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Regularity of the Boltzmann equation in convex domains

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Abstract A basic question about regularity of Boltzmann solutions in the presence of physical boundary conditions has been open due to characteristic nature of the boundary as well as the non-local mixing of the collision operator. Consider the Boltzmann equation in a strictly convex domain with the specular, bounce-back and diffuse boundary condition. With the aid of a distance function toward the grazing set, we construct weighted classical C^1 solutions away from the grazing set for all boundary conditions. For the diffuse boundary condition, we construct $W^{1,p}$ solutions for $1 < p < 2$ and weighted $W^{1,p}$ solutions for $2 \leq p \leq \infty$ as well.

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1 Introduction

Boundary effects play an important role in the dynamics of Boltzmann solutions of

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad F(0, x, v) = F_0(x, v), \tag{1.1}$$

where $F(t, x, v)$ denotes the particle distribution at time t, position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$ and F_0 denotes its initial datum. Throughout this paper the collision operator takes the form

$$\begin{aligned} Q(F_1, F_2) &:= Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2) \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u, \omega)[F_1(u')F_2(v') - F_1(u)F_2(v)]\mathrm{d}\omega\mathrm{d}u, \end{aligned} \tag{1.2}$$

where $u' = u + [(v-u)\cdot\omega]\omega, \ v' = v - [(v-u)\cdot\omega]\omega$.

Here, $B(v-u, \omega) = |v-u|^\kappa q_0\left(\frac{v-u}{|v-u|}\cdot\omega\right)$ and $0 \leq \kappa \leq 1$ (hard potential) and $0 \leq q_0\left(\frac{v-u}{|v-u|}\cdot\omega\right) \leq C\left|\frac{v-u}{|v-u|}\cdot\omega\right|$ (angular cutoff).

Despite extensive developments in the study of the Boltzmann equation, many basic questions regarding solutions in a physical bounded domain, such as their regularity, have remained largely open. This is partly due to the characteristic nature of boundary conditions in the kinetic theory. In [9], it is shown that in convex domains, Boltzmann solutions are continuous away from the grazing set. On the other hand, in [12], it is shown that singularity (discontinuity) does occur for Boltzmann solutions in a non-convex domain, and such singularity propagates precisely along the characteristics emanating from the grazing set. The boundary of the phase space is

$$\gamma := \{(x, v) \in \partial\Omega \times \mathbb{R}^3\}.$$

Let $n = n(x)$ be the outward normal direction at $x \in \partial\Omega$. We decompose γ as

$$\begin{aligned}
\gamma_- &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}, \quad (\textit{the incoming set}),\\
\gamma_+ &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v > 0\}, \quad (\textit{the outcoming set}),\\
\gamma_0 &= \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\}, \quad (\textit{the grazing set}).
\end{aligned} \tag{1.3}$$

In general the boundary condition is imposed only for the incoming set γ_- for general kinetic PDEs [1,2,7,9]. We consider, in this paper, the following basic boundary conditions:

(i) *Diffuse boundary condition*: With $c_\mu \int_{n(x)\cdot u>0} \mu(u)\{n(x) \cdot u\}\mathrm{d}u = 1$,

$$F(t, x, v) = c_\mu \mu(v) \int_{n(x)\cdot u>0} F(t, x, u)\{n(x) \cdot u\}\mathrm{d}u \quad \text{on} \quad (x, v) \in \gamma_-. \tag{1.4}$$

(ii) *Specular reflection boundary condition*:

$$\begin{aligned}
&F(t, x, v) = F(t, x, R_x v) \quad \text{on} \quad (x, v) \in \gamma_-,\\
&\text{where } R_x v := v - 2n(x)(n(x) \cdot v).
\end{aligned} \tag{1.5}$$

(iii) *Bounce-back reflection boundary condition*:

$$F(t, x, v) = F(t, x, -v) \quad \text{on} \quad (x, v) \in \gamma_-. \tag{1.6}$$

We denote

$$f := F/\sqrt{\mu},$$

where

$$\mu = e^{-\frac{|v|^2}{2}} \quad \text{(a global Maxwellian)}.$$

Throughout this paper we always assume a positive initial datum $F_0 = \sqrt{\mu} f_0 \geq 0$.

The function f satisfies

$$\partial_t f + v \cdot \nabla_x f = \Gamma_{\text{gain}}(f, f) - \nu(\sqrt{\mu} f) f. \tag{1.7}$$

Here

$$\begin{aligned}\nu(\sqrt{\mu} f)(v) &:= \frac{1}{\sqrt{\mu(v)}} Q_{\text{loss}}(\sqrt{\mu} f, \sqrt{\mu} f)(v) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v-u|^\kappa q_0 \left(\frac{v-u}{|v-u|} \cdot \omega\right) \sqrt{\mu(u)} f(u) \mathrm{d}\omega \mathrm{d}u,\end{aligned} \tag{1.8}$$

and the gain term of the nonlinear Boltzmann operator is given by

$$\begin{aligned}\Gamma_{\text{gain}}(f_1, f_2)(v) &:= \frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu} f_1, \sqrt{\mu} f_2)(v) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v-u|^\kappa q_0 \left(\frac{v-u}{|v-u|} \cdot \omega\right) \sqrt{\mu(u)} f_1(u') f_2(v') \mathrm{d}\omega \mathrm{d}u.\end{aligned} \tag{1.9}$$

The corresponding boundary conditions for f are the following ones:
(i) *Diffuse boundary condition*:

$$f(t, x, v) = c_\mu \sqrt{\mu(v)} \int_{n(x) \cdot u > 0} f(t, x, u) \sqrt{\mu(u)} \{n(x) \cdot u\} \mathrm{d}u, \quad \text{on } (x, v) \in \gamma_-. \tag{1.10}$$

(ii) *Specular reflection boundary condition*:

$$f(t, x, v) = f(t, x, R_x v), \quad \text{on } (x, v) \in \gamma_-. \tag{1.11}$$

(iii) *Bounce-back reflection boundary condition*:

$$f(t, x, v) = f(t, x, -v), \quad \text{on } (x, v) \in \gamma_-. \tag{1.12}$$

Throughout this paper we assume that Ω is a bounded open subset of $\mathbb{R}^3$ and there exists $\xi : \mathbb{R}^3 \to \mathbb{R}$ such that $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$, and $\partial\Omega = \{x \in \mathbb{R}^3 : \xi(x) = 0\}$. Moreover for all $x \in \bar{\Omega} = \Omega \cup \partial\Omega$ (therefore $\xi(x) \leq 0$) we assume the domain is *strictly convex*:

$$\sum_{i,j} \partial_{ij} \xi(x) \zeta_i \zeta_j \geq C_\xi |\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^3. \tag{1.13}$$

We assume that $\nabla\xi(x) \neq 0$ when $|\xi(x)| \ll 1$ and we define the *outward normal* as $n(x) \equiv \frac{\nabla\xi(x)}{|\nabla\xi(x)|}$.

Notation: We denote $||\cdot||_p$ the $L^p(\Omega\times\mathbb{R}^3)$ norm, while $|\cdot|_{\gamma,p}=|\cdot|_p$ is the $L^p(\partial\Omega\times\mathbb{R}^3;\mathrm{d}\gamma)$ norm, $|\cdot|_{\gamma_\pm,p}=|\cdot\mathbf{1}_{\gamma_\pm}|_{\gamma,p}$, $\mathrm{d}\gamma=|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v$ with the surface measure $\mathrm{d}S_x$ on $\partial\Omega$. For a function f of t,x, and v we denote $\|f(t)\|_p=\|f(t,\cdot,\cdot)\|_p$ and $|f(t)|_{\gamma_\pm,p}=|f(t,\cdot,\cdot)|_{\gamma_\pm,p}$. Denote $\langle v\rangle=\sqrt{1+|v|^2}$. The notation $X\lesssim Y$ is equivalent to $X\leq CY$, where C is a constant not depending on X and Y. We subscript this to denote dependence on parameters, thus $X\lesssim_a Y$ means $X\leq C_aY$. The notation $X\ll_a Y$ is equivalent to $X\leq C_aY$, where $C_a>0$ is sufficiently small. The notation $X\simeq_a Y$ means $X\lesssim_a Y$ and $Y\lesssim_a X$. Let A and B be $m\times n$ matrices and all entries of B are non-negative. Let A_{ij} be the $(i,j)-$entry of the matrix A. The notation $A\lesssim B$ means $|A_{ij}|\lesssim B_{ij}$ for all i,j.

