REGULAR RINGS AND PERFECT(OID) ALGEBRAS

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ABSTRACT. We prove a p-adic analog of Kunz's theorem: a p-adically complete noetherian ring is regular exactly when it admits a faithfully flat map to a perfectoid ring. This result is deduced from a more precise statement on detecting finiteness of projective dimension of finitely generated modules over noetherian rings via maps to perfectoid rings. We also establish a version of the p-adic Kunz's theorem where the flatness hypothesis is relaxed to almost flatness.

1. INTRODUCTION

This paper explores some homological properties of perfect(oid) algebras over commutative noetherian rings. A commutative ring of positive characteristic p is called perfect if its Frobenius endomorphism is an isomorphism. Perfectoid rings are generalizations of perfect rings to mixed characteristic (Definition 3.5). Their most important features for our work are: if A is a perfectoid ring, then $\sqrt{pA}$ is a flat ideal, $A/\sqrt{pA}$ is a perfect ring, and finitely generated radical ideals in A containing p have finite flat dimension (Lemma 3.7).

One of our main results is that over a noetherian local ring R, any perfectoid R-algebra A with $\mathfrak{m}A \neq A$ detects finiteness of homological dimension of R-modules. More precisely, given such an A, if a finitely generated R-module M satisfies $\operatorname{Tor}_j^R(A,M)=0$ for $j \gg 0$, then M has a finite free resolution by R-modules (Theorem 4.1). The crucial property of A that is responsible for this phenomenon is isolated in Theorem 2.1, which identifies a large class of modules that can detect finiteness of homological dimension over local rings.

As a consequence, we obtain a mixed characteristic generalization of Kunz's theorem, resolving a question from [9, Remark 5.5]; see also [3, pp. 6]. Recall that Kunz's theorem asserts that a noetherian ring R of characteristic p is regular if and only if the Frobenius map $R \to R$ is flat. One can reformulate this result as the following assertion: such an R is regular exactly when there exists a faithfully flat map $R \to A$ with A perfect. Our p-adic generalization is the following:

Theorem (see Theorem 4.7). *Let R be a noetherian ring such that p lies in the Jacobson radical of R (for example, R could be p-adically complete). Then R is regular if and only if there exists a faithfully flat map $R \to A$ with A perfectoid.*

Two algebras are of special interest: the absolute integral closure, R^+, of a domain R, and the perfection, R_{perf}, of a local ring R of positive characteristic. We prove that if R is an excellent local domain of positive characteristic and $\operatorname{Tor}_j^R(R^+,k)=0$ or $\operatorname{Tor}_j^R(R_{\mathrm{perf}},k)=0$

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for *some* $j \geq 1$, then R is regular; see Theorem 4.13, which contains also a statement about R^+ for local rings of mixed characteristic. A key input in its proof is that systems of parameters for R are weakly proregular on R^+ and on R_{perf} (Lemma 4.10).

Over a perfectoid ring A one has often to consider modules that are almost zero, meaning that they are annihilated by $\sqrt{pA}$. In particular, in the context of the theorem above, the more reasonable hypothesis on the map $R \to A$ is that it is almost flat, that is to say that $\mathrm{Tor}_i^R(-,A)$ is almost zero for $i \geq 1$. With this in mind, in Section 5, we establish the following, more natural, extension of the p-adic Kunz theorem in the almost setting.

Theorem (see Corollary 5.7)**.** *Let R be a noetherian p-torsionfree ring containing p in its Jacobson radical. If there exists a map $R \to A$ with A perfectoid that is almost flat and zero is the only R-module M for which $M \otimes_R A$ is zero, then R is regular.*

2. Criteria for finite flat dimension

Let S be a ring; throughout this work rings will be commutative but not always noetherian. The flat dimension of an S-module N is denoted $\operatorname{flat\,dim}_S(N)$; when S is noetherian and N is finitely generated, this coincides with the projective dimension of N. By a local ring $(R,\mathfrak{m},k)$ we mean that R is a ring with unique maximal ideal $\mathfrak{m}$ and residue field k.

Theorem 2.1. *Let $(R,\mathfrak{m},k)$ be a noetherian local ring and S an R-algebra containing an ideal J with $\mathfrak{m}S \subseteq J$ and $d := \operatorname{flat\,dim}_S(S/J)$ finite. Let U be an S-module with $JU \neq U$.*

If an R-module M has $\mathrm{Tor}_i^R(U,M) = 0$ for $i = s,\dots,s+d$ and some integer $s \geq 0$, then

$$\mathrm{Tor}_{s+d}^R(k,M) = 0\,.$$

In particular, if $\mathrm{Tor}_j^R(U,M) = 0$ for $j \geq s$, then $\mathrm{Tor}_j^R(k,M) = 0$ for $j \geq s+d$.

Proof. Set $V := (S/J) \otimes_S^{\mathbf{L}} U$, viewed as a complex of S-modules. The hypothesis is that $\mathrm{H}_i(U \otimes_R^{\mathbf{L}} M) = 0$ for $i = s,\dots,s+d$. Given the quasi-isomorphism of complexes

$$V \otimes_R^{\mathbf{L}} M \simeq (S/J) \otimes_S^{\mathbf{L}} (U \otimes_R^{\mathbf{L}} M)$$

it follows by, for example, a standard spectral sequence argument that $\mathrm{H}_{s+d}(V \otimes_R^{\mathbf{L}} M) = 0$. Since $\mathfrak{m}S \subseteq J$, the action of R on S/J and hence also on V, factors through $R/\mathfrak{m}$, that is to say, through k. Thus one has a quasi-isomorphism

$$V \otimes_R^{\mathbf{L}} M \simeq V \otimes_k^{\mathbf{L}} (k \otimes_R^{\mathbf{L}} M)$$

of complexes of R-modules. The Künneth isomorphism then yields the isomorphism below

$$0 = \mathrm{H}_{s+d}(V \otimes_R^{\mathbf{L}} M) \cong \bigoplus_j \mathrm{H}_j(V) \otimes_k \mathrm{H}_{s+d-j}(k \otimes_R^{\mathbf{L}} M)\,.$$

Since $\mathrm{H}_0(V) \cong U/JU$ is nonzero, by hypotheses, it follows that $\mathrm{H}_{s+d}(k \otimes_R^{\mathbf{L}} M) = 0$. □

Corollary 2.2. *Let S and U be as in Theorem 2.1. If a finitely generated R-module M satisfies $\mathrm{Tor}_i^R(U,M) = 0$ for $i = s,\dots,s+d$ and some integer $s \geq 0$, then M has a finite free resolution of length at most $s+d$.*

Proof. Since R is a noetherian local ring, $\operatorname{flat\,dim}_R(M) = \sup\{i \mid \mathrm{Tor}_i^R(k,M) \neq 0\}$ when M is finitely generated. Thus the desired result is a direct consequence of Theorem 2.1. □

A natural question that arises is whether $\mathrm{Tor}_s^R(U,M) = 0$ for some $s \geq 0$ ensures that M has a finite free resolution. This is indeed the case for $M = k$; the argument depends on a rigidity result, recalled below.

2.3. **Rigidity.** Let $(R,\mathfrak{m},k)$ be a noetherian local ring and N an R-module. Set

$$s(N) := \sup\{t \mid \mathrm{H}^t_{\mathfrak{m}}(\mathrm{Hom}_R(N,E)) \neq 0\}.$$

where E is the injective hull of the R-module k and $\mathrm{H}^*_{\mathfrak{m}}(-)$ denotes local cohomology with respect to $\mathfrak{m}$. The following statements hold:

(1) $s(N) \le \dim R$;
(2) If $\mathrm{Tor}^R_i(N,k) = 0$ for some $i \ge s(N)$, then $\mathrm{Tor}^R_j(N,k) = 0$ for all $j \ge i$.

Indeed (1) holds because $\mathrm{H}^i_{\mathfrak{m}}(-) = 0$ for $i > \dim R$; see [14, Theorem 3.5.7]. Part (2) is contained in [15, Proposition 3.3] due to Christensen, Iyengar, and Marley.

