COMPATIBLE SYSTEMS OF GALOIS REPRESENTATIONS ASSOCIATED TO THE EXCEPTIONAL GROUP E_6

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1. Introduction

In [Ser94], Serre raised the question of whether G_2 or E_8 was the motivic Galois group of a motive M over a number field, and one can evidently ask the same question for the other exceptional simple Lie groups. A slightly weaker version of this question asks for a motive M such that the associated p-adic Galois representations have algebraic monodromy group equal to the exceptional group in question. In this form, Serre's question was answered in the affirmative by Yun in [Yun14], who also dealt with the case of the exceptional group E_7. (A stronger version of the question for the group G_2 had previously been answered by Dettweiler and Reiter [DR10].) This left open the cases of E_6 or F_4. In [Pat16], the last author of this paper succeeded in constructing geometric Galois representations for the remaining exceptional groups using arguments inspired by Ramakrishna's lifting theorems [Ram02], at least for a set of primes p of density one (improved to all but finitely many in Theorem 1.2 of [Pat17]). While this answered a weak form of (the E_6 analogue of) Serre's question, the Galois representations constructed in [Pat16, Pat17] did not obviously come from motives M or from compatible systems of Galois representations (although that would certainly be a consequence of the Fontaine–Mazur conjectures [FM95]). The main goal of this paper is to remedy this lacuna for the group E_6.

Theorem 1.1. *Let F/F^+ be a totally imaginary CM field with maximal totally real subfield F^+. Let G denote the simply connected form of E_6, and fix a minuscule representation $G \to \mathrm{GL}_{27}$. Then there exists a strongly compatible system of Galois representations with coefficients in a number field M such that the representations*

$$r_\lambda : G_F \to G(\overline{M}_\lambda) \hookrightarrow \mathrm{GL}_{27}(\overline{M}_\lambda)$$

have images with Zariski closure $G(\overline{M}_\lambda)$ for all primes λ. Moreover, this compatible system is potentially automorphic and motivic in the sense that there is a CM extension H/F such that:

- *There a cuspidal automorphic representation π for GL_{27}/H such that $r_\lambda|_{G_H}$ is the compatible system of Galois representations associated to π.*
- *The compatible system r_λ satisfies the conclusion of the Fontaine–Mazur conjecture: there is a smooth projective variety X/F and integers i and j such that r_λ is a G_F-sub-representation of $H^i(X_{\overline{F}}, \overline{\mathbf{Q}}_l(j))$.*

G.B. was supported in part by NSF postdoctoral fellowship DMS-1503047, F.C. was supported in part by NSF Grant DMS-1701703, M.E. was supported in part by NSF Grant DMS-1601871, K.M.P. was supported in part by NSF Grant DMS-1803623, S.P. was supported in part by NSF Grant DMS-1700759.

The main idea of the paper is to follow the strategy of [Pat16], but to replace the lifting theorems inspired by Ramakrishna with those inspired by the work of Khare and Wintenberger [KW09a, KW09b], exploiting the modularity lifting theorems of [BLGGT14]. In order to do this, one must link G-representations with GL_n representations by choosing some (faithful) representation $r\colon G \to \mathrm{GL}_n$. The methods of [BLGGT14] require that the corresponding Galois representations have *distinct* Hodge–Tate weights. This imposes a strong restriction on the representation r, namely, that its formal character should be multiplicity-free. In particular, the method of this paper only applies to the exceptional groups G_2, E_6, and E_7, in their quasi-minuscule (G_2) or minuscule (E_6 and E_7) representations (we concentrate on E_6 because other methods are available in the other cases). As in [Pat16, Pat17], we require a seed representation $\overline{\rho} : G_F \to G(\overline{\mathbf{F}}_p)$ from which to construct geometric lifts. In [Pat16], suitable representations $\overline{\rho}$ came from composing representations associated to modular forms with the principal SL_2. These representations are not suitable for our purposes, because their composition with the minuscule representation is reducible. Instead, we construct a representation related to the action of the Weyl group of E_6 on the weight space of our representation. The fact that the representation we consider is irreducible in GL_n relies on the assumption that r is minuscule. The reason we succeed in controlling the monodromy groups at *all* primes is a consequence of elementary combinatorial properties of the formal character of E_6 (using ideas of Larsen and Pink [LP92]) together with our ability to exploit independence of p results in compatible systems of Galois representations associated to automorphic forms ([TY07, Shi11, Car12]).

We end the introduction with some remarks on what the methods of this paper cannot do.

Remark 1.2. Galois representations for Q versus imaginary quadratic fields. Our construction gives compatible systems of E_6-representations over any imaginary quadratic field F; these extend to representations of $G_{\mathbf{Q}}$ whose image is Zariski-dense in the L-group $G \rtimes \mathrm{Out}(G)$ of an outer form of E_6. We leave open the question as to whether actual E_6-systems exist over $\mathbf{Q}$, noting that the methods of this paper will not succeed in constructing them. Indeed, our methods require that the corresponding Galois representations have regular weight, and there do not exist any such compatible systems of Galois representations over $\mathbf{Q}$ (see Remark 5.1).

Remark 1.3. Motives versus motives with coefficients. We ultimately construct compatible systems of 27-dimensional Galois representations for a coefficient field L over which we have little control. One can ask the more refined question of whether there exists a motive M with coefficients over $\mathbf{Q}$ of type E_6 (Yun's result [Yun14] answer this question in the affirmative for E_8). One reason that this refinement is interesting is that it would have applications to the to the inverse Galois problem (see Corollary 6.1).

It seems to us that the Galois theoretic methods of either this paper or of [Pat16, Pat17] are unsuited to answering such a question. It is illustrative to consider the simpler case of GL_2. By constructing a geometric Galois representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Q}_p)$, one can hope to prove it is automorphic and hence associated to a modular form f, and thus to construct a corresponding motive M_f ([Sch90]). But it seems very hard to impose conditions on ρ to ensure that the form f has coefficients in $\mathbf{Q}$. For example, in weight 2 (on the modular form side) one would want to put conditions on ρ to ensure that it actually come from an elliptic curve rather than

an abelian variety of GL_2-type. In practice, we actually work in highly regular weight, and there is a certain amount of numerical evidence [Rob17] pointing to the fact that there may not exist any motives M at all with coefficients in $\mathbf{Q}$ and monodromy group GL_2 with Hodge–Tate weights $[0, k-1]$ when $k > 50$.

2. The mod p representation

In this section, we construct the mod p representations that we will lift in the next section using potential automorphy theorems.

Definition 2.1. Let G be the split simply-connected reductive group scheme over $\mathbf{Z}$ of type E_6. Fix a pinned based root datum of G, and let ${}^L G = G \rtimes \mathrm{Out}(G)$ with the non-trivial element τ of $\mathrm{Out}(G) = \mathbf{Z}/2\mathbf{Z}$ acting through the corresponding pinned automorphism. The induced action of τ on W_G is conjugation by the longest element w_0, since τ acts on T, the maximal torus of the pinning, by the opposition involution $-w_0$. Let $W = W_G \rtimes \mathrm{Out}(G)$, and continue to write τ for the non-trivial element of $\mathrm{Out}(G) \subset W$. We will write B for the Borel subgroup of G associated to the based root datum.

Remark 2.2. *There is an isomorphism $W \simeq W_G \times \mathbf{Z}/2\mathbf{Z}$ given by the identity on W_G and sending τ to $(w_0, 1)$.*

We fix, once and for all, a choice of minuscule representation $r_{\min} \colon G \to \mathrm{GL}_{27}$, writing $\Lambda_{\min}$ for the weights of T in $r_{\min}$. Now we explain how to extend $r_{\min}$ to ${}^L G$. Let $\mathcal{G}_{27} = (\mathrm{GL}_{27} \times \mathrm{GL}_1) \rtimes \mathbf{Z}/2\mathbf{Z}$, where the non-trivial element $\jmath \in \mathbf{Z}/2\mathbf{Z}$ acts via $\jmath(g, \mu)\jmath^{-1} = (\mu \cdot {}^t g^{-1}, \mu)$. The representation $g \mapsto r_{\min}(\tau g \tau^{-1})$ is isomorphic to the dual minuscule representation of G, so there exists $A \in \mathrm{GL}_{27}$ such that $r_{\min}(\tau g \tau^{-1}) = A \cdot {}^t r_{\min}(g)^{-1} A^{-1}$. Iterating, we see that $A \cdot {}^t A^{-1}$ commutes with $r_{\min}$, so must be a scalar: $A = {}^t A \cdot \varepsilon$. Clearly $\varepsilon \in \{\pm 1\}$, and since 27 is odd, we must in fact have $\varepsilon = 1$ by considering determinants.

We can now extend $r_{\min}$ to

$$r_{\min} \colon {}^L G \to \mathcal{G}_{27}$$

by $r_{\min}(\tau) = (A, \varepsilon, \jmath) = (A, 1, \jmath)$. The fact that $\varepsilon = 1$ has the following consequence, which we will need later:

Lemma 2.3. Let $\nu \colon \mathcal{G}_{27} \to \mathbf{G}_m$ be the character given by $\nu(g, a, 0) = a$, $\nu(\jmath) = -1$. Let $x \in {}^L G$ be any element with non-trivial projection to $\mathrm{Out}(G)$ (the case of interest will be x such that conjugation by x induces a Chevalley involution of G). Then $\nu \circ r_{\min}(x) = -1$.

Lemma 2.4. Let E^+/F^+ be a Galois extension of totally real fields whose Galois group is identified with a subgroup P of W_G, and let F/F^+ be a quadratic totally imaginary extension. Let $E = F.E^+$. Then the composite

$$\mathrm{Gal}(E/F^+) \xrightarrow{\sim} \mathrm{Gal}(E^+/F^+) \times \mathrm{Gal}(F/F^+) \xrightarrow{\sim} P \times \mathbf{Z}/2\mathbf{Z} \subset W_G \times \mathbf{Z}/2\mathbf{Z} \xrightarrow{\sim} W$$

sends complex conjugation to $(w_0, \tau) \in W = W_G \rtimes \mathrm{Out}(G)$. In this setting we will write ${}^L P$ for the image of $P \times \mathbf{Z}/2\mathbf{Z}$ in W.

Proof. This follows from the definition of the isomorphism $W_G \times \mathbf{Z}/2\mathbf{Z} \xrightarrow{\sim} W$ (Remark 2.2). □

We now construct the mod p Galois representation. Let P be a Sylow 3-subgroup of W_G; we note that $|W_G| = 2^7 \cdot 3^4 \cdot 5$. Let F/F^+ denote a fixed totally imaginary quadratic extension of our fixed totally real field F^+. Our starting point will be to construct P as a Galois group over F^+, with some local restrictions whose significance will become apparent when we apply potential automorphy theorems.

