the other hand, $L(s, \pi)$ might have a zero corresponding to a pole of $L_\infty(s, \pi)$; we call these zeros the *trivial* zeros of $L(s, \pi)$. Because the Selberg eigenvalue conjecture is not yet resolved for all $L(s, \pi)$, we might have trivial zeros in the critical strip. Unless specifically mentioned otherwise, we will always use $\rho = \beta + i\gamma$ to denote a nontrivial zero of $L(s, \pi)$. Note that neither 0 nor 1 can be a nontrivial zero of $L(s, \pi)$.

By hypothesis, $s^r(1 - s)^r L(s, \pi)L_\infty(s, \pi)$ is an entire function of order 1, and thus has a Hadamard product representation

$$s^r(1 - s)^r L(s, \pi)L_\infty(s, \pi) = e^{a_\pi + b_\pi s} \prod_\rho \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho}, \tag{3.1}$$

where ρ runs through the nontrivial zeros of $L(s, \pi)$. By taking the logarithmic derivative of both sides of (3.1), we see that

$$\sum_\rho \Big(\frac{1}{s - \rho} + \frac{1}{\rho}\Big) + b_\pi = \frac{L'}{L}(s, \pi) + \frac{L'_\infty}{L_\infty}(s, \pi) + \frac{r}{s} + \frac{r}{s - 1}. \tag{3.2}$$

Using (1.5) and the fact that $s^r(1 - s)^r L(s, \pi)L_\infty(s, \pi)$ is an entire function of order 1, one can prove that $\mathrm{Re}(b_\pi)$ equals the absolutely convergent sum $-\sum_\rho \mathrm{Re}(\rho^{-1})$. It

follows that

$$\sum_{\rho} \mathrm{Re}\Big(\frac{1}{s-\rho}\Big) = \mathrm{Re}\Big(\frac{L'}{L}(s,\pi_\infty) + \frac{L'}{L}(s,\pi) + \frac{r}{s-1} + \frac{r}{s}\Big). \tag{3.3}$$

LEMMA 3.1
We have

$$N_\pi(0,6) = \#\big\{\rho = \beta + i\gamma : |\gamma| \le 6\big\} \ge \frac{4}{15}\log C(\pi) + O(m). \tag{3.4}$$

Further, for any real number t, and any $0 < \eta \le 1$, we have

$$\sum_{\rho} \frac{1+\eta-\beta}{|1+\eta+it-\rho|^2} \le 2m\log C(\pi) + m\log\big(2+|t|\big) + 2\frac{m}{\eta} + O(m^2), \tag{3.5}$$

so that

$$\#\big\{\rho : \big|\rho - (1+it)\big| \le \eta\big\} \le 10m\eta\log C(\pi) + 5m\eta\log\big(2+|t|\big) + O(m^2). \tag{3.6}$$

Proof
These results all follow from the Hadamard formula (3.3). We start with (3.5) and (3.6). Apply (3.3) with $s = 1+\eta+it$. The left-hand side of (3.3) is

$$\sum_{\rho} \frac{(1+\eta-\beta)}{(1+\eta-\beta)^2 + (t-\gamma)^2} \ge \frac{1}{5\eta}\#\big\{\rho : \big|\rho-(1+it)\big| \le \eta\big\}. \tag{3.7}$$

The right-hand side of (3.3) is

$$\le \frac{1}{2}\log N_\pi + \frac{1}{2}\sum_{j=1}^{m} \mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1+\eta+it+\mu_\pi(j)}{2}\Big) + \sum_{n=1}^{\infty}\frac{|\lambda_\pi(n)|\Lambda(n)}{n^{1+\eta}} + \frac{r}{\eta} + r,$$

which after using (1.9), Stirling's formula, and the bound $r \le m$, is

$$\le 2m\log C(\pi) + m\log\big(2+|t|\big) + 2\frac{m}{\eta} + O(m^2).$$

From this estimate and (3.7), we conclude (3.5) and (3.6).

To prove (3.4), we begin by applying (3.3) with $s = \sigma \ge 3$. This gives

$$\sum_{\rho} \frac{(\sigma-\beta)}{(\sigma-\beta)^2+\gamma^2} = \log C(\pi) + O(m) + O\Big(\sum_{n=1}^{\infty}\frac{|\lambda_\pi(n)|\Lambda(n)}{n^3}\Big)$$
$$= \log C(\pi) + O(m).$$

Applying the above with $\sigma = 3$ and $\sigma = 4$, we obtain

$$\sum_{\rho}\Big(\frac{(3-\beta)}{(3-\beta)^2+\gamma^2}-\frac{13}{15}\frac{(4-\beta)}{(4-\beta)^2+\gamma^2}\Big)=\frac{2}{15}\log C(\pi)+O(m).$$

A small calculation shows that when $|\gamma|>6$ the terms on the left-hand side above are negative, and when $|\gamma|\le 6$, the corresponding term is $\le 1/(3-\beta)\le 1/2$. From this, (3.4) follows. □

We end this section by establishing Lemma 2.3.

Proof of Lemma 2.3

The proof is standard, based on the Hadamard factorization formula (see [19, Lemma 3.5]). Rearranging the expression for the logarithmic derivative of the Hadamard factorization formula for $L(s,\pi\times\widetilde{\pi})$ (see (3.3)), we must bound

$$\begin{aligned}&\mathrm{Re}\Big(-\frac{L'}{L}(1+\eta,\pi\times\widetilde{\pi})\Big)\\&\quad=\frac{1}{\eta}+\frac{1}{1+\eta}+\mathrm{Re}\Big(\frac{L'_\infty}{L_\infty}(1+\eta,\pi\times\widetilde{\pi})\Big)-\sum_{\rho\neq 0,1}\mathrm{Re}\Big(\frac{1}{1+\eta-\rho}\Big),\end{aligned}$$

where $\rho=\beta+i\gamma$ runs through the zeros of $s(1-s)L(s,\pi\times\widetilde{\pi})L_\infty(s,\pi\times\widetilde{\pi})$. Since $0<\beta<1$, we have

$$\mathrm{Re}\Big(\frac{1}{1+\eta-\rho}\Big)=\frac{1+\eta-\beta}{|1+\eta-\rho|^2}>0,$$

so that the contribution from zeros is negative and may be discarded. Moreover, by Stirling's formula and (1.8),

$$\begin{aligned}\mathrm{Re}\Big(\frac{L'_\infty}{L_\infty}(1+\eta,\pi\times\widetilde{\pi})\Big)&=-\sum_{|1+\eta+\mu_{\pi\times\widetilde{\pi}}(j)|<1}\mathrm{Re}\Big(\frac{1}{1+\eta+\mu_{\pi\times\widetilde{\pi}}(j)}\Big)\\&\quad+\frac{1}{2}\log C(\pi\times\widetilde{\pi})+O(m^2)\\&\le\frac{1}{2}\log C(\pi\times\widetilde{\pi})+O(m^2).\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{\lambda_{\pi\times\widetilde{\pi}}(n)\Lambda(n)}{n^{1+\eta}}\le\frac{1}{\eta}+\frac{1}{2}\log C(\pi\times\widetilde{\pi})+O(m^2),$$

completing our proof. □

4. Proof of Theorem 1.2

We prove the log-free zero density estimate of Theorem 1.2 by following Gallagher's treatment in [9], which is based on Turán's power sum method. For the sake of completeness, we show that the axiomatic framework given in (1.1) to (1.10) is sufficient to establish such a log-free zero density estimate.

