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RIGIDITY OF EXT AND TOR WITH COEFFICIENTS IN RESIDUE FIELDS OF A COMMUTATIVE NOETHERIAN RING

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Abstract Let $\mathfrak{p}$ be a prime ideal in a commutative noetherian ring R. It is proved that if an R-module M satisfies $\mathrm{Tor}_n^R(k(\mathfrak{p}), M) = 0$ for some $n \geqslant \dim R_{\mathfrak{p}}$, where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p}$, then $\mathrm{Tor}_i^R(k(\mathfrak{p}), M) = 0$ holds for all $i \geqslant n$. Similar rigidity results concerning $\mathrm{Ext}_R^*(k(\mathfrak{p}), M)$ are proved, and applications to the theory of homological dimensions are explored.

Keywords: Ext; Tor; rigidity; flat dimension; injective dimension

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1. Introduction

Flatness and injectivity of modules over a commutative ring R are characterized by vanishing of (co)homological functors and such vanishing can be verified by testing on cyclic R-modules. We discuss the flat case first, and in mildly greater generality: For any R-module M and integer $n \geqslant 0$, one has $\operatorname{flat\,dim}_R M < n$ if and only if $\mathrm{Tor}_n^R(R/\mathfrak{a}, M) = 0$ holds for all ideals $\mathfrak{a} \subseteq R$. When R is noetherian, and in this paper we assume that it is, it suffices to test on modules $R/\mathfrak{p}$ where $\mathfrak{p}$ varies over the prime ideals in R.

If R is local with unique maximal ideal $\mathfrak{m}$, and M is finitely generated, then it is sufficient to consider one cyclic module, namely the residue field $k := R/\mathfrak{m}$. Even if R is not local and M is not finitely generated, finiteness of $\operatorname{flat\,dim}_R M$ is characterized by vanishing of Tor with coefficients in fields, the residue fields $k(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ to be specific. While vanishing of $\mathrm{Tor}_*^R(k(\mathfrak{p}), M)$ for any one particular residue field does not imply that $\operatorname{flat\,dim}_R M$ is finite, one may still ask if vanishing of a single group $\mathrm{Tor}_n^R(k(\mathfrak{p}), M)$ implies vanishing of all higher groups, a phenomenon known as *rigidity*. While this does not hold in general (cf. Example 4.2), we prove that it does hold if n is sufficiently large; see Theorem 4.1 for the proof.

Theorem 1.1. *Let $\mathfrak{p}$ be a prime ideal in a commutative noetherian ring R and let M be an R-module. If one has $\operatorname{Tor}_n^R(k(\mathfrak{p}), M) = 0$ for some integer $n \geqslant \dim R_{\mathfrak{p}}$, then $\operatorname{Tor}_i^R(k(\mathfrak{p}), M) = 0$ holds for all $i \geqslant n$.*

As $\operatorname{flat\,dim}_R M < n$ holds if and only if one has $\operatorname{Tor}_i^R(k(\mathfrak{p}), M) = 0$ for all primes $\mathfrak{p}$ and all $i \geqslant n$, Theorem 1.1 provides for an improvement of existing characterizations of modules of finite flat dimension.

In parallel to the flat case, the injective dimension of an R-module M is less than n if $\operatorname{Ext}_R^n(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$. Moreover the injective dimension can be detected by vanishing *locally* of cohomology with coefficients in residue fields. That is, $\operatorname{inj\,dim}_R M < n$ holds if and only if one has $\operatorname{Ext}_{R_{\mathfrak{p}}}^n(k(\mathfrak{p}), M_{\mathfrak{p}}) = 0$ for all primes $\mathfrak{p}$. By the standard isomorphisms $\operatorname{Tor}_*^R(k(\mathfrak{p}), M) \cong \operatorname{Tor}_*^{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$ there is no local/global distinction for Tor vanishing. The following consequence of Proposition 3.2 is, therefore, a perfect parallel to Theorem 1.1.

Theorem 1.2. *Let $\mathfrak{p}$ be a prime ideal in a commutative noetherian ring R and let M be an R-module. If one has $\operatorname{Ext}_{R_{\mathfrak{p}}}^n(k(\mathfrak{p}), M_{\mathfrak{p}}) = 0$ for some integer $n \geqslant \dim R_{\mathfrak{p}}$, then $\operatorname{Ext}_{R_{\mathfrak{p}}}^i(k(\mathfrak{p}), M_{\mathfrak{p}}) = 0$ holds for all $i \geqslant n$.*

In contrast to the situation for Tor, the cohomology groups $\operatorname{Ext}_{R_{\mathfrak{p}}}^*(k(\mathfrak{p}), M_{\mathfrak{p}})$ and $\operatorname{Ext}_R^*(k(\mathfrak{p}), M)$ can be quite different, and it was only proved recently, in [**6**, Theorem 1.1], that the injective dimension of an R-module can be detected by vanishing *globally* of cohomology with coefficients in residue fields. That is, $\operatorname{inj\,dim}_R M < n$ holds if and only if one has $\operatorname{Ext}_R^i(k(\mathfrak{p}), M) = 0$ for all $i \geqslant n$ and all primes $\mathfrak{p}$. One advantage of this global vanishing criterion is that it also applies to complexes of modules; per Example 6.3 the local vanishing criterion does not. For the proof of the following rigidity result for $\operatorname{Ext}_R^*(k(\mathfrak{p}), M)$, see Remark 5.8.

Theorem 1.3. *Let $\mathfrak{p}$ be a prime ideal in a commutative noetherian ring R and let M be an R-module. If one has $\operatorname{Ext}_R^n(k(\mathfrak{p}), M) = 0$ for some integer $n \geqslant 2\dim R$, then $\operatorname{Ext}_R^i(k(\mathfrak{p}), M) = 0$ holds for all $i \geqslant n$.*

The case when $\mathfrak{p}$ is the maximal ideal of a local ring merits comment, for the bound on n in Theorems 1.2 and 1.3 differs by a factor of 2. The proof shows that it is sufficient to require $n \geqslant \dim R_{\mathfrak{p}} + \operatorname{proj\,dim}_R R_{\mathfrak{p}}$ in Theorem 1.3, and that aligns the two bounds in this special case. For a general prime $\mathfrak{p}$ however the number $\operatorname{proj\,dim} R_{\mathfrak{p}}$ may depend on the Continuum Hypothesis; see Osofsky [**14**].

In this introduction, we have focused on results that deal with rigidity of the Tor and Ext functors. In the text, we also establish results that track where vanishing of these functors starts, when indeed they vanish eventually.

Throughout R will be a commutative noetherian ring. Background material on homological invariants and local (co)homology is recalled in § 2. Rigidity results for Ext and Tor over local rings are proved in § 3, and applications to homological dimensions follow in §§ 4 and 5. The final section explores, by way of examples, the complicated nature of injective dimension of unbounded complexes.

2. Local homology and local cohomology

Our standard reference for definitions and constructions involving complexes is [2]. We will be dealing with graded modules whose natural grading is the upper one and also those whose natural grading is the lower one. Therefore, we set

$$\inf \mathrm{H}_*(M) := \inf\{i \mid \mathrm{H}_i(M) \neq 0\} \quad \text{and} \quad \inf \mathrm{H}^*(M) := \inf\{i \mid \mathrm{H}^i(M) \neq 0\}$$

for an R-complex M, and analogously we define $\sup \mathrm{H}_*(M)$ and $\sup \mathrm{H}^*(M)$. We often work in the derived category of R-modules, and write $\simeq$ for isomorphisms there. A morphism between complexes is a *quasi-isomorphism* if it is an isomorphism in homology; that is to say, if it becomes an isomorphism in the derived category.