Lemma 2.5. There exists a totally real Galois extension E^+/F^+ together with an equality $\mathrm{Gal}(E^+/F^+) = P$ (we write "=" to denote a fixed isomorphism) and all the primes of F^+ which ramify in E^+ are split in F/F^+.

Proof. The Scholz–Reichardt theorem ([Ser08, Thm 2.1.1]) guarantees the existence of a number field L with $\mathrm{Gal}(L/\mathbf{Q}) = P$, and, because $|P|$ is odd, such an extension will automatically be totally real. It suffices to show that we can construct such an extension ramified only at primes which are totally split in F (equivalently, in the Galois closure of F). This forces the intersection of L with the Galois closure of F to be unramified everywhere over $\mathbf{Q}$ and hence trivial, and thus $E^+ = L.F^+$ will have Galois group P over F and produce the desired extension. The result follows immediately by induction and from the following lemma, which is extremely close to [Ser08, Thm 2.1.3]:

Sublemma 2.6. Let $\widetilde{A} \to A$ be a central extension of a finite group A by $\mathbf{Z}/p\mathbf{Z}$, and assume that the exponent of $\widetilde{A}$ divides p^n. Let $L/\mathbf{Q}$ be Galois with Galois group A, and assume that every prime l which ramifies in L has the following properties:

(1) $l \equiv 1 \mod p^n$.
(2) The inertia group(s) of l in A coincides with the decomposition group(s) at l.
(3) l splits completely in F.

Then there exists an extension $\widetilde{L}/L$ which is Galois over $\mathbf{Q}$ with Galois group $\widetilde{A}$ and such that the primes l which ramify in $\widetilde{L}$ satisfy the same conditions as above.

The only difference between this statement and Theorem 2.1.3 of [Ser08] is the extra requirement that the primes l splits completely in F. There are two inductive steps in the proof of Theorem 2.1.3 of [Ser08], and we indicate the required argument to show that the new auxiliary prime q may be chosen to split completely in F.

Suppose first that $\widetilde{A} = A \times \mathbf{Z}/p\mathbf{Z}$ is a split extension. Pick any prime $q \equiv 1 \mod p^n$ which is totally split in the Galois closure of F and also totally split in the field $L(\zeta_{p^n}, \{l^{1/p}\}_{l \in \mathrm{Ram}(L/\mathbf{Q})})$ given by adjoining the pth roots of primes l which ramify in L. Then take $\widetilde{L}$ to be the composite of L and the sub-extension of $\mathrm{Gal}(\mathbf{Q}(\zeta_q)/\mathbf{Q})$ with Galois group $\mathbf{Z}/p\mathbf{Z}$.

Now suppose that $\widetilde{A}$ is a non-split extension. The argument in [Ser08] proceeds by first finding an extension $\widetilde{L}$ and then modifying $\widetilde{L}$ so that it is ramified at the same places as L. Hence the ramified primes l automatically satisfy the required splitting condition in F. The final step is to modify the field further so that it has property (2). This is achieved by choosing an auxiliary prime $q \equiv 1 \mod p$ along with a character $\chi : (\mathbf{Z}/q\mathbf{Z})^\times \to \mathbf{Z}/p\mathbf{Z}$ satisfying the following properties:

(1) The prime $q \equiv 1 \mod p^n$.
(2) For every prime l which ramifies in L, there is an equality $\chi(l) = c_l$ where c_l is determined from the extension L.

(3) The prime q splits completely in L.
(4) The prime q splits completely in F.

Only the last condition is new. The first three conditions are Čebotarev conditions in the field $L(\zeta_{p^n}, \{l^{1/p}\}_{l\in \mathrm{Ram}(L/\mathbf{Q})})$, whereas the fourth condition is a Čebotarev condition in the Galois closure of F. By construction, the primes l are totally split in the Galois closure of F. Hence the intersection of the Galois closure of F with $L(\zeta_{p^n}, \{l^{1/p}\}_l)$ must be contained inside $\mathbf{Q}(\zeta_{p^n})$. Since the first condition implies that q splits completely in $\mathbf{Q}(\zeta_{p^n})$, there is no obstruction to finding such primes q satisfying all four conditions using the Čebotarev density theorem provided there is no obstruction without the last hypothesis. But this is exactly what follows from the proof of Theorem 2.1.3 of [Ser08]. □

As in Lemma 2.4, let ${}^L P$ be the image of $P \times \mathbf{Z}/2\mathbf{Z}$ in W; explicitly, it is the product $P \times \langle (w_0, \tau)\rangle$ inside W. Our reason for working with the group P is the combination of the following two properties:

Lemma 2.7.

(1) The restriction of the extension

$$1 \to T(\mathbf{Z}) \to N_G(T)(\mathbf{Z}) \rtimes \mathrm{Out}(G) \xrightarrow{\pi} W_G \rtimes \mathrm{Out}(G) \to 1$$

to ${}^L P$ splits.
(2) P acts transitively on the set $\Lambda_{\min}$ of weights of the minuscule representation $r_{\min}$.

Proof. Since P is a 3-group while $T(\mathbf{Z})$ is an $\mathbf{F}_2$-module, the Hochschild–Serre spectral sequence

$$H^i(P, H^j(\langle (w_0, \tau)\rangle, T(\mathbf{Z}))) \implies H^{i+j}({}^L P, T(\mathbf{Z}))$$

degenerates at the E_2-page, and restriction induces an isomorphism

$$H^2({}^L P, T(\mathbf{Z})) \xrightarrow{\sim} H^2(\langle (w_0, \tau)\rangle, T(\mathbf{Z}))^P.$$

The image of our extension under this restriction isomorphism is trivial, because the element (w_0, τ) lifts to an order two element of $N(T)(\mathbf{Z}) \rtimes \mathrm{Out}(G)$ (see for instance [AH16, Lemma 3.1], noting that Z_G has order 3).

For the second part of the lemma, note that W_G acts transitively on $\Lambda_{\min}$, so the 3-part of the stabilizer of any element $\lambda \in \Lambda_{\min}$ has order 3 (recall $|W_G| = 2^7 \cdot 3^4 \cdot 5$ and $|\Lambda_{\min}| = 3^3$). It follows that the orbit of P on λ has order at least $|P|/3 = 3^3$, and thus (equality holds and) P acts transitively. (More generally, a finite group G acts transitively on a set X of p-power order if and only if a p-Sylow subgroup P acts transitively on X.) □

The work of Shafarevich on the inverse Galois problem for solvable groups implies that every split embedding problem with nilpotent kernel has a proper solution; we need some precise local control so will not be able to invoke this theorem, and unfortunately the following construction, guided by the demands of the local deformation theory as in [Pat16] and of automorphy lifting as in [BLGGT14], is somewhat technical.

Applying Lemma 2.7, let us fix a splitting of the extension

$$1 \to T(\mathbf{Z}) \to \pi^{-1}({}^L P) \xrightarrow{\pi} {}^L P \to 1$$

and write $s\colon \mathrm{Gal}(E/F^+) \to \pi^{-1}({}^L P)$ for the resulting lift. This representation is not yet suitable for potential automorphy theorems, so we modify it in the following Proposition.

We first establish some notation. For a fixed prime p, we write ϵ for the p-adic cyclotomic character and $\bar{\epsilon}$ for its mod p reduction. For a prime l exactly dividing $p-1$, let $\mathrm{pr}(l)$ be the projection from $\mathbf{F}_p^\times$ onto the l-torsion subgroup $\mathbf{F}_p^\times[l]$ restricting to the identity on $\mathbf{F}_p^\times[l]$. Finally, let $\bar{\epsilon}[l] = \mathrm{pr}(l) \circ \bar{\epsilon}\colon G_{F^+} \to \mathbf{F}_p^\times[l]$. For any homomorphism $\bar{\rho}\colon G_{F^+} \to G(\mathbf{F}_p)$ and any place v of F^+, let $\bar{\rho}_v := \bar{\rho}|_{G_{F_v^+}}$. Finally, for any homomorphism of groups $\rho\colon \Gamma \to \Gamma'$, and any $\rho(\Gamma)$-module M, let $\rho(M)$ denote M regarded as a Γ-module.

Proposition 2.8. Consider pairs of primes (l,p) such that:

- All primes above l split in F/F^+.
- p splits in $E/\mathbf{Q}$ and $p-1$ is divisible by l but not by l^2.

Then there exist infinitely many primes l such that there exist infinitely many pairs (l,p) such that there exists a homomorphism

$$\bar{\rho}\colon G_{F^+} \to T(\mathbf{F}_p)[l] \cdot \pi^{-1}({}^L P) \subset N_G(T)(\mathbf{F}_p) \rtimes \mathrm{Out}(G)$$

lifting our fixed identification $\mathrm{Gal}(E/F^+) = {}^L P$ and satisfying the following:

(1) The restriction $\overline{\rho}|_{G_F}$ is ramified only at places split in F/F^+.
(2) For any choice of complex conjugation $c \in G_{F^+}$, $\overline{\rho}(c)$ is a Chevalley involution of G, i.e. $\dim(\mathfrak{g}^{\mathrm{Ad}(\overline{\rho}(c))=1}) = \dim(G/B) = 36$.[1]
(3) For all places $v|p$ of F^+, fix any choice of integers $n_{v,\alpha}$ indexed by simple roots $\alpha \in \Delta = \Delta(G,B,T)$. Then $\overline{\rho}|_{G_{F_v^+}}$ is equal to
$$\prod_{\alpha \in \Delta} \alpha^\vee \circ \bar{\epsilon}[l]^{n_{v,\alpha}}.$$
(4) Let $\rho_G^\vee$ denote the half-sum of the positive coroots of (G,B,T); it lies in $X_\bullet(T)$ since $\#Z_G = 3$. There is a set S_{reg} of two primes q split in $E/\mathbf{Q}$ and of order l modulo p such that for some place $v|q$ of F^+, $\overline{\rho}|_{G_{F_v^+}}$ is unramified with Frobenius mapping to $\rho_G^\vee(q)$ (which lands inside the group $T(\mathbf{F}_p)[l]$ because $q^l \equiv 1 \pmod p$).

Moreover, in addition to satisfying the above conditions, we can choose $p > 56 = 2 \cdot (27+1)$, $l > h_G = 12$ (the Coxeter number of G), $\{n_{v,\alpha}\}$, and $\overline{\rho}$ such that

(1) For all $v|p$, the composite $r_{\min} \circ \overline{\rho}|_{G_{F_v^+}}$ is a direct sum of distinct powers of $\bar{\epsilon}[l]$.
(2) For all $v|p$ and for any Borel subgroup B (with Lie algebra $\mathfrak{b}$) containing T, the cohomology groups $H^0(G_{F_v^+}, \overline{\rho}_v(\mathfrak{g}/\mathfrak{b}))$ and $H^0(G_{F_v^+}, \overline{\rho}_v(\mathfrak{g}/\mathfrak{b})(1))$ both vanish. Moreover, for any Borel subgroup B_{27} containing the maximal torus of SL_{27} that stabilizes the weight spaces of $r_{\min}$, $H^0(G_{F_v^+}, (r_{\min} \circ \overline{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})) = 0$ and $H^0(G_{F_v^+}, (r_{\min} \circ \overline{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})(1)) = 0$.
(3) The composite $r_{\min} \circ \overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible.