Corollary 2.4. *Let S and U be as in Theorem 2.1. If $\mathrm{Tor}^R_i(U,k) = 0$ for some $i \ge \dim R$, then R is regular.*

Proof. By 2.3 one has $\mathrm{Tor}^R_j(U,k) = 0$ for all $j \ge i$. Theorem 2.1 then yields $\mathrm{Tor}^R_j(k,k) = 0$ for $j \gg 0$, and so R is regular; see by [14, Theorem 2.2.7]. □

The preceding result may be viewed as an generalization of the descent of regularity along homomorphisms of finite flat dimension. Indeed, if $(R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ is a local homomorphism with S regular and $\mathrm{flat\,dim}_R S$ finite, then Corollary 2.4 applies with $U := S$ and $J := \mathfrak{n}$ to yield that R is regular.

Remark 2.5. It should be clear from the proof of Theorem 2.1 that the ring S in the hypothesis may well be a differential graded algebra, U a dg S-module with $\mathrm{H}(k \otimes^{\mathbf{L}}_R U) \neq 0$, and M an R-complex; in that generality, the result compares with [6, Theorem 6.2.2].

3. Recollections on perfect and perfectoid rings

In this section, we recall the definition of perfect and perfectoid rings (with examples) and summarize their homological features most relevant to us. Fix a prime number p. In what follows $\mathbb{Z}_p$ denotes the p-adic completion of $\mathbb{Z}$.

3.1. **Perfect rings.** Let A be a commutative ring of positive characteristic p and $\varphi\colon A \to A$ the Frobenius endomorphism: $\varphi(a) = a^p$ for each $a \in A$. The ring A *perfect* if φ is bijective; such an A is reduced.

The *perfect closure* of a ring A is the colimit $A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$, denoted A_{perf}. It is easy to verify that A_{perf} is a perfect ring of characteristic p, and the map $A \to A_{\mathrm{perf}}$ is the universal map from A to such a ring. Moreover the kernel of the canonical map $A \to A_{\mathrm{perf}}$ is precisely $\sqrt{0}$, the nilradical of A; in particular, when A is reduced, we identify A as a subring of A_{perf}.

Each element x in a perfect ring A has a unique p^e-th root, for each $e \ge 1$. We set

$$(x^{1/p^\infty}) := \bigcup_{e=1}^{\infty} (x^{1/p^e})A$$

This ideal is reduced for it equals $\sqrt{xA}$.

Lemma 3.2. *Let S be a perfect ring of positive characteristic. For any set $x_1,\dots,x_n$ of elements of S, the ideal $J := (x_1^{1/p^\infty},\dots,x_n^{1/p^\infty})$ of S satisfies $\mathrm{flat\,dim}_S(S/J) \le n$.* □

This result is due to Aberbach and Hochster [2, Theorem 3.1]. We recall an elementary proof from [12, Lemma 3.16].

Proof. If R is any perfect ring of positive characteristic and $f \in R$, then $R/(f^{1/p^\infty})$ is reduced and hence also perfect. Thus, by induction, it is enough to treat the case when $n = 1$. Relabel $x = x_1$ for visual convenience. It suffices to check that the ideal $I := (x^{1/p^\infty}) \subset S$ is flat as an S-module. Consider the direct limit

$$M := \varinjlim \Big(S \xrightarrow{x^{1-\frac{1}{p}}} S \xrightarrow{x^{\frac{1}{p}-\frac{1}{p^2}}} S \to \cdots \to S \xrightarrow{x^{\frac{1}{p^n}-\frac{1}{p^{n+1}}}} S \to \dots \Big).$$

As M is a direct limit of free S-modules, it is flat. There is a natural map $M \to I$ determined by sending $1 \in S$ in the n-th spot of the diagram above to $x^{\frac{1}{p^n}} \in I$. It is enough to show that this map is an isomorphism. Surjectivity holds as all generators of I are hit.

As to injectivity, pick an element $\overline{g} \in M$ in the kernel. Then $\overline{g}$ lifts to an element $g \in S$ (viewed in the n-th copy of S in the diagram above for some n) such that $g \cdot x^{\frac{1}{p^n}} = 0$. But then $(g \cdot x^{\frac{1}{p^{n+1}}})^p = 0$ and so $g \cdot x^{\frac{1}{p^{n+1}}} = 0$, as S is reduced. As $\frac{1}{p^n} - \frac{1}{p^{n+1}} \geq \frac{1}{p^{n+1}}$, we get $g \cdot x^{\frac{1}{p^n}-\frac{1}{p^{n+1}}} = 0$, so g is killed by the transition map in the system above, whence $\overline{g} = 0$. □

Remark 3.3. An important property of perfect rings is that they admit canonical (and unique) lifts to $\mathbb{Z}_p$. Indeed, if A is a perfect ring of characteristic p, then the ring $W(A)$ of Witt vectors of A is a p-torsionfree and p-adically complete ring with $W(A)/p \cong A$. In fact, any such lift is uniquely isomorphic to $W(A)$: the functors $A \mapsto W(A)$ and $B \mapsto B/p$ implement an equivalence between the categories of perfect rings of characteristic p and the category of p-torsionfree and p-adically complete rings B with B/p being perfect. This perspective can help guess $W(A)$ in concrete situations. For instance, it follows that the p-adic completion of $\mathbb{Z}[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]$ coincides with the Witt vectors of $\mathbb{F}_p[x_1, \dots, x_d]_{\mathrm{perf}}$.

3.4. **Perfectoid rings.** Let us recall the notion of a perfectoid ring from [10, Definition 3.5]; this is sometimes referred to as *integral perfectoid* to emphasize its integral nature, and to contrast it with the perfectoid Tate rings that arise in the context of perfectoid spaces.

For any commutative ring A, consider the *tilt* $A^\flat$ of A defined by

$$A^\flat := \varprojlim_{x \mapsto x^p} A/p\,.$$

This ring is perfect of characteristic p and the projection map $A^\flat \to A/p$ is the universal map from a perfect ring to A/p. Set

$$A_{\mathrm{inf}}(A) := W(A^\flat)\,.$$

When A is p-adically complete, the projection map $A^\flat \to A/p$ lifts uniquely to a map $\theta : A_{\mathrm{inf}}(A) \to A$, called *Fontaine's θ-map*.

Definition 3.5. A ring A is *perfectoid* if the following conditions hold:

(1) A is p-adically complete.
(2) The Frobenius $A/p \to A/p$ is surjective.
(3) The kernel of Fontaine's map $\theta : A_{\mathrm{inf}}(A) \to A$ is principal.
(4) There exists an element $\varpi \in A$ with $\varpi^p = pu$ for a unit u.

The category of perfectoid rings is the full subcategory of all commutative rings spanned by perfectoid rings.

There is a more explicit characterization of perfectoid rings in terms of Teichmuller expansions that avoids directly contemplating the $A_{\mathrm{inf}}(-)$ construction. Recall that for any perfect ring B of characteristic p, each $f \in W(B)$ can be written *uniquely* as $\sum_{i=0}^\infty [b_i] p^i$

where $b_i \in B$ and the map $[\cdot] : B \to W(B)$ is the multiplicative (but not additive) Teichmuller section to the projection $W(B) \to B$; we refer to this presentation as the Teichmuller expansion of f. One then has the following characterization of perfectoidness:

Lemma 3.6. *A ring A is perfectoid if and only if there exists a perfect ring B of characteristic p and an isomorphism $W(B)/(\xi) \cong A$, where the coefficient of p in the Teichmuller expansion of ξ is a unit (i.e. ξ is primitive) and B is $(\overline{\xi})$-adically complete, where $\overline{\xi}$ is the image of ξ in B. For such a B, there is a unique identification $B \cong A^\flat$ compatible with the isomorphism $W(B)/(\xi) \cong A$. In particular, the element ϖ appearing in Definition 3.5 can be assumed to admit a compatible system $\{\varpi^{1/p^n}\}$ of p-power roots.*

The equivalence of this definition with Definition 3.5 can be deduced from the discussion in [10, §3] (and is presumably present in [19, §16]). The analogous characterization of perfectoid Tate rings can be found in [20, Proposition 1.1], [31, Theorem 3.6.5] and [37, Theorem 3.17]. An explicit reference is [33, Remark 8.6]. (We are grateful to Scholze for bringing [33] to our attention.) For the convenience of the reader, we sketch the proof.