1.1 Diffuse reflection BC

Theorem 1 *Assume that $0\leq\kappa\leq 1$ in (1.2) and $f_0\in W^{1,p}(\Omega\times\mathbb{R}^3)$ and $\|\nabla_x f_0\|_p+\|\nabla_v f_0\|_p+\|e^{\theta|v|^2}f_0\|_\infty<+\infty$ for any $\theta\in(0,1/4)$ and any fixed $1<p<2$, and the compatibility condition on $(x,v)\in\gamma_-$,*

$$f_0(x,v)=c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}f_0(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \tag{1.14}$$

Then there exists $T=T(\|e^{\theta|v|^2}f_0\|_\infty)>0$ such that $f\in L^\infty([0,T];W^{1,p}(\Omega\times\mathbb{R}^3))$ solves the Boltzmann equation (1.7) with diffuse boundary condition (1.10), and satisfies, for all $0\leq t\leq T$

$$\|\nabla_{x,v}f(t)\|_p^p+\int_0^t|\nabla_{x,v}f(s)|_{\gamma,p}^p\mathrm{d}s\lesssim_t\|\nabla_{x,v}f_0\|_p^p+P(\|e^{\theta|v|^2}f_0\|_\infty), \tag{1.15}$$

where P is some polynomial.

Furthermore, if $F_0=\mu+\sqrt{\mu}g_0$ with $\|e^{\theta|v|^2}g_0\|_\infty\ll 1$, then the unique bounded global-in-time solution $g(t)$ constructed in [9] satisfying (1.15), by changing f, f_0 to g, g_0 for any finite $t\geq 0$.

There can be no size restriction on initial data $F_0=\sqrt{\mu}f_0$. We need the exponential weight $e^{\theta|v|^2}$ to have local-in-time solutions (see Lemma 7). On the other hand, we also remark that from [3,9], the assumption $\|e^{\theta|v|^2}g_0\|_\infty\ll 1$ for $F_0=\mu+\sqrt{\mu}g_0$ without a mass constraint $\iint_{\Omega\times\mathbb{R}^3}g_0\sqrt{\mu}\mathrm{d}v\mathrm{d}x=0$ ensures a uniform-in-time bound as $\sup_{0\leq t\leq\infty}\|e^{\theta|v|^2}g(t)\|_\infty\lesssim\|e^{\theta|v|^2}g_0\|_\infty$ (not a decay). Due to these L^∞ uniform-in-time bounds we are able to obtain a global-in-time estimate for ∇g.

Moreover, we show that the estimate (1.15) in Theorem 1 for $p<2$ is indeed optimal even for the free transport equation $\partial_t f+v\cdot\nabla_x f=0$ with the diffuse boundary condition (Lemma 12). In fact, the boundary integral blows up at $p = 2$.

We now illustrate the main ideas in the proof of Theorem 1. Clearly, the t and v derivatives behave nicely for the diffuse boundary condition as for $(x, v) \in \gamma_-$,

$$\begin{aligned}\partial_t f(t,x,v) &= c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} \partial_t f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &= c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} [-u\cdot\nabla_x f + \Gamma_{\text{gain}}(f,f) \\ &\qquad -\nu(\sqrt{\mu} f) f]\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \end{aligned} \tag{1.16}$$

$$\nabla_v f(t,x,v) = c_\mu \nabla_v \sqrt{\mu(v)} \int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \tag{1.17}$$

where we have used (1.7) to express $\partial_t f = -u\cdot\nabla_x f + \Gamma_{\text{gain}}(f,f) - \nu(\sqrt{\mu} f) f$ in (1.16).

Let $\tau_1(x)$ and $\tau_2(x)$ be unit tangential vectors to $\partial\Omega$ satisfying $\tau_1(x)\cdot n(x) = 0 = \tau_2(x)\cdot n(x)$ and $\tau_1(x)\times\tau_2(x) = n(x)$. Define the orthonormal transformation from $\{n, \tau_1, \tau_2\}$ to the standard bases $\{e_1, e_2, e_3\}$ as $\mathcal{T}(x)n(x) = e_1,\ \ \mathcal{T}(x)\tau_1(x) = e_2,\ \mathcal{T}(x)\tau_2(x) = e_3$, and $\mathcal{T}^{-1} = \mathcal{T}^t$. Upon a change of variable: $u' = \mathcal{T}(x)u$, we have

$$n(x)\cdot u = n(x)\cdot \mathcal{T}^t(x)u' = n(x)^t\mathcal{T}^t(x)u' = [\mathcal{T}(x)n(x)]^t u' = e_1\cdot u' = u_1',$$

then

$$\begin{aligned} &c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &\quad = c_\mu\sqrt{\mu(v)}\int_{u_1'>0} f(t,x,\mathcal{T}^t(x)u')\sqrt{\mu(u')}\{u_1'\}\mathrm{d}u', \end{aligned}$$

so that we can further take tangential derivatives ∂_{τ_i} as, for $(x, v)\in\gamma_-$,

$$\begin{aligned} &\partial_{\tau_i} f(t,x,v) \\ &\quad = c_\mu\sqrt{\mu(v)}\int_{u_1'>0}\left\{\partial_{\tau_i} f(t,x,\mathcal{T}^t(x)u') + \nabla_v f(t,x,\mathcal{T}^t(x)u')\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}u'\right\} \\ &\qquad \times\sqrt{\mu(u')}\{u_1'\}\mathrm{d}u' \\ &\quad = c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_{\tau_i} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &\qquad + c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\nabla_v f(t,x,u)\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}\mathcal{T}(x)u\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \end{aligned} \tag{1.18}$$

The difficulty is always the control of the normal spatial derivative $\partial_n f$. From the general method of proving regularity in PDE with boundary conditions, it is natural to use the Boltzmann equation to solve the normal derivative $\partial_n f$ inside the region, in terms of $\partial_t f$, $\nabla_v f$, and $\partial_\tau f$ as:

$$\partial_n f(t,x,v) = -\frac{1}{n(x)\cdot v}\left\{\partial_t f + \sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i} f - \Gamma_{\text{gain}}(f,f) + \nu(\sqrt{\mu}f)f\right\}, \tag{1.19}$$

at least near $\partial\Omega$. We note that due to the nonlocal terms $\Gamma_{\text{gain}}(f,f)$ and $\nu(\sqrt{\mu}f)f$, it is strongly believed that $\partial_n f$ behaves as $\frac{1}{n(x)\cdot v}$ at the boundary. Unfortunately, this standard approach encounters a severe difficulty: $\frac{1}{n(x)\cdot v} \notin L^1_{loc}$ in the velocity space (a L^∞ bound is desirable for any $W^{1,p}$ estimate).

The first new ingredient of our approach is to use (1.19) *not* inside the domain, but at the boundary $\partial\Omega$. Using the diffuse boundary condition (1.10), (1.16), (1.17), (1.18) and (1.7), we can express $\partial_n f$ at $(x,v)\in\gamma_-$ as

$$\begin{aligned}
&\partial_n f(t,x,v)\\
&= -\frac{1}{n(x)\cdot v}\Bigg\{\sqrt{\mu(v)}\int_{n(x)\cdot u>0}[-u\cdot\nabla_x f + \Gamma_{\text{gain}}(f,f) - \nu(\sqrt{\mu}f)f]\\
&\qquad\qquad\times(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\quad+\sum_{i=1}^{2}(v\cdot\tau_i)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_{\tau_i} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\quad+\sum_{i=1}^{2}(v\cdot\tau_i)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\nabla_v f(t,x,u)\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}\mathcal{T}(x)u\\
&\qquad\qquad\times\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\quad-\Gamma_{\text{gain}}(f,f) + \nu(\sqrt{\mu}f)f\Bigg\}.
\end{aligned}\tag{1.20}$$

Due to the additional u integral in (1.20) and the crucial factor $|n(x)\cdot u|$ in the measure $\mathrm{d}\gamma$ on the boundary γ, it is clear that the singularity of $|\partial_n f|^p|n\cdot v|$ in (1.20) is roughly of the order

$$\frac{1}{\{n\cdot v\}^{p-1}},$$

so that its v-integration is precisely finite if $1 \le p < 2$, and indeed its v integration is *uniformly* bounded with respect to x.