Let $k \geq 1$ be a natural number, and let η be a real number with $1/\log(C(\pi)T) < \eta \leq 1/(200m)$. Let τ be a real number with $T \geq |\tau| \geq 200\eta$. Differentiating (3.2) k times we find, with $s = 1 + \eta + i\tau$,

$$\Big(\frac{L'}{L}(s,\pi)\Big)^{(k)} + \Big(\sum_{j=1}^{m} \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s+\mu_\pi(j)}{2}\Big)\Big)^{(k)} + (-1)^k k!\Big(\frac{r}{s^{k+1}} + \frac{r}{(s-1)^{k+1}}\Big)$$
$$= (-1)^k k! \sum_{\rho} \frac{1}{(s-\rho)^{k+1}}.$$

Since $\mathrm{Re}(\mu_\pi(j)) \geq -1 + 1/m$, we obtain

$$\frac{1}{2}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{s+\mu_\pi(j)}{2}\Big)\Big)^{(k)} = \frac{(-1)^{k+1}k!}{2^{k+1}} \sum_{n=0}^{\infty} \frac{1}{(n + (s+\mu_\pi(j))/2)^{k+1}} \ll m^{k+1}k!,$$

and, since $|\tau| \geq 200\eta$ and $r \leq m$, clearly

$$(-1)^k k!\Big(\frac{r}{s^{k+1}} + \frac{r}{(s-1)^{k+1}}\Big) \ll \frac{mk!}{(200\eta)^{k+1}}.$$

Thus, since $m \leq 1/(200\eta)$,

$$\frac{(-1)^k}{k!}\Big(\frac{L'}{L}(s,\pi)\Big)^{(k)} = O\Big(\frac{m}{(200\eta)^{k+1}}\Big) + \sum_{\rho} \frac{1}{(s-\rho)^{k+1}}. \tag{4.1}$$

Applying (3.5), we see that

$$\Big|\sum_{\substack{\rho \\ |s-\rho|\geq 200\eta}} \frac{1}{(s-\rho)^{k+1}}\Big| \leq \frac{1}{(200\eta)^{k-1}} \sum_{\rho} \frac{1}{|s-\rho|^2} \leq \frac{1}{(200\eta)^{k-1}} \frac{1}{\eta} \sum_{\rho} \frac{(1+\eta-\beta)}{|s-\rho|^2}$$
$$\ll \frac{1}{(200\eta)^k}\Big(m\log\big(C(\pi)T\big) + \frac{m}{\eta}\Big) \ll \frac{m\log(C(\pi)T)}{(200\eta)^k}.$$

Since $\eta \geq 1/\log(C(\pi)T)$, using this estimate in (4.1), we conclude that

$$\frac{(-1)^k}{k!}\Big(\frac{L'}{L}(s,\pi)\Big)^{(k)} = O\Big(\frac{m\log(C(\pi)T)}{(200\eta)^k}\Big) + \sum_{\substack{\rho \\ |s-\rho|\leq 200\eta}} \frac{1}{(s-\rho)^{k+1}}. \tag{4.2}$$

Equation (4.2) forms the starting point for the proof of Theorem 1.2. Using Turán's power sum method in [28], we shall obtain a lower bound for the right-hand side of (4.2) for a suitable k (which will eventually be of size about $\eta \log(C(\pi)T)$), provided there is a zero ρ with $|1 + i\tau - \rho| \le \eta$. On the other hand, we shall bound from above the left-hand side of (4.2) in terms of Dirichlet polynomials over prime powers. The interplay of these bounds will yield the theorem. We start with the lower bound, which will use the following result from Turán's method (see the Theorem in [28]).

LEMMA 4.1
Let $z_1, \dots, z_\nu \in \mathbb{C}$. If $K \ge \nu$, then there exists an integer $k \in [K, 2K]$ such that

$$|z_1^k + \cdots + z_\nu^k| \ge (|z_1|/50)^k.$$

LEMMA 4.2
Let η and τ be real numbers with $1/\log(C(\pi)T) < \eta \le 1/(200m)$ and $200\eta \le |\tau| \le T$. Suppose that $L(s,\pi)$ has a zero ρ_0 satisfying $|\rho_0 - (1 + i\tau)| \le \eta$. If $K > \lceil 2000m\eta \log(C(\pi)T) + O(m^2) \rceil$, then one has (recall $s = 1 + \eta + i\tau$)

$$\Big| \sum_{\substack{\rho \\ |s-\rho| \le 200\eta}} \frac{1}{(s-\rho)^{k+1}} \Big| \ge \Big(\frac{1}{100\eta}\Big)^{k+1},$$

for some integer $k \in [K, 2K]$.

Proof
By (3.6) we see that there are at most $2000m\eta \log(C(\pi)T) + O(m^2)$ zeros ρ satisfying $|s - \rho| \le 200\eta$. By applying Lemma 4.1 with z_1 there being $1/(s - \rho_0)$, which is $\ge 1/(2\eta)$ in size, the lemma follows. □

We now proceed to the upper bound.

LEMMA 4.3
Let η and τ be real numbers with $1/\log(C(\pi)T) < \eta \le 1/(200m)$ and $200\eta \le |\tau| \le T$. Let $K \ge 1$ be a natural number, and put $N_0 = \exp(K/(300\eta))$ and $N_1 = \exp(40K/\eta)$. With $s = 1 + \eta + i\tau$, we have for all $K \le k \le 2K$,

$$\Big| \frac{\eta^{k+1}}{k!} \Big(\frac{L'}{L}(s,\pi) \Big)^{(k)} \Big| \le \eta^2 \int_{N_0}^{N_1} \Big| \sum_{N_0 \le n \le u} \frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}} \Big| \frac{du}{u} + O\Big(\frac{m\eta \log(C(\pi)T)}{(110)^k} \Big).$$

Proof
By computing the kth derivative of the Dirichlet series for $\frac{L'}{L}(s,\pi)$, we find

$$\Big|\frac{\eta^k}{k!}\Big(\frac{L'}{L}(s,\pi)\Big)^{(k)}\Big| = \Big|\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)\Lambda(n)}{n^{1+\eta+i\tau}}\frac{(\eta\log n)^k}{k!}\Big|.$$

Put $j_k(u) = e^{-u}u^k/k!$, and split the sum over n into the ranges $n \in [N_0, N_1]$ and $n \notin [N_0, N_1]$. For $n \notin [N_0, N_1]$, we estimate trivially using the triangle inequality, and use partial summation in the range $n \in [N_0, N_1]$. Thus, the above is

$$\le \sum_{n\notin[N_0,N_1]}\frac{|\lambda_\pi(n)|\Lambda(n)}{n}j_k(\eta\log n) + \sum_{N_0\le n\le N_1}\frac{|\lambda_\pi(n)|\Lambda(n)}{n}j_k(\eta\log N_1)$$
$$+\int_{N_0}^{N_1}\Big|\frac{d}{du}j_k(\eta\log u)\Big|\Big|\sum_{N_0\le n\le u}\frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}}\Big|\,du. \tag{4.3}$$

Now $|\frac{d}{du}(j_k(\eta\log u))| = |-j_k(\eta\log u) + j_{k-1}(\eta\log u))|(\eta/u) \le \eta/u$, and so the integral in (4.3) is

$$\le \eta\int_{N_0}^{N_1}\Big|\sum_{N_0\le n\le u}\frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}}\Big|\frac{du}{u}.$$

Suppose that $n \le N_0$, in which case $\eta\log n \le K/300$. Since $k \in [K, 2K]$ and $k! \ge (k/e)^k$, we observe that

$$j_k(\eta\log n) = \frac{n^{-\eta}(\eta\log n)^k}{k!} \le n^{-\eta}\Big(\frac{e\eta\log n}{k}\Big)^k \le n^{-\eta/2}(110)^{-k}.$$

Next suppose that $n \ge N_1$, in which case $\eta\log n \ge 40K$. Since $e^{-u/2}u^k/k!$ is decreasing in the range $u > 2k$, we see that for $n \ge N_1$

$$j_k(\eta\log n) = n^{-\eta/2}\frac{e^{-\frac{1}{2}\eta\log n}(\eta\log n)^k}{k!} \le n^{-\eta/2}\frac{e^{-20K}(40K)^k}{k!}$$
$$\le n^{-\eta/2}e^{-20K}\Big(\frac{40eK}{k}\Big)^k \le n^{-\eta/2}(110)^{-k}.$$

The last estimate also implies that, for the sum over $N_0 \le n \le N_1$, one has $j_k(\eta\log N_1) \ll n^{-\eta/2}(110)^{-k}$. Therefore the sums appearing in (4.3) are bounded by

$$\ll \frac{1}{(110)^k}\sum_{n=1}^{\infty}\frac{|\lambda_\pi(n)|\Lambda(n)}{n^{1+\eta/2}} \ll \frac{m\log(C(\pi)T)}{(110)^k}$$

using (1.9). □

We now combine Lemma 4.2 and Lemma 4.3 to prove Theorem 1.2.

