KAUFFMAN STATES, BORDERED ALGEBRAS, AND A BIGRADED KNOT INVARIANT

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Abstract. We define and study a bigraded knot invariant whose Euler characteristic is the Alexander polynomial, closely connected to knot Floer homology. The invariant is the homology of a chain complex whose generators correspond to Kauffman states for a knot diagram. The definition uses decompositions of knot diagrams: to a collection of points on the line, we associate a differential graded algebra; to a partial knot diagram, we associate modules over the algebra. The knot invariant is obtained from these modules by an appropriate tensor product.

1. Introduction

The Alexander polynomial can be given a state sum formulation as a count of certain Kauffman states, each of which contributes a monomial in a formal variable t [6]. In [22], this description was lifted to knot Floer homology [23, 25]: knot Floer homology is given as the homology of a chain complex whose generators correspond to Kauffman states. The differentials in the complex, though, were not understood explicitly; they were given as counts of pseudo-holomorphic curves. A much larger model for knot Floer homology was described in [14], where the generators correspond to certain states in a grid diagram, and whose differentials count certain embedded rectangles in the torus. The grid diagram can be used to compute invariants for small knots [1, 4], but computations are limited by the size of the chain complex (which has $n!$ many generators for a grid diagram of size n).

The aim of this article is to construct and study an invariant of knots, $H^-(K)$, with the following properties.

(H-1) Letting $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, $H^-(K)$ is a bigraded module over the polynomial algebra $\mathbb{F}[U]$. That is, there is a vector space splitting $H^-(K) \cong \oplus_{d,s} H^-_d(K,s)$, and an endomorphism U of $H^-(K)$ with $U\colon H^-_d(K,s) \to H^-_{d-2}(K,s-1)$.
(H-2) If $\mathcal{D}$ is a diagram for K, with a marked edge, then $H^-(K)$ is obtained as the homology of a chain complex $C^-(\mathcal{D})$ associated to the diagram.
(H-3) The complex $C^-(\mathcal{D})$ is a bigraded chain complex over $\mathbb{F}[U]$, which is freely generated by the Kauffman states of $\mathcal{D}$.

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(H-4) The graded Euler characteristic of $H^-(K)$ is related to the symmetrized Alexander polynomial $\Delta_K(t)$ of the knot K, as follows: there is an identification of Laurent series in $\mathbb{Z}[t,t^{-1}]\!]$

$$\sum_{d,s}(-1)^d \dim H^-_d(K,s)t^s = \frac{\Delta_K(t)}{(1-t^{-1})}. \tag{1.1}$$

From its description, this invariant comes equipped with a great deal of algebraic structure, similar to the Khovanov and Khovanov-Rozansky categorifications of the Jones polynomial and its generalizations. The structure also makes computations of the invariant for large examples feasible. In this paper, we also give an algebraic proof of invariance, hence giving a self-contained treatment of this invariant.

Building on the present work, in [19], we generalize the constructions to define an invariant with more algebraic structure. In [17], we relate the constructions here and their generalizations from [19] with pseudo-holomorphic curve counting, to give an identification between these algebraically-defined invariants and a suitable variant of knot Floer homology [17]. See [21] for an expository paper overviewing this materal. A generalization of these constructions to links is given in [18].

1.1. **Decomposing knot diagrams.** The knot invariant $H^-(K)$ is constructed by decomposing a knot projection for K into elementary pieces, and using those pieces to put together a chain complex whose homology is $H^-(K)$.

In a little more detail, a *decorated knot diagram* $\mathcal{D}$ for K is an oriented, generic knot projection of K onto $\mathbb{R}^2$, together with a choice of a distinguished edge, which meets the infinite region. The projection gives a planar graph G whose vertices correspond to the double-points of the projection of K. Since G is four-valent, there are four distinct quadrants (bounded by edges) emanating from each vertex, each of which is a corner of the closure of some region of $\mathbb{R}^2\setminus G$. Let m denote the number of vertices of G. Clearly, G divides $\mathbb{R}^2$ into $m+2$ regions, one of which is the unbounded one.

Definition 1.1. *A* Kauffman state *(cf.* [6]*; see also Figure 1) for a decorated knot projection of K is a map $\mathbf{K}$ that associates to each vertex of G one of the four in-coming quadrants, subject to the following constraints:*

- *The quadrants assigned by $\mathbf{K}$ to distinct vertices are subsets of distinct bounded regions in $\mathbb{R}^2\setminus G$.*
- *The quadrants of the bounded region that meets the distinguished edge are not assigned by $\mathbf{K}$ to any of the vertices in G.*

Each Kauffman state sets up a one-to-one correspondence between vertices of G and the connected components of $\mathbb{R}^2\setminus G$ that do not meet the distinguished edge.

In [6], Kauffman describes the Alexander polynomial of a knot as a sum of monomials in t associated to every Kauffman state. We recall this description (with slight modifications to suit our purposes).

Definition 1.2. *Label the four quadrants around each crossing with 0, and $\pm\frac{1}{2}$, according to the orientations as specified in the second column of Figure 2. The* Alexander function *of a Kauffman state $\mathbf{K}$, $A(\mathbf{K})$, is the sum, over each crossing, of the contribution of the quadrant occupied by the state. The* Maslov function *of a*

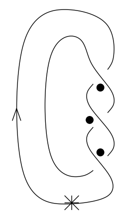

FIGURE 1. **Decorated knot projection for the left-handed trefoil.** The distinguished edge is marked with a star. We have illustrated one of the three Kauffman states for this projection.

Kauffman state $\mathbf{K}$ *is obtained similarly, with local contributions as specified in the third column of Figure 2.*

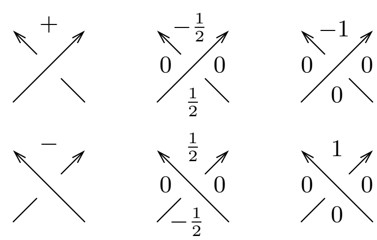

FIGURE 2. **Sign conventions for crossings, and local Alexander and Maslov contributions.** The first column illustrates the chirality of a crossing; the second the Alexander contribution of each quadrant; the third the Maslov contribution.

Let $\mathfrak{S} = \mathfrak{S}(\mathcal{D})$ denote the set of Kauffman states. Kauffman shows that the Alexander polynomial is computed by $\Delta_K(t) = \sum_{\mathbf{K}\in\mathfrak{S}}(-1)^{M(\mathbf{K})}t^{A(\mathbf{K})}$. (Kauffman's description is slightly different; he does not define the integer $M(\mathbf{K})$, only its parity, which suffices to compute the Alexander polynomial.)

In [22], we gave a description of knot Floer homology as the homology groups of a chain complex whose generators are Kauffman states, with bigradings given by the M and A functions defined above, and whose differential counts pseudo-holomorphic disks. In certain special cases (for example, for alternating knot diagrams, after a possible change of basis), these differentials could be computed explicitly; but in general, their computation remained elusive (compare [14]).

In the present paper, we define $H^-(K)$, which is the homology of a chain complex $C^-(\mathcal{D})$ associated to a diagram. Its generators are Kauffman states, and its differential is described algebraically. The construction will involve decompositions of the knot diagram $\mathcal{D}$, as follows.

Given a generic, oriented knot projection in the xy-plane, we will consider the intersection of $\mathcal{D}$ with half-planes $y \geq t$ and $y \leq t$, for generic values of t. Intersections with $y \geq t$ are called *upper partial knot diagrams*, and intersections with $y \leq t$ are called *lower partial knot diagrams*. The decorated edge of $\mathcal{D}$ will contain one of the points with minimal y value. See Figure 3.

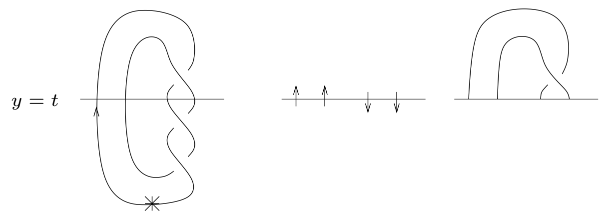

FIGURE 3. **Slicing the trefoil diagram.** The $y = t$ slice for the diagram on the left, gives the line with 4 marked points (and orientations) in the middle. The $y \geq t$ upper diagram is on the right.

Our knot invariant will be constructed out of invariants for upper and lower knot diagrams. In the spirit of bordered Floer homology [9], we will associate the following algebraic objects to this picture:

- an algebra associated to the intersection of the knot diagram $\mathcal{D}$ with generic lines $y = t$.
- a "type D structure" (in the sense of [9], recalled here in Section 2.4) associated to each (generic) upper diagram.
- a right $\mathcal{A}_\infty$-module associated to each (generic) lower diagram.

Given this data, the complex $C^-(\mathcal{D})$ is obtained, for generic t, as the tensor product (in the sense of [9], recalled here in Section 2.5) of the invariant associated to the $y \geq t$ upper diagram with the $y \leq t$ lower diagram, where the pairing is taken over the algebra associated to $y = t$ slice.

In more detail, the intersection of the knot diagram $\mathcal{D}$ with a generic horizontal line $y = t$ can be encoded as a line equipped with $2n$ points $p_1, \dots, p_{2n}$ (i.e. where the knot meets the slice), half of which are oriented upwards, and half of which are oriented downwards. Let $\mathcal{S}$ denote the subset of those points oriented upwards. We will define an algebra associated to this configuration. The points subdivide the line into $2n - 1$ bounded intervals, and two unbounded ones. There are certain distinguished *basic idempotents* in the algebra which correspond to choices of n of those bounded intervals. These algebras are constructed in Section 3.

We will describe certain natural modules over this algebra in terms of the following:

Definition 1.3. *An* upper Kauffman state *for an upper diagram $y \geq t$ is a pair $(\mathbf{K}, I)$ where*

- $\mathbf{K}$ *associates to each crossing in the upper diagram one of the four adjacent quadrants.*
- I *is a basic idempotent for the $y = t$ algebra.*

Moreover, these data are required to satisfy the compatibility conditions:

- *The quadrants assigned by $\mathbf{K}$ to distinct vertices are subsets of distinct bounded regions in the upper diagram $y \geq t$.*
- *The unbounded region meets none of the intervals in the idempotent I.*
- *Each bounded region in the upper diagram $y \geq t$ either:*

– *contains a quadrant associated to some vertex by* $\mathbf{K}$*, and meets none of the intervals in the idempotent* I*; or*
– *meets exactly one of the intervals in the idempotent* I*, and is not associated to any vertex by* $\mathbf{K}$*.*

It is easy to see that any Kauffman state can be restricted canonically to give an upper Kauffman state on any of its upper diagrams; whereas upper Kauffman states on an upper diagram might not extend to give a global Kauffman state.

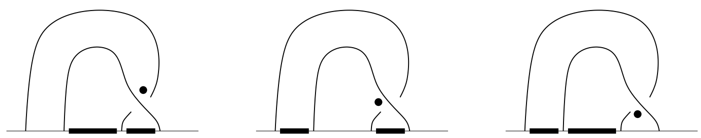

FIGURE 4. **Upper Kauffman states.** Here are all three upper Kauffman states for the upper diagram obtained from the trefoil diagram in Figure 3, with idempotent I represented by the pairs of darkened intervals.

The type D structure of an upper diagram is generated by its upper Kauffman states.

We will also describe bimodules for enlarging the upper knot diagram. Specifically, we will associate bimodules to crossings, caps, and cups, in Sections 5, 8, and 9 respectively. These bimodules are generated by certain localized Kauffman states, and the type D structure of an upper diagram is built as a tensor product of the bimodules associated to these various pieces. Tensoring together all the pieces gives the desired chain complex $C^-(\mathcal{D})$ whose generators are Kauffman states.

The topological invariance of $H^-(K)$ is established in Theorem 11.5. The proof proceeds by showing that $H^-(\mathcal{D})$ is invariant under planar isotopies of the knot diagram, and then locally verifying invariance under the three Reidemeister moves. In fact, invariance under Reidemeister moves 2 and 3 are part of the "braid relations" that the crossing bimodules satisfy (see Section 6), while Reidemeister 1 invariance is an easy computation.

The complex $C^-(\mathcal{D})$ is a chain complex over the polynomial ring in one generator U. We can set $U=0$ to get another chain complex $\widehat{C}(\mathcal{D})$ whose homology $\widehat{H}(K)$ is also a knot invariant.

Knot Floer homology [23] is an invariant with the above properties, according to [22]. The differentials appearing in the original definition of knot Floer homology involve analytical choices; algebraic constructions over a larger base ring were given in [15] and [2], and a chain complex with many more generators was given in [14]. The invariant $\widehat{H}(K)$ corresponds to the version of knot Floer homology denoted $\widehat{\mathrm{HFK}}(K)$ and $H^-(K)$ corresponds to another version of knot Floer homology, denoted $\mathrm{HFK}^-(K)$. In [17], we verify that $\widehat{H}(K)$ and $H^-(K)$ coincides with knot Floer homology $\widehat{\mathrm{HFK}}(K)$ and $\mathrm{HFK}^-(K)$.

The methods of this paper are conceptually similar to the computation of Heegaard Floer homology groups of three-manifolds by factoring mapping classes [11]; but the present constructions are ultimately algebraic in nature, as is the invariance proof we give here; compare [28]. Although the bimodules we write down here might seem *ad hoc* in nature; the holomorphic theory did give us guiding principles for what to look for; see Section 4.4. (See also [17])

1.2. **Organization.** In Section 2 we discuss the algebraic preliminaries, based on the algebra from of [12], using throughout the notions of bimodules of various types (DD, DA, and AA). In Section 3 we describe the algebras associated to knot diagram slices, along with a canonical (invertible) dualizing bimodule of type DD. In Section 4, we associate natural type DD bimodules to crossings, which are simple to describe. In Section 5 we construct the corresponding type DA bimodules, which are most useful to work with. These bimodules induce a braid group action on the category of modules over our algebras, as verified in Section 6. In Section 7, we construct the type DD bimodules associated to a critical point in the knot diagram, and verify the "trident relation", which describes how these bimodules interact with nearby crossing bimodules. The critical point bimodules have two kinds of corresponding type DA bimodules: the bimodule associated to a maximum, constructed in Section 8, and the bimodule associated to a minimum, constructed in Section 9. The theory is equipped with several symmetries, collected in Section 10. In Section 11, we construct the knot invariant from the constitutent bimodules, and verify its invariance properties. In Section 12 we verify a few basic properties of this invariant.

A bordered theory for tangles in the grid context was developed by Ina Petkova and Vera Vértesi [24]; compare also [10]. In a different direction, Rumen Zarev [27] constructed a "bordered sutured" invariant that can also be used to study knot Floer homology. Explicit computations of $\widehat{H}(K)$, for three-stranded pretzel knots, were done by Andrew Manion. A version of this construction for singular knots is also studied by Manion [13].

In [19], we give a slight variation on the present construction, together with a sign refinement. The techniques from that paper lead to efficient computations of $H^-(K)$ for large knots [20]. Similar to fast computations for Khovanov homology [3] for non-alternating projections, there are various cancelling differentials in the invariants associated to partial knot diagrams that allow for fast computer calculation. Details will be explained in [19]. As an illustration, we took the Gauss codes (which specify a knot up to reflection) for the knot K_3 from [5], to obtain a 91-crossing presentation of K. The Poincaré polynomial of $\widehat{H}(K)$, $P_K(m,t)=\sum_{d,s}\dim \widehat{H}_d(K,s)m^d t^s$, is given by

$$
\begin{aligned}
&2t^4m^6+t^3(7m^5+3m^3+m)+t^2(10m^4+m^3+16m^2+m+3)\\
&\quad+t(7m^3+2m^2+39m+2+3m^{-1})+(4m^2+2m+53+2m^{-1}+2m^{-2})\\
&\quad+t^{-1}(7m+2+39m^{-1}+2m^{-2}+3m^{-3})+t^{-2}(10+m^{-1}+16m^{-2}+m^{-3}+3m^{-4})\\
&\quad+t^{-3}(7m^{-1}+3m^{-3}+m^{-5})+2t^{-4}m^{-2}.
\end{aligned}
$$

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2. Algebraic preliminaries

We recall some algebraic preliminaries from bordered Floer homology. Further background on $\mathcal{A}_\infty$ algebras can be found in [7]. Most of this material (except Section 2.9) can be found, with more detail, in [12].

2.1. **Algebras.**

In this paper, we will be concerned with differential graded algebras $\mathcal{A}$ (DG algebras) in characteristic 2.

The DG algebra $\mathcal{A}$ is an abelian group equipped with a differential $\mu_1\colon \mathcal{A}\to\mathcal{A}$, and a multiplication map $\mu_2\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$, satisfying the usual compatibility conditions

$$\mu_1^2 = 0;$$
$$\mu_2\circ(\mu_1\otimes \mathrm{Id}+\mathrm{Id}\otimes\mu_1) = \mu_1\circ\mu_2;$$
$$\mu_2\circ(\mu_2\otimes \mathrm{Id}) = \mu_2\circ(\mathrm{Id}\otimes\mu_2).$$

(The latter two are the Leibniz rule and associativity rule respectively.) It is customary to abbreviate $\mu_1(a)$ by da and $\mu_2(a\otimes b)$ by $a\cdot b$.)

Our algebras are strictly unital; i.e. they are equipped with a distinguished multiplicative unit 1 which is a cycle. We will typically think of our algebras as defined over a ground ring $\mathbf{k}$, which in turn is a direct sum of finitely many copies of $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$, equipped with a vanishing differential. This means that there is a distinguished subalgebra of $\mathcal{A}$, identified with $\mathbf{k}$, whose unit 1 is also the unit in $\mathcal{A}$. Moreover, $\mathcal{A}$ as a bimodule over $\mathbf{k}$, μ_1 as a bimodule homomorphism, and $\mu_2\colon \mathcal{A}\otimes_{\mathbf{k}}\mathcal{A}\to\mathcal{A}$ is a bimodule homomorphism. (Our algebras will be typically not finitely generated over the ground ring $\mathbf{k}$.)

2.2. **Gradings.**

Our algebras will be equipped with gradings, a *Maslov grading*, which takes values in $\mathbb{Z}$, and an *Alexander multi-grading* which takes values in some Abelian group $\Lambda=\Lambda_{\mathcal{A}}$, compatible with the algebra actions as follows.

The algebra $\mathcal{A}$ is equipped with a direct sum splitting $\mathcal{A}=\bigoplus_{(d;\ell)\in\mathbb{Z}\oplus\Lambda}\mathcal{A}_{d;\ell}$. A non-zero element $a\in\mathcal{A}_{d;\ell}$ is called *homogeneous with grading* $(d;\ell)$; or simply *homogeneous with respect to the grading by* $\mathbb{Z}\oplus\Lambda$, when we do not wish to specify its actual grading. Similarly, if $a\in\bigoplus_{d\in\mathbb{Z}}\mathcal{A}_{d;\ell}$ for some fixed $\ell\in\Lambda$, we say that a is Λ-homogeneous (with grading ℓ). We require

$$\mu_1\colon \mathcal{A}_{d;\ell}\to\mathcal{A}_{d-1;\ell} \qquad \mu_2\colon \mathcal{A}_{d_1;\ell_1}\otimes\mathcal{A}_{d_2;\ell_2}\to\mathcal{A}_{d_1+d_2;\ell_1+\ell_2}.$$

In our present applications, the group $\Lambda_{\mathcal{A}}$ is $(\frac{1}{2}\mathbb{Z})^m\subset\mathbb{Q}^m$.

The algebras considered here will satisfy the following further condition:

Definition 2.1. *We say that the Alexander multi-grading on $\mathcal{A}$ is* positive over $\mathbf{k}$ *if the following two conditions hold:*

- $\mathbf{k}$ *is the set of algebra elements in $\mathcal{A}$ with Alexander multi-grading* 0.
- *For $a_1,\dots,a_\ell$ with Alexander multi-gradings $\lambda_1,\dots,\lambda_\ell$, if $\sum_{i=1}^{\ell}\lambda_i=0$, then each $\lambda_i=0$.*

In fact, in this paper $\mathbf{k}$ will consist of homogeneous elements with bigrading $(0;0)$.

2.3. **Modules.** We will consider several kinds of modules over our algebras. A *right differential module* over $\mathcal{A}$ is a right $\mathbf{k}$-module M, equipped with maps $m_1\colon M \to M$ and $m_2\colon M\otimes_{\mathbf{k}}\mathcal{A}\to M$ satisfying

$$\begin{aligned} m_1^2 &= 0; \\ m_2\circ(m_1\otimes \mathrm{Id}+\mathrm{Id}\otimes\mu_1) &= m_1\circ m_2; \\ m_2\circ(m_2\otimes\mathrm{Id}) &= m_2\circ(\mathrm{Id}\otimes\mu_2). \end{aligned}$$

We will consider modules that are *strictly unital*, meaning that $m_2(x,1)=x$ for all $x\in M$. In this case, m_2 is a right $\mathbf{k}$-module map $m_2\colon M\otimes_{\mathbf{k}}\mathcal{A}\to M$.

Weakening associativity, one naturally arrives at the notion of an $\mathcal{A}_\infty$ module M. A *right* $\mathcal{A}_\infty$ *module* over $\mathcal{A}$ is a right $\mathbf{k}$-module M, equipped with maps

$$m_i\colon M\otimes_{\mathbf{k}}\overbrace{\mathcal{A}\otimes_{\mathbf{k}}\cdots\otimes_{\mathbf{k}}\mathcal{A}}^{i-1}\to M$$

for $i\geq 1$, satisfying the following strict unitality conditions:

(1) $m_2(x\otimes 1)=x$ for all $x\in M$
(2) $m_i(x\otimes a_1\otimes\cdots\otimes a_{i-1})=0$ if $i>2$ and there is some $1\leq j\leq i-1$ with $a_j=1$;

and a compatibility condition which is perhaps best phrased in terms of the bar construction. (See Equation (2.1) below.) Define $\mathcal{T}^*(\mathcal{A})=\bigoplus_{i=0}^{\infty}\overbrace{\mathcal{A}\otimes_{\mathbf{k}}\cdots\otimes_{\mathbf{k}}\mathcal{A}}^{i}$, with the convention that the 0^{th} tensor product of $\mathcal{A}$ is $\mathbf{k}$, which is a chain complex, with a differential induced by μ_1 and μ_2; i.e.

$$\begin{aligned} \underline{d}(a_1\otimes\cdots\otimes a_i) &= \sum_{j=1}^{i} a_1\otimes\cdots\otimes a_{j-1}\otimes\mu_1(a_j)\otimes a_{j+1}\otimes\cdots\otimes a_i \\ &+\sum_{j=1}^{i-1} a_1\otimes\cdots\otimes a_{j-1}\otimes\mu_2(a_j\otimes a_{j+1})\otimes\cdots\otimes a_i, \end{aligned}$$

with the understanding that $\underline{d}$ vanishes on $\mathcal{T}^0(\mathcal{A})$. Consider $\mathcal{T}^*(M)=M\otimes_{\mathbf{k}}\mathcal{T}^*(\mathcal{A})$. The maps m_i $(i\geq 1)$ induce a map $\underline{m}\colon\mathcal{T}^*(M)\to\mathcal{T}^*(M)$ by the formula

$$\underline{m}(x\otimes a_1\otimes\cdots\otimes a_i)=\sum_{j=0}^{i} m_{j+1}(x\otimes a_1\otimes\cdots\otimes a_j)\otimes a_{j+1}\otimes\cdots\otimes a_i$$

The compatibility condition is equivalent to the condition that

$$\underline{m}(\underline{m}(x\otimes\underline{a}))+\underline{m}(x\otimes\underline{d}(a))=0; \qquad (2.1)$$

i.e. the map $\underline{d}_M\colon\mathcal{T}^*(M)\to\mathcal{T}^*(M)$ given by $\underline{d}_M(x\otimes\underline{a})=\underline{m}(x\otimes\underline{a})+x\otimes\underline{d}(\underline{a})$ is a differential.

Left differential modules and $\mathcal{A}_\infty$ modules are defined analogously (though in adherence with the conventions laid down in [9], our $\mathcal{A}_\infty$ modules will typically be right modules). A differential module is an $\mathcal{A}_\infty$ module with $m_i=0$ for all $i\geq 3$.

Gradings are as follows. The modules M we consider will typically have a Maslov grading by $\mathbb{Z}$, and a further Alexander multi-grading set S, which is a set S with an action by $\Lambda_{\mathcal{A}}$. That is, there is a direct sum splitting (as $\mathbf{k}$-modules) $M = \bigoplus_{d\in\mathbb{Z},s\in S} M_{d;s}$ and we assume that each summand $M_{d;s}$ is a finitely generated $\mathbf{k}$-module. The actions m_i will be graded as follows.

$$m_i \colon M_{d_0;s} \otimes \mathcal{A}_{d_1;\ell_1} \otimes \dots \mathcal{A}_{d_{i-1};\ell_{i-1}} \to M_{i-2+\sum_{j=0}^{i-1} d_j;s+\sum_{j=1}^{i-1}\ell_j}.$$

We will typically record the algebra $\mathcal{A}$ as a subscript for the $\mathcal{A}_\infty$-module M, writing $M_{\mathcal{A}}$ if M is a right $\mathcal{A}_\infty$-module, and ${}_{\mathcal{A}}M$ if M is a left $\mathcal{A}_\infty$-module.

Given two $\mathcal{A}_\infty$ modules $M_{\mathcal{A}}$ and $N_{\mathcal{A}}$, a *morphism* from $M_{\mathcal{A}}$ to $N_{\mathcal{A}}$ is a sequence of $\mathbf{k}$-module maps $\{\phi_i \colon M_{\mathcal{A}} \otimes_{\mathbf{k}} \overbrace{\mathcal{A}\otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} \mathcal{A}}^{i-1} \to N_{\mathcal{A}}\}_{i\geq 1}$. A morphism naturally induces a map $\underline{\phi} \colon M_{\mathcal{A}} \otimes \mathcal{T}^*(\mathcal{A}) \to N_{\mathcal{A}} \otimes \mathcal{T}^*(\mathcal{A})$ by

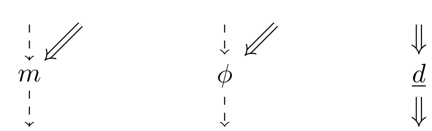
$$\underline{\phi}(x \otimes a_1 \otimes \dots \otimes a_n) = \sum_{i=0}^{n} \phi_{i+1}(x \otimes a_1 \otimes \dots \otimes a_i) \otimes a_{i+1} \otimes \dots \otimes a_n.$$

When this induced map is a chain map, we say that the morphism is a *homomorphism*. More generally, the space of morphisms can be given a differential, so that $d_{\mathrm{Mor}}(\underline{\phi}) = \underline{d}_N \circ \underline{\phi} + \underline{\phi} \circ \underline{d}_M$.

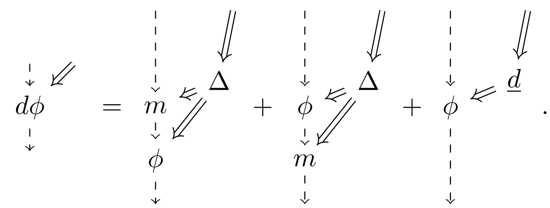
Let $\mathfrak{Mod}_{\mathcal{A}}$ resp. ${}_{\mathcal{A}}\mathfrak{Mod}$ denote the category of right resp. left $\mathcal{A}_\infty$ modules over $\mathcal{A}$. This is a differential category (so the morphism spaces are chain complexes). Specifically, given $M_{\mathcal{A}}, N_{\mathcal{A}} \in \mathfrak{Mod}_{\mathcal{A}}$, let $\mathrm{Mor}_{\mathcal{A}}(M,N)$ denote the chain complex whose elements are maps $\{\phi_i \colon M \otimes \mathcal{A}^{i-1} \to N\}_{i\geq 1}$, with differential given by

$$\begin{aligned}(d\phi)_i(x, a_1, \dots, a_{i-1}) &= \sum_{j=1}^{i} \phi_{i-j+1}(m_j^M(x, a_1, \dots, a_{j-1}), a_j, \dots, a_{i-1}) \\ &+ \sum_{j=1}^{i} m^N_{i-j+1}(\phi_j(x, a_1, \dots, a_{j-1}), a_j, \dots, a_{i-1}) \\ &+ \sum_{j=1}^{i-1} \phi_i(x, a_1, \dots, \mu_1(a_j), \dots, a_{i-1}) \\ &+ \sum_{j=1}^{i-2} \phi_{i-1}(x, a_1, \dots, \mu_2(a_j, a_{j+1}), \dots, a_{i-1}).\end{aligned}$$

There is a convenient graphical representation of formulas such as the one above. We represent elements of M by dashed arrows, elements of $\mathcal{T}^*(\mathcal{A})$ by doubled arrows, and various maps between them by labelled nodes. For instance, the map $m \colon M \otimes \mathcal{T}^*(\mathcal{A}) \to M$, the morphism $\phi \colon M \otimes \mathcal{T}^*(\mathcal{A}) \to N$, and the differential $\underline{d} \colon \mathcal{T}^*(\mathcal{A}) \to \mathcal{T}^*(\mathcal{A})$ are represented by the pictures



Using the map $\Delta\colon \mathcal{T}^*(\mathcal{A}) \to \mathcal{T}^*(\mathcal{A}) \otimes \mathcal{T}^*(\mathcal{A})$ defined by $\Delta(a_1 \otimes \cdots \otimes a_j) = \sum_{i=0}^{j}(a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_j)$, the differential of ϕ can be written:



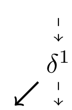
Let $M_{\mathcal{A}}$ be a right $\mathcal{A}_\infty$-module over $\mathbf{k}$, with a ($\mathbb{Z}$-valued) Maslov grading and an Alexander multi-grading with values in S. We can form the *opposite module* ${}_{\mathcal{A}}\overline{M}$, which is a space of maps from M to $\mathbb{F}$, also equipped with a Maslov grading and a grading by S,

$$ {}_{\mathcal{A}}\overline{M} = \bigoplus_{d \in \mathbb{Z}, s \in S} \mathrm{Hom}_{\mathbb{F}}(M_{-d;-s}, \mathbb{F}), \tag{2.2} $$

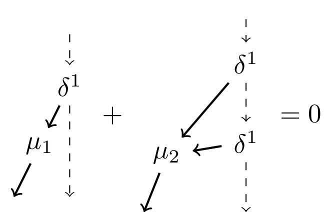
with action specified as follows. For fixed $\phi \in \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ and $a_1 \ldots, a_{i-1} \in \mathcal{A}$, let $\overline{m}_i(a_1 \otimes \ldots a_{i-1} \otimes \phi)$ be the homomorphism from M to $\mathbb{F}$ whose evaluation on x is given by $\phi(m_i(x \otimes a_{i-1} \otimes \cdots \otimes a_1))$. (Note that the tensor factors appearing here are over $\mathbb{Z}/2\mathbb{Z}$; when M is strictly unital, we can think of the tensor factors as taken over $\mathbf{k}$; with the understanding that in $a_{i-1} \otimes \cdots \otimes a_1$, the bimodule actions of $\mathbf{k}$ on the a_j are also opposites.)

2.4. **Type D structures.** A *left type D structure* over $\mathcal{A}$ is a left $\mathbf{k}$-module X, equipped with a $\mathbf{k}$-linear map $\delta^1\colon X \to \mathcal{A} \otimes_{\mathbf{k}} X$, satisfying the compatibility condition $(\mu_2 \otimes \mathrm{Id}_X) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathrm{Id}) \circ \delta^1 = 0$. A right type D structure is defined analogously.

The maps will be drawn



and the structure relation is drawn



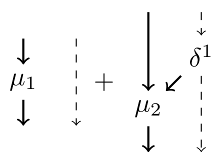
As in the case of modules, our type D structures will have a grading by $\mathbb{Z}$ (the Maslov grading), and an Alexander grading with set set $S = S_X$, a set with an action by $\Lambda_{\mathcal{A}}$. (In our case, this will typically be a quotient of $\Lambda_{\mathcal{A}}$.) Thus, X is given a direct sum splitting $X = \bigoplus_{d \in \mathbb{Z}; s \in S} X_{d;s}$, where each $X_{d;s}$ is a $\mathbf{k}$-module.

(In fact, in the cases of relevance to us, X will be finitely generated as a $\mathbf{k}$-module.) The actions respect these gradings, in the sense that

$$\delta^1\colon X_{d;s} \to \bigoplus_{d_0+d_1=d-1;\ell_0+s_1=s} \mathcal{A}_{d_0;\ell_0}\otimes X_{d_1;s_1}.$$

We abbreviate this, writing $\delta^1\colon X_{d;s}\to(\mathcal{A}\otimes X)_{d-1;s}$.

A left type D structure X induces a left differential module $\mathcal{A}\boxtimes X$ in a natural way. As a left $\mathcal{A}$-module, the space is $\mathcal{A}\otimes_{\mathbf{k}}X$; i.e. given $a\in\mathcal{A}$ and $b\otimes\mathbf{x}\in\mathcal{A}\otimes_{\mathbf{k}}X$, we define $m_2(a,b\otimes x)=\mu_2(a,b)\otimes x$. The operator $m_1=\mu_1\otimes\mathrm{Id}_X+(\mu_2\otimes\mathrm{Id}_X)\circ(\mathrm{Id}_{\mathcal{A}}\otimes\delta^1_X)$ which can be graphically represented by



induces differential on $\mathcal{A}\otimes_{\mathbf{k}}X=\mathcal{A}\boxtimes X$; i.e. $m_1^2=0$. It is easy to see that the differential and the $\mathcal{A}$-action satisfy a Leibniz rule; i.e. $\mathcal{A}\boxtimes X$ is a differential $\mathcal{A}$-module. (This is the special case of a more general construction: it is the tensor product of $\mathcal{A}$, viewed as a bimodule over itself, with the type D structure X.)

Conversely, let M be a left DG-module over $\mathcal{A}$ that splits as a left $\mathcal{A}$-module as a direct sum of modules that are isomorphic to left ideals in $\mathcal{A}$ by elements of $\mathbf{k}$; i.e.

$$M=\bigoplus_{g\in G}\mathcal{A}\cdot\mathbf{Y}(g) \tag{2.3}$$

for some finite set G, and a map $\mathbf{Y}\colon G\to\mathbf{k}$. There is a $\mathbf{k}$-submodule of M, $X=\bigoplus_{g\in G}\mathbf{k}\cdot\mathbf{Y}(g)$. Restricting the differential m_1 to X gives a map $\delta^1\colon X\to\mathcal{A}\otimes X=M$; the hypothesis that $m_1^2=0$ is equivalent to the condition that δ^1 determines a type D structure.

We will typically record the algebra as a superscript for a type D structure, writing ${}^{\mathcal{A}}X$ to denote a left type D structure X over $\mathcal{A}$. There is an analogous notion of right type D structures; for a right type D structure, we record the algebra (as a superscript) on the right, e.g. writing $X^{\mathcal{A}}$.

Analogously, we let ${}^{\mathcal{A}}\mathfrak{Mod}$ and $\mathfrak{Mod}^{\mathcal{A}}$ denote the category of left resp. right type D structures. For this category, $\mathrm{Mor}({}^{\mathcal{A}}P,\ {}^{\mathcal{A}}Q)$ is defined to be the chain complex of maps $h^1\colon P\to\mathcal{A}\otimes_{\mathbf{k}}Q$, where the differential is specified by

$$d(h^1)=(\mu_1^{\mathcal{A}}\otimes\mathrm{Id}_Q)\circ h^1+(\mu_2^{\mathcal{A}}\otimes\mathrm{Id}_Q)\circ(\mathrm{Id}_{\mathcal{A}}\otimes h^1)\circ\delta^1_P+(\mu_2^{\mathcal{A}}\otimes\mathrm{Id}_Q)\circ(\mathrm{Id}_{\mathcal{A}}\otimes\delta^1_Q)\circ h^1.$$

There is also a composition map, which is a chain map $\mathrm{Mor}({}^{\mathcal{A}}P,{}^{\mathcal{A}}Q)\otimes\mathrm{Mor}({}^{\mathcal{A}}Q,{}^{\mathcal{A}}R)\to\mathrm{Mor}({}^{\mathcal{A}}P,\ {}^{\mathcal{A}}R)$, defined by taking $f^1\otimes g^1$ to

$$(f\circ g)^1=(\mu_2\otimes\mathrm{Id}_Z)\circ(\mathrm{Id}_{\mathcal{A}}\otimes g^1)\circ f^1. \tag{2.4}$$

If ${}^{\mathcal{A}}X$ is a left type D structure, we can form the *opposite* type D structure as follows. As a $\mathbf{k}$-module, $\overline{X}^{\mathcal{A}}$ is a space of vector space maps from M to $\mathbb{F}$:

$$\overline{X}^{\mathcal{A}}=\bigoplus_{d\in\mathbb{Z};s\in S}\mathrm{Hom}_{\mathbb{F}}(X_{-d;-s},\mathbb{F}).$$

This inherits naturally the structure of a right $\mathbf{k}$-module: the action of $\iota \in \mathbf{k}$ on $\phi\colon M \to \mathbb{F}$ is the map that sends x to $\phi(\iota \cdot x)$. The requisite map $\overline{\delta}^1 : \overline{X}^{\mathcal{A}} \to \overline{X}^{\mathcal{A}} \otimes \mathcal{A}$ is adjoint to the map δ^1 for ${}^{\mathcal{A}}X$; i.e. given $\phi \in \overline{X}^{\mathcal{A}}$, define $\overline{\delta}^1(\phi) \in \mathrm{Hom}(X, \mathbb{F}) \otimes_{\mathbf{k}} \mathcal{A} \subset \mathrm{Hom}(X, \mathcal{A})$ to be the map that sends x to $\langle \delta^1 x, \phi \rangle$, where

$$\langle \cdot, \cdot \rangle \colon \mathcal{A} \otimes X \otimes \mathrm{Hom}(X, \mathbb{F}) \to \mathcal{A}$$

is induced by the evaluation map.

2.5. **Tensor products.** We recall the pairing between $\mathcal{A}_\infty$ modules and type D structures from [9], which in fact can be thought of as a model for the derived tensor product. (See for example [12, Proposition 2.3.18])

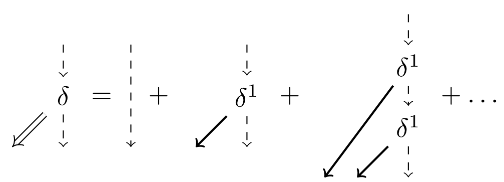
Fix first a type D structure ${}^{\mathcal{A}}X$. There are maps for integers $j \geq 0$ with

$$\delta^j \colon X \mapsto \overbrace{\mathcal{A} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{A}}^{j} \otimes_{\mathbf{k}} X, \tag{2.5}$$

with the following inductive definition:

- δ^0 is the identity map;
- δ^1 is as specified by the type D structure;
- and finally, $\delta^j = (\mathrm{Id}_{\mathcal{A}^{\otimes j-1}} \otimes \delta^1) \circ \delta^{j-1}$.

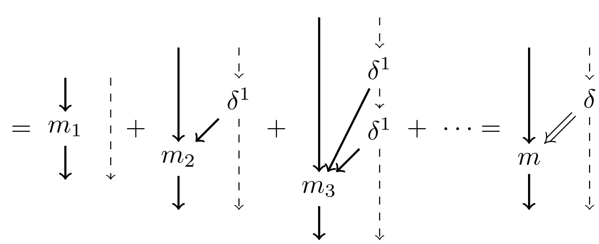
The sum $\sum_{j=0}^{\infty} \delta^j$ is notated



(Note that in general, the image of δ is contained in $(\prod_{m=0}^{\infty} \mathcal{T}^*(\mathcal{A})) \otimes X$.)

Recall that if M is a right $\mathcal{A}_\infty$-module and X is a left type D structure, then under suitable circumstances, we can form the tensor product $M \boxtimes X$. This is a chain complex whose underlying vector space is $M \otimes_{\mathbf{k}} X$, and with differential

$$\partial(p \otimes x) = \sum_{j=0}^{\infty} (m_{j+1} \otimes \mathrm{Id}_X) \circ (p \otimes \delta^j(x))$$



In general, the sum appearing in the definition of ∂ has infinitely many terms. The suitable circumstances needed to define ∂ are those where the above sum is finite. For instance, if the module M has the property that for all $p \in M$,

$m_{j+1}(p, a_1, \dots, a_j) = 0$ for all sufficiently large j and any sequence $a_1, \dots, a_j$, then the sum is guaranteed to be finite. Such an $\mathcal{A}_\infty$ module is called a *bounded* $\mathcal{A}_\infty$ *module*. Similarly, finiteness is guaranteed if X has the property that for all $x \in X$, there is a j with the property that $\delta^j x = 0$ for all sufficiently large j. Such a type D structure is called a *bounded type D structure*. To recapitulate, $M \boxtimes X$ exists if either $M_{\mathcal{A}}$ or ${}^{\mathcal{A}}X$ is bounded.

Let M be a right $\mathcal{A}_\infty$ module with Alexander grading set S and X a left type D structure with Alexander grading set T. Then the tensor product $M \boxtimes X$ is naturally graded by the product of $\mathbb{Z}$ (the Maslov grading) and the Alexander grading set $S \times_\Lambda T = (S \times T)/\Lambda_{\mathcal{A}}$.

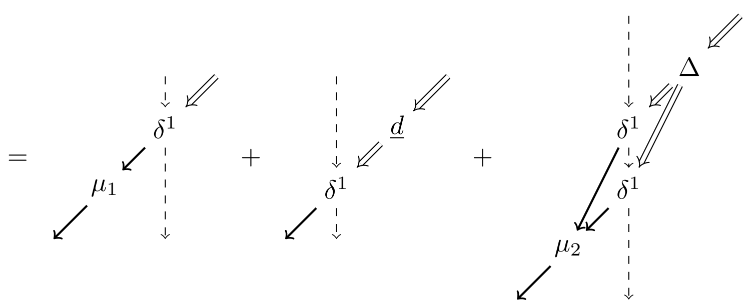
2.6. **Bimodules.** If $\mathcal{A}$ and $\mathcal{B}$ are two differential graded algebras over ground rings $\mathbf{j}$ and $\mathbf{k}$ respectively, a *left/left type DD bimodule* is a type D structure over the tensor product $\mathcal{A} \otimes \mathcal{B}$. In particular, it is a left module over $\mathbf{j} \otimes_{\mathbb{F}} \mathbf{k}$. A left/right type DD bimodule is a left/left type DD bimodule over $\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}$.

A left/right *type DA bimodule* is a $\mathbf{j} - \mathbf{k}$ bimodule equipped with maps for $i \geq 1$:

$$\delta^1_i \colon X \otimes_{\mathbf{k}} \overbrace{\mathcal{B} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{B}}^{i-1} \to \mathcal{A} \otimes_{\mathbf{j}} X,$$

satisfying the structure equation

$$\begin{aligned} 0 &= (\mu_1^{\mathcal{A}} \otimes \mathrm{Id}_X) \circ \delta^1_i(x \otimes a_1 \otimes \cdots \otimes a_{i-1}) \\ &+ \sum_{j=1}^{i-1} \delta^1_i(x \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes \mu_1^{\mathcal{B}}(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_{i-1}) \\ &+ \sum_{j=1}^{i-2} \delta^1_i(x \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes \mu_2^{\mathcal{B}}(a_j \otimes a_{j+1}) \otimes a_{j+2} \otimes \cdots \otimes a_{i-1}) \\ &+ \sum_{j=1}^{i} (\mu_2^{\mathcal{A}} \otimes \mathrm{Id}_X) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \delta^1_{i-j+1}) \circ (\delta^1_j(x \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes a_j \otimes \cdots \otimes a_{i-1}) \end{aligned}$$



Here, we think of $\mathcal{B}$ as the algebra of inputs, and $\mathcal{A}$ as the output algebra.

Example 2.2. *Fix an algebra $\mathcal{A}$ over $\mathbf{k}$. The identity bimodule ${}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}}$ is the type DA bimodule whose underlying $\mathbf{k}$-$\mathbf{k}$ bimodule is $\mathbf{k}$, so that the maps δ^1_j have the form*

$$\delta^1_j \colon \overbrace{\mathcal{A} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{A}}^{j-1} \to \mathcal{A}$$

and whose operations are given by $\delta^1_2(a) = a$, *and* $\delta^1_j = 0$ *for* $j = 1$ *and* $j > 2$.

Example 2.3. *Let* $\phi\colon \mathcal{B} \to \mathcal{A}$ *be a homomorphism of DG algebras over a base ring* $\mathbf{k}$. *This can be viewed as a bimodule* ${}^{\mathcal{A}}[\phi]_{\mathcal{B}}$ *with a single generator over* $\mathbf{k}$, *which we denote* $\mathbf{1}$, *with* $\delta^1_2(\mathbf{1}, b) = \phi(b) \otimes \mathbf{1}$ *and* $\delta^1_j = 0$ *for* $j \neq 2$.

Our DA bimodules ${}^{\mathcal{B}}X_{\mathcal{A}}$ will be bigraded as in Section 2.2. Specifically, there will be some set S with action by $\Lambda_{\mathcal{A}} \oplus \Lambda_{\mathcal{B}}$, and $X = \bigoplus_{(d;s)\in\mathbb{Z}\oplus S} X_{(d;s)}$. The actions respect these gradings as follows:

$$\delta^1_i\colon X_{d_1;s} \otimes_{\mathbf{k}} \mathcal{B}_{d_2;\ell_2} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{B}_{d_i;\ell_i} \to (\mathcal{A} \otimes X)_{i-2+\sum d_j; s+\sum_{j=2}^{i} \ell_j}.$$

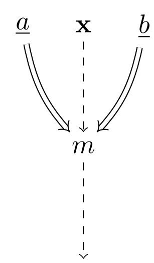
A left/right *type AA bimodule over* $\mathcal{A}$ *and* $\mathcal{B}$ is a left/right bimodule over $\mathbf{I}(\mathcal{A})$ and $\mathbf{I}(\mathcal{B})$, equipped with maps indexed by integers $i, j \geq 0$,

$$m_{i|1|j}\colon \overbrace{\mathcal{A} \otimes_{\mathbf{I}(\mathcal{A})} \otimes \cdots \otimes \mathcal{A}}^{i} \otimes_{\mathbf{I}(\mathcal{A})} \otimes X \otimes_{\mathbf{I}(\mathcal{B})} \overbrace{\mathcal{B} \otimes_{\mathbf{I}(\mathcal{B})} \otimes \cdots \otimes \mathcal{B}}^{j} \to X.$$

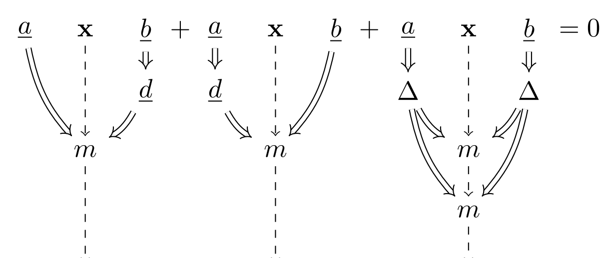
defined for all non-negative integers i and j, satisfying a structure equation which we will state shortly. The maps can $m_{i|1|j}$ can be assembled to form a map

$$m\colon \mathcal{T}^*(\mathcal{A}) \otimes_{\mathbf{I}(\mathcal{A})} \otimes M \otimes_{\mathbf{I}(\mathcal{B})} \mathcal{T}^*(\mathcal{B}) \to M$$

that is represented by the diagram



and satisfies the structure equation (for all $\underline{a} \in \mathcal{T}^*(\mathcal{A})$ and $\underline{b} \in \mathcal{T}^*(\mathcal{B})$):



As a basic example, if $\mathcal{A}$ is a DG algebra, we can view it as a bimodule over itself; in this case, we write ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$. The operation $m_{0|1|0}$ is the differential on the algebra, $m_{1|1|0}(a \otimes b) = a \cdot b$, and $m_{0|1|1}(b \otimes c) = b \cdot c$, for any $a, b, c \in \mathcal{A}$ (except now b is viewed as an element in the bimodule). All other operations vanish.

Bimodules have opposites, defined by the straightforward generalization of Equation (2.2). For example, the bimodule ${}_{\mathcal{A}}\overline{A}_{\mathcal{A}}$, the opposite bimodule of ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$, consists of maps from $\mathcal{A} \to \mathbb{F}$; more precisely given $(d, \ell) \in \mathbb{Z} \oplus \Lambda$,

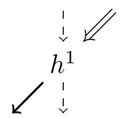
$$\overline{\mathcal{A}}_{(d,\ell)} = \mathrm{Hom}_{\mathbb{F}}(\mathcal{A}_{(-d,-\ell)}, \mathbb{F}). \tag{2.6}$$

This has operations

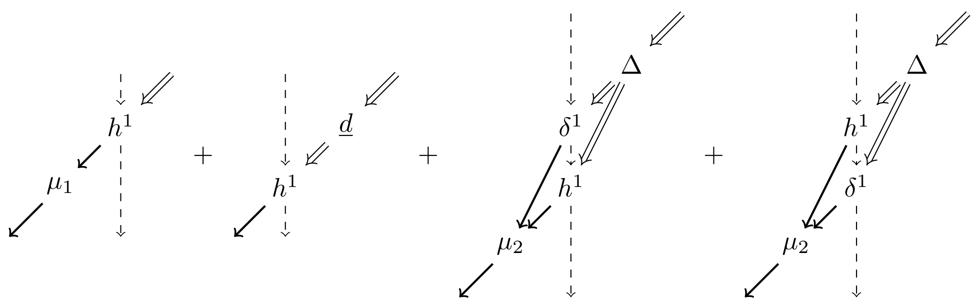
$$m_{0|1|0}(x \mapsto \psi(x)) = (x \mapsto \psi(dx))$$
$$m_{1|1|0}(a \otimes (x \mapsto \psi(x))) = (x \mapsto \psi(x \cdot a))$$
$$m_{0|1|1}((x \mapsto \psi(x)) \otimes b) = (x \mapsto \psi(b \cdot x))$$

A right/right type AA bimodule over $\mathcal{A}$ and $\mathcal{B}$ is a left/right bimodules over $\mathcal{A}^{op}$ (the "opposite algebra") and $\mathcal{B}$.

A *morphism* between type DA bimodules $h^1\colon {}^{\mathcal{A}}X_{\mathcal{B}} \to {}^{\mathcal{A}}Y_{\mathcal{B}}$ is a sequence of maps $\{h^1_j\colon X \otimes_{\mathbf{k}} \overbrace{\mathcal{B} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{B}}^{j-1} \to \mathcal{A} \otimes_{\mathbf{j}} Y\}_{j=1}^{\infty}$, abbreviated

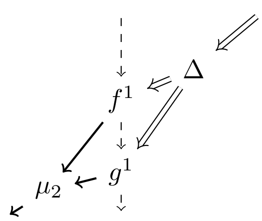


The differential of h^1 is the morphism dh^1 represented as the sum



A *homomorphism* is a morphism whose differential is zero. Two homomorphisms are *homotopic* if their difference is the differential of another morphism.

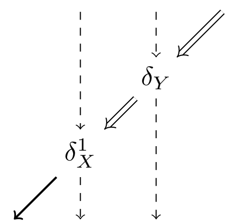
Morphisms can be composed; given morphisms $f^1\colon {}^{\mathcal{A}}X_{\mathcal{B}} \to {}^{\mathcal{A}}Y_{\mathcal{B}}$ and $g^1\colon {}^{\mathcal{A}}Y_{\mathcal{B}} \to {}^{\mathcal{A}}Z_{\mathcal{B}}$, the composition is defined by the picture



Morphisms, homomorphisms, and homotopies can be defined for bimodules of other types in a straightforward way; see [12, Section 2.2.4].

