A MOMENT METHOD FOR INVARIANT ENSEMBLES

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Abstract. We introduce a new moment method in Random Matrix Theory specifically tailored to the spectral analysis of invariant ensembles. Our method produces a classification of invariant ensembles which exhibit a spectral Law of Large Numbers and yields an explicit description of the limiting eigenvalue distribution when it exists. We discuss the future development and applications of this new moment method.

1. Introduction

Random Matrix Theory (RMT) is one of the most active research topics in contemporary probability theory. The goal of the subject is natural and compelling: given a random matrix, describe the statistical behavior of its eigenvalues. In addition to its intrinsic mathematical appeal, interest in RMT has been spurred by the scientific hypothesis that spectra of large random matrices yield models for complex systems comprised of many highly correlated components. Such systems are ubiquitous in mathematics and nature—particular examples include zeros of L-functions [7], energy levels of atomic nuclei [32], and arrival times of New York City subway trains [13]—but are not within the purview of classical scalar and vector-valued probability, whose limit theorems describe systems built from weakly correlated components.

The universe of random matrix models is too large to be studied all at once, and in practice it is parceled out into various paradigms. Among the most prominent of these is the *invariant paradigm*, which is populated by statistical ensembles of three types. A *real invariant ensemble* is a sequence

$$X^{(N)} = \begin{bmatrix} & \vdots & \\ \cdots & X_{ij}^{(N)} & \cdots \\ & \vdots & \end{bmatrix}_{i,j=1}^{N} \qquad N = 1, 2, 3, \ldots,$$

of random real selfadjoint matrices such that, for each $N \in \mathbb{N}$ and any $N \times N$ orthogonal matrix O, the distribution of $X^{(N)}$ coincides with that of $OX^{(N)}O^{-1}$. Similarly, a *complex invariant ensemble* is a sequence of random complex selfadjoint matrices each of which has distribution invariant under conjugation by unitary matrices, and a *quaternionic invariant ensemble* is a sequence of random quaternionic

selfadjoint matrices each of which has distribution invariant under conjugation by symplectic matrices. Conjugation invariance is a physically natural assumption (all coordinate systems are equivalent) which also has a natural probabilistic interpretation: it means that the distribution of $X^{(N)}$ is uniform conditional on its eigenvalues

$$E_1^{(N)} \geq \cdots \geq E_N^{(N)}.$$

Invariant ensembles of random matrices have been the focus of intense study for decades; see [8, 9, 11, 14] and references therein. However, these investigations have almost exclusively focused on a special class of invariant ensembles: those in which the distribution of $X^{(N)}$ is absolutely continuous with respect to Lebesgue measure, with density proportional to a function of the form $\exp(-\frac{\beta}{2}N \operatorname{Tr} V(X))$. Here Tr denotes the matrix trace, V is a sufficiently well-behaved real-valued function, and β is the *Dyson index*, which is equal to $1, 2$, or 4 according to whether the ensemble is real, complex, or quaternionic. The fixation on this special class of invariant ensembles stems from the fact that the joint density of eigenvalues is known explicitly, being proportional to $\exp(-\frac{\beta}{2}N^2\mathcal{H})$, where $\mathcal{H}$ is the Hamiltonian

$$\mathcal{H}(E_1, \ldots, E_N) = \frac{1}{N}\sum_{i=1}^{N} V(E_i) - \frac{1}{N^2}\sum_{i \neq j} \log|E_i - E_j|.$$

This formula furnishes a physical interpretation of the spectrum as a system of two-dimensional electrostatic charges living on a wire, with V playing the role of a confining potential. This system may be analyzed directly using a variety of powerful analytic techniques, leading to a detailed understanding of its macroscopic and microscopic statistics in the large N limit; see [9, 10] for the state of the art.

