Auxetics Abounding

Ciprian S. Borcea[1] and Ileana Streinu[2]

***Abstract*— Auxetic behavior refers to lateral widening upon stretching. Although a structural origin for this rather counter-intuitive type of deformation was often suggested, a theoretical understanding of the role of geometry in auxetic behavior has been a challenge for a long time. However, for structures modeled as periodic bar-and-joint frameworks, including atom-and-bond frameworks in crystalline materials, there is a complete geometric solution which opens endless possibilities for new auxetic designs. We construct a large family of three-dimensional auxetic periodic mechanisms and discuss the ideas involved in their design.**

I. INTRODUCTION

Most materials become laterally thinner when stretched. Auxetic behavior is the opposite and more intriguing type of response, which involves lateral widening. One of the most frequently used illustrations for auxetic behavior is the so-called reentrant honeycomb shown in Fig. 1. When stretched horizontally, the bar-and-joint structure widens vertically as well. A ninety degree rotation around a vertical axis allows the recombination of the planar pattern with itself and leads to one of the earliest examples of a three-dimensional framework with auxetic deformations [1].

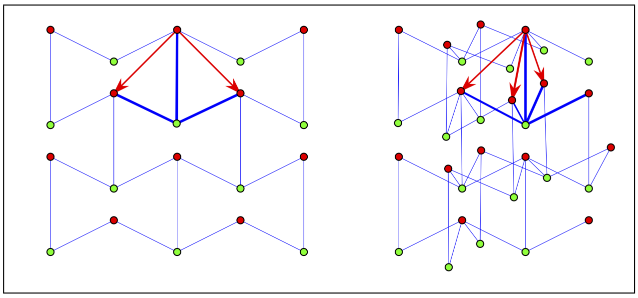

Fig. 1. A reentrant honeycomb and a related three-dimensional extension. The red arrows indicate a choice of two, respectively three independent generators for the corresponding periodicity lattice.

Intuitively, this may seem satisfactory, but a more persistent geometric inquiry would wonder about other possible periodic deformations, the number of parameters (or degrees of freedom) for local deformations and what is distinctive about auxetic deformations. One must be precise about the periodicity which is supposed to be preserved, since a relaxation in periodicity may allow new deformations and produce a larger local deformation space.

The problem of uncovering and expressing the role of geometry in auxetic behavior can be addressed successfully when auxetic deformations are formulated in strictly geometric language. *Geometric auxetics* [2] was introduced with the deliberate purpose of developing a mathematical theory of periodic auxetics, with no reliance on physical parameters, assumptions or heuristics. This approach uses the deformation theory of periodic bar-and-joint frameworks, as established in previous work of the authors [3], [4], [5], [6]. Since technical details can be found in the cited references, in this paper we would rather be less formal and place the emphasis on the chain of ideas which allow the geometrical approach to reach a complete structural understanding of periodic frameworks with auxetic deformations. We propose a rational classification scheme and introduce a methodically generated family of examples with one degree of freedom.

In Materials Science, auxetic studies extend over several areas and involve a multitude of techniques. This diverse landscape is surveyed in a number of recent reviews [7], [8], [9], [10], [11], [12], [13]. Periodic framework structures are sometimes referred to as lattices. The repertory of auxetic designs found in this category is rather confined and unsystematic. The following quotations are apt to show that this liability has been persistent.

The 2012 survey [14], p.4792, considers that "it has been a challenge to design 3D auxetic micro-/nano-structured materials". In [15], the authors note that "It seems timely to search in a systematic way for auxetic structures and their building principles". In [16], it is observed that "So far no systematic approach for generating auxetic cellular materials has been reported" and the recent review [17] estimates that "the rational design of metamaterials with a target property or functionality remains fiendishly difficult, and many designs so far have relied on luck and intuition".

Nevertheless, recent advances in digital manufacturing have augmented the interest in methodically designed auxetic metamaterials [18], [19], [20]. As intimated in the current title, our geometric results provide a systematic and open-ended resource of blueprints. The specific designs presented in this paper have one degree of freedom.

*This work was supported by the National Science Foundation (awards no. 1319389 and 1704285 to C.S.B., awards no. 1319366 and 1703765 to I.S.) and the National Institutes of Health (award 1R01GM109456 to C.S.B. and I.S.).

