

RIGID IDEALS IN GORENSTEIN RINGS OF DIMENSION ONE

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ABSTRACT. Motivated by a conjecture of Huneke and Wiegand concerning torsion in tensor products of modules over local rings, we investigate the existence of ideals I in a one-dimensional Gorenstein local ring R satisfying $\text{Ext}_R^1(I, I) = 0$.

1. INTRODUCTION

In their study [17] of torsion in tensor products of modules over noetherian, commutative rings, the first and third authors write, in the last line of page 473: *It seems reasonable to conjecture that if [a commutative, noetherian, ring] R is one-dimensional and Gorenstein and M is a finitely generated R -module such that $M \otimes_R M^*$ is torsionfree, then M^* is projective.* Here and throughout this paper we set $M^* = \text{Hom}_R(M, R)$. An R -module is *torsionfree* if no regular element of R kills a nonzero element of the module. Very little progress has been made on this conjecture, even when R is a domain, so for the moment we focus on that case. By a standard localization argument, to settle the conjecture above, it suffices to treat the case when R is a local ring; moreover, arguing as in the proof of [1, Lemma 3.1], we can assume M is torsionfree; since R is Gorenstein, this condition is equivalent to M being reflexive. Thus the roles of M and M^* are interchangeable, and we are left with the following:

Conjecture 1.1. Let R be a Gorenstein local domain of dimension one and M a finitely generated, torsionfree R -module such that $M \otimes_R M^*$ is torsionfree. Then M is free.

An R -module M is *rigid* provided $\text{Ext}_R^1(M, M) = 0$, that is to say, every self-extension of M splits. It is well-known that over a Gorenstein local domain R with $\dim R = 1$, a finitely generated torsionfree R -module M is rigid if and only if $M \otimes_R M^*$ is torsionfree; see Proposition 4.4. Hence Conjecture 1.1 is equivalent to:

Conjecture 1.2. Over a one-dimensional local Gorenstein domain, every finitely generated torsionfree rigid module is free.

Conjecture 1.1 holds for abstract hypersurfaces; see [17, Theorem 3.1]. Celikbas [8, Proposition 4.17] has shown that Conjecture 1.1 holds if R is a complete intersection and the module M is torsionfree and has bounded Betti numbers.

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Both conjectures fail without the hypothesis that R be one-dimensional; see Examples 2.2 and 2.3. A plausible way to extend Conjecture 1.1 to higher dimensions is to replace “torsionfree” with “reflexive”, but in fact that version of the conjecture reduces to the one-dimensional case; see, for example, [9, Proposition 8.6].

In this paper we concentrate chiefly on ideals in one-dimensional rings. The condition that R is a Gorenstein domain is relaxed to the condition that it is Cohen-Macaulay, and that the ideal in question is primary to the maximal ideal and reflexive. In such an R , an ideal is primary to the maximal ideal precisely when it contains a regular element, so if R is Gorenstein this implies the ideal is reflexive. Without such hypotheses, there are easy counterexamples; see Example 2.1.

The notation $(R, \mathfrak{m})$ means that R is a (noetherian, commutative) local ring with maximal ideal $\mathfrak{m}$. For such an R , we write $\text{embdim } R$ for embedding dimension, and $e(R)$ for the multiplicity. The minimal number of generators required for an R -module M is denoted $\nu_R M$. Our main results are summarized below:

Theorem 1.3. *Let R be a Cohen-Macaulay local ring with $\dim R = 1$. Let I be an $\mathfrak{m}$ -primary, reflexive ideal such that $I \otimes_R I^*$ is torsionfree. The ideal I is then principal under each of the following conditions:*

- (1) $\nu_R I \leq 2$ and $\nu_R I^* \leq 2$;
- (2) $e(R) \leq 8$;
- (3) $e(R) \leq \text{embdim } R + 2$;
- (4) R is a complete intersection and $e(R) \leq 10$;
- (5) R is a complete intersection and the pair (R, I) can be smoothed;
- (6) R is a complete intersection with $\text{embdim } R \leq 3$ and R/I is Gorenstein.

Parts (1) and (2) are Proposition 3.1 and Theorem 3.8 respectively. Part (3) is proved as Corollary 3.12; part (4) is contained in Theorem 5.13. Part (5) is the content of Theorem 5.12. As to (6), when $\text{embdim } R \leq 2$, that is to say, when R is a hypersurface, the desired conclusion is by [17, Theorem 3.7], while the case $\text{embdim } R = 3$ is covered by Corollary 5.11.

The paper is organized as follows. Section 2 contains some examples justifying the hypotheses in the conjectures. In Section 3 we prove a number of estimates on the number of generators of rigid ideals or modules for general one-dimensional rings, in terms of the maximum number of generators of any ideal containing the trace ideal of the module studied. These are used throughout the remainder of the paper, and in particular in Section 3 to prove parts (1)–(3) of Theorem 1.3. These results extend, and were inspired by, work of S. Goto, R. Takahashi, N. Taniguchi, and H. Le Truong [12, Theorem 1.4], which covers the case $e(R) \leq 6$.

The key result of Section 4 is a necessary and sufficient criterion for an ideal I in a one-dimensional Gorenstein ring to be rigid, in terms of the lengths of certain associated modules, especially the twisted conormal module, $I \otimes_R \omega_{R/I}$. This result is used in Section 5 to prove that rigid smoothable ideals are principal if the ambient ring is a complete intersection. This criterion leads to our results on rigidity for complete intersections of multiplicity at most 10.

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2. NOTATION AND EXAMPLES

Throughout the paper, $(R, \mathfrak{m}, k)$ is a local (for us, this means also commutative and noetherian) ring and $Q(R)$ denotes the ring of fractions of R . We deal only with finitely generated modules. An R -module M has *rank* if $M_{\mathfrak{p}}$ is free of constant rank r for all associated primes $\mathfrak{p}$ of R ; in this case, we write $\text{rank}_R(M) = r$. We say M is Cohen-Macaulay (abbreviated to CM) if it satisfies $\text{depth}_R M \geq \dim M$, and *maximal Cohen-Macaulay* (abbreviated to MCM) if $\text{depth}_R M \geq \dim R$. In both cases, equality holds if M is nonzero.

In this section we present some examples to justify the assumptions on the rings and modules in the conjectures stated in the introduction. The first one shows why the conjectures would fail without the assumption that R is a domain. Actually, we typically get by with the weaker assumption that the module in question is torsionfree, and also of positive rank, a condition that fails in this example.

Example 2.1. Let $R = \mathbb{C}[[x, y]]/(xy)$. Put $M = R/(x) \cong (y)$. Then $M \cong M^*$ and M is not free, but $M \otimes_R M^* \cong M$, which is torsionfree.

In dimensions greater than one, Conjecture 1.1 can fail without stronger depth conditions on the depth of the tensor product.

Example 2.2. For the ring $S = \mathbb{C}[[x, y, u, v]]/(xy - uv)$ and the ideal $I = (x, u)$, the tensor product $I \otimes_R I^*$ is isomorphic to the maximal ideal $\mathfrak{m}$ and hence is torsionfree (but not reflexive). See [17, Example 4.1].