The Dyson-type ensembles form a small island in the vast sea of all invariant ensembles. In this note, we suggest a new approach to the spectral analysis of general invariant ensembles, the implementation of which will both broaden and deepen current understanding of this important paradigm. This new approach is based on a simple observation: the distribution of any conjuagation-invariant random selfadjoint matrix is completely determined by the joint distribution of its diagonal matrix elements. This fundamental consequence of invariance seems to have been overlooked; certainly, it has never been exploited. Nevertheless, it is easily seen from the Fourier transform,

$$A \mapsto \mathbb{E}\big[e^{\mathbf{i}\operatorname{Tr} AX^{(N)}}\big].$$

Indeed, diagonalizing the selfadjoint matrix A, cyclic invariance of the trace and conjugation invariance of $X^{(N)}$ immediately imply that

$$\mathbb{E}\big[e^{\mathbf{i}\operatorname{Tr} AX^{(N)}}\big] = \mathbb{E}\big[e^{\mathbf{i}(a_1 X_{11}^{(N)} + \cdots + a_N X_{NN}^{(N)})}\big],$$

where $a_1, \ldots, a_N$ is any enumeration of the eigenvalues of A. Thus, everything one could hope to know about $X^{(N)}$ is encoded in the joint distribution of the real random variables $X_{11}^{(N)}, \ldots, X_{NN}^{(N)}$, which are identically distributed and exchangeable. This reduces the spectral analysis of invariant ensembles to extracting eigenvalue statistics from the joint distribution of diagonal matrix elements.

2. Statement of Results

For a general selfadjoint ensemble $X^{(N)}$, in the absence of an analytical description of the eigenvalues, one must interact with the spectrum via an appropriate system of observables. The random variables

$$p_d^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \left(E_i^{(N)}\right)^d, \quad d = 1, 2, 3, \ldots$$

which generate the algebra of symmetric polynomial functions of the eigenvalues, are a natural choice. Indeed, they are precisely the moments of the empirical eigenvalue distribution of $X^{(N)}$. In RMT, the term *moment method* refers to any technique which relates the distribution of spectral moments to the joint distribution of matrix elements. The prototypical moment method was introduced by Wigner in the 1950s, and is still widely used today; see [32]. Wigner's moment method applies to ensembles of random selfadjoint matrices whose matrix elements enjoy a high degree of independence; these *Wigner ensembles* constitute a random matrix paradigm which is, in a sense, orthogonal to the invariant paradigm. Wigner used his moment method to show that, under appropriate hypotheses, the spectral moments $p_d^{(N)}$ of a Wigner ensemble $X^{(N)}$ tend to deterministic limits $p_d^{(\infty)}$ as $N \to \infty$. This is the random matrix version of the Law of Large Numbers.

In this note, we present a new moment method specifically tuned to the invariant paradigm. As explained above, conjugation invariance means that we need only consider the joint distribution of diagonal matrix elements. We will say that an invariant ensemble $X^{(N)}$ is *smooth* if $X_{11}^{(N)}, \ldots, X_{NN}^{(N)}$ admit joint moments of all orders. In this case, thanks to exchangeability, it suffices to consider joint moments indexed by *Young diagrams*, which provide a convenient graphical representation of the unordered decompositions of a number into smaller numbers. For example, the diagram
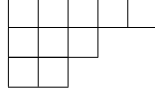
has 10 total cells and three rows of lengths 5, 3, and 2, and it represents the decomposition $10 = 5 + 3 + 2$. To a given Young diagram λ with $|\lambda| = d$ cells and $\ell(\lambda) = r$ rows $\lambda_1, \ldots, \lambda_r$, we associate the degree d joint moment of $X_{11}^{(N)}, \ldots, X_{rr}^{(N)}$ defined by

$$\mathbb{E}\big[\, \underbrace{X_{11}^{(N)} \cdots X_{11}^{(N)}}_{\lambda_1}, \ldots, \underbrace{X_{rr}^{(N)} \cdots X_{rr}^{(N)}}_{\lambda_r} \,\big],$$

and denote this joint moment $m_\lambda^{(N)}$. For example, in the case of the diagram above, we have

$$m_\lambda^{(N)} = \mathbb{E}\Big[\big(X_{11}^{(N)}\big)^5 \big(X_{22}^{(N)}\big)^3 \big(X_{33}^{(N)}\big)^2\Big].$$