[1]Ciprian S. Borcea is with the Department of Mathematics, Rider University, 2083 Lawrenceville Road, Lawrenceville, NJ 08648, USA borcea@rider.edu

[2]Ileana Streinu is with the Computer Science Department, Smith College, 10 Elm Street, Northampton, MA 01063, USA istreinu@smith.edu

II. WHAT IS AUXETIC?

The expression *lateral widening upon stretching* leaves much to the imagination. The approach based on elasticity theory considers *negative Poisson's ratios* the defining characteristic of auxeticity [7], [21], [22]. Geometrically, for periodic bar-and-joint frameworks, an alternative approach, based on auxetic paths, can be adopted. For a description of its rationale, it will be more convenient to start with the reverse formulation of *lateral shrinking under compression*. A periodic structure undergoing a deformation of this nature allows a comparison of any two moments, through the corresponding configurations of the periodicity lattice. This gives a linear operator and reversed auxeticity can be defined by the property that all these linear operators, from any moment to some subsequent moment, are *contractions* [2].

Let us recall that a linear operator T is a contraction when its *norm* is less or equal with one, that is:

$$||T|| = \sup_{||x|| \leq 1} ||Tx|| \leq 1. \tag{1}$$

This says that a contraction takes the unit ball to a subset of itself. Since the image of the unit ball by an invertible linear operator is some ellipsoid centered at the origin, the visual representation of a contraction operator (in dimension two) is aptly conveyed in Fig. 2.

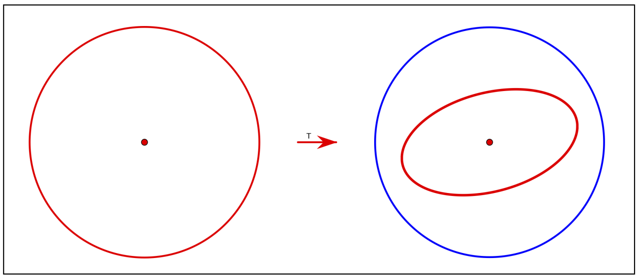

Fig. 2. A linear contraction operator $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ takes the unit disc to a subset of itself.

The evolution of the periodicity lattice is the relevant gauge for the transformation of the framework structure during deformation and a *lateral shrinking under compression* is adequately expressed through the *contraction operators* considered above. Furthermore, mathematically, the evolution of the periodicity lattice can be viewed as a curve of symmetric matrices: one chooses a set of independent generators for the lattice and follows the trajectory of their Gram matrix. This representation leads to an equivalent, but oftentimes more useful criterion for auxeticity: *a one-parameter deformation of a periodic framework is auxetic when the associated curve of Gram matrices has all its velocity vectors in the positive semidefinite cone* [2].

A significant consequence of this formulation is that semidefinite programming can be harnessed for algorithmic solutions of auxetic queries. In particular, the existence of auxetic infinitesimal deformations for a given periodic framework can be efficiently decided.

III. STRUCTURE REQUIRED FOR AUXETICITY

The resolution of the problem concerning the geometric underpinnings of auxetic behavior follows from the very same criterion of auxeticity. It involves, however, a new element: the use of *lattice coordinates* [5]. In crystallography, such coordinates are called *fractional coordinates*.

Let us inquire, before disclosing the solution, what form could be expected in a pertinent answer. The essential features of a periodic framework reside in its *periodicity group* represented by a *lattice of translations* and a system of vertices and edges, represented as joints and bars. This system needs description only *up to periodicity*, since the given *periodicity lattice* allows the completion of the infinite framework from any finite fragment which contains representatives for all vertices and edges. A planar periodic framework of Kagome type, for instance, is completely retrievable from a fragment with just five vertices and six edges, as shown in Fig. 3.

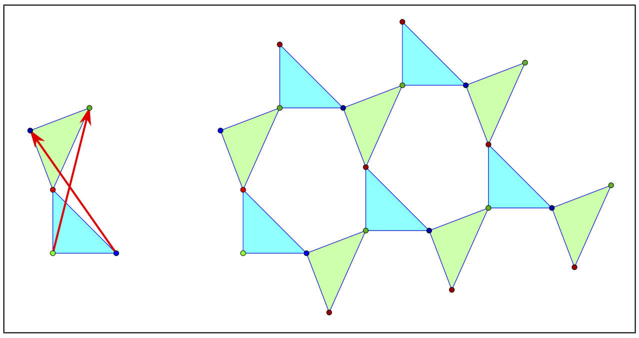

Fig. 3. A planar periodic framework of Kagome type. The essential information is shown on the left, with the two generators of the periodicity lattice marked as red arrows.

All vertices which are equivalent by periodicity to a given vertex form a *vertex orbit*. A connected periodic framework can also be retrieved from a complete set of representatives of *edge vectors*, when we know which vertex orbits are connected by each edge vector. The full framework is simply the result of repeated articulation of edge vectors, respecting the orbit markings for their endpoints.