Proof. For now let l be any odd prime such that the places of F^+ above l are all split in F/F^+, and E and $\mathbf{Q}(\mu_{l^2})$ are linearly disjoint over $\mathbf{Q}$ (eg, take l split in $E/\mathbf{Q}$); later in the argument we will require l to be larger than some absolute

[1] S.P. apologizes for the non-standard term "split Cartan involution" used in [Pat16], [Pat17].

bound depending only on the group G. Let p be any prime split in $E/\mathbf{Q}$ such that l, but not l^2, divides $p-1$ (such p exist by Čebotarev). Fix a set S_{reg} of two rational primes q that are split in $E/\mathbf{Q}$ and have order l modulo p: such q exist since E and $\mathbf{Q}(\mu_p)$ are linearly disjoint over $\mathbf{Q}$ (p splits in $E/\mathbf{Q}$). We will modify the original lift $s\colon \mathrm{Gal}(E/F^+) \to \pi^{-1}({}^L P)$ by an element of $H^1(G_{F^+}, T(\mathbf{F}_p[l]))$ so as to satisfy the local conditions (here G_{F^+} continues to act via the quotient $\mathrm{Gal}(E/F^+)$; note that $T(\mathbf{F}_p)[l]$ is τ-stable). Let $T(\mathbf{F}_p)[l] = \oplus W_i$ be the decomposition into irreducible $\mathbf{F}_l[P]$-modules. Recall that $(w_0, c) \in {}^L P$ acts on T by -1, so this is also a decomposition as ${}^L P$-module, and the splitting field $F^+(W_i)$ of the $\mathrm{Gal}(E/F^+)$-module W_i contains F for each i (since $l \neq 2$). Let Σ be the set of places of F^+ that are either split or ramified in F/F^+ (implicitly including the infinite places in the former condition); in particular, the action of G_{F^+} on $T(\mathbf{F}_p)[l]$ factors through $G_{F^+,\Sigma}$. For any finite subset T of Σ, [NSW00, Theorem 9.2.3(v)] implies that the restriction map

$$H^1(G_{F^+,\Sigma}, W_i) \to \bigoplus_T H^1(G_{F_v^+}, W_i)$$

is surjective for all i. Assembling the different i (with a common set T), the restriction map

$$H^1(G_{F^+,\Sigma}, T(\mathbf{F}_p)[l]) \to \bigoplus_T H^1(G_{F_v^+}, T(\mathbf{F}_p)[l])$$

is also surjective. We apply this observation to the following set T and the following local cohomology classes: let T be the union of the following sets of places of F^+:

- places which are ramified in F/F^+;
- places dividing p;
- places above the rational primes $q \in S_{\mathrm{reg}}$;
- an auxiliary place w lying above a rational prime r that splits completely in $E(\zeta_p)/\mathbf{Q}$.

Consider the following local classes in $H^1(G_{F_v^+}, T(\mathbf{F}_p)[l])$ for $v \in T$:

- trivial at places which ramify in F/F^+;
- the prescribed homomorphism $\prod_{\alpha\in\Delta} \alpha^\vee \circ \bar{\epsilon}[l]^{n_{v,\alpha}}$ for $v|p$,
- the unramified homomorphism $\mathrm{Fr}_v \mapsto \rho_G^\vee(q)$ for $v|q$ and $q \in S_{\mathrm{reg}}$, where $\rho_G^\vee$ denotes the half-sum of the positive coroots of G, which is in fact a cocharacter of G (we will only need this construction for one of the places above q).
- the unramified homomorphism $\mathrm{Fr}_w \mapsto t$, where t is any element of $T(\mathbf{F}_p)[l]$ such that the values $\lambda(t)$ for $\lambda \in \Lambda_{\min}$ are all distinct (for l sufficiently large, such t exist).

Let $\phi \in H^1(G_{F^+,\Sigma}, T(\mathbf{F}_p)[l])$ be a class with these local restrictions, and set $\overline{\rho} = \phi \cdot s$. We claim the conclusions of the proposition hold for this $\overline{\rho}$. The conditions at finite places are all evident from the construction, and the condition on complex conjugation is satisfied because any order two element $(\widetilde{w}_0, \tau) \in N_G(T)(\mathbf{F}_p)$ lifting $(w_0, \tau) \in W_G \rtimes \mathrm{Out}(G)$ gives a Chevalley involution of $\mathfrak{g}$: it acts by -1 on $\mathrm{Lie}(T)$, and it sends a root space $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$, so we get precisely $\dim(G/B) = \dim(\mathfrak{g}^{\mathrm{Ad}((\widetilde{w}_0,\tau))=1})$.

For the second list of assertions, note that for $v|p$, $r_{\min} \circ \overline{\rho}|_{G_{F_v^+}}$ is equal to

$$\bigoplus_{\lambda\in\Lambda_{\min}} \bar{\epsilon}[l]^{\sum_{\alpha\in\Delta} n_{v,\alpha}\langle\lambda,\alpha^\vee\rangle},$$

where $\Lambda_{\min}$ is the set of weights of $r_{\min}$. For varying λ, we want these exponents to be distinct modulo l. We simply choose the $(n_{v,\alpha})_\alpha$ so that the exponents are distinct in $\mathbf{Z}$, and then any l sufficiently large will do. To evaluate the cohomology groups appearing in the conclusion of the proposition, note that as a $G_{F_v^+}$-module, $\overline{\rho}_v(\mathfrak{g}/\mathfrak{b})$ is a direct sum of characters of the form

$$\bar{\epsilon}[l]^{\sum_{\alpha\in\Delta} n_{v,\alpha}\langle\beta,\alpha^\vee\rangle},$$

where β is a negative root of G. Clearly the choice of l can be modified if necessary to ensure that these exponents (which don't depend on p) are all (distinct, non-zero) integers in $(-l,l)$, and so regardless of how p is chosen the group $H^0(G_{F_v^+},\overline{\rho}_v(\mathfrak{g}/\mathfrak{b}))$ will vanish. (Since $\overline{\rho}_v$ factors through T, this argument in fact shows that $\overline{\rho}_v(\mathfrak{g}/\mathfrak{t})$ has no invariants; in particular, it applies to all Borel subgroups B containing T.) The vanishing of $H^0(G_{F_v^+},\overline{\rho}_v(\mathfrak{g}/\mathfrak{b})(1))$ is even more straightforward: the order of $\bar{\epsilon}$ is divisible by some prime other than l, whereas all powers of $\bar{\epsilon}[l]$ have order 1 or l. We can similarly deduce the vanishing of $H^0(G_{F_v^+},(r_{\min}\circ\overline{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27}))$ and $H^0(G_{F_v^+},(r_{\min}\circ\overline{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})(1))$, where $\mathfrak{b}_{27}$ is the Lie algebra of any Borel subgroup $B_{27}\subset \mathrm{SL}_{27}$ containing the maximal torus of SL_{27} characterized by the property that it stabilizes each of the weight spaces of $r_{\min}$. Then $(r_{\min}\circ\overline{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})$ is a direct sum of characters

$$\bar{\epsilon}[l]^{\sum_{\alpha\in\Delta} n_{v,\alpha}\langle\lambda_1-\lambda_2,\alpha^\vee\rangle},$$

for distinct weights $\lambda_1,\lambda_2\in\Lambda_{\min}$. These exponents are by construction non-zero modulo l, so we win, and the same argument as for $\overline{\rho}_v(\mathfrak{g}/\mathfrak{b})(1)$ applies to show $H^0(G_{F_v^+},(r_{\min}\circ\overline{\rho}_v)(\mathfrak{sl}_{27}/\mathfrak{b}_{27})(1))$ is zero as well.

The condition at the auxiliary prime w ensures absolute irreducibility of $r_{\min}\circ\overline{\rho}|_{G_{F(\zeta_p)}}$: P acts transitively on $\Lambda_{\min}$ by Lemma 2.7(2), so any non-zero submodule of $r_{\min}\circ\overline{\rho}|_{G_{F(\zeta_p)}}$ has non-zero projection to each weight space (recall that $F(\zeta_p)$ is linearly disjoint from E over F); but for a place $w'|w$ of $F(\zeta_p)$, the image of $G_{F(\zeta_p)_{w'}}$ in $T(\mathbf{F}_p)$ acts via distinct characters on the different weight spaces of $r_{\min}$. □

3. Lifting Galois Representations

Let $\overline{\rho}\colon G_{F^+}\to N_G(T)(\mathbf{F}_p)\rtimes\mathrm{Out}(G)\subset {}^LG(\mathbf{F}_p)$ be a homomorphism constructed as in Proposition 2.8. We would like to lift $\overline{\rho}$ to a homomorphism

$$\rho\colon G_{F^+}\to {}^LG(\overline{\mathbf{Z}}_p)$$

that belongs to a compatible system of representations, all having Zariski-dense image and appearing in the cohomology of an algebraic variety. To achieve Zariski-dense monodromy for a lift ρ, we follow the approach of [Pat16]: ensuring local Steinberg-type ramification at one auxiliary prime and sufficiently general Hodge–Tate cocharacter suffices. To produce the lift, and to put it in a compatible system, we compare deformation rings for $\overline{\rho}$ and $r_{\min}\circ\overline{\rho}\colon G_{F^+}\to\mathcal{G}_{27}(\mathbf{F}_p)$. We control a suitable deformation ring for $r_{\min}\circ\overline{\rho}$ using the method of Khare–Wintenberger and the automorphy lifting results of [BLGGT14].

Let S be a finite set of primes of F^+ which split in F containing all those where $\overline{\rho}$ is ramified (by the construction of $\overline{\rho}$, the primes of F^+ at which $\overline{\rho}$ is ramified split in F). We will enlarge the set S as necessary. Let F_S be the maximal extension (in $\overline{F}^+$) of F unramified outside (places above) S, and set $G_S=\mathrm{Gal}(F_S/F^+)$. We will be deforming G_S-representations. For each $v\in S$, fix an extension $\tilde{v}$ of v to F, and fix a member of the $G_{F,S}$-conjugacy class of homomorphisms $G_{F_{\tilde{v}}}\to G_{F,S}$.

Via the inclusion $G_{F,S} \subset G_S$, these choices specify what we mean by restricting $\overline{\rho}$ (or its lifts) to $G_{F_{\tilde{v}}}$. We do not review in detail the mechanics of the deformation theory in this setting, but instead refer the reader to [Pat16, §9.2] (for ${}^L G$) and [CHT08, §2] (for $\mathcal{G}_{27}$). We now specify local deformation conditions to define two global deformation functors, one for $\overline{\rho}$ and one for $r_{\min} \circ \overline{\rho}$.