However, in order to control $\nabla_v f$ and $\partial_\tau f$ for $p < 2$, a new difficulty arises. It is well-known from [3,9] that a crucial boundary estimate for diffuse boundary takes the form of a L^2-contraction:

$$\int_{\gamma_-} h^2 \mathrm{d}\gamma \leq \int_{\gamma_+} h^2 \mathrm{d}\gamma.$$

Unfortunately, this is not expected to be valid for $p \neq 2$, so it is impossible to absorb the incoming part $\gamma\_$ solely by the outgoing part γ_+ part.

Our second new ingredient is to split the γ_+ integral into the part near the grazing set γ_+^ε and the remaining part for $p \neq 2$ for our boundary representations for derivatives (1.16), (1.17), (1.18), and (1.20). For small $\varepsilon > 0$ we define γ_+^ε, the set of almost grazing velocities or large velocities

$$\gamma_+^\varepsilon = \{(x, v) \in \gamma_+ : v \cdot n(x) < \varepsilon \text{ or } |v| > 1/\varepsilon\}. \tag{1.21}$$

Denote $\partial = [\nabla_x, \nabla_v]$. We can roughly obtain

$$\begin{aligned}
\int_{\gamma_-} |\partial f|^p &\lesssim \int_{\partial\Omega} \left(\int_{n\cdot v>0} |\partial f| \mu^{1/4} \{n \cdot v\} \mathrm{d}v \right)^p + \text{good terms}, \\
&\lesssim \int_{\partial\Omega} \left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}} |\partial f| \mu^{1/4} \{n \cdot v\} \right)^p \\
&\quad + \int_{\partial\Omega} \left(\int_{\{v:(x,v)\in\gamma_+\setminus\gamma_+^\varepsilon\}} |\partial f| \mu^{1/4} \{n \cdot v\} \right)^p + \text{good terms}, \\
&\lesssim \sup_x \left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}} \mu^{q/4} \{n \cdot v\} \mathrm{d}v \right)^{p/q} \int_{\gamma_+^\varepsilon} |\partial f|^p \mathrm{d}\gamma \\
&\quad + \int_{\gamma_+\setminus\gamma_+^\varepsilon} |\partial f|^p \mathrm{d}\gamma + \text{good terms}.
\end{aligned}$$

It is important to realize that $\sup_x \left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}} \mu^{q/4} \{n \cdot v\} \mathrm{d}v \right)^{p/q}$ has a small measure of order ε, for $p > 1$, so that it can be absorbed by the outgoing part $\int_{\gamma_+}$. Fortunately, the outgoing boundary integral $\int_{\gamma_+\setminus\gamma_+^\varepsilon}$ can be further bounded by the integration in the bulk and initial data by Lemma 8 with a crucial time integration. On the other hand, such a process produces a large constant in the Gronwall estimates and leads to a growth in time. Of course, such an approach breaks down at $p = 1$. (See [10] for the estimates for $p = 1$.)

We introduce a crucial distance function towards the grazing set γ_0.

Definition 1 (*Kinetic Distance*) For $(x, v) \in \bar{\Omega} \times \mathbb{R}^3$,

$$\alpha(x, v) := |v \cdot \nabla\xi(x)|^2 - 2\{v \cdot \nabla^2\xi(x) \cdot v\}\xi(x),$$

where $v \cdot \nabla^2\xi(x) \cdot v = \sum_{i,j} v_i \partial_i \partial_j \xi(x) v_j$.

From (1.13), the kinetic distance $\alpha(x, v)$ vanishes if and only if $(x, v) \in \gamma_0$. Furthermore we combine the distance function with a strong decay factor $e^{-\varpi\langle v\rangle t}$ with $\varpi > 0$,

$$e^{-\varpi\langle v\rangle t}\alpha(x, v). \tag{1.22}$$

A direct computation yields

$$\begin{aligned}\{\partial_t + v \cdot \nabla_x\}[e^{-\varpi\langle v\rangle t}\alpha(x, v)] &= -\varpi\langle v\rangle e^{-\varpi\langle v\rangle t}\alpha(x, v)\\ &\quad - e^{-\varpi\langle v\rangle t} 2v\{v \cdot \nabla^3\xi(x) \cdot v\}\xi(x)\\ &\lesssim (-\varpi + O_\xi(1))\langle v\rangle e^{-\varpi\langle v\rangle t}\alpha(x, v),\end{aligned}$$

with the geometric contribution $O_\xi(1) = \frac{2v\{v \cdot \nabla^3\xi(x)\cdot v\}\xi}{\alpha\langle v\rangle}$ from (1.13). Throughout this paper we assume

$$\varpi \geq \max \frac{2\left|v\{v \cdot \nabla^3\xi(x) \cdot v\}\xi\right|}{\alpha\langle v\rangle}. \tag{1.23}$$

Remark that if ξ is quadratic (for example if the domain is a ball or an ellipsoid) then we are able to set $\varpi = 0$ and $\{\partial_t + v \cdot \nabla_x\}\alpha \equiv 0$.

Theorem 2 *Assume the compatibility condition* (1.14), $0 < \kappa \leq 1$.

For any fixed $2 \leq p < \infty$ *and* $\frac{p-2}{2p} < \beta < \frac{p-1}{2p}$, *if* $\|\alpha^\beta \nabla_{x,v} f_0\|_p + \|e^{\theta|v|^2} f_0\|_\infty < \infty$ *for some* $0 < \theta < \frac{1}{4}$, *then there exists* $T = T(\|e^{\theta|v|^2} f_0\|_\infty) > 0$ *such that* $e^{-\varpi\langle v\rangle t}\alpha^\beta \nabla_{x,v} f \in L^\infty([0, T]; L^p(\Omega \times \mathbb{R}^3))$ *and satisfying, for all* $0 \leq t \leq T$,

$$\begin{aligned}&\|e^{-\varpi\langle v\rangle t}\alpha^\beta \nabla_{x,v} f(t)\|_p^p + \int_0^t |e^{-\varpi\langle v\rangle t}\alpha^\beta \nabla_{x,v} f(s)|_{\gamma,p}^p \mathrm{d}s\\ &\quad\lesssim_t \|\alpha^\beta \nabla_{x,v} f_0\|_p^p + P(\|e^{\theta|v|^2} f_0\|_\infty),\end{aligned} \tag{1.24}$$

where P *is some polynomial. Furthermore, if* $F_0 = \mu + \sqrt{\mu} g_0$ *with* $\|e^{\theta|v|^2} g_0\|_\infty \ll 1$, *then the unique bounded global-in-time solution* $g(t)$ *constructed in* [9] *satisfying* (1.24), *by changing* f, f_0 *to* g, g_0 *for any finite* $t \geq 0$.

If $\|\alpha^{1/2}\nabla_{x,v} f_0\|_\infty + \|e^{\theta|v|^2} f_0\|_\infty < +\infty$ *for some* $0 < \theta < \frac{1}{4}$*, then* $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\nabla_{x,v} f \in L^\infty([0,T]; L^\infty(\Omega\times\mathbb{R}^3))$ *and satisfying, for all* $0 \le t \le T$,

$$\|e^{-\varpi\langle v\rangle t}\alpha^{1/2}\nabla_{x,v} f(t)\|_\infty \lesssim_t \|\alpha^{1/2}\nabla_{x,v} f_0\|_\infty + P(\|e^{\theta|v|^2} f_0\|_\infty). \quad (1.25)$$

If $\alpha^{1/2}\nabla f_0 \in C^0(\bar{\Omega}\times\mathbb{R}^3)$ *and*

$$v\cdot\nabla_x f_0 - \Gamma(f_0, f_0) = c_\mu\sqrt{\mu}\int_{n\cdot u>0}\{u\cdot\nabla_x f_0 - \Gamma(f_0, f_0)\}\sqrt{\mu}\{n\cdot u\}\mathrm{d}u, \quad (1.26)$$

is valid for γ_-*, then* $f \in C^1$ *away from the grazing set* γ_0*. Furthermore, if* $F_0 = \mu + \sqrt{\mu} g_0$ *with* $\|e^{\theta|v|^2} g_0\|_\infty \ll 1$*, then the unique bounded global-in-time solution* $g(t)$ *constructed in* [9] *satisfying* (1.25)*, by changing* f, f_0 *to* g, g_0 *for any finite* $t \ge 0$.