This brings us to the expectation that a geometric characterization of auxeticity could be expressed as a property of a complete set of edge vector representatives for the framework. This property must be found in what is distinctive in an auxetic deformation and we emphasized above that this is the fact that the associated curve of Gram matrices has all its velocity vectors in the *positive semidefinite cone*. Avoiding 'borderline situations', we may assume strictly auxetic deformations, when all velocity vectors are actually in the *positive definite cone*, that is, all velocity vectors are symmetric matrices with strictly positive eigenvalues.

Suppose a choice of independent generators for the periodicity lattice has been made and their Gram matrix $\omega(t)$ varies with time $t \in (-\epsilon, \epsilon)$ near the initial position $\omega = \omega(0)$. Lattice coordinates have a revealing role because the metric in these coordinates implicates directly the Gram matrices $\omega(t)$. At moment t, the squared Euclidean length $||e||^2$ of a vector e is given by the formula

$$||e||^2 = \langle \omega(t)e, e \rangle \quad (2)$$

Thus, if $e_{ij}(t)$ denotes some edge vector (between vertex orbits labeled by i and j), the fact that it maintains its Euclidean length during the deformation gives, by derivation at $t = 0$, the infinitesimal relation

$$\langle \dot{\omega} e_{ij}, e_{ij} \rangle + 2 \langle \omega e_{ij}, \dot{e}_{ij} \rangle = 0 \quad (3)$$

where $\dot{\omega} = \frac{d\omega}{dt}(0)$ and $\dot{e}_{ij} = \frac{de_{ij}}{dt}(0)$.

Now, for a strictly auxetic deformation, the velocity vector $\dot{\omega}$ must be a *positive definite symmetric matrix* and this means that $e_{ij} = e_{ij}(0)$ belongs to the **ellipsoid**

$$\langle \dot{\omega} x, x \rangle + 2 \langle \omega x, \dot{e}_{ij} \rangle = 0 \quad (4)$$

When we contemplate the system of equations of the form (3) for a complete set of edge vectors and notice that $\dot{e}_{ij}$ is the same for all edges from the vertex orbit i to the vertex orbit j, we see the following geometric manifestation of strict infinitesimal auxeticity [23].

Theorem. Let $e_{ij}, \ldots$ denote a complete set of edge vector representatives for a given periodic framework. Then, the framework has a strict infinitesimal auxetic deformation *if and only if* there is a family of homothetic ellipsoids such that:

(a) if the vertex orbits labeled i and j are connected by one or more edge orbits and all their edge vector representatives are shown as emanating from one chosen vertex in orbit i, this vertex and the endpoints of the edge vectors belong to an ellipsoid of the family, labeled ij;

(b) for any cycle of labels $i_1 i_2, i_2 i_3, \ldots, i_n i_1$, the sum of the vectors from the chosen vertex in orbit i_k to the center of the ellipsoid $i_k i_{k+1}$ $(n + 1 \equiv 1)$ vanishes.

Remarks. A homothety is the composition of a dilation with a translation. The ellipsoids ij in the theorem are homothetic because equations (4) have the same quadratic part. The 'cycle condition' in (b) follows from formulae for centers and the obvious relation $\dot{e}_{i_1 i_2} + \ldots + \dot{e}_{i_n i_1} = 0$. The center of the ellipsoid ij in (4) is at

$$c_{ij} = -\dot{\omega}^{-1} \omega(\dot{e}_{ij}) \quad (5)$$

Note also that, since Cartesian coordinates and lattice coordinates are related (at a given moment) by a linear transformation, the theorem reads the same in either system. Fig. 4 shows the three homothetic ellipses for a Kagome type framework. Auxetic behavior for this framework pattern was noticed and investigated in several earlier studies [6], [24], [25], [26].

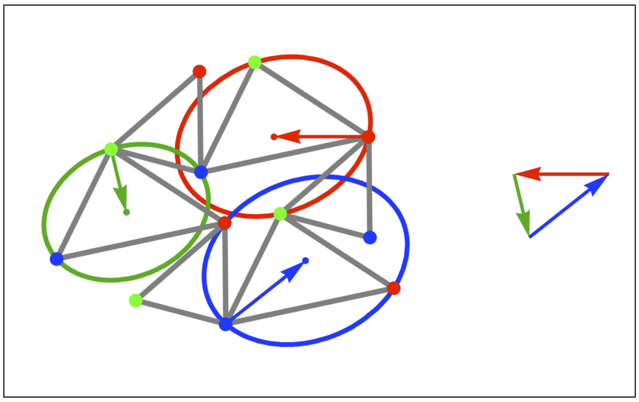

Fig. 4. A Kagome framework has one degree of freedom. The geometric manifestation of auxeticity is the existence of three homothetic ellipses satisfying a cycle condition.