2.7. **Tensor products of bimodules.** We recall here the tensor products of various types of bimodules; see [12, Section 2.3.2] for details. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be three differential graded algebras over base rings $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively, and and let ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}^{\mathcal{B}}Y_{\mathcal{C}}$ be type DA bimodules, their tensor product $X \boxtimes Y$ is a type DA

bimodule structure on the vector space $X \otimes Y$ (where the tensor product is taken over the ground ring of $\mathcal{B}$), with structure maps $\delta^1 \colon X \otimes_{\mathbf{j}} Y \to \mathcal{A} \otimes_{\mathbf{i}} X \otimes_{\mathbf{j}} Y$ which can be represented as



Here the map δ_Y is obtained by iterating δ^1_Y (as in Equation (2.5)).

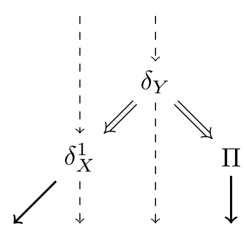
The following is immediate from the definition:

Lemma 2.4. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be four differential graded algebras, and fix type DA bimodules ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}^{\mathcal{B}}Y_{\mathcal{C}}$ and ${}^{\mathcal{C}}Z_{\mathcal{D}}$. Then, there is an isomorphism*

$$({}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}Y_{\mathcal{C}}) \boxtimes {}^{\mathcal{C}}Z_{\mathcal{D}} \cong {}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes ({}^{\mathcal{B}}Y_{\mathcal{C}} \boxtimes {}^{\mathcal{C}}Z_{\mathcal{D}}).$$

Given a type DA bimodule ${}^{\mathcal{A}}X_{\mathcal{B}}$ and a type DD bimodule ${}^{\mathcal{B}}Y^{\mathcal{C}}$, their tensor product ${}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}Y^{\mathcal{C}}$, when it makes sense, is a type DD bimodule over $\mathcal{A}$ and $\mathcal{C}$. The type DD structure map is described by

(2.7)



where $\Pi(b_1 \otimes \cdots \otimes b_j) = b_1 \cdots b_j$.

Of course, the sum implicit in the above description is not always finite; we describe a case where it is. (See also [12, Section 2.2.4].) Consider the map

$$\delta^j_Y \colon Y \to (\mathcal{B}^{\otimes j}) \otimes Y \otimes (\mathcal{C}^{\otimes j})$$

obtained by iterating δ^1 (i.e. so that the map δ_Y appearing in Equation (2.7) is given as $\delta_Y = \sum_{j=0}^{\infty} \delta^j_Y$).

Definition 2.5. *For fixed integer $j \geq 1$,* a length j $\mathcal{B}$-sequence out of $\mathbf{y} \in X$ *is any sequence of algebra elements $(b_1, \dots, b_j)$ in $\mathcal{B}$ with the property that, for a suitable choice of $\mathbf{z} \in X$ and sequence $(c_1, \dots, c_j)$ in $\mathcal{C}$, $(b_1 \otimes \cdots \otimes b_j) \otimes \mathbf{z} \otimes (c_1 \otimes \cdots \otimes c_j)$ appears with non-zero multiplicity in $\delta^j(\mathbf{y})$.*

Definition 2.6. *Fix DG algebras $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and bimodules ${}^{\mathcal{A}}X_{\mathcal{B}}$ and ${}^{\mathcal{B}}Y^{\mathcal{C}}$. We say that X is Y*-compatible over $\mathcal{B}$*, or when it is unambiguous, simply Y*-compatible *if for any non-zero $\mathbf{x} \otimes \mathbf{y} \in X \otimes Y$, if j is a sufficiently large integer, then for any length j $\mathcal{B}$-sequence out of $\mathbf{y}$ $(b_1, \dots, b_j)$, we have that $\delta^1_{j+1}(\mathbf{x}, b_1, \dots b_j) = 0$. Similarly, a morphism $\phi \in \mathrm{Mor}(X, X')$ is called Y*-compatible *if for any $\mathbf{x} \otimes \mathbf{y} \in X \otimes Y$, if j is sufficiently large, then for any length j $\mathcal{B}$-sequence out of $\mathbf{y}$ $(b_1, \dots, b_j)$, we have that $\phi^1_{j+1}(\mathbf{x}, b_1, \dots, b_j) = 0$. If X and X' are Y-compatible,* *we say that they are* Y-compatibly homotopy equivalent *if there are* Y*-compatible morphisms* $\phi\colon X \to X'$, $\psi\colon X' \to X$, $h\colon X \to X$, *and* $h'\colon X' \to X'$, *so that* $d\phi = 0$, $d\psi = 0$, $\psi \circ \phi = \mathrm{Id}_X + dh$ *and* $\phi \circ \psi = \mathrm{Id}_{X'} + dh'$.

Proposition 2.7. *Fix* ${}^{\mathcal{A}}X_{\mathcal{B}}$ *and* ${}^{\mathcal{B}}Y^{\mathcal{C}}$. *If* X *is* Y*-compatible, we can form the type* DD *bimodule* $X \boxtimes Y$, *as defined in Equation* (2.7). *Moreover, if* ${}^{\mathcal{A}}X'_{\mathcal{B}}$ *is also* Y*-compatible, and it is* Y*-compatibly homotopy-equivalent to* X, *then* $X \boxtimes Y$ *and* $X' \boxtimes Y$ *are homotopy equivalent type* DD *bimodules.*

Proof. It is straightforward to see that the Y-compatibility ensures that all the infinite sums appearing in the needed maps are all finite. □

For a very simple special case, suppose that ${}^{\mathcal{A}}X_{\mathcal{B}}$ has the property that for all sufficiently large j, $\delta^1_j = 0$; then X is Y-compatible for all ${}^{\mathcal{B}}Y^{\mathcal{C}}$.

Lemma 2.8. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be DG modules, and let* ${}^{\mathcal{A}}M_{\mathcal{B}}$ *be a type* DA *bimodule with the property that* $\delta^1_j = 0$ *for all sufficiently large* j. *Then,* M *is homotopy equivalent to a type* DA *bimodule with* $\delta^1_j = 0$ *for* $j > 2$.

Proof. Let M' be the bar resolution of M; see [7] or [12], ${}^{\mathcal{A}}M'_{\mathcal{B}} = {}^{\mathcal{A}}M_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathrm{Bar}^{\mathcal{B}} \boxtimes_{\mathcal{B}} \mathcal{B}_{\mathcal{B}}$. The desired homotopy equivalence is obtained from the homotopy equivalence ${}^{\mathcal{B}}\mathrm{Id}_{\mathcal{B}} \to {}^{\mathcal{B}}\mathrm{Bar}^{\mathcal{B}} \boxtimes_{\mathcal{B}} \mathcal{B}_{\mathcal{B}}$ by tensor product with the identity map on M. (Boundedness of M' ensures that this is a homotopy equivalence; compare [12, Lemma 2.3.19].) □

We have the following version of associativity (compare [12, Proposition 2.3.15]):

Lemma 2.9. *Let* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, *and* $\mathcal{D}$ *be four differential graded algebras;* ${}^{\mathcal{A}}X_{\mathcal{B}}$ *and* ${}_{\mathcal{C}}Z^{\mathcal{D}}$ *are bimodules of type* DA; *and* ${}^{\mathcal{B}}Y^{\mathcal{C}}$ *is of type* DD. *Suppose that* ${}^{\mathcal{A}}X_{\mathcal{B}}$ *and* ${}_{\mathcal{C}}Z^{\mathcal{D}}$ *are both bounded, in the sense that for sufficiently large* j, $\delta^1_j = 0$. *Then,*

$$({}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes\ {}^{\mathcal{B}}Y^{\mathcal{C}}) \boxtimes\ {}_{\mathcal{C}}Z^{\mathcal{D}} \simeq {}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes\ ({}^{\mathcal{B}}Y^{\mathcal{C}} \boxtimes\ {}_{\mathcal{C}}Z^{\mathcal{D}}).$$

Proof. This is clear if $\delta^1_j = 0$ for all $j > 2$ on ${}^{\mathcal{A}}X_{\mathcal{B}}$. We can reduce to this case by Lemma 2.8. □

2.8. **Taking homology of bimodules.** By the "homological perturbation lemma", $\mathcal{A}_\infty$ structures persist after taking homology. (See for example [7] for the case of algebras.) We will make use of the analogous result for bimodules.

We start with a standard result from homological algebra:

Lemma 2.10. *Let* $\mathbf{k} \cong \mathbb{F}^k$ *for some* k. *Let* Y *be an* S*-graded chain complex over* $\mathbf{k}$, *and* $Z = H(Y)$. *Then,* Z *is also an* S*-graded chain complex (with trivial differential), and there are* S*-graded chain maps* $f\colon Z \to Y$ *and* $g\colon Y \to Z$ *and an* S*-graded map* $T\colon Y \to Y$ *satisfying the identities*

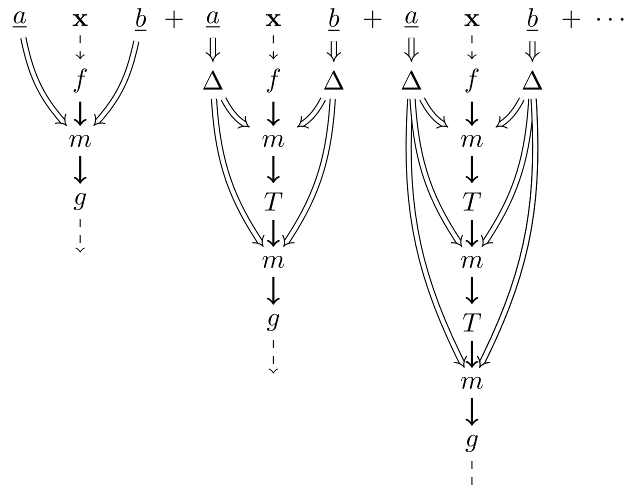
$$T \circ T = 0 \qquad T \circ f = 0 \qquad g \circ T = 0$$
$$g \circ f = \mathrm{Id}_Z \qquad f \circ g = \mathrm{Id}_Y + \partial \circ T + T \circ \partial.$$

We will use two versions of the homological perturbation lemma:

Lemma 2.11. *Let ${}_{\mathcal{A}}Y_{\mathcal{B}}$ be a strictly unital $\mathcal{A}_\infty$ bimodule with grading set S, let Z denote its homology. Let $f\colon Z \to Y$ be a homotopy equivalence of S-graded complexes of $\mathbf{k}$-modules as in Lemma 2.10. Then there is an $\mathcal{A}_\infty$ bimodule structure on Z, denoted ${}_{\mathcal{A}}Z_{\mathcal{B}}$, and an $\mathcal{A}_\infty$ homotopy equivalence $\phi\colon {}_{\mathcal{A}}Z_{\mathcal{B}} \to {}_{\mathcal{A}}Y_{\mathcal{B}}$ with $\phi_{0|1|0} = f$.*

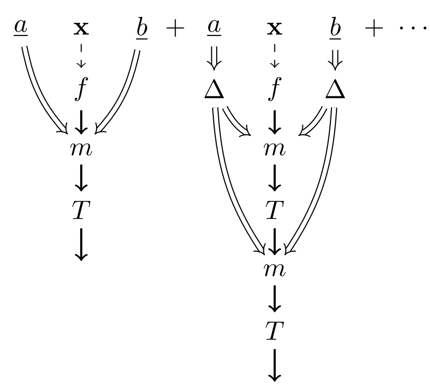
Proof. By hypothesis, we have maps $f\colon Z \to Y$ and $g\colon Y \to Z$ so that $f \circ g = \mathrm{Id} + \partial \circ T + T \circ \partial$. The differential on Z vanishes; i.e. $m_{0|1|0} = 0$. Operations $m_{i|1|j}$ with $i+j > 0$ are described by

$m^Z(\underline{a} \otimes \mathbf{x} \otimes \underline{b}) =$



where $a_1 \otimes \cdots \otimes a_i \in \underline{a} \in \mathcal{T}^*(\mathcal{A})$ and $b_1 \otimes \cdots \otimes b_j = \underline{b} \in \mathcal{T}^*(\mathcal{B})$; and each node laebelled by m contains an operation $m_{a|1|b}$ with $a + b > 0$; where the splitting operator Δ is generalized to have arbitrarily many outputs. With those operations, we have an $\mathcal{A}_\infty$-bimodule homomorphism defined by $\phi_{0|1|0} = f$ and $\phi_{i|1|j}$ with $i + j > 0$ specified by

$\phi(\underline{a} \otimes \mathbf{x} \otimes \underline{b}) =$



(again where each m-labelled node must have at least two inputs). With these definitions, it is straightforward to verify that ϕ is a homotopy equivalence of $\mathcal{A}_\infty$ bimodules; compare [7]. Strict unitality of both m^Z and ϕ follows immediately from these explicit descriptions of m^Z and ϕ, the conditions that $T^2 = f \circ T = T \circ g = 0$, and the strict unitality of Y. $\square$

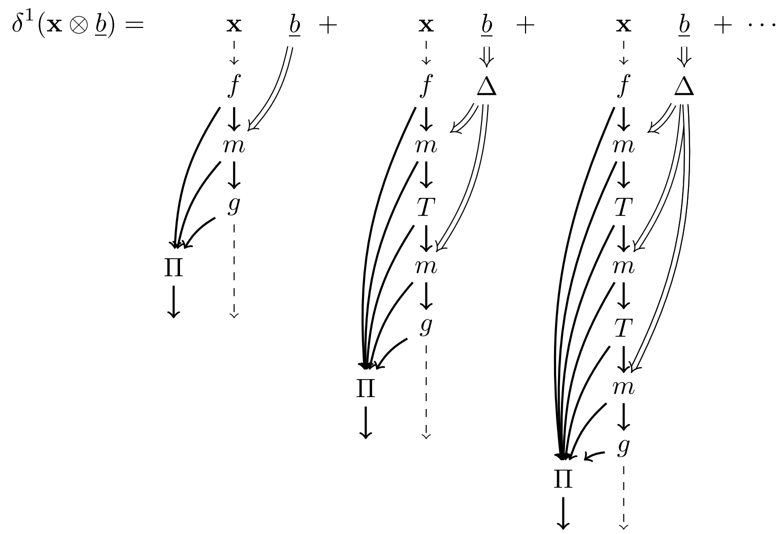


FIGURE 5. δ^1 **action on** Z

We will also use a variant for DA bimodules:

Lemma 2.12. *Let* ${}^{\mathcal{A}}Y_{\mathcal{B}}$ *be a strictly unital type* DA *bimodule with grading set* S*, and let* ${}^{\mathcal{A}}Z$ *be type* D *structure over* $\mathcal{A}$*. Suppose that there are type* D *structure homomorphisms* $f\colon {}^{\mathcal{A}}Z \to {}^{\mathcal{A}}Y$ *(i.e. as the notation suggests, we are forgetting here about the right* $\mathcal{B}$*-action) and* $g\colon {}^{\mathcal{A}}Y \to {}^{\mathcal{A}}Z$ *and a type* D *structure morphism* $T\colon {}^{\mathcal{A}}Y \to {}^{\mathcal{A}}Y$ *so that*

$$f \circ g = \mathrm{Id}_Z, \quad g \circ f = \mathrm{Id}_Y + dT, \quad T \circ T = 0.$$

(Here, $T \circ T$ *denotes the composite of type* D *structures; see Equation* (2.4)*.) Then* ${}^{\mathcal{A}}Z$ *can be turned into a strictly unital type* DA *bimodule, denoted* ${}^{\mathcal{A}}Z_{\mathcal{B}}$*; and there is an* $\mathcal{A}_\infty$ *homotopy equivalence* $\phi\colon {}^{\mathcal{A}}Z_{\mathcal{B}} \to {}^{\mathcal{A}}Y_{\mathcal{B}}$ *with* $\phi^1_1 = f$*.*

Proof. ${}^{\mathcal{A}}Z$ is already equipped with an action δ^1_1. For $j > 1$, operations δ^1_j on $\underline{b} = b_1 \otimes \cdots \otimes b_{j-1}$ are specified by in Figure 5 similarly, define $\phi^1_1 = f$, and for $j > 1$, ϕ^1 on $\underline{b} = b_1 \otimes \cdots \otimes b_{j-1}$ is as shown in Figure 6 $\square$

2.9. **Koszul duality.** Let $\mathcal{A}$ and $\mathcal{B}$ be two differential graded algebras Λ. Let ${}^{\mathcal{A}}X^{\mathcal{B}}$ be a type DD bimodule and ${}_{\mathcal{B}}Y_{\mathcal{A}}$ be a type AA bimodule. We say that these two bimodules are *quasi-inverses* if there are homotopy equivalences

$${}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \simeq {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}} \qquad \text{and} \qquad {}_{\mathcal{B}}Y_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \simeq {}_{\mathcal{B}}\mathrm{Id}^{\mathcal{B}};$$

and ${}^{\mathcal{A}}X^{\mathcal{B}}$ resp. ${}_{\mathcal{B}}Y_{\mathcal{A}}$ as a *quasi-invertible* type DD resp. type AA bimodule.

To see that X and Y are quasi-inverses, it suffices to exhibit homomorphisms $\phi^1\colon {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \to {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}}$ and $\psi^1\colon {}_{\mathcal{B}}Y_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \to {}_{\mathcal{B}}\mathrm{Id}^{\mathcal{B}}$ so that the maps

$$\mathrm{Id}_{\mathcal{A}} \boxtimes \phi^1\colon {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \to {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$$

$$\psi^1 \boxtimes \mathrm{Id}_{\mathcal{B}}\colon {}_{\mathcal{B}}Y_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}} \to {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}$$

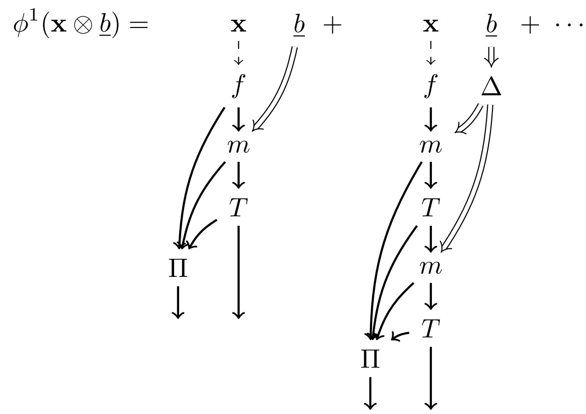


FIGURE 6. ϕ^1 **morphism**

induce isomorphisms on homology, according to [12, Corollary 2.4.4].

Definition 2.13. *Let* $\mathbf{k}$ *be a ring that is a finite direct sum of* $\mathbb{F}$*. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two DG algebras, both of which are positively graded over* $\mathbf{k}$ *(in the sense of Definition 2.1), with Alexander grading specified by the same Abelian group* $\Lambda = \Lambda_{\mathcal{A}} = \Lambda_{\mathcal{B}}$*. A bimodule* ${}^{\mathcal{A}}X^{\mathcal{B}}$ *is called a* Koszul dualizing bimodule *if it satisfies the following properties:*

(K-1) ${}^{\mathcal{A}}X^{\mathcal{B}}$ *is graded by* $\mathbb{Z}\oplus\Lambda$*, where the action of* $\Lambda_{\mathcal{A}}\oplus\Lambda_{\mathcal{B}}$ *is specified by the map* $\Lambda_{\mathcal{A}}\oplus\Lambda_{\mathcal{B}}\to\Lambda$ *given by* $(a,b)\mapsto(b-a)$*.*
(K-2) ${}^{\mathcal{A}}X^{\mathcal{B}}$ *has rank one; i.e. there is an isomorphism* ${}^{\mathcal{A}}X^{\mathcal{B}}\cong\mathbf{k}$ *as bigraded* $\mathbf{k}$*-*$\mathbf{k}$ *bimodules. (In particular,* ${}^{\mathcal{A}}X^{\mathcal{B}}$ *is supported in bigrading* $(0;0)$*).*
(K-3) ${}^{\mathcal{A}}X^{\mathcal{B}}$ *is quasi-invertible.*

If such a bimodule exists, $\mathcal{A}$ *and* $\mathcal{B}$ *are called* Koszul dual *to one another.*

For a Koszul dualizing bimodule, it follows that

$$\delta^1({}^{\mathcal{A}}X^{\mathcal{B}}) \subset \bigoplus_{\lambda\in\Lambda\setminus 0} \mathcal{A}_\lambda\otimes {}^{\mathcal{A}}X^{\mathcal{B}}\otimes\mathcal{B}_\lambda. \tag{2.8}$$

(Indeed, this conclusion holds even for ${}^{\mathcal{A}}X^{\mathcal{B}}$ as in the definition that do not satisfy Property (K-3).)

Definition 2.14. *If* ${}^{\mathcal{A}}X^{\mathcal{B}}$ *is a type DD bimodule, we can form the* candidate quasi-inverse module

$$^{\mathcal{A}}\mathrm{Mor}({}^{\mathcal{A}}X^{\mathcal{B}}\boxtimes{}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}, {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}}) \cong {}_{\mathcal{B}}\overline{B}_{\mathcal{B}}\boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}}\boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}. \tag{2.9}$$

Lemma 2.15. *For a type DD bimodule* ${}^{\mathcal{A}}X^{\mathcal{B}}$ *satisfying properties (K-1) and (K-2), the candidate quasi-inverse module has a subcomplex* $\overline{\mathbf{k}}\boxtimes{}^{\mathcal{B}}\overline{X}^{\mathcal{A}}\boxtimes\mathbf{k}$ *which is isomorphic to* $\mathbf{k}$*.*

Proof. For any DD bimodule, $\overline{\mathbf{k}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes \mathcal{A}$ is a subcomplex of the candidate quasi-inverse module. Comditions (K-1) and (K-2) ensures that Equation (2.8) holds, so the induced differential on $\overline{\mathbf{k}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes \mathcal{A}$ is simply the differential on $\mathcal{A}$; and therefore $\overline{\mathbf{k}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes \mathbf{k}$ is a subcomplex. Condition (K-2) now gives the desired identification of the subcomplex with $\mathbf{k}$. □

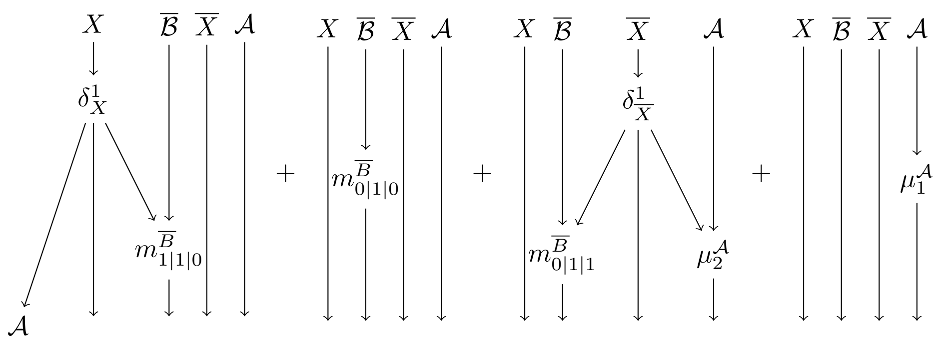
Lemma 2.16. *Fix* $\mathbf{k}$ *as above in Definition 2.13, and let* ${}^{\mathcal{A}}X^{\mathcal{B}}$ *be a type DD bimodule satisfying Conditions (K-1)-2.8 of Definition 2.13. Suppose that the inclusion map from* $\mathbf{k}$ *to the candidate quasi-inverse module coming from Lemma 2.15 induces an isomorphism on homology. Then the candidate quasi-inverse module is a quasi-inverse of* ${}^{\mathcal{A}}X^{\mathcal{B}}$*.*

Proof. Let ${}_{\mathcal{B}}Y_{\mathcal{A}}$ be the candidate quasi-inverse. Our goal is to show, under the stated hypotheses, that ${}_{\mathcal{B}}Y_{\mathcal{A}}$ is a quasi-inverse to ${}^{\mathcal{A}}X^{\mathcal{B}}$.

To this end, consider a DA bimodule by tensoring X with Y:

$$ {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} = {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\overline{B}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\overline{X}^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}. $$

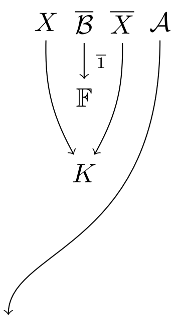
The differential on this bimodule is given pictorially by:



(Arrows that output $1 \in \mathcal{A}$, which would appear in the second, third, and fourth terms, are suppressed.) There is a natural map $h\colon {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \to {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}}$, defined by $x \otimes \psi \to \psi(x \otimes \mathbf{1})$, where here $\mathbf{1} \in \mathcal{B}$ is the unit; i.e. given $\mathbf{x} \in X$ and $\psi \in Y = {}^{\mathcal{A}}\mathrm{Mor}({}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}, {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}})$, $x \otimes \mathbf{1} \in {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}$, so $\psi(x \otimes \mathbf{1}) \in {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}}$. This natural map can be viewed as a DA morphism. Recall that a DA morphism ${}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \to {}^{\mathcal{A}}\mathrm{Id}_{\mathcal{A}}$ is specified by a sequence of maps indexed by $j \geq 1$:

$$ h^1_j\colon {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{A}} \otimes \overbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}^{j-1} \to \mathcal{A}. $$

The morphism under question has $h^1_j = 0$ for $j > 1$, and it was specified above in the case where $j = 1$. Pictorially, h is represented as follows:



Where here K denotes the Kronecker pairing, and $\overline{1}$ is dual to the map $1\colon \mathbb{F} \to \mathcal{B}$.

The verification that h is a DA bimodule homomorphism is straightforward.

So far, we have not used the hypothesis on the homology of ${}_{\mathcal{B}}Y_{\mathcal{A}}$ (which is needed to verify that h is a homotopy equivalence). From the hypothesis, there is a rank one bimodule ${}_{\mathcal{B}}Z_{\mathcal{A}}$; this is the bimodule structure induced on $\mathbf{k}$ from its quasi-isomorphism with ${}_{\mathcal{B}}Y_{\mathcal{A}}$ as in Lemma 2.11. Let $\phi\colon {}_{\mathcal{B}}Z_{\mathcal{A}} \to {}_{\mathcal{B}}Y_{\mathcal{A}}$ be the induced bimodule quasi-isomorphism. In more detail, denote the isomorphism $\mathbf{k} \cong X$ by $\iota \mapsto x_\iota$; taking its dual and using an isomorphism $\mathbf{k} \cong \overline{\mathbf{k}}$ induced from an $\mathbb{F}$-basis for $\mathbf{k}$ (which in our case is supplied by the basic idempotents), we get an isomorphism $\mathbf{k} \to \overline{X}$ denoted $\iota \mapsto \overline{x}_\iota$. Now, $\phi_{0|1|0}(\iota) = \overline{1} \otimes \overline{x}_\iota \otimes 1$.

We claim that on the bimodule ${}_{\mathcal{B}}Z_{\mathcal{A}}$, for all $k > 0$,

$$m_{k|1|0} = 0. \tag{2.10}$$

So see this, observe that, the image of $m_{k|1|0}(b_1, \dots, b_{k-1}, x)$ is supported in the portion in grading $\lambda(x) + \lambda(b_1) + \cdots + \lambda(b_{k-1})$ which, since $\lambda(x) = 0$ (because of Condition (K-2)), means that $\lambda(b_1) + \cdots + \lambda(b_{k-1}) = 0$. By positivity of the grading on $\mathcal{A}$, we conclude that each $\lambda(b_i) = 0$, and hence that each $b_i \in \mathbf{k}$; so these algebra elements have Maslov grading 0. Since Z is supported in Maslov grading 0, as well, we conclude from the grading conventions that $k = 1$, as required by Equation (2.10). Similar reasoning shows that for the bimodule homomorphism, for all $k > 0$,

$$\phi_{k|1|0} = 0. \tag{2.11}$$

To verify h is a DA bimodule quasi-isomorphism, consider the induced AA-bimodule homomorphism

$$(\mathrm{Id}_{\mathcal{A}} \boxtimes h)\colon {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{B}} = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\overline{\mathcal{B}}_{\mathcal{B}} \boxtimes {}_{\mathcal{B}}X^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}.$$

To verify that it is a quasi-isomorphism, we form the following composite

$${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Z_{\mathcal{A}} \xrightarrow{\mathrm{Id}_{\mathcal{A}\boxtimes X}\boxtimes\phi} \mathcal{A} \boxtimes X \boxtimes \overline{\mathcal{B}} \boxtimes \overline{X} \boxtimes \mathcal{A} \xrightarrow{\mathrm{Id}_{\mathcal{A}}\boxtimes h} {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}},$$

which we denote κ. From Equation (2.10), we conclude that the chain complex on the left is identified with $\mathcal{A}$. From Equation (2.11), we conclude that that $\kappa_{0|1|0}$ is the chain map $(\mathrm{Id}_{\mathcal{A}} \otimes h^1_1) \circ (\mathrm{Id}_{\mathcal{A}\boxtimes X} \otimes \phi_{0|1|0})$ that sends $a \otimes x_\iota \otimes \iota$ to a; in particular, it is an isomorphism of chain complexes. □

Remark 2.17. *In fact, if* ${}^{\mathcal{A}}X^{\mathcal{B}}$ *is quasi-invertible, the quasi-inverse is always given by the above bimodules by the argument from* [12, Proposition 9.2]*; i.e. there are quasi-isomorphisms:*

$$\begin{aligned} {}_{\mathcal{B}}Y_{\mathcal{A}} &\simeq \mathrm{Mor}_{\mathcal{B}}({}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}, {}_{\mathcal{B}}Y_{\mathcal{A}}) \\ &\simeq \mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}, {}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}Y_{\mathcal{B}}) \\ &\simeq \mathrm{Mor}^{\mathcal{A}}({}^{\mathcal{A}}X^{\mathcal{B}} \boxtimes {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}, {}^{\mathcal{A}}\,\mathrm{Id}_{\mathcal{A}}); \end{aligned}$$

the first of these is true for arbitrary ${}_{\mathcal{B}}Y_{\mathcal{A}}$*, the second uses the fact that* $X\boxtimes$ *induces an equivalence of categories, and the third uses the fact that* X *and* Y *are quasi-inverses.*

3. The algebras

We describe differential graded algebras used in the construction of our knot invariant. In fact, we will find it convenient to work with a more general construction $\mathcal{B}(m,k,\mathcal{S})$, where $0 \leq k \leq m+1$, and $\mathcal{S}$ is an arbitrary subset of $\{1,...,m\}$. The integer m is called the *index*. The integer k is the *number of occupied positions*; together with the index, it determines the base ring $\mathbf{k} = \mathbf{I}(\mathcal{B})$. When $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \{1,\dots,m\}$, then $\mathcal{B}(m,k,\mathcal{S}_1)$ will be a differential subalgebra of $\mathcal{B}(m,k,\mathcal{S}_2)$.

When $\mathcal{S} = \emptyset$, the differential on the algebra $\mathcal{B}(m,k,\emptyset) = \mathcal{B}(m,k)$ vanishes. In fact, this algebra is the quotient of a larger algebra $\mathcal{B}_0(m,k)$, which we define first.

When constructing the knot invariant for a knot with bridge number n, we will have $m = 2n$, $k = n$, and $\mathcal{S}$ will correspond to those (n) strands that are oriented upwards. In fact, for the purposes of the knot invariant, it would suffice to work in a summand corresponding to certain idempotents; see Remark 11.12. We have chosen to use the larger algebra in our constructions, as it satisfies a duality described in Section 3.8.

3.1. The algebra $\mathcal{B}_0(m,k)$.

We define $\mathcal{B}_0(m,k)$, which is a graded algebra over $\mathbb{F}[U_1,\dots,U_m]$, whose (Alexander multi-grading) set is $(\frac{1}{2}\mathbb{Z})^m$. (See Section 2.2.) Basic idempotents in $\mathcal{B}_0(m,k)$ correspond to *idempotent states*, or *I-states* for short, $\mathbf{x} = (x_1,\dots,x_k)$, which are increasing sequences of integers

$$0 \leq x_1 < ... < x_k \leq m. \tag{3.1}$$

The basic idempotent corresponding to $\mathbf{x}$ will be denoted by $\mathbf{I}_{\mathbf{x}}$. The elements $\mathbf{I}_{\mathbf{x}}$ are generators of a ring $\mathbf{k} = \mathbf{I}(m,k)$ satisfying:

$$\mathbf{I}_{\mathbf{x}} \cdot \mathbf{I}_{\mathbf{y}} = \begin{cases} \mathbf{I}_{\mathbf{x}} & \text{if } \mathbf{x} = \mathbf{y} \\ 0 & \text{if } \mathbf{x} \neq \mathbf{y}. \end{cases}$$

Remark 3.1. *Recall from Section 1 that these idempotents can be interpreted as marking intervals which are intersections of the regions in a knot diagram with the* $y = t$ *slice. In fact, the basic idempotents referred to there do not allow for the unbounded intervals, corresponding to replacing Equation* (3.1) *by the condition*

$$1 \leq x_1 < \cdots < x_k \leq m-1. \tag{3.2}$$

Our basic idempotents now will be as in Equation (3.1)*; we return to the restricted case in Section 12.*

The unit $\mathbf{1}$ in the algebra is given by the sum of the basic idempotents.

Given an I-state $\mathbf{x}$, define its *weight* $v^{\mathbf{x}} \in \mathbb{Z}^m$ by

$$v_i^{\mathbf{x}} = \#\{x \in \mathbf{x} \big| x \geq i\}. \tag{3.3}$$

Given two I-states $\mathbf{x}$ and $\mathbf{y}$, define their *minimal relative weight vector* $w^{\mathbf{x},\mathbf{y}} \in (\frac{1}{2}\mathbb{Z})^m$ to be given by

$$w_i^{\mathbf{x},\mathbf{y}} = \frac{1}{2}\big|v_i^{\mathbf{x}} - v_i^{\mathbf{y}}\big|.$$

$\mathcal{B}_0(m,k)$ is defined so that there is an identification of $\mathbb{F}[U_1,\dots,U_m]$-modules $\mathbf{I_x} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I_y} \cong \mathbb{F}[U_1,\dots,U_m]$; denote the identification by

$$\phi^{\mathbf{x},\mathbf{y}} \colon \mathbb{F}[U_1,\dots,U_m] \to \mathbf{I_x} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I_y}$$

A grading by $(\frac{1}{2}\mathbb{Z})^m$ on $\mathbf{I_x} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I_y}$ is specified by

$$w(\phi(U_1^{t_1} \dots U_m^{t_m})) = w^{\mathbf{x},\mathbf{y}} + (t_1,\dots,t_m), \tag{3.4}$$

for non-negative integers $t_1,\dots,t_m$.

Multiplication

$$\Big(\mathbf{I_x} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I_y}\Big) * \Big(\mathbf{I_y} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I_z}\Big) \to \Big(\mathbf{I_x} \cdot \mathcal{B}_0(m,k) \cdot \mathbf{I_z}\Big)$$

is the unique non-trivial, grading-preserving $\mathbb{F}[U_1,\dots,U_m]$-equivariant map. Explicitly, given I-states $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, if we define $g^{\mathbf{x},\mathbf{y},\mathbf{z}} = U^{t_1} \cdots U^{t_m}$, where $t_i = w_i^{\mathbf{x},\mathbf{y}} + w_i^{\mathbf{y},\mathbf{z}} - w_i^{\mathbf{x},\mathbf{z}}$, then for $a, b \in \mathbb{F}[U_1,\dots,U_m]$,

$$\phi^{\mathbf{x},\mathbf{y}}(a) * \phi^{\mathbf{y},\mathbf{z}}(b) = \phi^{\mathbf{x},\mathbf{z}}(a \cdot b \cdot g^{\mathbf{x},\mathbf{y},\mathbf{z}}).$$

Suppose that $\mathbf{x}$ is an I-state with $j-1 \in \mathbf{x}$ but $j \notin \mathbf{x}$. Then, we can form a new I-state $\mathbf{y} = \mathbf{x} \cup \{j\} \setminus \{j-1\}$, and let $R_j^{\mathbf{x}} = \phi^{\mathbf{x},\mathbf{y}}(1)$. We define

$$R_j = \sum_{\{\mathbf{x} \big| j-1 \in \mathbf{x}, j \notin \mathbf{x}\}} R_j^{\mathbf{x}},$$

so that

$$\mathbf{I_x} \cdot R_j = \begin{cases} R_j^{\mathbf{x}} & \text{if } j-1 \in \mathbf{x} \text{ and } j \notin \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, let $L_j^{\mathbf{y}} = \phi^{\mathbf{y},\mathbf{x}}(1)$, and $L_j = \sum_{\{\mathbf{y} \big| j \in \mathbf{y}, j-1 \notin \mathbf{y}\}} L_j^{\mathbf{y}}$. Less formally R_j moves one of the coordinates of an I-state from the $(j-1)^{st}$ position to the j^{th}, and L_j changes it back from j^{th} to $(j-1)^{st}$. The elements L_j and R_j are called the *left shifts* and *right shifts* respectively. See Figure 7.

For $i = 1,\dots,m$, U_i induces an algebra element in $\mathcal{B}_0$, defined by $\sum_{\mathbf{x}} \phi^{\mathbf{x},\mathbf{x}}(U_i)$. For notational simplicity, we also denote this induced element by U_i.

Proposition 3.2. *The weight function from Equation* (3.4) *descends to a grading* $w = (w_1,\dots,w_m)$ *on* $\mathcal{B}_0(m,k)$ *with values in* $(\frac{1}{2}\mathbb{Z})^m$.

Proof. It suffices to show that

$$w(\phi^{\mathbf{x},\mathbf{y}}(1) * \phi^{\mathbf{y},\mathbf{z}}(1)) = w(\phi^{\mathbf{x},\mathbf{y}}(1)) + w(\phi^{\mathbf{y},\mathbf{z}}(1)).$$

which follows from the definition of $g^{\mathbf{x},\mathbf{y},\mathbf{z}}$. ☐

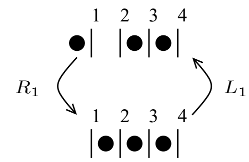


FIGURE 7. **Picture of algebra elements in $\mathcal{B}_0(4,3)$.** Idempotents $\mathbf{I}_{\{0,2,3\}}$ and $\mathbf{I}_{\{1,2,3\}}$ are pictured here; the algebra elements L_1 and R_1 which connect them are indicated (e.g. $\mathbf{I}_{\{0,2,3\}} \cdot R_1 \cdot \mathbf{I}_{\{1,2,3\}}$ is non-zero). Here, $\mathbf{I}_{\{0,2,3\}} \cdot R_1 \cdot L_1 = \mathbf{I}_{\{0,2,3\}} \cdot U_1$.

We have no further need to distinguish the product on the algebra $\mathcal{B}_0(m,k)$ from other products; so we will abbreviate $a * b$ by $a \cdot b$.

In the notation from Section 2.2, $\mathcal{B}_0(m,k)$ is graded by the Abelian group $\Lambda = (\frac{1}{2}\mathbb{Z})^m$. Recall that an element $a \in \mathcal{B}_0(m,k)$ that is supported in some fixed grading $\lambda \in \Lambda$ is called *homogeneous.*

Proposition 3.3. *The algebra $\mathcal{B}_0(m,k)$ is generated over $\mathbb{F}$ by the elements L_i, R_i, U_i, and the idempotents $\mathbf{I}_{\mathbf{x}}$.*

Proof. Let a be a homogeneous algebra elements with $\mathbf{I}_{\mathbf{x}} \cdot a \cdot \mathbf{I}_{\mathbf{y}} = a$. If $\mathbf{x} = \mathbf{y}$, then a factors as a product of U_i. Otherwise, suppose that some $i = x_t > y_t$; and indeed choose t minimal with this property. Then, it is easy to see that we can factor $a = L_i \cdot b$. Otherwise, there is some $y_t > x_t = i$, and we can choose t maximal with this property. Then, we can factor $a = R_{i+1} \cdot b$. In both cases, the total weight of b is smaller than that of a, so the result follows by induction on weight. $\square$

3.2. **The $\mathcal{B}(m,k)$.** The algebra $\mathcal{B}(m,k)$ is the quotient of $\mathcal{B}_0(m,k)$ by the relations

$$L_{i+1} \cdot L_i = 0 \tag{3.5}$$

$$R_i \cdot R_{i+1} = 0; \tag{3.6}$$

and also, if $\{x_1, ..., x_k\} \cap \{j-1, j\} = \emptyset$, then

$$\mathbf{I}_{\mathbf{x}} \cdot U_j = 0. \tag{3.7}$$

As we shall see, the relations Equations (3.5) and (3.6) guarantee that algebra elements in $\mathcal{B}(m,k)$ cannot move coordinates in the idempotents states by more than one unit. We formulate this quotient operation as follows:

Definition 3.4. *Let $\mathcal{J}$ be the (two-sided) ideal in $\mathcal{B}_0(m,k)$ that is generated by $L_{i+1} \cdot L_i$, $R_i \cdot R_{i+1}$, and $\mathbf{I}_{\mathbf{x}} \cdot U_j$, when $\{x_1, ..., x_k\} \cap \{j-1, j\} = \emptyset$. Let $\mathcal{B}(m,k)$ be the quotient of $\mathcal{B}_0(m,k)$ by this two-sided ideal.*

Note that $\mathcal{J}$ is generated by elements that are homogeneous with respect to the weights. Thus, the weights induce a grading on the quotient algebra $\mathcal{B}(m,k)$.

The ideal $\mathcal{J}$ can be understood concretely. To do so, we set up some notation.

Definition 3.5. *Two I-states* $\mathbf{x}$ *and* $\mathbf{y}$ *are said to be* far *if there is some $i = 1, \ldots, k$ with $|x_i - y_i| > 1$; otherwise they are called* close enough.

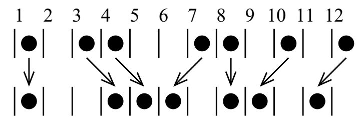


FIGURE 8. The ideal $\mathcal{I}(\mathbf{x},\mathbf{y})$ is generated by U_1U_2, U_3, U_6, U_8U_9, and U_{11}.

Given two I-states $\mathbf{x}$ and $\mathbf{y}$, we define an ideal $\mathcal{I}(\mathbf{x},\mathbf{y}) \subset \mathbb{F}[U_1,\dots,U_m]$. If the I-states $\mathbf{x}$ and $\mathbf{y}$ are far, let $\mathcal{I}(\mathbf{x},\mathbf{y}) = \mathbb{F}[U_1,\dots,U_m]$. If $\mathbf{x}$ and $\mathbf{y}$ are close enough, let $\mathcal{I}(\mathbf{x},\mathbf{y})$ be the ideal generated by monomials $U_{i+1}\cdots U_j$, taken over all $0 \leq i < j \leq m$ where i and j satisfy the following conditions:

- $i, j \in \{0,\dots,m\} \setminus (\mathbf{x}\cap\mathbf{y})$
- for all integers t with $i < t < j$, $t \in \mathbf{x}\cap\mathbf{y}$.
- $w_t^{\mathbf{x},\mathbf{y}} = 0$ for all $i+1 \leq t \leq j$.

Definition 3.6. *For $i < j$ as above, we call the interval $[i+1, j]$ a* generating interval for $\mathbf{x}$ and $\mathbf{y}$. *(Observe that a generating interval can have $i+1 = j$, and this corresponds to U_j.)*

Proposition 3.7. *For all I-states $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{J} \cdot \mathbf{I}_{\mathbf{y}} = \phi^{\mathbf{x},\mathbf{y}}(\mathcal{I}(\mathbf{x},\mathbf{y}))$.*

Proof. Let $\mathcal{I} = \bigoplus_{\mathbf{x},\mathbf{y}} \phi^{\mathbf{x},\mathbf{y}}(\mathcal{I}(\mathbf{x},\mathbf{y}))$. First we prove $\mathcal{I} \subseteq \mathcal{J}$; i.e. for any two I-states $\mathbf{x}$ and $\mathbf{y}$ we have $\phi^{\mathbf{x},\mathbf{y}}(\mathcal{I}(\mathbf{x},\mathbf{y})) \subset \mathcal{J}$.

If $\mathbf{x}$ and $\mathbf{y}$ are far, so that $\mathcal{I}(\mathbf{x},\mathbf{y}) = \mathbb{F}[U_1,\dots,U_m]$, then we claim that $\phi^{\mathbf{x},\mathbf{y}}(1)$ can be factored as a product $a \cdot R_t R_{t+1} \cdot b$ or $a \cdot L_t L_{t-1} \cdot b$ for some t. This is obvious, for example if there is an i with $x_i > y_i + 1$, then $\phi^{\mathbf{x},\mathbf{y}}(1)$ decomposes as $a \cdot L_t \cdot L_{t-1} \cdot b$, where $t = x_i$.

If $\mathbf{x}$ and $\mathbf{y}$ are not far, we show that for all generating intervals $[i+1, j]$ for $\mathbf{x}$ and $\mathbf{y}$, the algebra elements $\phi^{\mathbf{x},\mathbf{y}}(U_{i+1}\dots U_j)$ are in $\mathcal{J}$.

Consider first the case where $i+1 = j$. Then, $\phi^{\mathbf{x},\mathbf{y}}(U_j)$ can be factored as $a \cdot \mathbf{I}_{\mathbf{w}} \cdot U_j \cdot b$, where $j-1, j \notin \mathbf{w}$. In general, if $i < j$ is a generating interval for $\mathbf{x}$ and $\mathbf{y}$, $\phi^{\mathbf{x},\mathbf{y}}(U_{i+1}\cdots U_j)$ has a decomposition as

$$(a \cdot L_{i+1}\cdots L_{j-1}) \cdot \mathbf{I}_{\mathbf{w}} \cdot U_j \cdot R_{j-1}\cdots R_{i+1} \cdot b,$$

where $j-1, j \notin \mathbf{w}$. In these cases, $\mathbf{I}_{\mathbf{w}} \cdot U_j \in \mathcal{J}$. (For example, for the I-states $\mathbf{x}$ and $\mathbf{y}$ in Figure 8,

$$\begin{aligned}
\phi^{\mathbf{x},\mathbf{y}}(U_3) &= \phi^{\mathbf{x},\mathbf{y}}(1) \cdot \mathbf{I}_{\mathbf{y}} \cdot U_3 \cdot \phi^{\mathbf{y},\mathbf{y}}(1) & & \\
\phi^{\mathbf{x},\mathbf{y}}(U_{11}) &= \phi^{\mathbf{x},\mathbf{w}}(1) \cdot \mathbf{I}_{\mathbf{w}} \cdot U_{11} \cdot L_{12} & &\text{where } \mathbf{w} = \{1,4,5,6,7,8,9,12\} \\
\phi^{\mathbf{x},\mathbf{y}}(U_1U_2) &= \phi^{\mathbf{x},\mathbf{w}}(1) \cdot \mathbf{I}_{\mathbf{w}} \cdot U_2 \cdot \phi^{\mathbf{w},\mathbf{y}}(1) & &\text{where } \mathbf{w} = \{0,3,4,7,8,10,12\})
\end{aligned}$$

This completes the proof that $\mathcal{I} \subseteq \mathcal{J}$.

To prove that $\mathcal{J} \subseteq \mathcal{I}$, we first prove that $\mathcal{I}$ is an ideal; i.e. if a is an arbitrary algebra element, then $a \cdot \mathcal{I} \subseteq \mathcal{I}$; and also $\mathcal{I} \cdot a \subseteq \mathcal{I}$. In view of Proposition 3.3, it suffices to verify this in the case where $a = \mathbf{I}_{\mathbf{x}} \cdot U_i$, $\mathbf{I}_{\mathbf{x}} \cdot R_i$, or $\mathbf{I}_{\mathbf{x}} \cdot L_i$.

When $a = \mathbf{I}_{\mathbf{x}} \cdot U_i$, the result follows readily.

By symmetry, we can consider the case where $a = \mathbf{I_x} \cdot R_i \cdot \mathbf{I_y} \neq 0$. Let t denote the index with $x_t = i-1$ and $y_t = i$. We wish to show that $\mathbf{I_x} \cdot R_i \cdot \phi^{\mathbf{y},\mathbf{z}}(\mathcal{I}(\mathbf{y},\mathbf{z})) \subset \phi^{\mathbf{x},\mathbf{z}}(\mathcal{I}(\mathbf{x},\mathbf{z}))$. There are cases:

(1) If $\mathbf{x}$ and $\mathbf{z}$ are far, the statement is vacuously true.
(2) If $\mathbf{y}$ and $\mathbf{z}$ are far, but $\mathbf{x}$ and $\mathbf{z}$ are close enough, then $z_t = i-2$ and $g^{\mathbf{x},\mathbf{y},\mathbf{z}} = U_i$. Futhermore we have $i \notin \mathbf{x}$, $i-1 \notin \mathbf{z}$, and $w_i^{\mathbf{x},\mathbf{z}} = 0$. It follows that $U_i \in \mathcal{I}(\mathbf{x},\mathbf{z})$ and this finishes the argument in this subcase.
(3) If $\mathbf{y}$ and $\mathbf{z}$ are not far, and $\mathbf{x}$ and $\mathbf{z}$ are not far, as well, then there are two subcases according to the value of z_t.
   (3a) If $z_t = i$, the generating intervals for $\mathbf{x},\mathbf{z}$ are contained in the generating intervals for $\mathbf{y},\mathbf{z}$, and the containment is clear.
   (3b) If $z_t = i-1$, there can be at most one generating interval for $(\mathbf{y},\mathbf{z})$ which is not also a generating interval for $(\mathbf{x},\mathbf{z})$, and that is an interval which terminates in i. Since $g^{\mathbf{x},\mathbf{y},\mathbf{z}} = U_i$, it follows that
$$\mathbf{I_x} \cdot R_i \cdot \phi^{\mathbf{y},\mathbf{z}}(U_j U_{j+1} \cdots U_{i-1}) = \phi^{\mathbf{x},\mathbf{z}}(U_j U_{j+1} \cdots U_i).$$

The containment $\phi^{\mathbf{x},\mathbf{y}}(\mathcal{I}(\mathbf{x},\mathbf{y})) \cdot R_i \cdot \mathbf{I_z} \subset \phi^{\mathbf{x},\mathbf{z}}(\mathcal{I}(\mathbf{x},\mathbf{z}))$ follows symmetrically.

We must now verify that the definining relations for $\mathcal{J}$ are contained in $\mathcal{I}$. Clearly, if $\mathbf{I_x} \cdot R_i \cdot R_{i+1} \cdot \mathbf{I_y}$ is non-zero in $\mathcal{B}_0$, then $\mathbf{x}$ and $\mathbf{y}$ are too far, and so $\mathcal{I}(\mathbf{x},\mathbf{y}) = \mathbb{F}[U_1,\dots,U_m]$, proving the containment. The same argument works for $L_{i+1} \cdot L_i$. Finally, if $\mathbf{x} \cap \{i-1,i\} = \emptyset$, then U_i is a monomial corresponding to a generating interval for $(\mathbf{x},\mathbf{x})$, so $U_i \in \mathcal{I}(\mathbf{x},\mathbf{x})$. □

The following lemma will be useful later.