There can be no size restriction on initial data $F_0 = \sqrt{\mu} f_0$. On the other hand, we also remark that from [3,9], the assumption $\|e^{\theta|v|^2} g_0\|_\infty \ll 1$ for $F_0 = \mu + \sqrt{\mu} g_0$ without a mass constraint $\iint_{\Omega\times\mathbb{R}^3} g_0\sqrt{\mu}\mathrm{d}v\mathrm{d}x = 0$ ensures a uniform-in-time bound as $\sup_{0\le t\le\infty}\|e^{\theta|v|^2} g(t)\|_\infty \lesssim \|e^{\theta|v|^2} g_0\|_\infty$ (not a decay). Due to these L^∞ uniform-in-time bound we are able to a global-in-time estimate for the derivatives of g.

We remark that our results in this paper allow the initial conditions to be *singular near* γ_0. This is drastically different from the Vlasov case, in which regularity of the solutions is bounded by stronger vanishing condition for regular initial condition. Due to non-local interaction in the collision, such kind of regularity estimate is not expected. Instead, we show that *singular* behavior of the solution, i.e., weighted in α^β, can be controlled via *singular* behavior of its initial state.

We remark that for $\varpi > 0$, the derivatives of $f(t)$ behaves as $e^{\varpi\langle v\rangle t}$ so that in terms of solution $f(t)$, such an estimate not only creates an exponential growth in time, but also creates less integrability in velocity. Furthermore, when $\varpi > 0$, we crucially need a strong weight function $e^{\theta|v|^2}$ to balance such a factor $e^{-\varpi\langle v\rangle t}$, which produces a super exponential growth e^{t^2} in time. We suspect that it is impossible to obtain a uniform in time estimate especially when $\varpi \ne 0$.

The distance function α plays an important role in the study of regularity in convex domains for Vlasov equations [6,7,11], which can be controlled along the characteristics via the geometric Velocity lemma (Lemma 2). However, such an approach has not been successful in the study of Boltzmann equation due to the non-local nature of the Boltzmann collision operator, which mixes up different velocities so that their distance towards γ_0 can not be controlled.

In addition to the key boundary representation (1.16)–(1.20), we establish a delicate estimate for the interaction of $e^{-\varpi\langle v\rangle t}\alpha(x,v)$ with the collision kernel in (5.16) for $\beta < \frac{p-1}{2p}$ and $\partial = [\nabla_x, \nabla_v]$. An additional requirement $\beta > \frac{p-2}{2p}$ is needed to control the boundary singularity in (5.19). These estimates are sufficient to treat the case for $\beta < 1/2$, but unfortunately fail for the case $\beta = 1/2$, which accounts for the important C^1 estimate.

1.2 Dynamical non-local to local estimates

In order to establish the C^1 estimate, we employ the Lagrangian view point, estimating along the trajectory. The characteristics ODE of the Boltzmann equation (1.1) is

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = V(s), \quad \frac{\mathrm{d}V(s)}{\mathrm{d}s} = 0.$$

For $(x,v) \in \bar{\Omega} \times \mathbb{R}^3$ we define $t_{\mathbf{b}}(x,v)$ to be *the backward exit time* as

$$t_{\mathbf{b}}(x,v) = \inf\{\tau > 0 : x - sv \notin \Omega\}, \tag{1.27}$$

and $x_{\mathbf{b}}(x,v) = x - t_{\mathbf{b}} v$.

Lemma 1 *Let* $(t,x,v) \in [0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$, $\frac{1}{2} < \beta < \frac{3}{2}$, $0 < \kappa \le 1$ *and* $r \in \mathbb{R}$. *Let* $Z(s,x,v) \ge 0$ *be any bounded non-negative function in the phase space.*

(1) *Let* $x - (t-s)v$ *on* $s \in [t - t_{\mathbf{b}}(x,v), t]$. *For any* $\varepsilon > 0$, *there exists a large constant* $l \gg_{\xi} 1$ *which depends on the domain* Ω *such that*

$$\begin{aligned}
&\int_{t-t_{\mathbf{b}}(x,v)}^{t} \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}[\alpha(x-(t-s)v,u)]^{\beta}} \frac{\langle u\rangle^r}{\langle v\rangle^r} \\
&\quad \times Z(s,x,v)\mathrm{d}u\mathrm{d}s \\
&\lesssim \min\left\{ \frac{\varepsilon^{\frac{3}{2}-\beta}}{|v|^2\{\alpha(x,v)\}^{\beta-1}}, \frac{\{\alpha(x,v)\}^{\frac{1}{4}-\frac{\beta}{2}}|t_Z|^{\frac{3}{2}-\beta}}{|v|^{2\beta-1}} \right\} \\
&\quad \times \sup_{s\in[t-t_{\mathbf{b}}(x,v),t]} \{e^{-l\langle v\rangle(t-s)} Z(s,x,v)\} \\
&\quad + \frac{C_{\varepsilon}}{\{\alpha(x,v)\}^{\beta-1/2}} \int_{t-t_{\mathbf{b}}(x,v)}^{t} e^{-\frac{l}{2}\langle v\rangle(t-s)} Z(s,x,v)\mathrm{d}s,
\end{aligned} \tag{1.28}$$

where $t_Z = \sup\{s : Z(s,x,v) \neq 0\}$.

(2) *Let* $[X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)]$ *be the specular backward trajectory or the bounce-back trajectory in Definition 2.*

For any $\varepsilon > 0$*, there exists* $l \gg_{\xi} 1$ *such that*

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|V_{\mathbf{cl}}(s;t,x,v)-u|^2}}{|V_{\mathbf{cl}}(s;t,x,v)-u|^{2-\kappa}} \frac{\langle u\rangle^r}{\langle v\rangle^r} \frac{Z(s,x,v)}{[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)]^\beta} \mathrm{d}u\mathrm{d}s \\ \lesssim \frac{O(\varepsilon)}{\langle v\rangle\, [\alpha(x,v)]^{\beta-1/2}} \sup_{0\le s\le t} \left\{ e^{-\frac{l}{2}\langle v\rangle(t-s)} Z(s,x,v)\right\}. \end{aligned} \tag{1.29}$$

Even though one can not hope to control the regularity near γ_0 due to non-local nature of the collision operator as in the Vlasov theory, one can control its singular behavior (i.e. with weight α^β) thanks to such dynamical non-local to local estimates. The crucial gain of $\sqrt{\alpha}$, which only can be obtained for expected *singular* behavior with *negative* power of $\sqrt{\alpha}$, is due to a combination of two facts: the gain of power $\frac{1}{2}$ is due to a velocity average, and gain of the local behavior of $\sqrt{\alpha}$ is due to time integration and convexity. Even with $k = 1$, the gain of $\sqrt{\alpha}$ seems to be sharp.

The proof of such non-local to local estimates is a combination of analytical and geometrical arguments. The first part is a precise estimate of the u integration which is bounded via $\frac{1}{|v|^{2\beta-1}|\xi(x-(t-s)v)|^{\beta-1/2}}$. In this part of the proof we make use of a series of change of variables to obtain the precise power $\beta - \frac{1}{2}$. The second part is to relate $\frac{1}{|\xi(x-(t-s)v)|^{\beta-1/2}}$ back to $\frac{1}{\alpha}$. Clearly,

$$\frac{1}{|\xi(x-(t-s)v)|^2} \simeq \frac{1}{\alpha} \simeq \frac{1}{|v\cdot\nabla\xi(x-(t-s)v|^2 + |\xi(x-(t-s)v)\| v|^2},$$

when $|\xi(x-(t-s)v)\||v|^2$ is larger than $|v\cdot\nabla\xi(x-(t-s)v)|$. On the other hand, when $|v\cdot\nabla\xi(X_{\mathbf{cl}}(s))|$ dominates, the bound can only be achieved through a crucial use of time integration and the geometric Velocity lemma (see Lemma 2), by connecting

$$\mathrm{d}t \simeq \frac{\mathrm{d}\xi}{|v\cdot\nabla\xi|},$$

and recovering a power of α as in the bound of $\xi-$integration through the geometric Velocity Lemma (Lemma 2).