For $v \in S$, define the following local deformation conditions $\mathcal{P}_v$ on lifts of $\overline{\rho}|_{G_{F_{\tilde{v}}}}$:

- For v above the auxiliary primes $q \in S_{\mathrm{reg}}$ (see Proposition 2.8), we impose the Steinberg deformation condition as in [Pat16, §4.3], with respect to the Borel subgroup B of G specified by our based root datum. Note that by construction the order of $\bar{\epsilon}\colon G_{F_{\tilde{v}}} \to \mathbf{F}_p^\times$ is greater than $h_G - 1$ (since $l > h_G$ is the order of q modulo p).
- For $v|p$, we take an ordinary deformation condition as in [Pat16, §4.1]. To be precise, fix the following lift χ_T of $\overline{\rho}|_{I_{F_{\tilde{v}}}}$ to $T(\mathbf{Z}_p)$:

$$\chi_T = \prod_{\alpha \in \Delta} \alpha^\vee \circ (\epsilon^{\tilde{n}_{v,\alpha}} \cdot \chi^{-n_{v,\alpha}}), \tag{1}$$

  where the $\tilde{n}_{v,\alpha}$ are sufficiently general positive (positive ensures, as in [Pat16, Lemma 4.8], that our characteristic zero lifts are de Rham) integers congruent to $n_{v,\alpha}$ modulo $l-1$, and χ is the Teichmüller lift of $\bar{\epsilon} \cdot \bar{\epsilon}[l]^{-1}$.
- For all other primes $v \in S$, the inertial image $\overline{\rho}(I_{F_{\tilde{v}}})$ has order prime to p (indeed, $\overline{\rho}$ lands in the prime-to-p group $N_G(T)(\mathbf{F}_p) \rtimes \mathrm{Out}(G)$), and we take the minimal deformation condition of [Pat16, §4.4].

Lemma 3.1. For all $v \in S$, let $\mathcal{P}_v$ be the local condition just defined. Let $\mathcal{P} = \{\mathcal{P}_v\}_{v \in S}$, and let $\mathrm{Lift}_{\overline{\rho}}^{\mathcal{P}}$ be the associated global lifting functor associated to this collection of local conditions (see [Pat16, §9.2]).

(1) For all $v \in S$ not above p, the local lifting ring associated to the condition $\mathcal{P}_v$ just defined is formally smooth, and the associated deformation functor has tangent space L_v of dimension $\dim H^0(G_{F_{\tilde{v}}}, \overline{\rho}(\mathfrak{g}))$. For $v|p$, the same holds, except the tangent space has dimension $\dim H^0(G_{F_{\tilde{v}}}, \overline{\rho}(\mathfrak{g})) + \dim(G/B)$.

(2) The associated deformation functor $\mathrm{Def}_{\overline{\rho}}^{\mathcal{P}}$ is representable. Let $R_{\overline{\rho}}^{\mathcal{P}}$ be the representing object. For some integer δ, $R_{\overline{\rho}}^{\mathcal{P}}$ has a presentation as the quotient of a power series ring over $\mathbf{Z}_p$ in δ variables by an ideal generated by (at most) δ relations.

Proof. The local claims follow from [Pat16, §4.1, 4.3, 4.4]. The global claims follow from [Pat16, Proposition 9.2]), using that:

- the centralizer of $\overline{\rho}$ in $\mathfrak{g}$ is trivial;
- for all complex conjugations c, $\overline{\rho}(c)$ is a Chevalley involution of G;
- the local lifting rings have dimensions as computed in the first part of the lemma.

(We remark that the integer δ is the common dimension of the Selmer and dual Selmer groups associated to the global deformation functor.) □

Next we define an analogous deformation ring for $r_{\min} \circ \overline{\rho}\colon G_S \to \mathcal{G}_{27}(\mathbf{F}_p)$. Recall the character $\nu\colon \mathcal{G}_{27} \to \mathbf{G}_m$. The composition $\nu \circ (r_{\min} \circ \overline{\rho})$ is the non-trivial character of $\mathrm{Gal}(F/F^+)$, and we fix $\mu\colon G_{F^+} \to \mathbf{Z}_p^\times$ equal to its Teichmüller lift (we will consider lifts with this fixed character). To define a global deformation problem in the sense of [BLGGT14, §1.5] (see [CHT08] for more details), we must, for each $v \in S$, choose

an irreducible component $\mathcal{C}_v$ of the (generic fiber) lifting ring (in the case v not above p) $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}[1/p]$ or (in the case $v|p$) $\lim_K R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}},\{H\},K-\mathrm{ss}}[1/p]$, where here we follow the notation of [BLGGT14]: H is a collection (indexed by embeddings $F_{\tilde{v}} \to \mathbf{Q}_p$, but for us $F_{\tilde{v}} = \mathbf{Q}_p$) of multi-sets of Hodge numbers, K varies over finite extensions of $F_{\tilde{v}}$, and the lifting ring in question is the one constructed by Kisin ([Kis08]), whose characteristic zero points parametrize potentially semistable deformations, semistable over K, with the prescribed Hodge numbers; to be precise, it is the maximal reduced p-torsion-free quotient of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ whose $\overline{\mathbf{Q}}_p$-points satisfy these properties (see [BLGGT14, §1.4] for an overview). In both cases $v|p$ and $v \nmid p$, we then associate the lifting ring $R^{\mathcal{C}_v}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ given by the maximal reduced p-torsion-free quotient of $(v \nmid p)$ $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ or $(v|p)$ $\lim_K R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}},\{H\},K-\mathrm{ss}}$ that is, after inverting p, supported on the component $\mathcal{C}_v$. Namely, we take:

- For v above the auxiliary primes $q \in S_{\mathrm{reg}}$, recall that $\overline{\rho}|_{G_{F_{\tilde{v}}}}$ is unramified with (arithmetic) Frobenius $\mathrm{Fr}_{\tilde{v}}$ mapping to $\rho_G^\vee(q)$. Let $\varphi\colon \mathrm{PGL}_2 \to G$ be the principal homomorphism associated to our fixed pinning of G, so $\overline{\rho}(\mathrm{Fr}_v)$ equals $\varphi(\mathrm{diag}(q,1))$. The composite $r_{\min}\circ\varphi$ decomposes as $\mathrm{S}^{16}\oplus\mathrm{S}^8\oplus\mathrm{S}^0$ (see [Gro00, §7]), where we write S^i for the i^{th} symmetric power of the standard representation of SL_2. Consider the lift $\rho_{\tilde{v}}$ of $r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}$ given by $r_{16}\oplus r_8\oplus r_0$ where r_i is the tame unipotent representation on the S^i component given by the matrices
$$(r_i(\mathrm{Fr}_v))_{a,b} = \begin{cases} q^{i-2a+2} & \text{if } a=b,\\ 0 & \text{if } a\neq b,\end{cases}$$
and on a topological generator τ_v of tame inertia by
$$(r_i(\tau_v))_{a,b} = \begin{cases} p & \text{if } b=a+1,\\ 0 & \text{if } b\neq a+1.\end{cases}$$
A quick calculation shows that, for all finite extensions $K/F_{\tilde{v}}$, we have an equality $H^0(G_K,\mathrm{ad}(\rho_{\tilde{v}})(1))=0$, so $\rho_{\tilde{v}}$ is a robustly smooth point of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}[1/p]$, in the sense of [BLGGT14, §1.3]. In particular, $\rho_{\tilde{v}}$ lies on a unique irreducible component of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}[1/p]$, and we take $\mathcal{C}_v$ to be this component.
- For $v|p$, let the set H of Hodge–Tate weights be
$$\{h_\lambda = \sum_{\alpha\in\Delta}\tilde{n}_{v,\alpha}\langle\lambda,\alpha^\vee\rangle\}_{\lambda\in\Lambda_{\min}},$$
where $\Lambda_{\min}$ is the set of weights of $r_{\min}$ as before. Borel subgroups of SL_{27} containing the maximal torus T_{27} (the unique torus stabilizing the weight spaces in $r_{\min}$) are in bijection with orderings of the set $\Lambda_{\min}$; let B_{27} be the Borel defined by the ordering $\lambda>\lambda' \iff h_\lambda > h_{\lambda'}$. The ordinary deformation ring (again following the notation of [Pat16, §4.1]) associated to the Borel B_{27} and the lift $\chi_{T_{27}}\colon I_{F_{\tilde{v}}}\to T_{27}(\mathbf{Z}_p)$ of $r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}$ given by
$$\chi_{T_{27}} = r_{\min}\left(\prod_{\alpha\in\Delta}\alpha^\vee(\epsilon^{\tilde{n}_{v,\alpha}}\chi^{-n_{v,\alpha}})\right)$$

is formally smooth and receives by the universal property a canonical surjection from $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}},H,K-\mathrm{ss}}$ for $K = F_{\tilde{v}}(\chi)$ (recall that χ is the Teichmüller lift of $\bar{\epsilon}\cdot(\bar{\epsilon}[l])^{-1}$), and after inverting p it induces an isomorphism from a unique irreducible component of the source (the equality of dimensions follows from [Kis08] and [Pat16, §4.1]). We take $\mathcal{C}_v$ to be this component.

- For all other $v \in S$, we take the irreducible component of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ parametrizing minimal deformations in the sense of [Pat16, §4.4]; using the standard argument, this is a power series ring and induces a unique irreducible component $\mathcal{C}_v$ of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}[1/p]$.

In each case, we write $R^{\mathcal{C}_v}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ for the associated local lifting ring. Recall ([BLGGT14, §1.3]) that for v not above p, $R^{\mathcal{C}_v}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ is the maximal quotient of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ that is reduced, p-torsion-free, and after inverting p is supported on the component $\mathcal{C}_v$. For $v|p$, $R^{\mathcal{C}_v}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ is in general constructed similarly, but for us it is simply the formally smooth ordinary deformation ring produced by [Pat16, §4.1].

Lemma 3.2. For all $v \in S$, the representation $r_{\min}$ induces a map $R^{\mathcal{C}_v}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}} \to R^{\mathcal{P}_v}_{\overline{\rho}|_{G_{F_{\tilde{v}}}}}$.

Proof. For $v|p$, this follows directly from the definitions once we check that $r_{\min}(B) \subset B_{27}$. Let λ be the co-character $\sum_{\alpha\in\Delta}\tilde{n}_{v,\alpha}\alpha^{\vee}$. Since all $\tilde{n}_{v,\alpha}$ are positive, B is the locus of $g \in G$ where $\lim_{t\to 0}\mathrm{Ad}(\lambda(t))g$ exists. By construction, B_{27} is the locus where $\lim_{t\to 0}\mathrm{Ad}(r_{\min}\circ\lambda(t))g$ exists. The claim follows.