Each diagonal matrix element of $X^{(N)}$ can be decomposed as

$$X_{ii}^{(N)} = \sum_{j=1}^{N} U_{ij}^{(N)} E_j^{(N)} \overline{U}_{ij}^{(N)},$$

where $U^{(N)} = \left[U_{ij}^{(N)} \right]_{i,j=1}^{N}$ is a random matrix whose distribution is the Haar probability measure on $\mathrm{O}(N), \mathrm{U}(N)$, or $\mathrm{Sp}(N)$ according to whether $X^{(N)}$ is real, complex, or quaternionic. This decomposition of $X_{ii}^{(N)}$ can be substituted into the definition of $m_{\lambda}^{(N)}$, and the eigenvector information can be "integrated out" using the *Weingarten Calculus*, a unified set of tools for the evaluation of polynomial integrals on compact topological groups; see [4, 6, 5]. What remains is a presentation of $m_{\lambda}^{(N)}$ as the expectation of a certain polynomial in the spectral moments of $X^{(N)}$. This polynomial becomes more tractable if one applies a basic statistical principle, which may be traced to the nineteenth century astronomer Thorvald N. Thiele [19]: cumulants package the same information as moments in a more useful way. We thus trade $m_{\lambda}^{(N)}$ for the corresponding joint cumulant $c_{\lambda}^{(N)}$. For example, with λ as in the previous example, $c_{\lambda}^{(N)}$ is the coefficient of

$$\frac{a_1^5}{5!} \frac{a_2^3}{3!} \frac{a_3^2}{2!}$$

in the log of the characteristic function of the random vector $(X_{11}^{(N)}, X_{22}^{(N)}, X_{33}^{(N)})$. Execution of this strategy leads to the following result, which gives necessary and sufficient conditions for the emergence of a spectral Law of Large Numbers within the class of smooth invariant ensembles.

Theorem 2.1. *For any smooth invariant ensemble $X^{(N)}$, the following are equivalent:*

(1) *For each positive integer d, the random variable $p_d^{(N)}$ converges, in probability, to a constant $p_d^{(\infty)}$;*

(2) *For each Young diagram λ, the number $N^{|\lambda|-1} c_{\lambda}^{(N)}$ converges to a limit $c_{\lambda}^{(\infty)}$, and this limit vanishes if $\ell(\lambda) > 1$.*

Recalling that vanishing of mixed cumulants characterizes independence, Theorem 2.1 says that the moments of the empirical eigenvalue distribution of $X^{(N)}$ converge in probability to deterministic limits precisely when each diagonal element converges rapidly to a constant, and distinct diagonal elements rapidly decouple.

A detailed proof of Theorem 2.1 implementing the strategy described above will appear in [21]. This proof yields additional information, namely a precise relationship between the numerical sequences $p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \ldots$ and $c_1^{(\infty)}, c_2^{(\infty)}, c_3^{(\infty)}, \ldots$, where $c_d^{(\infty)}$ denotes $c_{\lambda}^{(\infty)}$ for λ a single row of d cells. It turns out that this relationship may be concisely described in the language of *Free Probability Theory*, a highly noncommutative probability theory developed by Voiculescu to address a famous unsolved problem in the theory of von Neumann algebras; see [31]. A fundamental tool in Free Probability is a bijective transform on sequences which plays the same role as the moment-to-cumulant transform in classical probability. More precisely, the R-transform of a given sequence $(m_1, m_2, m_3, \ldots)$ is the sequence $(f_1, f_2, f_3, \ldots) = R(m_1, m_2, m_3, \ldots)$ whose terms, known as *free cumulants*, are defined implicitly by

$$(1) \qquad m_d = \sum_{\pi \in \mathrm{NC}(d)} \prod_{B \in \pi} f_{|B|},$$

where the summation is over the lattice of noncrossing partitions of $\{1, \ldots, d\}$, and the product is over the blocks B of $\pi \in \mathrm{NC}(d)$. For more details on the R-transform and its role in Free Probability Theory, see see [23, 24, 25].

Theorem 2.2. *Let $X^{(N)}$ be a smooth invariant ensemble. Then*

$$R(p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \ldots) = \left(\frac{\gamma^0}{0!} c_1^{(\infty)}, \frac{\gamma^1}{1!} c_2^{(\infty)}, \frac{\gamma^2}{2!} c_3^{(\infty)}, \ldots \right),$$

where $\gamma = \beta/2$ is one half the Dyson index.