Proof of Theorem 1.2

We combine Lemma 4.2 and Lemma 4.3 to detect zeros near the line $\sigma = 1$. Let η and τ be real numbers with $1/\log(C(\pi)T) < \eta \le 1/(200m)$ and $200\eta \le |\tau| \le T$. In keeping with Lemma 4.2, we suppose that

$$K = 10^5 m^3 \eta \log\big(C(\pi)T\big) + O(m^2) \tag{4.4}$$

is sufficiently large, and put (as in Lemma 4.3) $N_0 = \exp(K/(300\eta))$ and $N_1 = \exp(40K/\eta)$. Suppose that $L(s,\pi)$ has a zero ρ_0 satisfying $|1 + i\tau - \rho_0| \le \eta$. Since K satisfies (4.4) and is sufficiently large, combining (4.2) with Lemma 4.2 we obtain, for some $k \in [K, 2K]$,

$$\Big|\frac{\eta^{k+1}}{k!}\Big(\frac{L'}{L}(s,\pi)\Big)^{(k)}\Big| \ge \Big(\frac{1}{100}\Big)^{k+1}\Big(1 - O\Big(\frac{\eta m \log(C(\pi)T)}{2^k}\Big)\Big) \ge \frac{1}{2(100)^{k+1}}.$$

On the other hand, by Lemma 4.3, we obtain (for all $k \in [K, 2K]$)

$$\Big|\frac{\eta^{k+1}}{k!}\Big(\frac{L'}{L}(s,\pi)\Big)^{(k)}\Big| \le \eta^2 \int_{N_0}^{N_1} \Big|\sum_{N_0 \le n \le u} \frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}}\Big|\frac{du}{u} + \frac{1}{4(100)^{k+1}},$$

where we bounded the error term $O((110)^{-k}(m\eta\log(C(\pi)T)))$ in Lemma 4.3 by $\frac{1}{4}(100)^{-k-1}$. Combining these two estimates, we conclude that if there is a zero ρ_0 with $|1 + i\tau - \rho_0| \le \eta$, then

$$1 \le 4(100)^{2K+1}\eta^2 \int_{N_0}^{N_1} \Big|\sum_{N_0 \le n \le u} \frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}}\Big|\frac{du}{u}.$$

Squaring the above estimate and using Cauchy–Schwarz, we obtain

$$\begin{aligned} 1 &\ll (100)^{4K}\eta^4\Big(\int_{N_0}^{N_1}\frac{du}{u}\Big)\Big(\int_{N_0}^{N_1}\Big|\sum_{N_0 \le n \le u}\frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}}\Big|^2\frac{du}{u}\Big) \\ &\ll (101)^{4K}\eta^3\int_{N_0}^{N_1}\Big|\sum_{N_0 \le n \le u}\frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}}\Big|^2\frac{du}{u}, \end{aligned}$$

since $\log(N_1/N_0) \ll K/\eta$. Because there are $\ll m\eta\log(C(\pi)T)$ zeros that satisfy $|1 + i\tau - \rho| \le \eta$, we may also recast the above estimate as (for $200\eta \le |\tau| \le T$)

$$\begin{aligned} &\frac{\#\{\rho = \beta + i\gamma : \beta \ge 1 - \eta/2, |\gamma - \tau| \le \eta/2\}}{m\eta\log(C(\pi)T)} \\ &\ll 101^{4K}\eta^3\int_{N_0}^{N_1}\Big|\sum_{N_0 \le n \le u}\frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}}\Big|^2\frac{du}{u}. \end{aligned}$$

Integrating both sides above over $200\eta \le |\tau| \le T$, we conclude that

$$\#\{\rho = \beta + i\gamma : \beta \ge 1 - \eta/2, 200\eta \le |\gamma| \le T\}$$
$$\ll 101^{4K}\eta^3 m \log\big(C(\pi)T\big) \int_{-T}^{T} \int_{N_0}^{N_1} \Big| \sum_{N_0 \le n \le u} \frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}} \Big|^2 \frac{du}{u}\, d\tau. \quad (4.5)$$

We now work on bounding the right-hand side of (4.5), which is clearly

$$\ll 101^{4K}\eta^3 m \log\big(C(\pi)T\big) \log(N_1/N_0) \max_{u \in [N_0, N_1]} \Big(\int_{-T}^{T} \Big| \sum_{N_0 \le n \le u} \frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}} \Big|^2 d\tau \Big)$$
$$\ll 102^{4K}\eta^2 m \log\big(C(\pi)T\big) \max_{u \in [N_0, N_1]} \Big(\int_{-T}^{T} \Big| \sum_{N_0 \le n \le u} \frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}} \Big|^2 d\tau \Big). \quad (4.6)$$

We bound the integral in the above display by an application of Plancherel, as in Gallagher (see [9, Theorem 1]): for $T \ge 1$ and any sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$, one has

$$\int_{-T}^{T} \Big| \sum_{n=1}^{\infty} a_n n^{-it} \Big|^2 dt \ll T^2 \int_0^{\infty} \Big| \sum_{n \in (w, we^{1/T}]} a_n \Big|^2 \frac{dw}{w}.$$

Applying Gallagher's bound, we deduce that for any $u \in [N_0, N_1]$,

$$\int_{-T}^{T} \Big| \sum_{N_0 \le n \le u} \frac{\lambda_\pi(n)\Lambda(n)}{n^{1+i\tau}} \Big|^2 d\tau \ll T^2 \int_0^{\infty} \Big| \sum_{\substack{x < n \le xe^{1/T} \\ N_0 \le n \le N_2}} \frac{\lambda_\pi(n)\Lambda(n)}{n} \Big|^2 \frac{dx}{x}$$
$$\ll T^2 \int_{N_0/e}^{N_1} \Big| \sum_{x < n \le xe^{1/T}} \big|\lambda_\pi(n)\big|\Lambda(n) \Big|^2 \frac{dx}{x^3}.$$

Appealing now to (1.10) (which applies because of (4.4)), we find that the above is

$$\ll_m T^2 \int_{N_0/e}^{N_1} \frac{x^2}{T^2} \frac{dx}{x^3} \ll_m \frac{K}{\eta}.$$

Using this in (4.6), we conclude that this quantity is bounded by

$$\ll 102^{4K} K\eta m \log\big(C(\pi)T\big) \ll 105^{4K}.$$

Inserting the above bound in (4.5), and noting that there are $\ll \eta m \log(C(\pi)T) \ll K$ zeros with $\beta > 1 - \eta/2$ and $|\gamma| \le 200\tau$, we obtain

$$N_\pi(1 - \eta/2, T) \ll 105^{4K}.$$

This estimate implies our theorem in the range $1/\log(C(\pi)T) \le 1-\sigma \le 1/(400m)$. In the range $1-\sigma \le 1/\log(C(\pi)T)$, we note that

$$N_\pi(\sigma, T) \le N_\pi(1 - 1/\log(C(\pi)T), T).$$

In the range $1-\sigma > 1/(400m)$, the theorem is trivial as there are $\ll mT\log(C(\pi)T)$ zeros with $\beta \in (0,1)$ and $|\gamma| \le T$. □

5. Proof of Theorem 1.1 and Corollary 1.3

Let $L(s,\pi) \in \mathcal{S}(m)$, and in proving the theorem, we may plainly suppose that $L(1/2,\pi) \neq 0$. Our starting point is Heath-Brown's argument to establish a sharp convexity bound for L-functions. This begins with a variant of Jensen's formula, connecting $\log|L(1/2,\pi)|$ with zeros lying in the critical strip $0 < \mathrm{Re}(s) < 1$. The Jensen formula that we need is

$$\begin{aligned}\log\bigl|(1/2)^r L(1/2,\pi)\bigr| &+ \sum_{\substack{\rho=\beta+i\gamma\\ 0<\beta<1}} \log\Bigl|\cot\Bigl(\frac{\pi}{2}\Bigl(\rho-\frac{1}{2}\Bigr)\Bigr)\Bigr| \\ &+ \sum_{\mathrm{Re}(\mu_\pi(j))<0} \log\Bigl|\cot\Bigl(\frac{\pi}{2}\Bigl(\mu_\pi(j)+\frac{1}{2}\Bigr)\Bigr)\Bigr| \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \log\bigl|L(1+it,\pi)L(it,\pi)t^r(1-it)^r\bigr| \frac{dt}{\cosh(\pi t)}. \end{aligned} \tag{5.1}$$

This may be established as in Heath-Brown [13], or applying [2, p. 207, Lemma 3.1] with $F(s) = (s-1)^r L(s,\pi)$ and $x = 1/2$. The proof is by conformally mapping the strip $z = x+iy$ with $0 < x < 1$ onto the unit disk $|\zeta| < 1$ by means of the substitution $\zeta = (e^{\pi i z} - i)/(e^{\pi i z} + i)$, and then using the usual Jensen formula for the unit disc.