The following paragraphs summarize the definitions and basic results on local cohomology and local homology, following [1, 13].

2.1. Local (co)homology

Let R be a commutative noetherian ring and $\mathfrak{a}$ an ideal in R. The right derived functor of the $\mathfrak{a}$-torsion functor $\varGamma_{\mathfrak{a}}$ is denoted $\mathsf{R}\varGamma_{\mathfrak{a}}$, and the *local cohomology* supported on $\mathfrak{a}$ of an R-complex M is the graded module

$$\mathrm{H}^*_{\mathfrak{a}}(M) := \mathrm{H}^*(\mathsf{R}\varGamma_{\mathfrak{a}}(M)).$$

There is a natural morphism $\mathsf{R}\varGamma_{\mathfrak{a}}(M) \to M$ in the derived category; M is said to be *derived* $\mathfrak{a}$*-torsion* when this map is an isomorphism. This is equivalent to the condition that $\mathrm{H}^*(M)$ is degreewise $\mathfrak{a}$-torsion; see [7, Proposition 6.12].

The left derived functor of the $\mathfrak{a}$-adic completion functor $\varLambda^{\mathfrak{a}}$ is denoted $\mathsf{L}\varLambda^{\mathfrak{a}}$ and the *local homology* of M supported on $\mathfrak{a}$ is the graded module

$$\mathrm{H}_*^{\mathfrak{a}}(M) := \mathrm{H}_*(\mathsf{L}\varLambda^{\mathfrak{a}}(M)).$$

There is a natural morphism $M \to \mathsf{L}\varLambda^{\mathfrak{a}}(M)$ in the derived category and we say M is *derived* $\mathfrak{a}$*-complete* when this map is an isomorphism. This is equivalent to the condition that for each i, the natural map $\mathrm{H}^i(M) \to \mathrm{H}_0^{\mathfrak{a}}(\mathrm{H}^i(M))$ is an isomorphism; see [7, Proposition 6.15].

The morphisms $\mathsf{R}\varGamma_{\mathfrak{a}}(M) \to M$ and $M \to \mathsf{L}\varLambda^{\mathfrak{a}}(M)$ induce isomorphisms

$$\begin{aligned} &\mathrm{Ext}^*_R(R/\mathfrak{a}, M) \cong \mathrm{Ext}^*_R(R/\mathfrak{a}, \mathsf{R}\varGamma_{\mathfrak{a}} M) \quad \text{and} \\ &\mathrm{Tor}_*^R(R/\mathfrak{a}, M) \cong \mathrm{Tor}_*^R(R/\mathfrak{a}, \mathsf{L}\varLambda^{\mathfrak{a}} M). \end{aligned} \tag{2.1}$$

Indeed, the first one holds because the functor $\mathsf{R}\varGamma_{\mathfrak{a}}$ is right adjoint to the inclusion of the $\mathfrak{a}$-torsion complexes (that is to say, complexes whose cohomology is $\mathfrak{a}$-torsion) into the derived category of R; see [13, Proposition 3.2.2]. Thus one has

$$\mathsf{R}\mathrm{Hom}(R/\mathfrak{a}, \mathsf{R}\varGamma_{\mathfrak{a}} M) \xrightarrow{\simeq} \mathsf{R}\mathrm{Hom}(R/\mathfrak{a}, M), \tag{2.2}$$

and this gives the first isomorphism. As to the second isomorphism, consider the commutative square in the derived category

$$\begin{tikzcd} \mathsf{R}\Gamma_{\mathfrak{a}} M \arrow[r] \arrow[d, "\simeq"'] & M \arrow[d] \\ \mathsf{R}\Gamma_{\mathfrak{a}} \mathsf{L}\Lambda^{\mathfrak{a}} M \arrow[r] & \mathsf{L}\Lambda^{\mathfrak{a}} M \end{tikzcd}$$

induced by the vertical morphism on right. The isomorphism on the left is part of Greenlees–May duality; see, for example, [1, Corollary (5.1.1)]. Applying the functor $R/\mathfrak{a} \otimes^{\mathsf{L}}_R (-)$ yields the commutative square

$$\begin{tikzcd} R/\mathfrak{a} \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}} M \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & R/\mathfrak{a} \otimes^{\mathsf{L}}_R M \arrow[d] \\ R/\mathfrak{a} \otimes^{\mathsf{L}}_R \mathsf{R}\Gamma_{\mathfrak{a}} \mathsf{L}\Lambda^{\mathfrak{a}} M \arrow[r, "\simeq"'] & R/\mathfrak{a} \otimes^{\mathsf{L}}_R \mathsf{L}\Lambda^{\mathfrak{a}} M \end{tikzcd}$$

The horizontal maps are isomorphisms by (2.2) and [7, Proposition 6.5]. Thus the vertical map on the right is also an isomorphism; in homology, this is the desired isomorphism.

Let $C(\mathfrak{a})$ denote the Čech complex on a set of elements that generate $\mathfrak{a}$. The values of the functors $\mathsf{L}\Lambda^{\mathfrak{a}}$ and $\mathsf{R}\Gamma_{\mathfrak{a}}$ on an R-complex M can then be computed as

$$\mathsf{L}\Lambda^{\mathfrak{a}}(M) = \mathsf{R}\mathrm{Hom}_R(C(\mathfrak{a}), M) \quad \text{and} \quad \mathsf{R}\Gamma_{\mathfrak{a}}(M) = C(\mathfrak{a}) \otimes^{\mathsf{L}}_R M.$$

See for example [1, Theorem (0.3) and Lemma (3.1.1)].

2.2. Depth and width

In the remainder of this section $(R, \mathfrak{m}, k)$ will be a local ring. This means that R is a commutative noetherian ring with unique maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$.

The *depth* and *width* of an R-complex M are defined as follows:

$$\operatorname{depth}_R M = \inf \operatorname{Ext}^*_R(k, M) \quad \text{and} \quad \operatorname{width}_R M = \inf \operatorname{Tor}^R_*(k, M).$$

One has $\operatorname{depth}_R M \geqslant \inf \mathrm{H}^*(M)$ and if $i = \inf \mathrm{H}^*(M)$ is finite, then equality holds if and only if $\operatorname{Hom}_R(k, \mathrm{H}^i(M)) \neq 0$. Similarly, one has $\operatorname{width}_R M \geqslant \inf \mathrm{H}_*(M)$ and if $j = \inf \mathrm{H}_*(M)$ is finite, then equality holds if and only if $k \otimes_R \mathrm{H}_j(M) \neq 0$.

If $\operatorname{flat\,dim}_R M$ is finite, then one has an equality

$$\operatorname{depth}_R M = \operatorname{depth} R - \sup \operatorname{Tor}^R_*(k, M). \tag{2.3}$$

This is an immediate consequence of [2, Lemma 4.4(F)]. For finitely generated modules it is the Auslander–Buchsbaum formula.

Similarly, if $\operatorname{inj\,dim}_R M$ is finite, then one has

$$\operatorname{width}_R M = \operatorname{depth} R - \sup \operatorname{Ext}^*_R(k, M). \tag{2.4}$$

This is a consequence of [2, Lemma 4.4(I)]. For finitely generated modules the equality above yields Bass' formula $\operatorname{inj\,dim}_R M = \operatorname{depth} R$.