Proof sketch. The "only if" direction is immediate from [10, Remark 3.11] as we can simply take $B = A^\flat$, so $W(B) = A_{\mathrm{inf}}(A)$. For the "if" direction, the formula $W(B)/(\xi) \cong A$ shows that $B/\overline{\xi} \cong A/p$, whence $B \cong A^\flat$ as B is perfect and $(\overline{\xi})$-adically complete; the formula $A \cong W(B)/(\xi)$ can then be rewritten as $A \cong A_{\mathrm{inf}}(A)/(\xi)$, which immediately gives the perfectoidness of A.

Finally, for $A = W(A^\flat)/(\xi)$, up to multiplication by units, we have $\xi = p - [a_0]u$ where $u \in W(A^\flat)^*$ and $a_0 \in A^\flat$. Set $\varpi \in A$ to be the image of $[a_0^{1/p}]$. Then $\varpi \in A$ satisfies (4) in Definition 3.5 and admits a compatible system of p-power roots $\varpi^{1/p^n} := [a_0^{1/p^{n+1}}]$. □

Perfectoid rings are reduced; this follows by combining [19, Corollary 16.3.61 (i)] with [10, Remark 3.8], or by arguing as in the first proof of [11, Proposition 4.18 (3)], or simply by [33, Lemma 8.9]. The most important feature of perfectoid rings for the purposes of this paper is that they are perfect modulo a flat ideal.

Lemma 3.7. *Let A be a perfectoid ring. Then the ideal $\sqrt{pA} \subset A$ is flat and $\overline{A} := A/\sqrt{pA}$ is a perfect ring of characteristic p. Moreover, if $J \subset A$ is any radical ideal containing p such that $J\overline{A}$ generated by n elements up to radicals, then* $\mathrm{flat\,dim}_A(A/\sqrt{J}) \leq n+1$.

The construction $A \mapsto \overline{A}$ gives a functor from perfectoid rings to perfect rings, left adjoint to the inclusion in the other direction (see Example 3.8 (1) below).

Proof. Write $A = W(A^\flat)/(\xi)$ and let ϖ be as in Lemma 3.6, with a compatible system of p-power roots. We first show that $A/(\varpi^{1/p^\infty})$ is perfect and thus reduced; since $(p) = (\varpi^p)$, this will imply that $\sqrt{pA} = (\varpi^{1/p^\infty})$ and hence that $A/\sqrt{pA}$ is perfect. Note that

$$A/(\varpi^{1/p^\infty}) \cong W(A^\flat)/(p - [a_0]u, [a_0^{1/p^\infty}]) \cong A^\flat/(a_0^{1/p^\infty}).$$

This ring is perfect as $A^\flat$ is perfect, proving the claim.

Next, we check that $\mathrm{flat\,dim}_A(\overline{A}) \leq 1$; this is equivalent to showing that $\sqrt{pA}$ is a flat ideal. Note that if A is p-torsionfree, then $\sqrt{pA} = (\varpi^{1/p^\infty})$ is a rising union of principal ideals generated by elements which are not zero divisors, and thus trivially flat. In general, as $\overline{A}$ has characteristic p, it is enough to check that p-complete Tor amplitude of $\overline{A} \in D(A)$

lies in $[-1,0]$, in the sense of [11, §4.1]. By [10, Lemma 3.13], the square

$$\begin{array}{ccc} W(A^\flat) & \longrightarrow & A \\ \downarrow & & \downarrow \\ W(\overline{A}) & \longrightarrow & \overline{A} \end{array}$$

is a Tor-independent pushout square. By base change, it is enough to check that p-complete Tor amplitude of $W(\overline{A}) \in D(W(A^\flat))$ lies in $[-1,0]$. As p is not a zero divisor on both $W(A^\flat)$ and $W(\overline{A})$, this reduces to checking that the Tor amplitude of $\overline{A}$ over $A^\flat$ lies in $[-1,0]$. But this follows from Lemma 3.2 since $\overline{A} = A^\flat/(a_0^{1/p^\infty})$.

The final assertion about flat dimensions now also follows from Lemma 3.2 and transitivity of flat dimensions. □

Example 3.8. Let us give some examples relevant to this paper.

(1) *Perfect rings.* A ring A of characteristic p is perfectoid if and only if it is perfect. The "if" direction is immediate: the kernel of $A_{\mathrm{inf}}(A) \cong W(A) \to A$ is generated by p, and we may take $\varpi = 0$. For the reverse implication, see [34, Example 3.11]. In terms of Lemma 3.6, these are exactly the perfectoid rings of the form $W(B)/(p)$ for a perfect ring B of characteristic p.

(2) *Absolute integral closures.* If A is an absolutely integrally closed domain, then its p-adic completion $\widehat{A}$ is perfectoid. This is clear from the previous example when A has characteristic p. In the mixed characteristic case, the ring A is a p-torsionfree ring that contains an element $\varpi \in A$ with $\varpi^p = p$. As A is absolutely integrally closed, the p-power map map $A/(\varpi) \to A/(\varpi^p)$ is readily checked to be an isomorphism. But then the same holds true for $\widehat{A}/(\varpi) \to \widehat{A}/(\varpi^p)$. The perfectoidness of $\widehat{A}$ follows from [10, Lemma 3.10].

(3) *Perfectoid polynomial rings.* The p-adic completion A of $\mathbb{Z}[p^{1/p^\infty}, x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]$ is perfectoid. In terms of the characterization in Lemma 3.6, one takes B to be the t-adic completion of $\mathbb{F}_p[t, x_1, \dots, x_d]_{\mathrm{perf}}$ and $\xi = p - [t]$.

(4) *Perfectoidification of unramified regular local rings.* Let $(R, \mathfrak{m}, k)$ be a complete noetherian regular local ring of mixed characteristic. Assume k is perfect and write $W = W(k)$ for the Witt vectors of k. Assume that R is unramified, i.e., $p \notin \mathfrak{m}^2$. By the Cohen structure theorem, we can write $R \cong W[\![x_2, \dots, x_d]\!]$. Let A be the p-adic completion of

$$W[p^{1/p^\infty}, x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}] \otimes_{W[x_1,\dots,x_d]} R\,.$$

By a variant of the previous example, one checks that the ring A is perfectoid and the map $R \to A$ is faithfully flat.

(5) *Perfectoidification of ramified regular local rings.* Fix $(R, \mathfrak{m}, k)$ and W be as in the first two sentences of the previous example. Assume that R is ramified, i.e., $p \in \mathfrak{m}^2$. By the Cohen structure theorem, we can write $R = W[\![x_1, \dots, x_d]\!]/(p - f)$ where $f \in (x_1, \dots, x_d)^2$. Let A be the p-adic completion of

$$W[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}] \otimes_{W[x_1,\dots,x_d]} R\,.$$

As observed by Shimomoto [38, Proposition 4.9], the ring A is perfectoid and the map $R \to A$ is faithfully flat (see also [4, Example 3.4.5], [9, Proposition 5.2]). In terms of

Lemma 3.6, one takes B to be the $(y_1,\dots,y_d)$-adic completion of $k[y_1,\dots,y_d]_{\mathrm{perf}}$ and $\xi = p - f([y_i])$ to get that $A \cong W(B)/(\xi)$ is perfectoid.

(6) *Modifications in characteristic p.* One can construct new perfectoid rings from old ones by changing their special fibers: if A is a perfectoid ring, and $B \to \overline{A}$ is any map of perfect rings of characteristic p (such as an inclusion) then $\widetilde{B} := A \times_{\overline{A}} B$ is a perfectoid ring. Indeed, if we write $A = W(A^\flat)/(\xi)$ for a primitive element ξ, then setting $B' := A^\flat \times_{\overline{A}} B$ gives a perfect ring of characteristic p such that $W(B') \cong W(A^\flat) \times_{\overline{A}} B$. Thus. as ξ maps to 0 in $\overline{A}$, it lifts uniquely to $W(B')$. One checks that $\widetilde{B} \cong W(B')/(\xi)$ is indeed perfectoid.

(7) *Completed localizations.* If A is a perfectoid ring and $S \subset A$ is a multiplicative set, then the p-adic completion $\widehat{S^{-1}A}$ of $S^{-1}A$ is perfectoid. In particular, the p-adically completed local rings of A are perfectoid.