As noted in the Introduction, Conjectures 1.1 and 1.2 are equivalent if $\dim R = 1$. The general statement is that when $\dim R = d \geq 1$ and M is an R -module that is maximal Cohen-Macaulay and locally free on the punctured spectrum of R , the R -module $M \otimes_R M^*$ is torsionfree precisely when $\text{Ext}_R^d(M, M) = 0$; this follows from a duality theorem due to Auslander [2]. Nevertheless, one may ask if Conjecture 1.2 remains valid in higher dimensions, but it does not, even if we assume the module in question is maximal Cohen-Macaulay.

Example 2.3. With $R = \mathbb{C}[[x, y]]/(xy)$ and $M = R/(x)$ as in Example 2.1, put $S = \mathbb{C}[[x, y, u, v]]/(xy - u^2 - v^2) \cong \mathbb{C}[[x, y, u, v]]/(xy - uv)$. The stable category of MCM S -modules is equivalent to that of MCM R -modules, by Knörrer periodicity [19]. One can check directly that M is a rigid R -module. Thus S , a three-dimensional Gorenstein domain, has a rigid MCM module that is not free.

Remark 2.4. The Gorenstein hypothesis in Conjecture 1.2 is essential. In fact, if R is any CM ring with canonical module ω_R , then ω_R is rigid, and it is free only if R is Gorenstein.

3. LOCAL RINGS OF SMALL MULTIPLICITY

We begin with a result, Proposition 3.1, proved in [13] and attributed to the first and third authors of this paper. We will sketch the proof here, since we have

replaced the hypothesis, in [13], that R is Gorenstein with the weaker one that I is reflexive. The proof is the same.

Proposition 3.1. *Let R be a local ring with $\dim R = 1$ and I a reflexive ideal containing a regular element. If both I and I^* are two-generated, but not principal, then the R -module $I \otimes_R I^*$ has nonzero torsion.*

Proof. Assume to the contrary that $I \otimes_R I^*$ is torsionfree. In the proof, we work with I^{-1} rather than I^* ; keep in mind that they are isomorphic. Also, we identify I^{-1} with the submodule $\{\alpha \in Q(R) \mid \alpha I \subseteq R\}$.

By hypothesis one has $\nu_R(I) = 2 = \nu_R(I^{-1})$. Using prime avoidance, write $I = (a, b)$ and $I^{-1} = (f, g)$, where a, b, f, g are units of $Q(R)$. Let $\varepsilon: I^{-1} \rightarrow R^2$ be the map that assigns α to $\begin{bmatrix} \alpha b \\ -\alpha a \end{bmatrix}$. It is easy to check that the sequence

$$0 \longrightarrow I^{-1} \xrightarrow{\varepsilon} R^2 \xrightarrow{\begin{bmatrix} a & b \end{bmatrix}} I \longrightarrow 0$$

is exact. There is a similar exact sequence

$$0 \longrightarrow I \xrightarrow{\sigma} R^2 \xrightarrow{\tau} I^{-1} \longrightarrow 0,$$

where $\sigma(c) = \begin{bmatrix} gc \\ -fc \end{bmatrix}$ and $\tau\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = fx + gy$. Splicing these exact sequences together in two ways, one obtains the following periodic minimal resolution for I^{-1} :

$$(3.1.1) \quad \mathbf{F}_\bullet \quad \cdots \longrightarrow R^2 \xrightarrow{\Psi} R^2 \xrightarrow{\Phi} R^2 \xrightarrow{\Psi} R^2 \xrightarrow{\Phi} R^2 \longrightarrow 0,$$

where, with $r = ga, s = gb, t = -fa$, and $u = -fb$, one has

$$\Phi = \begin{bmatrix} r & s \\ t & u \end{bmatrix} \quad \text{and} \quad \Psi = \begin{bmatrix} -u & s \\ t & -r \end{bmatrix}.$$

Since I is not invertible, the entries of these matrices are elements of the maximal ideal $\mathfrak{m}$. Let $J = -fI = (t, u)$. Since J and I are isomorphic R -modules, we see that $J \otimes_R I^{-1}$ is torsionfree.

Tensoring the exact sequence

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

with I^{-1} , we get an injection $\mathrm{Tor}_1^R(R/J, I^{-1}) \hookrightarrow J \otimes_R I^{-1}$. Since $\mathrm{Tor}_1^R(R/J, I^{-1})$ is killed by the regular element t and $J \otimes_R I^{-1}$ is torsionfree, we must have

$$(3.1.2) \quad \mathrm{H}_1((R/J) \otimes_R \mathbf{F}_\bullet) = \mathrm{Tor}_1^R(R/J, I^{-1}) = 0.$$

The maps in the complex $(R/J) \otimes_R \mathbf{F}_\bullet$ are given by the reductions

$$\overline{\Phi} = \begin{bmatrix} \bar{r} & \bar{s} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \overline{\Psi} = \begin{bmatrix} 0 & \bar{s} \\ 0 & -\bar{r} \end{bmatrix}.$$

of the matrixes Φ and Ψ modulo the ideal J . We will build a element $\alpha \in \ker \overline{\Phi} \setminus \mathrm{im} \overline{\Psi}$, to obtain the desired contradiction.

If $\bar{s} = 0$, we can take $\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so assume $\bar{s} \neq 0$. Since $\bar{s}$ is nilpotent and $\bar{r}$ is not a unit of $\overline{R} = R/J$, there is a positive integer n such that $\bar{s}^n \in \overline{R}\bar{r}$ but $\bar{s}^{n-1} \notin \overline{R}\bar{r}$. Write $\bar{s}^n = x\bar{r}$, where $x \in \overline{R}$. Then $\alpha = \begin{bmatrix} x \\ -\bar{s}^{n-1} \end{bmatrix}$ belongs to $\ker \overline{\Phi}$. If there were elements $y, z \in \overline{R}$ with $\overline{\Psi} \begin{bmatrix} y \\ z \end{bmatrix} = \alpha$, one would have $\bar{r}z = \bar{s}^{m-1}$, a contradiction. Thus $\alpha \notin \mathrm{im} \overline{\Psi}$. $\square$

In the context of two-generated ideals, we should mention the following result of Garcia-Sanchez and Leamer [11]: If a monomial ideal I in a complete intersection numerical semigroup ring R is two-generated and rigid, then I is principal.

Remark 3.2. Let R be a Cohen-Macaulay local ring and M a maximal Cohen-Macaulay R -module with positive rank. There is then an inequality

$$\nu_R(M) \leq e(R) \operatorname{rank}_R(M),$$

where $e(R)$ denotes the multiplicity of R ; for a proof of this assertion see, for example, [17, Lemma 1.6].

3.3. Trace. The *trace* of a module M is the image $\operatorname{tr}_R(M)$ of the canonical map

$$M^* \otimes_R M \longrightarrow R$$

that assigns an element $f \otimes x$ to $f(x)$.

Proposition 3.4. *Let R be a local ring with $\dim R = 1$ and M a nonzero finite R -module with positive rank. If $M \otimes_R M^*$ is torsionfree, then for each ideal $J \supseteq \operatorname{tr}_R(M)$ there are inequalities*

$$\nu_R(J) \leq e(R)(1 - \rho) + \frac{\nu_R(M \otimes_R M^*)}{\nu_R(M)^2} \leq e(R)(1 - \rho + \rho^2)$$

where $\rho = \operatorname{rank}_R(M)/\nu_R(M)$. In particular, if M is not free, then $\nu_R(J) \leq e(R) - 1$.