3. Applications

Theorems 2.1 and 2.2 together form a "skeleton key" result which can be used to recover many seemingly disparate theorems in RMT—and even some which seem unrelated to RMT—in a unified way. We illustrate this via several examples.

3.1. **Gaussian ensembles.** Let $X^{(N)}$ be an invariant ensemble such that $X_{11}^{(N)}, \ldots, X_{NN}^{(N)}$ are independent Gaussians of mean c_1 and variance $c_2 N^{-1}$, with c_1, c_2 constants. By Gaussianity, all pure cumulants of order higher than two vanish. Moreover, mixed cumulants are identically zero by independence. We thus have existence of the limits $c_\lambda^{(\infty)}$ for all Young diagrams λ. More precisely, we have

$$c_1^{(\infty)} = c_1, \ c_2^{(\infty)} = c_2,$$

and all other $c_\lambda^{(\infty)}$ are zero. Theorem 2.1 thus implies that each spectral moment $p_d^{(N)}$ of $X^{(N)}$ converges in probability to a deterministic limit $p_d^{(\infty)}$. Theorem 2.2 yields the R-transform of the limiting moment sequence

$$R(p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \ldots) = (c_1, \gamma c_2, 0, 0, \ldots).$$

There is a unique probability measure $\mu^{(\infty)}$ with this R-transform: the Wigner semicircle distribution with mean c_1 and variance γc_2. Thus we recover the *Wigner semicircle law* for Gaussian ensembles, which first appeared in [32]. For an extensive discussion of this result and its central role in RMT, we refer the reader to [8, 9, 10, 11].

3.2. **Wishart ensembles.** For each N, let $Z^{(N)}$ be a $p \times N$ random matrix whose matrix entries are iid Gaussians with mean 0 and variance αN^{-1}, with α constant. One then has an affiliated invariant ensemble, known as a *Wishart ensemble*, whose N^{th} member is $X^{(N)} = (Z^{(N)})^* (Z^{(N)})$. Mixed cumulants of the diagonal matrix elements $X_{11}^{(N)}, \ldots, X_{NN}^{(N)}$ are identically zero by independence. Moreover, it is easy to compute the pure cumulants of a single diagonal element: using the Wick formula, one obtains

$$c_d^{(N)} = p \gamma^{1-d} \left(\frac{\alpha}{N} \right)^d (d-1)!, \quad d \in \mathbb{N}.$$

Suppose now that $p = p_N$ grows with N in such a way that the limit

$$c = \lim_{N \to \infty} \frac{p}{N}$$

exists. Then, we have

$$c_d^{(\infty)} = \lim_{N \to \infty} N^{d-1} c_d^{(N)} = c \gamma^{1-d} \alpha^d (d-1)!.$$

Theorem 2.1 thus implies that each spectral moment $p_d^{(N)}$ of $X^{(N)}$ converges in probability to a deterministic limit $p_d^{(\infty)}$. Theorem 2.2 yields the R-transform of the limiting moment sequence:

$$R(p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \dots) = (c\alpha, c\alpha^2, c\alpha^3, \dots).$$

There is a unique probability measure $\mu^{(\infty)}$ with this R-transform: the Marčhenko-Pastur distribution with rate c and jump size α. This yields the famous *Marčhenko-Pastur law*, which first appeared in [20]. Gaussian Wishart ensembles are important in multivariate statistics, where they first appeared in [33]. In Free Probability, the Marčhenko-Pastur law arises naturally as a free version of the Poisson limit theorem, see [23, 25] for more on this.