(8) *Products.* An arbitrary product of perfectoid rings is perfectoid. This follows immediately from Definition 3.5 as the functor $A \mapsto A_{\inf}(A) = W(A^\flat) = W(\lim_{x\mapsto x^p} A/p)$ commutes with products.

Remark 3.9. All examples above involve perfectoid rings that either have characteristic p or were p-torsionfree. One can take products to obtain examples where neither of these properties holds true. Using Example 3.8 (6), one can also construct such examples which are closely related to products, but not themselves products. In fact, all examples have this flavor: if A is a perfectoid ring, then its maximal p-torsionfree quotient $A_{tf} := A/A[p^\infty]$ is a perfectoid p-torsionfree ring, and the natural maps give an exact sequence

$$0 \to A \to A_{tf} \times \overline{A} \to \overline{A_{tf}} \to 0$$

of A-modules by [33, Remark 8.8]. This realizes A as the fiber product $A_{tf} \times_{\overline{A_{tf}}} \overline{A}$. In other words, one can construct arbitrary perfectoid rings by "modifying" p-torsionfree perfectoid rings by the procedure of Example 3.8 (6).

4. Applications

In this section we prove the results dealing with perfect, and with perfectoid, algebras stated in the Introduction. Fix a prime p.

Theorem 4.1. *Let $(R,\mathfrak{m},k)$ be a noetherian local ring and A an R-algebra with $\mathfrak{m}A \neq A$. Assume that A is perfectoid (and thus k has characteristic p). Set $n = \dim(R)$.*

If M is an R-module with $\mathrm{Tor}_i^R(A,M) = 0$ for $s \le i \le s+n+1$ and integer $s \ge 0$, then

$$\mathrm{Tor}^R_{s+n+1}(k,M) = 0\,.$$

In particular, if $\mathrm{Tor}_j^R(A,M) = 0$ for $j \gg 0$, then $\mathrm{Tor}_j^R(k,M) = 0$ for $j \gg 0$.

Proof. Let $J \subset A$ the radical of the ideal generated by a system of parameters $x_1,\dots,x_n$ of R. Then $\mathfrak{m}A \subset J$. Moreover, since J is generated by n elements up to radicals and contains p, Lemma 3.7 implies that $\operatorname{flat\,dim}_A(A/J) \le n+1$. Theorem 2.1 applied with $S = A$, $U = A$ and $d = n+1$ then implies the result. □

Remark 4.2. If the perfectoid ring A appearing in Theorem 4.1 is either perfect or p-torsionfree, the hypothesis on the vanishing range can be improved slightly: it is sufficient to require vanishing of $\mathrm{Tor}_i^R(A,M) = 0$ when $s \le i \le s+n$ for an integer $s \ge 0$.

To see this, note that we can replace the sequence $x_1,\dots x_n$ in the proof above by a system of parameters for the image $\overline{R}$ of $R \to A$ without changing the proof. Now if A is perfect, then $\overline{R}$ has characteristic p, so we can replace the reference to Lemma 3.7 by a reference to Lemma 3.2, which results in an improvement of 1 in the range of vanishing. If A is p-torsionfree, so is $\overline{R}$, so we may take $x_1 = p$, in which case the ideal $J\overline{A}$ appearing in the proof above is generated by $n-1$ elements, up to radical; this again results in an improvement of 1. We do not know if this improvement of 1 is possible in general.

Remark 4.3. It will be evident from the proof that the result above holds under the weaker hypothesis that the R-algebra A factors through a perfectoid algebra. What is more, the only relevant property of perfectoid algebras that is needed is that it is of positive characteristic and perfect modulo a flat ideal.

There is an extension of Theorem 4.1 to the non-local case. It can be viewed an extension of Herzog's result [23, Satz 3.1] that, for a finitely generated R-module M, vanishing of Tor against high Frobenius twists of R implies finiteness of the flat dimension of M.

Corollary 4.4. *Let R be a noetherian ring with p in its Jacobson radical and let A be a perfectoid R-algebra such that* $\operatorname{Spec} A \to \operatorname{Spec} R$ *is surjective. Let M be an R-module satisfying* $\operatorname{Tor}_i^R(A,M) = 0$ *for $i \gg 0$.*

If M is finitely generated, or $p = 0$ in R and $\dim R$ *is finite, then* $\operatorname{flat\,dim}_R M$ *is finite.*

Proof. Fix a prime $\mathfrak{p} \subset R$ containing p. The hypothesis implies $\operatorname{Tor}_i^{R_\mathfrak{p}}(A_\mathfrak{p}, M_\mathfrak{p}) = 0$ for $i \gg 0$. Then $\mathfrak{p}A_\mathfrak{p} \neq A_\mathfrak{p}$, because $\operatorname{Spec} A \to \operatorname{Spec} R$ is surjective. The ring $A_\mathfrak{p}$ is perfect modulo a flat ideal (since the same was true for A), so Theorem 4.1 (see also Remark 4.3) applies and yields that, with $k(\mathfrak{p})$ the residue of the local ring $R_\mathfrak{p}$, one has

$$\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0 \quad \text{for all } i \gg 0. \tag{4.4.1}$$

Since p is in the Jacobson radical, this conclusion holds whenever $\mathfrak{p}$ is a maximal ideal.

When the R-module M is finitely generated, the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is finitely generated and the vanishing condition above implies that $\operatorname{flat\,dim}_{R_\mathfrak{p}}(M_\mathfrak{p})$ is finite, and since $\mathfrak{p}$ can be an arbitrary maximal ideal, it follows from a result of Bass and Murthy [7, Lemma 4.5] that $\operatorname{flat\,dim}_R M$ is finite.

Assume $p = 0$ in R and that $\dim R$ is finite. In particular (4.4.1) holds for each prime $\mathfrak{p}$ in R. It follows from [15, Theorem 4.1] that

$$\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0 \quad \text{for } i \geq \dim R + 1$$

and hence, again from *op. cit.*, that $\operatorname{flat\,dim}_R M$ is finite. □

Remark 4.5. In Corollary 4.4, the additional hypotheses on R, or on M, is necessary. For example, if R is a regular ring of positive characteristic and $M := \oplus_{\mathfrak{m} \in \operatorname{Max} R}(R/\mathfrak{m})$, one has $\operatorname{Tor}_i^R(R_{\mathrm{perf}}, M) = 0$ for each $i \geq 1$, since R_{perf} is a flat R-module. However one has $\operatorname{flat\,dim}_R M = \dim R$, and the latter can be infinite; see [35, Appendix A1].

4.6. **Regularity.** We can now prove a mixed characteristic generalization of Kunz's theorem [32, Theorem 2.1], answering a question in [9, Remark 5.5]; see also [3, pp. 6].

Theorem 4.7. *Let R be a noetherian ring with p in its Jacobson radical. If R is regular, then there exists a faithfully flat map $R \to A$ with A perfectoid.*

Conversely, fix a map $R \to A$ with A perfectoid. If $\operatorname{Spec} A \to \operatorname{Spec} R$ *is surjective and* $\operatorname{Tor}_i^R(A,A) = 0$ *for $i \gg 0$ (for example, if A is a faithfully flat R-algebra), then R is regular.*

Proof. Assume first that there exists an R-algebra A with the stated properties. Fix a maximal ideal $\mathfrak{m}$ of R. Since $p \in \mathfrak{m}$ holds, by hypothesis, arguing as in the proof of Corollary 4.4

(with $M = A$) one gets the vanishing of Tor below:

$$\mathrm{Tor}_i^R(k(\mathfrak{m}), A) \cong \mathrm{Tor}_i^{R_\mathfrak{m}}(k(\mathfrak{m}), A_\mathfrak{m}) = 0 \quad \text{for all } i \gg 0.$$

The isomorphism holds because the module on the left is $\mathfrak{m}$-local. Since $k(\mathfrak{m}) = R/\mathfrak{m}$, it is a finitely generated R-module, so another application of Corollary 4.4, now with $M = k(\mathfrak{m})$, implies that $\operatorname{flat\,dim}_R k(\mathfrak{m})$ is finite, and hence that $R_\mathfrak{m}$ is regular. Since this holds for each maximal ideal of R, one deduces that R is regular.