From [**9**, Definitions 2.3 and 4.3] one gets that the depth and width of an R-complex can be detected by vanishing of local (co)homology:

$$\operatorname{depth}_R M = \inf \mathrm{H}^*_{\mathfrak{m}}(M) \quad \text{and} \quad \operatorname{width}_R M = \inf \mathrm{H}_*^{\mathfrak{m}}(M).$$

Combining this with (2.1) and the isomorphisms

$$\mathsf{R}\varGamma_{\mathfrak{m}} \mathsf{L}\varLambda^{\mathfrak{m}}(M) \simeq \mathsf{R}\varGamma_{\mathfrak{m}}(M) \quad \text{and} \quad \mathsf{L}\varLambda^{\mathfrak{m}} \mathsf{R}\varGamma_{\mathfrak{m}}(M) \simeq \mathsf{L}\varLambda^{\mathfrak{m}}(M)$$

from [**1**, Corollary (5.1.1)] one gets equalities

$$\operatorname{depth}_R \mathsf{R}\varGamma_{\mathfrak{m}} M = \operatorname{depth}_R M = \operatorname{depth}_R \mathsf{L}\varLambda^{\mathfrak{m}} M \tag{2.5}$$

$$\operatorname{width}_R \mathsf{R}\varGamma_{\mathfrak{m}} M = \operatorname{width}_R M = \operatorname{width}_R \mathsf{L}\varLambda^{\mathfrak{m}} M. \tag{2.6}$$

For later use, we note that for each R-complex M there are inequalities

$$\sup \mathrm{H}^*_{\mathfrak{m}}(M) \leqslant \dim R + \sup \mathrm{H}^*(M) \quad \text{and} \quad \sup \mathrm{H}^*_{\mathfrak{m}}(M) \leqslant \sup \operatorname{Ext}^*_R(k, M). \tag{2.7}$$

The first is immediate as one has $\mathrm{H}^*_{\mathfrak{m}}(M) = \mathrm{H}^*(C(\mathfrak{m}) \otimes^{\mathsf{L}}_R M)$ where $C(\mathfrak{m})$ is the Čech complex on a system of parameters for R; the second is immediate once one recalls the isomorphism $\mathrm{H}^*_{\mathfrak{m}}(M) \cong \varinjlim_i \operatorname{Ext}^*_R(R/\mathfrak{m}^i, M)$.

The next result is a direct extension of [**16**, Proposition 2.1] by Simon. Concerning the last assertion: $\operatorname{Tor}^R_i(R, M) = \mathrm{H}_i(M) \neq 0$, so n cannot equal $\sup \mathrm{H}_*(M)$, unless both are infinite. However, for later applications it is convenient to have the statement in this form.

Lemma 2.1. *Let M be a derived $\mathfrak{a}$-complete R-complex with $\inf \mathrm{H}_*(M) > -\infty$ and n an integer. If $\operatorname{Tor}^R_n(R/\mathfrak{p}, M) = 0$ all prime ideals $\mathfrak{p} \supseteq \mathfrak{a}$, then $\operatorname{Tor}^R_n(-, M) = 0$.*

When in addition $n \geqslant \sup \mathrm{H}_*(M)$, one has $\operatorname{flat\,dim}_R M \leqslant n - 1$.

Proof. First we claim that for any finitely generated R-module L and integer i, if $\mathfrak{a} \operatorname{Tor}^R_i(L, M) = \operatorname{Tor}^R_i(L, M)$, then $\operatorname{Tor}^R_i(L, M) = 0$. Indeed, let F be a free resolution of L with each F_i finitely generated and equal to zero for $i < 0$. Let G be a semi-flat resolution of M with $G_i = 0$ for $i \ll 0$; this is possible as $\inf \mathrm{H}_*(M)$ is finite. Since M is derived $\mathfrak{a}$-complete, the complex $\varLambda^{\mathfrak{a}} G$, which computes $\mathsf{L}\varLambda^{\mathfrak{a}} M$, is quasi-isomorphic to M. Thus

$$\operatorname{Tor}^R_i(L, M) = \mathrm{H}_i(F \otimes_R \varLambda^{\mathfrak{a}} G).$$

Note that $F \otimes_R \varLambda^{\mathfrak{a}} G$ is a complex of $\mathfrak{a}$-adically complete R-modules; this is where we need that each F_i is finitely generated and that F_i and G_i are zero for $i \ll 0$. It remains to apply [**16**, Proposition 1.4].

For the stated result, it suffices to prove that the set

$$\{\mathfrak{b} \subset R \text{ an ideal} \mid \operatorname{Tor}^R_n(R/\mathfrak{b}, M) \neq 0\}$$

is empty. Suppose it is not. Pick a maximal element; say, $\mathfrak{q}$. We claim that this is a prime ideal. The argument is standard (see, for example, [**12**, 2.4]) and goes as follows. If the set is not empty, let $\mathfrak{q}'$ be an associated prime ideal of $R/\mathfrak{q}$, and $x \in R$ an element such that $\mathfrak{q}' = \{r \in R \mid xr \in \mathfrak{q}\}$. This then yields an exact sequence

$$0 \longrightarrow R/\mathfrak{q}' \xrightarrow{1 \mapsto x} R/\mathfrak{q} \longrightarrow R/((x) + \mathfrak{q}) \longrightarrow 0.$$

Since both $\mathfrak{q}'$ and $(x) + \mathfrak{q}$ strictly contain $\mathfrak{q}$, one obtains that

$$\operatorname{Tor}_n^R(R/\mathfrak{q}', M) = 0 = \operatorname{Tor}_n^R(R/((x) + \mathfrak{q}), M),$$

and hence $\operatorname{Tor}_n^R(R/\mathfrak{q}, M) = 0$, contradicting the choice of $\mathfrak{q}$. Thus $\mathfrak{q}$ is prime.

By hypothesis, $\mathfrak{q}$ does not contain $\mathfrak{a}$, so choose an element a in $\mathfrak{a}$ but not in $\mathfrak{q}$ and consider the exact sequence

$$0 \longrightarrow R/\mathfrak{q} \xrightarrow{a} R/\mathfrak{q} \longrightarrow R/((a) + \mathfrak{q}) \longrightarrow 0.$$

Noting that $\operatorname{Tor}_n^R(R/((a) + \mathfrak{q}), M) = 0$ by the choice of $\mathfrak{q}$, it follows from the exact sequence above that the map $\operatorname{Tor}_n^R(R/\mathfrak{q}, M) \xrightarrow{a} \operatorname{Tor}_n^R(R/\mathfrak{q}, M)$ is surjective. By the claim in the first paragraph, this implies that $\operatorname{Tor}_n^R(R/\mathfrak{q}, M) = 0$, which is a contradiction. $\square$

3. Local rings

In this section $(R, \mathfrak{m}, k)$ is a local ring. Note from (2.7) that in the next statement n cannot equal $\sup \mathrm{H}^*(M)$, but, as with Lemma 2.1, this formulation is convenient for later applications.

Lemma 3.1. *Let M be a derived $\mathfrak{m}$-torsion R-complex with $\inf \mathrm{H}^*(M) > -\infty$. If one has $\operatorname{Ext}_R^n(k, M) = 0$ for some integer $n \geqslant \sup \mathrm{H}^*(M)$, then $\operatorname{inj\,dim}_R M \leqslant n - 1$.*

Proof. Let I be the minimal semi-injective resolution of M. One has

$$\operatorname{Ext}_R^n(k, M) = \mathrm{H}^n(\operatorname{Hom}_R(k, I)) = \operatorname{Hom}_R(k, I^n).$$

As M is derived $\mathfrak{m}$-torsion, each module I^i is a direct sum of copies of the injective envelope of k, so $\operatorname{Ext}_R^n(k, M) = 0$ implies $I^n = 0$. It follows from the assumption on n and minimality of I that $I^i = 0$ holds for all $i \geqslant n$; in particular, one has $\operatorname{inj\,dim}_R M \leqslant n - 1$. $\square$

The result below extends (2.4); its proof would be significantly shorter under the additional hypothesis that $\inf \mathrm{H}_{\mathfrak{m}}^*(M)$ is finite.