Proof. We may assume R is Cohen-Macaulay, since otherwise every module with rank is free, and then it is easy to verify that the desired inequalities hold.

Set $\nu = \nu_R(M)$, and consider a presentation

$$0 \longrightarrow N \longrightarrow R^\nu \longrightarrow M \longrightarrow 0$$

Thus N is the first syzygy module of M . Since the ideal J contains $\operatorname{tr}_R(M)$, one has $\operatorname{Hom}_R(M, J) = M^*$, so applying $\operatorname{Hom}_R(-, J)$ to the exact sequence above yields exact sequences

$$(3.4.1) \quad 0 \longrightarrow M^* \longrightarrow J^\nu \longrightarrow X \longrightarrow 0$$

$$(3.4.2) \quad 0 \longrightarrow X \longrightarrow \operatorname{Hom}_R(N, J) \longrightarrow \operatorname{Ext}_R^1(M, J) \longrightarrow 0$$

Since M has rank, the R -module $\operatorname{Ext}_R^1(M, J)$ is torsion and hence (3.4.2) yields

$$\operatorname{rank}_R(X) = \operatorname{rank}_R(\operatorname{Hom}_R(N, J)) = \operatorname{rank}_R(N) = \nu - \operatorname{rank}_R(M)$$

For the second equality, we have used the fact that J contains $\operatorname{tr}_R(M)$ and hence is of rank one. This computation and Remark 3.2 yield the second inequality below

$$\begin{aligned} \nu \cdot \nu_R(J) &= \nu_R(J^\nu) \leq \nu_R(X) + \nu_R(M^*) \\ &\leq e(R)(\nu - \operatorname{rank}_R(M)) + \frac{\nu_R(M \otimes_R M^*)}{\nu} \\ &\leq e(R)(\nu - \operatorname{rank}_R(M)) + \frac{e(R) \operatorname{rank}_R(M)^2}{\nu} \end{aligned}$$

The first inequality is from (3.4.1). The last inequality is also by Remark 3.2; it applies as $M \otimes_R M^*$ is torsionfree. Dividing by ν yields the stated inequalities.

When M is not free $\operatorname{rank}_R M < \nu$ so $0 < \rho < 1$, which implies $1 - \rho + \rho^2 < 1$. $\square$

Remark 3.5. Proposition 3.4 leads one to consider the parabola $y = 1 - x + x^2$, which has its vertex at the point $(1/2, 3/4)$. In our context $x = \text{rank}_R(M)/\nu_R(M)$ and hence, assuming M is not free, one has $0 < x < 1$. In this range, the parabola satisfies $y < 1$.

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$. Let M be a finite R -module with positive rank. If there exist an element x in R and an integer $n \geq 1$ such that $\text{tr}_R(M) \subseteq \mathfrak{m}^n$ and $x\mathfrak{m}^n = \mathfrak{m}^{n+1}$, then $M \otimes_R M^*$ has nonzero torsion. This holds, in particular, when M has no nonzero free summands and R has minimal multiplicity.*

Proof. There are equalities

$$\nu(\mathfrak{m}^n) = \nu(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \lambda(\mathfrak{m}^n/x\mathfrak{m}^n) = e(\mathfrak{m}^n) = e(R)$$

where the last two hold as $\mathfrak{m}^n$ is Cohen-Macaulay of rank 1. The last statement of Proposition 3.4 then implies $M \otimes_R M^*$ has nonzero torsion.

When M has no nonzero free summands, $\text{tr}_R(M) \subseteq \mathfrak{m}$ holds, and when R has minimal multiplicity there exists an element x such that $x\mathfrak{m} = \mathfrak{m}^2$, so we are in a situation where the first part of the result applies. $\square$

Lemma 3.7. *Let R be a local ring and I an ideal containing a regular element. If the R -module $I \otimes_R I^*$ is torsionfree, then the canonical map $I \otimes_R I^* \rightarrow \text{tr}_R(I)$ is bijective, and hence*

$$\nu_R(I)\nu_R(I^*) = \nu_R(\text{tr}_R(I)).$$

When in addition $\dim R = 1$, one has $\nu_R(I^)\nu_R(I) \leq e(R)$.*

Proof. The map in question is surjective by definition, and generically an isomorphism since I contains a regular element. Since $I \otimes_R I^*$ is torsionfree it follows that the map is an isomorphism; this justifies the stated equality. It also implies that $\text{tr}_R(I)$ has rank one. When in addition $\dim R = 1$, the R -module $\text{tr}_R(I)$ is MCM, and 3.2 yields the desired inequality. $\square$

Theorem 3.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$. Let I be an $\mathfrak{m}$ -primary ideal that is reflexive and such that $I \otimes_R I^*$ is torsionfree. If $e(R) \leq 8$, then the ideal I is principal.*

Proof. Set $\nu := \nu_R(I)$ and $\xi := \nu_R(I^*)$; suppose $\nu \geq 2$, contrary to the desired conclusion. Since I is reflexive, we have $\xi \geq 2$ as well. If $\nu = 2 = \xi$, we apply Proposition 3.1. Otherwise, we may assume by symmetry that $\xi \geq 3$. The equality below is from Lemma 3.7.

$$\nu\xi = \nu_R(\text{tr}_R(I)) \leq e(R)\left(1 - \frac{1}{\nu}\right) + \frac{\nu\xi}{\nu^2}$$

The inequality is from the first inequality in Proposition 3.4 applied to $J = \text{tr}_R(I)$; note that the rank of I is one. This simplifies to an inequality

$$\xi(\nu + 1) \leq e(R),$$

which contradicts the assumption $e(R) \leq 8$. $\square$

Remark 3.9. Theorem 3.8 extends and was inspired by [12, Theorem 1.4], that states, in our notation, that $e > (\nu + 1)\xi \geq 6$ if I is not principal. The main new parts of our theorem are the cases of $e = 7, 8$.

Remark 3.10. It is important in Theorem 3.8 that we are tensoring I with its dual. For example, in [17, 4.3] the following example is given: let $R = k[[t^4, t^5, t^6]]$, $I = (t^4, t^5)R$, $J = (t^4, t^6)R$. Then R is a one-dimensional complete intersection (and so all ideals are reflexive), I and J are two-generated, but $I \otimes_R J$ is torsionfree. However, J is not isomorphic to the dual of I .

Proposition 3.4 can be significantly enhanced for reflexive ideals. Later on, we will apply this result when R is Gorenstein, in which case the hypothesis that I is reflexive becomes superfluous.

Proposition 3.11. *Let $(R, \mathfrak{m})$ be a CM local ring with $\dim R = 1$. Let I be an $\mathfrak{m}$ -primary reflexive ideal such that $I \otimes_R I^*$ is torsionfree. Set*

$$\delta := e(R) - \max\{\nu_R(J) \mid J \supseteq \operatorname{tr}_R(I)\}.$$

When I is not principal the following inequalities hold:

- (1) $\delta \geq \nu_R(I)$ and $\delta \geq \nu_R(I^*)$;
- (2) $\delta \geq (e(R) - \nu_R(I^*)) / \nu_R(I)$;
- (3) $\delta^2 \geq e(R) - \delta \geq \nu_R(I)\nu_R(I^*) \geq 6$.