Now if $z = x+iy$ is a complex number with $|x| \le 1/2$, then a small calculation gives

$$\log\bigl|\cot(\pi z/2)\bigr| = \frac{1}{2}\log\Bigl(\frac{\cosh(\pi y)+\cos(\pi x)}{\cosh(\pi y)-\cos(\pi x)}\Bigr) \ge \frac{\cos(\pi x)}{\cosh(\pi y)}, \tag{5.2}$$

where the last inequality follows because $\frac{1}{2}\log((1+t)/(1-t)) \ge t$ for $1 > t \ge 0$ by a Taylor expansion. From (5.2) and since $\mathrm{Re}(\mu_\pi(j)) > -1$, the terms $\log|\cot(\pi(\mu_\pi(j)+1/2)/2)|$ appearing in (5.1) are all nonnegative. Bounding the sum over zeros below using (5.2), we conclude that the left-hand side of (5.1) is at least

$$\log\bigl|L(1/2,\pi)\bigr| + \sum_{\substack{\rho=\beta+i\gamma\\ 0<\beta<1}} \frac{\sin(\pi\beta)}{\cosh(\pi\gamma)}. \tag{5.3}$$

Now we consider the right-hand side of (5.1). Using the functional equation to connect $L(it,\pi)$ with $L(1-it,\widetilde{\pi})$, and then using Stirling's formula, we obtain

$$\begin{aligned}\log\big|L(it,\pi)\big| &= \log\big|L(1-it,\widetilde{\pi})\big| + \frac{1}{2}\log N_\pi \\ &\quad + \sum_{j=1}^{m}\log\Big|\frac{\Gamma((1+\mu_{\widetilde{\pi}}(j)-it)/2)}{\Gamma((\mu_\pi(j)+it)/2)}\Big| + O(m) \\ &= \log\big|L(1+it,\pi)\big| + \frac{1}{2}\log N_\pi \\ &\quad + \frac{1}{2}\sum_{j=1}^{m}\log\big(1+\big|\mu_\pi(j)+it\big|\big) + O(m^2) \\ &\le \log\big|L(1+it,\pi)\big| + \frac{1}{2}\log C(\pi) + \frac{m}{2}\log\big(1+|t|\big) + O(m^2).\end{aligned}$$

Thus the right-hand side of (5.1) is bounded by

$$\begin{aligned}&\frac{1}{4}\log C(\pi) + \int_{-\infty}^{\infty}\Big(\log\big|t^r L(1+it,\pi)\big| + \frac{m}{4}\log\big(1+|t|\big) + O(m^2)\Big)\frac{dt}{\cosh(\pi t)} \\ &\quad = \frac{1}{4}\log C(\pi) + O(m^2) + \int_{-\infty}^{\infty}\log\big|t^r L(1+it,\pi)\big|\frac{dt}{\cosh(\pi t)}. \qquad (5.4)\end{aligned}$$

Since $|t^r L(1+it,\pi)|$ grows at most polynomially in $|t|$, and $1/\cosh(\pi t)$ decreases exponentially in $|t|$, we may see that

$$\begin{aligned}&\int_{-\infty}^{\infty}\log\big|t^r L(1+it,\pi)\big|\frac{dt}{\cosh(\pi t)} \\ &\quad = \lim_{\eta\to 0^+}\mathrm{Re}\Big(\int_{-\infty}^{\infty}\log\big(t^r L(1+\eta+it,\pi)\big)\frac{dt}{\cosh(\pi t)}\Big) \\ &\quad = \lim_{\eta\to 0^+}\mathrm{Re}\Big(\sum_{n=2}^{\infty}\frac{\lambda_\pi(n)\Lambda(n)}{n^{1+\eta}\log n}\int_{-\infty}^{\infty}n^{-it}\frac{dt}{\cosh(\pi t)}\Big) + O(m).\end{aligned}$$

Now

$$\int_{-\infty}^{\infty}n^{-it}\frac{dt}{\cosh(\pi t)} = \frac{1}{\cosh((\log n)/2)} = \frac{2}{\sqrt{n}+1/\sqrt{n}} = \frac{2}{\sqrt{n}} + O\Big(\frac{1}{n^{3/2}}\Big),$$

and therefore

$$\begin{aligned}&\int_{-\infty}^{\infty}\log\big|t^r L(1+it,\pi)\big|\frac{dt}{\cosh(\pi t)} \\ &\quad = 2\,\mathrm{Re}\Big(\sum_{n=2}^{\infty}\frac{\lambda_\pi(n)\Lambda(n)}{n^{3/2}\log n}\Big) + O\Big(\sum_{n=2}^{\infty}\frac{|\lambda_\pi(n)|\Lambda(n)}{n^{5/2}\log n} + m\Big)\end{aligned}$$

$$= 2\log\bigl|L(3/2,\pi)\bigr| + O(m).$$

Combining the above remarks with (5.3) and (5.4), we conclude that

$$\log\bigl|L(1/2,\pi)\bigr| \le \frac{1}{4}\log C(\pi) - \sum_{\substack{\rho=\beta+i\gamma\\ 0<\beta<1}} \frac{\sin(\pi\beta)}{\cosh(\pi\gamma)} + 2\log\bigl|L(3/2,\pi)\bigr| + O(m^2). \tag{5.5}$$

All this follows closely the work of Heath-Brown, except that we have kept a negative contribution from the zeros of $L(s,\pi)$, which we shall now bound from below.

Proof of Theorem 1.1
Plainly for any positive real number T, and any $1/2 \ge \delta > 0$, we have

$$\sum_{\substack{\rho=\beta+i\gamma\\ 0<\beta<1}} \frac{\sin(\pi\beta)}{\cosh(\pi\gamma)} \ge \sum_{\substack{\rho=\beta+i\gamma\\ |\gamma|\le T}} \frac{\sin(\pi\beta)}{\cosh(\pi T)}$$
$$\ge \frac{\sin(\pi\delta)}{\cosh(\pi T)} \sum_{\substack{\rho=\beta+i\gamma\\ \delta\le\beta\le 1-\delta\\ |\gamma|\le T}} 1 \ge \frac{2\delta}{\cosh(\pi T)} \sum_{\substack{\rho=\beta+i\gamma\\ \delta\le\beta\le 1-\delta\\ |\gamma|\le T}} 1.$$

The functional equation combined with complex conjugation shows that if $\beta+i\gamma$ is a zero, then so is $1-\beta+i\gamma$. Thus, choosing $T=6$ and invoking (3.4), we obtain

$$\sum_{\substack{\rho=\beta+i\gamma\\ \delta\le\beta\le 1-\delta\\ |\gamma|\le 6}} 1 = N_\pi(0,6) - 2N_\pi(1-\delta,6) \ge \frac{4}{15}\log C(\pi) - 2N_\pi(1-\delta,6) + O(m).$$

Therefore,

$$\sum_{\substack{\rho=\beta+i\gamma\\ 0<\beta<1}} \frac{\sin(\pi\beta)}{\cosh(\pi\gamma)} \ge \frac{2\delta}{\cosh(6\pi)}\Bigl(\frac{4}{15}\log C(\pi) - 2N_\pi(1-\delta,6)\Bigr) + O(m).$$