Proposition 3.2. *Let M be an R-complex. If $\operatorname{Ext}_R^n(k, M) = 0$ holds for some integer $n \geqslant \sup \mathrm{H}_{\mathfrak{m}}^*(M)$, then one has $\operatorname{Ext}_R^i(k, M) = 0$ for all $i \geqslant n$ and*

$$\sup \operatorname{Ext}_R^*(k, M) = \operatorname{depth} R - \operatorname{width}_R M.$$

Proof. Let J be the minimal semi-injective resolution of M and set $I := \varGamma_{\mathfrak{m}}(J)$. For every integer i one has

$$\mathrm{H}^i_{\mathfrak{m}}(M) = \mathrm{H}^i(I) \quad \text{and} \quad \mathrm{Ext}^i_R(k, M) = \mathrm{Hom}_R(k, J^i) = \mathrm{Hom}_R(k, I^i).$$

Let d be any integer with $n \geqslant d \geqslant \sup \mathrm{H}^*_{\mathfrak{m}}(M)$ and set $Z := \mathrm{Z}^d(I)$, the submodule of cycles in degree d. Vanishing of $\mathrm{H}^i_{\mathfrak{m}}(M)$ and the identifications above yield

$$\mathrm{Ext}^i_R(k, M) \cong \mathrm{Ext}^{i-d}_R(k, Z) \quad \text{for all } i \geqslant d.$$

In particular, one has $\mathrm{Ext}^{n-d}_R(k, Z) = 0$. Since Z is an $\mathfrak{m}$-torsion R-module, Lemma 3.1 and the isomorphisms above yield $\mathrm{Ext}^i_R(k, M) = 0$ for all $i \geqslant n$.

It remains to verify the claim about the supremum of $\mathrm{Ext}^*_R(k, M)$. To this end, let K be the Koszul complex on a minimal set of generators for $\mathfrak{m}$. It follows from [**9**, Definition 2.3] that one has the second equivalence below

$$\begin{aligned} \sup \mathrm{Ext}^*_R(k, M) = -\infty \iff & \mathrm{Ext}^*_R(k, M) = 0 \\ \iff & \mathrm{H}^*(K \otimes_R M) = 0 \\ \iff & \mathrm{width}_R M = \infty. \end{aligned}$$

The first one is by definition while the last one is by [**9**, Theorem 4.1]. We may thus assume that $s := \sup \mathrm{Ext}^*_R(k, M)$ and $w := \sup \mathrm{H}^*(M \otimes_R K)$ are integers. Because K is a bounded complex of finitely generated free R-modules, there is a quasi-isomorphism

$$\mathsf{RHom}_R(k, M) \otimes^{\mathsf{L}}_R K \simeq \mathsf{RHom}(k, M \otimes_R K).$$

From this and the fact that $\mathrm{Ext}^*_R(k, M)$ is a graded k-vector space, it follows that

$$s = \sup \mathrm{H}^*(\mathsf{RHom}_R(k, M) \otimes^{\mathsf{L}}_R K) = \sup \mathrm{Ext}^*_R(k, M \otimes_R K).$$

Let E be the minimal injective resolution of $M \otimes_R K$. Since $M \otimes_R K$ is derived $\mathfrak{m}$-torsion, one has $\varGamma_{\mathfrak{m}} E = E$. From $\mathrm{Ext}^*_R(k, M \otimes_R K) = \mathrm{Hom}^*_R(k, E)$ it thus follows that $E^s \neq 0$ and $E^i = 0$ for all $i > s$. On the other hand, as $w = \sup \mathrm{H}^*(E)$ the complex $E^{\geqslant w}$ is the minimal injective resolution of the module $W := \mathrm{Z}^w(E)$ of cycles in degree w, so one has $\mathrm{inj\,dim}_R W = s - w$.

It remains to show that $\mathrm{inj\,dim}_R W = \mathrm{depth}\, R$. Evidently one has $\mathrm{inj\,dim}_R W = \sup \mathrm{Ext}^*_R(k, W)$, so by (2.4) it suffices to show that W has width 0, that is to say, that $k \otimes_R W \neq 0$. But this is clear because $\mathrm{H}^w(E)$ is non-zero and annihilated by $\mathfrak{m}$, whence $\mathfrak{m}W \subseteq \mathrm{B}^w(E) \subsetneq W$. □

Proposition 3.3. *Let M be an R-complex with $\mathrm{Tor}^R_n(k, M) = 0$ for some integer n, and assume that one of the following conditions is satisfied:*

(1) $n \geqslant \sup \mathrm{H}^{\mathfrak{m}}_*(M)$ *and* $\inf \mathrm{H}_*(M) > -\infty$*;*

(2) $n \geqslant \sup \mathrm{H}^*_{\mathfrak{m}}(\mathrm{Hom}_R(M, E(k)))$ *where $E(k)$ is the injective envelope of k.*

One then has $\mathrm{Tor}^R_i(k, M) = 0$ for all $i \geqslant n$ and

$$\sup \mathrm{Tor}^R_*(k, M) = \mathrm{depth}\, R - \mathrm{depth}_R M.$$

Proof. Assume first that (1) is satisfied. From (2.1) one gets $\operatorname{Tor}_n^R(k, \mathsf{L}\Lambda^{\mathfrak{m}} M) = 0$. The complex $\mathsf{L}\Lambda^{\mathfrak{m}} M$ is $\mathfrak{m}$-adically complete and

$$\inf \mathrm{H}_*(\mathsf{L}\Lambda^{\mathfrak{m}} M) \geqslant \inf \mathrm{H}_*(M) > -\infty.$$

Thus Lemma 2.1 applies and yields that $\operatorname{flat\,dim}_R(\mathsf{L}\Lambda^{\mathfrak{m}} M)$ is at most $n-1$. Now (2.1) yields $\operatorname{Tor}_i^R(k, M) = 0$ for all $i \geqslant n$, and then (2.3) and (2.5) yield the second and third equalities below

$$\begin{aligned} \sup \operatorname{Tor}_*^R(k, M) &= \sup \operatorname{Tor}_*^R(k, \mathsf{L}\Lambda^{\mathfrak{m}} M) \\ &= \operatorname{depth} R - \operatorname{depth}_R(\mathsf{L}\Lambda^{\mathfrak{m}} M) \\ &= \operatorname{depth} R - \operatorname{depth}_R M. \end{aligned}$$

Assume now that the hypothesis in (2) holds, and set $(-)^\vee := \operatorname{Hom}_R(-, E(k))$. For each integer i, there is an isomorphism

$$\operatorname{Tor}_i^R(k, M)^\vee \cong \operatorname{Ext}_R^i(k, M^\vee).$$

The hypothesis and Proposition 3.2 yield $\operatorname{Tor}_i^R(k, M) = 0$ for all $i \geqslant n$ and

$$\sup \operatorname{Tor}_*^R(k, M) = \sup \operatorname{Ext}_R^*(k, M^\vee) = \operatorname{depth} R - \operatorname{width}_R M^\vee.$$

Finally one has $\operatorname{width}_R M^\vee = \operatorname{depth}_R M$; see [**9**, Proposition 4.4]. $\square$

The final result of this section fleshes out a remark made by Fossum *et al.* at the end of § 1 in [**8**]. They phrase it as statement about non-vanishing: If M is an R-module and $\operatorname{Ext}_R^n(k, M)$ is non-zero for some $n \geqslant \operatorname{depth} R + 1$ then one has $\operatorname{Ext}_R^i(k, M) \neq 0$ for all $i \geqslant n$. The formulation below makes for an easier comparison with Proposition 3.2.