Lemma 3.8. *Let $a \in \mathcal{B}(m,k)$ be homogeneous, and suppose that $w_i(a) \in \mathbb{Z}$. Then $a \cdot R_i \neq 0$ implies that $a \cdot U_i \neq 0$; similarly, $R_i \cdot a \neq 0$ implies that $U_i \cdot a \neq 0$.*

Proof. Suppose that $b = a \cdot R_i \neq 0$, and $b = \mathbf{I_x} \cdot b \cdot \mathbf{I_y}$. Since $w_i(a) \in \mathbb{Z}$ and $b \neq 0$, it follows that $w_i^{\mathbf{x},\mathbf{y}} = \frac{1}{2}$. Thus, it follows that i is not in any of the generating intervals for $\mathbf{x}$ and $\mathbf{y}$, so $b \cdot U_i \neq 0$, so $a \cdot U_i \neq 0$, as well. □

Proposition 3.9. *A homogeneous non-zero algebra element $a = \mathbf{I_x} \cdot a \cdot \mathbf{I_y} \in \mathcal{B}(m,k)$ is uniquely characterized by its initial (or terminal) idempotent and its weight.*

Proof. Fix initial and terminal idempotent states $\mathbf{x}$ and $\mathbf{y}$. By Proposition 3.7, $\mathbf{I_x} \cdot \mathcal{B}(m,k) \cdot \mathbf{I_y}$ is the quotient of the polynomial algebra $\mathbb{F}[U_1,\dots,U_m]$ by an ideal which is homogeneous with respect to the weights. An element of this quotient space in turn is uniquely determined by its various weights.

Note that the weight of a modulo 1 determines $w^{\mathbf{x},\mathbf{y}}$. It follows that $\mathbf{x}$ and $w(a)$ determines $\mathbf{y}$. □

3.3. **Defining $\mathcal{B}(m,k,\mathcal{S})$.** Let $\mathcal{S} \subset \{1,\dots,m\}$ be a subset. Define $\mathcal{B}(m,k,\mathcal{S})$ to be the differential graded algebra obtained by adjoining new algebra elements C_i for $i \in \mathcal{S}$ to $\mathcal{B}(m,k)$, which satisfy the following properties:

- The C_i commute with all other algebra elements in $\mathcal{B}(m,k,\mathcal{S})$.
- The square of C_i vanishes.
- The differential of C_i is U_i.

This construction can be done since the U_i are in the center of $\mathcal{B}(m,k,\mathcal{S})$ and $dU_i = 0$. More formally, if $\mathcal{S} = \{i_1, ..., i_n\}$, then $\mathcal{B}(m,k,\mathcal{S})$ is defined by the formula

$$\mathcal{B}(m,k,\mathcal{S}) = \frac{\mathcal{B}(m,k)[C_{i_1},\dots,C_{i_n}]}{\{C_j^2 = 0, dC_j = U_j\}_{j\in\mathcal{S}}}.$$

The algebra $\mathcal{B}(m,k,\mathcal{S})$ is equipped with a distinguished basis as an $\mathbb{F}$-vector space, with basis vectors corresponding to the following data:

- A pair of idempotents $\mathbf{x}$ and $\mathbf{y}$ that are not far,
- a monomial p in $\mathbb{F}[U_1,\dots,U_m]$ that is not divisible by any monomial associated to any generating interval for $\mathbf{x}$ and $\mathbf{y}$,
- a (possibly empty) subset J of $\mathcal{S}$.

The corresponding algebra element is $\phi^{\mathbf{x},\mathbf{y}}(p)\cdot\prod_{j\in J} C_j$. We call these basis vectors *pure algebra elements.*

3.4. **Gradings.** Note first since $\mathcal{B}(m,k)$ is obtained as a quotient of $\mathcal{B}_0(m,k)$ by a w-homogeneous ideal, the w-grading by $(\frac{1}{2}\mathbb{Z})^m$ descends to a grading on $\mathcal{B}(m,k)$. The w-gradings extend to $\mathcal{B}(m,k,\mathcal{S})$, by declaring $w_i(C_j) = 1$ if $i = j$ and 0 otherwise.

Explicitly, for $1 \leq i \leq m$, the i^{th} weight w_i of L_i and R_i is $1/2$, and w_i of U_i and C_i is 1, and w_i vanishes on L_j, R_j, U_j, and C_j with $j \neq i$. These functions each induce gradings on $\mathcal{B}(m,k,\mathcal{S})$; i.e. if a and b are homogenous elements with $a\cdot b\neq 0$, then $w_i(a\cdot b) = w_i(a) + w_i(b)$.

We will consider the following specialization, called the *Alexander grading*:

$$\mathrm{Alex}(a) = -\sum_{s\in\mathcal{S}} w_s(a) + \sum_{t\notin\mathcal{S}} w_t(a). \tag{3.8}$$

For $i = 1,\dots,m$, there is a filtration $\mathfrak{m}_i$ $\mathcal{B}(m,k,\mathcal{S})$ with values in $\{0,1\}$, specified by the function on pure algebra elements b so that $\mathfrak{m}_i(b) = 1$ if b is divisible by C_i and 0 otherwise. This extends to a filtration on the algebra:

$$\mathcal{B}(m,k,\mathcal{S}) = \mathcal{B}(m,k,\mathcal{S})_{\mathfrak{m}_i=0} \oplus \mathcal{B}(m,k,\mathcal{S})_{\mathfrak{m}_i=1},$$

where $\mathcal{B}(m,k,\mathcal{S})_{\mathfrak{m}_i=c}$ for $c = 0$ or 1 is the vector space spanned by generators $a \in \mathcal{B}(m,k,\mathcal{S})$ with $\mathfrak{m}_i(a) = c$. We call this a filtration, because the differential on the algebra d preseves $\mathcal{B}(m,k,\mathcal{S})_{\mathfrak{m}_i=0}$, but not $\mathcal{B}(m,k,\mathcal{S})_{\mathfrak{m}_i=1}$. Elements of $\mathcal{B}(m,k,\mathcal{S})$ that are contained entirely in $\mathcal{B}(m,k,\mathcal{S})_{\mathfrak{m}_i=0}$ or $\mathcal{B}(m,k,\mathcal{S})_{\mathfrak{m}_i=1}$ are called $\mathfrak{m}_i$-*homogeneous.* Note that $\sum_{i=1}^m \mathfrak{m}_i$ induces a $\mathbb{Z}$-grading on $\mathcal{B}(m,k,\mathcal{S})$ that drops by one under the differential. We will be interested in a different normalization of this, the *Maslov grading*, defined by

$$\mathfrak{m}(a) = \#(\ C_i \text{ in } a) - 2\sum_{s\in\mathcal{S}} w_s(a) = \sum_{i=1}^m \mathfrak{m}_i(a) - 2\sum_{s\in\mathcal{S}} w_s(a). \tag{3.9}$$

3.5. **Examples.** When $k = 0$, $\mathcal{B}(m,0)\cong\mathbb{F}$; where $1 = \mathbf{I}_\emptyset$. The elements L_i, R_i, and U_i are zero. When $k = m+1$, $\mathcal{B}(m,m+1) = \mathbb{F}[U_1,...,U_m]$; there is only one idempotent $1 = \mathbf{I}_{\{0,\dots,m\}}$.

Some other examples can be illustrated by the use of path algebras. Given a directed graph Γ, the *path algebra* is the $\mathbb{F}$-vector space generated by sequences of edges $e_1 * \cdots * e_n$ where the terminal vertex of e_i is the initial vertex of e_{i+1}. We include also trivial paths, based at any vertex. If two paths can be concatenated, then their product is the concatenation; otherwise it is zero.

For example, consider Figure 9, which is a graph with two vertices and four edges. Two of the edges (closed loops) are labelled by U, but they are not the same, as they have different initial points. Identifying $L{*}R$ and $R{*}L$ with the corresponding closed loops labelled by U, we obtain the algebra $\mathcal{B}(1,1)$.

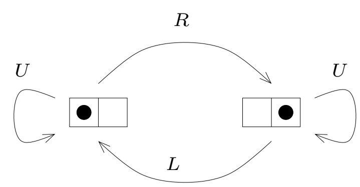


FIGURE 9. **Picture of $\mathcal{B}(1,1)$.** The two idempotents $\mathbf{I}_{\{0\}}$ and $\mathbf{I}_{\{1\}}$ are pictured, with arrows corresponding to algebra elements connecting idempotents. The algebra $\mathcal{B}(1,1)$ is the quotient of the pictured path algebra, where $R * L$ and $L * R$ are identified with the two closed loops labelled by U (that are distinguished by their starting points).

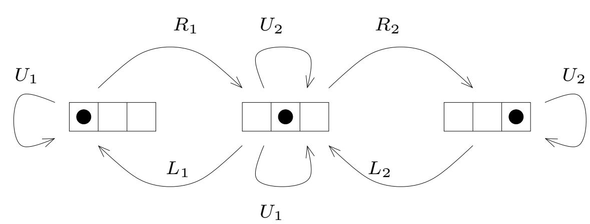


FIGURE 10. **Picture of $\mathcal{B}(2,1)$.** The three idempotents $\mathbf{I}_{\{0\}}$, $\mathbf{I}_{\{1\}}$, and $\mathbf{I}_{\{2\}}$ are pictured, with arrows corresponding to algebra elements connecting idempotents. The algebra $\mathcal{B}(2,1)$ can be thought of as a quotient of the pictured path algebra, divided out by relations $R_1 * R_2 = 0$, $L_2 * L_1 = 0$, $R_i * L_i = U_i$, $L_i * R_i = U_i$, $U_1 * U_2 = U_2 * U_1$.

3.6. **Symmetries in the algebras.** Consider the map $\rho\colon \{0,\dots,m\} \to \{0,\dots,m\}$ with $\rho(i) = m - i$. There is a map

$$\mathcal{R}\colon \mathcal{B}(m,k,\mathcal{S}) \to \mathcal{B}(m,k,\rho'_m(\mathcal{S})) \tag{3.10}$$

where $\rho'_m(i) = m + 1 - i$, characterized by the following properties:

- $\mathcal{R}(\mathbf{I}_{\mathbf{x}}) = \mathbf{I}_{\rho(\mathbf{x})}$.
- If $a \in \mathcal{B}(m,k)$ is non-zero and homogeneous with weights $(w_1(a),\dots,w_m(a))$, then $\mathcal{R}(a) = b$ is the non-zero element that is homogeneous with weights given by $w_i(b) = w_{m+1-i}(a)$.

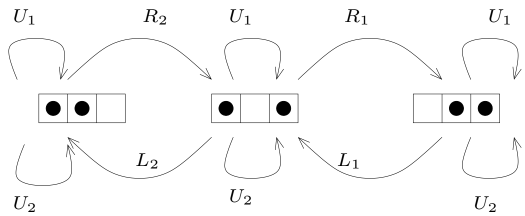


FIGURE 11. **Picture of $\mathcal{B}(2,2)$.** The three idempotents $\mathbf{I}_{\{0,1\}}$, $\mathbf{I}_{\{0,2\}}$, and $\mathbf{I}_{\{1,2\}}$ are pictured, with arrows corresponding to algebra generators. The algebra $\mathcal{B}(2,2)$ can be thought of as a quotient of the pictured path algebra by $R_i * L_i = U_i$, $L_i * R_i = U_i$, $U_1 * L_2 = L_2 * U_1$, $U_1 * R_2 = R_2 * U_1$, $U_2 * L_1 = L_1 * U_2$, and $U_2 * R_1 = R_1 * U_2$.

We extend this to $\mathcal{B}(m,k,\mathcal{S}) \supset \mathcal{B}(m,k)$ by requiring $\mathcal{R}(C_j \cdot a) = C_{m+1-j} \cdot \mathcal{R}(a)$. Note that for all $i = 1,\dots,m$ and $j \in \mathcal{S}$,

$$\mathcal{R}(L_i) = R_{m+1-i} \quad \mathcal{R}(R_i) = L_{m+1-i} \quad \mathcal{R}(U_i) = U_{m+1-i} \quad \mathcal{R}(C_j) = C_{m+1-j}$$

Clearly, this induced map $\mathcal{R}$ induces an isomorphism of algebras.

Another symmetry identifies the algebra with its "opposite" algebra. Specifically, the map $o(\mathbf{I_x}) = \mathbf{I_x}$ extends to an isomorphism of rings

$$o\colon \mathcal{B}(m,k,\mathcal{S}) \to \mathcal{B}(m,k,\mathcal{S})^{\mathrm{op}}, \tag{3.11}$$

with $o(L_i) = R_i$, $o(R_i) = L_i$, $o(U_i) = U_i$, and $o(C_j) = C_j$.

3.7. **Canonical DD bimodules.** Let

$$\mathcal{B}_1 = \mathcal{B}(m,k_1,\mathcal{S}_1), \quad \mathcal{B}_2 = \mathcal{B}(m,k_2,\mathcal{S}_2) \tag{3.12}$$

where $k_1 + k_2 = m+1$ and $\mathcal{S}_2 = \{1,...,m\} - \mathcal{S}_1$.

Note that there is a natural one-to-one correspondence between the I-states for $\mathcal{B}_1$ and $\mathcal{B}_2$: if $\mathbf{x} \subset \{0,\dots,m\}$ is a k_1-element subset, then its complement $\mathbf{x}'$ is a k_2-element subset of $\{0,\dots,m\}$. In this case, we say that $\mathbf{x}$ and $\mathbf{x}'$ are *complementary I-states*.

We define a DD bimodule $\mathcal{B}_1$ and $\mathcal{B}_2$ as follows. Let $\mathcal{K}$ be the $\mathbb{F}$-vector space whose generators $k_{\mathbf{x}}$ correspond to I-states for $\mathcal{B}(m,k_1,\mathcal{S}_1)$. We give $\mathcal{K}$ the structure of a left module over $\mathbf{I}(\mathcal{B}_1) \otimes \mathbf{I}(\mathcal{B}_2)$, determined by

$$(\mathbf{I_y} \otimes \mathbf{I_w}) \cdot k_{\mathbf{x}} = \begin{cases} k_{\mathbf{x}} & \text{if } \mathbf{x} = \mathbf{y} \text{ and } \mathbf{w} \text{ is complementary to } \mathbf{x} \\ 0 & \text{otherwise.} \end{cases}$$

The algebra element

$$A = \sum_{i=1}^{m} (L_i \otimes R_i + R_i \otimes L_i) + \sum_{s \in \mathcal{S}_1} C_s \otimes U_s + \sum_{t \in \mathcal{S}_2} U_t \otimes C_t \in \mathcal{B}_1 \otimes \mathcal{B}_2$$

specifies a map $\delta^1\colon \mathcal{K} \to \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{K}$ by $\delta^1(v) = A \otimes v$ (where the tensor product is taken over $\mathbf{I}(\mathcal{B}_1) \otimes \mathbf{I}(\mathcal{B}_2)$).

These data can be represented graphically, as follows. Take m vertical segments, and orient them arbitrarily. The upwards pointing segments specify $\mathcal{S}_2$ and the downwards pointing ones specify $\mathcal{S}_1$. We draw a horizontal arc crossing each of the vertical segment as a placeholder, and keep only half of each vertical segment above (resp. below) the horizontal arc if the segment is oriented upwards (resp. downwards). The horizontal arc is divided into $m+1$ intervals by the vertical arcs. An element of $k_{\mathbf{x}}$ is represented as a collection of $m+1$ dark dots corresponding to the intervals in the horizontal segment, each of distributed either above or below te horizontal segment. The set of intervals whose dots are below give $\mathbf{x}$. When illustrating generators of $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{K}$, we draw the algebra element from $\mathcal{B}_1$ below the diagram for $k_{\mathbf{x}}$ and the algebra element from $\mathcal{B}_2$ above. See Figure 12 for an illustration.

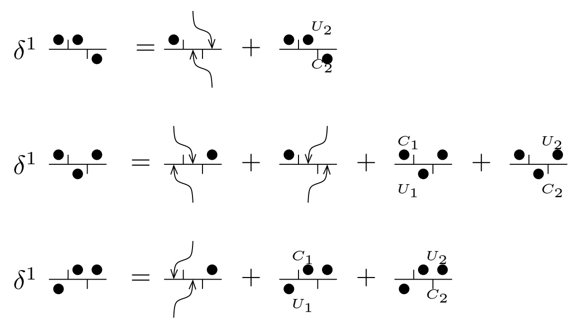


FIGURE 12. **The canonical DD bimodule.** In the left column, we have the three generators for the DD bimodule, with $\mathcal{B}_1 = \mathcal{B}(2,1,\{2\})$ and $\mathcal{B}_2 = \mathcal{B}(2,2,\{1\})$. They correspond to the I-states $\{2\}$, $\{1\}$, and $\{0\}$ respectively. To the right we have the non-zero terms in the differential. To read off the algebra element in $\mathcal{B}_1$, reflect the picture vertically. For example, if the three generators on the left are denoted E, F, and G, then the first equation expresses $\delta^1(E) = (L_2 \otimes R_2) \otimes F + (C_2 \otimes U_2) \otimes E$.

Lemma 3.10. *The map δ^1 satisfies the type DD structure relation.*

Proof. This is equivalent to the statement that $dA + A \cdot A = 0$, thought of as an element of $\mathcal{B}_1 \otimes \mathcal{B}_2$. We consider the four types of terms A: $L_i \otimes R_i$, $R_i \otimes L_i$, $C_i \otimes U_i$ when $i \in \mathcal{S}_1$, and $U_i \otimes C_i$ when $i \in \mathcal{S}_2$. In $dA + A \cdot A$, some of the terms cancel since U_i and C_i are in the center; other terms cancel when the indices i and j are sufficiently far ($|i-j| > 1$). When $|i-j| = 1$, terms of the first type and the second type cancel. When $i = j$, a term of the first and second type cancel with differentials of terms of the third or fourth type. □

The two out-going algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are graded by $\frac{1}{2}\mathbb{Z}^m$,

$$\mathbf{gr}_{\mathcal{B}_1}(a_1) = (w_1(a_1), \dots, w_m(a_1)) \qquad \mathbf{gr}_{\mathcal{B}_2}(a_2) = (w_1(a_2), \dots, w_m(a_2));$$

the canonical DD bimodule is also graded by $\frac{1}{2}\mathbb{Z}^m$ where $(v_1, v_2) \in \frac{1}{2}\mathbb{Z}^m \oplus \frac{1}{2}\mathbb{Z}^m$ acts on $w \in \frac{1}{2}\mathbb{Z}^m$ by $(v_1, v_2) \cdot w = (v_1 - v_2 + w)$. In fact, the module is supported in grading 0; i.e. for each term $a_1 \otimes a_2$ in A specifying δ^1, $\mathbf{gr}_{\mathcal{B}_2}(a_2) = \mathbf{gr}_{\mathcal{B}_1}(a_1)$.

3.8. **The canonical DD bimodule is invertible.**

3.8.1. *A candidate for the inverse module.* The main result is to prove that the bimodule defined in the previous section is invertible. Let

$$Y_{\mathcal{B}_1,\mathcal{B}_2} = \mathrm{Mor}^{\mathcal{B}_1}(({}_{\mathcal{B}_2}(\mathcal{B}_2)_{\mathcal{B}_2})\boxtimes_{\mathcal{B}_2}({}^{\mathcal{B}_1,\mathcal{B}_2}\mathcal{K}), {}^{\mathcal{B}_1}\mathbb{I}_{\mathcal{B}_1}).$$

This is naturally a $\mathcal{B}_2$-$\mathcal{B}_1$-bimodule (of type AA, with both actions on the right).

As a vector space, Y is spanned by elements of the form $(\overline{a}\big|b)$, where $\overline{a}\in\overline{\mathcal{B}_2}$ and $b\in\mathcal{B}_1$, where here $\overline{\mathcal{B}}$ is opposite bimodule to $\mathcal{B}$, thought of as a bimodule (as in Equation (2.6)), subject the restriction that the left idempotent of $\overline{a}$ is complementary to the left idempotent of b. Recall that $\overline{\mathcal{B}_2}$ is the $\mathcal{B}_2$-module consisting of maps from $\mathcal{B}_2$ to $\mathbb{F}$. This is a left-right $\mathcal{B}_2$-$\mathcal{B}_2$ bimodule by the rule

$$x\cdot\overline{a}\cdot y = (\xi\mapsto\overline{a}(y\cdot\xi\cdot x)),$$

for $x,y\in\mathcal{B}_2$ and $\overline{a}\in\overline{\mathcal{B}_2}$. We can take these generating vectors so that $\overline{a}$ is dual to a generating algebra element in $\mathcal{B}_2$; note that the right idempotent of a is the left idempotent of its dual $\overline{a}$.

The differential on Y has terms $(\overline{a}|b)\mapsto(L_i\cdot\overline{a}|R_i\cdot b)$, $(\overline{a}|b)\mapsto(R_i\cdot\overline{a}|L_i\cdot b)$, and furthermore $(\overline{a}|b)\mapsto(U_i\cdot\overline{a}|C_i\cdot b)$ if $i\in\mathcal{S}_1$; otherwise $(\overline{a}|b)\mapsto(C_i\cdot\overline{a}|U_i\cdot b)$. Finally, there are terms $(\overline{a}|C_i\cdot b)\mapsto(\overline{a}|U_i\cdot b)$ or $(\overline{U_i\cdot a}|b)\mapsto(\overline{C_i\cdot a}|b)$. The action by $\mathcal{B}_2$-$\mathcal{B}_1$ is given by

$$(\overline{a}|b)\cdot(b_2\otimes b_1) = (\xi\mapsto\overline{a}(b_2\cdot\xi)\big|b\cdot b_1)$$

We draw pictures of this action as follows. We draw a pair $(\overline{a}|b)$ where a and b are pure algebra elements, by drawing first a graphical representation of a (dual to $\overline{a}$) on top of a graphical representation for b. In this picture, the right idempotent of b is on the bottom, and the right idempotent of $\overline{a}$ is the initial state (on the top) of a. See Figure 13.

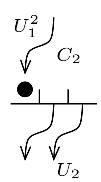


FIGURE 13. **Graphical representative of elements of Y.** Here we have a picture of $(\overline{L_1\cdot U_1^2\cdot C_2}|L_1\cdot L_2\cdot U_2)$ in Y, thought of as an element of $\overline{\mathcal{B}_2(2,1,\{1,2\})}\otimes\mathcal{B}_1(2,2,\emptyset)$.

3.8.2. *An example:* Consider $\mathcal{B}_1=\mathcal{B}(1,1,\{1\})$ and $\mathcal{B}_2=\mathcal{B}(1,1,\emptyset)$. The algebra $\mathcal{B}_2$ has generators L, R, and U, and idempotents $\mathbf{I}_{\{0\}}$ and $\mathbf{I}_{\{1\}}$. Moreover,

$$\mathbf{I}_{\{0\}}\cdot R\cdot\mathbf{I}_{\{1\}} = R \qquad \mathbf{I}_{\{1\}}\cdot L\cdot\mathbf{I}_{\{0\}} = L,$$

while $\mathcal{B}_1$ has the same generators, and an additional C. The bimodule decomposes into four summands, according to the right idempotent of $(\overline{a}|b)$.

In right idempotent $\mathbf{I}_{\{1\}}\otimes\mathbf{I}_{\{0\}}$, the complex further decomposes into summands. One of these summands contains the single element $(\overline{\mathbf{I}_{\{1\}}}|\mathbf{I}_{\{0\}})$. Another summand is the square

$$\begin{array}{ccccc} & & (\overline{L}|L) & & \\ & \nearrow & & \searrow & \\ (\overline{\mathbf{I}_{\{1\}}\cdot U}|\mathbf{I}_{\{0\}}) & & & & (\overline{\mathbf{I}_{\{1\}}}|U\cdot \mathbf{I}_{\{0\}}) \\ & \searrow & & \nearrow & \\ & & (\overline{\mathbf{I}_{\{1\}}}|C\cdot \mathbf{I}_{\{0\}}) & & \end{array} \tag{3.13}$$

See Figure 14 for a picture. (In fact, there are infinitely many different summands.)

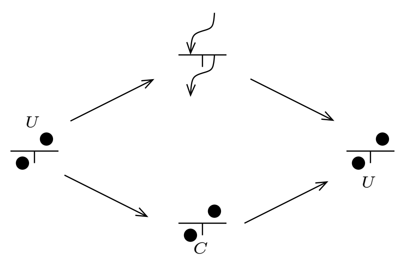


FIGURE 14. Terms in the differential of Y. We have drawn pairs of algebra elements; the top algebra element should be dualized to get the corresponding generator.

In right idempotent $\mathbf{I}_{\{0\}}\otimes\mathbf{I}_{\{0\}}$, there is a collection of acyclic complexes. For example, in the portion with total weight $1/2$, we have:

$$(\overline{R}|\mathbf{I}_{\{0\}}) \longrightarrow (\overline{\mathbf{I}_{\{0\}}}|L). \tag{3.14}$$

In the portion with total weight $3/2$, we have:

$$\begin{array}{ccccccc} & & & & (\overline{R}|C\cdot\mathbf{I}_{\{0\}}) & \longrightarrow & (\overline{\mathbf{I}_{\{0\}}}|L\cdot C) \\ & \nearrow & & \nearrow & \downarrow & & \downarrow \\ (\overline{R\cdot U}|\mathbf{I}_{\{0\}}) & \longrightarrow & (\overline{\mathbf{I}_{\{0\}}\cdot U}|L) & \longrightarrow & (\overline{R}|U\cdot\mathbf{I}_{\{0\}}) & \longrightarrow & (\overline{\mathbf{I}_{\{0\}}}|L\cdot U). \end{array} \tag{3.15}$$

3.8.3. *The candidate is the inverse.* We show now that the candidate inverse from Section 3.8.1 is indeed the inverse for the canonical type DD bimodule, by verifying the hypotheses of Lemma 2.16. Continuing notation from earlier, the algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are as in Equation (3.12); Y is as in Section 3.8.1, generated by pairs $(\overline{a}|b)$, with $a\in\mathcal{B}_2$ and $b\in\mathcal{B}_1$.

Proposition 3.11. *The rank of the homology group of Y is $\binom{m+1}{k}$; it is generated by the elements of the form $(\overline{\mathbf{I}_{\mathbf{x}'}}|\mathbf{I}_{\mathbf{x}})$, where $\mathbf{x}$ and $\mathbf{x}'$ are complementary idempotents.*

Lemma 3.12. *The candidate complex Y decomposes into a direct sum of complexes $C(Z,\mathbf{x},\mathbf{y})$, indexed by idempotent states $\mathbf{x}$ and $\mathbf{y}$ and $Z\in(\frac{1}{2}\mathbb{Z})^m$, where $C(Z,\mathbf{x},\mathbf{y})$ is the vector space generated by pairs $(\overline{a}|b)$ where a and b are pure algebra elements with $\mathbf{I}_{\mathbf{x}}\cdot a=a$ and $b\cdot\mathbf{I}_{\mathbf{y}}=b$, and $w(a)+w(b)=Z$.*

Proof. Since the left idempotent of a is the right idempotent of $\overline{a}$, the splitting by idempotents corresponds to the splitting of Y according to right idempotents. The fact that the splitting by Z is well defined follows immediately from the definition of ∂. (More abstractly, it is a formal consequence of the fact that δ^1 on the tautological DD bimodule is specified by an algebra element $A=\sum a_i\otimes b_i$ where the weight of a_i equals the weight of b_i.) ☐

Write the differential ∂ on Y as a sum of terms $\partial=\sum_{i=1}^m\partial_i$, where ∂_i involves terms on the i^{th} strand. More precisely, there is a differential $d_i\colon\mathcal{B}(m,k,\mathcal{S})\to\mathcal{B}(m,k,\mathcal{S})$ that vanishes unless $i\in\mathcal{S}$, in which case $d_i(C_i)=U_i$, but $d_i(C_j)=0$ for all $j\neq i$ and $d_i(L_j)=d_i(U_j)=d_i(R_j)=0$ for all $j=1,\dots,m$. Define

$$\partial_i(\overline{a}|b)=\begin{cases}(\overline{a}|d_ib)+(R_i\cdot\overline{a}|L_i\cdot b)+(L_i\cdot\overline{a}|R_i\cdot b)+(U_i\cdot\overline{a}|C_i\cdot b) & \text{if } i\in\mathcal{S}_1\\ (\overline{d}_i\overline{a}|b)+(R_i\cdot\overline{a}|L_i\cdot b)+(L_i\cdot\overline{a}|R_i\cdot b)+(C_i\cdot\overline{a}|U_i\cdot b) & \text{if } i\in\mathcal{S}_2,\end{cases}\tag{3.16}$$

where $\overline{d}_i\colon\overline{\mathcal{B}}_2\to\overline{\mathcal{B}}_2$ is dual to the differential $d_i\colon\mathcal{B}_2\to\mathcal{B}_2$.

Lemma 3.13. *The differential ∂ on Y can be written as $\partial=\sum_{i=1}^m\partial_i$, where $\partial_i^2=0$ and $\partial_i\circ\partial_j=\partial_j\circ\partial_i$ for $i\neq j$.*

Proof. This can be seen directly. A little more conceptually, on homogeneous generators $(\overline{a}|b)$ where a and b are pure algebra elements, the functions $w_j(a)$ for $j=1,\dots,m$ and

$$\mathfrak{m}_j(\overline{a}|b)=\begin{cases}\mathfrak{m}_j(b) & \text{if } j\in\mathcal{S}_1\\ 1-\mathfrak{m}_j(a) & \text{if } j\in\mathcal{S}_2,\end{cases}\tag{3.17}$$

for $j=1,\dots,m$ induce a filtration on $C(Z,\mathbf{x},\mathbf{y})$ by $(\frac{1}{2}\mathbb{Z})^m\times\{0,1\}^m$. Restricting to values with $j\neq i$, we have a filtration on $C(Z,\mathbf{x},\mathbf{y})$ with $(\frac{1}{2}\mathbb{Z})^{m-1}\times\{0,1\}^{m-1}$. The associated graded object is clearly $(C(Z,\mathbf{x},\mathbf{y}),\partial_i)$. ☐

Proposition 3.14. *Fix idempotents $\mathbf{x}$ and $\mathbf{y}$ and a total weight Z; suppose moreover that the weight Z is non-zero. Then there is a position i with the property that $H_*(C(Z,\mathbf{x},\mathbf{y}),\partial_i)=0$*

Proposition 3.14 is proved using the following lemma:

Lemma 3.15. *The chain complex $(C(Z,\mathbf{x},\mathbf{y}),\partial_i)$ splits into subcomplexes indexed by $(\frac{1}{2}\mathbb{Z})^{m-1}\times\{0,1\}^{m-1}$ that are spanned by $(\overline{a}|b)$, where a and b are pure algebra elements whose weights $w_j(b)$ with $j\neq i$ and values $\mathfrak{m}_j(\overline{a}|b)$ (as defined in Equation* (3.17) *above) for all $j\neq i$ are specified.*

Proof. This follows from the form of ∂_i: it does not involve any of the C_j with $j\neq i$, and it does not change the weight of b away from i. ☐

Lemma 3.16. *For fixed Z and $i \in \{1,\dots,m\}$, $\mathbf{x}$, $\mathbf{y}$ and $(\frac{1}{2}\mathbb{Z})^{m-1}\times\{0,1\}^{m-1}$, consider the corresponding subcomplex C of $(C(Z,\mathbf{x},\mathbf{y}),\partial_i)$ as in Lemma 3.15. Then $H(C)\neq 0$ precisely when there is a single element in C. When the homology of C is non-zero, there are pure algebra elements a and b, and a single generator $(\overline{a}|b)$ in C; and one of the following holds:*

- $w_i(a)=w_i(b)=0$
- $w_i(a)+w_i(b)=\frac{1}{2}$
- $w_i(a)+w_i(b)=1$ *and either* $a=a_0\cdot C_i$ *or* $b=b_0\cdot C_i$.

In particular, writing $z_i=w_i(a)+w_i(b)$, if $H(C(Z,\mathbf{x},\mathbf{y}),\partial_i)\neq 0$, then $z_i\leq 1$.

Proof. When $z_i=0$, the statement is clear; otherwise, there are cases, according to the local picture of the summand of $C(Z,\mathbf{x},\mathbf{y})$ near i. Specifically, if $C(Z,\mathbf{x},\mathbf{y})$ contains an element $(\overline{a}|b)$ with $a=\mathbf{I_x}\cdot a\cdot\mathbf{I_w}$, we consider separately the two cases:

(C-1) $i-1, i\in\mathbf{w}$.
(C-2) Exactly one of $i-1$ or i is in $\mathbf{w}$.

There is a third case, where neither $i-1,i\in\mathbf{w}$; but that is symmetric to Case (C-1) by a symmetry that exchanges roles of $\mathcal{B}_1$ and $\mathcal{B}_2$, and then dualizes C.

There are five further subcases of Case (C-1), as follows. if $i-1$ and i are the j^{th} and $j+1^{st}$ terms in the sequence $\mathbf{w}$, we further subdivide according to the placement of the j^{th} and the $j+1^{st}$ terms in the sequence of $\mathbf{x}$. Since $\mathbf{x}$ and $\mathbf{w}$ are not too far, the pair $\{j,j+1\}$ must be one of $\{i-2,i-1\}$, $\{i,i+1\}$, $\{i-2,i\}$, $\{i-1,i+1\}$, $\{i-1,i\}$, $\{i-2,i+1\}$.

The cases where $\{j,j+1\}=\{i-2,i\}$ and $\{i-1,i+1\}$ are exchanged by reflection of through a vertical axis (compare the map $\mathcal{R}$ from Equation (3.10)), as are $\{i-2,i-1\}$ and $\{i,i+1\}$. Dropping two symmetric cases, we arrive at the four pictures in the first three columns of Figure 15.

Similarly, in Case (C-2), if $i-1\in\mathbf{w}$ (which can be arranged after a vertical reflection) is the j^{th} term in $\mathbf{w}$ and i is the k^{th} one in the left idempotent of b, then we further subdivide according to the placement of the j^{th} and k^{th} terms in $\mathbf{x}$ and $\mathbf{y}$ respectively. These possibilities (after eliminating symmetric duplicates) are represented in the remaining pictures in Figure 15.

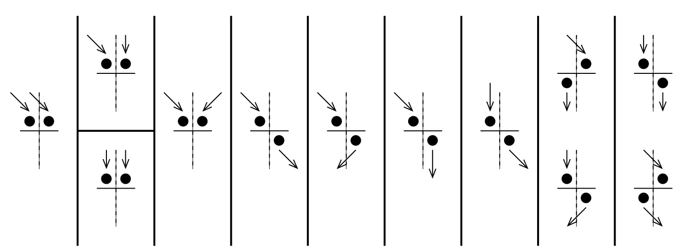


FIGURE 15. $C(Z,\mathbf{x},\mathbf{y})$ **cases.**

We consider now Cases (C-1) and (C-2) separately, and further subdivide them into subcases (that are related to the cases in Figure 15).

Case (C-1), with $z_i = 1/2$. (The first column in Figure 15.) There is now a single generator of this type, of the form $(\overline{a}|b)$ with $w_i(a) = 1/2$ and $w_i(b) = 0$. so the homology is one-dimensional.

Case (C-1), with $z_i \notin \mathbb{Z}$ and $z_i > 1/2$. (Again, this is illustrated in first column in Figure 15.) This has two further subcases, according to whether $i \in \mathcal{S}_1$ or $\mathcal{S}_2$ (i.e. whether $C_i \in \mathcal{B}_1$ or $\mathcal{B}_2$). If $i \in \mathcal{S}_2$, there are two generators of the complex, $(\overline{aU_i^k}|b)$ and $(\overline{aU_i^{k-1}C_i}|b)$ where $k = z_i - \frac{1}{2} \geq 1$, and a and b are fixed pure algebra elements with $w_i(a) = \frac{1}{2}$, $w_i(b) = 0$. To see that both types of terms appear, note that under the present hypotheses on $a = a \cdot \mathbf{I}_{\mathbf{w}}$ and $\mathbf{w}$, if $a \neq 0$ then $U_i \cdot a \neq 0$, as well. Clearly,

$$\partial_i(\overline{aU_i^k}|b) = (\overline{aU_i^{k-1}C_i}|b),$$

so the homology of the corresponding complex vanishes. When $i \notin \mathcal{S}_2$, the terms are of the form $(\overline{aU_i^n}|b)$ and $(\overline{aU_i^{n-1}}|C_i b)$, and the homology again vanishes.

Case (C-1), with $z_i \in \mathbb{Z}$. When $i \notin \mathcal{S}_2$, we either have a single element $(\overline{a}|C_i b)$ with $w_i(a) = w_i(b) = 0$ (as in the third column of Figure 15; this can also appear in the second column), or elements of the form $(\overline{aU_i^j}|b)$ and $(\overline{aU_i^j}|C_i b)$, with fixed a, b so that $w_i(a) = w_i(b) = 0$, where $j = z_i$ and the differential is given by

$$\partial_i(\overline{aU_i^j}|b) = (\overline{aU_i^{j-1}}|C_i \cdot b);$$

so the homology is trivial. The case where $i \in \mathcal{S}_2$ works similarly. We either have a single element element $(\overline{aC_i}|b)$, $w_i(a) = w_i(b) = 0$ or elements of the form $(\overline{aU_i^j}|b)$ and $(\overline{aU_i^{j-1}C_i}|b)$. Once again, the differentials cancel out the homology.

Case (C-2), with $z_i = \frac{1}{2}$. (This is the fifth or the eight column of Figure 15.) There are two possibly non-zero elements in the complex. These elements have the form $(\overline{a \cdot R_i}|b)$ and $(\overline{a}|L_i \cdot b)$, where $w_i(a) = w_i(b) = 0$. There are two subcases: in one subcase, only one of $a \cdot R_i$ or $L_i \cdot b$ is zero, so we have a single generator. (This case occurs in the fifth column of Figure 15.) In the other case, when both $a \cdot R_i$ and $L_i \cdot b$ are non-zero (which occurs now only in the eighth column of Figure 15), we have that $\partial_i(\overline{a \cdot R_i}|b) = (\overline{a}|L_i \cdot b)$, so the homology is trivial.

Case (C-2), with $z_i \notin \mathbb{Z}$, and $z_i = \frac{1}{2} + t > \frac{1}{2}$. (These cases are illustrated in the fifth and eighth columns of Figure 15.) Suppose that $i \notin \mathcal{S}_2$. We can assume that all the generators a corresponding summand of $C(Z, \mathbf{x}, \mathbf{y})$ have the following form $(\overline{aR_iU_i^j}|bU_i^k)$, $(\overline{aR_iU_i^j}|bC_iU_i^{k-1})$, $(\overline{aU_i^j}|L_ibU_i^k)$, and $(\overline{aU_i^j}|L_ibC_iU_i^{k-1})$, for fixed a and b with $w_i(a) = w_i(b) = 0$, and $j + k + \frac{1}{2} = z_i$.

Note that at least one of $aR_i \neq 0$ or $L_ib \neq 0$, otherwise the chain complex would be trivial. If $aR_i \neq 0$ and $L_ib = 0$, then there are exactly two elements in the chain complex, $(\overline{aR_iU_i^t}, b)$ and $(\overline{aU_i^{t-1}}, C_ib)$, and the differential cancels them. (Again, cases of this kind can occur in either the fifth or the eighth columns of Figure 15.) If $aR_i = 0$ and $L_ib \neq 0$, there are once again two terms, $(\overline{a}, L_ibU_i^t)$ and $(\overline{a}, L_ibC_iU_i^{t-1})$, and these two terms cancel in the differential.

In the remaining case where $aR_i \neq 0$ and $L_ib \neq 0$, we show that $H(C(Z, \mathbf{x}, \mathbf{y}), \partial_i) = 0$, using a further filtration on the complex, induced by the function on pure algebra elements a and b that associates to $(\overline{a}|b)$ the i^{th} weight of b, $w_i(b)$. This induces a grading on the vector space underlying $C(Z, \mathbf{x}, \mathbf{y}) = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \mathcal{F}_k$.

Writing $\partial_0^i(\overline{a}|b) = (\overline{a}|d_i b)$, we have that $\partial_i = \partial_0^i + L$, where $\partial_0^i\colon \mathcal{F}_k \to \mathcal{F}_k$, and $L\colon \mathcal{F}_k \to \bigoplus_{\ell>k} \mathcal{F}_\ell$. (We are using the form of ∂_i from Equation (3.16).) Thus, the associated graded complex on $(C(Z,\mathbf{x},\mathbf{y}),\partial_i)$ is equipped with the differential ∂_0^i, and we will show that its homology vanishes.

When $k\in\mathbb{Z}$, the complex $\mathcal{F}_k$ is spanned by $(\overline{aR_iU_i^j}, bU_i^k)$ and $(\overline{aR_iU_i^j}, bC_iU_i^{k-1})$, and they are connected by a differential in ∂_0^i. Similarly, when $k\geq 1$, the complex $\mathcal{F}_{k+\frac{1}{2}}$ contains the two elements when $(\overline{aU_i^j}, L_ibU_i^k)$, and $(\overline{aU_i^j}, L_ibC_iU_i^{k-1})$, and these elements are connected by a differential in ∂_0^i. Thus, $H(\mathcal{F}_k,\partial_0^i)=0$ except when $k=0$ and $1/2$; and $\mathcal{F}_0$ is generated by the single element $(\overline{aR_iU_i^t}|b)$ and $\mathcal{F}_{1/2}$ is generated by the element $(\overline{aU_i^t}|L_ib)$. Thus, by a simple filtration argument, $H(C(Z,\mathbf{x},\mathbf{y}),\partial^i)$ is computed as the homology of a two-dimensional vector space generated by these latter two generators $(\overline{aR_iU_i^t}|b)$ and $(\overline{aU_i^t}|L_ib)$; and these two elements are connected by a differential. This completes the verification that $H(C(Z,\mathbf{x},\mathbf{y}),\partial_i)=0$ in this case. See Equation (3.15) for an illustration when $z=3/2$ (and observe that for the diagram, the horizontal coordinate in the plane measures the filtration considered here). The case where $i\in\mathcal{S}_2$ works similarly: the homology of ∂_0^i is now supported in $\mathcal{F}_t$ and $\mathcal{F}_{t+1/2}$, with generators $(\overline{aR_i}|U_i^tb)$ and $(\overline{a}|L_iU_i^tb)$, with a cancelling differential.

Case (C-2), with $z_i=1$**.** Assume that $i\notin\mathcal{S}_2$. The chain complex $C(Z,\mathbf{x},\mathbf{y})$ is spanned by four vectors elements of the form

$$(\overline{a}|C_ib),\quad (\overline{a}|U_ib),\quad (\overline{aU_i}|b),\quad (\overline{aR_i}|R_i\cdot b),$$

where a and b are pure algebra elements with $w_i(a)=w_i(b)=0$. One of the following must hold:

- All four elements are non-zero (which can occur in the ninth column of Figure 15). In this case,
$$\partial_i(\overline{aU_i}|b) = (\overline{aR_i}|R_i\cdot b) + (\overline{a}|C_ib) \quad \partial_i(\overline{a}|C_ib) = (\overline{a}|U_ib) \quad \partial_i(\overline{aR_i}|R_i\cdot b) = (\overline{a}|U_ib),$$
and the complex has trivial homology. (See for example Equation (3.13).)
- $a\cdot R_i=0$ and $R_i\cdot b=0$, in which case also $aU_i=bU_i=0$, so three of the four vectors are zero, and the remaining vector $(\overline{a}|C_ib)$ generates the homology.
- $a\cdot R_i=0$ but $R_i\cdot b\neq 0$. In this case $U_i\cdot b\neq 0$ (by Lemma 3.8), so two of the above elements $(\overline{a}|C_ib)$ and $(\overline{a}|U_ib)$ are non-zero. These two elements are connected by a differential, and the homology is trivial.
- $a\cdot R_i\neq 0$ and $R_i\cdot b=0$. In this case, once again, two of the above elements are zero, and the remaining two are connected by a differential, so once again the homology is trivial.

The case where $i\in\mathcal{S}_2$ works similarly.

Case (C-2), with $z_i\in\mathbb{Z}$ **and** $z_i>1$**.** Let $(\overline{a}|b)$ be some non-trivial generator in the complex, where $i\in\mathbf{w}$, so $i+1\notin\mathbf{w}$. If either $a\cdot R_i=0$ or $R_i\cdot b=0$, then the argument works as in earlier cases (i.e. we have a pair of generators that cancel in homology). Otherwise, as in the case where $z_i\notin\mathbb{Z}$ and $z_i>\frac{1}{2}$, we consider the filtration by the weight w_i of the b component. If $i\notin\mathcal{S}_2$, then for $\mathcal{F}_k$ with $k\geq 1$, $H(\mathcal{F}_k,\partial_0^i)=0$, since the complex $\mathcal{F}_k$ has two terms in it that are connected by a differential in ∂_0^i. The remaining two terms are the generators of $\mathcal{F}_k$ with $k=0$ and $\frac{1}{2}$, which have the form $(\overline{aU_i^t}|b)$ and $(\overline{aU_i^{t-1}R_i}|R_i\cdot b)$; and these two terms are connected by a differential. The case where $i\in\mathcal{S}_2$ works similarly. □

Proof of Proposition 3.14. Let $f\colon\{1,\dots,m+1\}\to\mathbb{Z}$ be defined by

$$f(i)=\#\{j\big|j<i \text{ and } j\in\mathbf{x}\}+\#\{j\big|j<i \text{ and } j\in\mathbf{y}\}$$

It is easy to see that $i-1\leq f(i)\leq i+1$.

Choose n minimal so that the weight $z_n\neq 0$. There are three cases:

Case 1: $f(n)=n$**.** (Note that in this case, $z_i\in\mathbb{Z}$.) We claim that $H_*(C(Z,\mathbf{x},\mathbf{y}),\partial_n)=0$. By Lemma 3.16, we need only consider cases where $z_n=1$.

See Figure 16 when $n-1\in\mathbf{x}$; the cases where $n-1\in\mathbf{y}$ work similarly.

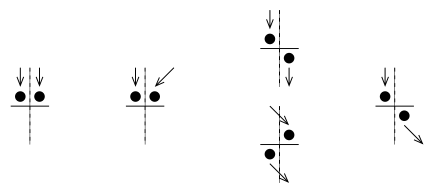


FIGURE 16. **Four cases where** $f(n)=n$**.** This is Case 1 from the proof of Proposition 3.14.

We claim that in all the cases from the figure, we will have an element of the form $(\overline{a\cdot U_n}|b)$ (where $w_n(a)=0$) in the subcomplex. This is clear because in each case, either n is not contained in a generating interval, or the generating interval containing n also contains $n-1$; but $w_{n-1}(a\cdot U_n)=0$. Thus, Lemma 3.16 completes the case.

Case 2: $f(n)=n+1$**.** (See Figure 17.) Again, we claim that $H_*(C(Z,\mathbf{x},\mathbf{y}),\partial_n)=0$. Note that in this case, $z_n\in\frac{1}{2}+\mathbb{Z}$; so by Lemma 3.16, we can assume $z_n=\frac{1}{2}$. The two possible chain complex generators are $(\overline{a}|L_n\cdot b)$ and $(\overline{a\cdot R_n}|b)$. Since by assumption $z_i=0$ for $i<n$, and clearly n is not the left endpoint of any generating interval (in either $\mathcal{B}_1$ or $\mathcal{B}_2$), it follows that both $aU_n\neq 0$ and $bU_n\neq 0$; and correspondingly $a\cdot R_n\neq 0$ and $L_n\cdot b\neq 0$; i.e. both generators are non-zero, so they cancel in homology.

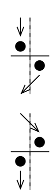


FIGURE 17. $f(n)=n+1$**.** This is Case 2 from the proof.

Case 3: $f(n)=n-1$**.** Since $f(m+1)=m+1$, we can find a minimal $j>n$ so that $f(j)=j$. Note that $z_{j-1}\equiv\frac{1}{2} \pmod 1$. We distinguish four further subcases.

Case 3a: $z_{j-1} > \frac{1}{2}$**.** By Lemma 3.16, $H_*(C(Z,\mathbf{x},\mathbf{y}),\partial_{j-1}) = 0$.

Case 3b: $j < m+1$ **and** $z_{j-1} = \frac{1}{2}$ **and** $z_j = 0$**.** We claim that

$$H_*(C(Z,\mathbf{x},\mathbf{y}),\partial_{j-1}) = 0.$$

The two possible generators are of the form $(\overline{a}|R_j \cdot b)$ and $(\overline{a \cdot L_j}|b)$. Since $z_j = 0$,

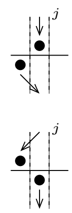


FIGURE 18. $f(n) = n-1$ **and** $z_{j-1} = \frac{1}{2}$**.**

it follows that $a \cdot U_{j-1} \neq 0$. and $b \cdot U_{j-1} \neq 0$, so both generators are non-zero, and cancel in homology.

Case 3c: $j = m+1$ **and** $z_m = \frac{1}{2}$**.** $H_*(C(Z,\mathbf{x},\mathbf{y}),\partial_m) = 0$, exactly as in Case 3b.

Case 3d: $j < m+1$ **and** $z_{j-1} = \frac{1}{2}$ **and** $z_j > 0$**.** We will show $H_*(C(Z,\mathbf{x},\mathbf{y}),\partial_j) = 0$. We have illustrated five cases in Figure 19. The remaining cases are symmetric,

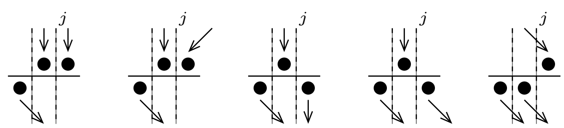


FIGURE 19. $f(n) = n-1$**,** $z_{j-1} = \frac{1}{2}$**,** $z_j = 1$**.**

obtained by switching the roles of a and b.

In the first four of these five cases, observe that the corresponding chain complex contains a non-zero element of the form $(\overline{a \cdot U_j}|b)$ with $w_j(a) = w_j(b) = 0$. Thus, by Lemma 3.16, the homology is trivial (since the generating intervals on $\mathcal{B}_2$ containing j also contain $j-1$). In the final case, the homology is also trivial by Lemma 3.16, since C_j is not present in the displayed generator. □

Proof of Proposition 3.11. Decompose $H(Y)$ into the summands $C(Z,\mathbf{x},\mathbf{y})$ as before. Suppose that the total weight Z is non-zero, and choose i as in Proposition 3.14. As in the proof of Lemma 3.13, there is a filtration on the complex $C(Z,\mathbf{x},\mathbf{y})$ of Y, whose associated graded object is $(C(Z,\mathbf{x},\mathbf{y}),\partial_i)$. It follows at once by an elementary spectral sequence argument that $H(C(Z,\mathbf{x},\mathbf{y}),\partial) = 0$ if Z is a non-zero weight vector.

It remains to consider summands where the weight is zero. These correspond to pairs of complementary idempotents, equipped with a vanishing differential. There are, of course, $\binom{m+1}{k}$ such complementary pairs, as claimed. □

Theorem 3.17. *The module Y is a quasi-inverse of the type DD bimodule $\mathcal{K}$.*

Proof. Note that $\mathcal{B}$ is positively graded over $\mathbf{k}$, using the grading set $\Lambda = (\frac{1}{2}\mathbb{Z})^m$. Conditions (K-1)-2.8 of Definition 2.13 are obvious from the construction of $\mathcal{K}$. The remaining condition of Lemma 2.16 is supplied by Proposition 3.11; so the result follows from Lemma 2.16. □

3.9. **Grading sets associated to one-manifolds.** Our knot invariant will be constructed by tensoring together bimodules with a particular kind of grading set. We formalize these grading sets presently, and study the boundedness needed for forming the tensor product.

Let W be an oriented disjoint union of finitely many intervals, equipped with a partition of its boundary $\partial W = Y_1 \cup Y_2$ into two sets of points. Let Y_i consist of m_i points. Let s_i denote the number of intervals in W that connect Y_i to itself, and s_0 denote the number of intervals that connect Y_1 to Y_2 in W. Let $\mathcal{S}_1$ be those points in Y_1 for which the oriented boundary of W appears with positive multiplicity in the oriented boundary of W, and let $\mathcal{S}_2$ be those points in Y_2 for which the oriented boundary of W appears with negative multiplicity in W. Choose any integer $0 \leq s \leq s_0 + 1$. Let $\mathcal{B}_1 = \mathcal{B}(m_1, s+s_1, \mathcal{S}_1)$, $\mathcal{B}_2(m_2, s+s_2, \mathcal{S}_2)$.