Conversely, assume that R is regular with $p \in J(R)$. We must construct a faithfully flat map $R \to A$ with A perfectoid. It clearly suffices to do this on each connected component of $\mathrm{Spec}(R)$, so we may assume R is a regular domain. When R has characteristic p, we may simply take $A = R_{\mathrm{perf}}$ by Kunz's theorem. Assume from now on that R is p-torsionfree.

Let us first explain how to construct the required cover when R is complete noetherian regular local ring of mixed characteristic $(0, p)$. By gonflement [13, IX, App., Theorem 1, Cor.], we may assume that the residue field of R is perfect. One can then perform the constructions in Examples 3.8 (4) and (5) to obtain the required covers.

It remains to globalize. As R is noetherian and $p \in J(R)$, each completion $\widehat{R_\mathfrak{m}}$ at a maximal ideal $\mathfrak{m}$ is flat over R and has mixed characteristic $(0, p)$. By the previous paragraph, for each maximal ideal $\mathfrak{m} \subset R$, we may choose a faithfully flat map $\widehat{R_\mathfrak{m}} \to A(\mathfrak{m})$ with perfectoid target. Consider the resulting map

$$R \to \prod_\mathfrak{m} \widehat{R_\mathfrak{m}} \to \prod_\mathfrak{m} A(\mathfrak{m}).$$

As R is noetherian, an arbitrary product of flat R-modules is flat, so the above map is flat. Moreover, it is also faithfully flat: the image of the induced map on $\mathrm{Spec}(-)$ is generalizing (by flatness) and hits all closed points (by construction), and hence must be everything. As a product of perfectoid rings is perfectoid, we have constructed the desired covers. $\square$

In fact, it is possible to give a more precise characterization of regularity in terms of perfectoids than that given in Theorem 4.7. For instance, it is enough to assume a single Tor-module vanishes in a sufficiently large degree.

Corollary 4.8. *Let R be a noetherian local ring with residue field k, and let A be an R-algebra with $\mathfrak{m}A \neq A$. Assume A is perfectoid. If $\mathrm{Tor}_i^R(A, k) = 0$ for some integer $i \geq \dim R$ (for example, if $\operatorname{flat\,dim}_R A$ is finite), then R is regular.*

Proof. The hypothesis implies $\mathrm{Tor}_j^R(A, k) = 0$ for $j \geq i$; this is by 2.3. Then Theorem 4.1 implies $\mathrm{Tor}_j^R(k, k) = 0$ for $j \gg 0$, so R is regular, by [14, Theorem 2.2.7]. $\square$

In the preceding result, we do not know if the requirement that $i \geq \dim R$ is necessary. Next we prove that, for special A, it is not. To this end we recall the notion of a proregular sequence introduced by Greenlees and May [17]. The treatment due by Schenzel [36] is better suited to our needs.

4.9. **Proregular sequences.** A sequence of elements $\underline{x} := x_1, \ldots, x_d$ in a ring S is *proregular* if for $i := 1, \ldots, d$ and integer $m \geq 1$, there exists an integer $n \geq m$ such that

$$((x_1^n, \ldots, x_{i-1}^n) :_S x_i^n) \subseteq ((x_1^m, \ldots, x_{i-1}^m) :_S x_i^{n-m})$$

It is not hard to verify that this property holds if $\underline{x}$ is a regular sequence, or if S is noetherian; see [36, pp. 167]. By [36, Lemma 2.7] such a sequence is *weakly proregular*, that is to say, for each m, there exists an integer $n \geq m$ such that the canonical map

$$\mathrm{H}_i(x_1^n, \ldots, x_d^n; S) \longrightarrow \mathrm{H}_i(x_1^m, \ldots, x_d^m; S)$$

on Koszul homology modules is zero for $i \geq 1$.

We care about these notions because of the following observation.

Lemma 4.10. *Let $\mathfrak{a}$ be an ideal in a noetherian ring R and S an R-algebra. If $\mathfrak{a}$ can be generated, up to radical, by a sequence whose image in S is weakly progregular, then $\mathrm{H}^i_{\mathfrak{a}}(I) = 0$ for $i \geq 1$ and any injective S-module I.*

Proof. By hypothesis, there exists a sequence $\underline{x}$ in R such that $\sqrt{\underline{x}} = \sqrt{\mathfrak{a}}$ and $\underline{x}S$, the image of the sequence $\underline{x}$ in S, is weakly proregular. Let C be the Čech complex on $\underline{x}$, so that $\mathrm{H}^i_{\mathfrak{a}}(M) = \mathrm{H}_i(C \otimes_R M)$ for any R-module M; see, for example, [14, Theorem 3.5.6]. Since I is an S-module has one $C \otimes_R I \cong (C \otimes_R S) \otimes_S I$. Since $C \otimes_R S$ is the Čech complex on $\underline{x}S$, the desired result then follows from [36, Theorem 3.2]. □

Absolute integral closure. Given a domain R, its absolute integral closure (that is to say, the its integral closure in an algebraic closure of it field of fractions), is denoted R^+. When R is of positive characteristic, R^+ contains a subalgebra isomorphic to R_{perf}.

When R has mixed characteristic, with residual characteristic p, the ideal $(p^{1/p^\infty})R^+$ is flat, and the quotient ring $R^+/(p^{1/p^\infty})R^+$ is of characteristic p and perfect. In the light of Remark 4.3, it follows that the conclusion of Corollary 4.8 also holds when R has mixed characteristic and the R-algebra A factors through R^+.

Proposition 4.11. *Let $\underline{x}$ be a system of parameters in an excellent local domain R.*
If R has positive characteristic, then $\underline{x}$ is weakly proregular in R_{perf} and in R^+.
If R has mixed characteristic and $\dim R \leq 3$, then $\underline{x}$ is weakly proregular in R^+.

Proof. By [36, Corollary 3.3], it suffices to verify that there is some choice of an s.o.p. that is (weakly) proregular on R_{perf}, or R^+, as the case maybe.

We treat first the case where R has positive characteristic. In this case, R^+ is a balanced big Cohen-Macaulay algebra, as proved by Hochster and Huneke [25, Theorem 1.1]; see also Huneke and Lyubeznik [28, Corollary 2.3]. Thus any s.o.p. for R, in particular, $\underline{x}$ is a regular sequence, and hence also a (weakly) proregular sequence, in R^+.

As to R_{perf}: Since R is excellent, it is a homomorphic image of an excellent Cohen-Macaulay local ring [30, Corollary 1.2] and hence it admits a p-standard s.o.p. [18, Definition 2.1 and Theorem 1.3]. In particular, R admits an s.o.p. $\underline{x} := x_1, \dots, x_d$ such that

$$((x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}}) :_R x_i^{n_i} x_j^{n_j}) = ((x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}}) :_R x_j^{n_j}) \qquad \text{for all } 1 \leq i \leq j \leq d.$$

In other words, $\underline{x}$ is a strong d-sequence; see [27, Definition 5.10]. We claim that such an $\underline{x}$ satisfies

$$((x_1^n, \dots, x_{i-1}^n) :_{R_{\mathrm{perf}}} x_i^n) \subseteq ((x_1^m, \dots, x_{i-1}^m) :_{R_{\mathrm{perf}}} x_i^{n-m}) \qquad \text{for all } m \geq 1 \text{ and } n > m.$$

To this end, since $R_{\mathrm{perf}} = \cup R^{1/p^e}$, it suffices to prove that

$$((x_1^{p^e n}, \dots, x_{i-1}^{p^e n}) :_R x_i^{p^e n}) \subseteq ((x_1^{p^e m}, \dots, x_{i-1}^{p^e m}) :_R x_i^{p^e n - p^e m}).$$

But the conditions on $\underline{x}$ imply

$$\begin{aligned} ((x_1^{p^e n}, \dots, x_{i-1}^{p^e n}) :_R x_i^{p^e n}) &= ((x_1^{p^e n}, \dots, x_{i-1}^{p^e n}) :_R x_i) \\ &\subseteq ((x_1^{p^e m}, \dots, x_{i-1}^{p^e m}) :_R x_i) \\ &= ((x_1^{p^e m}, \dots, x_{i-1}^{p^e m}) :_R x_i^{p^e n - p^e m}) \end{aligned}$$

This proves $\underline{x}$ is proregular, and hence also weakly proregular, on R_{perf}, as desired.