Proposition 3.4. *Let M be an R-complex. If $\operatorname{Ext}_R^n(k, M) = 0$ holds for some integer $n \geqslant \sup \mathrm{H}^*(M) + \operatorname{depth} R + 1$, then one has*

$$\operatorname{Ext}_R^i(k, M) = 0 \quad \textit{for every } i \textit{ in the range } \sup \mathrm{H}^*(M) + \operatorname{depth} R + 1 \leqslant i \leqslant n.$$

Proof. We may assume that $n > \sup \mathrm{H}^*(M) + \operatorname{depth} R + 1$ holds. It suffices to verify that when $\operatorname{Ext}_R^n(k, M)$ is zero, so is $\operatorname{Ext}_R^{n-1}(k, M)$.

Let $\boldsymbol{x}$ be a maximal regular sequence in R, set $S := R/(\boldsymbol{x})$ and $\mathfrak{n} := \mathfrak{m}/(\boldsymbol{x})$. Thus, $(S, \mathfrak{n}, k)$ is a local ring of depth 0; in particular, $(0 : \mathfrak{n})$, the socle of S, is non-zero. Thus, there exists a positive integer, say s, such that $(0 : \mathfrak{n})$ is contained in $\mathfrak{n}^s$ but not in $\mathfrak{n}^{s+1}$. Said otherwise, the composite of canonical maps

$$(0 : \mathfrak{n}) \longrightarrow \mathfrak{n}^s \longrightarrow \mathfrak{n}^s/\mathfrak{n}^{s+1}$$

is non-zero. Since the source and the target are k-vector spaces, this implies that k is a direct summand of $\mathfrak{n}^s$. It thus suffices to verify that $\operatorname{Ext}_R^{n-1}(\mathfrak{n}^s, M) = 0$.

The Koszul complex on $\boldsymbol{x}$ is a minimal free resolution of S over R, so one has $\operatorname{proj\,dim}_R S = \operatorname{depth} R$ and hence

$$\operatorname{Ext}_R^j(S, M) = 0 \text{ for } j \geqslant \operatorname{depth} R + 1 + \sup \mathrm{H}^*(M).$$

Given this, the exact sequence

$$0 \longrightarrow \mathfrak{n}^s \longrightarrow S \longrightarrow S/\mathfrak{n}^s \longrightarrow 0$$

yields an isomorphism

$$\operatorname{Ext}_R^{n-1}(\mathfrak{n}^s, M) \cong \operatorname{Ext}_R^n(S/\mathfrak{n}^s, M).$$

Since the length of $S/\mathfrak{n}^s$ is finite, $\operatorname{Ext}_R^n(k, M) = 0$ implies $\operatorname{Ext}_R^n(S/\mathfrak{n}^s, M) = 0$, and hence the isomorphism above yields $\operatorname{Ext}_R^{n-1}(\mathfrak{n}^s, M) = 0$, as desired. □

4. Flat dimension

Let R be a commutative noetherian ring, M be an R-complex, and n be an integer. Avramov and Foxby [**2**, Proposition 5.3.F] prove that $\operatorname{flat\,dim}_R M < n$ holds if and only if one has $\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0$ for every prime $\mathfrak{p}$ in R and all $i \geqslant n$. That is,

$$\operatorname{flat\,dim}_R M = \sup\{i \in \mathbb{Z} \mid \operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\}. \tag{4.1}$$

By way of the isomorphisms

$$\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \cong \operatorname{Tor}_i^R(k(\mathfrak{p}), M) \tag{4.2}$$

this result compares—or may be it is the other way around—to [**6**, Theorem 1.1]; see (5.1). Combining (4.1) and (4.2) with (2.3) one gets

$$\operatorname{flat\,dim}_R M = \sup_{\mathfrak{p} \in \operatorname{Spec} R} \{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}\} \tag{4.3}$$

for every R-complex M of finite flat dimension. For modules of finite flat dimension this equality is known from the work of Chouinard [**5**].

For rings of finite Krull dimension, the next theorem, which contains Theorem 1.1, represents a significant strengthening of (4.1).

Theorem 4.1. *Let R be a commutative noetherian ring and M be an R-complex. If for a prime ideal $\mathfrak{p}$ and $n \geqslant \dim R_\mathfrak{p} + \sup \mathrm{H}_*(M)$ one has $\operatorname{Tor}_n^R(k(\mathfrak{p}), M) = 0$, then*

$$\sup \operatorname{Tor}_*^R(k(\mathfrak{p}), M) = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} < n.$$

In particular, if there exists an integer $n \geqslant \dim R + \sup \mathrm{H}_*(M)$ such that

$$\operatorname{Tor}_n^R(k(\mathfrak{p}), M) = 0 \textit{ holds for every prime ideal } \mathfrak{p} \textit{ in } R,$$

then the flat dimension of M is less than n.

Proof. It suffices to prove the first claim; the assertion about the flat dimension of M is a consequence, given (4.1).

Fix $\mathfrak{p}$ and n as in the hypotheses. Given (4.2), this yields $\operatorname{Tor}_n^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0$. One has the following (in)equalities

$$\sup \mathrm{H}_*(M) \geqslant \sup \mathrm{H}_*(M_\mathfrak{p}) = \sup \mathrm{H}^* \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, E(k(\mathfrak{p}))).$$

Since $\dim R \geqslant \dim R_\mathfrak{p}$, it follows from (2.7) and Proposition 3.3 that

$$\sup \operatorname{Tor}_*^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} < n.$$

This is the desired result. $\square$

The next example shows that the constraint on n in Theorem 4.1 is needed.

Example 4.2. Let $(R, \mathfrak{m}, k)$ be a local ring, N a finitely generated Cohen–Macaulay R-module of dimension d, and set $M := \mathrm{H}_\mathfrak{m}^d(N)$. There are isomorphisms

$$\operatorname{Tor}_i^R(k, M) \cong \operatorname{Tor}_{i-d}^R(k, N) \quad \text{for all } i.$$

To see this, let F be the Čech complex on a maximal N-regular sequence $\boldsymbol{x}$. The complex $(\Sigma^d F) \otimes_R N$ is quasi-isomorphic to M, for $\inf \mathrm{H}_\mathfrak{m}^*(N) = \sup \mathrm{H}_\mathfrak{m}^*(N) = d$, by the hypothesis on N. Thus, there are quasi-isomorphisms

$$k \otimes_R^{\mathsf{L}} M \simeq k \otimes_R^{\mathsf{L}} ((\Sigma^d F) \otimes_R^{\mathsf{L}} N) \simeq \Sigma^d k \otimes_R^{\mathsf{L}} N.$$

The isomorphisms above follow. Thus one has

$$\operatorname{Tor}_i^R(k, M) = \begin{cases} 0 & \text{for } i < d, \\ N/\mathfrak{m}N & \text{for } i = d. \end{cases}$$

In particular, one has $\inf \operatorname{Tor}_*^R(k, M) = d$, while $\operatorname{Tor}_*^R(k(\mathfrak{p}), M) = 0$ for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$, since M is $\mathfrak{m}$-torsion.