Moreover the equality $\delta^2 = e(R) - \delta$ holds if and only if $\delta = \nu_R(I) = \nu_R(I^)$.*

Proof. Set $e := e(R)$, $\nu := \nu_R(I)$ and $\xi := \nu_R(I^*)$. Since I is reflexive, but not principal, we can assume that $2 \leq \nu \leq \xi$.

(2) From the first inequality in Proposition 3.4 applied with $M = I$ and J some ideal containing $\operatorname{tr}_R(I)$ with $\nu_R(J) = e - \delta$, one gets, after a rearrangement of terms, the desired inequality:

$$e \leq \nu\delta + \xi.$$

Since $e - \delta$ is the maximum of the number of generators of an ideal containing $\operatorname{tr}_R(I)$, Lemma 3.7 yields

$$e - \delta \geq \nu\xi$$

which is the second inequality in (3).

(1) It suffices to verify $\delta \geq \xi$. The first inequality below has been justified.

$$\nu\xi + \delta \leq e \leq \nu\delta + \xi;$$

The second inequality is from the already established part (2). It follows that $(\nu - 1)\xi \leq (\nu - 1)\delta$, and hence that $\xi \leq \delta$; recall $\nu \geq 2$.

(3) Since I is not principal, Proposition 3.1 yields that at least one of ν or ξ is greater than or equal to 3, and hence that $\nu\xi \geq 6$. It remains to verify that $\delta^2 \geq e - \delta$. The already verified inequalities $\delta \geq \xi$ and $\delta \geq \nu$ yield the first and the last of the following inequalities:

$$e - \delta \leq e - \xi \leq \nu\delta \leq \delta^2;$$

the one in the middle is from (2). It remains to observe that $e - \delta = \delta^2$ holds if and only if both equalities $\delta = \nu$ and $\delta = \xi$ hold. This completes the proof. $\square$

Here is a first application of Proposition 3.11.

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a CM local ring with $\dim R = 1$. Let I be an $\mathfrak{m}$ -primary reflexive ideal such that $I \otimes_R I^*$ is torsionfree. If $e(R) \leq \operatorname{embdim} R + 2$ holds, then I is principal.*

Proof. Assume I is not principal. We use the notation and conclusions of Proposition 3.11. Since I is not free, $\mathrm{tr}_R(I)$ is contained in the maximal ideal of R , and hence $\delta \leq e(R) - \mathrm{embdim} R \leq 2$, which contradicts $\delta^2 \geq 6$. $\square$

4. GORENSTEIN RINGS

From this section on the focus will be on one-dimensional Gorenstein local rings. Let $(R, \mathfrak{m}, k)$ be one such. Given nonzero R -submodules $A, B \subseteq Q(R)$, each containing a unit of $Q(R)$, we identify $\mathrm{Hom}_R(A, B)$ with

$$\mathcal{H}(A, B) := \{\alpha \in Q(R) \mid \alpha A \subseteq B\}.$$

In particular, $A^* = \mathcal{H}(A, R)$. If A is a subring of $Q(R)$ containing R , we usually write $\mathcal{C}_A$ (for “conductor”) instead of $\mathcal{H}(A, R)$. These considerations apply to an ideal I containing a regular element; in particular, we identify I^* with $\mathcal{H}(I, R)$, that is to say, with I^{-1} , and $\mathrm{End}_R(I)$ with the subring $\mathcal{H}(I, I)$ of $Q(R)$.

Lemma 4.1. *Let R be a Gorenstein ring and I a proper ideal containing a regular element. Set $S := \mathcal{H}(I, I)$. The following statements hold.*

- (1) $\mathcal{C}_S = \mathrm{tr}_R(I)$.
- (2) $\mathcal{H}(I^{-1}, I^{-1}) = S$.
- (3) $\mathcal{C}_S$ is the largest ideal of R whose endomorphism ring is S .
- (4) If A, B are R -submodules of $Q(R)$ with $A \cong B$, then $\mathcal{H}(A, A) = \mathcal{H}(B, B)$.

Proof. (1) $\mathrm{Hom} \otimes$ adjointness and the fact that R is Gorenstein yield

$$(\mathrm{tr}_R I)^* = (I^* I)^* = (I^* \otimes_R I)^* = \mathrm{Hom}_R(I, I^{**}) = S.$$

Therefore $\mathrm{tr}_R I = S^* = \mathcal{C}_S$, as claimed.

(2) For any element $\alpha \in Q(R) \setminus \{0\}$, it is easy to check that $\alpha I \subseteq I$ implies $\alpha I^{-1} \subseteq I^{-1}$, and conversely (by symmetry).

(3) Observe that $\mathrm{End}_R(S^*) = \mathrm{End}_R(S) = \mathrm{End}_S(S) = S$; this is by (2) and the fact that S is isomorphic to an ideal of R . Suppose next that J is an ideal of R with $\mathrm{End}_R(J) = S$. Then

$$S = \mathrm{Hom}_R(J, J) \subseteq \mathrm{Hom}_R(J, R) = J^*.$$

Therefore $\mathcal{C}_S = S^* \supseteq J^{**} = J$.

(4) There exists an element $\beta \in Q(R)$ such that $B = \beta A$. Then, for $\alpha \in Q(R)$ we have $\alpha A \subseteq A \iff \alpha \beta A \subseteq \alpha \beta A$, so $\mathcal{H}(A, A) = \mathcal{H}(B, B)$. $\square$

4.2. Two-generated ideals and Gorenstein ideals. In what follows we use some results on linkage of ideals. Recall that the ring R is assumed to be Gorenstein. Proper ideals I, J of R are *linked* provided there is regular sequence $\underline{x} = x_1, \dots, x_g$, contained in $I \cap J$ and such that

$$J = ((\underline{x}) :_R I) \quad \text{and} \quad I = ((\underline{x}) :_R J).$$

For details see [22, Section 2]. There is a correspondence between two-generated ideals and Gorenstein ideals. Here’s how it works [16, Proposition 2.5]:

Lemma 4.3. *Let R be a Gorenstein local ring and I a height-one unmixed ideal containing a regular element. Then I is two-generated if and only if it is linked to a height one Gorenstein ideal which is not principal; in this case any height-one Gorenstein ideal linked to I is isomorphic to I^* .*

Proof. Assume $\nu(I) = 2$; thus $I = (a, b)$, where we can assume that a is a regular element. Set $J := ((a) :_R b) = ((a) :_R I)$. Since I is unmixed, by [22, Section 2], one has $I = ((a) :_R J)$, so I and J are linked. Moreover $\omega_{R/J} \cong I/(a)$, which is cyclic, so J is a Gorenstein ideal of R .

Conversely, let J be a nonzero height one Gorenstein ideal of R that is not principal. Choose a regular element $a \in J$ and set $I := ((a) :_R J)$. Again, I and J are linked via (a) , and $I/(a) \cong \omega_{R/J}$, which is cyclic; therefore $I = (a, b)$ for some $b \in I$.