We can think of the Alexander multi-grading of $\mathcal{B}_i$ as taking values in $H^0(Y_i;\mathbb{Q})$: the weights of algebra elements are functions on the points in Y_i. The sum of grading groups $H^0(Y_1;\mathbb{Q}) \oplus H^0(Y_2;\mathbb{Q}) = H^0(\partial W;\mathbb{Q})$ act on $H^1(W, \partial W;\mathbb{Q})$, via the coboundary map $d^0\colon H^0(\partial W) \to H^1(W, \partial W)$.

Remark 3.18. *In fact, the grading on the algebras is supported in* $H^0(Y_i;\frac{1}{2}\mathbb{Z}) \subset H^0(Y_i;\mathbb{Q})$. *Also, the grading set for our modules is contained in* $H^1(W, \partial W;\frac{1}{4}\mathbb{Z})$; *compare Equation* (4.5).

Definition 3.19. *Fix* W *as above. A type DA bimodule* ${}^{\mathcal{B}_2}X_{\mathcal{B}_1}$ *is called* adapted to W *if it has a* $\mathbb{Z}$*-grading (compatible with the Maslov grading on the algebra) and an Alexander multi-grading, with grading set* $H^1(W, \partial W)$ *as described above, and it is finite dimensional (as a vector space).*

Proposition 3.20. *Let* W_1 *be a disjoint union of finitely many intervals joining* Y_1 *to* Y_2*; and let* W_2 *be a disjoint union of finitely many intervals joining* Y_2 *to* Y_3. *Suppose moreover that* $W_1 \cup W_2$ *has no closed components, i.e. it is a disjoint union of finitely many intervals joining* Y_1 *to* Y_3. *Given any two bimodules* ${}^{\mathcal{B}_2}X^1_{\mathcal{B}_1}$ *and* ${}^{\mathcal{B}_3}X^2_{\mathcal{B}_2}$ *adapted to* W_1 *and* W_2 *respectively, we can form their tensor product* ${}^{\mathcal{B}_3}X^2_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}X^1_{\mathcal{B}_1}$ *(i.e. the infinite sums in its definition are finite); and moreover, it is a bimodule that is adapted to* $W_1 \cup W_2$.

Proof. Recall that the grading set of $X^2 \boxtimes X^1$ is $H^1(W_2, Y_3 \cup Y_2) \oplus H^1(W_1, Y_2 \cup Y_1)$ modulo the coboundary of $H^0(Y_2)$, which is identified with $H^1(W_2 \cup W_1, Y_3 \cup Y_1)$. Clearly, the tensor product is finite-dimensional.

Next, we argue the necessary finiteness. Fix algebra elements $(a_1, \dots, a_\ell) \in \mathcal{B}_1$. Fix generators $\mathbf{x}_1, \mathbf{y}_1$ for X^1 and $\mathbf{x}_2, \mathbf{y}_2$ for X^2. Suppose that $(b_1 \otimes \dots \otimes b_j) \otimes \mathbf{y}_1$ appears in $\delta^j_i(\mathbf{x}_1, a_1, \dots, a_\ell)$. Since X^1 is graded, for each point i in Y_2 that is matched by W_1 to a point in Y_1, we have a constant K, depending only on the gradings of $(a_1, \dots, a_\ell)$ and $\mathbf{x}_1$ and $\mathbf{y}_1$, so that

$$|w_i(b_1 \otimes \dots \otimes b_j)| \leq K.$$

(In more detail, consider the component of W_1 that matched $i \in Y_2$ with some $i' \in Y_1$, and let ξ_i and η_i be the coefficient of $\mathbf{gr}(\mathbf{x}_1)$ and $\mathbf{gr}(\mathbf{y}_1)$ in that component. Since X_1 is graded,

$$w_i(b_1 \otimes \cdots \otimes b_j) + \eta_i = \xi_i + w_i(a_1 \otimes \cdots \otimes a_\ell).$$

Now let $K = |\xi_i + w_i(a_1 \otimes \cdots \otimes a_\ell) - \eta_i|$.) By the same reasonining, we can adjust K so that, for any two points i and i' in Y_2 that are matched in W_1,

$$|w_i(b_1 \otimes \cdots \otimes b_j) - w_{i'}(b_1 \otimes \cdots \otimes b_j)| \leq K.$$

Suppose that $c \otimes \mathbf{y}_2$ appears with non-zero multiplicity in $\delta^1_{j+1}(\mathbf{x}_2, b_1, \ldots, b_j)$, then for any point i' in Y_2 that is matched by W_2 to Y_3, we can further adjust K (depending now on $\mathbf{x}_2$ and $\mathbf{y}_2$) so that

$$|w_i(b_1 \otimes \cdots \otimes b_j) - w_{i'}(c)| \leq K.$$

By further adjusting K if necessary (depending only on $\mathbf{x}_2$ and $\mathbf{y}_2$), we can arrange that for any two points i and i' in Y_2 that are matched in W_2,

$$|w_i(b_1 \otimes \cdots \otimes b_j) - w_{i'}(b_1, \ldots, b_j)| \leq K.$$

Grading properties ensure that

$$\mathfrak{m}(\mathbf{x}_1) + \mathfrak{m}(\mathbf{x}_2) - \mathfrak{m}(\mathbf{y}_1) - \mathfrak{m}(\mathbf{y}_2) + \sum_{i=1}^{\ell} \mathfrak{m}(a_i) = \mathfrak{m}(c) + \ell - 1 \tag{3.18}$$

By the above considerations, for any point i in Y_2 that is contained in a path connected component of $W_2 \cup W_1$ that meets Y_1, there is a bound on $w_i(b_1 \otimes \cdots \otimes b_j)$ depending only on $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$, $\mathbf{y}_2$, and $(a_1, \ldots, a_\ell)$.

Suppose next that $i \in Y_2$ is contained in a component of $W_2 \cup W_1$ that meets Y_3 but not Y_1. Consider the inital point p of that arc, with respect to the orientation it inherits from W, and observe that $p \in \mathcal{S}_3$. The above considerations give a bound on $|w_i(b_1 \otimes \cdots \otimes b_j) - w_p(c)|$, again depending only on $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$, $\mathbf{y}_2$, and $(a_1, \ldots, a_\ell)$. Finally, observe that since $p \in \mathcal{S}_3$, Equation (3.9) shows that we can adjust K so that (depending on $|\mathcal{S}_2|$) with $|\mathfrak{m}(c) + 2w_p(c)| \leq K$. An upper bound on $\mathfrak{m}(c)$ is provided by Equation (3.18). Thus, we have an upper bound on $w_i(b_1 \otimes \cdots \otimes b_j)$ for any i in Y_2 that is contained in component of $W_2 \cup W_1$ that meets Y_3 but not Y_1.

Since $W_2 \cup W_1$ has no closed components, we have obtained a universal bound on any weight $w_i(b_1 \otimes \cdots \otimes b_j)$ for $i \in Y_2$. That upper bound implies also an upper bound on j; for if j could be arbitrarily large, we would be able to find arbitrarily large k so that $\delta^k(\mathbf{y}') = b_m \otimes \cdots \otimes b_{m+k} \otimes \mathbf{y}''$. (Here, $\delta^k \colon X^1 \to \mathcal{B}_2^{\otimes k} \otimes X^1$ is the map obtained by iterating δ^1.) But

$$\mathfrak{m}(\mathbf{y}') - k = \mathfrak{m}(b_m) + \cdots + \mathfrak{m}(b_{m+k}) + \mathfrak{m}(\mathbf{y}''),$$

so since X^1 is finitely generated, we conclude that $\mathfrak{m}(b_m) + \cdots + \mathfrak{m}(b_{m+k})$ can be arbitrarily large, contradicting the above obtained universal bound on the weight of $b_1 \otimes \cdots \otimes b_j$ at each point in Y_2, in view of Definition 3.9.

The resulting bound on j shows that the coefficient of $\mathbf{y}_1 \boxtimes \mathbf{y}_2$ in $\partial(\mathbf{x}_1 \boxtimes \mathbf{x}_2)$ is a sum of finitely many terms. The result now follows since $X^2 \boxtimes X^1$ is finitely generated. (See Figure 20.) ☐

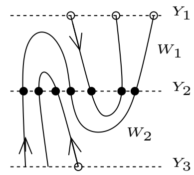


FIGURE 20. In the proof of Proposition 3.20, the total weights of the algebra elements at the circled points in Y_1 and Y_3 give bounds on the weights at points of Y_2 (drawn as dark dots) of the algebra elements in $b_1 \otimes \dots \otimes b_j$.

4. DD BIMODULES FOR CROSSINGS

Having defined the algebra associated to $y = t$ slice of the generic diagram, we turn now to the definitions of the modules associated to the partial knot diagrams. In Section 5, we will construct DA bimodules associated to special partial knot diagrams, consisting of a collection of vertical strands passing through the region $y_1 \leq y \leq y_2$ in the plane, containing exactly one crossing. Before doing this, we construct presently a simpler type DD bimodule associated to such a configuration.

4.1. **The DD bimodule of a positive crossing.** We describe first the DD bimodule $\mathcal{P}_i$ associated to a positive crossing between the i^{th} and $(i+1)^{st}$ strands.

Let $\tau\colon \{1, \dots, m\} \to \{1, \dots, m\}$ be the transposition that switches i and $i+1$. Let

$$\mathcal{B}_1 = \mathcal{B}(m, k_1, \mathcal{S}_1) \qquad \text{and} \qquad \mathcal{B}_2 = \mathcal{B}(m, k_2, \mathcal{S}_2), \tag{4.1}$$

where $k_1 + k_2 = m+1$, $|\mathcal{S}_1| + |\mathcal{S}_2| = m$, and $\mathcal{S}_1 \cap \tau(\mathcal{S}_2) = \emptyset$. We think of the algebra $\mathcal{B}_1$ as coming from above the crossing and algebra $\mathcal{B}_2$ as coming from below.

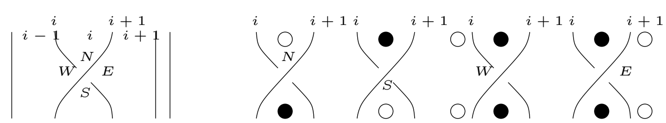


FIGURE 21. **Positive crossing DD bimodule generators.** The four generator types are pictured to the right.

As an $\mathbf{I}(m, k_2, \mathcal{S}_2)$-$\mathbf{I}(m, k_1, \mathcal{S}_1)$-bimodule, $\mathcal{P}_i$ is the submodule of $\mathbf{I}(m, k_2, \mathcal{S}_2) \otimes_{\mathbb{F}} \mathbf{I}(m, k_1, \mathcal{S}_1)$ generated by elements $\mathbf{I}_{\mathbf{y}} \otimes \mathbf{I}_{\mathbf{x}}$ where one of the following two conditions holds:

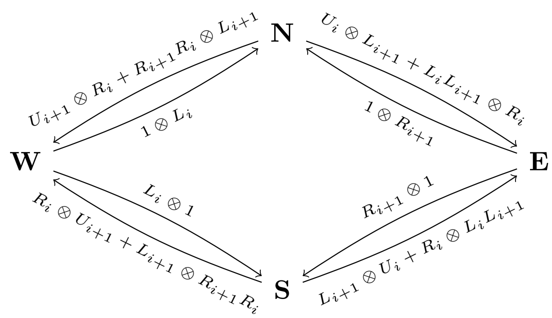
- $\mathbf{x} \cap \mathbf{y} = \emptyset$ or
- $\mathbf{x} \cap \mathbf{y} = \{i\}$ and $\{0, \dots, m\} \setminus (\mathbf{x} \cup \mathbf{y}) = \{i-1\}$ or $\{i+1\}$.

In a little more detail, generators correspond to certain pairs of idempotent states $\mathbf{x}$ and $\mathbf{y}$, where $|\mathbf{x}| = k_1$ and $|\mathbf{y}| = k_2$. They are further classified into four types, $\mathbf{N}$, $\mathbf{S}$, $\mathbf{W}$, and $\mathbf{E}$. For generators of type $\mathbf{N}$ the subsets $\mathbf{x}$ and $\mathbf{y}$ are complementary subsets of $\{0, \dots, m\}$, with $i \in \mathbf{y}$. For generators of type $\mathbf{S}$, $\mathbf{x}$ and $\mathbf{y}$ are complementary subsets of $\{0, \dots, m\}$ with $i \in \mathbf{x}$. For generators of type

$\mathbf{W}$, $i-1\notin\mathbf{x}$ and $i-1\notin\mathbf{y}$, and $\mathbf{x}\cap\mathbf{y}=\{i\}$. For generators of type $\mathbf{E}$, $i+1\notin\mathbf{x}$ and $i+1\notin\mathbf{y}$, and $\mathbf{x}\cap\mathbf{y}=\{i\}$.

The differential has the following types of terms:

(P-1) $R_j\otimes L_j$ and $L_j\otimes R_j$ for all $j\in\{1,\dots,m\}\setminus\{i,i+1\}$; these connect generators of the same type.
(P-2) $U_{\tau(j)}\otimes C_j$ if $j\in\mathcal{S}_1$ and $C_{\tau(j)}\otimes U_j$ if $j\notin\mathcal{S}_1$; these connect generators of the same type.
(P-3) Terms in the diagram below connect generators of different types:



(4.2)

Note that for a generator of type $\mathbf{E}$, the terms of Type (P-1) with $j=i+2$ vanish; while for one of type $\mathbf{W}$, the terms of Type (P-1) with $j=i-1$ vanish.

Proposition 4.1. *The bimodule* ${}^{\mathcal{B}_2,\mathcal{B}_1}\mathcal{P}_i$ *is a DD bimodule.*

Proof. The square of the differential, which we must verify vanishes, is obtained by either differentiating any of the terms of the Types (P-1)-(P-3) above or, multiplying together two of them.

We start by analyzing the terms of Types (P-3). Clearly, all of those terms have vanishing differential. The non-zero algebra elements obtained as products of pairs of such elements either connect generators of any type to itself, or it connects $\mathbf{N}$ and $\mathbf{S}$ or $\mathbf{W}$ and $\mathbf{E}$.

As an example, consider products of terms of Type (P-3) connecting $\mathbf{W}$ to itself. Those products that factor through $\mathbf{N}$ give terms

$$(1\otimes L_i)\cdot(U_{i+1}\otimes R_i+R_{i+1}R_i\otimes L_{i+1})=U_{i+1}\otimes U_i+R_{i+1}R_i\otimes L_iL_{i+1}$$

and those that factor through $\mathbf{S}$ give

$$(L_i\otimes 1)\cdot(R_i\otimes U_{i+1}+L_{i+1}\otimes R_{i+1}R_i)=U_i\otimes U_{i+1}+L_iL_{i+1}\otimes R_{i+1}R_i.$$

Note that $(R_{i+1}R_i\otimes L_iL_{i+1}+L_iL_{i+1}\otimes R_{i+1}R_i)\otimes\mathbf{W}$ vanishes for idempotent reasons: if $\mathbf{x}$ and $\mathbf{y}$ are pairs of idempotent states with $(\mathbf{I_x}\otimes\mathbf{I_y})\cdot\mathbf{W}\neq 0$, then

$$|\mathbf{x}\cap\{i-1,i,i+1\}|+|\mathbf{y}\cap\{i-1,i,i+1\}|=3.$$

The terms $U_i\otimes U_{i+1}$ and $U_{i+1}\otimes U_i$ cancel with the differentials of terms of Type (P-2) with $j=i$ and $i+1$.

Next consider products of terms of Type (P-3) connecting $\mathbf{N}$ to $\mathbf{S}$. The products that factor through $\mathbf{W}$ give terms:

$$(U_{i+1}\otimes R_i + R_{i+1}R_i\otimes L_{i+1})\cdot(L_i\otimes 1) = L_iU_{i+1}\otimes R_i + R_{i+1}U_i\otimes L_{i+1}.$$

Similarly, the products that factor through $\mathbf{E}$ give

$$(U_i\otimes L_{i+1} + L_iL_{i+1}\otimes R_i)\cdot(R_{i+1}\otimes 1) = R_{i+1}U_i\otimes L_{i+1} + L_iU_{i+1}\otimes R_i;$$

thus, these two factorizations give terms that cancel in pairs.

The cancellation of terms of Type (P-3) in pairs or with differentials of terms of Type (P-2) with $j=i$ and $i+1$ proceeds to other types of generators similarly.

Next, we consider products of pairs of terms of Type (P-1). When $j\neq i,i+1$, these terms cancel differentials of terms of Type (P-2) exactly as in the proof Lemma 3.10. When the generators are of type $\mathbf{E}$ and $\mathbf{W}$, there is a possible complication in this argument, since in that case, the corresponding term with $j=i+2$ or $i-1$ of Type (P-1) might vanish. For example, for terms of type $\mathbf{E}$, and $i+2\notin\mathcal{S}_1$, there is a possibly non-zero term $C_{i+2}\otimes U_{i+2}$ of Type (P-2), but the terms of Type (P-1) of the form $L_{i+2}\otimes R_{i+2}$ and $R_{i+2}\otimes L_{i+2}$ vanish. However, the differential of $C_{i+2}\otimes U_{i+2}$ vanishes in the idempotents of $\mathbf{E}$ (since if $\mathbf{I}_{\mathbf{y}}\otimes\mathbf{I}_{\mathbf{x}}$ is of type $\mathbf{E}$, then $i+1\notin\mathbf{x}$ and $i+1\notin\mathbf{y}$, and either $i+2\notin\mathbf{x}$ or $i+2\notin\mathbf{y}$; in either case, $(U_{i+2}\otimes U_{i+2})\cdot(\mathbf{I}_{\mathbf{y}}\otimes\mathbf{I}_{\mathbf{x}})=0$).

Consider next terms that are products of terms of Type (P-3) and those of Type (P-1). Typically, these are easily seen to cancel in pairs; the case where support of the Type (P-1) is immediately next to $\{i,i+1\}$ requires special care. Consider for example the terms of the form $L_{i+2}\otimes R_{i+2}$. Each term of Type (P-3) commutes with this term; but both product might be zero. For example, $L_i\otimes 1$ commutes with $(L_{i+2}\otimes R_{i+2})$. Also, if $a=L_iL_{i+1}\otimes R_i$, then $(L_{i+2}\otimes R_{i+2})\cdot a=0$ and $a\cdot(L_{i+2}\otimes R_{i+2})=0$.

The remaining terms of Type (P-2) and are easily seen to commute with each other and with terms of Type (P-3), giving the desired cancellation. □

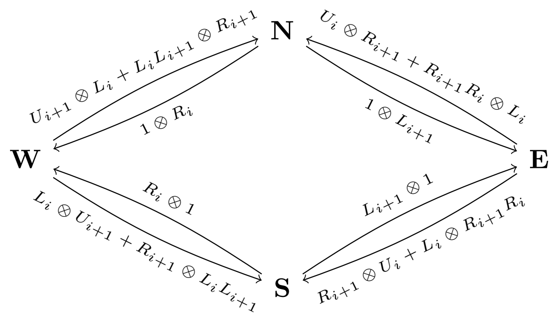
It is interesting to note that the bimodule has some symmetries; for instance,

$$^{\mathcal{B}_2,\mathcal{B}_1}\mathcal{P}_i \cong {}^{\mathcal{B}_1,\mathcal{B}_2}\mathcal{P}_i, \tag{4.3}$$

by a symmetry which switches the roles of $\mathbf{N}$ and $\mathbf{S}$, and fixes $\mathbf{W}$ and $\mathbf{E}$.

4.2. *DD* bimodule for a negative crossing.

We can define a type DD bimodule for a negative crossing. The generators are the same as for a positive crossing. Terms in the differential are also the same, except that those of Type (P-3) are replaced by the following:



(4.4)

Proposition 4.2. *The bimodule $\mathcal{N}_i$ is a DD bimodule.*

Proof. This follows exactly as in the proof of Proposition 4.1. Note that Diagram (4.4) is obtained from Diagram 4.2 by reversing all the arrows, and switching the roles of L_j and R_j. More formally, for $\mathcal{B}_1$ and $\mathcal{B}_2$ as in Equation (4.1), ${}^{\mathcal{B}_1,\mathcal{B}_2}\mathcal{N}_i$ is obtained from the opposite module of $\mathcal{P}_i$, $\overline{\mathcal{P}}_i^{\mathcal{B}_1,\mathcal{B}_2} = {}^{\mathcal{B}_1^{\mathrm{op}},\mathcal{B}_2^{\mathrm{op}}}\overline{\mathcal{P}}_i$ using the isomorphism $\mathcal{B}_i \cong \mathcal{B}_i^{\mathrm{op}}$ (denoted o in Equation (3.11)). □

4.3. **Gradings.** The bimodules $\mathcal{P}_i$ are graded by the set $S = \mathbb{Q}^m$ as follows Let $e_1, \dots, e_m$ be the standard basis for $\mathbb{Q}^m$. Let

(4.5)
$$\mathbf{gr}(\mathbf{N}) = \tfrac{e_i+e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{W}) = \tfrac{e_i-e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{E}) = \tfrac{-e_i+e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{S}) = \tfrac{-e_i-e_{i+1}}{4},$$

Now, if $(a \otimes b) \otimes Y$ appears in ∂X, then

$$\mathbf{gr}(X) = \mathbf{gr}(a) - \tau^{\mathbf{gr}}\mathbf{gr}(b) + \mathbf{gr}(Y), \tag{4.6}$$

where $\tau_i^{\mathbf{gr}}$ is the linear transformation acting by τ_i on the standard basis vectors, and $\mathbf{gr}(a) = (w_1(a), \dots, w_m(a))$. Verifying Equation (4.6) is straightforward, using Equation (4.2).

In the notation of Section 2.2, the grading set of $\mathcal{P}_i$ is half-integral valued functions on the the arcs in the diagram, thought of as an affine space for $\Lambda_{\mathcal{B}_1} \times \Lambda_{\mathcal{B}_2}$ in an obvious way.

Similarly, for $\mathcal{N}_i$, the gradings of the four generators $\mathbf{N}$, $\mathbf{S}$, $\mathbf{E}$, $\mathbf{W}$ are given by

(4.7)
$$\mathbf{gr}(\mathbf{N}) = \tfrac{-e_i-e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{W}) = \tfrac{-e_i+e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{E}) = \tfrac{e_i-e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{S}) = \tfrac{e_i+e_{i+1}}{4},$$

Note that the modules determine these gradings only up to an overall additive shift. The present gradings are consistent with conventions on the multivariable Alexander polynomial [6]; for further motivation, see Remark 7.3.

4.4. **Motivation.** The modules ${}^{\mathcal{B}_2,\mathcal{B}_1}\mathcal{P}_i$ and ${}^{\mathcal{B}_2,\mathcal{B}_1}\mathcal{N}_i$ came from considering a version of Lagrangian Floer homology in the four-punctured sphere, shown in Figure 22, taken with respect to a single β-circle and four α-arcs that go out to the punctures. Regions in the complement of this configuration are labelled by algebra elements (as shown in Figure 22, for the negative crossing).

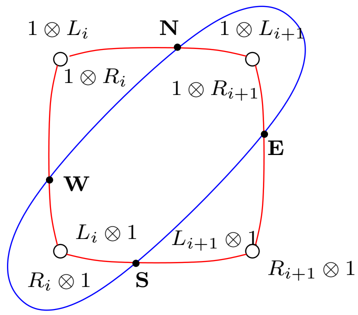


FIGURE 22. **Four-punctured sphere for a negative crossing.** The four punctures are connected by four arcs as shown; an auxiliary circle meets the four arcs in the points **N**, **W**, **S**, and **E**. Regions are labelled by algebra elements as shown.

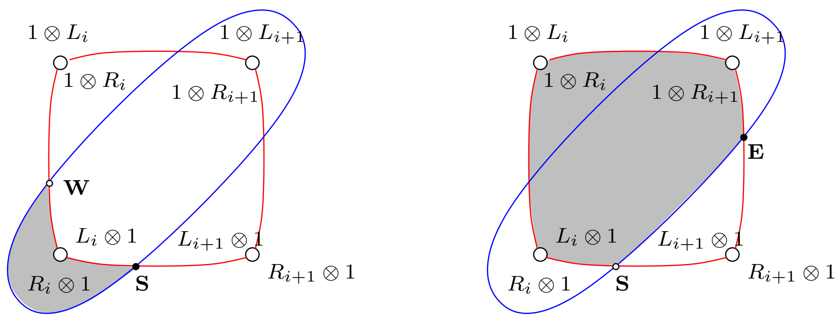


FIGURE 23. **Terms in Equation** (4.4) **as arising from disks.** The shaded region on the left represents the term $(R_i \otimes 1) \otimes \mathbf{W}$ in $\delta^1(\mathbf{S})$; the one on the right represents the term $(L_i \otimes R_{i+1}R_i) \otimes \mathbf{S}$ in $\delta^1(\mathbf{E})$.

The arrows in Equation (4.4) count rigid holomorphic bigons, and their contribution is obtained by multiplying the algebra elements specified by the regions; see Figure 23 for some examples. This relationship is central to [17].

5. DA BIMODULES ASSOCIATED TO CROSSINGS

Our goal here is to construct certain DA bimodules associated to crossings, which are in a suitable sense Koszul dual to the DD bimodules constructed in Section 4; see Lemma 6.2 for the precise duality statement.

For the construction, choose integers k and m with $0 \leq k \leq m+1$, and let $\mathcal{S} \subset \{1,\dots,m\}$ be arbitrary. Let $\tau_i \colon \{1,\dots,m\} \to \{1,\dots,m\}$ be the map that transposes i and $i+1$. Let $\mathcal{B}_1 = \mathcal{B}(m,k,\mathcal{S})$ and $\mathcal{B}_2 = \mathcal{B}(m,k,\tau(\mathcal{S}))$. (Note that $\mathcal{B}_1$ and $\mathcal{B}_2$ here denote different algebras than they did in Section 4.) The aim of the present section is to construct a type DA bimodule ${}^{\mathcal{B}_2}\mathcal{P}^i_{\mathcal{B}_1}$, which we think of as the bimodule associated to a region in the knot diagram $t_2 \leq y \leq t_1$ that contains exactly one "positive" crossing, and no local maxima or minima; and the crossing occurs between the i^{th} and $(i+1)^{st}$ strands, as shown on the left in Figure 24.

(Positivity here is meant with respect to a braid orientation, e.g. where all the strands are oriented upwards, which might differ from the orientation specified by $\mathcal{S}$.)

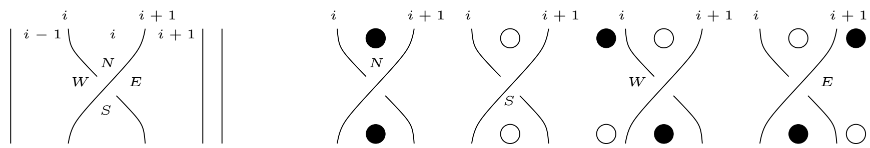


FIGURE 24. **Positive crossing DA bimodule generators.** The four generator types are pictured to the right.

Consider the submodule $\mathcal{P}^i$ of $\mathbf{I}(m,k) \otimes_{\mathbb{F}} \mathbf{I}(m,k)$, consisting of $\mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}$ where either $\mathbf{x} = \mathbf{y}$ or there is some $\mathbf{w} \subset \{1,\dots,i-1,i+1,\dots,m\}$ with $\mathbf{x} = \mathbf{w} \cup \{i\}$ and $\mathbf{y} = \mathbf{w} \cup \{i-1\}$ or $\mathbf{y} = \mathbf{w} \cup \{i+1\}$. Thus, there are once again four types of generators, of type $\mathbf{N}$, $\mathbf{S}$, $\mathbf{W}$, $\mathbf{E}$ as pictured in Figure 24; i.e.

$$\sum_{i\in\mathbf{x}} \mathbf{I}_{\mathbf{x}} \cdot \mathbf{N} \cdot \mathbf{I}_{\mathbf{x}} = \mathbf{N}, \qquad \sum_{i\notin\mathbf{x}} \mathbf{I}_{\mathbf{x}} \cdot \mathbf{S} \cdot \mathbf{I}_{\mathbf{x}} = \mathbf{S},$$

$$\sum_{\substack{i \notin \mathbf{x} \\ i-1 \in \mathbf{x}}} \mathbf{I}_{\{i\}\cup\mathbf{x}\setminus\{i-1\}} \cdot \mathbf{W} \cdot \mathbf{I}_{\mathbf{x}} = \mathbf{W}, \qquad \sum_{\substack{i+1 \in \mathbf{x} \\ i \notin \mathbf{x}}} \mathbf{I}_{\{i\}\cup\mathbf{x}\setminus\{i+1\}} \cdot \mathbf{E} \cdot \mathbf{I}_{\mathbf{x}} = \mathbf{E}.$$

This description has a geometric interpretation in terms of Kauffman states.

Definition 5.1. *A* partial knot diagram *is the portion of a knot diagram contained in the (x,y) plane with $t_2 \le y \le t_1$, so that the diagram meets the slices $y = t_1$ and $y = t_2$ generically. The knot projection divides the partial knot diagram into regions. A* partial Kauffman state *is a triple of data $(\mathbf{K}, \mathbf{y}, \mathbf{x})$, where $\mathbf{K}$ is a map that associates to each crossing one of its four adjacent regions; $\mathbf{x}$ is a collection of intervals in the intersection of the diagram with the $y = t_1$ slice; and $\mathbf{y}$ is a collection of intervals in the intersection of the diagram with the $y = t_2$ slice. The regions assigned to the crossings are called* occupied regions*; those that are not are called* unoccupied. *The data in a partial Kauffman state are further required to satisfy the following compatibility conditions:*

- *No two crossings are assigned to the same region.*
- *If a region R is occupied, then $\mathbf{x}$ contains all the intervals in $R \cap (y = t_1)$ and $\mathbf{y}$ contains none of the regions in $R \cap (y = t_2)$.*
- *If a region R is unoccupied, then either $\mathbf{x}$ contains all but one of the regions in $R \cap (y = t_1)$ and $\mathbf{y}$ contains none of the regions in $R \cap (y = t_2)$; or $\mathbf{x}$ contains all of the regions in $R \cap (y = t_1)$ and the region $\mathbf{y}$ contains exactly one of the regions of $R \cap (y = t_2)$.*

The compatibility conditions can be formulated more succinctly as follows: each region contains exactly one of the following: a quadrant assigned by $\mathbf{K}$, an interval in $R \cap (y = t_1)$ not contained in $\mathbf{x}$, or an interval in $R \cap (y = t_2)$ contained in $\mathbf{y}$.

The picture is simplified considerably when the partial knot diagram contains a single crossing. In that case, it is straightforward to see that the generators of the bimodule $\mathcal{P}^i$ defined above correspond to partial Kauffman states; $\mathbf{K}$ is one of

$\mathbf{N}$, $\mathbf{S}$, $\mathbf{W}$, or $\mathbf{E}$; $\mathbf{y}$ specifies the right idempotent of the generator and $\mathbf{x}$ the left idempotent of the generator.

The bimodules $\mathcal{P}^i$ are graded by the set $\mathbb{Q}^m$ in the following sense. Define gradings of the generators $\mathbf{N}$, $\mathbf{S}$, $\mathbf{E}$, and $\mathbf{W}$ as in Equation (4.5) (only now thinking of these as generators of $\mathcal{P}^i$ rather than $\mathcal{P}_i$). The actions δ^1_ℓ respect this grading, in the following sense. Suppose that X is some homogeneous generator, and $a_1,\dots,a_\ell\in\mathcal{B}_1$ are homogeneous elements, then $\delta^1_\ell(X,a_1,\dots,a_\ell)$ can be written as a sum of elements of the form $b\otimes Y$ where Y are homogeneous generators and $b\in\mathcal{B}_2$ is are homogeneous elements of the algebra, with

$$\mathbf{gr}(X)+\tau_i^{\mathbf{gr}}(\mathbf{gr}(a_1)+\dots+\mathbf{gr}(a_{\ell-1}))=\mathbf{gr}(b)+\mathbf{gr}(Y), \tag{5.1}$$

where here $\mathbf{gr}(a_i)$ and $\mathbf{gr}(b)$ denote the weight gradings on $\mathcal{B}_1$ and $\mathcal{B}_2$.

There is also a $\mathbb{Z}$-grading, specified by where the grading of a generator type coincides with the local Maslov contribution of the corresponding Kauffman state as in Figure 2. For instance, if all crossings are oriented upwards, then

$$M(\mathbf{N})=-1 \qquad M(\mathbf{S})=M(\mathbf{W})=M(\mathbf{E})=0. \tag{5.2}$$

The modules respect this $\mathbb{Z}$-grading: for X, Y, $a_1,\dots,a_{\ell-1}$ and b as above,

$$M(X)+M(a_1)+\dots+M(a_{\ell-1})+\ell-1=M(b)+M(Y). \tag{5.3}$$

The partial knot diagram with a single crossing in it determines in an obvious way a collection of arcs. Equation (5.1) can be interpreted as saying that the bimodule $\mathcal{P}^i$ is adapted to this one-manifold with boundary, in the sense of Definition 3.19.

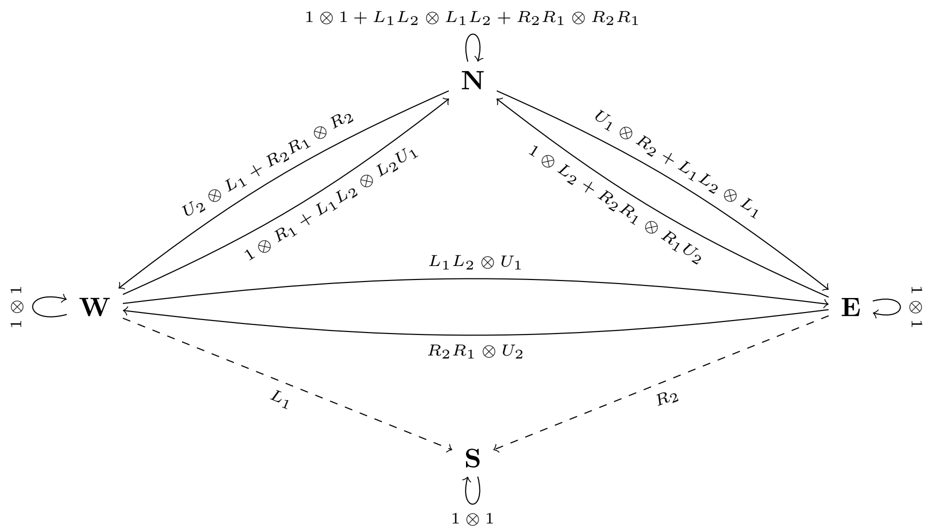
We consider first the case where $\mathcal{S}\cap\{i,i+1\}=\emptyset$, we will specify these actions by localizing to the crossing region, defining first bimodules over $\mathcal{B}(2)=\mathcal{B}(2,0)\oplus\mathcal{B}(2,1)\oplus\mathcal{B}(2,2)\oplus\mathcal{B}(2,3)$, and then extending them to $\mathcal{B}(m,k,\mathcal{S})$ (with $\mathcal{S}\cap\{i,i+1\}=\emptyset$) in Subsection 5.2. The local models when $\mathcal{S}\cap\{i,i+1\}$ is non-empty is defined in Section 5.3, and it is extended in the general case in 5.4.

5.1. **Local bimodule for a positive crossing.** We define first a type DA bimodule P over $\mathcal{B}(2)=\mathcal{B}(2,0)\oplus\mathcal{B}(2,1)\oplus\mathcal{B}(2,2)\oplus\mathcal{B}(2,3)$. Note that $\mathcal{B}(2)$ is defined over $\mathbf{I}(2)=\mathbf{I}(2,0)\oplus\mathbf{I}(2,1)\oplus\mathbf{I}(2,2)\oplus\mathbf{I}(2,3)\cong\mathbb{F}^8$.

As an $\mathbf{I}(2)-\mathbf{I}(2)$-module, P has four generators $\mathbf{N}$, $\mathbf{S}$, $\mathbf{W}$, and $\mathbf{E}$. As an $\mathbb{F}$-vector space, P is 12 dimensional, and all 12 basis vectors appear on the right hand sides of the following four expressions, which determine the $\mathbf{I}(2)-\mathbf{I}(2)$-module structure:

$$\begin{aligned}
\mathbf{N}&=\mathbf{I}_{\{1\}}\cdot\mathbf{N}\cdot\mathbf{I}_{\{1\}}+\mathbf{I}_{\{0,1\}}\cdot\mathbf{N}\cdot\mathbf{I}_{\{0,1\}}+\mathbf{I}_{\{1,2\}}\cdot\mathbf{N}\cdot\mathbf{I}_{\{1,2\}}+\mathbf{I}_{\{1,2,3\}}\cdot\mathbf{N}\cdot\mathbf{I}_{\{1,2,3\}}\\
\mathbf{S}&=\mathbf{I}_{\emptyset}\cdot\mathbf{S}\cdot\mathbf{I}_{\emptyset}+\mathbf{I}_{\{0\}}\cdot\mathbf{S}\cdot\mathbf{I}_{\{0\}}+\mathbf{I}_{\{2\}}\cdot\mathbf{S}\cdot\mathbf{I}_{\{2\}}+\mathbf{I}_{\{0,2\}}\cdot\mathbf{S}\cdot\mathbf{I}_{\{0,2\}}\\
\mathbf{W}&=\mathbf{I}_{\{1\}}\cdot\mathbf{W}\cdot\mathbf{I}_{\{0\}}+\mathbf{I}_{\{1,2\}}\cdot\mathbf{W}\cdot\mathbf{I}_{\{0,2\}}\\
\mathbf{E}&=\mathbf{I}_{\{1\}}\cdot\mathbf{E}\cdot\mathbf{I}_{\{2\}}+\mathbf{I}_{\{0,1\}}\cdot\mathbf{E}\cdot\mathbf{I}_{\{0,2\}}.
\end{aligned}$$

Next, we define the δ^1_1 and δ^1_2 actions on P. Some of these actions are specified in the following diagram:
(5.4)



Here, dashed arrows indicate δ^1_1 actions; while δ^1_k for $k > 1$ are indicated by solid arrows. For example, $\delta^1_1(\mathbf{W}) = L_1 \otimes \mathbf{S}$.

Further actions are obtained by the following *local extension rules*. For any $X \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}, \mathbf{S}\}$ and any pure algebra element $a \in \mathcal{B}(2)$,

$$\delta^1_2(X, U_1U_2 \cdot a) = U_1U_2 \cdot \delta^1_2(X, a). \tag{5.5}$$

and also:

- If $b \otimes Y$ appears with non-zero coefficient in $\delta^1_2(\mathbf{N}, a)$, then $(b \cdot U_2) \otimes Y$ appears with non-zero coefficient in $\delta^1_2(\mathbf{N}, a \cdot U_1)$ and $(b \cdot U_1) \otimes Y$ appears with non-zero coefficient in $\delta^1_2(\mathbf{N}, a \cdot U_2)$.
- If $b \otimes Y$ appears with non-zero coefficient in $\delta^1_2(\mathbf{W}, a)$, then $(U_2 \cdot b) \otimes Y$ appears with non-zero coefficient in $\delta^1_2(\mathbf{W}, U_1 \cdot a)$.
- If $b \otimes Y$ appears with non-zero coefficient in $\delta^1_2(\mathbf{E}, a)$, then $(U_1 \cdot b) \otimes Y$ appears with non-zero coefficient in $\delta^1_2(\mathbf{E}, U_2 \cdot a)$.

For example, the action $\delta^1_2(\mathbf{W}, 1) = \mathbf{W}$ combined with the local extension rules shows that $\delta^1_2(\mathbf{W}, U_1) = U_2 \otimes \mathbf{W} + L_1L_2 \otimes \mathbf{E}$.

The above rules uniquely specify the $\mathbb{F}$-linear map $\delta^1_2 \colon \mathcal{P} \otimes \mathcal{B}(2) \to \mathcal{B}(2) \otimes \mathcal{P}(2)$.

We wish next to specify δ^1_3. As in the case for δ^1_2, we define these on a basis. More formally, an algebra element is called *elementary* if it is of the form $p \cdot e$, where p is a monomial in U_1 and U_2, and

$$e \in \{1, \quad L_1, \quad R_1, \quad L_2, \quad R_2, \quad L_1L_2, \quad R_2R_1\}.$$

Having specified cations $\delta^1_2(X, a)$, where $X \in \{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$ and a is elementary. we turn to defining $\delta^1_3(X, a_1, a_2)$ where a_1 and a_2 are elementary.

Suppose that a_1 and a_2 are elementary algebra elements with $a_1 \otimes a_2 \neq 0$ (i.e. there is an idempotent state $\mathbf{x}$ so that $a_1 \cdot \mathbf{I}_{\mathbf{x}} \neq 0$ and $\mathbf{I}_{\mathbf{x}} \cdot a_2 \neq 0$); and suppose moreover that $U_1 \cdot U_2$ does not divide either a_1 nor a_2. In this case, $\delta^1_3(\mathbf{S}, a_1, a_2)$ is the sum of terms:

- $R_1 U_1^t \otimes \mathbf{E}$ if $(a_1, a_2) = (R_1, R_2 U_2^t)$ and $t \geq 0$
- $L_2 U_1^t U_2^n \otimes \mathbf{E}$ if $(a_1, a_2) \in$

$$\begin{array}{llll} \{(U_1^{n+1}, U_2^t), & (R_1 U_1^n, L_1 U_2^t), & (L_2 U_1^{n+1}, R_2 U_2^{t-1})\} & \text{when } 0 \leq n < t \\ \{(U_2^t, U_1^{n+1}), & (R_1 U_2^t, L_1 U_1^n), & (L_2 U_2^{t-1}, R_2 U_1^{n+1})\} & \text{when } 1 \leq t \leq n \end{array}$$

- $L_2 U_2^n \otimes \mathbf{W}$ if $(a_1, a_2) = (L_2, L_1 U_1^n)$ and $n \geq 0$
- $R_1 U_1^t U_2^n \otimes \mathbf{W}$ if $(a_1, a_2) \in$

$$\begin{array}{llll} \{(U_2^{t+1}, U_1^n), & (L_2 U_2^t, R_2 U_1^n), & (R_1 U_2^{t+1}, L_1 U_1^{n-1})\} & \text{when } 0 \leq t < n \\ \{(U_1^n, U_2^{t+1}), & (L_2 U_1^n, R_2 U_2^t), & (R_1 U_1^{n-1}, L_1 U_2^{t+1})\}, & \text{when } 1 \leq n \leq t \end{array}$$

- $L_2 U_1^t U_2^n \otimes \mathbf{N}$ if $(a_1, a_2) \in$

$$\begin{array}{llll} \{(U_1^{n+1}, L_2 U_2^t), & (R_1 U_1^n, L_1 L_2 U_2^t), & (L_2 U_1^{n+1}, U_2^t)\} & \text{when } 0 \leq n < t \\ \{(L_2 U_2^t, U_1^{n+1}), & (U_2^t, L_2 U_1^{n+1}) & (R_1 U_2^t, L_1 L_2 U_1^n)\} & \text{when } 1 \leq t \leq n \\ \{(L_2, U_1^{n+1})\} & & & \text{when } 0 = t \leq n \end{array}$$

- $R_1 U_1^t U_2^n \otimes \mathbf{N}$ if (a_1, a_2) is in the following list:

$$\begin{array}{llll} \{(U_2^{t+1}, R_1 U_1^n), & (L_2 U_2^t, R_2 R_1 U_1^n), & (R_1 U_2^{t+1}, U_1^n)\} & \text{when } 0 \leq t < n \\ \{(R_1 U_1^n, U_2^{t+1}) & (U_1^n, R_1 U_2^{t+1}), & (L_2 U_1^n, R_2 R_1 U_2^t)\} & \text{when } 1 \leq n \leq t \\ \{(R_1, U_2^{t+1})\} & & & \text{when } 0 = n \leq t. \end{array}$$

For example, $\delta^1_3(\mathbf{S}, U_1, U_2^2) = (L_2 U_1^2) \otimes \mathbf{E} + (R_1 U_1 U_2) \otimes \mathbf{W}$.

Extend this to the case where $U_1 U_2$ divides a_1 or a_2 or both, by requiring

$$\delta^1_3(Z, (U_1 U_2) \cdot a, b) = \delta^1_3(Z, a, (U_1 U_2) \cdot b) = (U_1 U_2) \cdot \delta^1_3(Z, a, b). \tag{5.6}$$

Proposition 5.2. *The operations δ^1_ℓ defined above give P the structure of a DA bimodule over $\mathcal{B}(2) = \mathcal{B}(2,0) \oplus \mathcal{B}(2,1) \oplus \mathcal{B}(2,2) \oplus \mathcal{B}(2,3)$, which is graded as in Equations* (5.1) *and* (5.3).

We prove the above proposition after some lemmas.

Lemma 5.3. *The actions δ^1_ℓ respect the gradings, as in Equation* (5.1).

Proof. In the present case, the gradings on $\mathcal{P}^i$ are specified by

$$\mathbf{gr}(\mathbf{N}) = \tfrac{e_i + e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{W}) = \tfrac{e_i - e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{E}) = \tfrac{-e_i + e_{i+1}}{4} \quad \mathbf{gr}(\mathbf{S}) = \tfrac{-e_i - e_{i+1}}{4},$$

Bearing this in mind, the lemma is a straighforward verification using the above defined actions δ^1_ℓ. □

Observe that if $b \otimes Y$ appears with non-zero multiplicity in $\delta^1_\ell(X, a_1, \dots, a_{\ell-1})$, and X is a fixed generator, and the $a_i \in \mathcal{B}(2)$ are elementary for $i = 1, \dots, \ell - 1$, then the output algebra b is uniquely specified in Equation (5.1).

Lemma 5.4. *Given a monomial p in U_1 and U_2 and an algebra element*

$$a \in \{p, \quad R_1 \cdot p, \quad L_2 \cdot p, \quad R_2 R_1 \cdot p, \quad L_1 L_2 \cdot p\},$$

there is a unique $X \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ and b so that $b \otimes \mathbf{N}$ appears with non-zero multiplicity in $\delta^1_2(X,a)$. Similarly, given a monomial p in U_1 and U_2 an algebra element from

$$a \in \{p, \qquad L_1 \cdot p, \qquad R_2 \cdot p\},$$

and a fixed state type $Y \in \{\mathbf{W}, \mathbf{E}\}$, there is a unique state type $X \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ and b so that $b \otimes Y$ appears with non-zero multiplicity in $\delta^1_2(X,a)$.

Proof. This follows from a straightforward inspection of Diagram (5.4) and the local extension rules. For example, if $Y = \mathbf{N}$, then if $a = p \cdot e$ where p is a monomial in U_1 and U_2, and $e \in \{1, L_1L_2, R_2R_1\}$, then $X = \mathbf{N}$. For $e = L_2$, write $p = U_1^x U_2^y$. If $x > y$, then $X = \mathbf{W}$, and if $x \leq y$, then $X = \mathbf{E}$. The case where $e = R_1$ works similarly. □

For any elementary a and $Y \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ with $a \otimes Y \neq 0$, define $I(a,Y)$ to be the generator $X \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ as defined in Lemma 5.4. Otherwise (i.e. if the pair (a,Y) is not covered by the lists of Lemma 5.4), define $I(a,Y) = 0$.

Lemma 5.5. *Fix elementary algebra elements a_1, a_2 in $\mathcal{B}(2)$, with $\mathbf{S} \otimes a_1 \otimes a_2 \neq 0$, an elementary b and $Y \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ so that $b \otimes Y \neq 0$ and*

$$\mathbf{gr}(\mathbf{S} \otimes a_1 \otimes a_2) = \mathbf{gr}(b \otimes Y). \tag{5.7}$$

Suppose moreover that $a_1 \cdot a_2 \neq 0$. Then the element $b \otimes Y$ appears with non-zero multiplicity in $\delta^1_3(\mathbf{S}, a_1, a_2)$ if and only if $I(a_2, Y) \neq 0$ and $I(a_1, I(a_2, Y)) \neq I(a_1 \cdot a_2, Y)$; (i.e. one of them is $\mathbf{E}$ and the other is $\mathbf{W}$).

Proof. This follows from a straightforward inspection of the definition of δ^1_3. □

We introduce a notational shorthand. The maps $\delta^1_i \colon P \otimes \overbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}^{i-1} \to \mathcal{B} \otimes P$ naturally extend to maps $\widetilde{\delta}^1_i \colon \mathcal{B} \otimes P \otimes \overbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}^{i-1} \to \mathcal{B} \otimes P$, by the rule

$$\delta^1_i(b \otimes x \otimes a_1 \otimes \cdots \otimes a_{i-1}) = b \cdot \delta^1_i(x \otimes a_1 \cdots \otimes \ldots a_{i-1}).$$

Fix elementary algebra elements $a_1, a_2 \in \mathcal{B}(2,k)$, so that $\mathbf{S} \otimes a_1 \otimes a_2 \neq 0$. When $k = 2$, $a_1 \cdot a_2$ is always non-zero. On the other hand, when $k = 1$, the actions $\delta^1_3(\mathbf{S}, a_1, a_2)$ are those for which $a_1 \cdot a_2 = 0$ and $b \otimes Y$ (as specified by Equation (5.7)) appears with non-zero multiplicity in $\widetilde{\delta}^1_2(\delta^1_2(\mathbf{S}, a_1), a_2)$.

Lemma 5.6. *With the above definition, if a_1, a_2, and a_3 are elementary, then*

$$\begin{aligned} &\widetilde{\delta}^1_2(\delta^1_3(\mathbf{S}, a_1, a_2), a_3) + \widetilde{\delta}^1_3(\delta^1_2(\mathbf{S}, a_1), a_2, a_3) \\ &\qquad + \delta^1_3(\mathbf{S}, a_1 \cdot a_2, a_3) + \delta^1_3(\mathbf{S}, a_1, a_2 \cdot a_3) = 0 \end{aligned} \tag{5.8}$$

Proof. The above actions δ^1_3 defined above vanish on $\mathcal{B}(2,0)$ and $\mathcal{B}(2,3)$, so Equation (5.8) is obvious. On $\mathcal{B}(2,1)$, the non-trivial δ^1_3 actions are

$$\begin{array}{ll} \delta^1_3(\mathbf{S}, R_1, R_2U_2^t) = R_1U_1^t \otimes \mathbf{E} & \delta^1_3(\mathbf{S}, L_2, L_1U_1^n) = L_2U_2^n \otimes \mathbf{W} \\ \delta^1_3(\mathbf{S}, L_2, U_1^{n+1}) = L_2U_2^n \otimes \mathbf{N} & \delta^1_3(\mathbf{S}, R_1, U_2^{t+1}) = R_1U_1^t \otimes \mathbf{N} \end{array}$$

with $k, n \geq 0$. For these, the verification of Equation (5.8) is straightforward.

In $\mathcal{B}(2,2)$, if $a_1 \otimes a_2 \otimes a_3 \neq 0$, then in fact $a_1 \cdot a_2 \cdot a_3 \neq 0$; so we wil henceforth assume that $a_1 \cdot a_2 \cdot a_3 \neq 0$. We treat two subcases separately, according to whether or not $a_1 = (U_1U_2)^\ell$.

Case 1: $a_1 = (U_1U_2)^\ell$. The first and the fourth terms in Equation (5.8) vanish, and the middle two agree by Equation (5.6).