Now, if in addition the inequality $d > \operatorname{depth} R$ holds, then $\operatorname{flat\,dim}_R M$ is infinite. To see this, apply Matlis duality $\operatorname{Tor}_i^R(k, M)^\vee \cong \operatorname{Ext}_R^i(k, M^\vee)$ and conclude from Proposition 3.4 that $\operatorname{Tor}_i^R(k, M)$ is non-zero for all $i \geqslant d$.

It remains to remark that such R and N exist: Let k be a field, d be a positive integer, and set

$$R := k[[x_1, \ldots, x_{d+1}]]/(x_1^2, x_1x_2, \ldots, x_1x_{d+1}).$$

This R is a local ring of dimension d and depth 0. The R-module $N = R/(x_1)$ is Cohen–Macaulay of dimension d.

5. Injective dimension

Let M be an R-complex, by [**6**, Theorem 1.1] one has

$$\operatorname{inj\,dim}_R M = \sup\{i \in \mathbb{Z} \mid \operatorname{Ext}_R^i(k(\mathfrak{p}), M) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\}. \tag{5.1}$$

In view of (4.2) this is a perfect parallel to the formula for flat dimension (4.1).

The equality $\operatorname{flat\,dim}_R M = \sup_{\mathfrak{p}\in\operatorname{Spec} R}\{\operatorname{flat\,dim}_{R_\mathfrak{p}} M_\mathfrak{p}\}$ is immediate from (4.1) and (4.2). The corresponding equality for the injective dimension only holds true under extra conditions, and the whole picture is altogether more complicated. If M satisfies $\inf \mathrm{H}^*(M) > -\infty$, then [**2**, Proposition 5.3.I] yields

$$\begin{aligned}\operatorname{inj\,dim}_R M &= \sup\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \neq 0 \text{ for some } \mathfrak{p} \in \operatorname{Spec} R\}\\ &= \sup_{\mathfrak{p}\in\operatorname{Spec} R}\{\operatorname{inj\,dim}_{R_\mathfrak{p}} M_\mathfrak{p}\}.\end{aligned} \tag{5.2}$$

Without the boundedness condition on $\mathrm{H}(M)$ the injective dimension may increase under localization; an example is provided in 6.3.

The next statement, which still requires homological boundedness, is folklore but not readily available in the literature.

Theorem 5.1. *Let R be a commutative noetherian ring and M an R-complex with $\inf \mathrm{H}^*(M) > -\infty$. If there exists an integer $n \geqslant \sup \mathrm{H}^*(M)$ such that*

$$\operatorname{Ext}^{n+1}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) = 0 \text{ holds for every prime ideal } \mathfrak{p} \text{ in } R,$$

then the injective dimension of M is at most n.

Proof. Let I be a minimal semi-injective resolution of M; as $\inf \mathrm{H}^*(M) > -\infty$ holds one has $I^n = 0$ for $n \ll 0$. For every integer i one has $I^i = \coprod_{\mathfrak{p}\in\operatorname{Spec} R} E(R/\mathfrak{p})^{(\mu_i(\mathfrak{p}))}$, and to prove that $\operatorname{inj\,dim}_R M$ is at most n it is sufficient to show that the index set $\mu_{n+1}(\mathfrak{p})$ is empty for every prime $\mathfrak{p}$. Fix $\mathfrak{p}$. Since $I_\mathfrak{p}$ is a complex of injectives with $(I_\mathfrak{p})^n = 0$ for $n \ll 0$, it is a minimal semi-injective resolution of $M_\mathfrak{p}$, so one has

$$\begin{aligned}0 &= \operatorname{Ext}^{n+1}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})\\ &= \mathrm{H}^{n+1}\operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), I_\mathfrak{p})\\ &= \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), (I_\mathfrak{p})^{n+1})\\ &= \operatorname{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(k(\mathfrak{p}))^{(\mu_{n+1}(\mathfrak{p}))}).\end{aligned}$$

It follows that $\mu_{n+1}(\mathfrak{p})$ is empty. □

The next result corresponds to (4.3). It removes a restriction on the boundedness of M in Yassemi's version [**17**, Theorem 2.10] of Chouinard's formula [**5**, Corollary 3].

Proposition 5.2. *For every R-complex M of finite injective dimension one has*

$$\operatorname{inj\,dim}_R M = \sup_{\mathfrak{p}\in\operatorname{Spec} R}\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p}\}.$$

Proof. Without loss of generality, we can assume that M is semi-injective with $M^i = 0$ for all $i > d := \operatorname{inj\,dim}_R M$. For every $u \leqslant d$ there is an exact sequence of semi-injective complexes

$$0 \longrightarrow M^{\geqslant u} \longrightarrow M \longrightarrow M^{\leqslant u-1} \longrightarrow 0 \tag{1}$$

with $\operatorname{inj\,dim}_R M^{\leqslant u-1} \leqslant u-1$ and $\operatorname{inj\,dim}_R M^{\geqslant u} = d$. The complex $M^{\geqslant u}$ is bounded, so (2.4) and (5.2) conspire to yield

$$d = \sup_{\mathfrak{p}\in \operatorname{Spec} R} \{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}^{\geqslant u}\}. \tag{2}$$

First we establish the inequality

$$\operatorname{inj\,dim}_R M \geqslant \sup_{\mathfrak{p}\in \operatorname{Spec} R} \{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}\}. \tag{3}$$

Let $\mathfrak{p}$ be a prime. There are inequalities

$$\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geqslant \inf \mathrm{H}_*(M_{\mathfrak{p}}) \geqslant \inf \mathrm{H}_*(M) = -\sup \mathrm{H}^*(M) \geqslant -d,$$

and without loss of generality one can assume that $\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is finite. Consider (1) for $u = -\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ and localize at $\mathfrak{p}$. The associated exact sequence of Tor groups yields $\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}^{\geqslant u}$, so the desired inequality $d \geqslant \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ follows from (2). It remains to prove that the equality holds for some prime.

Consider (1) for $u = d-1$ and choose by (2) a prime $\mathfrak{p}$ with

$$d = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}^{\geqslant d-1}.$$

The second inequality in the next display is (3) applied to the complex $M^{\leqslant d-2}$.

$$d-2 \geqslant \operatorname{inj\,dim}_R M^{\leqslant d-2} \geqslant \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}^{\leqslant d-2}.$$

Eliminating d and $\operatorname{depth} R_{\mathfrak{p}}$ between the two displays one gets the inequality

$$\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}^{\geqslant d-1} \leqslant \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}^{\leqslant d-2} - 2.$$

Finally, one gets $\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}^{\geqslant d-1}$ from the exact sequence of Tor groups associated to (1). $\square$

We now aim for a characterization of complexes of finite injective dimension that does not require homological boundedness. It is based on the following observation, of independent interest.