Suppose R is of mixed characteristic. When $\dim R \le 2$ once again R^+ is a big Cohen-Macaulay algebra, so the result follows. Assume $\dim R = 3$ and choose an s.o.p. of the form x, y, p. Since R^+ is normal, x, y is a regular sequence on it. It thus suffices to verify:

$$((x^n, y^n) :_{R^+} p^n) \subseteq ((x^m, y^m) :_{R^+} p^{n-m}) \quad \text{for all } m \ge 1 \text{ and } n > m.$$

Assume $zp^n \in (x^n, y^n)R^+$. It follows from a result of Heitmann's [21, Theorem 0.1], see also [22], that $p^\varepsilon z \in (x^n, y^n)R^+$ for *any* rational number ε. In particular, $pz \in (x^n, y^n)R^+$ and this implies the desired inclusion. □

Corollary 4.12. *Let $(R, \mathfrak{m}, k)$ be an excellent local domain and E the injective hull of k. If R has positive characteristic, then*

$$\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(R^+, E)) = 0 = \mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(R_{\mathrm{perf}}, E)) \qquad \text{for each } i \ge 1.$$

When R has mixed characteristic and $\dim R \le 3$, one has

$$\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(R^+, E)) = 0 \qquad \text{for each } i \ge 1.$$

Proof. By adjunction, the R^+-module $\mathrm{Hom}_R(R^+, E)$ and the R_{perf}-module $\mathrm{Hom}_R(R_{\mathrm{perf}}, E)$ are injective. Thus the desired result follows from Proposition 4.11 and Lemma 4.10. □

Aberbach and Li [1, Corollary 3.5] have proved parts (1) and (2) of the following result, using different methods.

Theorem 4.13. *Let $(R, \mathfrak{m}, k)$ be an excellent local domain. Then R is regular if any one of the following conditions.*

(1) *R has positive characteristic and $\mathrm{Tor}^R_i(R_{\mathrm{perf}}, k) = 0$ for some integer $i \ge 1$;*
(2) *R has positive characteristic and $\mathrm{Tor}^R_i(R^+, k) = 0$ for some integer $i \ge 1$;*
(3) *R has mixed characteristic, $\dim R \le 3$, and $\mathrm{Tor}^R_i(R^+, k) = 0$ for some $i \ge 1$.*

Proof. In all cases, it follows from Corollary 4.12 and 2.3 that $\mathrm{Tor}^R_j(R^+, k)$, respectively, $\mathrm{Tor}^R_j(R_{\mathrm{perf}}, k)$, is zero for each $j \ge i$. When R has positive characteristic, R^+ contains R_{perf}; in mixed characteristic, R^+ is perfect modulo a flat ideal. Therefore, in either case Theorem 4.1—see also Remark 4.3—implies $\mathrm{Tor}^R_j(k, k) = 0$ for $j \gg 0$ as desired. □

Here is a question suggested by part (3) above: If $(R, \mathfrak{m}, k)$ is a noetherian local domain of characteristic 0 and $\mathrm{Tor}^R_i(R^+, k) = 0$ for some $i \ge 1$, then is R regular?

5. Almost flatness

The goal of this section is to prove, for rings of mixed characteristic, the variations of Theorems 4.1 and 4.7 where the vanishing of Tor and the flatness hypotheses are relaxed to almost conditions. As before, throughout this section we fix a prime p; the notion of perfectoid is with respect to this prime. A module over a perfectoid ring A *almost zero* if it is killed by $\sqrt{pA}$; a map $R \to A$ is *almost flat* if $\mathrm{Tor}^R_i(-, A)$ is almost zero for each $i \ge 1$.

In what follows we will consider maps $R \to A$ with R noetherian and p-torsion free, and A perfectoid, satisfying the following:

5.1. **Valuative condition.** For every map $R \to V$ with V a p-torsionfree and p-adically complete rank 1 valuation ring, there exists an extension $V \to W$ of p-torsionfree and p-adically complete rank 1 valuation rings and a map $A \to W$ extending $R \to V \to W$.

See Remark 5.4 for an alternative description of this condition, and Proposition 5.6 for a sufficient, and perhaps easier to verify, condition under which it holds.

Compare the result below with Theorem 4.1, and also Remark 4.2.

Theorem 5.2. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring that is p-torsionfree and $p \in \mathfrak{m}$ holds. Let $R \to A$ be a map with A perfectoid and satisfying the valuative condition 5.1.*

If M is an R-module for which the A-module $\operatorname{Tor}_i^R(A, M)$ is almost zero for each integer $s \le i \le s + \dim R$, for some $s \ge 0$, then

$$\operatorname{Tor}^R_{s+\dim R+1}(k, M) = 0\,.$$

In particular, if $M = k$, then the ring R is regular.

The proof of this result, given further below, is a little more involved than that of Theorem 4.1. It will be clear from the proof that there is a version of the preceding result where almost zero is measured with respect to some fixed nonzero divisor in A, not necessarily p, that admits a compatible system of p-power roots. Moreover, it suffices that condition 5.1 holds for noetherian valuations V that dominate the maximal ideal $\mathfrak{m}$.

Here is an analogue of Theorem 4.7.

Theorem 5.3. *Let R be a noetherian p-torsionfree ring such that p lies in its Jacobson radical. Let $R \to A$ be a map with A perfectoid and satisfying the valuative condition 5.1. If $R \to A$ is almost flat, then R is regular.*

Proof. Fix a maximal ideal $\mathfrak{m}$ of R. We shall prove that $R_\mathfrak{m}$ is regular; as $\mathfrak{m}$ was arbitrary, the theorem follows. Since p is in the Jacobson radical of R, the residue field k at $\mathfrak{m}$ has characteristic p. Let $\widehat{A_\mathfrak{m}}$ denote the p-adic completion of $A_\mathfrak{m}$; this is a perfectoid ring. It is easy to verify that the valuative condition 5.1 is inherited by the induced map $R_\mathfrak{m} \to \widehat{A_\mathfrak{m}}$. Since k is of characteristic p and is $\mathfrak{m}$-local, for each i there are natural isomorphisms

$$\operatorname{Tor}_i^{R_\mathfrak{m}}(\widehat{A_\mathfrak{m}}, k) \cong \operatorname{Tor}_i^{R_\mathfrak{m}}(A_\mathfrak{m}, k) \cong \operatorname{Tor}_i^R(A, k)\,.$$

Since $R \to A$ is almost flat, it thus follows that the $\widehat{A_\mathfrak{m}}$-module $\operatorname{Tor}_i^{R_\mathfrak{m}}(\widehat{A_\mathfrak{m}}, k)$ is almost zero for $i \ge 1$. Thus, Theorem 5.2 applies and yields that $R_\mathfrak{m}$ is regular, as desired. □

Observe that, in contrast with the statement of Theorem 4.7, the preceding result makes no explicit hypothesis on the induced map of spectra of R and A. But in fact the valuative condition 5.1 can be described in terms of adic spectra.

Remark 5.4. Give a p-torsion free commutative ring B, let $\operatorname{Spa}(B[1/p], B)$ denote the adic spectrum of $(B[1/p], B)$ topologized using the p-adic topology on B; see Huber [26, Definition (iii)] and also [16, §10.3], keeping in mind that $\operatorname{Spa}(B[1/p], B)$ coincides with $\operatorname{Spa}(B[1/p], B^+)$, where B^+ is the integral closure of B in $B[1/p]$. The generic points of $\operatorname{Spa}(B[1/p], B)$ are in bijective correspondence with equivalence classes of maps $B \to V$ where V is a p-torsionfree and p-adically complete rank 1 valuation ring; the equivalence relation is generated by refinements of such V.

The valuative condition 5.1 is thus equivalent to the surjectivity on generic points of the induced map $\operatorname{Spa}(A[1/p], A) \to \operatorname{Spa}(R[1/p], R)$. For psychological ease, we remark that if a generic point $x \in \operatorname{Spa}(R[1/p], R)$ is the image of a point $y \in \operatorname{Spa}(A[1/p], A)$, then we can also find a generic point $y' \in \operatorname{Spa}(A[1/p], A)$ lifting x simply by setting y' to be the maximal generalization of y.