Case 2: $a_1 \neq (U_1U_2)^\ell$. In this case, $\delta^1_2(\mathbf{S}, a_1) = 0$, so the second term in Equation (5.8) is missing. Given $Y \in \{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$, let

$$\alpha = I(a_1 \cdot a_2, I(a_3, Y)), \qquad \beta = I(a_1 \cdot a_2 \cdot a_3, Y),$$
$$\gamma = I(a_1, I(a_2 \cdot a_3, Y)), \qquad \delta = I(a_1, I(a_2, I(a_3, Y))).$$

By Lemma 5.5, $b \otimes Y$ (where $\mathbf{gr}(\mathbf{S} \otimes a_1 \otimes a_2 \otimes a_3) = \mathbf{gr}(b \otimes Y)$) appears with non-zero multiplicity in:

- $\delta^1_3(\mathbf{S}, a_1 \cdot a_2, a_3)$ if the set $\{\alpha, \beta\}$ equals $\{\mathbf{W}, \mathbf{E}\}$
- $\delta^1_3(\mathbf{S}, a_1, a_2 \cdot a_3)$ if $\{\gamma, \beta\} = \{\mathbf{W}, \mathbf{E}\}$
- $\widetilde{\delta}^1_2(\delta^1_3(\mathbf{S}, a_1, a_2), a_3)$ if $\{\alpha, \delta\} = \{\mathbf{W}, \mathbf{E}\}$

Equation 5.8 states that either $\alpha = \beta = \gamma = \delta$ or exactly one of the three relations $(\alpha = \beta, \beta = \gamma, \alpha = \delta)$ holds. To establish this, we will need the following two observations about $I(a, Z)$:

Observation 1. If a is elementary and Z is a generator type, then $I(a, Z)$ is uniquely determined by a and whether or not $Z = \mathbf{N}$, except in the special case where $a = (U_1U_2)^\ell$ for some $\ell \geq 0$. For example, if $a = L_1 \cdot p$ or $R_2 \cdot p$, where p is a monomial in U_1 and U_2, then $I(a, Z) = \mathbf{N}$; if $a = U_1^x \cdot U_2^y$ and $Z \in \{\mathbf{W}, \mathbf{E}\}$, then $I(a, Z) = \mathbf{W}$ if $x > y$ and $I(a, Z) = \mathbf{E}$ if $x < y$.

The generator types $\{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ are further grouped into two classes, $\{\mathbf{N}\}$ and $\{\mathbf{W}, \mathbf{E}\}$; in this language, the Observation 1 can be phrased as saying that if $a \neq (U_1U_2)^\ell$, the class of Z and the algebra element a uniquely determines $I(a, Z)$.

Observation 2. If a_1 and a_2 are elementary and $a_1 \cdot a_2 \neq 0$, then the class of $I(a_1, I(a_2, Z))$ agrees with the class of $I(a_1 \cdot a_2, Z)$.

By the second observation, the class of $I(a_2 \cdot a_3, Y)$ agrees with the class of $I(a_2, I(a_3, Y))$, so by the first observation, $I(a_1, I(a_2 \cdot a_3, Y)) = I(a_1, I(a_2, I(a_3, Y)))$; i.e. $\gamma = \delta$, so if any two of the conditions $(\alpha = \beta, \beta = \gamma, \alpha = \delta)$ holds, the third one holds, verifying Equation (5.8) when $a_1 \neq (U_1U_2)^\ell$. $\square$

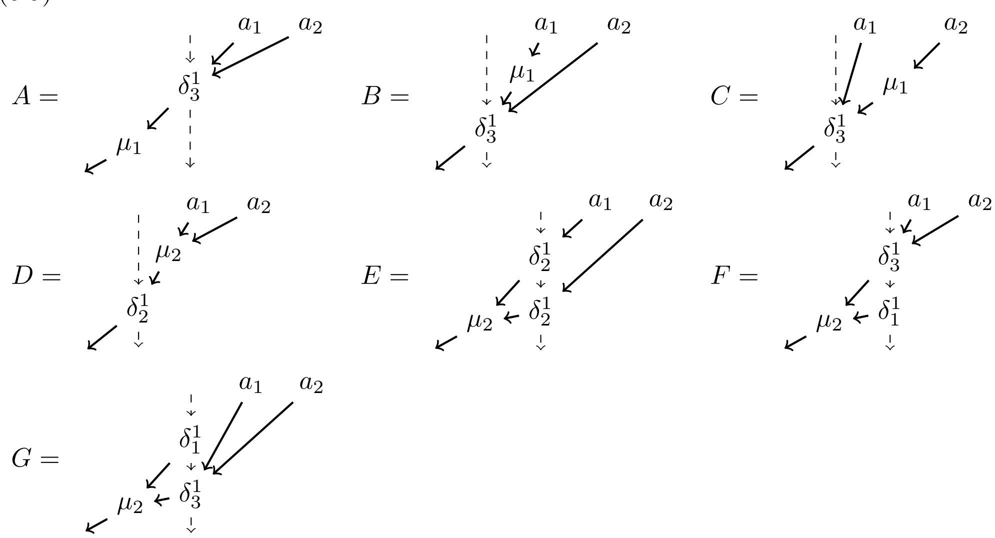
Proof of Proposition 5.2. Note that in $\mathcal{B}(2,3)$, there is only one non-trivial generator, and it is $\mathbf{N}$; so $\delta^1_\ell = 0$ for all $\ell > 2$. In fact, $\mathcal{P}^1$ in this case is simply the identity bimodule for $\mathcal{B}(2,3)$, and the $\mathcal{A}_\infty$ relation holds. A similar simplification occurs for $\mathcal{B}(2,0)$, where the algebra is one-dimensional, and $\mathbf{S}$ is the only module generator.

We turn to the more interesting cases, in the summands $\mathcal{B}(2,1)$ and $\mathcal{B}(2,2)$. Varying n, we verify the DA bimodule relation with n algebra inputs $a_1, \dots, a_n$ (and one module input), subdividing further each verification according to whether we are in $\mathcal{B}(2,1)$ or $\mathcal{B}(2,2)$.

When $n = 0$, the $\mathcal{A}_\infty$ relation is clear from the form of δ^1_1.

When $n=1$, terms of the form $\widetilde{\delta}^1_1\circ\delta^1_2$ cancel in pairs, except in the special case where $a_1=(U_1U_2)^\ell$, in which case a term of the form $\widetilde{\delta}^1_1\circ\delta^1_2$ cancels with another of the form $\widetilde{\delta}^1_2\circ\delta^1_1$.

When $n=2$, we find it convenient to label the terms in the $\mathcal{A}_\infty$ relation as follows:
(5.9)



So that the $\mathcal{A}_\infty$ relation on a DA bimodule with two inputs reads

$$A+B+C+D+E+F+G=0.$$

On the algebras we are considering presently, $\mu_1=0$, so $A=B=C=0$.

When $n=2$, since all actions are U_1U_2-equivariant, the case where at least one of a_1 or a_2 is $(U_1U_2)^\ell$ is straightforward: the only two possibly non-zero terms are D and E, and they cancel.

When $n=2$, and we are in the summand in $\mathcal{B}(2,2)$, and the starting and ending generator is in $\{\mathbf{E},\mathbf{W},\mathbf{N}\}$, the $\mathcal{A}_\infty$ relation follows from Lemma 5.5, and the observation that $a_1\cdot a_2\neq 0$. Specifically, in this case, $A=B=C=0$, and the δ^1_3 actions are defined so that the term in G cancels D and E.

When $n=2$ and we are in the summand in $\mathcal{B}(2,2)$, and the starting and ending generator is $\mathbf{S}$, then $A=B=C=G=0$, and we find that terms of type F cancel in pairs, except in the special cases where $a_1\cdot a_2=(U_1U_2)^\ell$, which are the cases where $D\neq 0$, in which case there is a single cancelling term of type F. This is verified by looking at the formulas defining δ^1_3. For example, consider an $\mathcal{A}_\infty$ relation with generator of type $\mathbf{S}$, $a_1=U_1^n$, and $a_2=U_2^t$. If $1\leq n\leq t$, there is a term of type F, factoring through $\mathbf{E}$, and this cancels against another term of type D, factoring through $\mathbf{W}$, when $n<t$; otherwise, it cancels against a contribution of type D.

When $n=2$ and we are in the summand in $\mathcal{B}(2,1)$ and the initial generator is of type $\{\mathbf{E},\mathbf{W},\mathbf{N}\}$, terms of type D and E cancel except in special cases where E contributes but $a_1\cdot a_2=0$, in which case there is a cancelling non-zero term of type G. When the initial generator is of type $\mathbf{S}$, and neither of a_1 nor a_2 equals

$(U_1U_2)^\ell$, the only possible non-zero term, which is of type F, vanishes thanks to the algebra; for example, in the $\mathcal{A}_\infty$ relation $\delta^1_3(\mathbf{S}, R_1, R_2U_2^t) = R_1U_1^t \otimes \mathbf{E}$, and $\delta^1_1(\mathbf{E}) = R_2 \otimes \mathbf{S}$, but $R_1 \cdot R_2 = 0$ in $\mathcal{B}(2,1)$.

The cases where $n = 3$, the $\mathcal{A}_\infty$ relation trivially holds except in the special cases where the initial generator is of type $\mathbf{S}$. That case was covered by Lemma 5.6.

The case where $n > 3$ is obvious. □

5.2. **Extension.** Fix integers k and m with $0 \leq k \leq m+1$, we extend the bimodule P to a bimodule ${}^{\mathcal{B}(m,k)}\mathcal{P}^i_{\mathcal{B}(m,k)}$, as follows.

Let $a \in \mathcal{B}_0(m,k)$ with $a = \mathbf{I}_{\mathbf{x}} \cdot a \cdot \mathbf{I}_{\mathbf{y}}$, and suppose that $\mathbf{x}$ and $\mathbf{y}$ are close enough. Suppose moreover that a is pure, in the sense that it corresponds to some monomial in $U_1, \dots, U_m$ under $\phi^{\mathbf{x},\mathbf{y}}$. We define the *type* of a, denoted $t(a)$, which is an expression in in U_1, U_2, R_1, L_1 R_2 and L_2, defined as follows:

$$t(a) = \begin{cases} R_2R_1U_1^{w_i(a)-\frac{1}{2}}U_2^{w_{i+1}(a)-\frac{1}{2}} & \text{if } w_i(a) \equiv w_{i+1}(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}} \\ & \text{and } v^{\mathbf{x}}_{i+1} < v^{\mathbf{y}}_{i+1} \\ L_1L_2U_1^{w_i(a)-\frac{1}{2}}U_2^{w_{i+1}(a)-\frac{1}{2}} & \text{if } w_i(a) \equiv w_{i+1}(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}} \\ & \text{and } v^{\mathbf{x}}_{i+1} > v^{\mathbf{y}}_{i+1} \\ R_2U_1^{w_i(a)}U_2^{w_{i+1}(a)-\frac{1}{2}} & \text{if } w_i(a) \in \mathbb{Z} \text{ and } w_{i+1}(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}}, \\ & \text{and } v^{\mathbf{x}}_{i+1} < v^{\mathbf{y}}_{i+1} \\ L_2U_1^{w_i(a)}U_2^{w_{i+1}(a)-\frac{1}{2}} & \text{if } w_i(a) \in \mathbb{Z} \text{ and } w_{i+1}(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}}, \\ & \text{and } v^{\mathbf{x}}_{i+1} > v^{\mathbf{y}}_{i+1} \\ R_1U_1^{w_i(a)-\frac{1}{2}}U_2^{w_{i+1}(a)} & \text{if } w_i(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}} \text{ and } w_{i+1}(a) \in \mathbb{Z}, \\ & \text{and } v^{\mathbf{x}}_{i} < v^{\mathbf{y}}_{i} \\ L_1U_1^{w_i(a)-\frac{1}{2}}U_2^{w_{i+1}(a)} & \text{if } w_i(a) \equiv \frac{1}{2} \pmod{\mathbb{Z}} \text{ and } w_{i+1}(a) \in \mathbb{Z}, \\ & \text{and } v^{\mathbf{x}}_{i} > v^{\mathbf{y}}_{i} \\ U_1^{w_i(a)}U_2^{w_{i+1}(a)} & \text{if } w_i(a) \text{ and } w_{i+1}(a) \text{ are integers.} \end{cases}$$

Similarly, there is a map t from generators of $\mathcal{P}^i$ to the four generators of P, that remembers only the type ($\mathbf{N}$, $\mathbf{S}$, $\mathbf{W}$, $\mathbf{E}$) of the generator of $\mathcal{P}^i$.

Definition 5.7. *For $X \in \mathcal{P}^i$, an integer $\ell \geq 1$, and a sequence of algebra elements $a_1, \dots, a_{\ell-1}$ in $\mathcal{B}_0(m,k)$ with specified weights, so that there exists a sequence of idempotent states $\mathbf{x}_0, \dots, \mathbf{x}_\ell$ with*

- $X = \mathbf{I}_{\mathbf{x}_0} \cdot X \cdot \mathbf{I}_{\mathbf{x}_1}$
- $a_t = \mathbf{I}_{\mathbf{x}_t} \cdot a_t \cdot \mathbf{I}_{\mathbf{x}_{t+1}}$ *for* $t = 1, \dots, \ell-1$
- $\mathbf{x}_t$ *and* $\mathbf{x}_{t+1}$ *are close enough (for* $t = 0, \dots, \ell-1$*),*

define $D_\ell(X, a_1, \dots, a_{\ell-1}) \in \mathcal{B}_0(m,k) \otimes \mathcal{P}^i$ as the sum of pairs $b \otimes Y$ where $b \in \mathcal{B}_0(m,k)$ and Y is a generator of $\mathcal{P}^i$, satisfying the following conditions:

- *the weights of b and Y satisfy*

$$\mathbf{gr}(X) + \tau_i^{\mathbf{gr}}(\mathbf{gr}(a_1) + \dots + \mathbf{gr}(a_{\ell-1})) = \mathbf{gr}(b) + \mathbf{gr}(Y) \tag{5.10}$$

- *There are generators X_0 and Y_0 with the same type (i.e. with the same label $\{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$) as X and Y respectively, so that $t(b) \otimes Y_0$ appears with non-zero multiplicity in $\delta^1_\ell(X_0, t(a_1), \dots, t(a_{\ell-1}))$.*

Clearly, $D_\ell = 0$ for $\ell > 3$. Setting $\delta^1_1 = D_1$, we get

$$\delta^1_1(\mathbf{W}) = L_1 \otimes \mathbf{S}, \quad \delta^1_1(\mathbf{E}) = R_2 \otimes \mathbf{S}, \quad \delta^1_1(\mathbf{N}) = \delta^1_1(\mathbf{S}) = 0.$$

Lemma 5.8. *Suppose that any $a_t \in \mathcal{I}(\mathbf{x}_t, \mathbf{x}_{t+1})$ (the ideal studied in Proposition 3.7), then the projection of $D_\ell(X, a_1, \dots, a_{\ell-1})$ to $\mathcal{B}(m,k) \otimes \mathcal{P}^i$ vanishes; i.e. the maps D_ℓ induce well-defined maps*

$$\delta^1_\ell \colon \mathcal{P}^i \otimes \overbrace{\mathcal{B}(m,k) \otimes \cdots \otimes \mathcal{B}(m,k)}^{\ell-1} \to \mathcal{B}(m,k) \otimes \mathcal{P}^i,$$

for all $\ell = 1, 2, 3$.

Proof. For the δ^1_1 operations, the result is obvious.

The operation δ^1_2 can be written as a sum of 16 terms, corresponding to the terms in the labels on Figure 5.4.

For example, consider $\delta^1_2(\mathbf{E}, a \cdot U^\ell_{i+1})$, where $w_i(a) = w_{i+1}(a) = 0$ and $\ell > 0$. According to Equation 5.4, this can be written as a sum of two terms,

$$R_{i+1}R_i \cdot a \cdot U_i^{\ell-1} \otimes \mathbf{W} + a \cdot U_i^\ell \otimes \mathbf{E}.$$

We check that for any idempotent states $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{E} \cdot \mathbf{I}_\mathbf{x} \neq 0$ and $\mathbf{I}_\mathbf{x} \cdot a \cdot U^\ell_{i+1} \cdot \mathbf{I}_\mathbf{x} \neq 0$, if $\mathbf{I}_\mathbf{x} \cdot a \cdot U^\ell_{i+1} \in \mathcal{J}$, then the two output terms also vanish. If a is divisible by a monomial corresponding to a generating interval for $(\mathbf{x}, \mathbf{y})$ that is disjoint from $\{i, i+1\}$, each output elements are also divisible by such monomials, and the claim is clear. It remains to check the claim when $a \cdot U_{i+1}$ is the monomial corresponding to a generating interval $U_{i+1} \cdots U_\beta$ for the incoming element. In this case, the output is divisible by $U_{i+2} \cdots U_\beta$, which corresponds to a generating interval in both output algebras.

The other non-trivial checks include the following:

$$\begin{aligned}
\delta^1_2(\mathbf{W}, U_\alpha \cdots U_i) &= (U_\alpha \cdots U_{i-1}U_{i+1}) \otimes \mathbf{W} + L_iL_{i+1}(U_\alpha \cdots U_{i-1}) \otimes \mathbf{E} \\
\delta^1_2(\mathbf{N}, U_{i+1} \cdots U_\beta \cdot L_i) &= (U_i \cdots U_\beta) \otimes \mathbf{W} + L_iL_{i+1}U_i(U_{i+2} \cdots U_\beta) \otimes \mathbf{E} \\
\delta^1_2(\mathbf{N}, U_\alpha \cdots U_{i-1} \cdot L_i) &= (U_\alpha \cdots U_i) \otimes \mathbf{W} + L_iL_{i+1}(U_\alpha \cdots U_{i-1}) \otimes \mathbf{E} \\
\delta^1_2(\mathbf{W}, U_\alpha \cdots U_i \cdot L_{i+1}) &= U_\alpha \cdots U_{i-1} \cdot L_i \cdot L_{i+1} \otimes \mathbf{N} \\
\delta^1_2(\mathbf{W}, U_\alpha \cdots U_{i-1} \cdot R_i) &= U_\alpha \cdots U_{i-1} \otimes \mathbf{N}
\end{aligned}$$

where the monomials $U_\alpha \cdots U_i$ and $U_{i+1} \cdots U_\beta$ (with $\alpha \leq i$ and $\beta \geq i+2$) correspond to generating intervals for the input, for suitable choices of idempotents; it is then straightforward to check that the outputs are also zero. The remaining non-trivial checks are symmetric to the above ones (under the symmetry exchanging $\mathbf{W}$ and $\mathbf{E}$, and L_i with R_{i+1}).

For δ^1_3, we must show that if a_1 or $a_2 \in \mathcal{J}$, then the output $D_3(X, a_1, a_2)$ projects to zero in $\mathcal{B}(m,k) \otimes \mathcal{P}^i$. In cases where $\min(w_i(a_1)+w_i(a_2), w_{i+1}(a_1)+w_{i+1}(a_2)) = \frac{1}{2}$, i.e. when $(t(a_1), t(a_2)) \in \{(R_1, R_2U_2^t), (L_2, L_1U_1^n), (L_2, U_1^{n+1}), (R_1, U_2^{t+1})\}$, are considered separately. In all these cases, the generating intervals for a_1 and a_2 do not contain i or $i+1$, so the result is obvious.

For the remaining cases, the following stronger assertion holds: if a_1 and a_2 satisfy

$$\min(w_i(a_1) + w_i(a_2), w_{i+1}(a_1) + w_{i+1}(a_2)) > \frac{1}{2}, \tag{5.11}$$

and $a_1 \cdot a_2 \in \mathcal{J}$ (i.e. $a_1 \cdot a_2$ projects to zero in $\mathcal{B}(m,k)$), and with $\mathbf{I_x} \cdot a_1 = a_1$ then $D_3(X, a_1, a_2) \in \mathcal{J}$. For instance, the terms in $\delta^1_3(\mathbf{S}, a_1, a_2)$ in $\mathcal{B}(2)$ that output $L_2 U_1^t U_2^n \otimes \mathbf{E}$ (all of which satisfy Equation (5.11)) give rise to actions with $a_1, a_2 \in \mathcal{B}(k,m)$ with $\mathbf{I_x} \cdot a_1 \cdot a_2 = a_1 \cdot a_2$ with

$$i \notin \mathbf{p}, \quad w_i(a_1 \cdot a_2) = n+1 \geq 1, \quad w_{i+1}(a_1 \cdot a_2) = t \geq 1$$

and output algebra element $b = \mathbf{I_x} \cdot b \cdot \mathbf{I_y}$ with

$$i \in \mathbf{y}, \quad i+1 \notin \mathbf{y} \quad w_i(b) = t \geq 1$$

It is easy to see that if $a_1 \cdot a_2$ is divisible by a monomial corresponding to a generating interval of the form $U_\alpha \cdots U_i$, then so is b; checking cases where $a_1 \cdot a_2$ is divisible by other generating intervals is even more straightforward. Remaining cases where the output contains $\mathbf{W}$ work similarly. Cases where the output contains $\mathbf{N}$ are easily verified, as well. The stronger statement (following Equation (5.11)) now follows. □

For example, $\delta^1_2(\mathbf{N}, R_{i+2}R_{i+1}) = R_{i+2}R_{i+1}R_i \otimes \mathbf{W} + R_{i+2}U_i \otimes \mathbf{E}$.

As another example, choose R_j with $j \neq i$ or $i+1$. Then, for all generators X, $\delta^1_2(X, R_j) = R_j \otimes X$. (Note that in this example, $t(R_j) = 1$.)

As another example, $\delta^1_3(\mathbf{S}, R_i, R_{i+2}R_{i+1}) = \delta^1_3(\mathbf{S}, R_iR_{i+2}, R_{i+1}) = R_iR_{i+2} \otimes \mathbf{E}$.

Let $\delta^1_\ell = 0$ for all $\ell \geq 4$.

It will be useful to have the following characterization of δ^1_3:

Lemma 5.9. *The operation $\delta^1_3(X, a_1, a_2)$ contains only those terms $b \otimes Y$ where both Equation (5.10) holds, and one of the following two conditions holds:*

- *(C-1) $(t(a_1), t(a_2)) \in \{(R_1, R_2U_2^t), (L_2, L_1U_1^n), (L_2, U_1^{n+1}), (R_1, U_2^{t+1})\}$ and $t(b) \otimes t(Y)$ appears with non-zero multiplicity in $\delta^1_3(t(X), t(a_1), t(a_2))$;*
- *(C-2) $a_1 \cdot a_2 \neq 0$, $I(a_2, Y) \neq 0$ and $I(t(a_1), I(t(a_2), t(Y))) \neq I(t(a_1) \cdot t(a_2), t(Y))$.*

Proof. In the proof of Lemma 5.8, we showed that if we are not in Case (C-1), then Equation (5.11) holds, and so we conclude that $a_1 \cdot a_2 \neq 0$. Thus, our description of δ^1_3 follows from Lemma 5.5. □

Proposition 5.10. *The above maps give $\mathcal{P}^i$ the structure of a type DA bimodule over $\mathcal{B}(m,k)$-$\mathcal{B}(m,k)$.*

Proof. The proof of Proposition 5.2 adapts, with a few remarks. In the verification of the $\mathcal{A}_\infty$ relation with two algebra inputs, that proof decomposed according to whether we were working in $\mathcal{B}(2,1)$ or $\mathcal{B}(2,2)$. In the present case, when $(t(a_1), t(a_2)) \in \{(R_1, R_2U_2^t), (L_2, L_1U_1^n), (L_2, U_1^{n+1}), ($ the $\mathcal{A}_\infty$ relation holds exactly as it did in $\mathcal{B}(2,1)$. Otherwise , if $a_1 \cdot a_2 \neq 0$, the proof of the $\mathcal{A}_\infty$ relation for $\mathcal{B}(2,2)$ applies, using Lemma 5.9. Finally, if $a_1 \cdot a_2 = 0$, then by Lemma 5.9, the term involving δ^1_3 (F or G) vanishes. The term D vanishes by Lemma 5.8. The verification of the $\mathcal{A}_\infty$ relation in $\mathcal{B}(2,2)$ shows that the remaining possible non-zero term, which is of type E, has the same contribution as a term of type F, G, or D, all of which contribute 0.

Consider next the case of three algebra inputs a_1, a_2, and a_3. When $a_1 \cdot a_2 \cdot a_3 = 0$, the verification (now in the proof of Lemma 5.6) works as it did in $\mathcal{B}(2,1)$. In the remaining cases, the earlier proof of the $\mathcal{A}_\infty$ relation in $\mathcal{B}(2,2)$ (contained in the proof of Lemma 5.6) still applies, as it hinges on the description of δ^1_3 (Lemma 5.5) which still holds in this case (according to Lemma 5.9). □

5.3. **Adding C_i, locally.** We extend the local bimodule P defined over $\mathcal{B}(2)$ to a bimodule ${}^{\mathcal{B}(2,\tau(\mathcal{S}))}P_{\mathcal{B}(2,\mathcal{S})}$, where $\mathcal{S}$ is a non-empty subset of $\{1,2\}$.

In the local module P, there were δ^1_2 actions connecting generators of types $\{\mathbf{N},\mathbf{W},\mathbf{E}\}$ to $\{\mathbf{N},\mathbf{W},\mathbf{E}\}$ or $\mathbf{S}$ to $\mathbf{S}$. We extend these to similar actions so that

$$\delta^1_2(X, C_1\cdot a) = C_2\cdot \delta^1_2(X,a) \quad \delta^1_2(X, C_2\cdot a) = C_1\cdot \delta^1_2(X,a) \tag{5.12}$$

when the formulas make sense; i.e. the first holds when $1\in\mathcal{S}$ and the second when $2\in\mathcal{S}$. Similarly, we extend the previous δ^1_3 actions so that

$$\begin{aligned} C_2\cdot\delta^1_3(\mathbf{S},a_1,a_2) &= \delta^1_3(\mathbf{S},C_1\cdot a_1,a_2) = \delta^1_3(\mathbf{S},a_1,C_1\cdot a_2)\\ C_1\cdot\delta^1_3(\mathbf{S},a_1,a_2) &= \delta^1_3(\mathbf{S},C_2\cdot a_1,a_2) = \delta^1_3(\mathbf{S},a_1,C_2\cdot a_2). \end{aligned}$$

For example, $\delta^1_3(C_1R_1, C_2R_2) = \delta^1_3(C_1C_2R_1, R_2) = R_1C_1C_2\otimes\mathbf{E}$.

We specify further δ^1_2 actions from $\mathbf{S}$ to $\{\mathbf{N},\mathbf{W},\mathbf{E}\}$:

$$\begin{array}{ll}
\delta^1_2(\mathbf{S},C_2) = R_1\otimes\mathbf{W} & \\
\delta^1_2(\mathbf{S},C_1) = L_2\otimes\mathbf{E} & \delta^1_2(\mathbf{S},C_1C_2) = C_2R_1\otimes\mathbf{W} + C_1L_2\otimes\mathbf{E}\\
\delta^1_2(\mathbf{S},U_1C_2) = U_1L_2\otimes\mathbf{E} & \delta^1_2(\mathbf{S},U_1C_1C_2) = U_1C_2L_2\otimes\mathbf{E}\\
\delta^1_2(\mathbf{S},C_1U_2) = R_1U_2\otimes\mathbf{W} & \delta^1_2(\mathbf{S},C_1C_2U_2) = C_1R_1U_2\otimes\mathbf{W}\\
\delta^1_2(\mathbf{S},R_1C_2) = R_1\otimes\mathbf{N} & \delta^1_2(\mathbf{S},R_1C_1C_2) = R_1C_2\otimes\mathbf{N}\\
\delta^1_2(\mathbf{S},L_2C_1) = L_2\otimes\mathbf{N} & \delta^1_2(\mathbf{S},C_1L_2C_2) = C_1L_2\otimes\mathbf{N}\\
\delta^1_2(\mathbf{S},R_1C_1U_2) = R_1U_2\otimes\mathbf{N} & \delta^1_2(\mathbf{S},R_1C_1U_2C_2) = R_1C_1U_2\otimes\mathbf{N}\\
\delta^1_2(\mathbf{S},U_1L_2C_2) = L_2U_1\otimes\mathbf{N} & \delta^1_2(\mathbf{S},U_1C_1L_2C_2) = U_1L_2C_2\otimes\mathbf{N}
\end{array}$$

These are extended to commute with multiplication by U_1U_2 (as in Equation (5.5)).

By Equation (5.12), the actions $\delta^1_2(\mathbf{S},1) = \mathbf{S}$ gives rise to actions $\delta^1_2(\mathbf{S},C_2) = C_1\otimes\mathbf{S}$. The first action listed above, i.e. the relation $\delta^1_2(\mathbf{S},C_2) = R_1\otimes\mathbf{W}$, is now a consequence of this fact, together with the $\mathcal{A}_\infty$ relation with module element $\mathbf{S}$, the single algebra input C_2, and output module element $\mathbf{S}$. The second action on the above list follows symmetrically. The actions on the left column of rows three, five, and seven respectively are forced by the following δ^1_3 actions (with $\mathcal{S}=\emptyset$)

$$\delta^1_3(\mathbf{S},U_1,U_2) = L_2U_1\otimes\mathbf{E}$$
$$\delta^1_3(\mathbf{S},L_2,U_1) = L_2\otimes\mathbf{N}$$
$$\delta^1_3(\mathbf{S},R_1U_2,U_1) = R_1U_2\otimes\mathbf{N}$$

and the $\mathcal{A}_\infty$ relations with inputs $(\mathbf{S},U_1,C_2)$, $(\mathbf{S},L_2,C_1)$ and $(\mathbf{S},R_1U_2,C_1)$ respectively. The remaining actions on the left column follow symmetrically. The actions in the second column follow from actions from the first column column of the form $(\mathbf{S},da)$ and $\mathcal{A}_\infty$ relations with inputs $(\mathbf{S},a)$.

Lemma 5.11. *For any $\mathcal{S}\subset\{1,2\}$, the above actions induce a type DA bimodule structure on ${}^{\mathcal{B}(2,\tau(\mathcal{S}))}P_{\mathcal{B}(2,\mathcal{S})}$.*

Proof. It suffices to prove the lemma in the case where $\mathcal{S}=\{1,2\}$. For example, if $\mathcal{S}=\{1\}$, the outputs of the δ^1_j actions lie in the subalgebra $\mathcal{B}(2,\{2\})$.

We consider the $\mathcal{A}_\infty$ relation with n incoming algebra elements $a_1,\dots,a_n$.

It is a straightforward verification to check that the actions defined above are consistent with the $\mathcal{A}_\infty$ relation with $n=1$ input.

When $n=2$, and the incoming generator is not $\mathbf{S}$, the $\mathcal{A}_\infty$ relation follows from Equation (5.12) together with the $\mathcal{A}_\infty$ relation with $\mathcal{S}=\emptyset$.

When $n=2$ and the incoming generator is $\mathbf{S}$, label the terms in the $\mathcal{A}_\infty$ relation as in Equation (5.9). The key point to verifying this relation now is that when differentiating a_1 or a_2 (in terms B or C above), the U_1 or U_2-power can change by at most one so the corresponding δ^1_3 action with a_1 and a_2 is non-zero when the action with (da_1,a_2) and (a_1,da_2) is. However, there are borderline cases where this change in the U_1 or the U_2 power is enough to to turn off one of those actions. These are precisely the cases either the product a_1a_2 acts non-trivially (i.e. where D contributes), or there is an iterated δ^1_2 (i.e. E contributes).

For example, consider the case where $a_1=U_1^nC_1$, and $a_2=U_2^t$ with $n,t\geq 0$, and consider the terms where the output generator has type $\mathbf{E}$. We have the following non-zero terms in the $\mathcal{A}_\infty$ relation:

$$A\neq 0\Leftrightarrow 0\leq n-1<t,\quad B\neq 0\Leftrightarrow 0\leq n<t,\qquad D\neq 0\Leftrightarrow n=t\geq 0,$$
$$E\neq 0\Leftrightarrow n=0,t\geq 0;$$

Thus, for all choices of $n,t\geq 0$, there are either no non-zero terms or exactly two, and so the $\mathcal{A}_\infty$ relation holds. Other cases where $n=2$ and the initial generator is of type $\mathbf{S}$ work similarly.

Consider the case where $n=3$. Recall that the algebra has a filtration in $\{0,1\}\times\{0,1\}$ given by the functions $(\mathfrak{m}_1,\mathfrak{m}_2)$, with the property that if a is a pure algebra element not divisible by C_j, then $\mathfrak{m}_j(a)=0$ and $\mathfrak{m}_j(C_ja)=1$. Clearly, the operations δ^1_ℓ respect this filtration, in the sense that if a $(\mathfrak{m}_1,\mathfrak{m}_2)$-homogeneous element $b\otimes Y$ appears with non-zero multiplicity in $\delta^1_\ell(a_1,\dots,a_{\ell-1})$, and each a_i is homogeneous (with respect to μ), then for $j=1,2$,

$$(\mathfrak{m}_1(b),\mathfrak{m}_2(b))\leq\sum_{k=1}^{\ell-1}(\mathfrak{m}_1(a_k),\mathfrak{m}_2(a_k)).$$

For the terms that preserve this $\{0,1\}\times\{0,1\}$-filtration, the $\mathcal{A}_\infty$ relation with 3 inputs is an immediate consequence of Proposition 5.10. Consider next terms that drop filtration level by one. Note that each δ^1_ℓ action of this type has $\ell=2$ and end in the span of $\{\mathbf{N},\mathbf{W},\mathbf{E}\}$, while all δ^1_3 actions start from $\mathbf{S}$. It follows that there are no such terms that appear in the $\mathcal{A}_\infty$ relation with 3 inputs. (It also follows that there are no terms of this kind in the $\mathcal{A}_\infty$ relation which drop by more than 1.)

For $n>3$, the $\mathcal{A}_\infty$ relation is easy. □

5.4. **The general case of $\mathcal{P}^i$.** Let $\mathcal{B}_1=\mathcal{B}(m,k,\mathcal{S})$, where $0\leq k\leq m+1$ and $\mathcal{S}\subset\{1,\dots,m\}$ is arbitrary; and let $\mathcal{B}_2=\mathcal{B}(m,k,\tau(\mathcal{S}))$. In cases where $\mathcal{S}\cap\{i,i+1\}\neq\emptyset$, we must modify our earlier constructions as follows.

Extend the type $t(a)$ to be a monomial in $U_1, U_2, L_1, L_2, C_1, C_2, R_1, R_2$, so that

$$\begin{aligned} t(C_i \cdot a) &= C_1 \cdot t(a) \quad && \text{if } C_i \cdot a \neq 0 \\ t(C_{i+1} \cdot a) &= C_2 \cdot t(a) \quad && \text{if } C_{i+1} \cdot a \neq 0; \end{aligned}$$

and $t(a)$ is as defined before when a is a pure algebra element not divisible by C_i or C_{i+1}.

Let $t(\mathcal{S}) \subset \{1,2\}$ be the set with $1 \in t(\mathcal{S})$ iff $i \in \mathcal{S}$ and $2 \in t(\mathcal{S})$ iff $i+1 \in \mathcal{S}$.

Definition 5.12. *For $X \in \mathcal{P}^i$ and a sequence of pure algebra elements $a_1, \dots, a_{\ell-1}$ in $\mathcal{B}_0(m,k,\mathcal{S})$, so that there exist a sequence of idempotent states $\mathbf{x}_0, \dots, \mathbf{x}_\ell$ with*

- $X = \mathbf{I}_{\mathbf{x}_0} \cdot X \cdot \mathbf{I}_{\mathbf{x}_1}$
- $a_t = \mathbf{I}_{\mathbf{x}_t} \cdot a_t \cdot \mathbf{I}_{\mathbf{x}_{t+1}}$ *for $t = 1, \dots, \ell$*
- $\mathbf{x}_t$ *and $\mathbf{x}_{t+1}$ are close enough (for $k = 0, \dots, \ell-1$),*

define $D_\ell(X, a_1, \dots, a_{\ell-1}) \in \mathcal{B}_0(m,k,\tau(\mathcal{S})) \otimes \mathcal{P}^i$ as the sum of pairs $b \otimes Y$ where $b \in \mathcal{B}_0(m,k,t(\tau(\mathcal{S})))$ and Y is a generator of $\mathcal{P}^i$, satisfying the the conditions of Definition 5.7, with the understanding that now $\delta^1_\ell(X_0, t(a_1), \dots, t(a_{\ell-1})$ is computed using the crossing over ${}^{\mathcal{B}(2,t\circ\tau(\mathcal{S}))}P_{\mathcal{B}(2,t(\mathcal{S}))}$; and the additional condition that for any $k \neq i$ or $i+1$, and any $m = 1, \dots, \ell-1$,

$$D_\ell(X, a_1, \dots, C_k \cdot a_m, \dots, a_{\ell-1}) = C_k \cdot D_\ell(X, a_1, \dots, a_m, \dots, a_{\ell-1}).$$

Lemma 5.13. *If any $a_t \in \mathcal{I}(\mathbf{x}_t, \mathbf{x}_{t+1})$, then the projection of $D_\ell(X, a_1, \dots, a_{\ell-1})$ to $\mathcal{B}(m,k) \otimes \mathcal{P}^i$ vanishes; i.e. the maps D_ℓ induce well-defined maps*

$$\delta^1_\ell \colon \mathcal{P}^i \otimes \overbrace{\mathcal{B}(m,k,\mathcal{S}) \otimes \cdots \otimes \mathcal{B}(m,k,\mathcal{S})}^{\ell-1} \to \mathcal{B}(m,k,\tau(\mathcal{S})) \otimes \mathcal{P}^i,$$

for all $\ell = 2, 3$.

Proof. Lemma 5.8 takes care of most of this; we must check further the additional δ^1_2 actions from $\mathbf{S}$ to $\{\mathbf{N}, \mathbf{W}, \mathbf{E}\}$ listed in the beginning of this subsection; but this is straightforward. ☐

Proposition 5.14. *The above maps give $\mathcal{P}^i$ the structure of a type DA bimodule over $\mathcal{B}(m,k,\tau(\mathcal{S}))$-$\mathcal{B}(m,k,\mathcal{S})$.*

Proof. This follows easily from Proposition 5.10, handling terms with C_i inputs as in Lemma 5.11. ☐

5.5. **The negative crossing.** Consider the map $\mathcal{R} \colon \mathcal{B}(m,k,\mathcal{S}) \to \mathcal{B}(m,k,\mathcal{S})$ from Section 3.6. Recall that $o(a \cdot b) = o(b) \cdot o(a)$, $o(\mathbf{I}_\mathbf{x}) = \mathbf{I}_\mathbf{x}$, $o(L_t) = R_t$ $o(R_t) = L_t$ $o(U_t) = U_t$ and $o(C_j) = C_j$ for all $t = 1, \dots, m$ and $j \in \mathcal{S}$.

Let $\mathcal{N}^i$ be generated by the same generators $\{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$ as before. If $\delta^1_1(X) = b \otimes Y$ in $\mathcal{P}^i$, then $\delta^1_1(Y) = o(b) \otimes X$ in $\mathcal{N}^i$ If $\delta^1_2(X,a) = o(b) \otimes Y$ in $\mathcal{P}^i$, then $\delta^1_2(Y, o(a))) = o(b) \otimes X$ in $\mathcal{N}^i$ If $\delta^1_3(X, a_1, a_2) = b \otimes Y$ in $\mathcal{P}^i$, then $\delta^1_3(Y, o(a_2), o(a_1)) = o(b) \otimes X$.

More succinctly, the opposite module of $\mathcal{P}^i$ is identified with

$$_{\mathcal{B}_2}(\overline{\mathcal{P}}^i)^{\mathcal{B}_1} \cong {}^{\mathcal{B}_1^{\mathrm{op}}}\overline{\mathcal{P}}^i_{\mathcal{B}_2^{\mathrm{op}}} = {}^{\mathcal{B}_1}\mathcal{N}^i_{\mathcal{B}_2},$$

under the identification of $\mathcal{B}_t^{\mathrm{op}} \cong \mathcal{B}_t$ for $t = 1, 2$ (Equation (3.11)).

For example, in $\mathcal{N}^i$, we have actions

$$\delta^1_1(\mathbf{S}) = R_i \otimes \mathbf{W} + L_{i+1} \otimes \mathbf{E}$$
$$\delta^1_2(\mathbf{E}, R_i) = R_{i+1}R_i \otimes \mathbf{N}$$
$$\delta^1_3(\mathbf{W}, R_iU_i^n, R_{i+1}) = R_{i+1}U_{i+1}^n \otimes \mathbf{S}.$$

Proposition 5.15. *The above maps give $\mathcal{N}^i$ the structure of a type DA bimodule over $\mathcal{B}(m,k,\tau(\mathcal{S}))$-$\mathcal{B}(m,k,\mathcal{S})$.*

Proof. This is a formal consequence of the above definition. □

5.6. **Gradings and partial Kauffman states.** As noted earlier, $\mathcal{P}^i$ is adapted to the underlying manifold. Thus, its grading set takes values in $H^1(W,\partial W)$. We can specialize to a simply $\mathbb{Q}$-graded-graded setting as follows.

Recall (Equation (3.8)) that the algebra has an Alexander grading with values in $\mathbb{Q}$, obtained from the map $\phi\colon \frac{1}{2}\mathbb{Z}^m \to \frac{1}{2}\mathbb{Z}$ defined by

$$\phi(e_i) = \begin{cases} -1 & \text{if } i \in \mathcal{S} \\ 1 & \text{if } i \notin \mathcal{S}. \end{cases}$$

There is an induced grading on the bimodule, given by

$$A(X) = \phi(\mathbf{gr}(X)); \tag{5.13}$$

which we can think of, more abstractly, as the evaluation of the grading, thought of as an element of $H^1(W,\partial W)$, against the element $[W,\partial] \in H_1(W,\partial W)$ induced by the orientation of W.)

It is now an immediate consequence of Equation (5.1) that

$$A(X,a_1,\dots,a_\ell) = A(X) + A(a_1) + \dots + A(a_\ell) = A(b) + A(Y),$$

if $b \otimes Y$ appears with non-zero multiplicity in $\delta^1_{\ell+1}(X,a_1,\dots,a_\ell)$.

Proposition 5.16. *The $\mathbb{Q}$-valued Alexander grading on generators from Equation (5.13) for the crossing bimodules is computed by the local Kauffman contributions displayed in Figure 2.*

Proof. This a straightforward check using Equation (5.13), (4.5), and (4.7), considering all the possible orientations of the braidlike positive and negative crossing. □

6. Braid relations

We prove that the DA bimodules $\mathcal{P}^i$ and $\mathcal{N}^i$ satisfy the following braid relations:

Theorem 6.1. *Fix m, k with $0 \leq k \leq m+1$, i with $1 \leq i \leq m-1$, and $\mathcal{S} \subset \{1,\dots,m\}$. Let $\mathcal{B}_1 = \mathcal{B}(m,k,\mathcal{S})$ and $\mathcal{B}_2 = \mathcal{B}(m,k,\tau_i(\mathcal{S}))$. Then,*

$$^{\mathcal{B}_1}\mathcal{P}^i_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\mathcal{N}^i_{\mathcal{B}_1} \simeq {}^{\mathcal{B}_1}\mathrm{Id}_{\mathcal{B}_1} \simeq {}^{\mathcal{B}_1}\mathcal{N}^i_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\mathcal{P}^i_{\mathcal{B}_1} \tag{6.1}$$

Given $j \neq i$ with $1 \leq j \leq m-1$, let $\mathcal{B}_3 = \mathcal{B}(m,k,\tau_j\tau_i(\mathcal{S}))$ and $\mathcal{B}_4 = \mathcal{B}(m,k,\tau_j(\mathcal{S}))$. If $|i-j| > 1$

$$^{\mathcal{B}_3}\mathcal{P}^j_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\mathcal{P}^i_{\mathcal{B}_1} \simeq {}^{\mathcal{B}_3}\mathcal{P}^i_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\mathcal{P}^j_{\mathcal{B}_1}. \tag{6.2}$$

while if $j=i+1$*, let* $\mathcal{B}_5=\mathcal{B}(m,k,\tau_i\tau_{i+1}\tau_i(\mathcal{S}))$ *and* $\mathcal{B}_6=\mathcal{B}(m,k,\tau_i\tau_{i+1}(\mathcal{S}))$*; then,*

$$ {}^{\mathcal{B}_5}\mathcal{P}^{i}_{\mathcal{B}_3}\boxtimes{}^{\mathcal{B}_3}\mathcal{P}^{i+1}_{\mathcal{B}_2}\boxtimes{}^{\mathcal{B}_2}\mathcal{P}^{i}_{\mathcal{B}_1}\simeq{}^{\mathcal{B}_5}\mathcal{P}^{i+1}_{\mathcal{B}_6}\boxtimes{}^{\mathcal{B}_6}\mathcal{P}^{i}_{\mathcal{B}_4}\boxtimes{}^{\mathcal{B}_4}\mathcal{P}^{i+1}_{\mathcal{B}_1}. \tag{6.3}$$

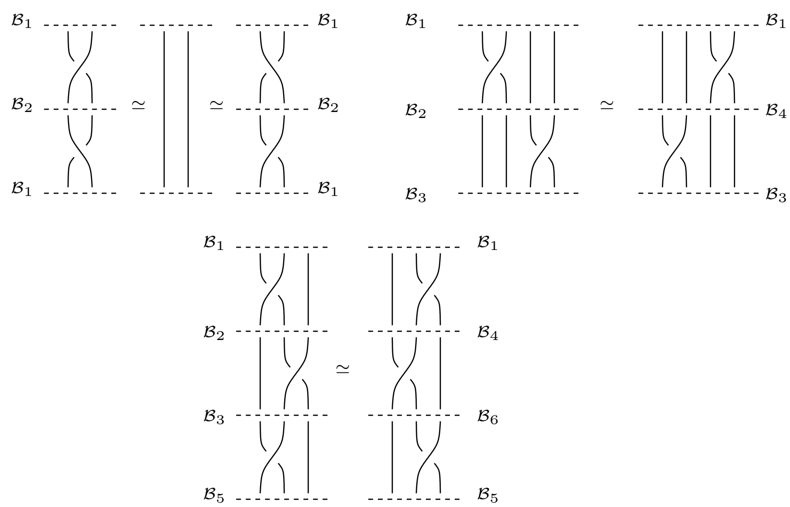


Figure 25. Bimodules and algebras appearing in Theorem 6.1.

We will prove the above with the help of the following:

Lemma 6.2. *Fix* m*,* k *with* $0\leq k\leq m+1$*,* i *with* $1\leq i\leq m-1$*, and* $\mathcal{S}\subset\{1,\dots,m\}$*. Let* $\mathcal{B}_1=\mathcal{B}(m,k,\mathcal{S})$*,* $\mathcal{B}_2=\mathcal{B}(m,k,\tau_i(\mathcal{S}))$*,* $\mathcal{B}_1'=\mathcal{B}(m,m+1-k,\{1,\dots,m\}\setminus\mathcal{S})$*. Let* ${}^{\mathcal{B}_2}\mathcal{P}^i_{\mathcal{B}_1}$ *be the DA bimodule from Section 5, and let* ${}^{\mathcal{B}_1,\mathcal{B}_1'}\mathcal{K}$ *be the canonical type DD bimodule from Section 3.7. Then,*

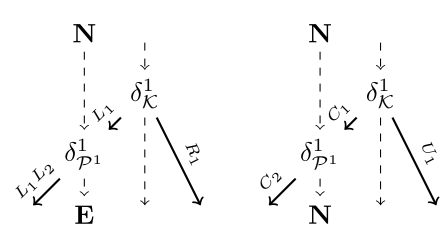
$$ {}^{\mathcal{B}_2}\mathcal{P}^{i}_{\mathcal{B}_1}\boxtimes{}^{\mathcal{B}_1,\mathcal{B}_1'}\mathcal{K}\simeq{}^{\mathcal{B}_2,\mathcal{B}_1'}\mathcal{P}_i \tag{6.4}$$

$$ {}^{\mathcal{B}_2}\mathcal{N}^{i}_{\mathcal{B}_1}\boxtimes{}^{\mathcal{B}_1,\mathcal{B}_1'}\mathcal{K}\simeq{}^{\mathcal{B}_2,\mathcal{B}_1'}\mathcal{N}_i \tag{6.5}$$

where the type DD bimodule appearing on the right is the one defined in Section 7.

Proof. We start by verifying Equation (6.4); and for simplicity, we assume $i=1$. The computation depends on $\mathcal{S}\cap\{1,2\}$.

We start with the case where $\{1,2\}\subset\mathcal{S}$. In fact, in this case, $\mathcal{P}^i\boxtimes\mathcal{K}=\mathcal{P}_i$. The arrows connecting $\mathbf{N}$, $\mathbf{W}$, and $\mathbf{E}$ are all induced from δ_2 actions on $\mathcal{P}^i$. For example, the differential from $\mathbf{N}$ to $\mathbf{E}$ labelled by $L_1L_2\otimes R_1$ from Figure 4.2 is obtained by pairing the term $L_1\otimes R_1$ term in the canonical DD bimodule pairs with the action $\delta^1_2(\mathbf{N},L_1)=L_1L_2\otimes\mathbf{E}$ in the $\mathcal{P}^1$, as shown on the left diagram:



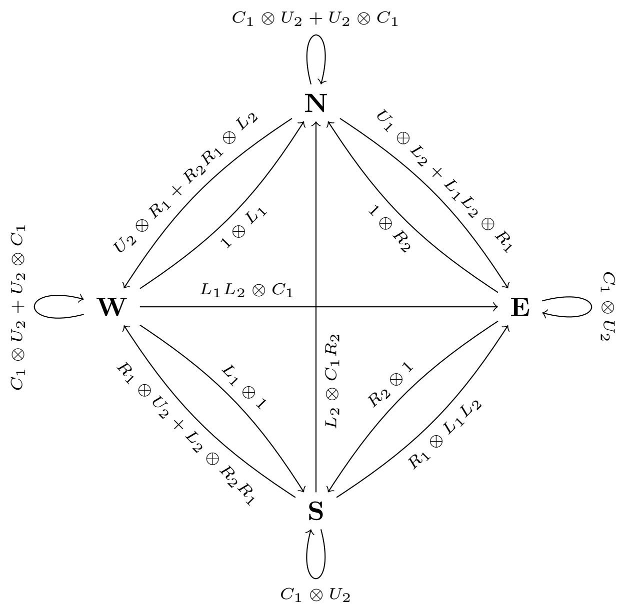
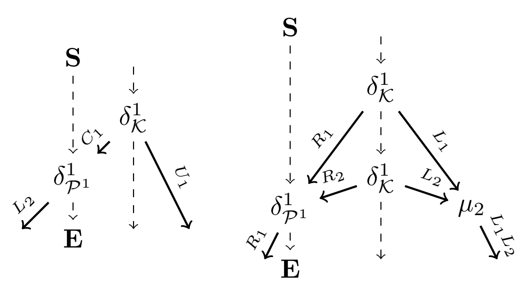


FIGURE 26. $\mathcal{P}^1 \boxtimes \mathcal{K}$ **when** $\mathcal{S} \cap \{1,2\} = \{2\}$

On the right, we demonstrate how to construct the term $(C_2 \otimes U_1) \otimes \mathbf{N}$ in $\delta^1(\mathbf{N})$ (of Type (P-2)).

The differentials into $\mathbf{S}$ come from the δ^1_1 action on $\mathcal{P}^i$, and the differentials out of $\mathbf{S}$ come from δ^1_2 and δ^1_3 actions on $\mathcal{P}^i$. For example, the two terms out of $\mathbf{S}$ involving the generator $\mathbf{E}$ arise as follows:



(Note that since $\mathcal{K}$ bimodule is a left/left bimodule, the multiplication appearing on the right is taking place in $(\mathcal{B}'_1)^{\mathrm{op}}$.)

Terms of Type (P-1) are easily constructed from pairing differentials in the identity DD bimodule with the part of $\mathcal{P}^i$ that behaves like an identity bimodule.