Inserting this lower bound into (5.5), we obtain Theorem 1.1. □

Proof of Corollary 1.3
Choose $\delta = 10^{-8}m^{-3}(\log\log C(\pi))/\log C(\pi)$. Then Theorem 1.2 gives

$$N_\pi(1-\delta,6) \ll_m \sqrt{\log C(\pi)}.$$

Inserting this bound in Theorem 1.1, the corollary follows. □

6. Proof of Theorem 2.4

We fix a nonnegative smooth function Φ supported in $(-2, 2)$, say, and write

$$\check{\Phi}(s) = \int_{-\infty}^{\infty} \Phi(y) e^{sy}\, dy. \tag{6.1}$$

Thus $\check{\Phi}(s)$ is an entire function of s, and by integrating by parts many times, we obtain, for any integer $k \geq 0$,

$$\big|\check{\Phi}(s)\big| \ll_{\Phi,k} \frac{e^{2|\operatorname{Re}(s)|}}{|s|^k}. \tag{6.2}$$

Let $T \geq 1$ be a real parameter, and note that by Mellin (or Fourier) inversion one has (for any positive real number x, and any real c)

$$T\Phi(T \log x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \check{\Phi}(s/T) x^{-s}\, ds. \tag{6.3}$$

Recall that

$$L(s, \pi \times \widetilde{\pi}) = \sum_{n \geq 1} \frac{a_{\pi\times\widetilde{\pi}}(n)}{n^s} = \prod_p L_p(s, \pi \times \widetilde{\pi}),$$

with

$$L_p(s, \pi \times \widetilde{\pi}) = \prod_{j_1=1}^{m} \prod_{j_2=1}^{m} \Big(1 - \frac{\alpha_{j_1,j_2,\pi\times\widetilde{\pi}}(p)}{p^s}\Big)^{-1} = 1 + \sum_{j=1}^{\infty} \frac{a_{\pi\times\widetilde{\pi}}(p^j)}{p^{js}}. \tag{6.4}$$

The Rankin–Selberg L-function $L(s, \pi \times \widetilde{\pi})$ has nonnegative coefficients, converges in $\operatorname{Re}(s) > 1$, and extends to the complex plane with a simple pole at $s = 1$.

Our proof of the Brun–Titchmarsh result Theorem 2.4 will be based on an application of the Selberg sieve. To pave the way for this, given a square-free number d we need an asymptotic formula for

$$\sum_{d|n} a_{\pi\times\widetilde{\pi}}(n) \Phi\Big(T \log \frac{n}{x}\Big),$$

which we establish in the following lemma.

LEMMA 6.1

Let $\pi \in \mathcal{A}(m)$, and Φ be as above. Let $d \geq 1$ be a square-free integer. For any $x > 0$ and $T \geq 1$, we have

$$\sum_{d|n} a_{\pi\times\widetilde{\pi}}(n) \Phi\Big(T \log \frac{n}{x}\Big) = \kappa g(d) \frac{x}{T} \check{\Phi}(1/T) + O_m\big(x^{\frac{1}{2}} C(\pi \times \widetilde{\pi}) d^{m^2} T^{m^2}\big),$$

where

$$\kappa = \operatorname*{Res}_{s=1} L(s, \pi \times \widetilde{\pi}), \qquad \textit{and} \qquad g(d) = \prod_{p|d} \big(1 - L_p(1, \pi \times \widetilde{\pi})^{-1}\big).$$

Proof

Using (6.3), we may write (for any real number $c > 1$)

$$\sum_{d|n} a_{\pi\times\widetilde{\pi}}(n) \Phi\Big(T \log \frac{n}{x}\Big) = \frac{1}{2\pi i T} \int_{c-i\infty}^{c+i\infty} \check{\Phi}(s/T) x^s \sum_{d|n} \frac{a_{\pi\times\widetilde{\pi}}(n)}{n^s}\, ds.$$

The Dirichlet series appearing above has nonnegative coefficients and converges in the region $\mathrm{Re}(s) > 1$, and matches the Rankin–Selberg L-function $L(s, \pi \times \widetilde{\pi})$ except for the Euler factors at primes p dividing d. Indeed, by multiplicativity, we may write

$$\sum_{d|n} \frac{a_{\pi\times\widetilde{\pi}}(n)}{n^s} = \prod_{p\nmid d} L_p(s, \pi \times \widetilde{\pi}) \prod_{p|d} \Big(\sum_{j=1}^{\infty} \frac{a_{\pi\times\widetilde{\pi}}(p^j)}{p^{js}}\Big) = L(s, \pi \times \widetilde{\pi}) g_d(s, \pi \times \widetilde{\pi}),$$

where

$$g_d(s, \pi \times \widetilde{\pi}) = \prod_{p|d} \big(1 - L_p(s, \pi \times \widetilde{\pi})^{-1}\big) = \prod_{p|d} \Big(1 - \prod_{j_1, j_2 = 1}^{m} \Big(1 - \frac{\alpha_{j_1, j_2, \pi\times\widetilde{\pi}}(p)}{p^s}\Big)\Big). \tag{6.5}$$

Thus the integral above equals

$$\frac{1}{2\pi i T} \int_{c-i\infty}^{c+i\infty} \check{\Phi}(s/T) L(s, \pi \times \widetilde{\pi}) g_d(s, \pi \times \widetilde{\pi}) x^s\, ds. \tag{6.6}$$

We evaluate (6.6) by moving the line of integration to $\mathrm{Re}(s) = 1/2$. We encounter a simple pole at $s = 1$, and the residue here is the main term appearing in our lemma (note that $g(d) = g_d(1, \pi \times \widetilde{\pi})$). To bound the integral on the line $\mathrm{Re}(s) = 1/2$, using the Phragmén–Lindelöf principle and Stirling's formula (using that on the line $\mathrm{Re}(s) = 5/2$ we have $L(s, \pi \times \widetilde{\pi}) \ll 1$), we find

$$\Big|L\Big(\frac{1}{2} + it, \pi \times \widetilde{\pi}\Big)\Big| \ll C(\pi \times \widetilde{\pi}) \big(2 + |t|\big)^{m^2}.$$

Further, since $|\alpha_{j_1, j_2, \pi\times\widetilde{\pi}}(p)| \le p$ for all j_1, j_2 and p, from the definition (6.5) it follows that

$$\Big|g_d\Big(\frac{1}{2} + it, \pi \times \widetilde{\pi}\Big)\Big| \le \prod_{p|d} \big(1 + (1 + p^{1/2})^{m^2}\big) \le (2d)^{m^2}.$$

Thus the integral on the $\mathrm{Re}(s) = 1/2$ line is

$$\ll \frac{\sqrt{x}}{T} C(\pi \times \widetilde{\pi})(2d)^{m^2} \int_{-\infty}^{\infty} (2+|t|)^{m^2} \Big| \check{\Phi}\Big(\frac{1}{T}\Big(\frac{1}{2}+it\Big)\Big)\Big| \, dt.$$

Using (6.2) with $k = 0$ for $|t| \le T$, and $k = m^2 + 2$ for $|t| > T$, we see that the above is

$$\ll_{m,\Phi} \frac{\sqrt{x}}{T} C(\pi \times \widetilde{\pi})\, d^{m^2} \int_{-\infty}^{\infty} (2+|t|)^{m^2} \min\Big(1, \frac{T^{m^2+2}}{(2+|t|)^{m^2+2}}\Big)\, dt$$

$$\ll_{m,\Phi} \sqrt{x} C(\pi \times \widetilde{\pi}) d^{m^2} T^{m^2}. \qquad \square$$

From Lemma 6.1 and an application of the Selberg sieve, we shall obtain the following proposition.