3.3. **Sum of independent ensembles.** Let $X^{(N)}$ and $Y^{(N)}$ be independent smooth invariant ensembles, and suppose it is known that the spectral moments of each converge in probability to deterministic limits:

$$p_d(X^{(N)}) \to x_d \quad \text{and} \quad p_d(Y^{(N)}) \to y_d.$$

From this data, we obtain a new smooth invariant ensemble defined by setting $Z^{(N)} := X^{(N)} + Y^{(N)}$ for each $N \in \mathbb{N}$. Given a Young diagram λ, the relationship between the corresponding joint cumulants of the diagonal matrix elements of $Z^{(N)}, X^{(N)}, Y^{(N)}$ is, by independence, simply

$$c_\lambda(Z^{(N)}) = c_\lambda(X^{(N)}) + c_\lambda(Y^{(N)}).$$

Now, since the spectral moments of $X^{(N)}$ and $Y^{(N)}$ are known to converge to deterministic limits, Theorem 2.1 implies existence of the limits

$$c_\lambda(X) := \lim_{N \to \infty} N^{|\lambda|-1} c_\lambda(X^{(N)}),$$
$$c_\lambda(Y) := \lim_{N \to \infty} N^{|\lambda|-1} c_\lambda(Y^{(N)}),$$

with these limits vanishing if $\ell(\lambda) > 1$. We thus obtain

$$c_\lambda(Z) = \lim_{N \to \infty} N^{|\lambda|-1} c_\lambda(Z^{(N)}) = c_\lambda(X) + c_\lambda(Y),$$

so that by Theorem 2.1 we have convergence in probability of the spectral moments $p_d(Z^{(N)})$ to deterministic limits z_d. Moreover, by Theorem 2.2,

$$R(z_1, z_2, z_3, \dots) = R(x_1, x_2, x_3, \dots) + R(y_1, y_2, y_3, \dots).$$

Thus we recover the fundamental fact that the R-transform of the sum of two independent invariant ensembles is, asymptotically, the sum of their R-transforms. This fact, which is analogous to the classical statement that the log-Fourier transform of a sum of two independent scalar random variables is the sum of their log-Fourier transforms, was discovered by Voiculescu in his first paper linking random matrices to Free Probability Theory [30]. In Free Probability, the R-transform plays the role of the log-Fourier transform in the sense that it linearizes the addition of free random variables; that this holds for large independent random matrices is a manifestation of the more general fact that independent invariant ensembles are *asymptotically free*, see [22, 24, 25].

3.4. **Compressed ensembles.** Let $X^{(N)}$ be a smooth invariant ensemble, and suppose it is known that the spectral moments $p_d^{(N)}$ of $X^{(N)}$ converge in probability to deterministic limits $p_d^{(\infty)}$. For any choice of $t \in (0,1)$, one obtains a new smooth invariant ensemble whose N^{th} member $X_{\lfloor tN \rfloor}^{(N)}$ is the $\lfloor tN \rfloor \times \lfloor tN \rfloor$ principal submatrix of $X^{(N)}$. Let $p_{d,t}^{(N)}$ denote the spectral moments of $X_{\lfloor tN \rfloor}^{(N)}$. For any Young diagram λ, the corresponding joint cumulants $c_\lambda^{(N)}$ and $c_{\lambda,t}^{(N)}$ of the diagonal matrix elements of $X^{(N)}$ and $X_{\lfloor tN \rfloor}^{(N)}$ are well-defined and equal for $\lfloor tN \rfloor \geq \ell(\lambda)$. By Theorem 2.1, the limit

$$c_\lambda^{(\infty)} = \lim_{N \to \infty} N^{|\lambda|-1} c_\lambda^{(N)}$$

exists and vanishes if $\ell(\lambda) > 1$. Moreover,

$$\begin{aligned} c_{\lambda,t}^{(\infty)} &= \lim_{N \to \infty} (\lfloor tN \rfloor)^{|\lambda|-1} c_{\lambda,t}^{(N)} \\ &= t^{|\lambda|-1} \lim_{N \to \infty} N^{|\lambda|-1} c_\lambda^{(N)} \\ &= t^{|\lambda|-1} c_\lambda^{(\infty)}. \end{aligned}$$

Thus, Theorem 2.1 implies that each $p_{d,t}^{(N)}$ converges to a deterministic limit $p_{d,t}^{(\infty)}$, and Theorem 2.2 yields

$$R(p_{1,t}^{(\infty)}, p_{2,t}^{(\infty)}, p_{3,t}^{(\infty)}, \dots) = \left(\frac{(\gamma t)^0}{0!} c_1^{(\infty)}, \frac{(\gamma t)^1}{1!} c_2^{(\infty)}, \frac{(\gamma t)^2}{2!} c_3^{(\infty)}, \dots \right).$$

This reproduces the so-called *free compression* principle for corners of random matrices, see [23].