For $v|q$, we write $R^{\mathcal{P}_v}_{\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ for the Steinberg lifting ring; recall that under our hypotheses it is a power series ring over $\mathbf{Z}_p$ (in $\dim(\mathfrak{g})$ variables). We have surjections

$$R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}} \to R^{\square}_{\overline{\rho}|_{G_{F_{\tilde{v}}}}} \to R^{\mathcal{P}_v}_{\overline{\rho}|_{G_{F_{\tilde{v}}}}}.$$

Recall that $R^{\mathcal{C}_v}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ is the maximal reduced p-torsion-free quotient of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ whose $\overline{\mathbf{Q}}_p$-points lie on $\mathcal{C}_v$. Since $R^{\mathcal{P}_v}_{\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ is a reduced, p-torsion-free quotient of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}$, it suffices to show that every $\overline{\mathbf{Q}}_p$-point of $R^{\mathcal{P}_v}_{\overline{\rho}|_{G_{F_{\tilde{v}}}}}$ lies on the same irreducible component of $R^{\square}_{r_{\min}\circ\overline{\rho}|_{G_{F_{\tilde{v}}}}}[1/p]$ as the representation $\rho_{\tilde{v}}$ (defined above) that characterizes the component $\mathcal{C}_v$. As $R^{\mathcal{P}_v}_{\overline{\rho}|_{G_{F_{\tilde{v}}}}}[1/p]$ is itself irreducible, it suffices to check this for a single $\overline{\mathbf{Q}}_p$-point, which we will do for the unramified lift $\rho^{\mathrm{un}}_{\tilde{v}}(\mathrm{Fr}_{\tilde{v}}) = \rho^{\vee}_G(q)$. This claim follows from applying [BLGGT14, Lemma 1.3.5] to $\rho_{\tilde{v}}$, whose associated graded for the obvious filtration is canonically identified with $r_{\min}\circ\rho^{\mathrm{un}}_{\tilde{v}}$.

For the other ramified primes $v \in S$, the lemma is evident. □

Now consider the global deformation problem (in the sense of [BLGGT14, §1.5]; see [CHT08] for details)

$$\mathcal{S} = (F/F^+, S, \{\tilde{v}\}_{v\in S}, \mathbf{Z}_p, r_{\min}\circ\overline{\rho}, \mu, \{\mathcal{C}_v\}_{v\in S}).$$

Since $r_{\min} \circ \overline{\rho}|_{G_F}$ is absolutely irreducible, this deformation functor is pro-represented by some $R^{\mathcal{S}}_{r_{\min} \circ \overline{\rho}}$.

Lemma 3.3. The representation $r_{\min}$ induces a surjection $R^{\mathcal{S}}_{r_{\min} \circ \overline{\rho}} \to R^{\mathcal{P}}_{\overline{\rho}}$.

Proof. There is an induced map $R^{\mathcal{S}}_{r_{\min} \circ \overline{\rho}} \to R^{\mathcal{P}}_{\overline{\rho}}$ by Lemma 3.2. It is a surjection because the G_{F^+}-module $\overline{\rho}(\mathfrak{g})$ is a direct summand of $\mathrm{ad}(r_{\min} \circ \overline{\rho})$ (indeed, these representations factor through representations of a finite prime-to-p group), so the associated map on tangent spaces is injective; dually, the map on co-tangent spaces is surjective, and we conclude by Nakayama's lemma. □

Finally, we can invoke the main results of [BLGGT14] to deduce that $R^{\mathcal{P}}_{\overline{\rho}}$ has a $\overline{\mathbf{Q}}_p$-point ρ such that $r_{\min} \circ \rho$ is potentially automorphic:

Theorem 3.4. *For sufficiently general choice of lifts $\tilde{n}_{v,\alpha}$ as in Equation 1, the representation $\overline{\rho}\colon G_{F^+} \to {}^L G(\mathbf{F}_p)$ constructed in Proposition 2.8 admits a geometric lift $\rho\colon G_{F^+} \to {}^L G(\overline{\mathbf{Z}}_p)$ such that:*

1. *The Zariski closure of the image of ρ is ${}^L G$.*
2. *The composite $r_{\min} \circ \rho|_{G_F}$ is potentially automorphic in the sense of* [BLGGT14].
3. *The composite $r_{\min} \circ \rho|_{G_F}$ belongs to a compatible system of l-adic representations: there exist a number field M and a strictly pure — in the sense of* [BLGGT14, §5.1] *— compatible system*
$$r_\lambda\colon G_F \to \mathrm{GL}_{27}(\overline{M}_\lambda)$$
indexed over all finite places λ of M. In particular, the restriction of r_λ to I_v for v above the auxiliary primes $q \in S_{\mathrm{reg}}$ is unipotent with Jordan blocks of size 1, 9, and 17 as long as v has residue characteristic different from λ.

Proof. By the proof of [BLGGT14, Theorem 4.3.1] (see especially the last paragraph), $R^{\mathcal{S}}_{r_{\min} \circ \overline{\rho}}$ is a finite $\mathbf{Z}_p$-module. Lemma 3.3 then implies that $R^{\mathcal{P}}_{\overline{\rho}}$ is a finite $\mathbf{Z}_p$-module. We have already seen in Lemma 3.1 that it has dimension at least one, so we conclude that $R^{\mathcal{P}}_{\overline{\rho}}(\overline{\mathbf{Z}}_p)$ is non-empty. Let ρ be an element of $R^{\mathcal{P}}_{\overline{\rho}}(\overline{\mathbf{Z}}_p)$. Then:

- By [BLGGT14, Theorem 4.5.1], the composite $r_{\min} \circ \rho|_{G_F}$ is potentially automorphic (in the sense of [BLGGT14]).
- The Zariski closure G_ρ of the image of $\rho|_{G_F}$ is G: by [Pat16, Lemma 7.8], it suffices to show
  - G_ρ is reductive;
  - G_ρ contains a regular unipotent element of G; and
  - for some $v|p$, $\rho|_{G_{F_{\tilde{v}}}}$ is B-ordinary, and, for all simple roots α, $\alpha \circ \rho|_{I_K} = \epsilon^{r_\alpha}$ for some finite extension $K/F_{\tilde{v}}$ and for distinct integers r_α.

  Reductivity is immediate since ρ is irreducible. The third condition follows by taking the integers $\{\tilde{n}_{v,\alpha}\}_{\alpha \in \Delta}$ in the definition of the local condition $\mathcal{P}_v$ (see the discussion preceding Lemma 3.1) to be sufficiently general. Finally, to show that the image of ρ contains a regular unipotent element, we check that for $v|q$, the tame inertia in $\rho|_{G_{F_{\tilde{v}}}}$ acts by a regular unipotent. Lemma 3.5 below shows that this would follow from the corresponding claim that $r_{\min} \circ \rho(I_{F_{\tilde{v}}})$ contains a unipotent element with Jordan blocks of dimension 17, 9, and 1. Let π be the automorphic representation of $\mathrm{GL}_{27}(\mathbf{A}_{F'})$, for

a suitable finite extension F'/F, witnessing the potential automorphy of $r_{\min} \circ \rho$, and let v' be a place of F' above v. By local-global compatibility at $l \neq p$ (Proved in general by [Car12], but known for odd dimensional representations by previous work of [HT01, TY07, Shi11]), the (Frobenius semi-simple) Weil–Deligne representation associated to $r_{\min} \circ \rho|_{G_{F'_{v'}}}$ is isomorphic to the image of $\pi_{v'}$ under the local Langlands correspondence. It follows (eg, using [BLGGT14, Lemma 1.3.2(1)]) that $r_{\min} \circ \rho|_{G_{F_{\tilde{v}}}}$ lies on a unique irreducible component of $R^{\square}_{r_{\min} \circ \rho|_{G_{F_{\tilde{v}}}}}$. By construction, $r_{\min} \circ \rho|_{G_{F_{\tilde{v}}}}$ and $r_{\min} \circ \rho_{\tilde{v}}$ lie on the same irreducible component $\mathcal{C}_v$ of $R^{\square}_{r_{\min} \circ \overline{\rho}|_{G_{F_{\tilde{v}}}}}$, and since they both lie on a unique component, [BLGGT14, Lemma 1.3.4(2)] implies their inertial restrictions are isomorphic. The result follows.

The claim that $r_{\min} \circ \rho$ can be put in a compatible system, follows from [BLGGT14, Theorem 5.5.1], and the claim concerning the restriction to I_v for $v|q$ follows as above from local-global compatibility at $l \neq p$.

□

In the next section, we will show that in fact all members of the compatible system $\{r_\lambda\}_\lambda$ have algebraic monodromy group equal to G.

Finally, here is the Lie-theoretic lemma promised in the proof of Theorem 3.4.

Lemma 3.5. Let G be an almost-simple algebraic group over an algebraically closed field of characteristic zero, and let T be a maximal torus in G. Fix a system of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$ of $\mathfrak{g}$ with respect to T. Let $\lambda \in X^\bullet(T)$ be a dominant (with respect to Δ) weight, with corresponding highest weight representation $r_\lambda \colon G \to \mathrm{Aut}(V_\lambda)$. Let $N \in \mathfrak{g}$ be a regular nilpotent element. Then for any nilpotent element $X \in \mathfrak{g}$, $r_\lambda(X)$ is conjugate to $r_\lambda(N)$ if and only if X is regular nilpotent.

Proof. It is well-known that the regular nilpotent elements of $\mathfrak{g}$ form a single G-orbit, so one direction is clear. By the theorems of Jacobson-Morosov, Kostant, and Mal'cev ([CM93, Theorem 3.3.1, 3.4.10, 3.4.12]), there is an $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ in $\mathfrak{g}$ containing X, this triple is unique up to conjugacy, and its conjugacy class is determined by that of H. Replacing $\{X, Y, H\}$ by a conjugate, we may assume that $H \in \mathrm{Lie}(T)$ and $\alpha(H) \in \{0, 1, 2\}$ for all $\alpha \in \Delta$ (see [CM93, §3.5]). When we likewise extend N to an $\mathfrak{sl}_2$-triple with semi-simple element H_N, and assume after conjugating that $H_N \in \mathrm{Lie}(T)$ is dominant, we have $\alpha(H_N) = 2$ for all $\alpha \in \Delta$. Since the conjugacy class of H determines that of X, X is regular if and only if $\alpha(H) = 2$ for all $\alpha \in \Delta$. We claim that the highest eigenvalue of H on V_λ is enough to determine whether X is regular. For each weight μ of V_λ, write

$$\mu = \sum_{\alpha \in \Delta} b_\alpha(\mu)\alpha$$

for some $b_\alpha(\mu) \in \mathbf{Q}$. Because λ is the highest weight of V_λ, we have $b_\alpha(\lambda) \geq b_\alpha(\mu)$ for all other weights μ occurring in V_λ. The maximal eigenvalue of H on V_λ is thus $\sum_{\alpha \in \Delta} b_\alpha(\lambda)\alpha(H)$. Assuming that $b_\alpha(\lambda) > 0$ for all α, it follows easily that this maximal eigenvalue is *strictly* less than $\sum_{\alpha \in \Delta} b_\alpha(\lambda) \cdot 2$ if X is not regular. If $b_\alpha(\lambda) \leq 0$, then (by symmetry) $b_\alpha(\mu) = 0$ for every weight μ of V_λ. But (assuming $\lambda \neq 0$) this contradicts the simplicity of $\mathfrak{g}$. □

4. Controlling the image in the compatible system

Theorem 3.4 provides the existence of a compatible system $\{r_\lambda\}$ of G_F representations with the property that the geometric monodromy group at one prime is precisely E_6. Our goal in this section is to use known properties of compatible systems ([LP92]) together with the additional properties our compatible system satisfies at the auxiliary primes S_{reg} to ensure that the monodomy group is E_6 at *all* primes λ.