PROPOSITION 6.2

Keep the notations of Lemma 6.1. Then for any $x > 0$, $T \ge 1$, and $z \gg_m C(\pi \times \widetilde{\pi})^4$, we have

$$\sum_{\substack{n \\ p|n \Longrightarrow p > z}} a_{\pi \times \widetilde{\pi}}(n) \Phi\Big(T \log \frac{n}{x}\Big)$$

$$\le \frac{3x}{T \log z} \check{\Phi}(1/T) + O_{m,\Phi}\big(x^{\frac{1}{2}} C(\pi \times \widetilde{\pi}) T^{m^2} z^{2m^2+3}\big).$$

Proof

As mentioned already, this follows from a standard application of Selberg's sieve and Lemma 6.1; see, for example, Theorem 7.1 of [8]. Using Theorem 7.1 of [8] and (6.3) there (with $D = z^2$ in their notation), we find

$$\sum_{\substack{n \\ p|n \Longrightarrow p > z}} a_{\pi \times \widetilde{\pi}}(n) \Phi\Big(T \log \frac{n}{x}\Big)$$

$$\le \kappa \frac{x}{T} \check{\Phi}(1/T) \Big(\sum_{\substack{d \,|\, \prod_{p \le z} p \\ d \le z}} \prod_{p|d} \frac{g(p)}{1-g(p)} \Big)^{-1}$$

$$+ O_{m,\Phi}\Big(x^{\frac{1}{2}} C(\pi \times \widetilde{\pi}) T^{m^2} \sum_{d \le z^2} d^{m^2} \tau_3(d)\Big). \tag{6.7}$$

Here $\tau_3(d)$ is the number of ways of writing d as a product of three natural numbers. Because $\tau_3(d) \ll_\epsilon d^\epsilon$, we may trivially bound the error term in inequality (6.7) by $\ll_{m,\Phi} x^{\frac{1}{2}} C(\pi \times \widetilde{\pi}) T^{m^2} z^{2m^2+3}$.

For the first sum, we observe from the definitions of $g(p)$ and $L_p(s, \pi \times \widetilde{\pi})$ that

$$\sum_{\substack{d \mid \prod_{p \le z} p \\ d \le z}} \prod_{p \mid d} \frac{g(p)}{1-g(p)} \ge \sum_{\substack{n \le z \\ n \text{ square-free}}} \prod_{p \mid n} \sum_{j=1}^{\infty} \frac{a_{\pi \times \widetilde{\pi}}(p^j)}{p^j} \ge \sum_{n \le z} \frac{a_{\pi \times \widetilde{\pi}}(n)}{n}. \tag{6.8}$$

Let Φ_1 be a nonnegative smooth function supported on $[0,1]$, with $\Phi_1(t) = 1$ for $\epsilon \le t \le 1-\epsilon$ and $\Phi_1(t) \le 1$ for $0 \le t \le 1$. Then appealing to Lemma 6.1 with $d = 1$ and $T = 1$ there, we obtain that

$$\begin{aligned} \sum_{y \le n \le ey} \frac{a_{\pi \times \widetilde{\pi}}(n)}{n} &\ge \frac{1}{ey} \sum_{n} a_{\pi \times \widetilde{\pi}}(n) \Phi_1\Big(\log \frac{n}{y}\Big) \\ &= \frac{1}{e}\big(e - 1 + O(\epsilon)\big)\kappa + O_m\big(y^{-\frac{1}{2}} C(\pi \times \widetilde{\pi})\big). \end{aligned}$$

Dividing the interval $[\sqrt{z}, z]$ into blocks of the form $[y, ey]$, it follows that

$$\sum_{\sqrt{z} \le n \le z} \frac{a_{\pi \times \widetilde{\pi}}(n)}{n} \ge \frac{\kappa}{3} \log z + O_m\big(z^{-\frac{1}{4}} C(\pi \times \widetilde{\pi})\big).$$

Therefore, if $z \gg_m C(\pi \times \tilde{\pi})^4$, then

$$\sum_{n \le z} \frac{a_{\pi \times \widetilde{\pi}}(n)}{n} \ge 1 + \sum_{\sqrt{z} < n \le z} \frac{a_{\pi \times \widetilde{\pi}}(n)}{n} \ge \frac{1}{3}(1 + \kappa \log z).$$

Using this bound in (6.8) and then in (6.7), and noting that for all $\kappa > 0$ one has $\kappa/(1 + \kappa \log z) \le 1/\log z$, the proposition follows. □

Proof of Theorem 2.4

Since Theorem 2.4 follows from (1.11) for $m = 1$, we may assume below that $m \ge 2$. Suppose that $x \gg_m C(\pi \times \widetilde{\pi})^{36m^2}$, and that $1 \le T \le x^{1/9m^2}$. Take $z = x^{1/9m^2}$, and Φ to be a smooth nonnegative function supported in $(-\epsilon, 1+\epsilon)$ with $\Phi(t) = 1$ for $0 \le t \le 1$. An application of Proposition 6.2 gives

$$\sum_{\substack{x < n \le x e^{1/T} \\ p \mid n \Longrightarrow p > z}} a_{\pi \times \widetilde{\pi}}(n) \ll_m \frac{x}{T \log z} + x^{\frac{1}{2}} C(\pi \times \widetilde{\pi}) T^{m^2} z^{2m^2+3} \ll_m \frac{x}{T \log x}.$$

The left-hand side above includes all prime powers p^k in $(x, xe^{1/T}]$ with $p > z$, and so we conclude that

$$\sum_{\substack{x < p^k \le x e^{1/T} \\ k \le 9m^2}} a_{\pi \times \widetilde{\pi}}(p^k) \ll_m \frac{x}{T \log x}. \tag{6.9}$$

In Theorem 2.4, we are interested in bounding $\lambda_{\pi\times\widetilde{\pi}}(p^k)$ in place of $a_{\pi\times\widetilde{\pi}}(p^k)$ above. Note from (1.2) that for any given prime p, we have the formal identity

$$\exp\Big(\sum_{k=1}^{\infty}\frac{\lambda_{\pi\times\widetilde{\pi}}(p^k)}{k}X^k\Big)=1+\sum_{k=1}^{\infty}a_{\pi\times\widetilde{\pi}}(p^k)X^k.$$

Expanding both sides and comparing coefficients, from the nonnegativity of $\lambda_{\pi\times\widetilde{\pi}}(p^k)$ and $a_{\pi\times\widetilde{\pi}}(p^k)$, we deduce that

$$a_{\pi\times\widetilde{\pi}}(p^k)\geq\frac{\lambda_{\pi\times\widetilde{\pi}}(p^k)}{k}. \tag{6.10}$$

From (6.9) and (6.10), it follows that

$$\sum_{\substack{x<n=p^k\leq xe^{1/T}\\ k\leq 9m^2}}\lambda_{\pi\times\widetilde{\pi}}(n)\Lambda(n)\ll_m\frac{x}{T}.$$

To complete the proof of Theorem 2.4 it remains lastly to bound the contribution of primes powers p^k with $k>9m^2$. Since there are very few such prime powers, it will be enough to use a crude bound on $\lambda_{\pi\times\widetilde{\pi}}(p^k)$. From (2.6), one obtains $\lambda_{\pi\times\widetilde{\pi}}(p^k)\leq m^2p^{k(1-1/m^2)}$, and so

$$\sum_{\substack{x<n=p^k\leq xe^{1/T}\\ k>9m^2}}\lambda_{\pi\times\widetilde{\pi}}(n)\Lambda(n)\ll_m x^{1-1/m^2}\sum_{\substack{p^k\leq ex\\ k>9m^2}}\Lambda(m)\ll_m x^{1-8/9m^2}\ll_m\frac{x}{T}.$$

This finishes the proof of Theorem 2.4. □

Appendix. An inequality on Rankin–Selberg coefficients

Farrell Brumley[1]

Let π, π' be irreducible unitary generic representations of $\mathrm{GL}_m(\mathbb{Q}_p)$ and $\mathrm{GL}_{m'}(\mathbb{Q}_p)$, respectively. Let $L(s,\pi\times\pi')$ be the local Rankin–Selberg L-factor. Write its logarithm as

$$\log L(s,\pi\times\pi')=\sum_{f\geq 1}\frac{\lambda_{\pi\times\pi'}(p^f)}{fp^{fs}}.$$

Our aim is to prove the following inequality.