Consider next the case where $\mathcal{S} \cap \{1,2\} = \{2\}$. Then, $\mathcal{P}^1 \boxtimes \mathcal{K}$ is given as in Figure 26, and outside actions (i.e. of Type (P-1) and (P-2) with $j \neq 1,2$). Consider the map $h^1 \colon \mathcal{P}^1 \boxtimes \mathcal{K} \to \mathcal{P}_1$

$$h^1(X) = \begin{cases} \mathbf{S} + (L_2 \otimes C_1) \otimes \mathbf{E} & \text{if } X = \mathbf{S} \\ X & \text{otherwise.} \end{cases}$$

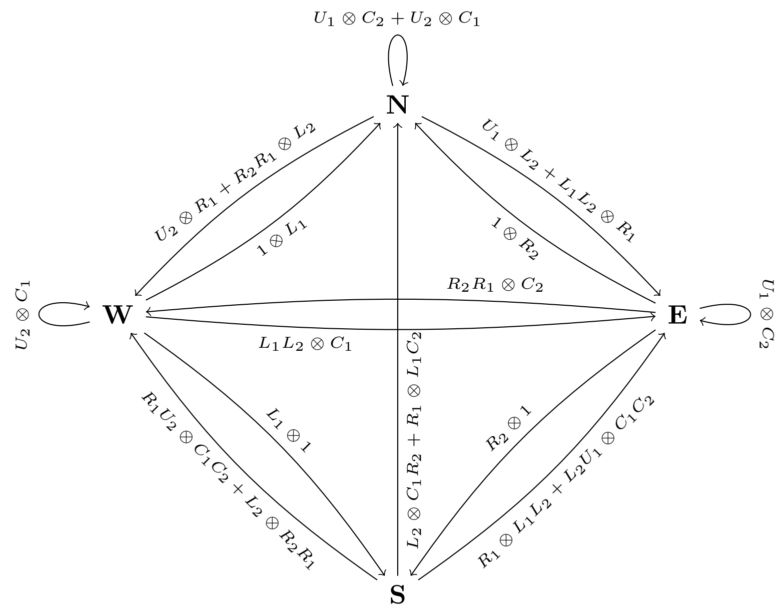


FIGURE 27. $\mathcal{P}^1 \boxtimes \mathcal{K}$ **when** $\mathcal{S} \cap \{1,2\} = \emptyset$**.**

Let $g^1 \colon \mathcal{P}_1 \to \mathcal{P}^1 \boxtimes \mathcal{K}$ be given by the same formula. It is easy to verify that h^1 and g^1 are homomorphisms of type DD structures, $h^1 \circ g^1 = \mathrm{Id}$, and $g^1 \circ h^1 = \mathrm{Id}$.

The case where $\mathcal{S} \cap \{1,2\} = \{1\}$ works similarly.

When $\mathcal{S} \cap \{1,2\} = \emptyset$, $\mathcal{P}^1 \boxtimes \mathcal{K}$ is given as in Figure 27

Consider the map $h^1 \colon \mathcal{P}^1 \boxtimes \mathcal{K} \to \mathcal{P}_1$

$$h^1(X) = \begin{cases} \mathbf{S} + (L_2 \otimes C_1) \cdot \mathbf{E} + (R_1 \otimes C_2) \cdot \mathbf{W} & \text{if } X = \mathbf{S} \\ X & \text{otherwise.} \end{cases}$$

Let $g^1 \colon \mathcal{P}_1 \to \mathcal{P}^1 \boxtimes \mathcal{K}$ be given by the same formula. It is easy to verify that h^1 and g^1 are homomorphisms of type DD structures, $h^1 \circ g^1 = \mathrm{Id}$, and $g^1 \circ h^1 = \mathrm{Id}$.

Equation (6.4) in cases where $i \neq 1$ works the same way. Equation (6.5) can be proved similarly. (Or alternatively, it can be seen as a consequence of Equation (6.4) and the symmetry of the bimodules, phrased in terms of the opposite algebras.) $\square$

Lemma 6.3. *Equation* (6.1) *holds.*

Proof. Since $(\mathcal{P}^i \boxtimes \mathcal{N}^i) \boxtimes \mathcal{K} \simeq \mathcal{P}^i \boxtimes (\mathcal{N}^i \boxtimes \mathcal{K}) \simeq \mathcal{P}^i \boxtimes \mathcal{N}_i$ (by associativity of $\boxtimes$ and Lemma 6.2), the verification that $\mathcal{P}^i \boxtimes \mathcal{N}^i \simeq \mathrm{Id}_{\mathcal{B}_1}$, will follow from the identity

$$\mathcal{P}^i \boxtimes \mathcal{N}_i \simeq \mathcal{K}, \tag{6.6}$$

which we verify presently. For notational simplicity, we assume that $i = 1$.

The verification of Equation (6.6) can be divided into cases, according to $\mathcal{S} \cap \{1,2\}$. Consider first the case where $\mathcal{S} \cap \{1,2\} = \{1,2\}$. Classify the generators of $\mathcal{P}^i \boxtimes \mathcal{N}_i$ into types, labeled XY, where $X, Y \in \{\mathbf{N}, \mathbf{S}, \mathbf{W}, \mathbf{E}\}$, where the first symbol X

denotes the generator type in $\mathcal{P}^i$ and Y denotes the generator type in $\mathcal{N}_i$, as illustrated in Figure 28.

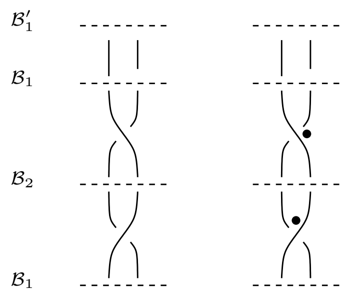


FIGURE 28. Tensoring bimodules on the left; at the right, the generator type of $\mathbf{NE} = \mathbf{N} \boxtimes \mathbf{E} \boxtimes \mathbf{1}$.

By a straightforward computation, we find the following kinds of differential: the outside actions, $C_1 \otimes U_1$, $C_2 \otimes U_2$, and additional arrows indicated on the diagram on the left of Figure 29. Canceling arrows (i.e. setting $\mathbf{N} = \mathbf{NN} + (1 \otimes R_1)\mathbf{WS} + (1 \otimes L_2)\mathbf{ES}$ and $\mathbf{S} = \mathbf{SS}$; and noting that the quotient complex is acyclic) gives the diagram on the right (in addition to the $C_i \otimes U_i$ arrows for $i = 1, 2$ and the outside arrows). This is, in fact, the canonical DD bimodule, verifying Equation (6.6).

Similarly, if $\mathcal{S} \cap \{1, 2\} = \{2\}$, we compute the bimodule to be as given in Figure 30. Again, we have suppressed the outside arrows, and additional arrows of the form $C_2 \otimes U_2$ that connect every generator to themselves. By contrast, we did include self-arrows of the form $U_1 \otimes C_1$, since not all generator types have these kinds of

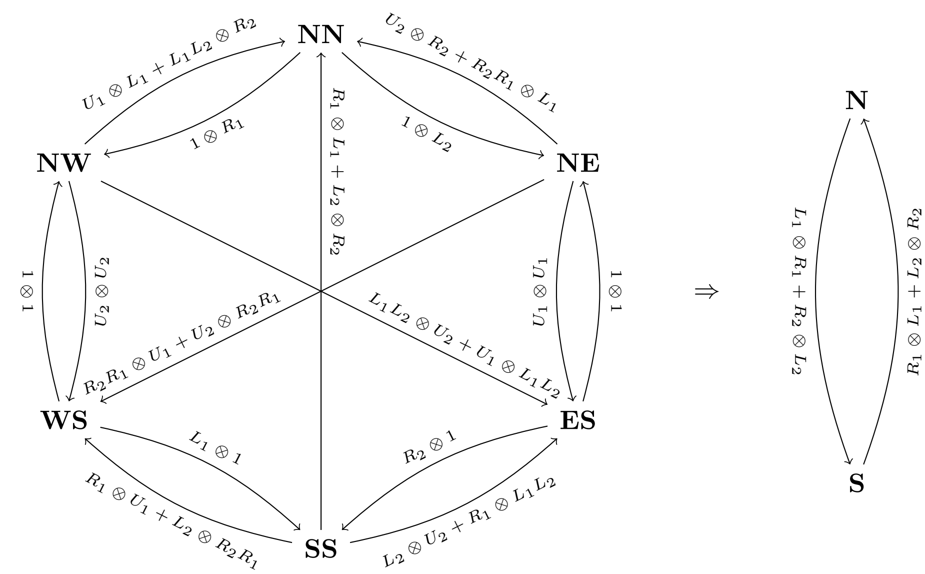


FIGURE 29. **Arrows in $\mathcal{P}^i \boxtimes \mathcal{N}_i$ when $\mathcal{S} \cap \{1, 2\} = \{1, 2\}$.**

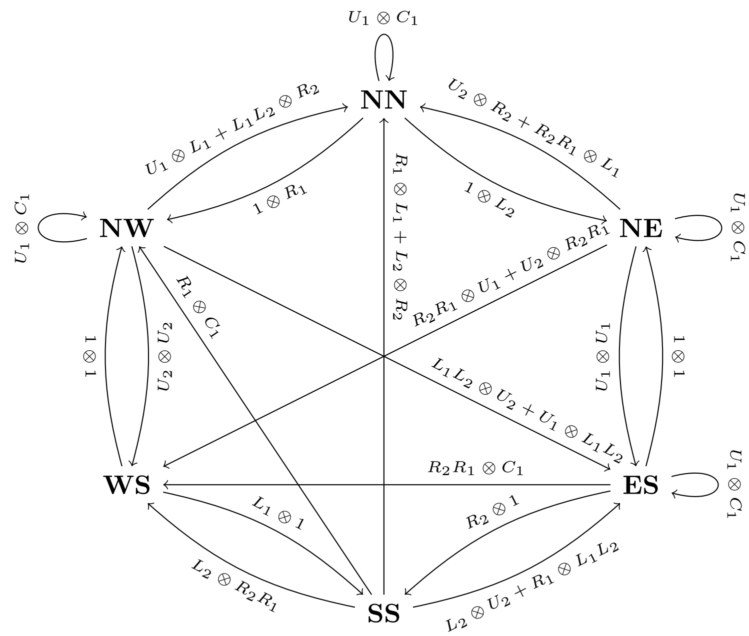


FIGURE 30. **Arrows in $\mathcal{P}^i \boxtimes \mathcal{N}_i$ when $\mathcal{S} \cap \{1,2\} = \{2\}$.**

terms. We can reduce to the previous case by changing basis; replacing **SS** by **SS**$+(R_1 \otimes C_1) \otimes$ **WS**. A similar computation works for $\mathcal{S} \cap \{1,2\} = \{1\}$, only now the reduction to the previous case involves the change of basis **SS**$+(L_2 \otimes C_2) \otimes$**ES**. Finally, when $\mathcal{S} \cap \{1,2\} = \emptyset$, we get a bimodule which can be reduced to the earlier case by the basis change **SS** $+ (R_1 \otimes C_1) \otimes$ **WS** $+ (L_2 \otimes C_2) \otimes$ **ES**.

Once again, cancelling arrows we find that this gives the identity DD bimdoule. The other two computations needed to verify Equation (6.6) work similarly.

Having verified that $\mathcal{P}^i \boxtimes \mathcal{N}^i \simeq {}^{\mathcal{B}_1}\mathrm{Id}_{\mathcal{B}_1}$, the verification that $\mathcal{N}^i \boxtimes \mathcal{P}^i \simeq {}^{\mathcal{B}_1}\mathrm{Id}_{\mathcal{B}_1}$ is a formality:

$$\begin{aligned}\mathrm{Id} = \overline{\mathrm{Id}} &\simeq \overline{{}^{\mathcal{B}_1}\mathcal{P}^i_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\mathcal{N}^i_{\mathcal{B}_1}} \simeq {}_{\mathcal{B}_1}\overline{\mathcal{N}}^{i\,\mathcal{B}_2} \boxtimes {}_{\mathcal{B}_2}\overline{\mathcal{P}}^{i\,\mathcal{B}_1} \\ &= {}^{\mathcal{B}_1^{\mathrm{op}}}\overline{\mathcal{P}}^i{}_{\mathcal{B}_2^{\mathrm{op}}} \boxtimes {}^{\mathcal{B}_2^{\mathrm{op}}}\overline{\mathcal{N}}^i{}_{\mathcal{B}_1^{\mathrm{op}}} = {}^{\mathcal{B}_1}\mathcal{N}^i_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\mathcal{P}^i_{\mathcal{B}_2}\end{aligned}$$

□

Lemma 6.4. *Equation* (6.2) *holds.*

Proof. In view of Lemma 6.2 and the invertibility of $\mathcal{K}$, it suffices to show that

$$\mathcal{P}^j \boxtimes \mathcal{P}_i \sim \mathcal{P}^i \boxtimes \mathcal{P}_j.$$

Note that generators for both bimodules in Equation (6.2) correspond to partial Kauffman states in the partial knot diagrams contianing the two crossings. Clearly, these generators are independent of the order; to see that the bimodules are independent of the order, it suffices to construct a third, more symmetric bimodule (where we think of the two crossings as appearing in the same level), that is quasi-isomorphic to both.

Generators for this bimodule will also correspond to partial Kauffman states. There will in general be 16 generator types, labelled by pairs of letters among **N**, **S**, **W**, and **E**, corresponding to the four local choices at each of the two crossings. The computation is mostly straightforward. In the special case where the two crossings are adjacent, there are only 15 generator types: there are no generators of type **EW**. This occurs, for example , when $i = 1$ and $j = 3$; see Figure 31. In this case, the DD bimodule is as specified in Figure 32 (where, as usual, we have suppressed here the distant arrows and the additional arrows of the form $U_\tau(i)\otimes C_i$ or $C_\tau(i)\otimes U_i$). □

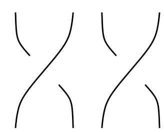


FIGURE 31. When the two crossings appearing in Equation (6.2) are adjacent, there are only 15 types of partial Kauffman states; a partial Kauffman state cannot associate **E** to the crossing on the left and **W** to the one on the right.

Lemma 6.5. *Equation* (6.3) *holds.*

Proof. We describe the case where $i = 1$. We first compute ${}^{\mathcal{B}_3}\mathcal{P}^2_{\mathcal{B}_2}\boxtimes {}^{\mathcal{B}_2,\mathcal{B}_1'}\mathcal{P}_1$. Again, generators correspond to partial Kauffman states, which we now label as an ordered pair, showing which crossing is associated to which region; see Figure 33. When $\{1,2,3\}\subset\mathcal{S}$, after cancelling arrows, we find that the bimodule is given as in Figure 34, along with the usual outside arrows and self-arrows of the form $C_i\otimes U_{\tau_2\tau_1(i)}$ or $U_i\otimes C_{\tau_2\tau_1(i)}$, depending on whether or not $i\in\mathcal{S}$. When $\{1,2,3\}\cap\mathcal{S}$ is a proper subset of $\{1,2,3\}$, we need to apply additional homotopies to obtain this bimodule, as in the proof of Lemma 6.2.

Tensoring this on the left with ${}^{\mathcal{B}_5}\mathcal{P}^i_{\mathcal{B}_3}$, and following the labelling conventions from Figure 35, we find that the bimodule is homotopic to one of the following form from Figure 36, along with the usual outside arrows and self-arrows.

Tensoring in the other order gives the same bimodule; this can be seen quickly from a symmetry on the answer. The picture after the Reidemeister move can be realized as a rotation by 180° (i.e. rotate the leftmost picture in Figure 35). There is a corresponding action on the algebra, exchanging the two tensor factors of the algebra, and further exchanging R_i and L_{4-i}. The corresponding symmetry of the bimodule can be realized by rotating the description from Figure 36 by 180°. The fact that the bimodule is fixed by the symmetry implies the claimed invariance under Reidemeister 3 moves. □

6.1. **Other symmetries in the crossing bimodule.** The canonical DD bimodules commute with the action of the braid group, in the following sense.

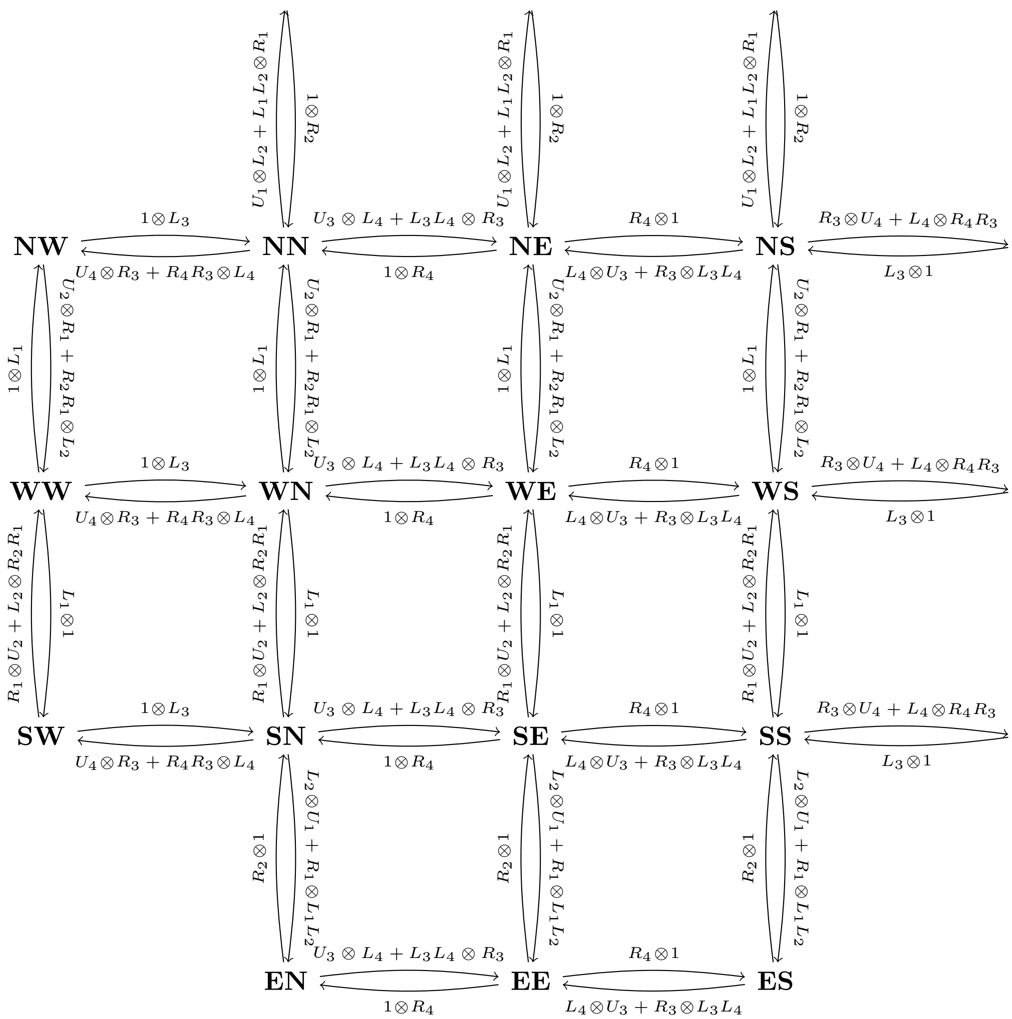


FIGURE 32. $\mathcal{P}^i \boxtimes \mathcal{P}_j$. This is drawn on the torus; e.g. the arrows that point off to the right wrap around to the left.

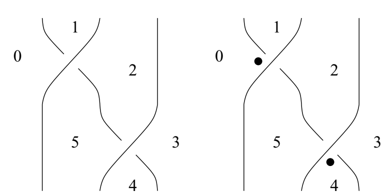


FIGURE 33. At the right, we have shown a generator of type SW; or equivalently, X_{04}.

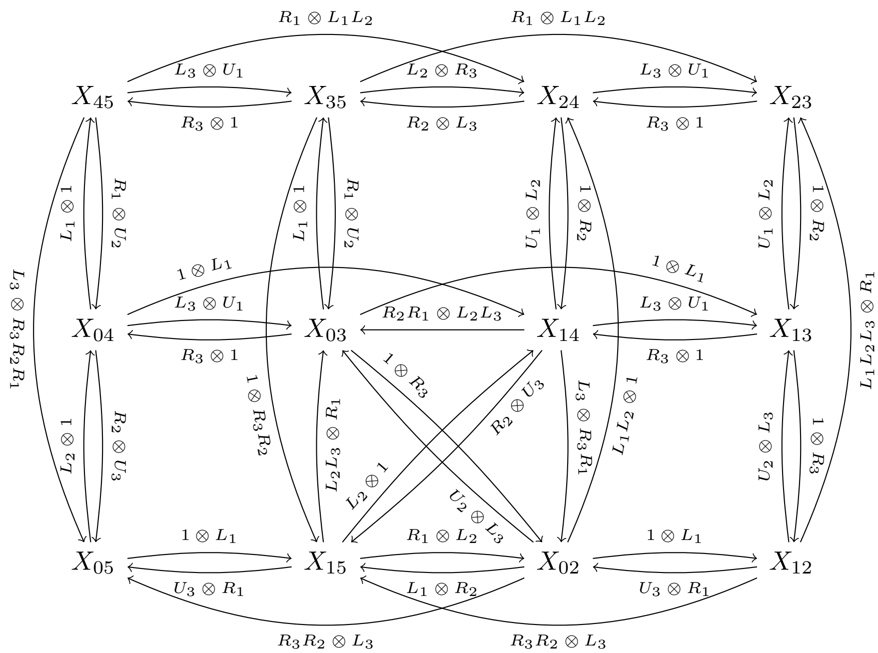


FIGURE 34. ${}^{\mathcal{B}_3}\mathcal{P}^2_{\mathcal{B}_2} \boxtimes^{\mathcal{B}_2,\mathcal{B}'_1} \mathcal{P}_1$ **when** $\{1,2,3\} \subset \mathcal{S}$

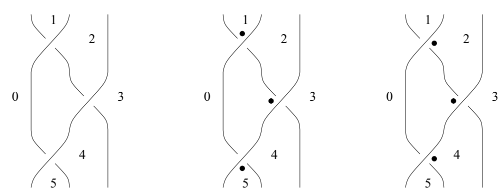


FIGURE 35. We illustrate here generators of type Y_{15} and Y_{24} respectively.

Lemma 6.6. *Fix* $0 \leq k \leq m+1$ *and an arbitrary subset* $\mathcal{S} \subset \{1,\dots,m\}$*, and let*

$$\begin{array}{ll} \mathcal{B}_1 = \mathcal{B}(m,k,\mathcal{S}), & \mathcal{B}_2 = \mathcal{B}(m,k,\tau_i(\mathcal{S})) \\ \mathcal{B}'_1 = \mathcal{B}(m,m+1-k,\{1,\dots,m\} \setminus \mathcal{S}) & \mathcal{B}'_2 = \mathcal{B}(m,m+1-k,\{1,\dots,m\} \setminus \tau_i(\mathcal{S})) \end{array}$$

There is an equivalence ${}^{\mathcal{B}_2}\mathcal{P}^i_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1,\mathcal{B}'_1}\mathcal{K} \simeq {}^{\mathcal{B}'_1}\mathcal{P}^i_{\mathcal{B}'_2} \boxtimes {}^{\mathcal{B}'_2,\mathcal{B}_2}\mathcal{K}$.

Proof. This is an immediate consequence of Lemma 6.2, and the symmetry of the $\mathcal{P}_i$ from Equation (4.3). □

7. THE DD BIMODULE OF A CRITICAL POINT

In Sections 8 and 9, we will construct DA bimodules ${}^{\mathcal{B}_2}\Omega_{\mathcal{B}'_1}$ and ${}^{\mathcal{B}_1}\mho_{\mathcal{B}'_2}$ (where the algebras will be made precise shortly) associated to a region in the knot diagram where there are no crossings and a single critical point, which can be a maximum (as in Section 8) or a minimum (as in Section 9). In the present section, we will

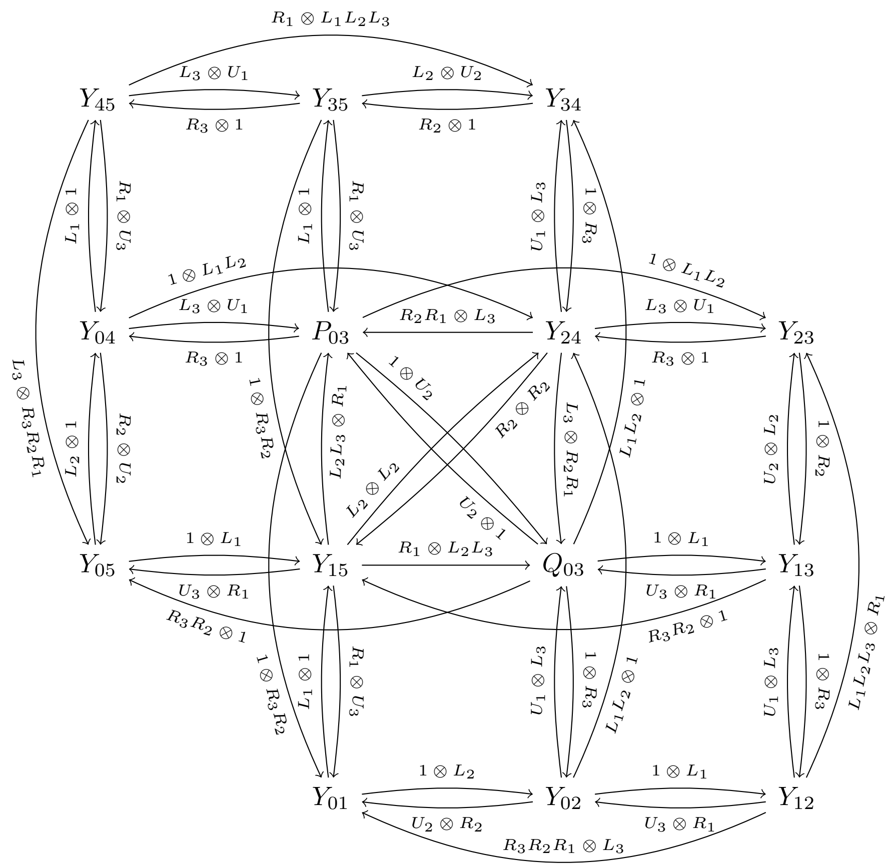


FIGURE 36. ${}^{\mathcal{B}_5}\mathcal{P}^i_{\mathcal{B}_3} \boxtimes^{\mathcal{B}_3} \mathcal{P}^2_{\mathcal{B}_2} \boxtimes^{\mathcal{B}_2,\mathcal{B}'_1} \mathcal{P}_1$

construct a type DD bimodule, called *the type DD bimodule for a critical point*, ${}^{\mathcal{B}_1,\mathcal{B}_2}\mathcal{E}$ which is related to the aforementioned type DA bimodules via equivalences

$$ {}^{\mathcal{B}_2}\Omega_{\mathcal{B}'_1} \boxtimes {}^{\mathcal{B}'_1,\mathcal{B}_1}\mathcal{K} \simeq {}^{\mathcal{B}_2,\mathcal{B}_1}\mathcal{E} \quad \text{and} \quad {}^{\mathcal{B}_2}\mho_{\mathcal{B}'_1} \boxtimes {}^{\mathcal{B}'_1,\mathcal{B}_1}\mathcal{K} \simeq {}^{\mathcal{B}_2,\mathcal{B}_1}\mathcal{E}. $$

(See Propositions 8.4 and 9.5 for precise statements.)

Fix integers c, k, and m with $1 \leq c \leq m+1$ and $0 \leq k \leq m+1$. The algebras appearing in the bimodule for a critical point are specified as follows. Let $\phi_c \colon \{1,\dots,m\} \to \{1,\dots,m+2\}$ be the function

$$ \phi_c(j) = \begin{cases} j & \text{if } j < c \\ j+2 & \text{if } j \geq c. \end{cases} \tag{7.1} $$

Let $\mathcal{B}_1 = \mathcal{B}(m,k,\mathcal{S}_1)$ and $\mathcal{B}_2 = \mathcal{B}(m+2, m+2-k, \mathcal{S}_2)$, where $\mathcal{S}_1 \subset \{1,\dots,m\}$ and $\mathcal{S}_2 \subset \{1,\dots,m+2\}$, so that $\phi_c(\mathcal{S}_1) \cap \mathcal{S}_2 = \emptyset$ and $|\mathcal{S}_1| + |\mathcal{S}_2| = m+1$ and $|\mathcal{S}_2 \cap \{c, c+1\}| = 1$; equivalently, $\mathcal{S}_2 = \phi_c(\{1,\dots,m\} \setminus \mathcal{S}_1) \cup \{c\}$ or $\mathcal{S}_2 = \phi_c(\{1,\dots,m\} \setminus \mathcal{S}_1) \cup \{c+1\}$.

In this section, we construct the type DD bimodule for a critical point, denoted $\mathcal{E}_c = {}^{\mathcal{B}_1,\mathcal{B}_2}\mathcal{E}_c$, where here c (which we will sometimes drop from the notation) indicates where the critical point occurs.

Having specified the algebras, we now describe the underlying vector space for $\mathcal{E}_c$. We call an idempotent state $\mathbf{y}$ for $\mathcal{B}_2$ an *allowed idempotent state for* $\mathcal{B}_2$ if

$$|\mathbf{y}\cap\{c-1,c,c+1\}|\leq 2 \qquad \text{and} \qquad c\in\mathbf{y}.$$

There is a map ψ from allowed idempotent states $\mathbf{y}$ for $\mathcal{B}_2$ to idempotent states for $\mathcal{B}_1$, where $\mathbf{x}=\psi(\mathbf{y})\subset\{0,\dots,m\}$ is characterized by

$$|\mathbf{y}\cap\{c-1,c,c+1\}|+|\mathbf{x}\cap\{c-1\}|=2 \qquad \text{and} \qquad \phi_c(\mathbf{x})\cap\mathbf{y}=\emptyset. \tag{7.2}$$

As a vector space, $\mathcal{E}_c$ is spanned by vectors that are in one-to-one correspondence with allowed idempotent states for $\mathcal{B}_2$. The bimodule structure, over the rings of idempotents $\mathbf{I}(\mathcal{B}_1)$ and $\mathbf{I}(\mathcal{B}_2)$, is specified as follows. If $\mathbf{P}=\mathbf{P}_\mathbf{y}$ is the generator associated to the idempotent state $\mathbf{y}$, then for idempotent states $\mathbf{x}$ and $\mathbf{z}$ for $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively,

$$(\mathbf{I}_\mathbf{x}\otimes\mathbf{I}_\mathbf{z})\cdot\mathbf{P}_\mathbf{y}=\left\{\begin{array}{ll}\mathbf{P}_\mathbf{y} & \text{if } \mathbf{y}=\mathbf{z} \text{ and } \mathbf{x}=\psi(\mathbf{y})\\ 0 & \text{otherwise.}\end{array}\right. .$$

To specify the differential, consider the element $A\in\mathcal{B}_1\otimes\mathcal{B}_2$

$$\begin{aligned} A = {} & (1\otimes L_cL_{c+1})+(1\otimes R_{c+1}R_c)+\sum_{j=1}^m R_j\otimes L_{\phi(j)}+L_j\otimes R_{\phi(j)} \\ & +\left\{\begin{array}{ll}(1\otimes C_cU_{c+1}) & \text{if } c\in\mathcal{S}_2\\ (1\otimes U_cC_{c+1}) & \text{if } c+1\in\mathcal{S}_2,\end{array}\right\}+\sum_{j=1}^m\left\{\begin{array}{ll}C_j\otimes U_{\phi(j)} & \text{if } j\in\mathcal{S}_1\\ U_j\otimes C_{\phi(j)} & \text{if } j\notin\mathcal{S}_1\end{array}\right\}. \end{aligned} \tag{7.3}$$

where we have dropped the subscript c from $\phi_c=\phi$. Let

$$\delta^1(\mathbf{P}_\mathbf{y})=(\mathbf{I}_{\psi(\mathbf{y})}\otimes\mathbf{I}_\mathbf{y})\cdot A\otimes\sum_{\mathbf{z}}\mathbf{P}_\mathbf{z},$$

where the latter sum is taken over all allowed idempotent states $\mathbf{z}$ for $\mathcal{B}_2$.

Lemma 7.1. *The space* ${}^{\mathcal{B}_1,\mathcal{B}_2}\mathcal{E}_c$ *defined above, and equipped with the map*

$$\delta^1\colon\mathcal{E}_c\to(\mathcal{B}_1\otimes\mathcal{B}_2)\otimes\mathcal{E}_c,$$

specified above, is a type DD *bimodule over* $\mathcal{B}_1$ *and* $\mathcal{B}_2$.

Proof. The proof is a straightforward adaptation of Lemma 3.10. □

It is helpful to understand $\mathcal{E}_c$ a little more explicitly. To this end, we classify the allowed idempotents for $\mathcal{B}_2$ into three types, labelled $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$:

- $\mathbf{y}$ is of type $\mathbf{X}$ if $\mathbf{y}\cap\{c-1,c,c+1\}=\{c-1,c\}$,
- $\mathbf{y}$ is of type $\mathbf{Y}$ if $\mathbf{y}\cap\{c-1,c,c+1\}=\{c,c+1\}$,
- $\mathbf{y}$ is of type $\mathbf{Z}$ if $\mathbf{y}\cap\{c-1,c,c+1\}=\{c\}$.

There is a corresponding classification of the generators $\mathbf{P}_\mathbf{y}$ into $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$, according to the type of $\mathbf{y}$; see Figure 37.

With respect to this decomposition, terms in the differential are of the following four types:

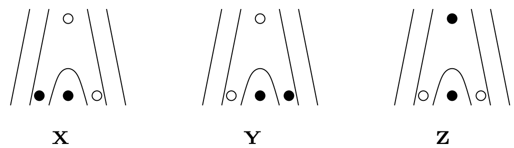
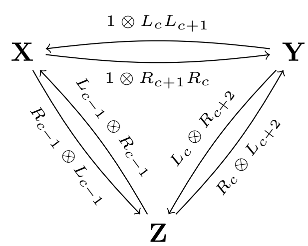


FIGURE 37. *DD* **bimodule of a critical point.** Three generator types are illustrated.

- (P-1) $R_j \otimes L_{\phi(j)}$ and $L_j \otimes R_{\phi(j)}$ for all $j \in \{1, \dots, m\} \setminus \{c-1, c\}$; these connect generators of the same type.
- (P-2) $C_j \otimes U_{\phi(j)}$ if $j \in \mathcal{S}_1$ and $U_j \otimes C_{\phi(j)}$ if $j \in \{1, \dots, m\} \setminus \mathcal{S}_1$
- (P-3) $1 \otimes U_c C_{c+1}$ if $c+1 \in \mathcal{S}_2$ or $1 \otimes C_c U_{c+1}$ if $c \in \mathcal{S}_2$.
- (P-4) Terms in the diagram below connect generators of different types.



(7.4)

With the understanding that if $c = 1$, then the terms containing L_{c-1} or R_{c-1} are missing; similarly, if $c = m+1$, the terms containing R_{c+2} and L_{c+2} are missing.

7.1. **Grading sets.** For any generator P of $\mathcal{E}_c$, if $(a_2 \otimes a_1) \otimes Q$ appears with non-zero multiplicity in $d^1(P)$, then for all $i = 1, \dots, m$, $w_i(a_1) = w_{\phi_c(i)}(a_2)$ and $w_c(a_2) = w_{c+1}(a_2)$. This is obvious from the form of the bimodule (Equation (7.3)). In the language of Section 3.9, we can express this by saying that the bimodule is supported in grading 0 in $H_1(W, \partial W)$, where W is the partial knot diagram with a single critical point in it.

7.2. **Critical points and crossings.** Later, we will study in detail how the bimodules of critical points interact with the bimodules associated to crossings. For the time being, we will content ourselves with the following special case, where the critical point and the crossing are connected to each other.

To set notation, let $\psi_c \colon \{1, \dots, m\} \to \{1, \dots, m+2\}$ be the function

$$\psi_c(j) = \begin{cases} j & \text{if } j < c \\ c+1 & \text{if } j = c \\ j+2 & \text{if } j > c; \end{cases}$$

i.e. $\psi_c = \tau_{c+1} \circ \phi_c = \tau_c \circ \phi_{c+1}$.

Lemma 7.2. *Fix integers* $0 \le k \le m+1$, *and an arbitrary subset* $\mathcal{S}_1 \subset \{1, \dots, m\}$, *and let* $\mathcal{B}_1 = \mathcal{B}(m, k, \mathcal{S}_1)$, $\mathcal{B}_2 = \mathcal{B}(m+2, m+2-k, \mathcal{S}_2)$ $\mathcal{B}_3 = \mathcal{B}(m+2, m+2-k, \mathcal{S}_3)$, $\mathcal{B}_4 = \mathcal{B}(m+2, m+2-k, \mathcal{S}_4)$ *where*

$$\mathcal{S}_2 = \phi_{c+1}(\{1, \dots, m\}) \setminus \mathcal{S}_1) \cup \{c+1\} \qquad \textit{or} \qquad \mathcal{S}_2 = \phi_{c+1}(\{1, \dots, m\} \setminus \mathcal{S}_1) \cup \{c+2\},$$

$\mathcal{S}_3 = \tau_c(\mathcal{S}_2)$ and $\mathcal{S}_4 = \tau_{c+1}(\mathcal{S}_3)$. There is homotopy equivalence of graded bimodules:

$$ {}^{\mathcal{B}_3}\mathcal{P}^c_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2,\mathcal{B}_1}\mathcal{E}_{c+1} \simeq {}^{\mathcal{B}_3}\mathcal{N}^{c+1}_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4,\mathcal{B}_1}\mathcal{E}_c \tag{7.5}$$

Remark 7.3. *The gradings on the crossing bimodules were chosen so that Equation* (7.5) *holds as a bigraded module map.*

The above lemma is a straightforward computation, using the following *trident bimodule* ${}^{\mathcal{B}_1,\mathcal{B}_3}\mathcal{T}$.

Generators correspond to pairs of idempotents $\mathbf{x}$ and $\mathbf{y}$ for $\mathcal{B}_1$ and $\mathcal{B}_3$ with the following properties:

- Letting, then $\psi_c(\mathbf{x}) \cap \mathbf{y} = \emptyset$.
- $|\mathbf{y}| = |\mathbf{x}| + 1$
- $|\mathbf{y} \cap \{c, c+1\} \geq 1$
- if $|\mathbf{y} \cap \{c, c+1\}| = 2$ then either $c-1 \notin \mathbf{x}$ and $c-1 \notin \mathbf{y}$ or $c \notin \mathbf{x}$ and $c+2 \notin \mathbf{y}$

We separate these pairs into four types:

- $(\mathbf{x}, \mathbf{y})$ is of Type $\mathbf{P}$ if $c-1 \notin \mathbf{x}$ and $c-1 \notin \mathbf{y}$ (so $\{c, c+1\} \subset \mathbf{y}$)
- $(\mathbf{x}, \mathbf{y})$ is of Type $\mathbf{Q}$ if $c \notin \mathbf{x}$ and $c+2 \notin \mathbf{y}$ (so $\{c, c+1\} \subset \mathbf{y}$)
- $(\mathbf{x}, \mathbf{y})$ is of Type $\mathbf{X}$ if $c \notin \mathbf{y}$
- $(\mathbf{x}, \mathbf{y})$ is of Type $\mathbf{Y}$ if $c+1 \notin \mathbf{y}$

See Figure 38 for a picture. (Note that these generator types correspond to the partial Kauffman states in the sense of Definition 5.1 of the diagram containing the maximum and the crossing, with the understanding that the incoming idempotent is complementary to idempotent state coming from the top of the diagram.)

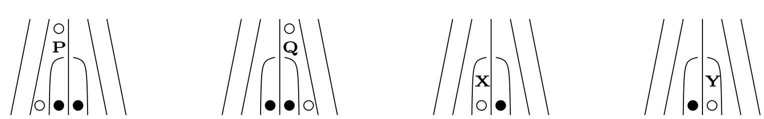
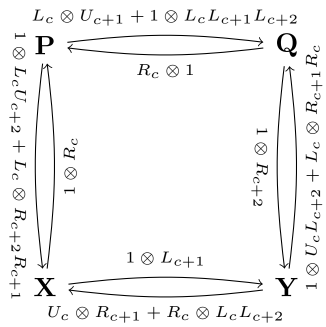


FIGURE 38. **The four generator types of the trident bimodule.**

Let $\mathbf{T}_{\mathbf{x},\mathbf{y}}$ be the corresponding generator of $\mathcal{T}$. As a left module over $\mathbf{I}(\mathcal{B}_1) \otimes \mathbf{I}(\mathcal{B}_3)$, the action is specified by $(\mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}) \cdot \mathbf{T}_{\mathbf{x},\mathbf{y}} = \mathbf{T}_{\mathbf{x},\mathbf{y}}$. The differential has the following types of terms:

(1) $R_j \otimes L_j$ and $L_j \otimes R_j$ for all $j \in \{1, \dots, m\} \setminus \{c\}$; these connect generators of the same type.
(2) $C_j \otimes U_{\psi_c(j)}$ if $j \in \mathcal{S}_1 \setminus \{c\}$ and $U_j \otimes C_{\psi_c(j)}$ if $j \in \{1, \dots, m\} \setminus (\mathcal{S}_1 \cup \{c\})$; these connect generators of the same type.
(3) $C_c \otimes U_{c+1}$ if $c \in \mathcal{S}_1$ and $U_c \otimes C_{c+1}$ if $c \notin \mathcal{S}_1$

(4) Terms in the diagram below connect generators of different types:



(7.6)

Proof of Lemma 7.2. A straightforward computation identifies both sides with the trident bimodule specified above, after a possible homotopy (as in the proof of Lemma 6.2). The fact that the map respects gradings follows quickly from the grading conventions; see Equations (4.5) and(4.7). □

8. The DA bimodule associated to a maximum

We will describe now the type DA bimodule ${}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}_1'}$ of a region in the knot diagram where there are no crossings and a single local maximum, which connects the c^{th} and the $(c{+}1)^{st}$ outgoing strand. The algebras are specified as follows. Fix integers c, k, and m with $1 \leq c \leq m+1$ and $0 \leq k \leq m+1$., and let

$$\mathcal{B}_1' = \mathcal{B}(m,k,\mathcal{S}_1) \qquad \text{and} \qquad \mathcal{B}_2 = \mathcal{B}(m+2,k+1,\mathcal{S}_2),$$

where $\mathcal{S}_1' \subset \{1,\dots,m\}$ is arbitrary, and $\mathcal{S}_2 \subset \{1,\dots,m+2\}$ is given by

$$\mathcal{S}_2 = \phi_c(\mathcal{S}_1) \cup \{c\} \text{ or } \mathcal{S}_2 = \phi_c(\mathcal{S}_1) \cup \{c+1\}.$$

The two cases as corresponding to the two possible orientations of the strand with the maximum; see Figure 39.

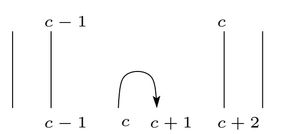


Figure 39. **Picture of maximum.** Here the maximum is oriented from left to right, so $\mathcal{S}_2 \cap \{c, c+1\} = c$.

As in Section 7, an *allowed idempotent state for* $\mathcal{B}_2$ is an idempotent state $\mathbf{y}$ for $\mathcal{B}_2$ with $c \in \mathbf{y}$ and $|\mathbf{y} \cap \{c-1, c+1\}| \leq 1$. There is a map ψ' from allowed idempotent states for $\mathcal{B}_2$ to idempotent states for $\mathcal{B}_1'$, given by

$$\psi'(\mathbf{x}) = \begin{cases} \phi^{-1}(\mathbf{y}) & \text{if } c+1 \notin \mathbf{y} \\ \phi^{-1}(\mathbf{y}) \cup \{c-1\} & \text{if } c+1 \in \mathbf{y} \end{cases}$$

Observe that $\psi'(\mathbf{y}) = \{0,\dots,m\} \setminus \psi(\mathbf{y})$, where ψ is the map from Section 7 (c.f. Equation (7.2)).

A basis for the underlying vector space of ${}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}_1'}$ is specified by the allowed idempotent states for $\mathcal{B}_2$. The bimodule structure, over the rings of idempotents $\mathbf{I}(\mathcal{B}_1')$ and $\mathbf{I}(\mathcal{B}_2)$, is specified as follows. If $\mathbf{Q}_{\mathbf{y}}$ is the generator associated to the allowed idempotent state $\mathbf{y}$, then for idempotent states $\mathbf{x}$ and $\mathbf{z}$ for $\mathcal{B}_1'$ and $\mathcal{B}_2$ respectively,

$$\mathbf{I}_{\mathbf{z}} \cdot \mathbf{Q}_{\mathbf{y}} \cdot \mathbf{I}_{\mathbf{x}} = \begin{cases} \mathbf{Q}_{\mathbf{y}} & \text{if } \mathbf{y} = \mathbf{z} \text{ and } \mathbf{x} = \psi'(\mathbf{y}) \\ 0 & \text{otherwise.} \end{cases}$$

The map $\delta^1_1 \colon {}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}_1'} \to \mathcal{B}_2 \otimes_{\mathbf{I}(\mathcal{B}_2)} {}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}_1'}$ is given by

$$\mathbf{Q}_{\mathbf{y}} \mapsto \mathbf{I}_{\mathbf{y}} \cdot \left(R_{c+1}R_c + L_c L_{c+1} + \left\{ \begin{array}{ll} U_c C_{c+1} & \text{if } c+1 \in \mathcal{S}_2 \\ C_c U_{c+1} & \text{if } c \in \mathcal{S}_2, \end{array} \right\} \right) \otimes \sum_{\mathbf{z}} \mathbf{Q}_{\mathbf{z}}.$$

where the sum is taken over all allowed idempotents $\mathbf{z}$ for $\mathcal{B}_2$.

We split the bimodule ${}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}_1'} \cong \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z}$ according to the types of the corresponding idempotents as defined in Section 7; i.e.

$$\mathbf{X} = \bigoplus_{\{\mathbf{y} \big| \mathbf{y} \cap \{c-1,c,c+1\} = \{c-1,c\}\}} \mathbf{Q}_{\mathbf{y}} \qquad \mathbf{Y} = \bigoplus_{\{\mathbf{y} \big| \mathbf{y} \cap \{c-1,c,c+1\} = \{c,c+1\}\}} \mathbf{Q}_{\mathbf{y}}$$

$$\mathbf{Z} = \bigoplus_{\{\mathbf{y} \big| \mathbf{y} \cap \{c-1,c,c+1\} = \{c\}\}} \mathbf{Q}_{\mathbf{y}}.$$

With respect to this splitting, δ^1_1 can be expressed as follows. If $c \in \mathcal{S}_2$, then

$$\begin{aligned} \delta^1_1(\mathbf{X}) &= C_c U_{c+1} \otimes \mathbf{X} + R_{c+1}R_c \otimes \mathbf{Y} \\ \delta^1_1(\mathbf{Y}) &= C_c U_{c+1} \otimes \mathbf{Y} + L_c L_{c+1} \otimes \mathbf{X} \\ \delta^1_1(\mathbf{Z}) &= C_c U_{c+1} \otimes \mathbf{Z}; \end{aligned}$$

whereas if $c+1 \in \mathcal{S}_2$,

$$\begin{aligned} \delta^1_1(\mathbf{X}) &= U_c C_{c+1} \otimes \mathbf{X} + R_{c+1}R_c \otimes \mathbf{Y} \\ \delta^1_1(\mathbf{Y}) &= U_c C_{c+1} \otimes \mathbf{Y} + L_c L_{c+1} \otimes \mathbf{X} \\ \delta^1_1(\mathbf{Z}) &= U_c C_{c+1} \otimes \mathbf{Z}; \end{aligned}$$

To define δ^1_2, it is helpful to have the following:

Lemma 8.1. *Let $\mathbf{x}'$ and $\mathbf{y}'$ be two idempotents for $\mathcal{B}_1'$ that are close enough, and $\mathbf{x}$ be an allowed idempotent state for $\mathcal{B}_2$ with $\psi'(\mathbf{x}) = \mathbf{x}'$. Then, there is a uniquely associated allowed idempotent state $\mathbf{y}$ with $\psi'(\mathbf{y}) = \mathbf{y}'$ so that there is a surjective map $\Phi_{\mathbf{x}} \colon \mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}_1' \cdot \mathbf{I}_{\mathbf{y}'} \to \mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{y}}$ that maps the portion of $\mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}_1' \cdot \mathbf{I}_{\mathbf{y}'}$ with weights $(w_1', \dots, w_m')$ surjectively onto the portion of $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{y}}$ with $w_{\phi_c(i)} = w_i'$ and $w_c = w_{c+1} = 0$, and that satisfies the relations $\Phi_{\mathbf{x}}(U_i \cdot a) = U_{\phi_c(i)} \cdot \Phi_{\mathbf{x}}(a)$ and $\Phi_{\mathbf{x}}(C_j \cdot a) = C_{\phi_c(j)} \cdot \Phi_{\mathbf{x}}(a)$ for any $i \in 1, \dots, m$, $j \in \mathcal{S}_1'$, and $a \in \mathbf{I}_{\mathbf{x}'} \cdot \mathcal{B}_1' \cdot \mathbf{I}_{\mathbf{y}'}$.*

Proof. Recall the weight $v^{\mathbf{x}}$ of idempotents, as defined in Equation (3.3). Since $\mathbf{x}'$ and $\mathbf{y}'$ are close enough, then there is some integer $j \in \mathbb{Z}$ so that exactly one of the following holds:

(1) $(v^{\mathbf{x}'}_{c-1}, v^{\mathbf{x}'}_c) = (j,j)$ and $(v^{\mathbf{y}'}_{c-1}, v^{\mathbf{y}'}_c) = (j, j-1)$
(2) $(v^{\mathbf{x}'}_{c-1}, v^{\mathbf{x}'}_c) = (j,j)$ and $(v^{\mathbf{y}'}_{c-1}, v^{\mathbf{y}'}_c) = (j,j)$

(3) $(v^{\mathbf{x}'}_{c-1}, v^{\mathbf{x}'}_c) = (j,j)$ and $(v^{\mathbf{y}'}_{c-1}, v^{\mathbf{y}'}_c) = (j+1,j)$
(4) $(v^{\mathbf{x}'}_{c-1}, v^{\mathbf{x}'}_c) = (j,j-1)$ and $(v^{\mathbf{y}'}_{c-1}, v^{\mathbf{y}'}_c) = (j-1,j-2)$
(5) $(v^{\mathbf{x}'}_{c-1}, v^{\mathbf{x}'}_c) = (j,j-1)$ and $(v^{\mathbf{y}'}_{c-1}, v^{\mathbf{y}'}_c) = (j-1,j-1)$
(6) $(v^{\mathbf{x}'}_{c-1}, v^{\mathbf{x}'}_c) = (j,j-1)$ and $(v^{\mathbf{y}'}_{c-1}, v^{\mathbf{y}'}_c) = (j,j-1)$
(7) $(v^{\mathbf{x}'}_{c-1}, v^{\mathbf{x}'}_c) = (j,j-1)$ and $(v^{\mathbf{y}'}_{c-1}, v^{\mathbf{y}'}_c) = (j,j)$
(8) $(v^{\mathbf{x}'}_{c-1}, v^{\mathbf{x}'}_c) = (j,j-1)$ and $(v^{\mathbf{y}'}_{c-1}, v^{\mathbf{y}'}_c) = (j+1,j)$

In each of these cases, we claim that there is a unique allowed idempotent state $\mathbf{y}$ for $\mathcal{B}_2$ with $\psi'(\mathbf{y}) = \mathbf{y}'$ so that

$$(v^{\mathbf{y}}_{c-1}, v^{\mathbf{y}}_c) = (v^{\mathbf{x}}_{c-1}, v^{\mathbf{x}}_c). \tag{8.1}$$

To specify $\mathbf{y}$, it suffices to specify its type, as we do in the eight cases listed above:

(1) $\mathbf{x}$ is of type $\mathbf{Z}$; $\mathbf{y}$ is of type $\mathbf{Y}$.
(2) $\mathbf{x}$ is of type $\mathbf{Z}$ and $\mathbf{y}$ is of type $\mathbf{Z}$
(3) $\mathbf{x}$ is of type $\mathbf{Z}$; $\mathbf{y}$ is of type $\mathbf{X}$.
(4) $\mathbf{x}$ is of type $\mathbf{X}$ and $\mathbf{y}$ is of type $\mathbf{Y}$.
(5) $\mathbf{x}$ is of type $\mathbf{X}$ and $\mathbf{y}$ is of type $\mathbf{Z}$
(6) $\mathbf{x}$ and $\mathbf{y}$ are both of type $\mathbf{X}$ or both of type $\mathbf{Y}$
(7) $\mathbf{x}$ is of type $\mathbf{Y}$ and $\mathbf{y}$ is of type $\mathbf{Z}$
(8) $\mathbf{x}$ is of type $\mathbf{Y}$ and $\mathbf{y}$ is of type $\mathbf{X}$.