Let M denote the coefficient field of our compatible system. For a pair of reductive groups $\mathbf{G}_i$ over $\overline{M}_\lambda$ together with faithful irreducible representations ρ_i, recall that $(\mathbf{G}_1, \rho_1)$ is *similar* to $(\mathbf{G}_2, \rho_2)$ if and only if, for some maximal tori $T_i \subset \mathbf{G}_i$, the pairs $(T_i, \rho_i|_{T_i})$ are isomorphic (cf. §5.1 of [LP92]). We would like to compare the following two faithful representations of reductive groups over $\overline{M}_\lambda$:

(1) The monodromy group G_λ of r_λ over $\overline{M}_\lambda$ (that is, the Zariski closure of the image of r_λ over $\overline{M}_\lambda$) with its tautological representation r_λ,
(2) The group $G|_{\overline{M}_\lambda}$, where G is (as above) the split simply-connected reductive group scheme over $\mathbf{Z}$ of type E_6, and $r_{\min}$ is the minuscule representation.

Lemma 4.1. The monodromy group G_λ for each prime λ of M has the following properties:

(1) The component group of G_λ is is trivial.
(2) The rank of G_λ is 6.
(3) If T_λ is a maximal torus in G_λ, and $T|_{\overline{M}_\lambda}$ is a maximal torus in $G|_{\overline{M}_\lambda}$, then there is an isomorphism
$$(T_\lambda, r_\lambda|_{T_\lambda}) \simeq (T|_{\overline{M}_\lambda}, r_{\min}|_{T|_{\overline{M}_\lambda}}).$$
(4) If $G_\lambda = G_\lambda^\circ$ acts irreducibly, then (G_λ, r_λ) is similar to $(G|_{\overline{M}_\lambda}, r_{\min})$.
(5) If $G_\lambda = G_\lambda^\circ$ acts irreducibly, then G_λ is isomorphic to $G|_{\overline{M}_\lambda}$ and r_λ is isomorphic to $r_{\min}$.
(6) There exists a unipotent element in the image with Jordan blocks of size 1, 9, and 17.

Proof. The first three properties (1), (2), (3) involve quantities which are constant in a compatible system, c.f. Propositions 6.12 and 6.14 of [LP92]. We choose λ so that $r_\lambda = r_{\min} \circ \rho_F$ as in Theorem 3.4(1). Then, by Theorem 3.4(1), we deduce that $(G_\lambda, r_\lambda) \simeq (G|_{\overline{M}_\lambda}, r_{\min})$ in this case. Claim (4) follows immediately from (3) and the definition of similarity recalled above. Claim (5) follows from Theorem 5.6 of [LP92], noting that E_6 does not occur in the explicit list of groups which gives rise to any of the basic similarity relations of §5 of *ibid.* The nilpotent operator of the Weil–Deligne representation of r_λ at the auxiliary prime $v|q$ for $q \in S_{\mathrm{reg}}$ decomposes (by Theorem 3.4 (3)) into Jordan blocks of size 1, 9, and 17, *assuming* that the residual characteristic of q is different from that of λ. Yet S_{reg} was chosen (for this purpose!) to consist of two primes, so this holds for at least one prime v, which gives part (6). □

We now show that Conditions (2), (3), and (6) of Lemma 4.1 are sufficient to imply — purely by representation theoretic methods — that G_λ acts irreducibly, which by Lemma 4.1(5) will prove the claims in Theorem 1.1 concerning the monodromy groups of $\{r_\lambda\}$. In light of the existence of the unipotent element whose

existence is guaranteed by Lemma 4.1 (6), it suffices to show that G_λ cannot act faithfully on a direct sum of representations of dimension

$$27 = 26 + 1 = 18 + 9 = 17 + 10 = 17 + 9 + 1$$

unless $G_\lambda = E_6$ and the representation is irreducible.

4.1. **The Formal Character of $\mathfrak{e}_6$.** The root lattice Φ of E_6 consists of 72 roots; it may be given as $\Phi^+ \cup \Phi^-$, where the positive roots Φ^+ are given explicitly in $\mathbf{R}^6$ by the $2\binom{5}{2} = 20$ vectors $e_i \pm e_j$ for $2 \leq i < j \leq 6$, and the $2^4 = \binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16$ vectors

$$\left(\frac{\sqrt{3}}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}\right)$$

where there are an even number of minus signs. If $2\rho = \sum_{\alpha \in \Phi^+} \alpha$, then $\rho = (4\sqrt{3}, 4, 3, 2, 1, 0)$. The root lattice is not self-dual, but has discriminant 3. A weight μ corresponding to a choice of minuscule representation is given by

$$\mu = \left(\frac{2\sqrt{3}}{3}, 0, 0, 0, 0, 0\right).$$

The 27 weights Σ of the corresponding minuscule representation may be obtained from μ from the orbit of the Weyl group; all 27 such weights may be obtained by applying at most 2 reflections in the roots of Φ to μ. The 27 roots can also be described explicitly in this basis; there are 10 roots of the form $-\sqrt{3}/3 e_1 \pm e_i$ for $i = 2, \ldots, 5$, and $2^4 = \binom{5}{1} + \binom{5}{3} + \binom{5}{5} = 16$ roots of the form

$$\left(\frac{\sqrt{3}}{6}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}, \frac{\pm 1}{2}\right)$$

where there are an *odd* number of minus signs, and finally μ itself. These three sets (of orders $10 + 16 + 1 = 27$) are precisely the orbits under the stabilizer $W_{G,\mu}$ of μ in the Weyl group W_G.

Lemma 4.2. Of the $\binom{27}{3} = 2925$ collections of 3 vectors in Σ, exactly 45 such triples generate a subspace of dimension 2, and they all consist of a triple of weights (μ, μ', μ'') with $\mu + \mu' + \mu'' = 0$.

Proof. This can be checked explicitly using `magma` given the explicit description of Σ given above. □

We also recall the following facts (cf. [Lur01]) which are also elementary to verify by direct computation. (These facts are included purely for the reader's enjoyment and are not used in the sequel, except perhaps tangentially in the proof of Lemma 5.2.) If Λ is the weight lattice, then Σ injects into $V = \Lambda/2\Lambda$, which acquires the structure of a quadratic space via the map $q(\mu) = \frac{1}{2}\langle \mu, \mu \rangle$ (note that Λ is an even lattice). The pairing $\langle x, y \rangle = q(x + y) - q(x) - q(y)$ is preserved by the Weyl group W_G, which may be identified with the corresponding orthogonal group $O(V, q)$ ([Lur01, Remark 4.3.4]). The triples $\{\mu, \mu', \mu''\}$ of Lemma 4.2 lie inside a maximal isotropic subspace $U \subset V$ of dimension 2. The stabilizer of U (and of a triple) is a subgroup of W_G of index 45, which correspondingly acts

transitively on the set of 45 triples. The stabilizer is also isomorphic to the Weyl group of F_4.

Lemma 4.3. The quotient of T_λ which acts faithfully on any sub-representation of dimension ≥ 4 has rank at least 3, while T_λ itself acts faithfully on any sub-representation of dimension 26. In particular, if G_λ has a sub-representation of dimension ≥ 4 or 26, then the quotient of G_λ which acts faithfully on this subrepresentation has rank at least 3 or 6 respectively.

Proof. Recall (in the notation of Lemma 4.1) that we have an isomorphism $(T_\lambda, r_\lambda|_{T_\lambda}) \simeq (T|_{\overline{M}_\lambda}, r_{\min}|_{T|_{\overline{M}_\lambda}})$. Hence it suffices to prove the corresponding claims for the formal character of the minuscule representation of E_6. For the first claim, it suffices to show that any four weights appearing in the sub-representation must generate a sublattice of the weight lattice of rank at least 3. Assume otherwise. Then certainly any two of these weights must generate a sublattice of rank at most 2. By Lemma 4.2, such triples of weights are all of the form (μ, μ', μ'') where $\mu + \mu' + \mu'' = 0$. But (among four distinct weights) such a relation can hold for at most one such triple. Hence we obtain a contradiction. The second claim follows from the fact that the sum of all 27 weights in Σ is zero, while none of the weights in Σ is zero. □

We now note the following:

Lemma 4.4. Suppose that $\mathfrak{g}$ is a reductive Lie algebra with a faithful irreducible representation of dimension d for $d \in \{17, 26, 18\}$. Then $\mathfrak{g} = \mathfrak{h}$ or $\mathfrak{h} \oplus \mathfrak{t}$, where $\mathfrak{h}$ is semi-simple and $\mathfrak{t}$ is a rank one torus. Furthermore, assuming that $\mathfrak{h}$ is simple when $d = 18$, then $\mathfrak{h}$ is one of the following:

$\mathfrak{h}$	d	rank($\mathfrak{h}$)
$\mathfrak{sl}_2$	17	1
$\mathfrak{so}_{17}$	17	8
$\mathfrak{sl}_{17}$	17	16
$\mathfrak{sl}_2$	26	1
$\mathfrak{f}_4$	26	4
$\mathfrak{so}_{13} \times \mathfrak{sl}_2$	26	7
$\mathfrak{sl}_{13} \times \mathfrak{sl}_2$	26	13
$\mathfrak{sp}_{26}$	26	13
$\mathfrak{so}_{26}$	26	13
$\mathfrak{sl}_{26}$	26	25
$\mathfrak{sl}_2$	18	1
$\mathfrak{sp}_{18}$	18	9
$\mathfrak{so}_{18}$	18	9
$\mathfrak{sl}_{18}$	18	17

Proof. It suffices to classify all small (of dimension at most 27) representations of the simple Lie groups; one may (and we did) compute these using the Weyl character formula. One can also deduce this result immediately for groups of rank ≤ 8 by consulting [**?**, Table 1, p.25–83]. Groups of rank > 8 can also be handled directly by hand, but they also play no role in the argument below because we are always able to immediately dismiss any groups of rank $> 6 = \text{rank}(E_6)$. □

Let us now return to the possible cases in which our 27 dimensional Galois representation is reducible, and consider the corresponding monodromy groups.