Remark 5.5. The main reason to use the valuative condition 5.1 in formulating Theorem 5.2 is that nonzero finitely generated ideals in a valuation ring cannot contain elements of arbitrarily small valuation. This provides an easy way to test whether certain modules not almost zero (which is a *stronger* statement than merely requiring them to be nonzero); see the paragraph following Claim 1 in the proof of Theorem 5.2. The restriction to rank 1 valuations ensures that we may replace the ring A appearing in the statement of Theorem 5.2 with an almost isomorphic one without affecting the hypotheses on A.

Proof of Theorem 5.2. The assumptions on $R \to A$ are stable under replacing A with an almost isomorphic perfectoid ring (such as A_{tf}). Thus, we may assume A is p-torsionfree.

Set $d := \dim R$ and choose elements $f_1, \dots, f_d$ in $\mathfrak{m}$ that generate it up to radical.

We begin by replacing the images of the f_i's in A by elements that admit p-power roots as follows: Choose $g_1 = \varpi^p$, so that $(g_1) = (p) = (f_1)$; see Lemma 3.6. For $i \geq 2$, choose elements $h_i \in A^\flat$ lifting $f_i \in A/(g_1)$ and set $g_i = h_i^\sharp$. Then each g_i admits a compatible system $\{g_i^{1/p^n}\}$ of p-power roots. Moreover, by construction we have

$$\big(g_i^{1/p^n}\big)^{p^n} \equiv f_i \mod (g_1) \quad \text{for } i \geq 2.$$

In particular, there is an equality $(f_1, \dots, f_d) = (g_1, \dots, g_d)$ of ideals of A.

The key step will be to justify the following

Claim 1. When $\operatorname{Tor}^R_{s+d+1}(k, M) \neq 0$ holds, there is an containment of ideals

$$(g_1^{1/p^\infty}) \subseteq (g_1, \dots, g_d)A = (f_1, \dots, f_d)A \subset A. \tag{5.5.1}$$

Given this we complete the proof by checking that (5.5.1) is not compatible with the valuative condition. Choose a map $A \to W$ to a p-adically complete and p-torsionfree rank 1 valuation ring W such that the image of f_i in W is not invertible; to construct such a map, one first does it for R—where it exists since R is p-torsionfree and (f_i) is not the unit ideal [29, Theorem 6.4.3]—and then invokes condition 5.1. As $(g_1) = (p)$ and p is a pseudouniformizer in W, elements of (g_1^{1/p^∞}) give elements of W with arbitrarily small valuation. On the other hand, the ideal $(f_1, \dots, f_d) \subseteq W$ is finitely generated and non-unital by construction, so it cannot contain elements of arbitrarily small valuation. In particular, it cannot contain (g_1^{1/p^∞}), contradicting (5.5.1). See Proposition 5.6 for an alternative denouement.

Now we take up the task of proving Claim 1. To that end for each integer $n \geq 1$ set

$$A_n := \mathrm{K}(g_1^{1/p^n}, \dots, g_d^{1/p^n}; A),$$

the Koszul complex over A on the elements $g_1^{1/p^n}, \dots, g_d^{1/p^n}$.

Claim 2. For each n the A-module $\operatorname{Tor}^R_{s+d+1}(A_n, M)$ is almost zero.

This is a straightforward verification using the fact that A_n can be constructed as an iterated mapping cone (d of them are required) starting with A, and our hypothesis that $\operatorname{Tor}^R_i(A, M)$ is almost zero for $s \leq i \leq s+d$.

In the next steps we will exploit the fact that each A_n has a structure of a strict graded-commutative dg (differential graded) A-algebra; namely, it is an exterior algebra over A on indeterminates $y_{n,1}, \dots, y_{n,d}$ of degree one with differential defined by the assignment $y_{n,i} \mapsto g_i^{1/p^n}$. For each integer $n \geq 1$, writing one gets a morphism of dg A-algebras

$$A_n \to A_{n+1} \quad \text{where} \quad y_{n,i} \mapsto \big(g_i^{\frac{1}{p^n} - \frac{1}{p^{n+1}}}\big) y_{n+1,i}\,.$$

Then $A_\infty := \operatorname{colim}_n A_n$ is a strict graded-commutative dg A-algebra, and the structure maps $A_n \to A_\infty$ are morphisms of dg A-algebras.

Claim 3. The dg A-algebra A_∞ satisfies

$$\mathrm{H}_i(A_\infty) = \begin{cases} A/(g_1^{1/p^\infty}, \dots, g_d^{1/p^\infty}) & \text{for } i = 0 \\ 0 & \text{for } i \geq 1. \end{cases}$$

To begin with, set $B := A/(\sqrt{pA})$; this ring is of characteristic p and is perfect; see Lemma 3.7. Since A is p-torsion free, the dg A-algebra

$$0 \to A \xrightarrow{p^{1/p^n}} A \to 0$$

is quasi-isomorphic to its homology module in degree zero, namely, $A/(p^{1/p^n})$. Thus the colimit, as $n \to \infty$, of these dg algebras is quasi-isomorphic to $A/(p^{1/p^\infty}) = B$. Since $g_1 = p$ and colimits commute with tensor products, it follows that A_∞ is quasi-isomorphic to the colimit, B_∞, of dg B-algebras

$$B_n := K(g_2^{1/p^n}, \dots, g_d^{1/p^n}; B)$$

where the maps $B_n \to B_{n+1}$ are defined as for the A_n. It thus suffices to prove that the homology of B_∞ is concentrated in degree 0, where it is $B/(g_2^{1/p^\infty}, \dots, g_d^{1/p^\infty})$. This is essentially the content of Lemma 3.2. Indeed, as in the proof of Lemma 3.2 one reduces to the case of a single element, g, in B. Consider $F := 0 \to (g^{1/p^\infty}) \xrightarrow{\subset} B \to 0$, viewed as a dg B-algebra concentrated in degrees 0 and 1, and the morphism $B_n \to F$ of dg B-algebras

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \xrightarrow{g^{1/p^n}} & B & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle 1 \mapsto g^{1/p^n}} & & \Big\| & & \\ 0 & \longrightarrow & (g^{1/p^\infty}) & \xrightarrow{\subset} & B & \longrightarrow & 0 \end{array}$$

It is clear that these morphisms are compatible with the morphisms $B_n \to B_{n+1}$, and so yield a morphism $B_\infty \to F$ of dg B-algebras. This an isomorphism: this is clear in degree 0 zero, whilst in degree 1 it was verified in the proof of Lemma 3.2.

This completes the proof of the claim. Observe that $A/(g_1^{1/p^\infty}, \dots, g_d^{1/p^\infty}) \cong (A/\mathfrak{m}A)_{\mathrm{perf}}$.

In the remainder of the proof we use some basic facts about (strict graded-commutative) semifree dg R-algebras, referring to Avramov [5] for details. Let $R[X]$ be a resolvent of k viewed as an R-algebra; in particular, the R-algebra $R[X]$ is the strict graded-commutative polynomial ring on a graded set of indeterminates $X := \{X_i\}_{i \geqslant 1}$. Since R is noetherian, one can choose X such that set X_i is finite for each i; see [5, Proposition 2.1.10].

Claim 3 implies that the canonical surjection $A_\infty \to \mathrm{H}_0(A_\infty)$ is a quasi-isomorphism of dg A-algebras, and hence also of dg R-algebras. By construction $\mathfrak{m}\mathrm{H}_0(A_\infty) = 0$ so the induced morphism $R \to \mathrm{H}_0(A_\infty)$ factors through the surjection $R \to k$. Since $R[X]$ is semifree, it then follows from [5, Proposition 2.1.9] that there is a commutative square

$$\begin{array}{ccc} R[X] & \xrightarrow{\varphi} & A_\infty \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \simeq} \\ k & \longrightarrow & \mathrm{H}_0(A_\infty) \end{array}$$

of dg R-algebras. Recall that A_∞ is constructed as a colimit of the A_n.

Claim 4. For $n \gg 0$, the morphism $\varphi\colon R[X] \to A_\infty$ of dg R-algebras factors through A_n; that is to say, there is a morphism $\varphi_n\colon R[X] \to A_n$ of dg R-algebras such that its composition with $A_n \to A_\infty$ is φ.