The most striking feature is that not only our estimates retain the local structure for α, but they *gain* a $\sqrt{\alpha}$ order of regularity. Such a precise gain of regularity is exactly enough to balance out the singularity in α appeared in $\nabla_{x,v}X_{\mathbf{cl}}(s;t,x,v)$ and $\nabla_{x,v}V_{\mathbf{cl}}(s;t,x,v)$ in both the specular and bounce-back cycles. In order to squeeze out a small constant for $|v| \gg 1$, we need to use the decay of $e^{-l\langle v\rangle(t-s)}$. This requires a precise regrouping of the cycles according to the time scale of

$$t|v| \simeq 1.$$

Within such an important time scale, $V_{\mathbf{cl}}(s;t,x,v)$ stays almost *invariant* due to the Velocity Lemma (see Lemma 2). We then are able to obtain a precise estimate for the number of bounces within $t|v| \simeq 1$ and extract a small constant from $e^{-l\langle v\rangle(t-s)}$ for $t-s \geq \frac{1}{|v|}$. On the other hand, for $t-s \leq \frac{1}{|v|}$, the small constant comes from Lemma 1.

1.3 Specular reflection BC

Recall the specular reflection boundary condition in (1.11) and the specular cycles in Definition 2. Our main theorem is as follow.

Theorem 3 *Assume* $F_0 = \sqrt{\mu} f_0 \geq 0$ *and* $f_0 \in W^{1,\infty}(\Omega \times \mathbb{R}^3)$ *and* $0 < \kappa \leq 1$ *for* $1 < \beta < \frac{3}{2}$, $0 < \theta < \frac{1}{4}$, *and* $b \in \mathbb{R}$,

$$\left\| \frac{\alpha^{\beta-\frac{1}{2}}}{\langle v\rangle^b} \nabla_x f_0 \right\|_\infty + \left\| \frac{|v|^2 \alpha^{\beta-1}}{\langle v\rangle^b} \nabla_v f_0 \right\|_\infty + \left\| e^{\theta|v|^2} f_0 \right\|_\infty < \infty,$$

and the compatibility condition

$$f_0(x,v) = f_0(x, R_x v) \quad \text{on } (x,v) \in \gamma_-. \tag{1.30}$$

Then for all $0 \leq t \leq T$ *with* $T = T(\|e^{\theta|v|^2} f_0\|_\infty) > 0$

$$\begin{aligned} &\left\| e^{-\varpi\langle v\rangle t} \frac{\alpha^\beta}{\langle v\rangle^{b+1}} \nabla_x f(t) \right\|_\infty + \left\| e^{-\varpi\langle v\rangle t} \frac{|v| \alpha^{\beta-\frac{1}{2}}}{\langle v\rangle^b} \nabla_v f(t) \right\|_\infty \\ &\quad \lesssim_{\xi,t} \left\| \frac{\alpha^{\beta-\frac{1}{2}}}{\langle v\rangle^b} \nabla_x f_0 \right\|_\infty + \left\| \frac{|v|^2 \alpha^{\beta-1}}{\langle v\rangle^b} \nabla_v f_0 \right\|_\infty + P\left(\left\| e^{\theta|v|^2} f_0 \right\|_\infty \right), \end{aligned} \tag{1.31}$$

where P *is some polynomial. Furthermore, if* Ω *is real analytic* (ξ *is real analytic on* $\mathbb{R}^3$) *and* $F_0 = \mu + \sqrt{\mu} g_0$ *with* $\|e^{\theta|v|^2} g_0\|_\infty \ll 1$, *then the unique bounded global-in-time solution* $g(t)$ *constructed in* [9] *satisfying* (1.31), *by changing* f, f_0 *to* g, g_0 *for any finite* $t \geq 0$.

Furthermore, if $f_0 \in C^1$ *and*

$$v \cdot \nabla_x f_0(x,v) = R_x v \cdot \nabla_x f_0(x, R_x v) \quad \text{on} \quad (x,v) \in \gamma_-. \tag{1.32}$$

then $f \in C^1$ *away from the grazing set* γ_0.

There can be no size restriction on initial data $F_0 = \sqrt{\mu} f_0$. We remark from the local existence theorem, $T > 0$. The analyticity is a crucial assumption using a linear L^2-decay to conclude $L^\infty-$decay toward the Maxwellian in [5]

($L^2 - L^\infty$ interpolation argument) in order to have global-in-time solutions (See recent work [13]).

We also remark that the specular theorem is drastically different from the diffusive theorem: in addition to the loss of moments, there is a loss of regularity of α with respect to the initial data. This makes it impossible to use the continuity argument to choose small time interval to close the estimates. We need to use large ϖ in $e^{-\varpi\langle v\rangle t}$ to extract a small constant to close, which requires extra precise estimates. We note that in 3D case, $\beta > 1/2$, due to the failure of the proof of the non-local to local estimates for the critical $\beta = 1/2$ (Lemma 1).

On the other hand, in 2D, due to boundedness of $\partial_{v_3} f$ from x_3-invariance, we are able to estimate $\partial_v \Gamma_{\text{gain}}$ for the critical case $\beta = 1/2$ (Lemma 19). More precisely we consider the 2D specular problem for $f(t, x_1, x_2, v_1, v_2, v_3)$ solving

$$\partial_t f + v_1 \partial_{x_1} f + v_2 \partial_{x_2} f = \Gamma_{\text{gain}}(f, f) - \nu(\sqrt{\mu} f) f, \tag{1.33}$$

where v_3 is a parameter.

Theorem 4 *Assume a stronger cut-off assumption on q_0 of (1.2)*

$$\left|\nabla_v q_0\left(\frac{v-u}{|v-u|}\cdot\omega\right)\right| \Big/ \left|\frac{v-u}{|v-u|}\cdot\omega\right| \lesssim 1. \tag{1.34}$$

Assume $f_0 \in W^{1,\infty}$ with (1.11). Assume that

$$\sup_{0<t\leq T}\left\{\|e^{\theta|v|^2} f(t)\|_\infty + \|\partial_{v_3} f(t)\|_\infty\right\} \leq c_{T,\zeta,f_0} < +\infty,$$

then

$$\sup_{0\leq t\leq T}\left\{\left\|e^{-\varpi\langle\bar{v}\rangle t}\frac{\alpha}{1+|\bar{v}|^2}\nabla_x f(t)\right\|_\infty + \left\|e^{-\varpi\langle\bar{v}\rangle t}\frac{|\bar{v}|}{\langle\bar{v}\rangle}\sqrt{\alpha}\nabla_v f(t)\right\|_\infty\right\}$$
$$\lesssim_{T,\Omega,L}\left\|\frac{\alpha^{1/2}}{\langle\bar{v}\rangle}\nabla_{\bar{x}} f_0\right\|_\infty + \left\|\frac{|\bar{v}|^2}{\langle v\rangle}\nabla_{\bar{v}} f_0\right\|_\infty + \|\partial_{v_3} f\|_\infty + P\left(\left\|e^{\theta|v|^2} f_0\right\|_\infty\right),$$

where P is some polynomial. If $f_0 \in C^1$ and the compatibility conditions (1.30) and (1.32) are satisfied, then there exists f solves (1.33) and C^1 away from the grazing set γ_0. Furthermore, if $\|e^{\theta|v|^2} f_0\|_\infty \ll 1$, and $\partial\Omega$ (therefore ξ) is real analytic, then T can be arbitrarily large.

We remark that powers of singularity α and $\sqrt{\alpha}$ are barely missed in 3D case (borderline case).

In additional to the dynamical non-local to local estimate, the second important ingredient for the specular reflection BC is the following crucial estimate for the derivatives of specular cycles $[X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)]$ in Definition 2.

Theorem 5 *There exists $C = C(\Omega) > 0$ such that for all $(s;t,x,v) \in \mathbb{R} \times \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^3$ with $s \neq t^\ell$ for $\ell = 1, 2, \ldots, \ell_*$*

$$\begin{aligned}
|\nabla_x X_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)} \frac{|v|}{\sqrt{\alpha(x,v)}},\\
|\nabla_v X_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)} \frac{1}{|v|},\\
|\nabla_x V_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)} \frac{|v|^3}{\alpha(x,v)},\\
|\nabla_v V_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)} \frac{|v|}{\sqrt{\alpha(x,v)}}.
\end{aligned} \tag{1.35}$$

Our estimates are optimal in terms of the order of $\frac{1}{\alpha}$, and $e^{C|v|(t-s)}$ relates to the $|v|$ growth in the Velocity lemma (Lemma 2). We remark that these precise orders of singularity, play a critical role for our design of the anisotropic norms in Theorem 3. In fact, if $|\nabla_x X_{\mathbf{cl}}(s;t,x,v)| \simeq \frac{1}{\alpha}$, it would have been too singular for the half power gain of α from the dynamical non-local to local estimates (Lemma 1), and our method should fail. Moreover, it is also crucial to have precise $|v|$ growth in both $|\nabla_x X_{\mathbf{cl}}(s;t,x,v)|$ and $|\nabla_x V_{\mathbf{cl}}(s;t,x,v)|$ to be controlled by $e^{-\varpi\langle v\rangle t}$.