3.5. **Random lozenge tilings.** Let

$$(b_1^{(N)}, \dots b_N^{(N)}), \quad N = 1, 2, 3, \dots$$

be a sequence of random integer vectors such that $b_1^{(N)} > \cdots > b_N^{(N)}$ for each $N \in \mathbb{N}$. This random data gives rise to a sequence $\Omega^{(N)}$ of random planar domains via the following construction. Fix a coordinate system in the plane whose axes meet at a $120°$ angle. We specify $\Omega^{(N)}$ by constructing its boundary, which consists of two piecewise linear components, one deterministic and one random. The deterministic component of $\partial\Omega^{(N)}$ is simply the horizontal axis in the plane. The random component is built in three steps. First, construct the line parallel to the lower boundary passing through the point $(0, N)$. Second, affix N outward-facing unit triangles to this line such that the midpoints of their bases have horizontal coordinates $b_1^{(N)} > \cdots > b_N^{(N)}$. Finally, erase the bases of these triangles. We will refer to $\Omega^{(N)}$ as the *sawtooth domain* of rank N with boundary conditions $(b_1^{(N)}, \dots, b_N^{(N)})$.

A *lozenge* is a unit rhombus in the plane whose sides are parallel to one of the coordinate axes, or to the line bisecting the obtuse angle between them. Lozenges are thus divided into three classes: left-leaning, right-leaning, and vertical. Each instance of the random domain $\Omega^{(N)}$ can be tiled with lozenges in finitely many ways, an example being given in Figure 1. Consequently, we may consider a random tiling $\mathsf{T}^{(N)}$ of $\Omega^{(N)}$ whose distribution is uniform conditional on its boundary conditions. For each instance of $\mathsf{T}^{(N)}$, and any integer $1 \leq k \leq N$, the horizontal line through $(0, k)$ passes through exactly k vertical tiles, as in Figure 1. Moreover, the entire tiling can be reconstructed given only the locations of the vertical tiles.
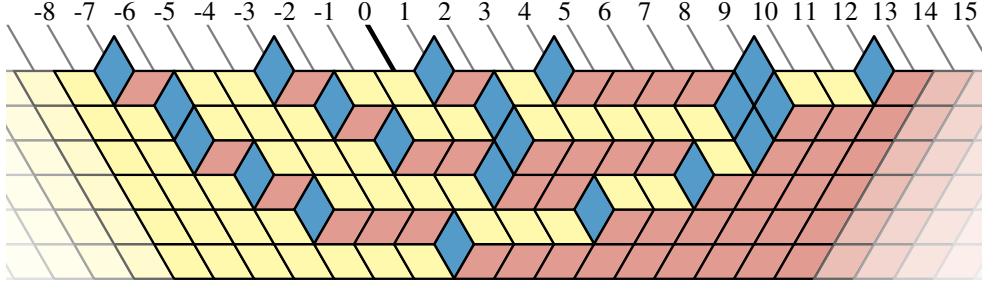
FIGURE 1. A lozenge tiling of a sawtooth domain of rank 6.

The positions of the vertical tiles on adjacent lines interlace, a feature which is reminiscent of Cauchy interlacing for the eigenvalues of a selfadjoint matrix and its principal submatrices.