The crucial point is that since each complex A_n is zero outside (homological) degrees $[0,d]$, so is their colimit A_∞. In particular, $\varphi(X_i) = 0$ for $i \geq d+1$ for degree reasons, and φ is completely determined by its values on the X_i for $1 \leq i \leq d$. Since each X_i is finite, it clear that φ lifts to a map of R-algebras $\varphi_n\colon R[X] \to A_n$ for some $n \geq 1$. Moreover, increasing n if needed we can ensure that the commutator $[\partial, \varphi_n]$ vanishes on the X_i, that is to say, φ_n is also a morphism of complexes, and hence a morphism of dg R-algebras.

Claim 5. For $n \gg 0$ there is an isomorphism of graded A-modules

$$\operatorname{Tor}^R(A_n, M) \cong \mathrm{H}(A_n) \otimes_k \operatorname{Tor}^R(k, M)\,.$$

By each n as in Claim 4, the R-module structure on A_n extends, via φ_n, to that of a dg module over $R[X]$, so one has

$$A_n \otimes^{\mathbf{L}}_R M \simeq A_n \otimes^{\mathbf{L}}_{R[X]} (R[X] \otimes^{\mathbf{L}}_R M)$$

as complexes of A-modules. Since $\mathrm{H}(R[X]) = k$ is a field, the Künneth map

$$\mathrm{H}(A_n) \otimes_{\mathrm{H}(R[X])} \mathrm{H}(R[X] \otimes^{\mathbf{L}}_R M) \longrightarrow \mathrm{H}(A_n \otimes^{\mathbf{L}}_{R[X]} (R[X] \otimes^{\mathbf{L}}_R M))$$

is an isomorphism. Combining the preceding two isomorphisms yields

$$\operatorname{Tor}^R(A_n, M) = \mathrm{H}(A_n \otimes^{\mathbf{L}}_R M) \cong \mathrm{H}(A_n) \otimes_k \operatorname{Tor}^R(k, M)\,.$$

This completes the proof of Claim 5.

Proof of Claim 1: Assume to the contrary that $\operatorname{Tor}^R_{s+d+1}(k, M) \neq 0$. Fix $n \gg 0$ so that Claim 5 applies. Then the A-module $\operatorname{Tor}^R_{s+d+1}(A_n, M)$ contains $\mathrm{H}_0(A_n)$ as a direct summand. Claim 2 implies that $\mathrm{H}_0(A_n)$ is almost zero. Since $\mathrm{H}_0(A_n) \cong A/(g_1^{1/p^n}, \dots, g_d^{1/p^n})$, by construction, this fact translates to

$$\sqrt{pA} = (g_1^{1/p^\infty})A \subseteq (g_1^{1/p^n}, \dots, g_d^{1/p^n})\,.$$

Raising to a sufficiently large power and observing that (g_1^{1/p^∞}) is idempotent, this gives

$$(g_1^{1/p^\infty})A \subseteq (g_1, \dots, g_r)A = (f_1, \dots, f_r)A.$$

This completes the proof of Claim 1 and so also that of the theorem. ◻

The next result gives a way to check the valuative condition 5.1.

Proposition 5.6. *Let R be a noetherian p-torsionfree ring containing p in its Jacobson radical. Let $R \to A$ be a map, with A perfectoid, that is almost flat and* 0 *is the only R-module M for which $M \otimes_R A = 0$. Then the following statements hold.*

(1) *$R \to A$ satisfies the valuative condition 5.1.*
(2) *$\sqrt{pA} \not\subseteq \mathfrak{m}A$ for any maximal ideal $\mathfrak{m}$ of R.*
(3) 0 *is the only R-module M for which $M \otimes_R A$ is almost zero.*

Proof. We proved that (2) follows from (1) as part of the proof of Theorem 5.3. Here we establish (2) first, and then deduce (1) and (3) from it.

(2) This part does not use the hypothesis that $R \to A$ is almost flat. Let $\mathfrak{p}$ be a minimal prime of $\widehat{R_\mathfrak{m}}$, the $\mathfrak{m}$-adic completion of $R_\mathfrak{m}$ and set $S := \widehat{R_\mathfrak{m}}/\mathfrak{p}$. For $d = \dim S$ one gets that

$$\mathrm{H}^d_\mathfrak{m}(S \otimes_R A) \cong \mathrm{H}^d_\mathfrak{m}(S) \otimes_R A \neq 0\,;$$

here we have used the hypothesis that $(-)\otimes_R A$ is faithful on modules. Therefore $S\otimes_R A$ is a solid S-algebra in the sense of Hochster; see [24, Corollary 2.4].

Contrary to the desired result, suppose $\sqrt{pA}\subseteq \mathfrak{m}A$, and consider the element ϖ from Definition 3.5. Since $\varpi^p = pu$, for some unit u in A, one has that $\sqrt{\varpi A}\subseteq \mathfrak{m}A$. Therefore, for each $e\geq 0$ the element ϖ^{1/p^e} is in $\mathfrak{m}(S\otimes_R A)$ and hence $\varpi\in \mathfrak{m}^{p^e}(S\otimes_R A)$; this implies that p is the same ideal. By definition [24, (1.2)], this implies that p is contained in the solid closure of $\mathfrak{m}^{p^e}S$, and hence in the integral closure of $\mathfrak{m}^{p^e}$, by [24, Theorem 5.10]. This is a contradiction for p is not zero in S; see [29, Proposition 5.3.4].

(3) Suppose $M\neq 0$, so there is an embedding $R/I\subseteq M$ for some nonunit ideal I of R. Since $R\to A$ is almost flat, when $M\otimes_R A$ is almost zero, so is $(R/I)\otimes_R A$, that is to say, $(p^{1/p^\infty})A\subseteq IA$; this contradicts (2). Note that (2) is the special case $M=R/\mathfrak{m}$ of (3).

(1) At this point we can assume $R\to A$ is almost flat and that 0 is the only R-module M for which $M\otimes_R A$ almost zero. These hypotheses remain unchanged, and it suffices to verify the conclusion, when we replace A by any almost isomorphic ring, and by doing so we can assume that A is also p-torsion free. Then, for any integer $n\geq 1$ the map $R/p^n\to A/p^n$ is almost flat, and has the property that $M\otimes_{R/p^n}(A/p^n)$ almost zero implies $M=0$. These observations will be used below.

Fix a map $R\to V$ with V a p-adically complete and p-torsionfree rank 1 valuation ring. Set $B:=A\otimes_R V$ and let $\widehat{B}$ denote the p-adic completion of B. It will be enough to show that $\widehat{B}[1/p]\neq 0$; then, for any prime $\mathfrak{q}$ in $\widehat{B}$ not containing p, there exists a p-adic rank 1 valuation on the domain $\widehat{B}/\mathfrak{q}$, for each maximal of this quotient contains p, and any such valuation extends V; see also [26, Proposition 3.6].

Assume towards contradiction that $\widehat{B}[1/p]=0$. Then the Banach open mapping theorem shows that for some $m\geq 0$ one has $p^m\cdot\widehat{B}=0$, that is to say, $\widehat{B}\simeq\widehat{B}/p^m$; see [8]. Since $\widehat{B}/p^m\simeq B/p^m$ and the transition maps in the tower $\{B/p^n\}$ limiting to $\widehat{B}$ are surjective, it follows that $B/p^{m+1}\simeq B/p^m$ via the natural map. In other words, the surjective map $V/p^{m+1}\to V/p^m$ becomes an isomorphism after applying $-\otimes_{R/p^n}A/p^n$ for $n\geq m+1$. It then follows that $p^mV/p^{m+1}V\otimes_{R/p^n}A/p^n$ is almost zero and hence that $p^mV/p^{m+1}V=0$, which is absurd as V is p-torsionfree and p-adically complete. □

Here is a more intuitive formulation of the p-adic Kunz theorem in the almost setting.

Corollary 5.7. *Let R be a noetherian p-torsionfree ring containing p in its Jacobson radical. If there exists a map $R\to A$ with A perfectoid that is almost flat and zero is the only R-module M for which $M\otimes_R A$ is zero, then R is regular.*

Proof. This is a direct consequence of Proposition 5.6(1) and Theorem 5.3. □

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