Suppose there is a constituent of dimension 17. The rank must be bounded by 6. From Lemma 4.4, it follows that the Lie algebra of the monodromy group on this summand must be $\mathfrak{sl}_2$ or $\mathfrak{sl}_2 \times \mathfrak{t}$. But the rank of these algebras is at most 2, which violates Lemma 4.3. Suppose there is a constituent of dimension 26. Then by Lemma 4.3, the rank of the Lie algebra of this representation is exactly 6, and hence the rank of $\mathfrak{h}$ with $\mathfrak{g} = \mathfrak{h}$ or $\mathfrak{h} \times \mathfrak{t}$ is 6 or 5. Since there are no such groups of this rank with irreducible representations of dimension 26 by Lemma 4.4, we once more derive a contradiction. Hence the only remaining possibility is that the 27 dimensional representation decomposes into two irreducible pieces of dimensions 9 and 18, corresponding to a decomposition of weights $\Sigma = \Sigma_9 \cup \Sigma_{18}$. On the other hand, we know that the 18 dimensional representation must have a unipotent element with Jordan blocks of size $1 + 17$. The tensor product of two Jordan blocks of size m and n with $n \leq m$ decomposes into blocks of size $n+m-1$, $n+m-3, \ldots, n-m+1$. In particular, it must be the case that the 18-dimensional representation does not factor into a product of smaller dimensional representations, since otherwise there could not be a Jordan block of a unipotent element of size as larger as 17. Hence the monodromy group on this representation must have a simple Lie algebra (up to a torus). Again, we deduce from Lemma 4.4 and using rank considerations that $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{sl}_2 \times \mathfrak{t}$, once more contradicting Lemma 4.3.

We conclude:

Corollary 4.5. Let $\{r_\lambda \colon G_F \to \mathrm{GL}_{27}(\overline{M}_\lambda)\}_\lambda$ be the compatible system produced in Theorem 3.4. Then for all λ, the Zariski closure of the image of r_λ is isomorphic to $G|_{\overline{M}_\lambda}$ under the minuscule representation $r_{\min}$.

5. E_6 motives over CM fields

We conclude by showing that r_λ is a sub-representation of the cohomology of some smooth projective variety over F, thus completing the proof of Theorem 1.1. Note that, according to the Tate conjecture, we expect that such a compatible family should be cut out by correspondences over F, and thus arise from a motive M over F. We do not have any idea how to prove this. On the other hand, we do know that the compatible family r_λ becomes automorphic over a CM extension $H/F/\mathbf{Q}$, and (since we are in highly regular weight), using standard methods combined with the work of Shin [Shi11], one can associate a motive M over H whose associated p-adic Galois representations $\{r_\lambda|_{G_H}\}$ have monodromy group E_6. More precisely, we should say that one *expects* to be able to associate such a motive where the correspondences cutting out M arise from Hecke operators. In practice, we take a shortcut and deduce from [Shi11] the weaker claim that the Galois representations over G_H (and thus over G_F by restriction of scalars) came from cohomology. We apologize for the omission and leave it as an exercise to the more responsible reader to write down the correct argument.

To set up all the required notation would be quite cumbersome, so we will simply use the notation of [Shi11], giving precise references to where the relevant terms are defined. We hope that a reader with a copy of [Shi11] at hand can easily follow this argument. Fix an isomorphism $\iota_l \colon \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$; it is implicit in all of the constructions of [Shi11]. By Theorem 3.4, there is a CM extension H/F and a cuspidal automorphic representation Π^0 of $\mathrm{GL}_{27}(\mathbf{A}_H)$ such that

- $(\Pi^0)^\vee \cong (\Pi^0)^c$.

- $R_l(\Pi^0) \cong r_\lambda|_{G_H} \cdot \epsilon^{\frac{1-n}{2}}$, in the notation of [Shi11, Theorem 7.5]. (Note that n is odd, and we have taken into account a standard twist in the association of Galois representations to cohomological automorphic representations, as in [Shi11]; the relation between the conjugate self-dualities on the automorphic and Galois sides is explicitly stated in [BLGGT14, Theorem 2.1.1]. Of course it will suffice to show this Tate twist of r_λ appears in cohomology.)
- $[H^+ : \mathbf{Q}] \geq 2$, and H contains a quadratic imaginary field (we can simply enlarge an initial choice of H to ensure these conditions).

Set $n = 27$, for ease of reference to [Shi11]; we will recall the construction of $R_l(\Pi^0)$ and see as a result that after some further base-change that there is an explicit description of this Galois representation in the cohomology of a unitary similitude group Shimura variety. We begin with two reductions. Let E be an imaginary quadratic field not contained in H satisfying the four bulleted conditions in Step (II) of the proof of [Shi11, Theorem 7.5]. Replace H by HE and Π^0 by $\mathrm{BC}_{HE/H}(\Pi^0)$. Then *having made this replacement* the triple (E, H, Π^0) satisfies the six bulleted conditions at the beginning of Step (I) of the proof of [Shi11, Theorem 7.5]. Let H' be an imaginary quadratic extension of H^+ satisfying the three bulleted conditions (defining the set denoted $\mathcal{F}(H)$ — but note our H is Shin's F) in Step (I) of [Shi11, Theorem 7.5]. Then replace H by HH' and Π^0 by $\mathrm{BC}_{HH'/H}(\Pi^0)$. Again having made this replacement, Proposition 7.4 of [Shi11] now applies to the triple (E, H, Π^0). There is a Hecke character ψ of $\mathbf{A}_E^\times/E^\times$ such that, setting $\Pi = \psi \otimes \Pi^0$ (an automorphic representation of the group $\mathbb{G}_n(\mathbf{A}) \cong \mathrm{GL}_1(\mathbf{A}_E) \times \mathrm{GL}_n(\mathbf{A}_H)$ of [Shi11, §3.1]), we have (in the notation of [Shi11, Corollary 6.8])

$$R_l(\Pi^0) := R'_l(\Pi) := \widetilde{R}'_l(\Pi) \otimes \mathrm{rec}_{l,\iota_l}(\psi^c)|_{G_H},$$

where (see [Shi11, 5.5, 6.23], and note that the group G no longer denotes E_6, but rather the unitary similitude group defined in [Shi11, §5.1]!)

$$C_G \cdot \widetilde{R}'_l(\Pi) = \sum_{\pi^\infty \in \mathcal{R}_l(\Pi)} R^{n-1}_{\xi,l}(\pi^\infty)^{\mathrm{ss}}.$$

In our case, the integer $C_G = \tau(G)\ker^1(\mathbf{Q}, G)$ (defined in [Shi11, Theorem 6.1]) is 2: this is explained in [Tay12, p. 411–412]. Moreover, $\widetilde{R}'_l(\Pi)$ is irreducible (since the image of $r_\lambda|_{\Gamma_H}$ is Zariski-dense in E_6), and $\mathcal{R}_l(\Pi)$ in fact contains at least two elements: following [Tay12, p. 413], there will be two automorphic representations of $G(\mathbf{A}_\mathbf{Q})$, differing by a twist but having isomorphic base-changes to $\mathrm{GL}_n(\mathbf{A}_H)$, that contribute to $\mathcal{R}_l(\Pi)$ (these are denoted $\widetilde{\pi}$ and $\widetilde{\pi} \otimes (\delta_{A/\mathbf{Q}} \circ \nu)$ in [Tay12, p. 413]). We fix one such $\widetilde{\pi}^\infty \in \mathcal{R}_l(\Pi)$. It follows that $\widetilde{R}'_l(\Pi) \cong R^{n-1}_{\xi,l}(\widetilde{\pi}^\infty)$ (no semisimplification necessary because these are irreducible representations).

Now recall the decomposition ([Shi11, 5.5])

$$H^{n-1}(\mathrm{Sh}, \mathcal{L}_\xi) = \bigoplus_{\pi^\infty} \pi^\infty \otimes R^{n-1}_{\xi,l}(\pi^\infty).$$

At some finite level U, we deduce that $\widetilde{R}'_l(\Pi)$ is contained in $H^{n-1}(\mathrm{Sh}_U \times_H \overline{H}, \mathcal{L}_\xi)$ (it is even a direct summand cut out by Hecke operators). Finally, letting $\mathcal{A}_U \to \mathrm{Sh}_U$ denote the universal abelian scheme (arising from the PEL moduli problem), and letting $\mathcal{A}_U^{(m)}$ denote its m-fold fiber product over Sh_U (for any integer $m \geq 1$), then

there are integers m_ξ and t_ξ (see [HT01, p. 98]) such that

$$H^{n-1}(\mathrm{Sh}_U \times_H \overline{H}; \mathcal{L}_\xi) \cong \varepsilon \cdot H^{n-1+m_\xi}(\mathcal{A}_U^{(m_\xi)} \times_H \overline{H}, \overline{\mathbf{Q}}_l)(t_\xi),$$

where ε is a suitable idempotent projector. We recall that our assumption $[H^+ : \mathbf{Q}] \geq 2$ implies that Sh_U, and therefore $\mathcal{A}_U^{(m_\xi)}$, are smooth projective varieties over H. Thus $(r_\lambda \cdot \epsilon^{\frac{1-n}{2}})|_{G_H} \otimes \mathrm{rec}_{l,\iota l}(\psi^c)^{-1}|_{G_H}$ is a sub-representation of the cohomology of the smooth projective variety $\mathcal{A}_U^{(m_\xi)}$ over H. Possibly replacing H by a finite extension, we can find a product of CM abelian varieties A/H such that $\mathrm{rec}_{l,\iota l}(\psi^c)|_{G_H}$ is a sub-representation of $H^i(A \times_H \overline{H}, \overline{\mathbf{Q}}_l)(j)$ for some integers i and j ([DMOS82, IV. Proposition D.1]). We conclude that $r_\lambda|_{G_H}$ is a sub-representation of $H^r(X \times_H \overline{H}, \overline{\mathbf{Q}}_l)(s)$ for some smooth projective variety X/H (namely, $X = A \times_H \mathcal{A}_U^{(m_\xi)}$). By Frobenius reciprocity and irreducibility of r_λ, r_λ (as G_F-representation) is a sub-representation of $\mathrm{Ind}_{G_H}^{G_F}\left(H^r(X \times_H \overline{H}, \overline{\mathbf{Q}}_l)(s)\right)$. This induction is just the cohomology of X regarded as a variety over F (i.e., via $X \to \mathrm{Spec}(H) \to \mathrm{Spec}(F)$), so the proof of Theorem 1.1 is complete.

Remark 5.1. One may reasonably ask whether there exist E_6 motives (or strongly compatible systems) over $\mathbf{Q}$. We do not know the answer. There is, however, a technical obstruction for applying the methods of this paper. To use automorphic methods, the Hodge–Tate weights of $r_{\min} \circ \rho_\lambda$ must be distinct. However, there cannot exist such a compatible system over $\mathbf{Q}$ (or any totally real field), since, by Corollary 5.4.3 of [BLGGT14], this would imply that the trace of complex conjugation representation must be ± 1, and, as the following lemma shows, there are no such involutions in $E_6(\mathbf{C}) \subset \mathrm{GL}_{27}(\mathbf{C})$.