We remark that $|\nabla_x X_{\mathbf{cl}}(s;t,x,v)| \simeq \frac{1}{\sqrt{\alpha}}$ is unexpected, even after one bounce we would have $\nabla_x x^1 \simeq \frac{1}{\sqrt{\alpha}}$ and it is natural to expect $\nabla_x X_{\mathbf{cl}}(s;t,x,v)$ picks up additional power of $\frac{1}{\sqrt{\alpha}}$ in the accumulation of $\frac{1}{\sqrt{\alpha}}$ number of bounces. However, via direct computations in 2D disk, we discover that even though

$$\nabla_x t^\ell \simeq \frac{1}{\alpha}, \quad \text{and} \quad \nabla_x x^\ell \simeq \frac{1}{\alpha},$$

but surprisingly

$$\nabla_x X_{\mathbf{cl}}(s;t,x,v) = \nabla_x[x^\ell - (t^\ell - s)v^\ell] \lesssim \frac{1}{\sqrt{\alpha}}\ !$$

Clearly, certain cancellations take place in the disk, which is difficult to even expect for general domains.

The proof of our Theorem 5 is split into 10 steps, and it is the most delicate proof throughout this paper. We first remark that, due to the 'discontinuous behaviors' of the normal component of $v \cdot n$ at each specular reflection, it is impossible to apply the standard techniques for ODE to estimate

$|\nabla_{x,v}X_{\mathbf{cl}}(s;t,x,v)|$ and $|\nabla_{x,v}V_{\mathbf{cl}}(s;t,x,v)|$. We have to develop different strategies to overcome several analytical difficulties to finally complete the proof.

Topological obstruction and moving frames: It turns out that we only need to consider the most delicate case in which all the bounces are almost grazing and staying near the boundary for $\mathbf{r}^\ell = \frac{|v^\ell \cdot n|}{|v^\ell|} \ll 1$. It is important for us to introduce the spherical co-ordinate system to cover the whole cycle and transform it into the ODE (6.13). Unfortunately, due to the 'hair-ball' theorem in Topology, such a change of coordinate system (or any change of coordinates) can not be smooth everywhere in the 2D surface $\partial\Omega$. In the case of a ball, all the trajectories are confined in a plane, so that one may choose a single chart to cover the whole trajectories. However, in other convex domains except the ball case, with large t, the specular trajectories are extremely complicated, which can reach almost every point on $\partial\Omega$. Hence, choosing a single chart is all but impossible. On the other hand, a 'sudden' change of a chart may create new order of singularity of α from the matrix $\mathcal{P}$ as in (6.50), which will ruin the estimates. It is therefore important to design a 'continuous' changes of charts associated with the almost grazing bounces. Given $n(x)$, we need to construct another globally defined, orthogonal, and continuous vector field. This would have been impossible if we were to seek it only in the physical space, in light of the 'hair-ball' theorem. The key observation is that, we need continuity not from just $\partial\Omega$, but from the phase space $\partial\Omega\times\mathbb{R}^3$. In fact, for almost grazing bounces, the velocity field v is almost perpendicular to $n(x)$, which provides a natural choice for construction of the desired moving frames. These continuous moving frames cost manageable errors for each bounce, which are controlled by the next method.
Matrix method for normal parts of $\nabla_{x,v}X_{\mathbf{cl}}(s)$ and $\nabla_{x,v}V_{\mathbf{cl}}(s)$: With such a well-defined moving charts, via the chain rule, one can represent $\nabla_{x,v}X_{\mathbf{cl}}(s;t,x,v)$ and $\nabla_{x,v}V_{\mathbf{cl}}(s;t,x,v)$ via a multiplication of Jacobian matrices $(t^\ell,x^\ell,v^\ell) \to (t^{\ell-1},x^{\ell-1},v^{\ell-1})$ in the spherical coordinate system. The 'matrix method' refers to the study of each discrete Jacobian matrix and precise estimates of their multiplication ($\frac{1}{\sqrt{\alpha}}$ of them!). One important step is to bound such a matrix by $J(\mathbf{r}^\ell)$ in (6.40) which can be diagonalized as $J(\mathbf{r}^\ell) = \mathcal{P}^{-1}\Lambda\mathcal{P}$, with a diagonal matrix Λ. Based on the crucial cancellation property (6.46), we can extract a crucial second order of $\mathbf{r}^\ell \ll 1$ appeared in $J(\mathbf{r}^\ell)$. Therefore, over the interval $t|v| \simeq 1$, we are able to estimate $\Pi_{\ell=1}^{\frac{1}{\sqrt{\alpha}}} J(\mathbf{r}^\ell) \lesssim \frac{1}{\sqrt{\alpha}}$. Together with $\frac{1}{\sqrt{\alpha}}$ from the initial bounce, we expect $\frac{1}{\alpha}$-singularity for both $\nabla_{x,v}X_{\mathbf{cl}}(s;t,x,v)$ and $\nabla_{x,v}V_{\mathbf{cl}}(s;t,x,v)$ as in (6.59). Even though such estimate is too singular for our purpose we can improve it. Upon a closed inspection,

$$|\nabla_x \mathbf{X}_\perp(s;t,x,v)| \lesssim \frac{1}{\sqrt{\alpha}},$$

for the normal component of $X_{\mathbf{cl}}(s)$. This is based on the fact $v_\perp^\ell \simeq \sqrt{\alpha}$ via the Velocity lemma [9, Lemma 1]. Unfortunately, the estimate of $\nabla_x \mathbf{X}_\parallel(s;t,x,v)$ is like $\frac{1}{\alpha}$ which is still too singular.

ODE method for tangential parts of $\nabla_{x,v} X_{\mathbf{cl}}(s)$ *and* $\nabla_{x,v} V_{\mathbf{cl}}(s)$: To improve such an estimate, we observe that given the estimates for the normal parts $[\mathbf{X}_\perp(s;t,x,v), \mathbf{V}_\perp(s;t,x,v)]$, the sub-system of ODE for $[\mathbf{X}_\parallel(s;t,x,v), \mathbf{V}_\parallel(s;t,x,v)]$, enjoys much better property. In fact, at each specular reflection, $[\mathbf{X}_\parallel(s;t,x,v), \mathbf{V}_\parallel(s;t,x,v)]$ are continuous, unlike the normal velocity $\mathbf{V}_\perp(s;t,x,v)$. Upon integrating over time as $\mathbf{V}_\perp(s;t,x,v) = \dot{\mathbf{X}}_\perp(s;t,x,v)$ (position $\mathbf{X}_\perp(s;t,x,v)$ is still continuous at specular reflection), we are able to derive an integral equations of $[\mathbf{X}_\parallel(s;t,x,v), \mathbf{V}_\parallel(s;t,x,v)]$ without broken into small discontinuous pieces (6.67) at each specular reflection. In other words, we can use the standard ODE theory to estimate these tangential parts. Our ODE method refers such ODE (Gronwall) estimates (6.63) which lead to the final conclusion of the theorem.

With such crucial estimates, we are able to design anisotropic norms in terms of singularity of $\frac{1}{\alpha}$. Thanks to $\int_0^t \int_u \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} \frac{1}{\alpha(X(s),u)^\beta} \lesssim \alpha^{-\beta+1/2}$ and $\int_0^t \int_u \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} \frac{1}{\alpha(X(s),u)^{\beta-1/2}} \lesssim \alpha^{-\beta+1}$ from the dynamical non-local to local estimates for $\beta > 1$, we have exact cancellations of the power of α in the coefficients on the right hand side, and we are able to close the estimates. For $|v|$ either small or large, more careful analysis is needed. In particular, it is important to use the weight function of $e^{-\varpi\langle v\rangle t}$ in (1.22) to control both the growth in Theorem 5 as well as $|v|$ in front of $\nabla_x X_{\mathbf{cl}}$ and $\nabla_v V_{\mathbf{cl}}$ to control singularity of $|v|$ in (1.35).

1.4 Bounce-back reflection BC

We recall the bounce-back reflection boundary condition (1.12) and the bounce-back cycles in Definition 2.