With I two-generated and linked to a Gorenstein ideal J as above, we have

$$I^* \cong I^{-1} = \frac{1}{a}J \quad \text{and} \quad J^* \cong J^{-1} = \frac{1}{a}I.$$

This justifies the assertions. We note that if I is a principal ideal, then any ideal linked to I is also principal. $\square$

There is a useful reinterpretation of Conjecture 1.1 in term of rigidity: A module M over a ring R is *rigid* if $\text{Ext}_R^1(M, M) = 0$. The following result is well-known; we record it for ease of reference. We write $\Omega_R^n M$ for the n^{th} syzygy (defined only up to projective summands) of M ; when R is Gorenstein and M is MCM, this makes sense for all $n \in \mathbb{Z}$.

In the next result, the hypothesis on the rank of M can be relaxed to the condition that the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is free for each minimal prime $\mathfrak{p}$ of R .

Proposition 4.4. *Let R be a Gorenstein local ring with $\dim R = 1$ and M a finite torsionfree R -module with positive rank. The following conditions are equivalent:*

- (1) $M \otimes_R M^*$ is torsionfree;
- (2) $\text{Tor}_1^R(\Omega^{-1}M, M^*) = 0$;
- (3) M is rigid.

Proof. For R as in the statement, a nonzero finitely generated module is torsionfree if and only if it is MCM. Thus (1) $\iff$ (3) is a special case of [18, Theorem 5.9].

(1) $\iff$ (2) By definition of $\Omega^{-1}M$, there is an exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow \Omega^{-1}M \longrightarrow 0$$

where F is finite free. Tensoring with M^* yields an exact sequence

$$0 \longrightarrow \text{Tor}_1^R(\Omega^{-1}M, M^*) \longrightarrow M \otimes_R M^* \xrightarrow{f} F \otimes_R M^*$$

Since M^* is torsionfree, so is $F \otimes_R M^*$, and since M has rank so does M^* , and hence $\text{Tor}_1^R(\Omega^{-1}M, M^*)$ is torsion. It follows that the latter module is zero precisely when $M \otimes_R M^*$ is torsionfree. $\square$

In view of the preceding result Conjecture 1.1 translates to: Over a Gorenstein local ring of dimension one, any finitely generated rigid module is free. In the remainder of this section we once again focus on the case of an ideal. The new ingredient will be its endomorphism ring. Given an ideal I containing a regular element, when needed its endomorphism ring $\text{End}_R(I)$ will be viewed as a subring of the integral closure of R ; see the remarks at the beginning of Section 4.

The observation below is due to Bass [4, Section 7].

Remark 4.5. Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 1$ and I an $\mathfrak{m}$ -primary ideal. If $\text{End}_R(I) = R$, then I is principal.

Indeed, we assume R is not regular, that is to say, $\mathfrak{m}$ is not principal. Then the endomorphism ring E of $\mathfrak{m}$ is equal to $\mathfrak{m}^*$. Similarly, if I is not principal, then $I^* = \text{Hom}_R(I, \mathfrak{m})$, which is an E -module. Therefore $I = I^{**}$ is an E -module too, so $E \subseteq R$. Applying $(-)^*$ to the exact sequence

$$0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow k \rightarrow 0$$

we see that $E/R \cong \text{Ext}_R^1(k, R) \cong k$, so E is a proper extension of R .

The next observation is contained in [12, Lemma 2.2, Proposition 2.3]; see also [21, Theorem 6.4]. A simpler proof is available in our special case.

Proposition 4.6. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 1$ and I an $\mathfrak{m}$ -primary ideal. If I is rigid and the ring $\text{End}_R(I)$ is Gorenstein, then I is principal.*

Proof. Set $S := \text{End}_R(I)$, viewed as a subring of the integral closure of R . Remark 4.5 implies that I is a principal ideal of S , whence $I \cong S$: Even if S is not local, we simply apply the remark to each local ring of S , and then use the Chinese Remainder Theorem.

The canonical surjection $\pi: I \otimes_R I^* \twoheadrightarrow I \otimes_S I^*$ is bijective, because the injection $I \otimes_R I^* \rightarrow Q(R) \otimes_R I \otimes_R I^*$ factors through π . It follows that

$$\nu_R(I)\nu_R(I^*) = \nu_R(I \otimes_S I^*) = \nu_R(S \otimes_S I^*) = \nu_R(I^*).$$

Canceling the factor $\nu_R(I^*)$, we get the equation $\nu_R(I) = 1$. $\square$

Remark 4.7. It follows from Proposition 4.6 that if all rings between R and its integral closure are Gorenstein then there are no non-principal rigid ideals—but much more is known in this case; see [4].

Remark 4.8. Let I be an ideal in R that contains a regular element and is rigid; set $S := \text{End}_R(I)$. Applying $\text{Hom}_R(-, R)$ and $\text{Hom}_R(I, -)$, respectively, to the exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$, induces the rows of the following commutative diagram of R -modules:

$$(4.8.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & I^* & \longrightarrow & \text{Ext}_R^1(R/I, R) \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \gamma \\ 0 & \longrightarrow & S & \longrightarrow & I^* & \longrightarrow & \text{Hom}_R(I, R/I) \longrightarrow \text{Ext}_R^1(I, I) = 0 \end{array}$$

The vertical map on the left is the canonical homothety; it is one-to-one because I contains a regular element. The map γ is induced by the other two vertical arrows.

The Snake Lemma applied to the diagram above implies that γ is surjective and identifies $\ker \gamma$ with S/R . One thus gets an exact sequence

$$(4.8.2) \quad 0 \rightarrow S/R \rightarrow \text{Ext}_R^1(R/I, R) \rightarrow \text{Hom}_{R/I}(I/I^2, R/I) \rightarrow 0.$$

Suppose now that $(R, \mathfrak{m}, k)$ is one dimensional, Gorenstein. The canonical module of R/I is given by

$$(4.8.3) \quad \omega_{R/I} = \text{Ext}_R^1(R/I, R);$$

see, for example, [6, Theorem 3.3.7]. Moreover, we have

$$(4.8.4) \quad \begin{aligned} \operatorname{Hom}_{R/I}(I/I^2 \otimes_{R/I} \omega_{R/I}, \omega_{R/I}) &= \operatorname{Hom}_{R/I}(I/I^2, \operatorname{Hom}_{R/I}(\omega_{R/I}, \omega_{R/I})) \\ &= \operatorname{Hom}_{R/I}(I/I^2, R/I). \end{aligned}$$

Therefore the exact sequence (4.8.2) becomes

$$(4.8.5) \quad 0 \longrightarrow S/R \longrightarrow \omega_{R/I} \longrightarrow ((I/I^2) \otimes_{R/I} \omega_{R/I})^\vee \longrightarrow 0,$$

where $(-)^\vee$ denotes the $\omega_{R/I}$ -dual $\operatorname{Hom}_{R/I}(-, \omega_{R/I})$.

Denote the length of an R -module M by $\lambda_R(M)$.