IV. AUXETIC DESIGN

According to the structure theorem presented above, we can recognize strictly auxetic capabilities in a given periodic framework by identifying a family of homothetic ellipsoids adapted in a particular fashion to the bundles of edge vector representatives for pairs of vertex orbits. Auxetic design proceeds by arranging first a family of homothetic ellipsoids and *then* fitting into them some bundles of edge vectors which will assemble into a periodic framework.

A. Classification criteria

At this point, a few mathematical notions will be useful for organizing the design enterprise. In the abstract, any periodic framework has an underlying graph $G = (V, E)$, with vertex set V and edge set E. At the abstract level, periodicity is expressed by a group of automorphisms Γ which is isomorphic with the periodicity lattice of the framework and operates on the graph the way periodicity translations operate on the framework. This language allows the use of the quotient operation G/Γ, which produces the *quotient multigraph*, whose vertices V/Γ correspond with the vertex orbits of the framework and whose edges E/Γ correspond with the edge orbits of the framework. There might be loops and multiple edges in the quotient graph because periodic frameworks may have several edges between pairs of orbits.

Periodic frameworks are always assumed to have a *finite* quotient graph G/Γ, with $n = |V/\Gamma|$ vertex orbits and $m = |E/\Gamma|$ edge orbits under periodicity. For example, the planar framework in Figure 1 has $n = 2$ and $m = 3$, while the spatial version has $n = 2$ and $m = 5$. Kagome type frameworks have $n = 3$, $m = 6$ and the quotient graph may be represented as a triangle with doubled edges.

Since all period vectors must increase in length in a strictly auxetic deformation, frameworks with such capabilities cannot have loops in the quotient graph. Thus, when multiple edges are replaced by a single edge, one obtains a simple graph called the *reduced quotient graph*. It may be observed now that the quotient graph and its reduced version provide the natural labeling for the family of homothetic ellipsoids involved in our theorem: there is an ellipsoid for every edge of the reduced quotient graph and vanishing cycle conditions must hold on all cycles.

With awareness of the fact that strict auxeticity is preserved under linear transformations [27] and the possibility of choosing a linear transformation which takes a given family of homothetic ellipsoids to a family of spheres, the content of the structure theorem can be condensed in a single *geometric diagram* consisting of a simple graph presented with spheres drawn over each edge, with diameter given by that edge and circumscribing the corresponding bundle of framework edge vectors representatives emanating from one endpoint [23].

This background review explains our reasons for introducing a *typology of periodic auxetics based on reduced quotient graphs* and *degrees of freedom*.

It should be observed that too many degrees of freedom trivialize the question about auxetic capabilities since, in dimension two, most periodic frameworks with at least three degrees of freedom and, in dimension three, most periodic frameworks with at least six degrees of freedom, would have plenty of auxetic deformations [2]. The thresholds three and six are the dimensions of the space of 2×2, respectively 3×3 symmetric matrices. Thus, the case of a single degree of freedom is of highest interest.

We shall presently illustrate our auxetic design methodology by constructing a family of three-dimensional examples for the most basic case of a reduced quotient graph represented by a segment and a single degree of freedom. We assume independent edge constraints and this implies the following general relation between the number f of degrees of freedom, the number n of vertex orbits and the number m of edge orbits [3].

$$f = 3n + 3 - m \tag{6}$$

For $n = 2$ and $f = 1$, this gives $m = 8$.

B. Twenty one auxetic blueprints

Our reduced quotient graph has two vertices connected by an edge, hence the quotient multigraph has two vertices connected by $m = 8$ edges. This means we have to depict a single sphere with a chosen diameter and place eight vectors emanating from one endpoint of the diameter and reaching to eight other points on the sphere. These eight vectors will be a complete set of framework edge vector representatives.

The eight points on the sphere cannot be chosen arbitrarily since the vectors between them must be periods of the resulting framework and actually span the periodicity lattice. Thus, we better *start with eight lattice points* on a sphere and choose some diameter endpoint afterwards.

We consider the cube with vertices at $(\pm 1, \pm 1, \pm 1)$ and the six neighboring cubes over its faces. There are $6 \times 4 = 24$ unshared vertices at equal distance from the origin. Actually, the sphere of radius $\sqrt{11}$, centered at the origin, intersects the integer lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ exactly in these twenty four points. We are going to choose configurations of eight points amongst these twenty four lattice points, subject to the following conditions:

(i) the free vectors given by all pairs of points in the configuration span a rank three lattice (that is, there are three independent vectors among them) and all are *primitive (indivisible)* vectors in the spanned lattice;

(ii) the 8 points impose independent conditions on quadrics, that is, the family of quadrics passing through them is exactly one dimensional (a pencil).

Remarks. For a generic choice of the ninth point on the sphere, the primitivity condition in (i) implies that the associated periodic framework does not have self-intersections, while condition (ii) implies exactly one degree of freedom.