Lemma 5.2. There are two conjugacy classes of non-trivial involutions in $E_6(\mathbf{C})$ with traces -5 and 3 respectively.

Proof. This lemma is "well known" in the sense that at least one of the authors of this paper claims to know it. In lieu of a reference, we give the following computational proof, which we explain in as conceptual terms as possible. Let $X_\bullet = X_\bullet(T)$ denote the cocharacter group of G, and $X^\bullet = X^\bullet(T)$ the character group. We may assume any involution lies in T and thus can be considered as an element of $X_\bullet \otimes \mu_2 \simeq (X^\bullet/2X^\bullet)^\vee$. We may identify the latter group with the $\mathbf{F}_2$-dual $V^\vee$ of $V = \Lambda/2\Lambda$. This is a quadratic space, thus self-dual, and so there is a W_G-equivariant isomorphism $V^\vee \simeq V$. There are $63 = 2^6 - 1$ non-zero elements of V. These are divide into two sets depending on whether the vectors are isotropic or not. The action of W_G correspondingly has two orbits of size 27 and 36 respectively. This is almost true by pure thought, since the orbits must have order greater than 2 (V is an irreducible W_G-representation) and the only non-trivial permutation representations of W_G of degree less than 40 and greater than 2 are of orders 27 and 36 respectively ([**?**]). In fact, one can identify these orbits of size 27 and 36 are precisely the images of the set of weights and positive roots, respectively. If μ is the highest weight vector corresponding to $r_{\min}$, let $W_{G,\mu}$ the stabilizer of μ in W_G. Since $r_{\min}$ is minuscule, the corresponding weights are precisely those of the form $w\mu$ for $w \in W_G/W_{G,\mu}$. Hence the trace of an involution corresponding to $v^\vee$ turns out to be equal to

$$\sum_{w \in W_G/W_{G,\mu}} (-1)^{\langle v^\vee, w\mu \rangle}$$

By computing this sum explicitly over all $v^\vee$, we see that the traces of the two non-trivial involutions in the minuscule representation are -5 and 3 for precisely 27 and 36 of the $v^\vee$ respectively. For example, for the first sum, we have already noted that the stabilizer W_μ acts on the roots with orbits of size 1, 10, and 16 respectively given explicitly before Lemma 4.2; the corresponding signs in the sum are $+1$, $+1$, and -1 respectively, and hence the sum is $1+10-16=-5$. The second sum can be computed directly from the first by averaging over the W_G action and then using the fact that, for any fixed $v^\vee$, one has

$$\sum_{v\in V}(-1)^{\langle v^\vee, v\rangle}=0.$$

Alternatively, if $W_{G,\alpha}$ is the stabilizer of the image of a root $\alpha\in V$ (and hence of index 36 in W_G), then the orbits of $W_{G,\alpha}$ on the weights are of size 15 and 12, and the corresponding signs are $+1$ and -1 and the trace is $15-12=+3$.

As a confirmation of these numbers, we can appeal to the following variation. There exists a cover (the extended Weyl group of Tits [**?**]) $\widetilde{W_G}$ of W_G by $V=(\mathbf{Z}/2\mathbf{Z})^6$ which lives as a subgroup of $E_6(\mathbf{C})$. The group has an explicit description as the $\mathbf{Z}$-points of the normalizer N of a (split) maximal torus T of G. The group $\widetilde{W_G}$ admits a 27 dimensional representation via the minuscule representation of E_6 which turns out to be absolutely irreducible. Then one can check (using `magma`) from the character table that the two conjugacy classes of involution in $V=\ker(\widetilde{W_G}\to W_G)$ (of orders 27 and 36 respectively) have traces -5 and 3 under both irreducible 27 dimensional representations of $\widetilde{W_G}$. □

6. Complements

We note the following application of Theorem 1.1 to the inverse Galois problem. Let us write $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ for the $\mathbf{F}_l$-points of the (simply connected) form of E_6 that we have been considering. The group $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ is the Schur cover of the simple Chevalley group of type E_6 (with Schur multiplier of order $(3,l-1)$), which we denote below by $E_6(\mathbf{F}_l)$. The groups $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ and $E_6^{\mathrm{sc}}(\mathbf{F}_l).2$ are known to occur as Galois groups over $\mathbf{Q}$ for $p\equiv 4,5,6,9,16,17 \mod 19$ (these are primes of order 9 in $\mathbf{F}_{19}^\times$) by [Mal88, Thm 2.3] as a consequence of the rigidity method. In contrast, we can prove that $E_6^{\mathrm{sc}}(\mathbf{F}_l).2$ is a Galois group over $\mathbf{Q}$ for a positive density of primes $\equiv 1 \mod 19$ (by taking $E=\mathbf{Q}(\zeta_{19})$ below).

Corollary 6.1. Let F be an imaginary quadratic field. Then, for a set S of primes l of positive density, there exists a number field $L/\mathbf{Q}$ containing F with $\mathrm{Gal}(L/\mathbf{Q})={}^LG(\mathbf{F}_l)=E_6^{\mathrm{sc}}(\mathbf{F}_l).2$ and $\mathrm{Gal}(L/F)=E_6^{\mathrm{sc}}(\mathbf{F}_l)$. Moreover, one may assume that all the primes in S split completely in any finite extension $E/\mathbf{Q}$.

Proof. We combine the previous result with Theorem 3.17 of [Lar95], to deduce that $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ is the Galois group of the kernel of ρ_λ over F for a relative density one set of primes l which split completely in the coefficient field M of the compatible system. In particular, the intersection of S with the set of primes split completely in $E/\mathbf{Q}$ has positive density. Since the representation $\overline{r}_\lambda: G_F\to \mathrm{GL}_{27}(\mathbf{F}_l)$ is, by construction, conjugate self-dual, it extends to a representation of $G_{\mathbf{Q}}$ to $\mathcal{G}_{27}(\mathbf{F}_l)$ whose kernel therefore has Galois group ${}^LG(\mathbf{F}_l)$. □

Remark 6.2. Note that Corollary 6.1 remains true if one replaces $E_6^{\mathrm{sc}}(\mathbf{F}_l).2$ and $E_6^{\mathrm{sc}}(\mathbf{F}_l)$ by $E_6(\mathbf{F}_l).2$ and $E_6(\mathbf{F}_l)$, for the obvious reason that the latter groups are quotients of the former groups.

6.1. **Acknowledgments.** This work traces its origins to a discussion following a number theory seminar given by one of us (S.P.) at the University of Chicago in February 2016, where all the authors of this paper were in attendance. We would like to thank Sug Woo Shin for answering questions related to his paper [Shi11] relevant for the discussion in §5. Finally, we would like to thank the referee for a number of useful suggestions which helped to improve this manuscript.

G.B. was supported in part by NSF postdoctoral fellowship DMS-1503047, F.C. was supported in part by NSF Grant DMS-1701703, M.E. was supported in part by NSF Grant DMS-1601871, K.M.P. was supported in part by NSF Grant DMS-1803623, S.P. was supported in part by NSF Grant DMS-1700759.

References

[AH16] Jeffrey Adams and Xuhua He, *Lifting of elements of Weyl groups*, 2016.

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609. MR 3152941

[Bou68] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[Car12] Ana Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413. MR 2972460

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR 2470687 (2010j:11082)

[CM93] David H. Collingwood and William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin, 1982. MR 654325 (84m:14046)

[DR10] Michael Dettweiler and Stefan Reiter, *Rigid local systems and motives of type G_2*, Compos. Math. **146** (2010), no. 4, 929–963, With an appendix by Michael Dettweiler and Nicholas M. Katz. MR 2660679

[FM95] Jean-Marc Fontaine and Barry Mazur, *Geometric Galois representations*, Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995, pp. 41–78. MR 1363495

[Gro00] Benedict H. Gross, *On minuscule representations and the principal* SL_2, Represent. Theory **4** (2000), 225–244 (electronic). MR 1795753

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR MR1876802 (2002m:11050)

[Kis08] Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546. MR 2373358

[KW09a] Chandrashekhar Khare and Jean-Pierre Wintenberger, *Serre's modularity conjecture. I*, Invent. Math. **178** (2009), no. 3, 485–504. MR 2551763

[KW09b] ———, *Serre's modularity conjecture. II*, Invent. Math. **178** (2009), no. 3, 505–586. MR 2551764

[Lar95] M. Larsen, *Maximality of Galois actions for compatible systems*, Duke Math. J. **80** (1995), no. 3, 601–630. MR 1370110

[LP92] M. Larsen and R. Pink, *On l-independence of algebraic monodromy groups in compatible systems of representations*, Invent. Math. **107** (1992), no. 3, 603–636. MR 1150604 (93h:22031)

[Lur01] Jacob Lurie, *On simply laced Lie algebras and their minuscule representations*, Comment. Math. Helv. **76** (2001), no. 3, 515–575. MR 1854697

[Mal88] Gunter Malle, *Exceptional groups of Lie type as Galois groups*, J. Reine Angew. Math. **392** (1988), 70–109. MR 965058

[NSW00] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2000. MR MR1737196 (2000j:11168)

[Pat16] Stefan Patrikis, *Deformations of Galois representations and exceptional monodromy*, Inventiones mathematicae **205** (2016), no. 2, 269–336.

[Pat17] ———, *Deformations of Galois representations and exceptional monodromy, II: raising the level*, Math. Ann. **368** (2017), no. 3-4, 1465–1491. MR 3673661

[Ram02] Ravi Ramakrishna, *Deforming Galois representations and the conjectures of Serre and Fontaine-Mazur*, Ann. of Math. (2) **156** (2002), no. 1, 115–154. MR 1935843

[Rob17] David Roberts, *Newforms with rational coefficients*, to appear in the Ramanujan Journal, 2017.

[Sch90] A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), no. 2, 419–430. MR 1047142

[Ser94] Jean-Pierre Serre, *Propriétés conjecturales des groupes de Galois motiviques et des représentations l-adiques*, Motives (Seattle, WA, 1991) (1994).

[Ser08] ———, *Topics in Galois theory*, second ed., Research Notes in Mathematics, vol. 1, A K Peters, Ltd., Wellesley, MA, 2008, With notes by Henri Darmon. MR 2363329

[Shi11] Sug Woo Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741. MR 2800722

[Tay12] Richard Taylor, *The image of complex conjugation in l-adic representations associated to automorphic forms*, Algebra Number Theory **6** (2012), no. 3, 405–435. MR 2966704

[TY07] Richard Taylor and Teruyoshi Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493. MR 2276777

[Yun14] Zhiwei Yun, *Motives with exceptional Galois groups and the inverse Galois problem*, Invent. Math. **196** (2014), no. 2, 267–337. MR 3193750

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