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PROPOSITION A.1
For every $f \geq 1$, we have

$$\left|\lambda_{\pi\times\pi'}(p^f)\right| \leq \sqrt{\lambda_{\pi\times\tilde{\pi}}(p^f)\lambda_{\pi'\times\tilde{\pi}'}(p^f)} \leq \frac{1}{2}\left(\lambda_{\pi\times\tilde{\pi}}(p^f) + \lambda_{\pi'\times\tilde{\pi}'}(p^f)\right).$$

The model computation is when π and π' are both unramified. In this case, the proposition is immediate from the well-known expression for the local Rankin–Selberg L-factor

$$L(s,\pi\times\pi') = \prod_{j=1}^{m}\prod_{k=1}^{m'}\left(1-\alpha_\pi(p,j)\alpha_{\pi'}(p,k)p^{-s}\right)^{-1} \tag{A.1}$$

in terms of the Satake parameters $\alpha_\pi(p,j)$ and $\alpha_{\pi'}(p,k)$. From this, it follows that the coefficients $\lambda_{\pi\times\pi'}(p^f)$ are given by

$$\lambda_{\pi\times\pi'}(p^f) = \sum_{j=1}^{m}\sum_{k=1}^{m'}\alpha_\pi(p,j)^f\alpha_{\pi'}(p,k)^f = \lambda_\pi(p^f)\lambda_{\pi'}(p^f). \tag{A.2}$$

Similarly, in the unramified situation, $\lambda_{\pi\times\tilde{\pi}}(p^f) = |\lambda_\pi(p^f)|^2$ and $\lambda_{\pi'\times\tilde{\pi}'}(p^f) = |\lambda_{\pi'}(p^f)|^2$. Thus, $|\lambda_{\pi\times\pi'}(p^f)| = \sqrt{\lambda_{\pi\times\tilde{\pi}}(p^f)\lambda_{\pi'\times\tilde{\pi}'}(p^f)}$, and the proposition follows from the inequality $|AB| \leq \frac{|A|^2+|B|^2}{2}$ of geometric and arithmetic means. The proof of Proposition A.1 follows along the same lines, but we shall need a more explicit description of the Rankin–Selberg local L-factors. The main issue is that, contrary to the unramified case, the local roots of the Rankin–Selberg convolution are not simply the products of the local roots of the standard L-function.

A.1. Description of local Rankin–Selberg factor
In this section, we describe the local Rankin–Selberg L-function $L(s,\pi\times\pi')$ in terms of representation theoretic data. The main identity is (A.6) below. We follow closely the exposition in [27, Appendix A], where the case when $\pi' \simeq \tilde{\pi}$ was explicated.

We begin by realizing π as a Langlands quotient

$$\pi = J\left(G, P; \tau_1[\sigma_1], \ldots, \tau_r[\sigma_r]\right). \tag{A.3}$$

Here $G = \mathrm{GL}_m(\mathbb{Q}_p)$, P is a standard parabolic of G corresponding to the partition $(m_1,\ldots,m_r)$ of m, τ_j is a tempered representation of $\mathrm{GL}_{m_j}(\mathbb{Q}_p)$, the real numbers σ_j satisfy $\sigma_1 \geq \cdots \geq \sigma_r$, and $\tau[\sigma]$ denotes the representation $\tau \otimes |\det|^\sigma$. Similar notation holds for π'. Then

$$L(s,\pi\times\pi') = \prod_{j=1}^{r}\prod_{k=1}^{r'} L(s+\sigma_j+\sigma'_k, \tau_j\times\tau'_k). \tag{A.4}$$

Next we use the fact that tempered representations of $\mathrm{GL}_m(\mathbb{Q}_p)$ are fully induced representations from discrete series. Moreover, discrete series themselves can be constructed as generalized Speh representations, obtained through an induction procedure from supercuspidals as follows. For any discrete series representation δ on $\mathrm{GL}_m(\mathbb{Q}_p)$, there is a divisor $d \mid m$ and a unitary supercuspidal representation ρ on $\mathrm{GL}_d(\mathbb{Q}_p)$ such that δ is isomorphic to the unique square-integrable subquotient of the representation

$$\bigtimes_{\nu=1}^{n} \rho\big[\nu - (n+1)/2\big]$$

induced from the standard Levi

$$\underbrace{\mathrm{GL}_d(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_d(\mathbb{Q}_p)}_{n},$$

where $n = m/d$.

We apply this for every τ_j appearing in (A.3) to obtain integers $d_j \mid m_j$, $n_j = m_j/d_j$ and unitary supercuspidals ρ_j on $\mathrm{GL}_{d_j}(\mathbb{Q}_p)$. We proceed similarly for π'. Using induction by stages (to combine the reduction of tempered representations τ to discrete series δ with the reduction of discrete series δ to supercuspidals ρ), we obtain

$$L(s, \pi \times \pi') = \prod_{j=1}^{r} \prod_{k=1}^{r'} \prod_{\nu=1}^{\min(n_j, n_k')} L\Big(s + \sigma_j + \sigma_k' + \frac{n_j + n_k'}{2} - \nu, \rho_j \times \rho_k'\Big). \quad \text{(A.5)}$$

We now organize the ρ_j and ρ_k' into twist-equivalence classes. Let

(1) $\underline{J} = [J_1, \dots, J_A]$ be a set partition of $\{1, \dots, r\}$;
(2) $\underline{K} = [K_1, \dots, K_B]$ be a set partition of $\{1, \dots, r'\}$;
(3) $\{\varrho_1, \dots, \varrho_L\}$ be a set of unitary twist-inequivalent supercuspidal representations ϱ_ℓ of a general linear group over $\mathbb{Q}_p$,

with the property that

(1) for every $a \in \{1, \dots, A\}$, there is a $\ell = \ell(a) \in \{1, \dots, L\}$, and for every $j \in J_a$, there is $t_j \in \mathbb{R}$ such that $\rho_j \simeq \varrho_\ell[it_j]$;
(2) for every $b \in \{1, \dots, B\}$, there is a $\ell' = \ell'(b) \in \{1, \dots, L\}$, and for every $k \in K_b$, there is $t_k' \in \mathbb{R}$ such that $\rho_k' \simeq \varrho_{\ell'}[it_k']$;
(3) the assignments $a \mapsto \ell(a)$ and $b \mapsto \ell'(b)$ are injective.

In this way, for any $a \in \{1, \dots, A\}$, the set $\{\rho_j : j \in J_a\}$ consists of all those ρ_j appearing in (A.5) which are twist equivalent to some given $\varrho_{\ell(a)}$. We may assume, if we wish, that the set $\{\varrho_1, \dots, \varrho_L\}$ is minimal for this property. Setting $s_j = \sigma_j + it_j$, $s_k' = \sigma_k' + it_k'$, and

$$L_{J_a, K_b}(s) = \prod_{j \in J_a} \prod_{k \in K_b} \prod_{\nu=1}^{\min(n_j, n_k')} L\Big(s + s_j + s_k' + \frac{n_j + n_k'}{2} - \nu, \varrho_{\ell(a)} \times \varrho_{\ell'(b)}\Big),$$

we obtain the following expression

$$L(s,\pi\times\pi')=\prod_{a=1}^{A}\prod_{b=1}^{B}L_{J_a,K_b}(s).$$

Now, many of the factors in the above product are simply 1. Indeed, for supercuspidal representations ϱ on $\mathrm{GL}_d(\mathbb{Q}_p)$ and ϱ' on $\mathrm{GL}_{d'}(\mathbb{Q}_p)$, the local factor $L(s,\varrho\times\varrho')$ is 1 unless ϱ is twist equivalent to ϱ' (in which case $d=d'$). Otherwise, when $\varrho'=\varrho[\sigma]$, we have

$$L\big(s,\varrho\times\varrho[\sigma]\big)=L(s+\sigma,\varrho\times\varrho)=(1-p^{-e(\sigma+s)})^{-1},$$

where e is the torsion number for ϱ. (The torsion number is the order of the finite cyclic group of characters $\chi=|\det|^u$ such that $\varrho\otimes\chi\simeq\varrho$.)