Lemma 5.3. *Let M be an R-complex and $\mathfrak{m}$ a maximal ideal in R. The localization maps $\rho\colon M \to M_{\mathfrak{m}}$ and $\sigma\colon R \to R_{\mathfrak{m}}$ induce quasi-isomorphisms*

$$\mathsf{RHom}_R(k(\mathfrak{m}),\rho)\colon \mathsf{RHom}_R(k(\mathfrak{m}),M) \xrightarrow{\simeq} \mathsf{RHom}_R(k(\mathfrak{m}),M_{\mathfrak{m}}) \quad \textit{and}$$

$$k(\mathfrak{m}) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(\sigma,M)\colon k(\mathfrak{m}) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(R_{\mathfrak{m}},M) \xrightarrow{\simeq} k(\mathfrak{m}) \otimes^{\mathsf{L}}_R M.$$

Proof. In the derived category of R, consider the distinguished triangle

$$M \xrightarrow{\rho} M_{\mathfrak{m}} \longrightarrow C \longrightarrow \Sigma M.$$

The induced morphism $k(\mathfrak{m}) \otimes^{\mathsf{L}}_R \rho$ is a quasi-isomorphism, so $k(\mathfrak{m}) \otimes^{\mathsf{L}}_R C$ is acyclic. Then $\mathsf{RHom}_R(k(\mathfrak{m}),C)$ is also acyclic, by [**3**, Theorem 4.13], whence the map $\mathsf{RHom}_R(k(\mathfrak{m}),\rho)$ is a quasi-isomorphism, as claimed.

In the same vein, the distinguished triangle $R \to R_{\mathfrak{m}} \to D \to \Sigma R$ induces a distinguished triangle

$$\mathsf{RHom}_R(D, M) \longrightarrow \mathsf{RHom}_R(R_{\mathfrak{m}}, M) \xrightarrow{\mathsf{RHom}_R(\sigma, M)} M \longrightarrow \Sigma\,\mathsf{RHom}_R(D, M).$$

By adjunction $\mathsf{RHom}_R(k(\mathfrak{m}), \mathsf{RHom}_R(\sigma, M))$ is a quasi-isomorphism, so that

$$\mathsf{RHom}_R(k(\mathfrak{m}), \mathsf{RHom}_R(D, M))$$

is acyclic. Thus the complex $k(\mathfrak{m}) \otimes^{\mathsf{L}}_R \mathsf{RHom}_R(D, M)$ is acyclic by [**3**, Theorem 4.13], which justifies the second of the desired quasi-isomorphisms. □

Proposition 5.4. *Let M be an R-complex and $\mathfrak{m}$ a maximal ideal in R. If one has $\operatorname{Ext}^n_R(k(\mathfrak{m}), M) = 0$ for some $n \geqslant \dim R_{\mathfrak{m}} + \sup \mathrm{H}^*(M_{\mathfrak{m}})$, then $\operatorname{Ext}^i_R(k(\mathfrak{m}), M) = 0$ holds for all $i \geqslant n$ and there are equalities*

$$\begin{aligned}\sup \operatorname{Ext}^*_R(k(\mathfrak{m}), M) &= \operatorname{depth} R_{\mathfrak{m}} - \operatorname{width}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \\ &= \operatorname{depth} R_{\mathfrak{m}} - \operatorname{width}_{R_{\mathfrak{m}}} \mathsf{RHom}_R(R_{\mathfrak{m}}, M).\end{aligned}$$

Proof. The first isomorphism below is by Lemma 5.3; the second is by adjunction.

$$\operatorname{Ext}^*_R(k(\mathfrak{m}), M) \cong \operatorname{Ext}^*_R(k(\mathfrak{m}), M_{\mathfrak{m}}) \cong \operatorname{Ext}^*_{R_{\mathfrak{m}}}(k(\mathfrak{m}), M_{\mathfrak{m}}).$$

In view of these isomorphisms and the assumption on n, Proposition 3.2 now yields

$$\sup \operatorname{Ext}^*_R(k(\mathfrak{m}), M) = \operatorname{depth} R_{\mathfrak{m}} - \operatorname{width}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} < n.$$

It remains to observe that the width of $M_{\mathfrak{m}}$ and $\mathsf{RHom}_R(R_{\mathfrak{m}}, M)$ coincide, by the second quasi-isomorphism in Lemma 5.3. □

Corollary 5.5. *Let R be an artinian ring and M an R-complex. If there exists an integer $n \geqslant \sup \mathrm{H}^*(M)$ such that*

$$\operatorname{Ext}^n_R(k(\mathfrak{p}), M) = 0 \text{ holds for every prime ideal } \mathfrak{p} \text{ in } R,$$

then the injective dimension of M is less than n.

Proof. Every prime ideal $\mathfrak{p}$ in R is maximal and there are inequalities

$$n \geqslant \sup \mathrm{H}^*(M) \geqslant \sup \mathrm{H}^*(M_{\mathfrak{p}}) = \dim R_{\mathfrak{p}} + \sup \mathrm{H}^*(M_{\mathfrak{p}}).$$

Thus the claim follows from (5.1) and Proposition 5.4. □

Remark 5.6. In the sequel we require the invariant

$$\operatorname{splf} R = \sup\{\operatorname{proj\,dim}_R F \mid F \text{ is a flat } R\text{-module}\}.$$

Every flat R-module is projective if and only if R is artinian, so $\operatorname{splf} R > 0$ holds if $\dim R > 0$, and from work of Jensen [**11**, Proposition 6] and Raynaud and Gruson [**15**, thm. II.3.2.6] one gets the upper bound $\operatorname{splf} R \leqslant \dim R$. A result of Gruson and

Jensen [**10**, Theorem 7.10] yields another bound on the invariant $\operatorname{splf} R$: If R has cardinality at most $\aleph_m$ for some natural number m, then one has $\operatorname{splf} R \leqslant m+1$. Thus for countable rings, and for 1-dimensional rings, the bound in the theorem below is $n \geqslant \dim R + \sup \mathrm{H}^*(M)$.

Theorem 5.7. *Let R be a commutative noetherian ring with $\dim R \geqslant 1$ and M an R-complex. If there exists an integer $n \geqslant \dim R - 1 + \operatorname{splf} R + \sup \mathrm{H}^*(M)$ such that*

$$\operatorname{Ext}_R^n(k(\mathfrak{p}), M) = 0 \text{ holds for every prime ideal } \mathfrak{p} \text{ in } R,$$

then the injective dimension of M is less than n.

Proof. Fix a prime ideal $\mathfrak{p}$ and consider the $R_\mathfrak{p}$-complex $N := \mathsf{RHom}_R(R_\mathfrak{p}, M)$. One has

$$\mathrm{H}^i(N) = \operatorname{Ext}_R^i(R_\mathfrak{p}, M) = 0 \quad \text{for } i > \operatorname{proj\,dim}_R R_\mathfrak{p} + \sup \mathrm{H}^*(M),$$

and standard adjunction yields

$$\operatorname{Ext}_R^*(k(\mathfrak{p}), M) \cong \operatorname{Ext}_{R_\mathfrak{p}}^*(k(\mathfrak{p}), N).$$

Since $R_\mathfrak{p}$ is a flat R-module, $\operatorname{proj\,dim}_R R_\mathfrak{p}$ is at most $\operatorname{splf} R$. Given this, (2.7) yields

$$\mathrm{H}^i_{\mathfrak{p}R_\mathfrak{p}}(N) = 0 \quad \text{for } i > \dim R_\mathfrak{p} + \operatorname{splf} R + \sup \mathrm{H}^*(M).$$

Thus, if $n \geqslant \dim R_\mathfrak{p} + \operatorname{splf} R + \sup \mathrm{H}^*(M)$ holds, then Proposition 3.2 yields

$$\sup \operatorname{Ext}_R^*(k(\mathfrak{p}), M) = \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} N < n.$$

The same equality also holds when $n < \dim R_\mathfrak{p} + \operatorname{splf} R + \sup \mathrm{H}^*(M)$, for then the assumption on n forces $\dim R_\mathfrak{p} = \dim R$, so $\mathfrak{p}$ is a maximal ideal and so Proposition 5.4 applies. Now the desired conclusion follows from (5.1). □

Remark 5.8. As noted in Remark 5.6, one has $\operatorname{splf} R \leq \dim R$. Thus Theorem 1.3 is a consequence of Corollary 5.5 and Theorem 5.7.