Our setting has an obvious cubic symmetry and we will be looking for configurations which are not equivalent under this symmetry. An algorithmic search leads to *twenty one octet configurations*, up to symmetry. A sample is shown in Fig. 5, left. The other twenty octets are shown in Fig. 6.

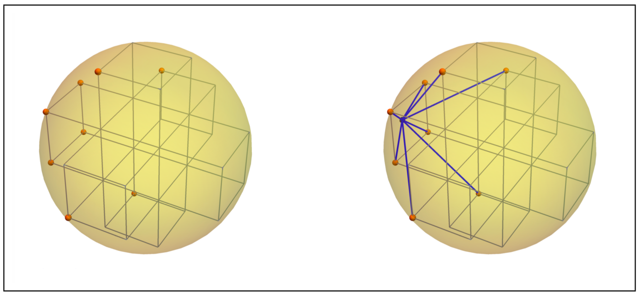

Fig. 5. A configuration of eight lattice points on a sphere next to an auxetic framework design determined by choosing a ninth point on the sphere with its eight connecting edges.

For a given octet configuration, an auxetic design is obtained by choosing a ninth point on the sphere (in a generic position). The eight vectors emanating from this point and ending at the eight points of the configuration define the framework edge vector representatives. As mentioned earlier, the periodicity lattice is the lattice generated by all vectors between pairs of points in the octet. This information is sufficient for generating the corresponding periodic framework.

For intuitive purposes, we illustrate several fragments of the periodic framework resulting from the design provided in Fig. 5. Each edge orbit is shown in a particular color. The periodicity lattice is $2\mathbb{Z}^3$ and a fundamental domain (also called a *unit cell*) is shown as a cube (of edge length 2). The depicted positioning has all framework vertices in the interior of the cubical cells, hence there are two vertices in each cell.

Fig. 7 shows a few periodicity translates of the eight bars emanating from the vertex corresponding to the ninth point on the sphere. This rendition shows only full bars. Segments of other bars entering the depicted cells are not shown.