Proposition 4.9. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 1$ and I an $\mathfrak{m}$ -primary ideal. We have the following equality:*

$$(4.9.1) \quad \lambda_R(\operatorname{Ext}_R^1(I, I)) + \lambda_R(R/I) = \lambda_R(S/R) + \lambda_R((I/I^2) \otimes_{R/I} \omega_{R/I}).$$

In particular, the ideal I is rigid if and only if

$$\lambda_R(R/I) - \lambda_R(S/R) = \lambda_R((I/I^2) \otimes_{R/I} \omega_{R/I})$$

if and only if

$$\lambda((\operatorname{tr}_R I)/I) = \lambda_R((I/I^2) \otimes_R \omega_{R/I}).$$

Proof. As R is Gorenstein and I contains a regular element, $\operatorname{Ext}_R^1(I, R) = 0$ holds. Thus, applying $\operatorname{Hom}_R(I, -)$ to the exact sequence $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ yields an exact sequence

$$0 \rightarrow S \rightarrow I^* \rightarrow \operatorname{Hom}_{R/I}(I/I^2, R/I) \rightarrow \operatorname{Ext}_R^1(I, I) \rightarrow 0.$$

Given (4.8.4) this translates to an exact sequence

$$(4.9.2) \quad 0 \rightarrow S/R \rightarrow I^*/R \rightarrow ((I/I^2) \otimes_{R/I} \omega_{R/I})^\vee \rightarrow \operatorname{Ext}_R^1(I, I) \rightarrow 0.$$

The top row of the commutative diagram (4.8.1) shows that $I^*/R \cong \omega_{R/I}$. Thus

$$\lambda_R(I^*/R) = \lambda_R(R/I).$$

Also, by duality, we have

$$\lambda_R(((I/I^2) \otimes_{R/I} \omega)^\vee) = \lambda_R((I/I^2) \otimes_{R/I} \omega).$$

Using these equalities and counting lengths in (4.9.2) yields (4.9.1). $\square$

Corollary 4.10. *Let R be a Gorenstein local ring R with $\dim R = 1$ and $I = (a, b)$ an ideal such that a and b are regular elements. Then I is rigid if and only if*

$$((a) : b) \cap ((b) : a) = ((a) : b)((b) : a).$$

Proof. Set $J := ((a) : b)$ so that I and J are linked via (a) . One checks that

$$bJ = b((a) :_R b) = (a) \cap (b) = a((b) : a).$$

Therefore

$$\operatorname{tr}_R J = \operatorname{tr}_R I = (a, b)I^{-1} = a \frac{1}{a} J + b \frac{1}{a} J = ((a) :_R b) + ((b) :_R a).$$

Now, by Proposition 4.9 J is rigid if and only if

$$\lambda_R\left(\frac{\operatorname{tr}_R J}{J}\right) = \lambda_R\left(\frac{J}{J^2}\right),$$

if and only if

$$\lambda_R\left(\frac{((a) :_R b) + ((b) :_R a)}{((a) :_R b)}\right) = \lambda_R\left(\frac{((a) :_R b)}{((a) :_R b)^2}\right),$$

if and only if

$$\lambda_R\left(\frac{((b) :_R a)}{((a) :_R b) \cap ((b) :_R a)}\right) = \lambda_R\left(\frac{b((a) :_R b)}{b((a) :_R b)^2}\right),$$

if and only if

$$\lambda_R\left(\frac{((b) :_R a)}{((a) :_R b) \cap ((b) :_R a)}\right) = \lambda_R\left(\frac{a((b) :_R a)}{a((b) :_R a)((a) :_R b)}\right),$$

if and only if

$$\lambda_R\left(\frac{((b) :_R a)}{((a) :_R b) \cap ((b) :_R a)}\right) = \lambda_R\left(\frac{((b) :_R a)}{((b) :_R a)((a) :_R b)}\right),$$

if and only if

$$((a) :_R b) \cap ((b) :_R a) = ((a) :_R b)((b) :_R a). \quad \square$$

Remark 4.11. Corollary 4.10 is rather remarkable, for it shows that the existence of a two-generated rigid ideal is *equivalent* to the existence of isomorphic ideals I and J such that R/I and R/J are Gorenstein and $\mathrm{Tor}_1^R(R/I, R/J) = 0$. It seems unlikely that such ideals exist.

The preceding results highlight the importance of studying endomorphism rings of rigid ideals. The result below, though not strictly necessary in the sequel, anticipates the key role linkage plays in the next section.

Lemma 4.12. *Let I and J be ideals of a Noetherian ring S . Assume each contains a regular element, and identify their endomorphism rings A and B , respectively, with subrings of the integral closure of S . If I and J are linked, then $A = B$.*

Proof. We have $J = ((\underline{x} :_S I))$ and $I = ((\underline{x} :_S J))$, where $\underline{x}$ is a regular sequence. Suppose $t \in A$, that is, t is in the integral closure of S and $tI \subseteq I$. If $z \in J$, we have $tzI = z(tI) \subseteq zI \subseteq (\underline{x})$. This shows that $tz \in ((\underline{x}) :_R I) = J$; thus $t \in B$. We have shown that $A \subseteq B$. By symmetry, the two endomorphism rings coincide. $\square$

5. SMOOTHABLE IDEALS

From now on the focus will be on complete intersection rings. In this section we verify Conjecture 1.1 for ideals that deform to smoothable ideals. To this end we recollect some basic properties of twisted conormal modules.

5.1. The twisted conormal module. Consider a pair (Q, J) where Q is a Gorenstein local ring and $J \subset Q$ an ideal such that Q/J is Cohen-Macaulay. The hypotheses on Q and J imply that $\mathrm{Ext}_Q^d(Q/J, Q)$, where $d = \mathrm{height} I$, is the canonical module of Q/J . Motivated by Proposition 4.9, we consider the Q/J -module

$$C(Q, J) := (J/J^2) \otimes_{Q/J} \omega_{Q/J},$$

which Buchweitz [7] calls the *twisted conormal module*. We are particularly interested in the situation when Q/J has finite length, and satisfies an inequality

$$(5.1.1) \quad \lambda_Q(C(Q, J)) \geq (\dim Q) \lambda_Q(Q/J).$$

In this case $\omega_{Q/J}$ is the injective hull of k as a module over Q/J . Thus, by Matlis duality, the preceding inequality is equivalent to

$$\lambda_Q(\operatorname{Hom}_{Q/J}(J/J^2, Q/J)) \geq (\dim Q)\lambda_Q(Q/J).$$

While such an inequality does not hold in general—see Example 5.15—the main agenda in this section is to identify various classes of ideals for which it does. We begin with the following simple observation.

Remark 5.2. In the context of (5.1.1), since Q/J has finite length one has

$$\nu_{Q/J}(J/J^2) = \nu_Q(J) \geq \dim Q$$

Thus when J is generated by a regular sequence, the Q/J -module J/J^2 , and hence also its dual, is free and so (5.1.1) holds; in fact, we get an equality. This observation can be extended to cover the case when Q is regular and J can be deformed to a ideal that is generically complete intersection; see Proposition 5.10.

Our interest in (5.1.1) is explained by the next result.