Let us associate to the random tiling $\mathsf{T}^{(N)}$ a finite sequence of unitarily invariant random Hermitian matrices defined by

$$X^{(k,N)} = U_k \begin{bmatrix} b_{k1}^{(N)} & & \\ & \ddots & \\ & & b_{kk}^{(N)} \end{bmatrix} U_k^{-1}, \quad 1 \le k \le N,$$

where U_k is a Haar-distributed random $k \times k$ unitary matrix and $b_{k1}^{(N)} > \cdots > b_{kk}^{(N)}$ are the horizontal coordinates of the centroids of the vertical tiles on the horizontal line through $(0, k)$. It is tempting to hope that the distribution of $X_k^{(N)}$ coincides with the distribution of the $k \times k$ principal submatrix of $X_N^{(N)}$. In this case, the joint distribution of the diagonal matrix elements $X_{11}^{(k,N)}, \ldots, X_{kk}^{(k,N)}$ of $X^{(k,N)}$ would coincide with the joint distribution of the first k diagonal matrix elements $X_{11}^{(N,N)}, \ldots, X_{kk}^{(N,N)}$ of $X^{(N,N)}$ and we would be in the setting of the previous example. This is not quite the case—the k-dimensional random vector $\left(X_{11}^{(k,N)}, \ldots, X_{kk}^{(k,N)} \right)$ has the same distribution as

$$\left(X_{11}^{(N,N)}, \ldots, X_{kk}^{(N,N)} \right) - \sum_{i=1}^{N-k} (Z_{i1}, \ldots, Z_{ik}),$$

where

$$Z^{(k,N)} = \begin{bmatrix} Z_{11} & \cdots & Z_{1k} \\ \vdots & \cdots & \vdots \\ Z_{N-k,1} & \cdots & Z_{N-k,k} \end{bmatrix}$$

is an $(N-k) \times k$ random matrix independent of $\mathsf{T}^{(N)}$ whose entries Z_{ij} are iid uniformly random samples from the unit interval $[0, 1]$. For a proof of this, see [26]. Suppose it is known that the spectral moments $p_d^{(N)}$ of $N^{-1}X^{(N,N)}$ converge in probability to deterministic limits $p_d^{(\infty)}$, these numbers being the moments of the "limit profile" of $N^{-1}\Omega^{(N)}$. Fix $t \in (0,1)$, and let $p_{d,t}^{(N)}$ denote the spectral moments of $N^{-1}X^{(\lfloor tN \rfloor, N)}$. For any Young diagram λ, the corresponding joint

cumulants $c_\lambda^{(N)}$ and $c_{\lambda,t}^{(N)}$ of $N^{-1}X^{(N,N)}$ and $N^{-1}X^{(\lfloor tN \rfloor, N)}$ are well-defined for $\lfloor tN \rfloor \geq \ell(\lambda)$. By the above, the relation between these joint cumulants is

$$c_{\lambda,t}^{(N)} = c_\lambda^{(N)} - \delta_{1,\ell(\lambda)}(N - \lfloor tN \rfloor)c_{|\lambda|}(N^{-1}Z),$$

where Z is a single uniformly random sample from the unit interval. By Theorem 2.1, the limit

$$c_\lambda^{(\infty)} = \lim_{N \to \infty} N^{|\lambda|-1}c_\lambda^{(N)}$$

exists and vanishes if $\ell(\lambda) > 1$. We thus obtain

$$c_{\lambda,t}^{(\infty)} = \lim_{N \to \infty}(\lfloor tN \rfloor)^{|\lambda|-1}c_{\lambda,t}^{(N)}$$

$$= t^{|\lambda|-1}c_\lambda^{(\infty)} - \delta_{1,\ell(\lambda)}(1-t)t^{|\lambda|-1}c_{|\lambda|}(Z),$$

so that $p_{d,t}^{(N)}$ converges in probability to a deterministic limit $p_{d,t}^{(\infty)}$, by Theorem 2.1. The numbers

$$p_{d,t}^{(\infty)}, \quad d \in \mathbb{N}, \ t \in (0,1),$$

are the moments of the "limit shape" of $N^{-1}\mathsf{T}^{(N)}$. From Theorem 2.2, we obtain

$$R(p_{1,t}^{(\infty)}, p_{2,t}^{(\infty)}, p_{3,t}^{(\infty)}, \dots) = \left(\frac{t^0}{0!}c_1^{(\infty)}, \frac{t^1}{1!}c_2^{(\infty)}, \frac{t^2}{2!}c_3^{(\infty)}, \dots \right)$$

$$- (1-t)\left(\frac{t^0}{0!}u_1, \frac{t^1}{1!}u_2, \frac{t^2}{2!}u_3, \dots \right),$$

where $u_1, u_2, u_3, \dots$ is the cumulant sequence of Z (these numbers are Bernoulli numbers). This provides a description of the limit shape of $N^{-1}\mathsf{T}^{(N)}$ in terms of the limit profile of $N^{-1}\Omega^{(N)}$.