Recall that there are graded identifications

$$\phi^{\mathbf{x}',\mathbf{y}'}\colon \mathbb{F}[U_1,\dots,U_m] \to \mathbf{I}_{\mathbf{x}'}\cdot \mathcal{B}_0(m,k)\cdot \mathbf{I}_{\mathbf{y}'}$$
$$\phi^{\mathbf{x},\mathbf{y}}\colon \mathbb{F}[U_1,\dots,U_{m+2}] \to \mathbf{I}_{\mathbf{x}}\cdot \mathcal{B}_0(m+2,k+1)\cdot \mathbf{I}_{\mathbf{y}}.$$

The ring map

$$\varphi\colon \mathbb{F}[U_1,\dots,U_m] \to \mathbb{F}[U_1,\dots,U_{m+2}]$$

with $\varphi(1)=1$ and $\varphi(U_i) = U_{\phi(i)}$ induces a map

$$\mathbf{I}_{\mathbf{x}'}\cdot \mathcal{B}_0(m,k)\cdot \mathbf{I}_{\mathbf{y}'} \to \mathbf{I}_{\mathbf{x}}\cdot \mathcal{B}_0(m+2,k+1)\cdot \mathbf{I}_{\mathbf{y}}$$

that maps the portion of $\mathbf{I}_{\mathbf{x}'}\cdot \mathcal{B}_0(m,k)\cdot \mathbf{I}_{\mathbf{y}'}$ with weights fixed at $(w'_1,\dots,w'_m)$ onto the portion of $\mathbf{I}_{\mathbf{x}}\cdot \mathcal{B}_0(m+2,k+1)\cdot \mathbf{I}_{\mathbf{y}}$ with weights fixed at $w_i = w'_{\phi(i)}$ and $w_c = w_{c+1} = 0$.

In the eight above cases, we check that $\varphi(\mathcal{I}(\mathbf{x}',\mathbf{y}'))$ is mapped into $\mathcal{I}(\mathbf{x},\mathbf{y})$. In all cases other than Case 6, there are no generating intervals for $(\mathbf{x}',\mathbf{y}')$ that contain $c-1$ in their interior. Moreover, the images of these generating intervals under ϕ_c are generating intervals for $(\mathbf{x},\mathbf{y})$. In Case 6, there is a generating interval $[p,q]$ with $p < c-1 < q$. If, furthermore, $\mathbf{x}$ and $\mathbf{y}$ are of type $\mathbf{X}$, then $[c+2,\dots,q+2]$ is a generating interval for $(\mathbf{x},\mathbf{y})$, so $\mathcal{I}(\mathbf{x}',\mathbf{y}')$ is still mapped into $\mathcal{I}(\mathbf{x},\mathbf{y})$. Similarly, if $\mathbf{x}$ and $\mathbf{y}$ are of type $\mathbf{Y}$, $[p,\dots,c-1]$ is a generating interval for $(\mathbf{x},\mathbf{y})$, so once again $\mathcal{I}(\mathbf{x}',\mathbf{y}')$ is mapped into $\mathcal{I}(\mathbf{x},\mathbf{y})$.

It follows that φ induces the map $\Phi_{\mathbf{x}}$ on $\mathcal{B}(m,k)\subset \mathcal{B}'_1$, which we can extend so that $\Phi_x(a\cdot C_j) = C_{\phi(j)}\cdot \Phi_{\mathbf{x}}(a)$ for all $j\in \mathcal{S}'_1$ to get the map required by the lemma. ∎

Lemma 8.2. *Suppose that $\mathbf{x}'_1$, $\mathbf{x}'_2$, and $\mathbf{x}'_3$ are three idempotent states for $\mathcal{B}'_1$, so that $\mathbf{x}'_i$ is close enough to $\mathbf{x}'_{i+1}$ for $i=1,2$; and choose any $\mathbf{x}_1$ so that $\psi(\mathbf{x}_1) = \mathbf{x}'_1$.*

Let $\mathbf{x}_2$ *be the idempotent state associated to* $(\mathbf{x}_1, \mathbf{x}'_1, \mathbf{x}'_2)$ *as in Lemma 8.1, and then* $\mathbf{x}_3$ *be associated to* $(\mathbf{x}_2, \mathbf{x}'_2, \mathbf{x}'_3)$ *by that lemma. Then, there is a commutative diagram:*

$$\begin{array}{ccc}
(\mathbf{I}_{\mathbf{x}'_1}\cdot\mathcal{B}'_1\cdot\mathbf{I}_{\mathbf{x}'_2})\otimes(\mathbf{I}_{\mathbf{x}'_2}\cdot\mathcal{B}'_1\cdot\mathbf{I}_{\mathbf{x}'_3}) & \xrightarrow{\mu'_2} & \mathbf{I}_{\mathbf{x}'_1}\cdot\mathcal{B}'_1\cdot\mathbf{I}_{\mathbf{x}'_3} \\
\Big\downarrow{\scriptstyle \Phi_{\mathbf{x}_1}\otimes\Phi_{\mathbf{x}_2}} & & \Big\downarrow{\scriptstyle \Phi_{\mathbf{x}_1}} \\
(\mathbf{I}_{\mathbf{x}_1}\cdot\mathcal{B}_2\cdot\mathbf{I}_{\mathbf{x}_2})\otimes(\mathbf{I}_{\mathbf{x}_2}\cdot\mathcal{B}_2\cdot\mathbf{I}_{\mathbf{x}_3}) & \xrightarrow{\mu_2} & \mathbf{I}_{\mathbf{x}_1}\cdot\mathcal{B}_2\cdot\mathbf{I}_{\mathbf{x}_3},
\end{array}$$

where μ'_2 *and* μ_2 *are multiplications on* $\mathcal{B}'_1$ *and* $\mathcal{B}_2$ *respectively.*

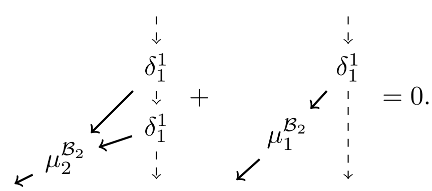
Proof. Commutativity of this diagram is an immediate consequence of the fact that Φ preserves the weights, which in turn determine the multiplication on the algebras. □

If $a = \mathbf{I}_{\mathbf{x}'}\cdot a\cdot\mathbf{I}_{\mathbf{y}'} \in \mathcal{B}(m,k)\subset\mathcal{B}'_1$ is a non-zero algebra element, and $\mathbf{x}$ is any allowed idempotent with $\psi'(\mathbf{x}) = \mathbf{x}'$, let $\delta^1_2(\mathbf{Q}_{\mathbf{x}}, a) = \Phi_{\mathbf{x}}(a)\otimes\mathbf{Q}_{\mathbf{y}}$, where $\mathbf{y}$ is the idempotent from Lemma 8.1 associated to $(\mathbf{x},\mathbf{x}',\mathbf{y}')$; i.e. $\Phi_{\mathbf{x}}(a)\cdot\mathbf{I}_{\mathbf{y}} = \Phi_{\mathbf{x}}(a)$.

Theorem 8.3. *These actions give* ${}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}'_1}$ *the structure of a DA bimodule that is adapted to the corresponding partial knot diagram (containing a single maximum).*

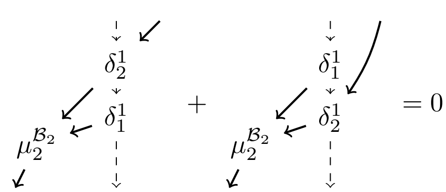
Proof. Consider the case where $c+1\in\mathcal{S}_2$; the case where $c\in\mathcal{S}_2$ works similarly.

The $\mathcal{A}_\infty$ relation with no incoming algebra elements has the form

(8.2)


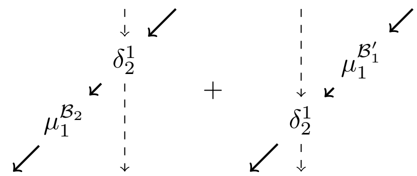
The terms that change type from $\mathbf{X}$ to $\mathbf{Y}$ and back to $\mathbf{X}$ (contributing on the left to Equation (8.2)) contribute $(R_{c+1}\cdot R_c\cdot L_c\cdot L_{c+1})\otimes\mathbf{X} = U_c\cdot U_{c+1}\otimes\mathbf{X}$; which cancels with $d(C_{c+1}\cdot U_c)\otimes\mathbf{X}$ (contributing on the right to Equation (8.2)). A similar argument works for generators of type $\mathbf{Y}$. Since $U_cU_{c+1}\otimes\mathbf{Z} = 0$, Equation (8.2) now follows.

The $\mathcal{A}_\infty$ relation with one algebra input follows from the equations

(8.3)


and

(8.4)



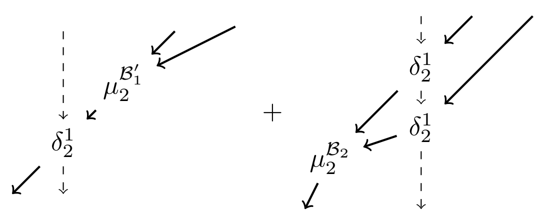
$= 0.$

We verify Equation (8.3) for incoming algebra elements $a = \mathbf{I}_{\mathbf{x}'} \cdot a \cdot \mathbf{I}_{\mathbf{y}'}$. For the terms in δ^1_1 that preserve type, $U_c C_{c+1}$, both have the same contribution: multiplication by $U_c \cdot C_{c+1}$ induces an endomorphism of $\mathbf{I}_{\mathbf{x}} \cdot \mathcal{B}_2 \cdot \mathbf{I}_{\mathbf{y}}$ that maps the portion with $w_c = w_{c+1} = 0$ injectively into $w_c = w_{c+1} = 1$ (since the generating interval for $(\mathbf{x}, \mathbf{y})$ containing c or $c+1$ contains both c and $c+1$).

Consider next the terms ($L_c L_{c+1}$ and $R_{c+1} R_c$) where δ^1_1 changes the type of the idempotent. When δ^1_2 changes the type of the idempotent, as well, both terms in Equation (8.3) vanish, since in that case the outgoing algebra element moves too far. Finally, consider the the case where δ^1_2 preserves the idempotent, and suppose the incoming generator is of the form $Q_{\mathbf{x}_1}$ where $\mathbf{x}_1$ is of type $\mathbf{X}$ and $\psi'(\mathbf{x}_1) = \mathbf{x}$. Let $\mathbf{x}_2$ be the allowed algebra element of type $\mathbf{Y}$ with $\psi'(\mathbf{x}_2) = \mathbf{y}$, so that $\mathbf{I}_{\mathbf{x}_1} \cdot L_c L_{c+1} = L_c L_{c+1} \cdot \mathbf{I}_{\mathbf{x}_2}$. The cancellation of the corresponding terms in Equation (8.3) now follows from the easily verified identity $\Phi_{\mathbf{x}_1}(a) \cdot L_c L_{c+1} = L_c L_{c+1} \cdot \Phi_{\mathbf{x}_2}(a)$. The case where the incoming generator is of type $\mathbf{Y}$ follows from the similar equation: $\Phi_{\mathbf{x}_2}(a) \cdot R_{c+1} R_c = R_{c+1} R_c \cdot \Phi_{\mathbf{x}_1}(a)$.

Equation (8.4) follows from the fact that $C_{\phi_c(j)} \cdot \Phi_{\mathbf{x}} = \Phi_{\mathbf{x}} \cdot C_j$ for all $j \in \mathcal{S}_1$.

The relation

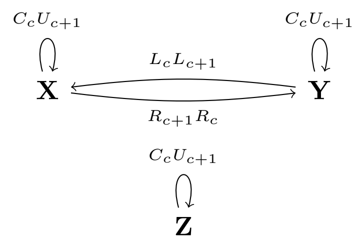


$= 0.$

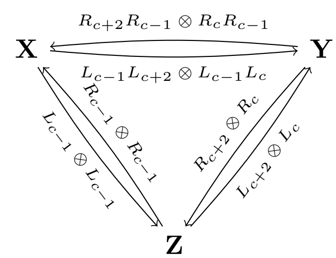
is equivalent to Lemma 8.2. Since $\delta^1_\ell = 0$ for $\ell > 2$, the $\mathcal{A}_\infty$ relations now follow.

To verify the grading, note that all the algebra output in the bimodule satisfy $w_c(b) = w_{c+1}(b)$. Combined with the grading properties from Lemma 8.1, it follows that Ω^c is graded by $H^1(W, \partial W)$, where W is specified by the partial knot diagram, and the module is thought of as supported in grading 0. To verify the Maslov grading, note that all the algebra outputs appearing in δ^1_1 have Maslov grading -1. We can think of Ω^c as supported in Maslov grading 0. It follows readily that Ω^c is adapted to W, as in Definition 3.19. ☐

It is convenient to summarize this as follows: when $c+1 \in \mathcal{S}_2$, the δ^1_1 actions are specified by the following diagram:



and the δ^1_2 actions are specified by the diagram



once we include outside actions $L_i \otimes L_{\phi_c(i)}$, $R_i \otimes R_{\phi(i)}$, $U_i \otimes U_{\phi(i)}$ and $C_j \otimes C_{\phi_c(j)}$ for $i \in \{1,\dots,m\}$ and $j \in \mathcal{S}_1$, with the further understanding that δ^1_2 is extended to be multiplicative in the incoming algebra elements.

Proposition 8.4. *Ω^c is dual to the module $\mathcal{E}_c$ from Section 7, in the following sense. Let $\mathcal{B}_1 = \mathcal{B}(m,k,\mathcal{S}_1)$, where $\mathcal{S}_1 \subset \{1,\dots,m\}$; $\mathcal{B}'_1 = \mathcal{B}(m,m+1-k,\mathcal{S}'_1)$, where $\mathcal{S}'_1 = \{1,\dots,m\}\setminus\mathcal{S}_1$; and $\mathcal{B}_2 = \mathcal{B}(m,m+1-k,\mathcal{S}_2)$, where $\mathcal{S}_2 = \mathcal{S}'_1 \cup \{c\}$ or $\mathcal{S}'_1 \cup \{c+1\}$. Then, ${}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}'_1} \boxtimes {}^{\mathcal{B}'_1,\mathcal{B}_1}\mathcal{K} \simeq {}^{\mathcal{B}_2,\mathcal{B}_1}\mathcal{E}_c$.*

Proof. This is straightforward to check using the definitions. □

8.1. **A special case.** When there is a single maximum, and no other strands, we define a type D structure over the algebra $\mathcal{B}_2 = \mathcal{B}(2,1,\{1\})$ or $\mathcal{B}(2,1,\{2\})$, ${}^{\mathcal{B}_2}\Omega$. This type D structure has one generator $\mathbf{Z}$ with $\mathbf{I}_{\{1\}}\cdot\mathbf{Z} = \mathbf{Z}$, and $\delta^1(\mathbf{Z}) = C_1U_2 \otimes \mathbf{Z}$ or $\delta^1(\mathbf{Z}) = U_1C_2 \otimes \mathbf{Z}$, according to how the strand is oriented. Obviously, this can be thought of as a degenerate case of the earlier construction, where the incoming algebra $\mathcal{B}'_1 = \mathcal{B}(0,0,\emptyset) \cong \mathbb{F}$.

8.2. **Partial Kauffman states and grading sets.** Formally, the generators can be thought of as corresponding to partial Kauffman states. There are no crossings, and it has the following types of regions:

- one of these regions does not meet the top slice, and so it must be "unoccupied", and so its only intersection with the bottom slice must be occupied,
- one of these regions meets the top slice in one interval and the bottom in two: thinking of this region as "occupied" gives the generator of type $\mathbf{Z}$; thinking of it as unoccupied gives the other two generator types $\mathbf{X}$ and $\mathbf{Y}$,
- all other regions meet both the top and the bottom slice in one interval apiece; they can be either occupied or unoccupied.

The bimodules can be graded by $\frac{1}{2}\mathbb{Z}^m$ exactly as in the case of Section 7. The fact that δ^1_2 respects this grading is contained in Lemma 8.1; the fact that δ^1_1 respects it is clear. As in Section 7.1, the grading set of can be thought of as $\frac{1}{2}\mathbb{Z}$-valued functions on the arcs in the partial knot diagram. This grading is consistent with the grading of $\mathcal{K}$ by $\frac{1}{2}\mathbb{Z}^m$, combined with the induced grading on the tensor product, Proposition 8.4.

9. The minimum

Fix integers $0 \leq k \leq m+1$ and some $1 \leq c \leq m+1$, and let

$$\mathcal{B}_1 = \mathcal{B}(m+2, k+1, \mathcal{S}_1) \qquad \text{and} \qquad \mathcal{B}_2 = \mathcal{B}(m, k, \mathcal{S}_2) \tag{9.1}$$

and where $\mathcal{S}_2 \subset \{1, \dots, m\}$ is arbitrary and $\mathcal{S}_1 = \phi_c(\mathcal{S}_2) \cup \{c\}$ or $\mathcal{S}_1 = \phi_c(\mathcal{S}_2) \cup \{c+1\}$, where ϕ_c is the function from Equation (7.1). We will describe a bimodule ${}^{\mathcal{B}_2}\mho^c_{\mathcal{B}_1}$ that will correspond to introducing a new minimum (or cup) in the diagram that connects the incoming strands c and $c+1$. (We will be primarily interested in the case where $m = 2n$, $k = n = |\mathcal{S}|$.) Note that we follow the convention that $\mathcal{B}_1$ is the incoming algebra and $\mathcal{B}_2$ is the outgoing algebra; thus the notation for $\mathcal{B}_1$ and $\mathcal{B}_2$ is opposite to the one used in Section 8.

9.1. Description of the bimodule when $c = 1$.

We start by describing ${}^{\mathcal{B}_2}\mho^c_{\mathcal{B}_1}$ when $c = 1$ and $2 \in \mathcal{S}_1$ (and so $1 \notin \mathcal{S}_1$).

A *preferred idempotent state* for $\mathcal{B}_1 = \mathcal{B}(m+2, k+1, \mathcal{S}_2)$ is an idempotent state $\mathbf{x}$ with $\mathbf{x} \cap \{0,1,2\} \in \{\{0\},\{2\},\{0,2\}\}$. We define a map ψ from preferred idempotent states of $\mathcal{B}_1$ to idempotent states of $\mathcal{B}_2$, as follows. Given preferred idempotent state $\mathbf{x}$ for $\mathcal{B}_1$, order the components $\mathbf{x} = \{x_1, \dots, x_{k+1}\}$ so that $x_1 < \dots < x_{k+1}$. Define

$$\psi(\mathbf{x}) = \begin{cases} \{0, x_3 - 2, \dots, x_{k+1} - 2\} & \text{if } |\mathbf{x} \cap \{0,1,2\}| = 2 \\ \{x_2 - 2, \dots, x_{k+1} - 2\} & \text{if } |\mathbf{x} \cap \{0,1,2\}| = 1 \end{cases}$$

Generators of the DA bimodule $\mho^1 = {}^{\mathcal{B}_2}\mho^1_{\mathcal{B}_1}$ correspond to preferred idempotent states, and the bimodule structure over the idempotent algebras is as follows. If $\mathbf{x}$ is a preferred idempotent state, let $\mathbf{T_x}$ be its corresponding generator. Then,

$$\mathbf{I_y} \cdot \mathbf{T_x} \cdot \mathbf{I_z} = \begin{cases} \mathbf{T_x} & \text{if } \mathbf{x} = \mathbf{z} \text{ and } \mathbf{y} = \psi(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

The bimodule structure is expressed in terms of the following oriented graph Γ. The vertices of Γ correspond to words $\{C_2, L_1C_2, R_2, U_1^t, L_1U_1^t\}_{t\geq 0}$, with the understanding that $U_1^0 = 1$. The graph comes equipped with the following oriented edges, labelled by words of the form $\{1, L_1, U_1^n, U_1^tC_2, R_1U_1^tC_2\}_{t\geq 0, n>0}$:

- An edge labelled 1 from R_2 to R_2.
- An edge labelled 1 from L_1C_2 to L_1C_2.
- An edge labelled L_1C_2 from 1 to L_1C_2.
- An edge labelled L_1 from C_2 to L_1C_2.
- An edge labelled R_2 from from 1 to R_2.
- An edge labelled U_2 from 1 to C_2.
- An edge labelled L_2 from R_2 to C_2.
- For each $n > 0$, an edge labelled U_1^n from L_1C_2 to L_1U^{n-1}.

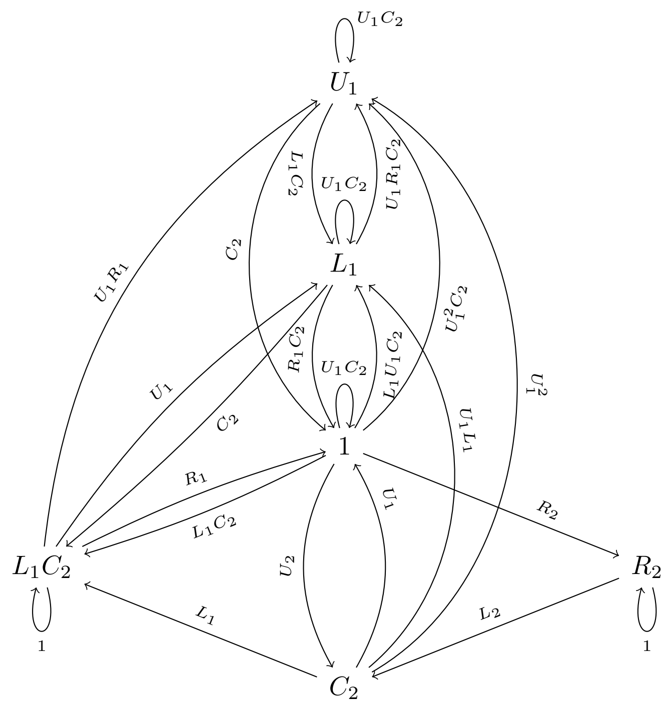


FIGURE 40. **Operation graph for a minimum when $2 \in \mathcal{S}_1$.**

- For each $t \geq 0$, an edge labelled $R_1 U_1^t$ from L_1C_2 to U_1^t.
- For each $s \geq 0$ and $t \geq 0$ so that $s + t > 0$, an edge labelled $C_2 U_1^s$ from U_1^t to U_1^{s+t-1}; and another with the same label from $U_1^t L_1$ to $U_1^{s+t-1} L_1$.
- For each $s \geq 0$ and $t \geq 0$ with $s + t > 0$ an edge labelled $C_2 U_1^s L_1$ from U_1^t to $U_1^{s+t-1} L_1$.
- For each $s \geq 0$ and $t \geq 0$, an edge labelled $C_2 U_1^s R_1$ from $U_1^t L_1$ to U_1^{s+t}.
- For each $n \geq 1$, an edge labelled U_1^n from C_2 to U_1^{n-1}.
- For each $n \geq 1$, an edge labelled $U_1^n L_1$ from C_2 to $U_1^n L_1$.

See Figure 40 for an illustration.

Fix a sequence $a_1, \dots, a_{\ell-1}$ of pure algebra elements in $\mathcal{B}(m+2, k+1, \mathcal{S}_1)$. We call the sequence an *preferred sequence* if the following conditions are satisfied:

(℧-1) there are idempotent states $\mathbf{x}_1, \dots, \mathbf{x}_\ell$ so that $\mathbf{I}_{\mathbf{x}_i} \cdot a_i \cdot \mathbf{I}_{\mathbf{x}_{i+1}} = a_i$ for $i = 1, \dots, \ell - 1$,

(℧-2) The idempotent states $\mathbf{x}_1$ and $\mathbf{x}_\ell$ are preferred idempotent states.

(℧-3) Each C_j with $j \in \mathcal{S}_1 \setminus \{2\}$ divides at most one of the a_k.

(℧-4) There is an oriented path $e_1, \dots, e_\ell$ in Γ beginning at C_2L_1 if $0 \in \mathbf{x}_1$ and at R_2 if $0 \notin \mathbf{x}_1$. The label on the edge e_i, thought of as an algebra element in $\mathcal{B}(m+2, k+1, \mathcal{S}_1)$, has $w_1(a_i) = w_1(e_i)$ and $w_2(a_i) = w_2(e_i)$; and a_i is divisible by C_2 if and only if e_i is divisible by C_2, none of the internal vertices are at R_2 or L_1C_2, and the terminal vertex is at L_1C_2 or R_2.

For a preferred sequence, there is at most one pure non-zero element $b \in \mathcal{B}_2$ characterized by the following properties

(PS-1) $b = \mathbf{I}_{\psi(\mathbf{x}_1)} \cdot b$
(PS-2) For $j \in \mathcal{S}_2$, C_j divides b if and only if C_{j+2} divides some a_k
(PS-3) For $i = 1, \dots, m$, $w_i(b) = \sum_{j=1}^{\ell-1} w_{i+2}(a_j)$.

Define maps $\delta^1_\ell \colon {}^{\mathcal{B}_2}\mho^1_{\mathcal{B}_1} \otimes \overbrace{\mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_1}^{\ell-1} \to \mathcal{B}_2 \otimes {}^{\mathcal{B}_2}\mho^1_{\mathcal{B}_1}$ by specifying them on sequences $a_1, \dots, a_{\ell-1} \in \mathcal{B}(m+2, k+1, \mathcal{S}_1)$ for which there are idempotent states $\mathbf{x}_1, \dots, \mathbf{x}_\ell$ with $\mathbf{I}_{\mathbf{x}_i} \cdot a_i \cdot \mathbf{I}_{\mathbf{x}_{i+1}} = a_i$ and the a_i are pure. For such a sequence, the operation $\delta^1_\ell(\mathbf{T}_{\mathbf{x}_1}, a_1, \dots, a_{\ell-1})$ is non-zero only if the sequence is a preferred sequence, and we define $\delta^1_\ell(\mathbf{T}_{\mathbf{x}_1}, a_1, \dots, a_{\ell-1}) = b \otimes \mathbf{T}_{\mathbf{x}_\ell}$, where b is the algebra element specified by the sequence. For $\ell = 1$, we define $\delta^1_1 = 0$.

For example, sequences $(a_1, \dots, a_{\ell-1})$ for which $\delta^1_\ell(\mathbf{T}, a_1, \dots, a_\ell)$ is non-zero (for suitably chosen $\mathbf{T}$) include the sequences

$$(L_2, L_1), \quad (R_1, R_2), \quad (L_2, U_1, R_2), \quad (U_1, C_2); \tag{9.2}$$

the first comes from a path from R_2 to L_1C_2; the second comes from a path from L_1C_2 to R_2, the third goes from R_2 to itself, and the last goes from L_1C_2 to itself). The loops labelled 1 also give rise to the following actions: if $\mathbf{x}_1$ and $\mathbf{x}_2$ are preferred idempotent states, then:

$$\begin{aligned}
\delta^1_2(\mathbf{T}_{\mathbf{x}_1}, R_i) &= R_{i-2} \otimes \mathbf{T}_{\mathbf{x}_2} && \text{if } i > 2 \text{ and } \mathbf{I}_{\mathbf{x}_1} \cdot R_i = R_i \cdot \mathbf{I}_{\mathbf{x}_2} \\
\delta^1_2(\mathbf{T}_{\mathbf{x}_1}, L_i) &= L_{i-2} \otimes \mathbf{T}_{\mathbf{x}_2} && \text{if } i > 2 \text{ and if } \mathbf{I}_{\mathbf{x}_1} \cdot L_i = L_i \cdot \mathbf{I}_{\mathbf{x}_2} \\
\delta^1_2(\mathbf{T}_{\mathbf{x}_1}, U_i) &= U_{i-2} \otimes \mathbf{T}_{\mathbf{x}_1} && \text{if } i > 2 \\
\delta^1_2(\mathbf{T}_{\mathbf{x}_1}, C_j) &= C_{j-2} \otimes \mathbf{T}_{\mathbf{x}_1} && \text{if } 2 < j \in \mathcal{S}_1.
\end{aligned}$$

As an illustration, we list some other preferred sequences:

$$(U_1R_1, L_1C_2, C_2), \qquad (L_1, \overbrace{U_1, U_2, \dots, U_1, U_2}^{n}, L_2), \qquad (U_1^n, \overbrace{C_2, \dots, C_2}^{n}).$$

So far, we described the case where $2 \in \mathcal{S}_1$. In cases where $1 \in \mathcal{S}_1$, we modify the earlier construction slightly as follows: in the description of the graph Γ, switch the roles of U_1 and U_2, L_1 and R_2, R_1 and L_2, C_2 and C_1. To read off the actions, allowed idempotent states with $0 \in \mathbf{x}$ correspond to the starting vertex L_1, and those with $0 \notin \mathbf{x}$ correspond to R_2C_1. With this adjustment, the bimodule is defined as before. See Figure 41 for a picture.

Proposition 9.1. *The above maps δ^1_ℓ on ${}^{\mathcal{B}_2}\mho^1_{\mathcal{B}_1}$ satisfy the DA bimodule relations; and $\mho^1$ is adapted to the one-manifold underlying its knot diagram (Definition 3.19).*

Proposition 9.1 will be proved after we give an alternative construction of $\mho^1$, later in this section. The significance of $\mho^1$ is formulated in the following:

Lemma 9.2. *$\mho^1$ is dual to $\mathcal{E}_1$, in the following sense. Fix arbitrary integers $0 \le k < m$, a subsequence $\mathcal{S}_2 = \{u_1, \dots, u_\ell\} \subset \{1, \dots, m\}$, and let*

$$\mathcal{S}_1 = \{1, u_1 + 2, \dots, u_\ell + 2\} \quad \textit{or} \quad \mathcal{S}_1 = \{2, u_1 + 2, \dots, u_\ell + 2\},$$

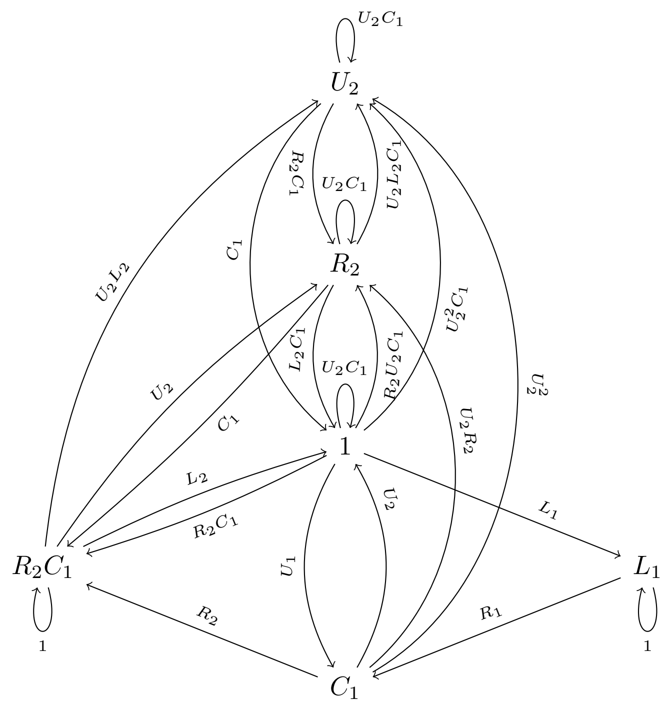


FIGURE 41. **Operation graph for a minimum when** $1 \in \mathcal{S}_1$**.**

$\mathcal{S}_3 = \{1, \dots, m+2\} \setminus \mathcal{S}_1$*;* $\mathcal{B}_1 = \mathcal{B}(m+2, k+1, \mathcal{S}_1)$*,* $\mathcal{B}_2 = \mathcal{B}(m, k, \mathcal{S}_2)$*, and* $\mathcal{B}_3 = \mathcal{B}(m+2, m+1-k, \mathcal{S}_3)$*. Then, there is an equivalence* ${}^{\mathcal{B}_2}\mho_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1,\mathcal{B}_3}\mathcal{K} \simeq {}^{\mathcal{B}_2,\mathcal{B}_3}\mathcal{E}_1$*.*

Proof. The generators of $\mho^1$ corresponding to idempotent states with $\mathbf{x} \cap \{0,1,2\} = \{2\}$, $\{0\}$, and $\{0,2\}$ respectively induce generators in ${}^{\mathcal{B}_2}\mho_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1,\mathcal{B}_3}\mathcal{K}$ corresponding to generators of Ω of type $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$, in the notation of Section 8.

If $2 \in \mathcal{S}_1$, we can pair the actions from Equation (9.2) (counting the last one twice) with the differential from $\mathcal{K}$ to give the differentials

$$\begin{aligned} &\mathbf{X} \to (1 \otimes L_1 L_2) \otimes \mathbf{Y}, \quad \mathbf{Y} \to (1 \otimes R_2 R_1) \otimes \mathbf{X}, \quad \mathbf{Y} \to (1 \otimes C_1 U_2) \otimes \mathbf{Y} \\ &\mathbf{X} \to (1 \otimes C_1 U_2) \otimes \mathbf{X}, \quad \mathbf{Z} \to (1 \otimes C_1 U_2) \otimes \mathbf{Z} \end{aligned}$$

appearing in the description of ${}^{\mathcal{B}_2,\mathcal{B}_3}\mathcal{E}_1$. The loops labelled 1 induce δ^1_2 actions by the part of the algebra with $w_1 = w_2 = 0$. These actions give rise to the remaining terms for the differential in $\mathcal{E}_1$. The case where $1 \in \mathcal{S}_1$ works similarly. $\square$

9.2. **An alternative construction.** We describe a $\mathcal{B}_2 - \mathcal{B}_1$ DG bimodule M that is quasi-isomorphic to the bimodule ${}^{\mathcal{B}_2}\mho^c_{\mathcal{B}_1}$ with $c = 1$. Proposition 9.1 will be an immediate consequence of this construction. We continue with the hypothesis that $2 \in \mathcal{S}_1$, returning to the other case at the end of the subsection.

Let $\mathbf{I} = \sum_{\{\mathbf{x} \mid x_1 = 1\}} \mathbf{I}_{\mathbf{x}} \in \mathcal{B}_1$. There is a natural inclusion of $\phi\colon \mathcal{B}_2 \to \mathbf{I} \cdot \mathcal{B}_1 \cdot \mathbf{I}$, whose image consists of the portion of $\mathbf{I} \cdot \mathcal{B}_1 \cdot \mathbf{I}$ with $w_1 = w_2 = 0$. In particular, $\phi(R_i) = \mathbf{I} \cdot R_{i+2}$ and $\phi(L_i) = \mathbf{I} \cdot L_{i+2}$. Thus, we can think of $\mathbf{I} \cdot \mathcal{B}_1$ as a left module for $\mathcal{B}_2$; it is also a right module for $\mathcal{B}_1$. Consider the bimodule

$$_{\mathcal{B}_2}M_{\mathcal{B}_1} = \mathbf{I}\cdot\mathcal{B}_1/L_1L_2\cdot\mathcal{B}_1,$$

thought of as a $\mathcal{B}_2$-$\mathcal{B}_1$-module, i.e., with $m_{1|1|0}(b,a) = \phi(b)\cdot a$ and $m_{0|1|1}(a,a') = a\cdot a'$; and equipped with the equipped with the endomorphism $\partial = d + U_1C_2$.

In the next lemma, we describe the left $\mathcal{B}_2$-module strcture on M, using the following notation. Fix an algebra element $a\in\mathcal{B}_1$, and consider $\mathbf{I}\cdot a$. This generates a left $\mathbf{I}(\mathcal{B}_2)$-module, under the identification $\mathbf{I}(\mathcal{B}_2)\cong \mathbf{I}\cdot\mathbf{I}(\mathcal{B}_2)\cdot\mathbf{I}$. Thus, we can form a new left $\mathcal{B}_2$-module $\mathcal{B}_2\otimes_{\mathbf{I}(\mathcal{B}_2)}\mathbf{I}(\mathcal{B}_2)\cdot\mathbf{I}a$, which we abbreviate $\mathcal{B}_2\otimes_{\mathbf{I}(\mathcal{B}_2)} a$. In general, there is a quotient map of left $\mathcal{B}_2$-modules $\mathcal{B}_2\otimes_{\mathbf{I}(\mathcal{B}_2)} a\to\mathcal{B}_2\cdot a$.

Lemma 9.3. *Consider $_{\mathcal{B}_2}M_{\mathcal{B}_1}$ equipped with endomorphism $m_{0|1|0}=\partial$ as a left module over $\mathcal{B}_2$. There is an isomorphism $_{\mathcal{B}_2}M\cong\bigoplus_{a_i}\mathcal{B}_2\otimes_{\mathbf{I}(\mathcal{B}_2)} a_i$, where the a_i are chosen from the generating set*

$$\{R_2U_2^t,\quad U_2^n,\quad C_2R_2U_2^t,\quad C_2U_2^t,\qquad L_1U_1^t,\quad U_1^t,\quad C_2U_1^t\}_{t\geq 0,n>0}. \tag{9.3}$$

Proof. Consider any $a=\mathbf{I}_{\mathbf{x}}\cdot a\cdot\mathbf{I}_{\mathbf{y}}$ with $x_1=1$, and with non-trivial projection to $\mathbf{I}\cdot\mathcal{B}_1/L_1L_2\cdot\mathcal{B}_1$. If $y_1=0$, then $a=L_1U_1^t\cdot a_2$ with $w_1(a_2)=w_2(a_2)=0$; indeed, in this case, $a=a_2'L_1U_1^t$ (where a_2' and a_1' differ in their initial idempotent). If $y_1=2$, then $a=b\cdot R_2U_2^t$ or $b\cdot R_2U_2^tC_2$. Finally, if $y_1=1$, then $a=b\cdot U_2^t$ or $a=b\cdot U_1^t$. This shows that the generating sets are as enumerated above.

So far, we have identified the $\mathcal{B}_2$-orbits $_{\mathcal{B}_2}M\cong\bigoplus_{a_i}\mathcal{B}_2{\cdot}a_i$, where the a_i are as listed above. For each of the above summands, the identification $\mathcal{B}_2\cdot a_i\cong\mathcal{B}_2\otimes_{\mathbf{I}(\mathcal{B}_2)} a_i$ is equivalent to the statement that if $\mathbf{x}$ is an idempotent state with $x_1=1$, $b=\mathbf{I}{\cdot}b{\cdot}\mathbf{I}$ is an element of $\mathcal{B}_1$ with $w_1(b)=w_2(b)=0$, and $\mathbf{I}_{\mathbf{x}}\cdot b\cdot a_i=0$, then $\mathbf{I}_{\mathbf{x}}\cdot b=0$. But this follows from Proposition 3.7, together with the fact that that if $\mathbf{x}$ is an idempotent state for $\mathcal{B}_1$ with $x_1=1$ and $\mathbf{y}$ is another idempotent state, then no generating interval (in the sense of Definition 3.6) can contain exactly one of 1 or 2, whereas none of the a_i enumerated above is divisible by U_1U_2. □

According to the above lemma, $_{\mathcal{B}_2}M_{\mathcal{B}_1}$ can be thought of as a type DA bimodule with generating set specified in the lemma (see Equation (2.3)); i.e. there is a type DA bimodule $^{\mathcal{B}_2}X_{\mathcal{B}_1}$ so that $_{\mathcal{B}_2}M_{\mathcal{B}_1}=\mathcal{B}_2\boxtimes{}^{\mathcal{B}_2}X_{\mathcal{B}_1}$. There is a much smaller model for this type DA bimodule using the following:

Lemma 9.4. *The inclusion map $(\mathcal{B}_2\cdot L_1C_2)\oplus(\mathcal{B}_2\cdot R_2)\subset {}_{\mathcal{B}_2}M_{\mathcal{B}_1}$ induces an isomorphism in homology.*

Proof. In view of Lemma 9.3, we can define $H\colon M\to M$ by specifying

$$\begin{aligned}
H(b_2\cdot R_2U_2^n) &= b_2R_2U_2^{n-1}C_2\\
H(b_2\cdot U_2^n) &= b_2U_2^{n-1}C_2\\
H(b_2\cdot C_2U_1^n) &= b_2U_1^{n-1}\\
H(b_2) = H(b_2\cdot R_2) &= H(b_2C_2R_2U_2^t) = H(b_2\cdot L_1U_1^t)\\
&= H(b_2\cdot C_2R_2U_2^t) = H(b_2\cdot L_1C_2U_1^t) = 0
\end{aligned}$$

for all $n>0$, $t\geq 0$, and $b_2\in\mathcal{B}_2$ (thought of as the subalgebra of $\mathbf{I}\cdot\mathcal{B}_1\cdot\mathbf{I}$ with $w_1=w_2=0$). It is straightforward to verify that $\mathrm{Id}+\partial\circ H+H\circ\partial$ is projection onto the image of $(\mathcal{B}_2 \cdot L_1C_2) \oplus (\mathcal{B}_2 \cdot R_2) \subset {}_{\mathcal{B}_2}M_{\mathcal{B}_1}$. For example, if $a = b_2 \cdot C_2U_1^n$ with $n > 0$, then $\partial \circ H(a) = a + (db_2)U_2^{n-1}$, and $H \circ \partial(a) = (db_2)U_1^{n-1}$; so $(\mathrm{Id} + \partial \circ H + H \circ \partial)(a) = 0$. □

Proof of Proposition 9.1. We continue with the hypothesis that $2 \in \mathcal{S}_1$. By Lemma 9.4 (and the homological perturbation lemma, in the form of Lemma 2.12) $(\mathcal{B}_2 \cdot C_2L_1) \oplus (\mathcal{B}_2 \cdot R_2)$ inherits an $\mathcal{A}_\infty$ bimodule structure quasi-isomorphic to M. To this end, note that the maps H appearing in Lemma 9.4 are $\mathcal{B}_2$-linear, and so they can be thought of as morphisms of type D structures as in Lemma 2.12.

The induced DA bimodule structure on $(\mathcal{B}_2 \cdot C_2L_1) \oplus (\mathcal{B}_2 \cdot R_2)$ coincides with the bimodule structure on ${}^{\mathcal{B}_2}\mho^1_{\mathcal{B}_1}$. In a little more detail, the labels on the edges $e_1, \dots, e_{\ell-1}$, associated to the algebra elements $a_1, \dots, a_{\ell-1}$ are chosen according to the following rules:

- labels for the possible edges e_1 out of $v_1 = R_1$ or L_1C_2 are chosen so that, according to Lemma 9.3 we have $b_1 \in \mathcal{B}_2$ with $v_1 \cdot a_1 = b_1 \cdot v_2$ (thinking of v_i as certain left $\mathcal{B}_2$-module generators of M from Equation (9.3));
- labels for the edges e_i with $1 < i < \ell - 1$ from v_i to v_{i+1} (thought of as certain elements of $\mathcal{B}_1$) are chosen so that, according to Lemma 9.3, there is $b_i \in \mathcal{B}_2$ with $v_i \cdot a_i = b_i \cdot w_i$, and $v_{i+1} = H(w_i)$
- For the choices of final edges $e_{\ell-1}$ into (with $v_\ell = R_1$ or L_1C_2), once again the labels were chosen so that for some $b_{\ell-1} \in \mathcal{B}_2$, $v_{\ell-1} \cdot a_{\ell-1} = b_{\ell-1} \cdot v_\ell$.

Thus, as in Lemma 2.12, the induced DA bimodule structure associates to the sequence $a_1, \dots, a_{\ell-1}$, is the product $b = b_1 \cdots b_{\ell-1}$, which in turn is the output element associated to the pure sequence in our definition of δ^1_ℓ from Section 9.1.

Observe that for all sequences $a_1 \otimes \cdots \otimes a_{\ell-1}$ appearing in the $\mathcal{A}_\infty$ operations, $\sum_{i=1}^{\ell-1} w_1(a_i) = \sum_{i=1}^{\ell-1} w_2(a_i)$. This, together with Property (PS-3) amounts to the statement that $\mho^i$ is graded by the set $H^1(W, \partial W)$. Observe also that for each preferred sequence $a_1, \dots, a_{\ell-1}$, we have that $\sum_{i=1}^{\ell} \mathfrak{m}_2(a_i) - 2w_2(a_i) = 1 - \ell$. (To see this, note that the change in the vertical coordinate of an arrow labelled by an algebra element a in Figure 40 is given by $\mathfrak{m}_2(a) - 2w_2(a) - 1$.) This, together with Property (PS-2) ensures that the operations respect a $\mathbb{Z}$-valued Maslov grading. Obviously, $\mho^i$ is finite dimensional. We have thus verified that $\mho^i$ is adapted to the one-manifold underlying the partial knot diagram with a left-most minimum.

When $1 \in \mathcal{S}_1$, the above discussion applies with straightforward notational changes, switching the roles of U_1 and U_2, L_1 and R_2, R_1 and L_2, C_2 and C_1. For example, the analogue of Lemma 9.3 holds, where now a_i are chosen from

$$\{L_1U_1^t, \quad U_1^n, \quad C_1L_1U_1^t, \quad C_1U_1^t, \qquad R_2U_2^t, \quad U_2^t, \quad C_1U_2^t\}_{t \geq 0, n > 0}.$$

The analogue of Lemma 9.4 gives a model generated by $(\mathcal{B}_2 \cdot R_2C_1) \oplus (\mathcal{B}_2 \cdot L_1)$, with $\mathcal{A}_\infty$ structure obtained from the graph Γ with the corresponding modification. □

9.3. **The bimodule of a minimum with arbitrary** c**.** Having defined ${}^{\mathcal{B}_2}\mho^c_{\mathcal{B}_1}$ for $c = 1$, we can inductively define it for arbitrary c by the relation

$$({9.4}) \qquad {}^{\mathcal{B}_2}\mho^c_{\mathcal{B}_1} = {}^{\mathcal{B}_2}\mho^{c-1}_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\mathcal{P}^c_{\mathcal{B}_3} \boxtimes {}^{\mathcal{B}_3}\mathcal{P}^{c-1}_{\mathcal{B}_1};$$

with algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ chosen as in the beginning of the section; and $\mathcal{B}_3 = \mathcal{B}(m+2, k+1, \tau_{c-1}(\mathcal{S}_1))$ and $\mathcal{B}_4 = \mathcal{B}(m+2, k+1, \tau_c \circ \tau_{c-1}(\mathcal{S}_1))$. See Figure 42.

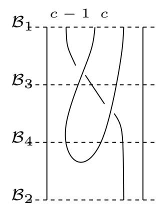


FIGURE 42. To define $\mho^c$, tensor $\mho^{c-1}$ with $\mathcal{P}^c$ and $\mathcal{P}^{c-1}$, as shown.

Proposition 9.5. *Choose $\mathcal{B}_1$ and $\mathcal{B}_2$ as in Equation* (9.1)*, and let $\mathcal{B}'_1 = \mathcal{B}(m+2, m+2-k, \{1,\dots,m+2\} \setminus \mathcal{S}_1)$. Then ${}^{\mathcal{B}_2}\mho^c_{\mathcal{B}_1}$ is a type DA bimodule that is adapted to the one-manifold underlying its partial knot diagram, and it is dual to $\mathcal{E}_c$, in the sense that*

$$ {}^{\mathcal{B}_2}\mho^c_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1,\mathcal{B}'_1}\mathcal{K} \simeq {}^{\mathcal{B}_2,\mathcal{B}'_1}\mathcal{E}_c. \tag{9.5}$$

Proof. The tensor product defining $\mho^c$ is well defined thanks to Proposition 3.20, and induction on c.

Equation (9.5) is verified by induction on c, and the basic case $c=1$ is Lemma 9.2. For the inductive step, we compute:

$$\begin{aligned}
{}^{\mathcal{B}_2}\mho^c_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1,\mathcal{B}'_1}\mathcal{K} &\simeq {}^{\mathcal{B}_2}\mho^{c-1}_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\mathcal{P}^c_{\mathcal{B}_3} \boxtimes \left({}^{\mathcal{B}_3}\mathcal{P}^{c-1}_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1,\mathcal{B}'_1}\mathcal{K}\right) \\
&\simeq {}^{\mathcal{B}_2}\mho^{c-1}_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\mathcal{P}^c_{\mathcal{B}_3} \boxtimes \left({}^{\mathcal{B}'_1}\mathcal{P}^{c-1}_{\mathcal{B}'_3} \boxtimes {}^{\mathcal{B}'_3,\mathcal{B}_3}\mathcal{K}\right) \\
&\simeq {}^{\mathcal{B}_2}\mho^{c-1}_{\mathcal{B}_4} \boxtimes \left({}^{\mathcal{B}'_1}\mathcal{P}^{c-1}_{\mathcal{B}'_3} \boxtimes \left({}^{\mathcal{B}_4}\mathcal{P}^c_{\mathcal{B}_3} \boxtimes {}^{\mathcal{B}_3,\mathcal{B}'_3}\mathcal{K}\right)\right) \\
&\simeq {}^{\mathcal{B}_2}\mho^{c-1}_{\mathcal{B}_4} \boxtimes \left({}^{\mathcal{B}'_1}\mathcal{P}^{c-1}_{\mathcal{B}'_3} \boxtimes \left({}^{\mathcal{B}'_3}\mathcal{P}^c_{\mathcal{B}'_4} \boxtimes {}^{\mathcal{B}'_4,\mathcal{B}_4}\mathcal{K}\right)\right) \\
&\simeq {}^{\mathcal{B}'_1}\mathcal{P}^{c-1}_{\mathcal{B}'_3} \boxtimes {}^{\mathcal{B}'_3}\mathcal{P}^c_{\mathcal{B}'_4} \boxtimes \left({}^{\mathcal{B}_2}\mho^{c-1}_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4,\mathcal{B}'_4}\mathcal{K}\right) \\
&\simeq {}^{\mathcal{B}'_1}\mathcal{P}^{c-1}_{\mathcal{B}'_3} \boxtimes {}^{\mathcal{B}'_3}\mathcal{P}^c_{\mathcal{B}'_4} \boxtimes {}^{\mathcal{B}'_4,\mathcal{B}_2}\mathcal{E}_{c-1} \\
&\simeq {}^{\mathcal{B}'_1}\mathcal{P}^{c-1}_{\mathcal{B}'_3} \boxtimes {}^{\mathcal{B}'_3}\mathcal{N}^{c-1}_{\mathcal{B}'_1} \boxtimes {}^{\mathcal{B}'_1,\mathcal{B}_2}\mathcal{E}_c \\
&\simeq {}^{\mathcal{B}'_1,\mathcal{B}_2}\mathcal{E}_c,
\end{aligned}$$

using associativity of $\boxtimes$ (Lemmas 2.4 and 2.9), Lemma 6.6, the trident relation (Lemma 7.2), the inductive hypothesis, and the fact that $\mathcal{P}$ and $\mathcal{N}$ are inverses (Equation (6.1)). Lemma 2.9 applies, since the bimodules $\mathcal{P}^k$ for any k have $\delta^1_j = 0$ for all $j > 3$. The steps (skipping associativity) are illustrated in Figure 43. □

9.4. **The terminal Type A module.** We have described the DA bimodule associated to a local (but not global) minimum. For the global minimum, we give the following different construction. Note that the algebra immediately above the global minimum has $\mathcal{B} = \mathcal{B}(2,1,\{2\})$ or $\mathcal{B}(2,1,\{1\})$, depending on the orientation on the global minimum (left to right or right to left). Consider first the case where

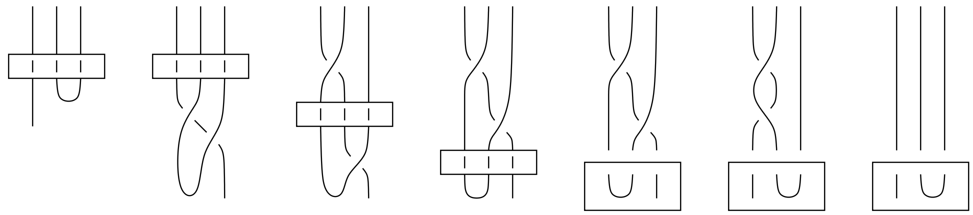


FIGURE 43. Pictures of bimodules appearing in the inductive step of Proposition 9.1. Boxed components correspond to type DD bimodules, the rest correspond to type DA bimodules.