We define

$$\partial_t f(0) = \partial_t f_0 := -v \cdot \nabla_x f_0 + \Gamma_{\text{gain}}(f_0, f_0) - \nu(\sqrt{\mu} f_0) f_0. \tag{1.36}$$

Theorem 6 *Assume* $f_0 \in W^{1,\infty}(\Omega \times \mathbb{R}^3)$ *and* $0 < \kappa \le 1$ *in* $0 < \theta < \frac{1}{4}$,

$$\|\langle v\rangle \nabla_x f_0\|_\infty + \|\nabla_v f_0\|_\infty + \|e^{\theta|v|^2} \partial_t f_0\|_\infty + \|e^{\theta|v|^2} f_0\|_\infty < +\infty,$$

and the compatibility conditions

$$f_0(x,v) = f_0(x,-v), \quad v\cdot\nabla_x f_0(x,v) = -v\cdot\nabla_x f_0(x,-v) \quad on \quad \gamma_-. \tag{1.37}$$

Then there is $T = T(\|e^{\theta|v|^2} f_0\|_\infty) > 0$ *so that for all* $0 \le t \le T$

$$\begin{aligned}&\|e^{-\varpi\langle v\rangle t}\frac{\alpha}{\langle v\rangle^2}\nabla_x f(t)\|_\infty + \|e^{-\varpi\langle v\rangle t}\frac{|v|\alpha^{1/2}}{\langle v\rangle^2}\nabla_v f(t)\|_\infty + \|e^{\theta|v|^2}\partial_t f(t)\|_\infty \\ &\quad\lesssim_{\xi,t} \|\langle v\rangle\nabla_x f_0\|_\infty + \|\nabla_v f_0\|_\infty + P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty) + P(\|e^{\theta|v|^2} f_0\|_\infty),\end{aligned} \tag{1.38}$$

for some polynomial P.

Moreover, if $f_0 \in C^1$ *then* $f \in C^1$ *away from the grazing set* γ_0. *Furthermore, if* $F_0 = \mu + \sqrt{\mu} g_0$ *with* $\|e^{\theta|v|^2} g_0\|_\infty \ll 1$, *then the unique bounded global-in-time solution* $g(t)$ *constructed in* [9] *satisfying* (1.38), *by changing* f, f_0 *to* g, g_0 *for any finite* $t \ge 0$.

There can be no size restriction on initial data $F_0 = \sqrt{\mu} f_0$. We remark that the bounce-back case enjoys explicit expressions of $\partial_{\mathbf{e}} X_{\mathbf{cl}}(s;t,x,v)$ and $\partial_{\mathbf{e}} V_{\mathbf{cl}}(s;t,x,v)$ for $\partial_{\mathbf{e}} \in \{\partial_t, \nabla_x, \nabla_v\}$. Since $\partial_x t^\ell \lesssim \frac{1}{\alpha}$ and $\partial_x x^\ell \lesssim \frac{1}{\sqrt{\alpha}}$, a new difficulty arises in the estimate

$$\partial_x X_{\mathbf{cl}}(s;t,x,v) \lesssim \frac{1}{\alpha},$$

which is too singular to control by our non-local to local estimates (Lemma 1). Roughly speaking, the new difficulty is exactly the opposite to the specular case : ∂x^ℓ and ∂v^ℓ are in desired form but not $\partial_x X_{\mathbf{cl}}(s;t,x,v)$! The key idea is to make a change of variable to transform (see (7.1))

$$\partial_x X_{\mathbf{cl}}(s;t,x,v) \lesssim v^\ell \partial_x t^\ell + \partial_x x^\ell,$$

while $\partial_x t^\ell$ captures the worst singularity of $\frac{1}{\alpha}$. Fortunately, $\partial_x t^\ell$ is paired with $\partial_t f$, which is bounded, from the time-invariance of the problem and we are able to close the estimate.

1.5 Non-existence of $\nabla^2 f$ up to the boundary

In the appendix, we demonstrate that, our estimates can not be valid for higher order derivatives. Otherwise, if $\partial^2 f$ were to exist up to the boundary, we observe that from taking second derivatives of the Boltzmann equation:

$$v_n \partial_n^2 f = -\partial_{tn} f - (\partial_n v_n)\partial_n f - \sum_{i=1}^{2} \partial_n (v_{\tau_i})\partial_{\tau_i} f$$
$$-\sum_{i=1}^{2} v_{\tau_i \partial_{n\tau_i}} f - \nu(F)\partial_n f + \partial_n K(f) + \partial_n \Gamma_{\text{gain}}(f, f).$$

If $|\partial_n f| \geq \frac{1}{\sqrt{\alpha}}$ and $\partial_n K(f)$ behaves as $K(\partial_n f)$ then at the boundary, we have

$$|\partial_n f| \geq \frac{1}{|v_n|} \notin L^1_{loc}(\mathbb{R}^1),$$

so that $\partial_n K(f)$ is not defined. Since $|\partial_n f|$ is expected to behave, from (1.19), as bad as $\frac{1}{\sqrt{\alpha}}$ for all diffusive, specular and bounce-back cases, we are able to identify initial conditions such that $|\partial_n f| \geq \frac{1}{|v_n|}$ for some future time.

2 Preliminary

Before the trajectory hits the boundary, $t - s < t_{\mathbf{b}}(x, v)$, we have $[X(s; t, x, v), V(s; t, x, v)] = [x - (t - s)v, v]$ with the initial condition $[X(t; t, x, v), V(t; t, x, v)] = [x, v]$. On the other hand, when the trajectory hits the boundary we define the generalized characteristics as follows:

Definition 2 [9]
Let $(x, v) \notin \gamma_0$ and $(t^0, x^0, v^0) = (t, x, v)$.

(i) Define the specular cycles, $\ell \geq 1$,

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^\ell - t_{\mathbf{b}}(x^\ell, v^\ell), x_{\mathbf{b}}(x^\ell, v^\ell), v^\ell - 2n(x^\ell)(v^\ell \cdot n(x^\ell))).$$

(ii) Define the bounce-back cycles, $\ell \geq 1$,

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^\ell - t_{\mathbf{b}}(x^\ell, v^\ell), x_{\mathbf{b}}(x^\ell, v^\ell), -v^\ell).$$

Then for $\ell \geq 1$

$$t^\ell = t^1 - (\ell - 1)t_{\mathbf{b}}(x^1, v^1), \quad x^\ell = \frac{1 - (-1)^\ell}{2} x^1$$
$$+ \frac{1 + (-1)^\ell}{2} x^2, \quad v^{\ell+1} = (-1)^{\ell+1} v.$$

(iii) For both specular and bounce-back cycles we define the backward trajectory as

$$X_{\mathbf{cl}}(s;t,x,v) = \sum_{\ell} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\left\{x^{\ell} - (t^{\ell}-s)v^{\ell}\right\},$$
$$V_{\mathbf{cl}}(s;t,x,v) = \sum_{\ell} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)v^{\ell}.$$

Note that if $G(t,x,v)$ solves $\partial_t G + v\cdot\nabla_x G = 0$ with a boundary condition (either specular, or bounce-back boundary condition) then

$$G(t,x,v) = G(s, X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)),$$

where $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ is defined respectively [9].

The following crucial invariant property of α under operator $v\cdot\nabla_x$ is the key for our analysis.