Corollary 5.3. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 1$ and I an $\mathfrak{m}$ -primary rigid ideal. If $C(R, I)$ satisfies the inequality in (5.1.1), then I is principal.*

Proof. If I is rigid and $C(R, I)$ satisfies the inequality in (5.1.1), it follows from Proposition 4.9 that $R = \operatorname{End}_R(I)$, and so Remark 4.5 yields that I is principal. $\square$

A special case of this result seems worth highlighting.

Corollary 5.4. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 1$ and I an $\mathfrak{m}$ -primary rigid ideal. If the R/I -module $\operatorname{Hom}_{R/I}(I/I^2, R/I)$ contains a submodule isomorphic to R/I , then I is principal.*

Proof. The hypothesis implies the inequality below

$$\lambda_R C(R, I) = \lambda_R \operatorname{Hom}_{R/I}(I/I^2, R/I) \geq \lambda_R(R/I)$$

the equality is by Matlis duality. It remains to recall Corollary 5.3. $\square$

5.5. Linkage. Ideals I, J in a Gorenstein ring are said to be *in the same linkage class* provided there is a sequence $I = I_0, I_1, \dots, I_n = J$ of ideals such that for each $0 \leq r \leq n-1$ the ideals I_r and I_{r+1} are linked, in the sense of 4.2. An ideal is *licci* provided it is in the linkage class of a complete intersection ideal, that is to say one that can be generated by a regular sequence.

The result below, attributed to Buchweitz in [14], asserts that for ideals of finite projective dimension the validity of (5.1.1) is invariant under linkage.

Proposition 5.6. *Let (R, I) be a pair as in 5.1 with R/I of finite length and of finite projective dimension. If $J \subset R$ is in the linkage class of I , then*

$$\lambda_R(C(R, I)) - (\dim R)\lambda_R(R/I) = \lambda_R(C(R, J)) - (\dim R)\lambda_R(R/J).$$

Remarks in place of a proof. See the proof of [14, Theorem (2.4)], in particular, the second display after the statement of Theorem (2.5). Also, it is shown in the proof of [14, Theorem (2.2)] that finite projective dimension is preserved by linkage, so that all ideals in the chain of links from I to J have finite projective dimension. Moreover, all of them are $\mathfrak{m}$ -primary and hence have grade equal to the common lengths of the regular sequences forming the links. $\square$

5.7. Deformations. The pair (R, I) *deforms* to (Q, J) , where Q is a local ring and J is an ideal in Q , if there exists a sequence $\underline{x}$ in Q that is regular both on Q and Q/J , and there are isomorphisms

$$\frac{Q}{(\underline{x})} \cong R \quad \text{and} \quad \frac{Q}{J + (\underline{x})} \cong \frac{R}{I}.$$

We say R *deforms* to Q if $(R, 0)$ deforms to $(Q, 0)$, that is to say, $R \cong Q/\underline{x}$ for a Q -regular sequence $\underline{x}$. In these cases R is Gorenstein if and only if Q is Gorenstein, and R/I is CM if and only if Q/J is CM; thus $C(R, I)$ is defined if and only if $C(Q, J)$ is defined. These remarks will be used without further comment.

The proof of the following result is standard, so is omitted.

Lemma 5.8. *When (R, I) deforms to (Q, J) there is a natural isomorphism of R -modules $C(R, I) \cong R \otimes_Q C(Q, J)$.* $\square$

5.9. Smoothability. As before, let R be a Gorenstein ring and $I \subset R$ an ideal. We say that I is *smoothable* provided the pair (R, I) deforms, in the sense of 5.7, to (Q, J) where J is generically complete intersection, that is to say, for each prime $\mathfrak{q}$ in Q associated to J , the ideal $J_{\mathfrak{q}}$ is generated by a regular sequence. Observe that the ring Q is also Gorenstein. When in addition, Q is a quotient of a regular local ring, the pair (Q, J) , and hence also the pair (R, I) , can be further deformed to a pair (Q', J') , where J' is reduced; see [15, Theorem 3.10]. This remark reconciles our notion of smoothability with the one introduced in [16, Definition 4.2].

Proposition 5.10. *Let (R, I) be a pair as in 5.1 with R/I of finite length. Then $C(R, I)$ satisfies the inequality in (5.1.1) when any of the following conditions holds:*

- (1) *R deforms to Q and with K the inverse image of I under the quotient map $Q \twoheadrightarrow R$, the Q/K -module $C(Q, K)$ satisfies (5.1.1);*
- (2) *(R, I) is smoothable;*
- (3) *I is licci.*

Proof. In either case $R \cong Q/(\underline{x})$, where $\underline{x}$ is a Q -regular sequence, say of length c .

(1) Let ω denote the canonical module of $S := R/I$. Since $I = K/(\underline{x})$ there is an exact sequence

$$\frac{K^2 + (\underline{x})}{K^2} \otimes_S \omega \longrightarrow \frac{K}{K^2} \otimes_S \omega \longrightarrow \frac{I}{I^2} \otimes_S \omega \longrightarrow 0,$$

of S -modules. Noting that S^c maps onto $(K^2 + (\underline{x}))/K^2$ and that $Q/K = R/I$, one gets an exact sequence of S -modules:

$$\omega^c \longrightarrow C(Q, K) \longrightarrow C(R, I) \longrightarrow 0.$$

This yields the first of the following (in)equalities

$$\begin{aligned} \lambda(C(R, I)) &\geq \lambda(C(Q, K)) - c\lambda(\omega) \\ &\geq (\dim Q)\lambda(S) - c\lambda(S) \\ &= (\dim Q - c)\lambda(S) \\ &= (\dim R)\lambda(S). \end{aligned}$$

The second inequality is by our hypothesis that $C(Q, K)$ satisfies (5.1.1), and the fact that $\lambda(\omega) = \lambda(S)$, which is a consequence of Matlis duality. The equalities are clear. This gives the desired conclusion.

(2) By hypothesis, (R, I) deforms to (Q, J) with J generically complete intersection. In particular, both (Q/J) -modules J/J^2 and $\omega_{Q/J}$ have rank, equal to height J and 1, respectively. Therefore one gets equalities

$$\text{rank}_{Q/J}(\text{C}(Q, J)) = \dim Q - \dim(Q/J) = \dim R - \dim(R/I) = \dim R,$$

where the first and the last one hold because Q and R are Gorenstein, and the second one holds because (Q, J) is a deformation of (R, I) . This computation will be used further below. For the remainder of the proof, we employ the multiplicity symbol $e(\underline{x}; -)$ on Q/J -modules. This is valid since $\underline{x}$ is a (regular) system of parameters for Q/J ; recall that R/I has finite length. From Lemma 5.8 and [6, Theorem 4.7.10(1)] one gets the first inequality below.

$$\begin{aligned} \lambda_Q(\text{C}(R, I)) &\geq e(\underline{x}, \text{C}(Q, J)) \\ &= e(\underline{x}, Q/J) \text{rank}_{Q/J}(\text{C}(Q, J)) \\ &= e(\underline{x}, Q/J)(\dim R) \\ &= \lambda(R/I)(\dim R) \end{aligned}$$

The first equality is by [6, Corollary 4.7.9], which applies since $\text{C}(Q, J)$ has rank, equal to $\dim R$. The other ones are clear.