Confer the following result and Proposition 5.2.

Corollary 5.9. *For every R-complex of finite injective dimension one has*

$$\operatorname{inj\,dim}_R M = \sup_{\mathfrak{p} \in \operatorname{Spec} R} \{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)\}.$$

Proof. Given Lemma 5.3, the desired equality is restatement of Proposition 5.2 in case R is artinian. If R is not artinian, then one has $\dim R \geqslant 1$ and the equality is immediate from (5.1) and the last display in the proof of Theorem 5.7. □

Remark 5.10. Let R be a complete local domain of positive dimension. One has $\operatorname{width}_{R_{(0)}} R_{(0)} = 0$, but the complex $\mathsf{RHom}_R(R_{(0)}, R)$ is acyclic, see [**3**, Example 4.20], so $\operatorname{width}_{R_{(0)}} \mathsf{RHom}_R(R_{(0)}, R) = \infty$. We do not know how the numbers $\operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p}$ and $\operatorname{width}_{R_\mathfrak{p}} \mathsf{RHom}_R(R_\mathfrak{p}, M)$ from Proposition 5.2 and Corollary 5.9 compare in general.

6. Examples

In this section we describe examples to illustrate that, for complexes whose cohomology is not bounded below, finiteness of injective dimension does not behave well under localization or passage to torsion subcomplexes. This builds on [4].

Remark 6.1. Let R be a ring. A complex I of injective R-modules is semi-injective if and only if for each (equivalently, for some) integer n the quotient complex $I^{\leqslant n}$ is semi-injective. This is immediate from the exact sequence of complexes

$$0 \longrightarrow I^{>n} \longrightarrow I \longrightarrow I^{\leqslant n} \longrightarrow 0$$

since $I^{>n}$ is always semi-injective.

Remark 6.2. Let $(R, \mathfrak{m}, k)$ be a local ring and E be the injective envelope of k. One has $(E^{\mathbb{N}})_{\mathfrak{p}} \neq 0$ for every prime ideal in R. Indeed, the claim is trivial if R is artinian. If R is not artinian, then one can choose an element $e = (e_n)_{n\in\mathbb{N}}$ in $E^{\mathbb{N}}$ with $\mathfrak{m}^n \subseteq (0 : e_n) \not\supseteq \mathfrak{m}^{n-1}$. The map $R \to E^{\mathbb{N}}$ given by $1 \mapsto e$ is injective by Krull's intersection theorem, so R is a submodule of $E^{\mathbb{N}}$.

Example 6.3. Let $(R, \mathfrak{m}, k)$ be a local ring such that $(0 : x) = (x)$ holds for some $x \in \mathfrak{m}$; set $S = R/(x)$. The complex

$$\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow 0$$

concentrated in non-negative degrees has homology S in degree 0 and zero elsewhere. Dualizing with respect to E, the injective envelope of k over R, yields a complex

$$I := 0 \longrightarrow E \xrightarrow{x} E \xrightarrow{x} E \xrightarrow{x} \cdots$$

of injective R-modules. It is the minimal injective resolution of $E_S := \operatorname{Hom}_R(S, E)$ over R. By periodicity, every injective syzygy of E_S is E_S. Consider the complex $J = \prod_{n>0} \Sigma^n I$, which is a semi-injective resolution of $\prod_{n>0} \Sigma^n E_S$.

Claim. *The complex* $M := J^{\leqslant 0}$ *has injective dimension* 0, *whereas for each prime ideal* $\mathfrak{p} \neq \mathfrak{m}$, *one has that* $\operatorname{inj\,dim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ *is infinite.*

Indeed, since J is semi-injective, so is M, by Remark 6.1. Since the cohomology module $\mathrm{H}^0(M) \cong (E_S)^{\mathbb{N}}$ is non-zero, it follows that $\operatorname{inj\,dim}_R M = 0$ holds.

Fix a prime ideal $\mathfrak{p} \neq \mathfrak{m}$. For $i < 0$ one has $\mathrm{H}^i(M) = \mathrm{H}^i(J) = E_S$ and, therefore, $\mathrm{H}^i(M_{\mathfrak{p}}) = \mathrm{H}^i(M)_{\mathfrak{p}} = 0$. This justifies the first quasi-isomorphism in the computation below; the rest are standard.

$$\begin{aligned}\operatorname{RHom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) &\simeq \operatorname{RHom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), ((E_S)^{\mathbb{N}})_{\mathfrak{p}}) \\ &\simeq \operatorname{RHom}_R(R/\mathfrak{p}, (E_S)^{\mathbb{N}})_{\mathfrak{p}}\end{aligned}$$

$$
\begin{aligned}
&\simeq (\mathsf{R}\mathrm{Hom}_R(R/\mathfrak{p}, E_S)^{\mathbb{N}})_{\mathfrak{p}} \\
&\simeq (\mathrm{Hom}_R(R/\mathfrak{p}, I)^{\mathbb{N}})_{\mathfrak{p}} \\
&\cong \left(\left(\coprod_{i\geqslant 0} \Sigma^{-i} \mathrm{Hom}_R(R/\mathfrak{p}, E)\right)^{\mathbb{N}}\right)_{\mathfrak{p}} \\
&\cong \coprod_{i\geqslant 0} \Sigma^{-i} (\mathrm{Hom}_R(R/\mathfrak{p}, E)^{\mathbb{N}})_{\mathfrak{p}}.
\end{aligned}
$$

The first isomorphism holds because $x = 0$ in $R/\mathfrak{p}$, for $x^2 = 0$, and hence the induced differential on the complex $\mathrm{Hom}_R(R/\mathfrak{p}, I)$ is zero.

The module $E_{R/\mathfrak{p}} := \mathrm{Hom}_R(R/\mathfrak{p}, E)$ is the injective envelope of k over the domain $R/\mathfrak{p}$. The computation above shows that for every $i \geqslant 0$ there is an isomorphism as $R/\mathfrak{p}$-modules

$$\mathrm{Ext}^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) \cong ((E_{R/\mathfrak{p}})^{\mathbb{N}})_{(0)}.$$

Thus Remark 6.2 yields $\mathrm{Ext}^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$ for all $i \geqslant 0$; hence $\mathrm{inj\,dim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is infinite, as claimed.

Example 6.4. Let k be a field and $R := k[|x, y|]/(x^2)$. Since $(0 : x) = (x)$, we are in the situation considered in the previous example. Let M be the complex of injectives with injective dimension zero constructed there. We claim that the injective dimension of the complexes M_y and $\Gamma_{(y)}M$ are infinite.

Indeed, observe that $M_y \cong M_{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime ideal (x) of R, so $\mathrm{inj\,dim}_R M_y$ is infinite, by the claim in the previous example. Since $C(y) \otimes_R M$ is quasi-isomorphic to $\Gamma_{(y)}M$ and there is an exact sequence

$$0 \longrightarrow \Sigma^{-1} M_y \longrightarrow C(y) \otimes_R M \longrightarrow M \longrightarrow 0,$$

it follows that the injective dimension of $\Gamma_{(y)}M$ is infinite as well.

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