We deduce that

$$L(s,\pi\times\pi')=\prod_{(a,b)\in\Delta}L_{J_a,K_b}(s),$$

where

$$\Delta=\big\{(a,b)\in\{1,\dots,A\}\times\{1,\dots,B\}:\ell(a)=\ell'(b)\big\}.$$

Let $\ell:\Delta\to\{1,\dots,L\}$ be the map sending (a,b) to $\ell(a,b):=\ell(a)=\ell'(b)$; it is injective. If e_ℓ denotes the torsion number of ϱ_ℓ, then

$$L_{J_a,K_b}(s)=\prod_{j\in J_a}\prod_{k\in K_b}\prod_{\nu=1}^{\min(n_j,n'_k)}(1-p^{-e_{\ell(a,b)}(s+s_j+s'_k+\frac{n_j+n'_k}{2}-\nu)})^{-1}.$$

Setting $z_j=p^{-s_j-n_j/2}$ and $z'_k=p^{-s'_k-n'_k/2}$, we obtain the formula

$$L(s,\pi\times\pi')=\prod_{(a,b)\in\Delta}\prod_{j\in J_a}\prod_{k\in K_b}\prod_{\nu=1}^{\min(n_j,n'_k)}\big(1-(p^\nu z_j z'_k)^{e_{\ell(a,b)}}p^{-e_{\ell(a,b)}s}\big)^{-1}.\quad(\mathrm{A.6})$$

We now give some examples to show that formula (A.6) can be specialized to recover known cases.

Example 6

When $\pi'=\tilde{\pi}$, we have $r=r'$, $\underline{J}=\underline{K}$ (so that $A=B=L$), and the subset Δ is the diagonal copy of $\{1,\dots,A\}$ inside $\{1,\dots,A\}\times\{1,\dots,A\}$. Letting $\underline{F}=[F_1,\dots,F_L]$ denote the set partition $\underline{J}=\underline{K}$ of $\{1,\dots,r\}$, we recover in this case the formula

$$L(s,\pi\times\tilde{\pi})=\prod_{l=1}^{L}\prod_{j,k\in F_l}\prod_{\nu=1}^{\min(n_j,n_k)}\left(1-(p^{\nu}z_j\overline{z}_k)^{e_l}p^{-e_l s}\right)^{-1} \tag{A.7}$$

of [27, (A.12)].

Example 7

When π and π' are both principal series representations, we have $r=m$, $r'=m'$, and $n_j\equiv n'_k\equiv 1$. If, furthermore, π and π' are both unramified, then $\underline{J}=[J_1]$, where $J_1=\{1,\dots,m\}$ and $\underline{K}=[K_1]$, where $K_1=\{1,\dots,m'\}$. Thus $A=B=L=1$, and ℓ sends $(1,1)$ to 1. Set $\alpha_\pi(p,j)=p^{s_j}$ and $\alpha_{\pi'}(p,k)=p^{s'_k}$, so that $pz_jz'_k=\alpha_\pi(p,j)\alpha_{\pi'}(p,k)$. Then (A.6) simplifies to the expression (A.1).

A.2. Proof of Proposition A.1

Let $\mathcal{L}$ denote the image of the injective map $\ell:\Delta\to\{1,\dots,L\}$. Throughout this section, we shall write $(a,b)\in\Delta$ for the preimage of $\ell\in\mathcal{L}$. We may rewrite (A.6) as

$$L(s,\pi\times\pi')=\prod_{\ell\in\mathcal{L}}L_\ell(s,\pi\times\pi'),$$

where

$$L_\ell(s,\pi\times\pi')=\prod_{\nu\geq 1}\prod_{\substack{j\in J_a\\ n_j\geq\nu}}\prod_{\substack{k\in K_b\\ n'_k\geq\nu}}\left(1-(p^{\nu}z_jz'_k)^{e_\ell}p^{-e_\ell s}\right)^{-1}.$$

Letting $\log L_\ell(s,\pi\times\pi')=\sum_{f\geq 1}\frac{\lambda_{\ell,\pi\times\pi'}(f)}{fp^{e_\ell fs}}$, we obtain

$$\lambda_{\ell,\pi\times\pi'}(f)=\sum_{\nu\geq 1}p^{e_\ell\nu f}\Big(\sum_{\substack{j\in J_a\\ n_j\geq\nu}}z_j^{e_\ell f}\Big)\Big(\sum_{\substack{k\in K_b\\ n'_k\geq\nu}}z_k'^{\,e_\ell f}\Big). \tag{A.8}$$

Example 8

We let $\pi'=\tilde{\pi}$, and we use the notation of Example 6. Then the identity (A.8) reduces to

$$\lambda_{\ell,\pi\times\tilde{\pi}}(f)=\sum_{\nu\geq 1}p^{e_\ell\nu f}\Big|\sum_{\substack{j\in F_\ell\\ n_j\geq\nu}}z_j^{e_\ell f}\Big|^2, \tag{A.9}$$

which recovers the same expression in the proof of [27, Lemma A.1].

Example 9

When π and π' are both unramified, formula (A.8) reduces to

$$\lambda_{\ell,\pi\times\pi'}(f) = p^f \sum_{j=1}^{m} z_j^f \sum_{k=1}^{m'} {z'_k}^f = \sum_{j=1}^{m}\sum_{k=1}^{m'} \alpha_\pi(p,j)^f \alpha_{\pi'}(p,k)^f = \lambda_{\pi\times\pi'}(p^f).$$

Applying the Cauchy–Schwarz inequality in (A.8) we get

$$\begin{aligned}\big|\lambda_{\ell,\pi\times\pi'}(f)\big|^2 &\le \Big(\sum_{\nu\ge1} p^{e_\ell \nu f}\Big|\sum_{\substack{j\in J_a\\ n_j\ge\nu}} z_j^{e_\ell f}\Big|^2\Big)\Big(\sum_{\nu\ge1} p^{e_\ell\nu f}\Big|\sum_{\substack{k\in K_a\\ n'_k\ge\nu}} {z'_k}^{e_\ell f}\Big|^2\Big)\\ &= \lambda_{\ell,\pi\times\tilde\pi}(f)\lambda_{\ell,\pi'\times\tilde\pi'}(f), \end{aligned} \tag{A.10}$$

in view of (A.9).

Now from

$$\begin{aligned}\sum_{f\ge1} p^{-fs}\Big(\frac{\lambda_{\pi\times\pi'}(p^f)}{f}\Big) &= \log L(s,\pi\times\pi')\\ &= \sum_{\ell\in\mathfrak{L}} \log L_\ell(s,\pi\times\pi')\\ &= \sum_{\ell\in\mathfrak{L}}\sum_{f\ge1} p^{-e_\ell fs}\Big(\frac{\lambda_{\ell,\pi\times\pi'}(f)}{f}\Big)\\ &= \sum_{f\ge1} p^{-fs}\sum_{\substack{\ell\in\mathfrak{L}\\ e_\ell\mid f}}\Big(\frac{\lambda_{\ell,\pi\times\pi'}(f/e_\ell)}{f/e_\ell}\Big),\end{aligned}$$

we deduce

$$\lambda_{\pi\times\pi'}(p^f) = \sum_{\substack{\ell\in\mathfrak{L}\\ e_\ell\mid f}} e_\ell \lambda_{\ell,\pi\times\pi'}(f/e_\ell). \tag{A.11}$$

Using this and (A.10) we find, by Cauchy–Schwarz,

$$\begin{aligned}\big|\lambda_{\pi\times\pi'}(p^f)\big| &\le \sum_{\substack{\ell\in\mathfrak{L}\\ e_\ell\mid f}} e_\ell\big|\lambda_{\ell,\pi\times\pi'}(f/e_\ell)\big|\\ &\le \Big(\sum_{\substack{\ell\in\mathfrak{L}\\ e_\ell\mid f}} e_\ell\lambda_{\ell,\pi\times\tilde\pi}(f/e_\ell)\Big)^{1/2}\Big(\sum_{\substack{\ell\in\mathfrak{L}\\ e_\ell\mid f}} e_\ell\lambda_{\ell,\pi'\times\tilde\pi'}(f/e_\ell)\Big)^{1/2}.\end{aligned}$$

From (A.11) we recognize the right-hand side as $\sqrt{\lambda_{\pi\times\tilde\pi}(p^f)\lambda_{\pi'\times\tilde\pi'}(p^f)}$, proving Proposition A.1. □

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