(3) This follows from Proposition 5.6 and Remark 5.2. $\square$

Corollary 5.11. *Let $(R, \mathfrak{m})$ be a complete intersection local ring with $\dim R = 1$ and I an $\mathfrak{m}$ -primary rigid ideal. If $\text{embdim } R \leq 3$ and R/I is Gorenstein, then I is principal.*

Proof. We may assume that R is complete. By hypothesis, one can find a regular local ring Q of dimension three mapping onto R . The preimage K in Q of the ideal I is also of codimension three, and Q/K is Gorenstein. An argument due to J. Watanabe, implicit in the proof of [23, Theorem], shows K is licci. Thus $\text{C}(R, I)$ satisfies (5.1.1), by Proposition 5.10(3). It remains to recall Corollary 5.3. $\square$

Theorem 5.12. *Let $(R, \mathfrak{m})$ be a complete intersection local ring with $\dim R = 1$ and I an $\mathfrak{m}$ -primary rigid ideal. Let $\pi: Q \twoheadrightarrow R$ be a surjective local homomorphism with Q a regular local ring and set $K := \pi^{-1}(I)$.*

If I is rigid and $\text{C}(Q, K)$ satisfies the inequality in (5.1.1), in particular, if (Q, K) is licci or smoothable, then I is principal.

Proof. By Proposition 5.10(1) the R/I -module $\text{C}(R, I)$ also satisfies (5.1.1), so the desired result is a consequence of Corollary 5.3. $\square$

Here is an application of the preceding results.

Theorem 5.13. *Let $(R, \mathfrak{m})$ be a complete intersection local ring with $\dim R = 1$ and I an $\mathfrak{m}$ -primary rigid ideal. If $e(R) \leq 10$, then I is principal.*

Proof. Given Theorem 3.8 we only have to treat the case when $e(R)$ is 9 or 10. We assume that I is not principal. We consider the integer

$$\delta := e(R) - \max\{\nu_R(J) \mid J \supseteq \text{tr}_R(I)\}.$$

Since I is not principal, from Proposition 3.11 one gets inequalities

$$6 \leq \nu_R(I)\nu_R(I^*) \leq e(R) - \delta \leq \delta^2,$$

Since $e(R) \leq 10$, these imply that δ is one of $\{3, 4\}$, and in either case one has equality on the left, so that $\nu_R(I) = 3$ and $\nu_R(I^*) = 2$, or vice versa. We can assume without loss of generality that $\nu_R(I^*) = 2$, so I is Gorenstein, by Lemma 4.3. Then, given Corollary 5.11, one can assume also that $\text{embdim } R \geq 4$.

By passing to completions, if necessary, we can ensure that R is of the form $Q/(f_1, \dots, f_c)$, with Q a regular local ring and $\mathbf{f}$ a Q -regular sequence contained in $\mathfrak{n}^2$, where $\mathfrak{n}$ is the maximal ideal of Q ; note that $c = \text{embdim } Q - 1$. Let K be the inverse image of I under the surjection $Q \rightarrow R$. We will prove that K is licci, contradicting Theorem 5.12. To that end note that if $d_1, \dots, d_c$ are nonnegative integers such that $f_i \in \mathfrak{n}^{d_i} \setminus \mathfrak{n}^{d_i+1}$, then it follows from [5, Chapter VIII, §7, Proposition 7] that

$$e(R) \geq d_1 \cdots d_c.$$

Since $d_i \geq 2$ and $c = \text{embdim } R - 1 \geq 3$, the hypothesis $e(R) \leq 10$ implies $c = 3$, so that $\text{embdim } R = 4$, and that $d_i = 2$ for each i . In particular, the number of generators of $\mathfrak{m}^2$ is 7; here $\mathfrak{m}$ is the maximal ideal of R . This will be used later on.

Assume $e(R) = 9$, so that $\delta = 3$. Since the number of generators of $\mathfrak{m}^2$ is 7 and $\delta = 3$, it follows from the definition of δ that $\text{tr}_I(R)$ contains an element not in $\mathfrak{m}^2$. Write $I^* = (a, b)$ for elements $a, b \in R$. Then $I = ((a) : b)$, and the trace ideal is $((a) : b) + ((b) : a)$. Since the trace ideal is not contained in $\mathfrak{m}^2$, without loss of generality we may assume that $((a) : b)$ is not contained in $\mathfrak{m}^2$. Choose an element $c \in ((a) : b)$ such that $c \notin \mathfrak{m}^2$. Write $cb = da$. Then $I = ((a) : b) = ((c) : d)$, so after changing notation we may assume that $a \notin \mathfrak{m}^2$. We have shown that $Q/K \cong R/I$ is Gorenstein and that K is not contained in the square of the maximal ideal of Q . Lift a to Q and by abuse of notation, call that element a as well. Then $Q/(a)$ is a three-dimensional regular local ring, and $K/(a)$ is a grade-three Gorenstein ideal. As noted in the proof of Corollary 5.11, such an ideal is licci. It follows that K is also licci: do the links in $Q/(a)$ and lift back to Q by taking the same regular sequences and throwing in a . This is the desired result.

Assume $e(R) = 10$. When $\delta = 3$ inequality (2) in Proposition 3.11 is violated for

$$\nu_R(I^*)\delta + \nu_R(I) = 2(3) + 3 = 9.$$

Therefore $\delta = 4$. As in the case of $e(R) = 9$ and $\delta = 3$ we deduce that $\text{tr}_R(I)$ is not in $\mathfrak{m}^2$, so that K is licci. $\square$

Our thanks to the editor for the following observation.

Remark 5.14. The preceding results have implications also for the case $e(R) = 11$, namely, if in addition $\text{embdim } R \leq 3$, then there are no nontrivial $\mathfrak{m}$ -primary rigid ideals. Indeed, assuming I is a rigid $\mathfrak{m}$ -primary ideal that is not principal, one can argue as in the first paragraph of the proof of Theorem 5.13 to deduce that

$$3 \leq \delta \leq 11 - \nu_R(I)\nu_R(I^*).$$

This implies that one of I or I^* is two-generated; without loss of generality, suppose $\nu_R(I) = 2$. Then I is linked to an $\mathfrak{m}$ -primary ideal J that is Gorenstein and not principal; see Lemma 4.3. Since I is rigid so is J , and that contradicts Corollary 5.11, since $\text{embdim } R \leq 3$.

Example 5.15. D. Eisenbud and B. Ulrich provided us with examples that show that inequality (5.1.1) does not hold in general, even when Q is a regular ring and J is a Gorenstein ideal.

One such example is obtained by letting Q be the polynomial ring (say over $\mathbb{C}$) in six variables, and J the Gorenstein ideal with dual socle element a general cubic. The Hilbert function Q/J is then 1, 6, 6, 1. The length of J/J^2 , and hence also of $C(Q, J)$, is 76 so that

$$\lambda_Q(C(Q, J)) - (\dim Q)\lambda_Q(Q/J) = 76 - 6 \cdot 14 = -8.$$

This example is part of a family appearing in the work of J. Emsalem and A. Iarrobino [10], in which they study smoothing of algebras via their tangent spaces.

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