The convergence of random lozenge tilings to a deterministic limit shape has been extensively studied from a statistical physics viewpoint, beginning with the paper [3], which studied random lozenge tilings of a hexagon; these coincide with tilings of a sawtooth domain having all its teeth in two "clumps." A limit shape theorem for lozenge tilings of sawtooth domains whose number of sides remains fixed as $N \to \infty$ was obtained by Petrov [28]. Limit shapes for lozenge tilings of sawtooth domains $\Omega^{(N)}$ whose number of sides grows without bound as N increases, so that the limit profile of $\Omega^{(N)}$ need not be the restriction of Lebesgue measure to finitely many intervals, were treated by Bufetov and Gorin [2]. The fact that limit shapes for random tilings tend to be governed by random matrix limit laws has been known since the seminal work of Johansson [15]; however, the identification of lozenge tilings of sawtooth domains with invariant ensembles at finite N via the above construction, and the subsequent derivation of a limit shape theorem via a moment method analysis of the corresponding invariant ensemble, is new.

4. Conclusion

We have outlined a new moment method in Random Matrix Theory specifically tailored to the invariant ensembles. The method is based on the observation that, if the distribution of a random selfadjoint matrix is invariant under conjugation, then it is completely determined by the joint distribution of its diagonal matrix elements, which form a family of real, identically distributed, exchangeable random variables. When these random variables admit joint moments of all orders, the Weingarten calculus may be used to recognize them as statistics of spectral moments. This

leads to a characterization of smooth invariant ensembles which exhibit a spectral Law of Large Numbers, and a formula for the limiting spectral moments when they exist. The utility of these results was illustrated via a number of significant examples from RMT, as well as an example from 2D statistical physics which is not a priori related to random matrices. There is much more that can be done with this idea; for example, one can address fluctuations of the eigenvalues of smooth invariant ensembles, thereby producing a Central Limit Theorem to accompany Theorem 2.1, or carry out a moment method analysis of the largest eigenvalue in smooth invariant ensembles analogous to Soshnikov's analysis of the spectral edge in Wigner ensembles; see [29]. Another extremely interesting direction is the generalization of this moment method to the multimatrix setting. These directions will be the subject of future work.

Let us conclude with further discussion of how our moment method for invariant ensembles fits into the existing literature. First, in ergodic theory, a beautiful paper of Olshanski and Vershik analyzed *infinite* random selfadjoint matrices with conjugation invariant law; see [27]. It was shown there that the distribution of any such matrix is completely determined by the distribution of a single diagonal matrix element, with distinct diagonal matrix elements being independent. This is the $N = \infty$ version of Theorem 2.1. In the Free Probability community, Collins studied the distribution of the single random variable $X_{11}^{(N)}$ in complex invariant ensembles with deterministic eigenvalues using an early version of Weingarten Calculus, and found a connection with free cumulants and the R-transform; see [4]. Building on Collins' work, the sum of two independent real or complex invariant ensembles was analyzed by Guionnet and Maida, who used large deviation methods to obtain results related to our third example; see [12]. In the Integrable Probability community, Bufetov and Gorin obtained a counterpart of (one direction of) Theorem 2.1 for a certain class of discrete particle systems; see [2]. Their results, which were inspired by work of Borodin, Bufetov, and Olshanski in asymptotic representation theory [1], have in turn been a source of inspiration to us. The work of Bufetov and Gorin is not based in RMT; instead, its technical backbone is the asymptotic analysis of certain families of symmetric functions via determinantal formulas and steepest descent. Using their technology, Bufetov and Gorin obtained a Law of Large Numbers for the random lozenge tilings model discussed above. This lozenge tiling model is a special case of the celebrated dimer model, for which a comprehensive limit theory has been developed by Kenyon, Okounkov, and Sheffield, see [16, 17, 18] and references therein.

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