$\mathcal{B} = \mathcal{B}(2,1,\{2\})$. Since the outgoing algebra has no strands (and so it can be thought of as $\mathbb{F}$), the we construct a type A module over the incoming algebra.

As a vector space, $t\mho_{\mathcal{B}(2,1,\{2\})}$ is a two-dimensional, with generators $\mathbf{X}$ and $\mathbf{Y}$. The right $\mathbf{I}(\mathcal{B}(2,1,\{2\}))$-module structure is determined by

$$\mathbf{X} = \mathbf{X}\cdot \mathbf{I}_{\{1\}} \qquad \mathbf{Y} = \mathbf{Y}\cdot \mathbf{I}_{\{0\}}. \tag{9.6}$$

The right module structure is determined by the formulas:

$$\mathbf{X}\cdot L_1 = \mathbf{Y} \qquad \mathbf{Y}\cdot R_1 = \mathbf{X} \qquad \mathbf{X}\cdot C_2 = \mathbf{Y}\cdot C_2 = 0. \tag{9.7}$$

The idempotent relations imply that $\mathbf{X}\cdot R_2 = 0$. It follows that

$$\mathbf{X}\cdot L_1 U_1^t = \mathbf{Y} \qquad \mathbf{Y}\cdot R_1 U_1^t = \mathbf{X} \qquad \mathbf{X}\cdot U_1^t = \mathbf{X} \qquad \mathbf{Y}\cdot U_1^t = \mathbf{Y},$$

for all $t\geq 0$; and U_2 acts trivially.

Defining $M(\mathbf{X}) = M(\mathbf{Y}) = 0$, it follows that $t\mho$ has a $\mathbb{Z}$-valued Maslov grading.

Unlike the modules encountered thus far, $t\mho$ is not graded by Alexander set, which in this case is $\frac{1}{2}\mathbb{Z}$. However, it is filtered by it, in the sense that if a is a homogeneous element with $m_2(X,a)\neq 0$, then $w_2(a) - w_1(a)\leq 0$.

An analogous construction works for $\mathcal{B} = \mathcal{A}(1,\{1\})$. In that case, $\mathbf{X} = \mathbf{X}\cdot\mathbf{I}_{\{1\}}$, $\mathbf{Y} = \mathbf{Y}\cdot\mathbf{I}_{\{2\}}$; and actions are determined by $\mathbf{X}\cdot R_2 = \mathbf{Y}$, $\mathbf{Y}\cdot L_2 = \mathbf{X}$, $\mathbf{X}\cdot C_1 = \mathbf{Y}\cdot C_1 = 0$. Again, this module is filtered.

Both versions of $t\mho$ have an associated graded object, with two generators $\mathbf{X}$ and $\mathbf{Y}$ satisfying Equation (9.6). The actions by L_1 L_2, R_1, R_2, U_1, U_2 are all 0; the action by C_1 or C_2 (whichever is in the algbera) is also 0.

Proposition 9.6. *Let Y be a type D structure over $\mathcal{B} = \mathcal{B}(2,1,\{1\})$ or $\mathcal{B}(2,1,\{2\})$, with equipped with a $\mathbb{Z}$-valued Maslov grading and a $\frac{1}{2}\mathbb{Z}$-valued Alexander grading. Then, the tensor product $t\mho\boxtimes Y$ is naturally $\mathbb{Z}$-graded (by the Maslov grading) and it is filtered by the Alexander grading. Its associated graded object coincides with the tensor product $\widehat{t\mho}\boxtimes Y$.*

Proof. The sums in the tensor product are finite since $t\mho$ is bounded. The Maslov gradings on Y and $t\mho$ induce Maslov gradings on the tensor product as usual, and the Alexander grading on Y induces an Alexander function on $t\mho\boxtimes Y$. Since $\widehat{t\mho}$ is the associated graded object for $t\mho$, it follows that the associated graded object for $t\mho\boxtimes Y$ is $\widehat{t\mho}\boxtimes Y$. ∎

10. Symmetries

We note some symmetries in the algebras and bimodules described above.

One symmetry, which can be visualized as rotation through a vertical axis, is induced by an isomorphism between the algebras. That is, consider the isomorphism $\mathcal{R}\colon \mathcal{B}(m,k,\mathcal{S})\to\mathcal{B}(m,k,\rho(\mathcal{S}))$ defined in Equation (3.10). As in Example 2.3, this map gives a DA bimodule, denoted $[\mathcal{R}] = {}^{\mathcal{B}(m,k,\rho(\mathcal{S}))}[\mathcal{R}]_{\mathcal{B}(m,k,\mathcal{S})}$.

The bimodules for crossings and critical points are symmetric under this vertical rotation, according to the following:

Lemma 10.1. *Fix integers c, k, and m, with $1\leq c\leq m+1$ and $0\leq k\leq m+1$; and fix $\mathcal{S}_1\subset\{1,\dots,m\}$, $\mathcal{S}_2\subset\{1,\dots,m+2\}$ so that $\mathcal{S}_2=\phi_c(\mathcal{S}_1)\cup\{c\}$ or $\mathcal{S}_2=\phi_c(\mathcal{S}_1)\cup\{c+1\}$ (with ϕ_c as in Equation (7.1)). Let $\mathcal{B}_1=\mathcal{B}(m,k,\mathcal{S}_1)$, $\mathcal{B}_2=\mathcal{B}(m,k,\mathcal{S}_2)$, $\mathcal{B}'_1=\mathcal{B}(m,k,\rho'_m(\mathcal{S}_1))$, and $\mathcal{B}'_2=\mathcal{B}(m,k,\rho'_{m+2}(\mathcal{S}_2))$ (using notation from Section 3.6). The following identities hold:*

$$ {}^{\mathcal{B}'_2}[\mathcal{R}]_{\mathcal{B}_2}\boxtimes{}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}_1}\simeq{}^{\mathcal{B}'}\Omega^{m-c}_{\mathcal{B}'_1}\boxtimes{}^{\mathcal{B}'_1}[\mathcal{R}]_{\mathcal{B}_1}\qquad {}^{\mathcal{B}'_1}[\mathcal{R}]_{\mathcal{B}_1}\boxtimes{}^{\mathcal{B}_1}\mho^c_{\mathcal{B}_2}\simeq{}^{\mathcal{B}'_1}\mho^{m-c}_{\mathcal{B}'_2}\boxtimes{}^{\mathcal{B}'_2}[\mathcal{R}]_{\mathcal{B}'}. $$

Also, for any $i=1,\dots,m-1$, let $\mathcal{B}_3=\mathcal{B}(m,k,\tau(\mathcal{S}_1))$ and $\mathcal{B}'_3=\mathcal{B}(m,k,\rho'_m(\tau(\mathcal{S}_1))$ we have identities

$$ {}^{\mathcal{B}'_3}[\mathcal{R}]_{\mathcal{B}_3}\boxtimes{}^{\mathcal{B}_3}\mathcal{P}^i_{\mathcal{B}_1}\simeq{}^{\mathcal{B}'_3}\mathcal{P}^{m-i}_{\mathcal{B}'_1}\boxtimes{}^{\mathcal{B}'_1}[\mathcal{R}]_{\mathcal{B}_1}\qquad {}^{\mathcal{B}'_3}[\mathcal{R}]_{\mathcal{B}_3}\boxtimes{}^{\mathcal{B}_3}\mathcal{N}^i_{\mathcal{B}_1}\simeq{}^{\mathcal{B}'_3}\mathcal{N}^{m-i}_{\mathcal{B}'_1}\boxtimes{}^{\mathcal{B}'_1}[\mathcal{R}]_{\mathcal{B}_1} $$

Proof. It is easy to see that $[\mathcal{R}]\boxtimes\mathcal{K}\cong\mathcal{K}\boxtimes[\mathcal{R}]$.

The above results then follow from the invertibility of the DD bimodule, and the easily verified identities $[\mathcal{R}]\boxtimes\mathcal{E}_c\cong\mathcal{E}_{m+2-c}\boxtimes[\mathcal{R}]$ and $[\mathcal{R}]\boxtimes\mathcal{P}_i\cong\mathcal{P}_{m-i}\boxtimes[\mathcal{R}]$, where the latter isomorphism preserves $\mathbf{N}$ and $\mathbf{S}$ and switches $\mathbf{E}$ and $\mathbf{W}$. □

Recall that there is an algebra isomorphism $\mathcal{B}(m,k,\mathcal{S})\cong\mathcal{B}(m,k,\mathcal{S})^{\mathrm{op}}$ (Equation (3.11)). An arbitrary DA bimodule ${}^{\mathcal{B}_2}X_{\mathcal{B}_1}$ has an *opposite module* $\overline{X}$, of the form ${}_{\mathcal{B}_1}X^{\mathcal{B}_2}\cong{}^{\mathcal{B}_2^{\mathrm{op}}}\overline{X}_{\mathcal{B}_1^{\mathrm{op}}}$. If $\mathcal{B}_i=\mathcal{B}(m_i,k_i,\mathcal{S}_i)$ for $i=1,2$, we can view the opposite module X^{op} as a bimodule over the same two algebras, $X^{\mathrm{op}}=[o]\boxtimes\overline{X}\boxtimes[o]$.

Proposition 10.2. *Under the identification $\mathcal{B}(m,k,\mathcal{S})\cong\mathcal{B}(m,k,\mathcal{S})^{\mathrm{op}}$ from Equation (3.11), there are identifications of bimodules $\Omega_c^{\mathrm{op}}\simeq\Omega_c$, $\mho_c^{\mathrm{op}}\simeq\mho_c$, $(\mathcal{P}^i)^{\mathrm{op}}\simeq\mathcal{N}_i$, and $(\mathcal{N}^i)^{\mathrm{op}}\simeq\mathcal{P}_i$.*

Proof. We claim that $\mathcal{E}_c^{\mathrm{op}}\simeq\mathcal{E}_c$. The identification switches the roles of $\mathbf{X}$ and $\mathbf{Y}$, and fixes $\mathbf{Z}$. For each pair of generators in Equation (7.4), there are two arrows, and the symmetry switches those two arrows; observe that if one arrow is labelled by $a\otimes b$, then the other is labelled by $o(a)\otimes o(b)$.

The identity $(\mathcal{P}_i)^{\mathrm{op}}\simeq\mathcal{N}_i$ follows from the definition of $\mathcal{N}_i$ given in Section 5.5. □

11. Construction and invariance of the invariant

11.1. Topological preliminaries.

Definition 11.1. *A* pointed knot diagram *is a projection of an oriented knot diagram in S^2, with a marked point on it. A* pointed Reidemeister move *is a Reidemeister move supported in a complement of the marked point.*

The following result is well known; see [8, Section 3].

Proposition 11.2. *Any two pointed knot diagrams for the same knot can be connected by a sequence of isotopies (in S^2) and pointed Reidemeister moves (in S^2).*

Proof. By Reidemeister's theorem [26], any two knot diagrams for the same knot can be connected by a sequence of Reidemeister moves. To see that we can arrange to make those moves disjoint from the marked point p, it suffices to show that p can be moved through a crossing by a sequence of pointed Reidemeister moves. Suppose we are attempting to move a crossing across p, situated immediately south of the crossing. Viewing the knot projection as supported in S^2, we can equivalently move the strand north across the knot diagram, bringing it back north through the point at infinity, as shown in Figure 44. This move can be realized as a sequence of Reidemeister 2 and 3 moves disjoint from p. □

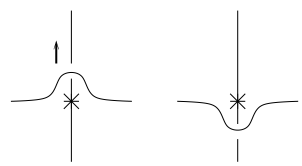


FIGURE 44. Moving an overpass across the basepoint is equivalent to a sequence of Reidemeister moves away from the basepoint.

Consider a pointed knot diagram in the plane, and consider the projection to the y axis. We say that the diagram is in *bridge position* if the following properties hold:

- All critical points are either minima or maxima.
- The minima, maxima, and crossings project to distinct points in the y axis.
- The global minimum is the marked point.

There are two kinds of diagrams in bridge position, depending on the orientation of the global minimum; i.e. left to right (LR) or right to left (RL).

Definition 11.3. *A* bridge move *connecting two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ in bridge position is any of the following types of moves:*

- *Creation of a single local maximum and local minimum; or the cancellation of such a pair. This is called a* pair creation *or* pair annihilation*.*
- *Sliding a maximum or non-global minimum through a crossing, called a* trident move
- *Commutations between distant pairs of special points (each of which is a maxmimum, non-global minimum, or a crossing), called a* commutation move*.*

In the above description, "distant" refers to special points not on the same strand, which are covered by the other two kinds of moves. All of the above moves are pointed isotopies; they leave the global minimum in place.

If $\mathcal{D}$ is in bridge position, there is a pointed diagram $\mathcal{D}'$ in the sphere, by adding a point at infinity adjacent to the marked point. We say that $\mathcal{D}$ *induces* $\mathcal{D}'$.

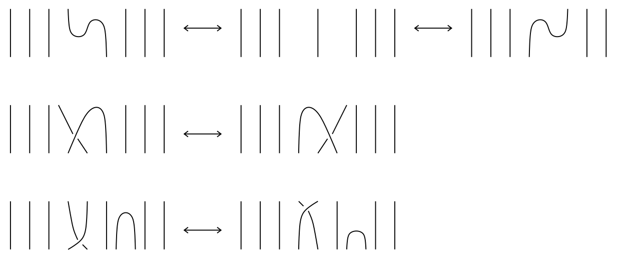


FIGURE 45. **Bridge moves.** Top row: annihilation and creation of a pair of local minimum and a maximum; second row: a trident move. (The three other trident moves are obtained by changing the two crossings, reflecting through a horizontal line, or both.) Third row: commutations of a maximum and a distant crossing.

Lemma 11.4. *Any pointed knot diagram in the sphere is induced from a planar knot diagram in bridge position. Moreover if two planar knot diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same type (i.e. LR or RL) induce (planar) isotopic knot diagrams in the sphere, then they can be connected by a sequence of bridge moves as above.*

Proof. Consider the basepoint on the knot projection, and place a second point $p \in S^2$ so that the point on the knot with minimal distance to p is the basepoint. There is a choice here: the orientation on the knot distiguishes the two sides where p can be placed; e.g. thinking of the knot locally as the x axis, oriented from left to right, p could either be slightly above or slightly below the x axis. Let a be a minimal arc from p to the basepoint in K. Stereographic projection from p gives a planar knot diagram, which we rotate so that a is taken to the portion of the y axis with $y \leq t$. If p was slightly below resp. above the distinguished point, the resulting diagram is of type LR resp. RL. Perturbing this map slightly (e.g. by moving p slightly) we can arrange that the resulting diagram is in bridge position.

Performing this construction in a one-parameter family, we get a one-parameter family of planar diagrams, for which the global minimum is the marked point. In a generic one-parameter family, the following can occur: two critical points create or cancel, a crossing crosses a critical point, or any two distant special points project to the same y coordinate. Crossing this codimension one locus, the diagram undergoes a pair creation or annihilation, a trident move, or a commutation move. $\square$

11.2. **Constructing the invariant.** Choose a knot planar knot diagram $\mathcal{D}$ in bridge position, and slice it up $t_1 < \cdots < t_k$ so that the following conditions hold:

- for $i = 1, \ldots, k-1$, the interval $[t_i, t_{i+1}]$ contains the projection onto the y axis of exactly one crossing or critical point
- for $i = 1, \ldots, k$, t_i is not the projection of any crossing or critical point
- there are no crossings or critical points whose y value is greater than t_k (and so $[t_{k-1}, t_k]$ contains the global maximum)
- there are no crossings below t_1, and the only critical point whose y value is smaller than t_1 is the global minium.

For $i = 1, \dots, k-1$, we have seen how to associate a type DA bimodule to the portion of the diagram that projects into $[t_i, t_{i+1}]$; it is either of the form $\mathcal{P}^j$, $\mathcal{N}^j$, Ω^c or $\mho^c$. The top portion $[t_{k-1}, t_k]$ contains the global maximum, where the incoming algebra has no strands, so the DA bimodule is, in fact, simply a type D module (see Section 8.1). The bottom portion $[-\infty, t_1]$ has only the global minimum, so the outgoing algebra has no strands; there we attach the terminal type A module $t\mho$ from Section 9.4. Note that for each partial knot diagram, the assoicated bimodule is adapted to the one-manifold underlying the partial knot diagram; and the hypothesis that our diagram is a indeed one for a knot ensures that the tensor products can always be taken; see Proposition 3.20. It is also worth noting that the generating sets for all the partial knot diagrams correspond to partial Kauffman states; this property is evidently preserved by $\boxtimes$.

The chain complex $\mathcal{C}(\mathcal{D})$ is now obtained as an iterated tensor product (via $\boxtimes$) of these pieces. It inherits a filtration, whose associated graded object, $\widehat{C}(\mathcal{D})$, can be computed by exchanging $t\mho$ with the simpler terminal module $\widehat{t\mho}$.

Theorem 11.5. *The filtered homotopy type of $\mathcal{C}(\mathcal{D})$ is an oriented knot invariant.*

The quasi-isomorphism type of the chain complex $\mathcal{C}(\mathcal{D})$ does not depend on the order in which the type DA bimodules are tensored together (Lemma 2.4). Thus, to check the invariance of $\mathcal{C}(\mathcal{D})$, it suffices to check identities between DA bimodules associated to the bridge moves, which we do in the next section. The proof of Theorem 11.5 is then given in Subsection 11.4.

11.3. **Invariance under bridge moves.** We order bridge moves as follows:

1. Commutations of distant crossings
2. Trident moves
3. Critical points commute with distant crossings
4. Commuting distant critical points
5. Pair creation and annihilation.

We verify the invariance of our invariant under bridge moves in the above order.

Bridge moves involve the interactions of two consecutive pieces in the chopped up knot diagram, The above procedure associates bimodules to each of those consecutive pieces which we tensor together, and the bridge moves are verified by verifying identities between the bimodule associated to the two bimodules tensored together and the tensor of the two bimodules after the bridge move (except in pair creation or annihilation, which states a relation between the tensor of two bimodules and a third bimodule, the identity bimodue). Thus, for example, commutations of distant positive crossings asserts the relation $\mathcal{P}^i \boxtimes \mathcal{P}^j \sim \mathcal{P}^j \boxtimes \mathcal{P}^i$ when $|i-j| > 1$, which was already verified in the verification of the braid relations (Theorem 6.1).

Consider next trident moves. There are four types of trident moves: the one illustrated in the second row of Figure 45, the one obtained by changing the crossings in the picture, the one obtained by mirroring the picture through a horizontal line, and the one obtained by changing the crossings in the horizontally reflected picture.

Lemma 11.6. *The DA bimodules associated to the two pictures before and after a trident move are quasi-isomorphic; i.e. the following four identities hold,* *corresponding to the four kinds of trident moves:*

$$ {}^{\mathcal{B}_3}\mathcal{P}^c_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\Omega^{c+1}_{\mathcal{B}_1} \simeq {}^{\mathcal{B}_3}\mathcal{N}^{c+1}_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}_1} \tag{11.1} $$

$$ {}^{\mathcal{B}_3}\mathcal{P}^{c+1}_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\Omega^c_{\mathcal{B}_1} \simeq {}^{\mathcal{B}_3}\mathcal{N}^c_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\Omega^{c+1}_{\mathcal{B}_1} \tag{11.2} $$

$$ {}^{\mathcal{B}_3}\mho^{c+1}_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\mathcal{P}^c_{\mathcal{B}_1} \simeq {}^{\mathcal{B}_3}\mho^c_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\mathcal{N}^{c+1}_{\mathcal{B}_1} \tag{11.3} $$

$$ {}^{\mathcal{B}_3}\mho^c_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\mathcal{P}^{c+1}_{\mathcal{B}_1} \simeq {}^{\mathcal{B}_3}\mho^{c+1}_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\mathcal{N}^c_{\mathcal{B}_1} \tag{11.4} $$

Proof. All four identities follow from Lemma 7.2, together with symmetries of the bimodules, as follows.

Since the $\mathcal{K}$ is invertible (Theorem 3.17), Equation (11.1) follows from $\mathcal{P}^c \boxtimes \Omega^{c+1} \boxtimes \mathcal{K} \simeq \mathcal{N}^{c+1} \boxtimes \Omega^c \boxtimes \mathcal{K}$; which follows from Proposition 8.4 and Lemma 7.2.

Applying horizontal rotation to Equation (11.1) (with $m - c - 1$ in place of c), together with Lemma 10.1, Equation (11.2) follows from Equation (11.1)

Similarly, observe that

$$ \begin{aligned} {}^{\mathcal{B}_3}\mho^{c+1}_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2}\mathcal{P}^c_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1,\mathcal{B}_1'}\mathcal{K} &\simeq {}^{\mathcal{B}_3}\mho^{c+1}_{\mathcal{B}_2} \boxtimes ({}^{\mathcal{B}_1'}\mathcal{P}^c_{\mathcal{B}_2'} \boxtimes {}^{\mathcal{B}_2',\mathcal{B}_2}\mathcal{K}) \\ &\simeq {}^{\mathcal{B}_1'}\mathcal{P}^c_{\mathcal{B}_2'} \boxtimes ({}^{\mathcal{B}_3}\mho^{c+1}_{\mathcal{B}_2} \boxtimes {}^{\mathcal{B}_2,\mathcal{B}_2'}\mathcal{K}) \\ &\simeq {}^{\mathcal{B}_1'}\mathcal{P}^c_{\mathcal{B}_2'} \boxtimes {}^{\mathcal{B}_2',\mathcal{B}_3}\mathcal{E}_{c+1} \end{aligned} $$

(using Lemma 6.6 twice, Lemma 2.9, Proposition 9.5); and by the same logic,

$$ {}^{\mathcal{B}_3}\mho^c_{\mathcal{B}_4} \boxtimes {}^{\mathcal{B}_4}\mathcal{N}^{c+1}_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1,\mathcal{B}_1'}\mathcal{K} \simeq {}^{\mathcal{B}_1'}\mathcal{N}^{c+1}_{\mathcal{B}_4'} \boxtimes {}^{\mathcal{B}_4',\mathcal{B}_3}\mathcal{E}_c. $$

Thus, Equation (11.3) follows from Lemma 7.2.

Equation (11.4) follows from Equation (11.3) by horizontal rotation. □

Lemma 11.7. *The DA bimodules for positive crossings commute with those for local maxima and minima, in the sense that if* $|c + 1 - i| > 0$*, then (with* ϕ_c *as in Equation* (7.1)*)*

$$ \mathcal{P}^{\phi_c(i)} \boxtimes \Omega^c \simeq \Omega^c \boxtimes \mathcal{P}^i \tag{11.5} $$

$$ \mathcal{P}^i \boxtimes \mho^c \simeq \mho^c \boxtimes \mathcal{P}^{\phi_c(i)} \tag{11.6} $$

Proof. Direct computation, as in the proof of Lemma 6.4, shows that

$$ \mathcal{P}^{\phi_c(i)} \boxtimes \mathcal{E}_c \simeq \Omega^c \boxtimes \mathcal{P}_i. \tag{11.7} $$

Both stated equations now follow from Equation (11.7), the invertibility of $\mathcal{K}$, and Propositions 8.4 and 9.5; compare the proof of Lemma 11.6. □

Lemma 11.8. *The DA bimodules for crossings commute with those for distant critical points.*

Proof. Lemma 11.7 handles positive crossings. Multiplying Equation (11.5) on both sides by $\mathcal{N}^i$ and using the fact that $\mathcal{N}^i$ and $\mathcal{P}^i$ are inverses of one another (Equation (6.1)), it follows that negative crossings also commute with local maxima; Equation (11.6) shows that negative crossings commute with local minima. □

Lemma 11.9. *The DA bimodules for pairs of distant critical points commute.*

Proof. Fix $i<j$. The lemma states identifications

$$\begin{array}{ll}\Omega^i\boxtimes\Omega^{j-1}\simeq\Omega^{j+1}\boxtimes\Omega^i & \mho^i\boxtimes\mho^{j+1}\simeq\mho^{j-1}\boxtimes\mho^i\\ \Omega^{j-1}\boxtimes\mho^i\simeq\mho^i\boxtimes\Omega^{j+1} & \mho^{j+1}\boxtimes\Omega^i\simeq\Omega^i\boxtimes\mho^{j-1}.\end{array}$$

To verify the first identity, we tensor on the right with $\mathcal{K}$, to reduce to the identity between two type DD bimodules (using Proposition 8.4):

$$\Omega^i\boxtimes\mathcal{E}_{j-1}\simeq\Omega^{j+1}\boxtimes\mathcal{E}_i, \tag{11.8}$$

which is easy to verify in the spirit of the proof of Lemma 6.4. Tensoring on the right with $\mathcal{K}$, the second on the right,

$$\begin{aligned}{}^{\mathcal{B}_3}\mho^i_{\mathcal{B}_2}\boxtimes{}^{\mathcal{B}_2}\mho^{j+1}_{\mathcal{B}_1}\boxtimes{}^{\mathcal{B}_1,\mathcal{B}_1'}\mathcal{K}&\simeq{}^{\mathcal{B}_3}\mho^i_{\mathcal{B}_2}\boxtimes{}^{\mathcal{B}_2,\mathcal{B}_1'}\mathcal{E}_{j+1}\\&\simeq{}^{\mathcal{B}_3}\mho^i_{\mathcal{B}_2}\boxtimes\left({}^{\mathcal{B}_1'}\Omega^{j+1}_{\mathcal{B}_2'}\boxtimes{}^{\mathcal{B}_2,\mathcal{B}_2'}\mathcal{K}\right)\\&\simeq{}^{\mathcal{B}_1'}\Omega^{j+1}_{\mathcal{B}_2'}\boxtimes\left({}^{\mathcal{B}_3}\mho^i_{\mathcal{B}_2'}\boxtimes{}^{\mathcal{B}_2,\mathcal{B}_2'}\mathcal{K}\right)\\&\simeq{}^{\mathcal{B}_1'}\Omega^{j+1}_{\mathcal{B}_2'}\boxtimes{}^{\mathcal{B}_2',\mathcal{B}_3}\mathcal{E}_i\end{aligned}$$

Similarly, $\mho^{j-1}\boxtimes\mho^i\boxtimes\mathcal{K}\simeq\Omega^i\boxtimes\mathcal{E}_{j-1}$, so the second identity also follows from Equation (11.8).

Consider the third identity, now with $i=1$. Tensoring on the right with $\mathcal{K}$, we reduce to the easily verified identity $\Omega^{j-1}\boxtimes\mathcal{E}_1\simeq\mho^1\boxtimes\mathcal{E}_{j+1}$. (Compare Equation (11.8).) The cases where $i>1$ now follow from the case where $i=1$, the inductive definition of $\mho^i$ (Equation (9.4)), and Lemma 11.7.

The fourth identity follows from the third and Lemma 10.1. □

Lemma 11.10. *The DA bimodules are invariant under pair creation and annihilations, in the sense that* $\mho^c\boxtimes\Omega^{c+1}\simeq\mathrm{Id}\simeq\mho^{c+1}\boxtimes\Omega^c$

Proof. Rotating through a vertical axis, i.e. using Lemma 10.1, we see that $\mho^c\boxtimes\Omega^{c+1}\simeq\mathrm{Id}$ implies also that $\mathrm{Id}\simeq\mho^{c+1}\boxtimes\Omega^c$. The verification of $\mho^c\boxtimes\Omega^{c+1}\simeq\mathrm{Id}$ can be reduced to the case where $c=1$ using Reidemeister 2 moves, as follows. First, introduce a sequence of $c-1$ positive crossings that carry the c^{th} strand to the far left, and let P be the corresponding bimodule; then let N be its inverse, i.e. $N\boxtimes P\simeq\mathrm{Id}$. (Clearly, N is obtained from P by reversing all the crossings and taking them in reverse order.) (cf. Equation (6.1)). In view of Lemma 11.6, a sequence of trident moves identifies $P\boxtimes\mho^c\boxtimes\Omega^{c+1}\simeq\mho^1\boxtimes\Omega^2\boxtimes P$. Thus, if $\mho^1\boxtimes\Omega^2\simeq\mathrm{Id}$, we can conclude that $\mho^c\boxtimes\Omega^{c+1}\simeq N\boxtimes P\boxtimes\mho^c\boxtimes\Omega^{c+1}\simeq N\boxtimes\mho^1\boxtimes\Omega^2\boxtimes P\simeq N\boxtimes P\simeq\mathrm{Id}$.

It remains now to verify that $\mho^1\boxtimes\Omega^2\simeq\mathrm{Id}$. We claim that the generators of ${}^{\mathcal{B}_1}\mho^1_{\mathcal{B}_2}\boxtimes^{\mathcal{B}_2}\Omega^2_{\mathcal{B}_1}$ correspond to the idempotents in the algebra $\mathcal{B}_1$; see Figure 46. Specifically, $\Omega^2_{\mathcal{B}}$, the generator is constrained to be either of type $\mathbf{Y}$ or $\mathbf{Z}$; while in $\mho^1$, the generator is either of type $\mathbf{X}$ or $\mathbf{Z}$. Thus, in the tensor product, we divide the generators into four types $\mathbf{X}\boxtimes\mathbf{Y}$, $\mathbf{X}\boxtimes\mathbf{Z}$, $\mathbf{Z}\boxtimes\mathbf{Y}$, and $\mathbf{Z}\boxtimes\mathbf{Z}$. We claim that

$$\delta^1_2(\mathbf{X}\boxtimes\mathbf{Y},L_1)=L_1\otimes(\mathbf{Z}\boxtimes\mathbf{Z})\quad\delta^1_2(\mathbf{Z}\boxtimes\mathbf{Z},R_1)=R_1\otimes(\mathbf{X}\boxtimes\mathbf{Y}), \tag{11.9}$$

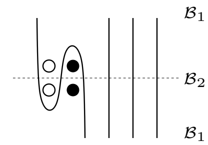


FIGURE 46. **Generators of ${}^{\mathcal{B}_1}\mho^1_{\mathcal{B}_2} \boxtimes^{\mathcal{B}_2} \Omega^2_{\mathcal{B}_1}$.** Generators correspond to the idempotents in the incoming algebra $\mathcal{B}_1$; the constraints on the intermediate generators are as shown.

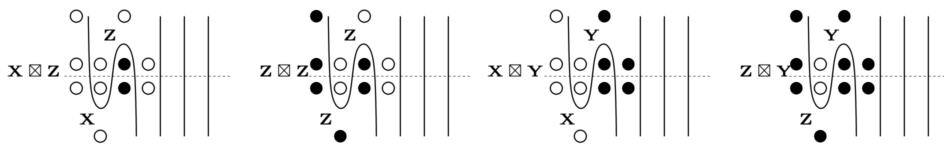
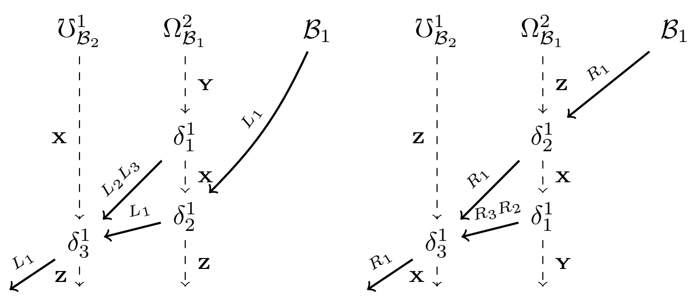


FIGURE 47. **Generator types for ${}^{\mathcal{B}_1}\mho^1_{\mathcal{B}_2} \boxtimes^{\mathcal{B}_2} \Omega^2_{\mathcal{B}_1}$.**

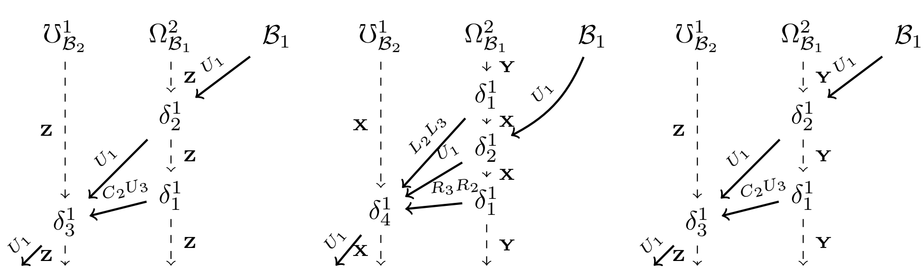
as can be seen from the following diagrams:



If the leftmost strand is downwards, then furthermore

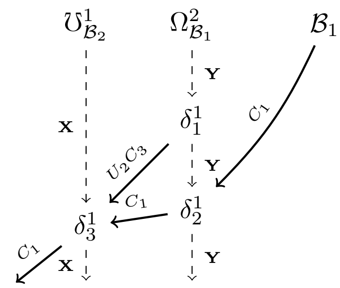
$$\delta^1_2(\mathbf{p} \boxtimes \mathbf{q}, U_1) = U_1 \otimes (\mathbf{p} \boxtimes \mathbf{q}) \tag{11.10}$$

for all choices of $\mathbf{p} \in \{\mathbf{X}, \mathbf{Z}\}$ and $\mathbf{q} \in \{\mathbf{Y}, \mathbf{Z}\}$: when $\mathbf{p} \boxtimes \mathbf{q} = \mathbf{X} \boxtimes \mathbf{Z}$, both sides vanish; for the other cases:

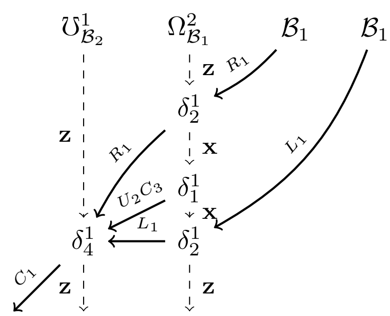


With these computations in hand, it follows that $\mho^1 \boxtimes \Omega^2 \boxtimes \mathcal{K} = \mathcal{K}$, so $\mho^1 \boxtimes \Omega^2 \simeq \mathrm{Id}$.

If the leftmost strand is upwards, then instead of Equation (11.10), we claim that $\delta^1_2(\mathbf{X}\boxtimes\mathbf{q},C_1)=C_1\otimes(\mathbf{X}\boxtimes\mathbf{q})$ for $\mathbf{q}\in\{\mathbf{Y},\mathbf{Z}\}$; for example,



We also claim that $\delta^1_3(\mathbf{Z}\boxtimes\mathbf{Z},R_1,L_1)=C_1\otimes\mathbf{Z}\boxtimes\mathbf{Z}$, since



Again, it is straightforward to verify now that $\mho^1\boxtimes\Omega^2\simeq\mathrm{Id}$. □

11.4. **The invariance proof.** We can now assemble the pieces to prove Theorem 11.5; and tie it with the discussion from the introduction.

Proof of Theorem 11.5. The tensor product description gives a chain complex thanks to a combination of Proposition 3.20, and finally Proposition 9.6 when attaching the final stage, to get a filtered complex. Combining Lemma 11.4 with the invariance of the bimodules under bridge moves (Theorem 6.1, Lemmas 11.6, 11.7, 11.8, 11.10) shows that (Alexander-)filtered chain homotopy type of $\mathcal{C}(K)$ depends only on the pointed knot diagram for K, and the orientation type of the global minimum, i.e. LR or RL of Lemma 11.4. (Keep here in mind that homotopy equivalences of DA bimodules induce homotopy equivalences under tensor product.) Choose for definiteness the orientation type LR.

Next, we appeal to Proposition 11.2 to see that we have a knot invariant.

Invariance under Reidemeister (1) moves follows from the easily checked relation $\mathcal{P}_c\boxtimes\Omega^c_r\simeq\Omega^c_\ell$. (Here Ω^c_r and Ω^c_ℓ are bimodules associated to maxima, with opposite orientations on the new strand.) Invariance under Reidemeister (2) and (3) moves now follows from the braid relations, Theorem 6.1.

Arguing in the same manner for RL, we obtain another conceivably different knot invariant. A straightforward computation shows that

$$t\mho_{\mathcal{B}(2,1,\{2\})}=t\mho_{\mathcal{B}(2,1,\{1\})}\boxtimes{}^{\mathcal{B}(2,1,\{1\})}\mathcal{P}^1_{\mathcal{B}(2,1,\{2\})}.$$

It follows that both RL and LR give the same knot invariant.

The Alexander filtration, takes values in the H^1 of the knot modulo a neighborhood of the global minimum, can be turned into a rational number by evaluating against the orientation class of the knot. This evaluation is computed by the local formula from Figure 2 according to Proposition 5.16. Since it is the exponent of t in the contribution of the corresponding Kauffman state to the Alexander polynomial (compare [6]), it follows that the Alexander filtration takes values in $\mathbb{Z}$. □

Corollary 11.11. *The homology of $\widehat{C}(\mathcal{D})$ is a bigraded invariant of the underlying oriented knot, whose Euler characteristic agrees with the Alexander polynomial.*

Proof. Invariance follows from Theorem 11.5, since the homology of the associated graded object is invariant under filtered homotopy equivalences. The Euler characteristic computation follows from the correspondence between the generators and Kauffman states, and the observation that that $(-1)^{M(\mathbf{x})}t^{A(\mathbf{x})}$ (see Figure 2) is the monomial associated to a Kauffman state in the computation of the Alexander polynomial from [6]. □

It is a straightforward matter to go from the filtered homotopy type to the invariant over a polynomial algebra described in the introduction; see [16, Chapter 14]. Explicitly, replace the filtered module $t\mho$ with an Alexander graded module $t\mho^-$ over a polynomial algebra $\mathbb{F}[v]$. For $\mathcal{B}(2,1\{2\})$, replace Equation (9.7) with

$$\mathbf{X}\cdot L_1 = v\cdot \mathbf{Y} \qquad \mathbf{Y}\cdot R_1 = v\cdot\mathbf{X} \qquad \mathbf{X}\cdot C_2 = \mathbf{Y}\cdot C_2 = 0$$

(and making the analogous construction for $\mathcal{B}(2,1,\{1\})$). In this construction the base algebra now is $\mathbb{F}[v]$, and v is given gradings $A(v)=1/2$ and $M(v)=0$. Replacing $t\mho$ with $t\mho^-$ and forming the same tensor product as before, we arrive at a bigraded complex C^- over $\mathbb{F}[v]$. Since the Alexander grading of generators is integral, we can restrict this to a complex over $\mathbb{F}[U]$ with $U=v^2$. According to Theorem 11.5, the homology of the associated graded object, thought of as a bigraded module $H'(\vec{K})$ over $\mathbb{F}[U]$, is an oriented knot invariant. Letting $H^-_d(\vec{K},s) = H'_{d-2s}(\vec{K},-s)$, we arrive at the bigraded invariant from the introduction. Setting $v=0$ (and so $U=0$) clearly recaptures $\widehat{C}$. Since U drops Alexander grading by one and Maslov grading by 2 on $H^-(K)$, Equation (1.1) follows from the fact that the graded Euler characteristic of $\widehat{C}$ is the Alexander polynomial.

Remark 11.12. *We have described $t\mho$ as having two generators. In fact, to define our knot invariants, it suffices to work with the submodule $t\mho\cdot\mathbf{I}_{\{1\}}$, since the outputs of our D modules are all contained in the subalgebras of $\mathcal{B}(m,k,\mathcal{S})$ $\left(\sum_{\mathbf{x}|0,m\not\in\mathbf{x}}\mathbf{I}_{\mathbf{x}}\right)\cdot\mathcal{B}(m,k,\mathcal{S})\cdot\left(\sum_{\mathbf{x}|0,m\not\in\mathbf{x}}\mathbf{I}_{\mathbf{x}}\right)$.*

12. First properties

We will discuss efficient computations of these invariants in [19]; but there are some easy computations that can be done readily by hand. As a trivial example, the unknot has one-dimensional $\widehat{H}$, supported in bigrading $(0,0)$, since it has a diagram with a single Kauffman state in it. More generally:

Proposition 12.1. *If K is an alternating knot, then $\widehat{H}(K)$ is determined by the signature $\sigma = \sigma(K)$ of K and the symmetrized Alexander polynomial $\Delta_K(t) = \sum_i a_i \cdot t^i$ by $\widehat{H}_d(K,s) = \mathbb{F}^{|a_s|}$ if $d = s + \frac{\sigma}{2}$, and 0 otherwise.*

Proof. As in [22], this result is simply a consequence of the local contributions to the Alexander and Maslov gradings (pictured in Figure 2). Those formulas show that for an alternating diagram $M - A = \frac{\sigma}{2}$. The rest now follows from the interpretation of the Alexander polynomial in terms of Kauffman states [6]. □

We list a few of the properties that follow immediately from the constructions from this paper. The methods of this paper are best suited for $\widehat{H}$; analogous results for H^- will be established in a follow-up paper [19].

Proposition 12.2. *If K and K' are mirror knots, then $\widehat{H}_d(K,s) \cong \widehat{H}_{-d}(K',-s)$.*

Proof. According to Lemma 10.2, the complex of $\widehat{C}(K')$ is computed by tensoring together the opposites of the various bimodules used to compute $\widehat{C}(K)$. It follows that $\widehat{C}(K)$ and $\widehat{C}(K')$ are dual complexes. Since we are working over the field $\mathbb{F}$, the proposition now follows from the universal coefficient theorem. □

Proposition 12.3. *Let K_1 and K_2 be two knots. Then, $\widehat{H}(K_1 \# K_2)$ is the bi-graded tensor product of $\widehat{H}(K_1)$ and $\widehat{H}(K_2)$.*

Before turning to the proof, we start with some more general properties.

Let $\mathcal{D}$ be a knot diagram with a single global minimum, oriented up and to the left. Let $\mathcal{B} = \mathcal{B}(2,1\{1\})$ and let ${}^{\mathcal{B}}\mathcal{C}(\mathcal{D})$ denote the type D structure associated to the diagram with minimum removed.

Lemma 12.4. *Let $\mathcal{B} = \mathcal{B}(2,1,\{1\})$ and $\mathcal{B}' = \mathbf{I}_{\{1\}} \cdot \mathcal{B} \cdot \mathbf{I}_{\{1\}}$. The output algebra for ${}^{\mathcal{B}}\mathcal{C}(\mathcal{D})$ is contained in $\mathcal{B}'$; i.e. if ${}^{\mathcal{B}}[i]_{\mathcal{B}'}$ denotes the bimodule associated to the inclusion $i\colon \mathcal{B}' \to \mathcal{B}$, then we have a type D structure ${}^{\mathcal{B}'}\mathcal{C}(\mathcal{D})$ so that ${}^{\mathcal{B}}[i]_{\mathcal{B}'} \boxtimes {}^{\mathcal{B}'}\mathcal{C}(\mathcal{D}) = {}^{\mathcal{B}}\mathcal{C}(\mathcal{D})$.*

Proof. For m,k with $0 \leq k \leq m+1$ and $\mathcal{S} \subset \{1,\dots,m\}$, let $\mathcal{B}'(m,k,\mathcal{S}) \subset \mathcal{B}(m,k,\mathcal{S})$, be the subalgebra $\left(\sum_{\mathbf{x}\big|0,m\not\in\mathbf{x}} \mathbf{I}_{\mathbf{x}}\right) \cdot \mathcal{B}(m,k,\mathcal{S}) \cdot \left(\sum_{\mathbf{x}\big|0,m\not\in\mathbf{x}} \mathbf{I}_{\mathbf{x}}\right)$. We claim that all of the DA bimodules Ω^c, $\mho^c$, $\mathcal{P}^i$, $\mathcal{N}^i$ have the property that their restriction to $\mathcal{B}' \subset \mathcal{B}$ (with appropriate decorations) have their output algebras in the corresponding $\mathcal{B}' \subset \mathcal{B}$. Moreover, the global maximum has output algebra contained in $\mathcal{B}'$ (c.f. Subsection 8.1). □

Recall that $\mathcal{B}' = \mathbf{I}_{\{1\}} \cdot \mathcal{B}(2,1,\{1\}) \cdot \mathbf{I}_{\{1\}}$ is $\mathbb{F}[C_1,U_1,U_2]/C_1^2 = 0$, with $dC_1 = U_1$, with gradings

$$(w_1(U_1), w_2(U_1)) = (1,0), \quad (w_1(C_1), w_2(C_1)) = (1,0), \quad (w_1(U_2), w_2(U_2)) = (0,1).$$

It has a subalgebra $\mathcal{B}'' \subset \mathcal{B}'$ isomorphic to $\mathbb{F}[E_1,U_2]/E_1^2 = 0$ (with vanishing differential), and $(w_1(E_1), w_2(E_1)) = (1,1)$ and $(w_1(U_2), w_2(U_2)) = (0,1)$. The subalgebra is specified by $E_1 = C_1 U_2$.

Lemma 12.5. *With $\mathcal{B} = \mathcal{B}(2,1,\{1\})$, the type D module ${}^{\mathcal{B}}\mathcal{C}(\mathcal{D})$ is homotopy equivalent to a type D module over $\mathcal{B}'' \subset \mathcal{B}$.*

Proof. Use Lemma 12.4 to restrict to $\mathcal{B}'$. This result then follows from homological perturbation theory, since the inclusion $\mathcal{B}'' \subset \mathcal{B}'$ is a homotopy equivalence. □

Proof of Proposition 12.3. Consider a connected sum diagram for K_1 and K_2, where the connected sum region is taken to be the global minimum and the next minimum above it, as pictured in Figure 48.

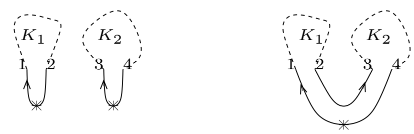


FIGURE 48. **Connected sums.**

The invariant associated to the disjoint union of K_1 and K_2 (missing the two minima) is a type D structure $\mathcal{C}(K_1 \cup K_2)$ over $\mathcal{B}(4,2,\{1,3\})$. Arguing as in Lemma 12.4, the output algebra of this D module is contained in $\mathbf{I}_{\{1,3\}} \cdot \mathcal{B}(4,2,\{1,3\}) \cdot \mathbf{I}_{\{1,3\}}$. Note that there is a natural identification

$$\left(\mathbf{I}_{\{1\}} \cdot \mathcal{B}(2,1,\{1\}) \cdot \mathbf{I}_{\{1\}}\right) \otimes \left(\mathbf{I}_{\{1\}} \cdot \mathcal{B}(2,1,\{1\}) \cdot \mathbf{I}_{\{1\}}\right) \cong \mathbf{I}_{\{1,3\}} \cdot \mathcal{B}(4,2,\{1,3\}) \cdot \mathbf{I}_{\{1,3\}};$$

and under this identification, $\mathcal{C}(K_1 \cup K_2) = \mathcal{C}(K_1) \otimes \mathcal{C}(K_2)$.

Consider the type A structure N over $\mathcal{B}(4,2,\{1,3\})$ with one generator $\mathbf{X}$ satisfying $\mathbf{X} \cdot \mathbf{I}_{\{1,3\}} = \mathbf{X}$, and trivial action by all other pure algebra elements. Clearly,

$$N \boxtimes (\mathcal{C}(K_1 \cup K_2)) \cong \widehat{C}(K_1) \otimes \widehat{C}(K_2) \tag{12.1}$$

Consider next the type A structure P over $\mathcal{B}(4,2,\{1,3\})$ obtained by tensoring together a minimum with a global minimum to obtain the type A structure pictured on the right in Figure 48. We restrict to the idempotent type $P \cdot \mathbf{I}_{\{1,3\}}$.

It is easy to see that P has actions $m_2(\mathbf{X}, \mathbf{I}_{\{1,3\}}) = \mathbf{X}$. Also, if $m_{\ell+1}(\mathbf{X}, a_1, \dots, a_\ell) \neq 0$, then $\sum_{i=1}^{\ell} w_1(a_i) = 0 = \sum_{i=1}^{\ell} w_4(a_i)$ and

$$\sum_{i=1}^{\ell} w_2(a_i) = \sum_{i=1}^{\ell} w_3(a_i). \tag{12.2}$$

Let $\mathcal{B}' = \mathbf{I}_{\{1,3\}} \cdot \mathcal{B}(4,2,\{1,3\}) \cdot \mathbf{I}_{\{1,3\}}$, and $\mathcal{B}''$ be the subalgebra isomorphic to $\mathbb{F}[E_1, E_3, U_2, U_4]/E_1^2 = E_2^2 = 0$, were $E_1 = C_1 U_2$ and $E_2 = C_3 U_4$. Lemma 12.5 gives type D structures ${}^{\mathcal{B}''(2,1,\{1\})}Q_1$ and ${}^{\mathcal{B}''(2,1,\{1\})}Q_2$ so that

$${}^{\mathcal{B}(2,1,\{1\})}\mathcal{C}(K_1) \simeq Q_1 \qquad \text{and} \qquad {}^{\mathcal{B}(2,1,\{1\})}\mathcal{C}(K_2) \simeq Q_2;$$

and so ${}^{\mathcal{B}''}(Q_1 \otimes Q_2) \simeq {}^{\mathcal{B}'}\mathcal{C}(K_1 \cup K_2)$.

Since $w_1(E_1) > 0$, we see that the action by E_1 on P is trivial. Next, we claim that any output algebra element in Q_1 of the form U_2^t with $t \geq 0$ must pair with some action on P with $w_2 > 0$; but such an action must also involve an output from Q_2 with $w_3 > 0$ (by Equation (12.2)), and so it must come from some multiple of

E_3. But any sequence containing a non-trivial factor of E_3 in it acts trivially on P, since $w_4(E_3) > 0$. Similarly, outputs from Q_2 containing a non-trivial multiple of E_3 or U_4 act trivially since w_4 of both of those algebra elements are trivial.

We have shown that the outputs from $Q_1 \otimes Q_2$ giving non-zero differentials when paired with P consist only of the trivial output (1), showing that

$$N \boxtimes (Q_1 \otimes Q_2) \cong P \boxtimes (Q_1 \otimes Q_2) = \widehat{C}(K_1 \# K_2).$$

Combining this with Equation (12.1), we get an isomorphism of chain complexes $\widehat{C}(K_1 \# K_2) \cong \widehat{C}(K_1) \otimes \widehat{C}(K_2)$; and the proposition follows readily. □

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