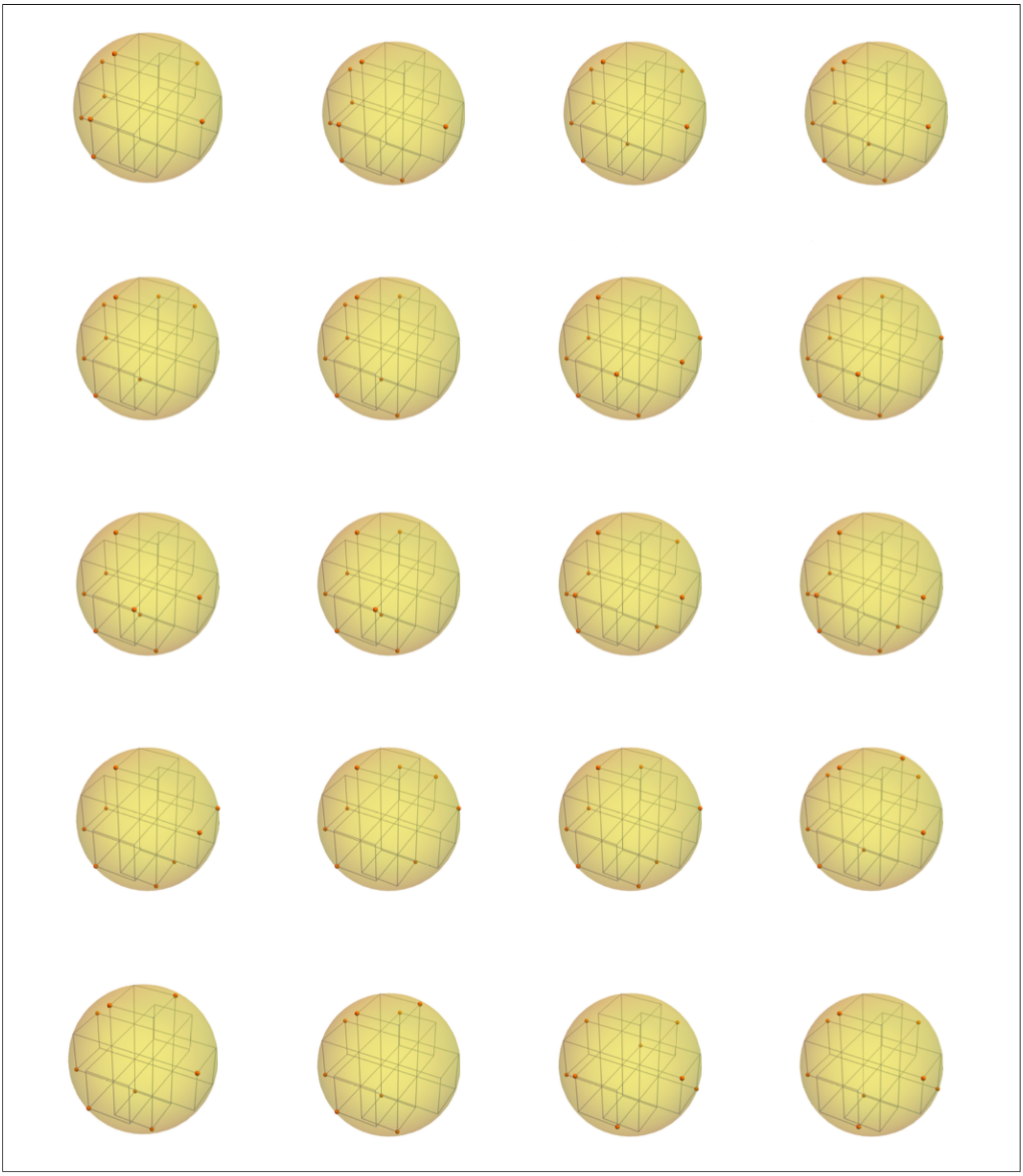

Fig. 6. The other twenty octets. Each octet leads to a distinct type of periodic framework after choosing a ninth point on the sphere. The eight connecting edges from this point to the octet points form a complete set of edge representatives for the periodic framework.

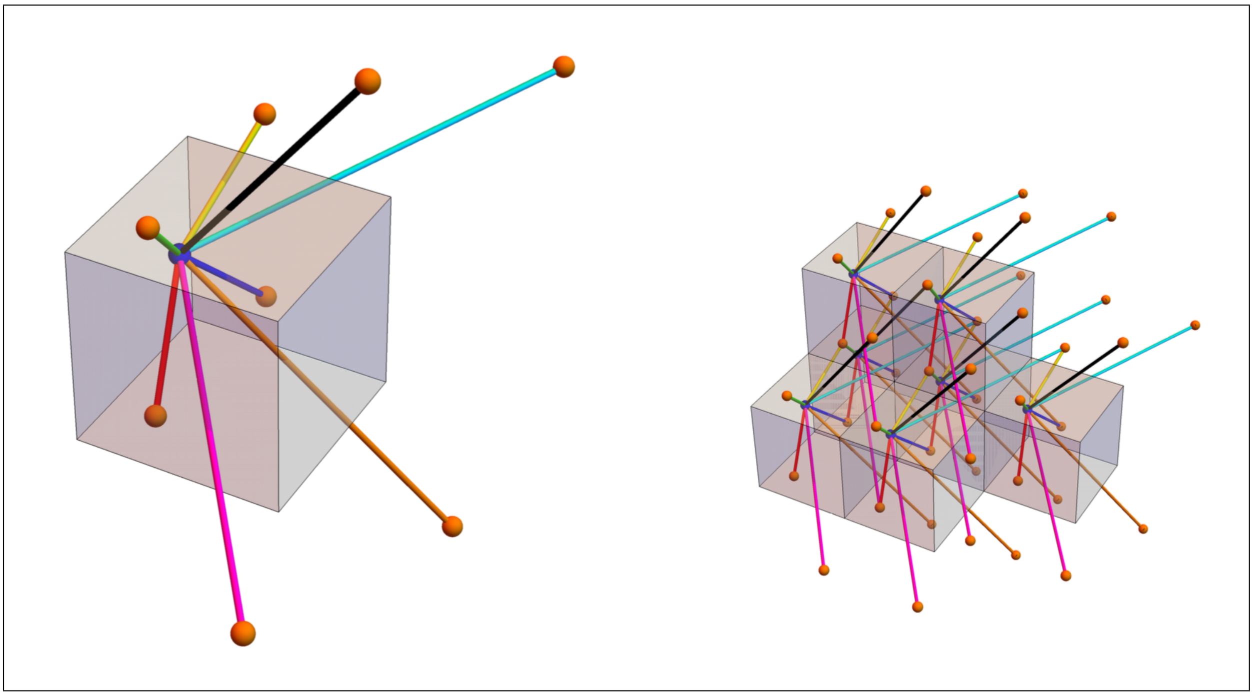

Fig. 7. A positioning of the cubic unit cell and a scaled view of a few periodicity translates of the eight bars for the design specified in Fig. 5.

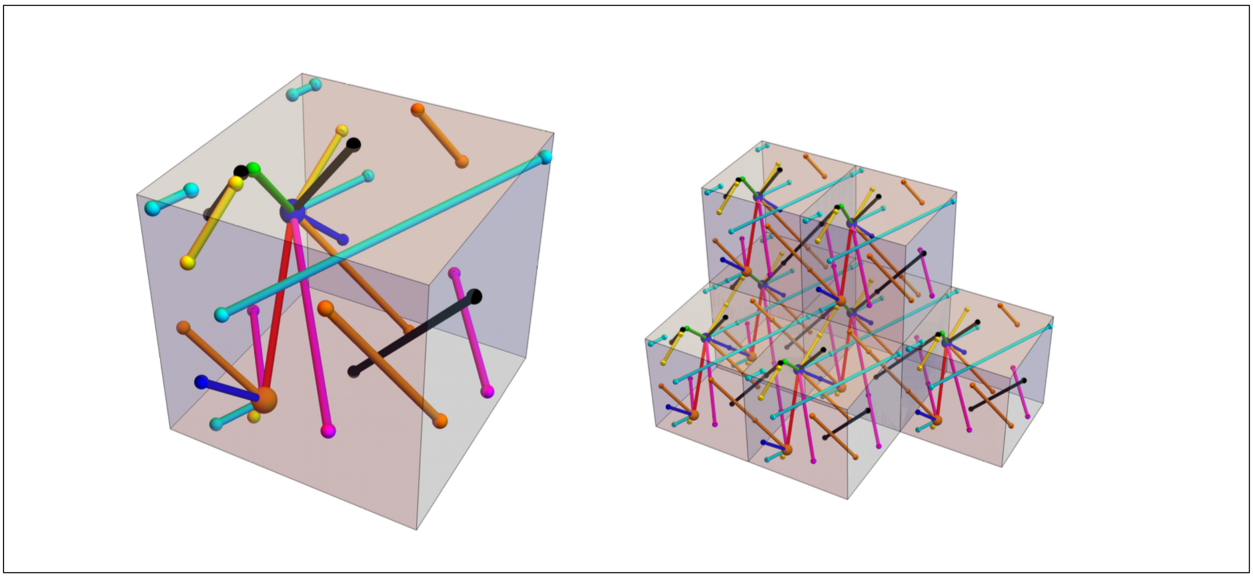

Fig. 8. A unit cell next to a scaled cluster of its replicas. The points where various edges cross the walls of the cubical cells are marked by nodules.

For a representation of what goes through a single unit cell or several contiguous cells, we refer to Fig. 8. This rendition uses nodules (small balls) on edges when cell walls are crossed. Nodules should not be confused with the larger balls representing framework vertices.

V. DISCUSSION

The reader may wonder why we didn't begin our considerations with a lattice unit cube and its circumscribed sphere. It does provide an octet of lattice points on a sphere and our procedure would yield an auxetic design. However, the eight vertices of a cube do not impose independent conditions on quadrics: there is a net of quadrics through the octet, spanned by the three pairs of planes corresponding to parallel faces of the cube. The associated periodic frameworks have *two degrees of freedom*. Particular configurations with this type of design have been considered in [8], [28], [29].

The presence of more than one degree of freedom entails the existence of non-auxetic deformation trajectories and the selection of an auxetic response, amongst mixed possibilities, would require some form of control or guidance.

This aspect enhances the relevance of *one degree of freedom* auxetic designs. Our twenty one blueprints refer to the most basic case of periodic frameworks with two vertex orbits ($n = 2$, $m = 8$) and represent just the tip of the iceberg.

If we drop the restriction about self-intersection (which, in applications, may be avoided by various adaptations, such as slightly curved bars) the design possibilities immediately run into the thousands. Theoretically, the possibilities are endless, since the virtual catalog of eight lattice points on spheres is infinite. And then, there's an infinity of other choices for the reduced quotient graph adopted at the start of the process [23].

VI. CONCLUSIONS

We have shown that a longstanding open problem in auxetics, namely, the geometry required for auxetic behavior, can be solved for structures modeled as periodic bar-and-joint frameworks. The solution is mathematical and relies on a strictly geometric definition of auxetic deformations. Our structural characterization of auxeticity and the rigorous design methodology derived from it reveal unsuspected convexity features and offer a virtually unlimited supply of auxetic blueprints. This should be seen in the context of rapid progress in digital manufacturing, which has made structure the dominant element of metamaterial design.

REFERENCES

[1] R. F. Almgren, An isotropic three dimensional structure with Poisson's ratio = -1, J. Elasticity, 1985, 15, 427-430.

[2] C. S. Borcea and I. Streinu, Geometric auxetics, Proc. Royal Society A, 2015, 471, 20150033, https://doi.org/10. 1098/rspa.2015.0033.

[3] C. S. Borcea and I. Streinu, Periodic frameworks and flexibility, Proc. Royal Soc. A, 2010, 466, 2633-2649.

[4] C. S. Borcea and I. Streinu, I. Minimally rigid periodic graphs, Bulletin LMS, 2011, 43, 1093-1103. doi:10.1112/blms/bdr044

[5] C. S. Borcea and I. Streinu, Frameworks with crystallographic symmetry, Philosophical Transactions of the Royal Society A, 2014, 372, 20120143. http://dx.doi.org/10.1098/rsta.2012.0143

[6] C. S. Borcea and I. Streinu, Liftings and stresses for planar periodic frameworks, Discrete and Computational Geometry, 2015, 53, 747-782.

[7] G. N. Greaves, A. L. Greer, R. Lakes and T. Rouxel, Poisson's ratio and modern materials, Nature Materials, 2011, 10, 823837.

[8] J. C. A. Elipe and A. D. Lantada. Comparative study of auxetic geometries by means of computer-aided design and engineering, Smart Materials and Structures, 2012, 21, 105004.

[9] Y. Prawoto. Seeing auxetic materials from the mechanics point of view: A structural review on the negative Poisson's ratio, Comput. Materials Science, 2012, 58, 140-153.

[10] K. K. Saxena, R. Das and E. P. Calius, Three decades of auxetics research: Materials with negative Poisson's ratio: A review, Advanced Engineering Materials, 2016, 18, 1847-1870.

[11] K. Bertoldi, Harnessing instabilities to design tunable architected cellular materials, Annu. Rev. Mater. Res., 2017, 47, 51-61.

[12] R. S. Lakes, Negative-Poisson's-ratio materials: Auxetic solids. Annu. Rev. Mater. Res., 2017, 47, 63-81.

[13] H. M. A. Kolken and A. A. Zadpoor, Auxetic mechanical metamaterials, RSC Advances, 2017, 7, 5111-5129.

[14] J.-H. Lee, J. P. Singer and E. L. Thomas, Micro-/nano-structured mechanical metamaterials. Advanced Materials, 2012, 24:4782-4810.

[15] H. Mitschke, V. Robins, K. Mecke, and G. E. Schröder-Turk, Finite auxetic deformations of plane tessellations, Proceedings of the Royal Society A, 2013, 469(20120465).

[16] C.Körner and Y. Liebold-Ribeiro, A systematic approach to identify cellular auxetic materials, Smart Materials and Structures, 2015, 24, 025013 (10pp).

[17] K. Bertoldi, V. Vitelli, J. Christensen and M. van Hecke, Flexible mechanical metamaterials, Nature Reviews Materials, 2017, 2, 17066.

[18] J. Christensen, M. Kadic, O. Kraft and M. Wegener, Vibrant times for mechanical metamaterials, MRS Communications, 2015, 5, 453-462.

[19] P. M. Reis, H. M. Jaeger and M. van Hecke, Designer matter: A perspective, Extreme Mechanics Letters, 2015, 5, 25-29.

[20] A. A. Zadpoor, Mechanical meta-materials, Mater. Horiz., 2016, 3, 371-381.

[21] R. Lakes, Foam structures with a negative Poisson's ratio, Science, 1987, 235, 1038-1040.

[22] K. E. Evans, M. A. Nkansah, I. J. Hutchinson, and S. C. Rogers, Molecular network design, Nature, 1991, 353, 124-125.

[23] C. S. Borcea and I. Streinu, Periodic auxetics: Structure and design, Quarterly Journal of Mechanics and Applied Mathematics, https://doi.org/10.1093/qjmam/hbx028 . Preprint at: https://arxiv.org/abs/1708.03960.

[24] J. N. Grima and K. E. Evans, Self expanding molecular networks, Chemical Communications, 2000, 1531-1532.

[25] J.N. Grima, E. Chetcuti, E. Manicaro, D. Attard, M. Camilleri, R. Gatt and K.E. Evans, On the auxetic properties of generic rotating rigid triangles, Proc. Royal Society A, 2012, 468, 810-830.

[26] K. Sun, K. A. Souslov, X. Mao and T. C. Lubensky, Surface phonons, elastic response, and conformal invariance in twisted kagome lattices, Proc. Nat. Acad. Sci. 2012, 109, 12369-12374.

[27] C. S. Borcea and I. Streinu, New principles for auxetic periodic design, SIAM J. Appl. Algebra Geometry, 2017, 1, 442-458. DOI. 10.1137/16M1088259.

[28] T-C. Lim, A 3D auxetic material based on intersecting double arrowheads, Phys. Status Solidi B, 2016, 253, 1252-1260, DOI 10.1002/pssb.201600015.

[29] Q. Gao, L. Wang, Z. Zhou, Z. D. Ma, C. Wang and Y. Wang, Theoretical, numerical and experimental analysis of three-dimensional double-V honeycomb, Materials and Design, 2018, 139, 380-391.