Lemma 2 (Velocity lemma, Lemma 1 of [9]) *Along the backward trajectory we define*

$$\alpha(s;t,x,v) := \alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)).$$

Then there exists $\mathcal{C} = \mathcal{C}(\xi) > 0$ *such that, for all* $0 \le s_1, s_2 \le t$,

$$e^{-\mathcal{C}|v||s_1-s_2|}\alpha(s_1;t,x,v) \le \alpha(s_2;t,x,v) \le e^{\mathcal{C}|v||s_1-s_2|}\alpha(s_1;t,x,v).$$

Proof The proof is basically same as the proof of Lemma 1 of [9] but the definition of α is slightly different. By an explicit computation, we have

$$\begin{aligned} v\cdot\nabla_x\alpha &= 2v\cdot\nabla\xi(x)[v\cdot\nabla^2\xi\cdot v] - 2v\cdot\nabla\xi(x)[v\cdot\nabla^2\xi\cdot v] \\ &\quad - 2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi(x) \\ &= -2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi(x) = O_\xi(1)|v|^3|\xi(x)| \\ &= O_\xi(1)|v|\alpha(x,v), \end{aligned} \tag{2.1}$$

where we used $\{v\cdot\nabla^2\xi(x)\cdot v\} \simeq |v|^2$ from (1.13). Therefore there exists $\mathcal{C} = \mathcal{C}_\xi > 0$ such that

$$-\mathcal{C}|v|\alpha(x,v) \le v\cdot\nabla_x\alpha(x,v) \le \mathcal{C}|v|\alpha(x,v).$$

Since

$$\frac{d}{ds}\alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) = v\cdot\nabla_x\alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)),$$

we conclude the lemma. □

Lemma 3 *Let* Ω *be convex as in* (1.13). *Then for* $(x, v) \in \gamma_+$

$$t_{\mathbf{b}}(x, v) \gtrsim_\Omega \frac{\sqrt{\alpha(x, v)}}{|v|^2}, \tag{2.2}$$

and

$$t_{\mathbf{b}}(x, v) \lesssim_\Omega \frac{\sqrt{\alpha(x, v)}}{|v|^2}. \tag{2.3}$$

Proof Recall Lemma 2. For (2.2) it suffices to prove $t_{\mathbf{b}}(x, v) \gtrsim_\Omega \frac{|n(x)\cdot v|}{|v|^2}$. For $x \in \partial\Omega$ there exists $0 < \delta \ll 1$ such that

$$\sup_{\substack{y\in\partial\Omega \\ |x-y|<\delta}} \frac{|(x-y)\cdot n(x)|}{|x-y|^2} \lesssim \max_{\substack{y\in\partial\Omega \\ |x-y|<\delta}} |\nabla^2\xi(x)|.$$

If $|x-y| \geq \delta$ then $\frac{|(x-y)\cdot n(x)|}{|x-y|^2} \leq \delta^{-2}|(x-y)\cdot n(x)| \lesssim_{\delta,\Omega} 1$. By the compactness of Ω and $\partial\Omega$, we have $|(x-y)\cdot n(x)| \lesssim |x-y|^2$ for all $x, y \in \partial\Omega$. Taking the inner product of $x - x_{\mathbf{b}}(x, v) = t_{\mathbf{b}}(x, v)v$ with $n(x)$, we have

$$\begin{aligned} t_{\mathbf{b}}(x, v)|v\cdot n(x)| &= |(x - x_{\mathbf{b}}(x, v))\cdot n(x)| \\ &\lesssim |x - x_{\mathbf{b}}(x, v)|^2 = C_\Omega |v|^2 |t_{\mathbf{b}}(x, v)|^2, \end{aligned}$$

and this proves (2.2).

For (2.3) it suffices to show $t_{\mathbf{b}}(x, v) \lesssim_\xi \frac{|n(x)\cdot v|}{|v|^2}$. Since $\xi(x) = 0 = \xi(x - t_{\mathbf{b}}(x, v)v)$ for $(x, v) \in \gamma_+$, we have

$$\begin{aligned} 0 = \xi(x - t_{\mathbf{b}}(x, v)v) &= \xi(x) + \int_0^{t_{\mathbf{b}}(x,v)} [-v\cdot\nabla_x\xi(x - sv)]\mathrm{d}s \\ &= [-v\cdot\nabla_x\xi(x)]t_{\mathbf{b}}(x, v) + \int_0^{t_{\mathbf{b}}(x,v)}\int_0^s \{v\cdot\nabla_x^2\xi(x-\tau v)\cdot v\}\mathrm{d}\tau\mathrm{d}s. \end{aligned}$$

By the convexity of ξ in (1.13) we have $[v\cdot\nabla_x\xi(x)]t_{\mathbf{b}}(x, v) \geq \frac{(t_{\mathbf{b}}(x,v))^2}{2} C_\xi |v|^2$, and therefore this proves (2.3). $\square$

We need a version of Gronwall's inequality for matrices:

Lemma 4 *Let* $a(\tau), b(\tau), f(\tau), g(\tau) \geq 0$ *for all* $0 \leq \tau \leq t$, *and satisfy*

$$\begin{bmatrix} a(\tau) \\ b(\tau) \end{bmatrix} \lesssim \begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix} \begin{bmatrix} \int_\tau^t a(\tau')\mathrm{d}\tau' \\ \int_\tau^t b(\tau')\mathrm{d}\tau' \end{bmatrix} + \begin{bmatrix} g(t-\tau) \\ h(t-\tau) \end{bmatrix}$$

then

$$\begin{bmatrix} a(\tau) \\ b(\tau) \end{bmatrix} \lesssim e^{C(\tau-t)} \begin{bmatrix} 1 & |\tau-t| \\ |v|^2|\tau-t| & 1 \end{bmatrix} \begin{bmatrix} g(0) \\ h(0) \end{bmatrix} + \int_t^\tau e^{C(\tau-\tau')} \begin{bmatrix} 1 & |\tau-\tau'| \\ |v|^2|\tau-\tau'| & 1 \end{bmatrix} \begin{bmatrix} g(t-\tau') \\ h(t-\tau') \end{bmatrix} \mathrm{d}\tau'. \quad (2.4)$$

Proof First we consider A^ε, B^ε solving, for $\varepsilon > 0$,

$$\begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} = C \begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix} \begin{bmatrix} \int_\tau^t A^\varepsilon(\tau')\mathrm{d}\tau' \\ \int_\tau^t B^\varepsilon(\tau')\mathrm{d}\tau' \end{bmatrix} + \begin{bmatrix} g(t-\tau) \\ h(t-\tau) \end{bmatrix} + \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}. \quad (2.5)$$

We claim that

$$\begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} \lesssim e^{C(\tau-t)} \begin{bmatrix} 1 & |\tau-t| \\ |v|^2|\tau-t| & 1 \end{bmatrix} \begin{bmatrix} g(0)+\varepsilon \\ h(0)+\varepsilon \end{bmatrix} + \int_t^\tau e^{C(\tau-\tau')} \begin{bmatrix} 1 & |\tau-\tau'| \\ |v|^2|\tau-\tau'| & 1 \end{bmatrix} \begin{bmatrix} g(t-\tau')+\varepsilon \\ h(t-\tau')+\varepsilon \end{bmatrix} \mathrm{d}\tau'. \quad (2.6)$$

We consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix}$. Denote

$$r_1 := \frac{1+\sqrt{5}}{2},\ r_2 := \frac{1-\sqrt{5}}{2},\ r_3 := \frac{1}{\sqrt{5}}.$$

Then we diagonalize this matrix as

$$\begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_1|v| & r_2|v| \end{bmatrix} \begin{bmatrix} r_1|v| & 0 \\ 0 & r_2|v| \end{bmatrix} \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix}.$$

Denote $\begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} := \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix}$ and rewrite the equations as

$$\frac{d}{d\tau} \begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} = C \begin{bmatrix} r_1|v| & 0 \\ 0 & r_2|v| \end{bmatrix} \begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} + \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} g(t-\tau)+\varepsilon \\ h(t-\tau)+\varepsilon \end{bmatrix}.$$

Directly we compute

$$\begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix} = \begin{bmatrix} e^{Cr_1|v|(\tau-t)}\mathcal{A}^\varepsilon(t) \\ e^{C_\xi r_2|v|(\tau-t)}\mathcal{B}^\varepsilon(t) \end{bmatrix}$$
$$+\int_t^\tau \begin{bmatrix} e^{Cr_2|v|(\tau-\tau')} & 0 \\ 0 & e^{Cr_2|v|(\tau-\tau')} \end{bmatrix} \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} g(t-\tau')+\varepsilon \\ h(t-\tau')+\varepsilon \end{bmatrix} \mathrm{d}\tau'.$$

Then

$$\begin{bmatrix} A^\varepsilon(\tau) \\ B^\varepsilon(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r_1|v| & r_2|v| \end{bmatrix} \begin{bmatrix} \mathcal{A}^\varepsilon(\tau) \\ \mathcal{B}^\varepsilon(\tau) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ r_1|v| & r_2|v| \end{bmatrix} \begin{bmatrix} e^{Cr_1|v|(\tau-t)} & 0 \\ 0 & e^{Cr_2|v|(\tau-t)} \end{bmatrix} \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} A^\varepsilon(t) \\ B^\varepsilon(t) \end{bmatrix}$$
$$+\int_t^\tau \begin{bmatrix} 1 & 1 \\ r_1|v| & r_2|v| \end{bmatrix} \begin{bmatrix} e^{Cr_1|v|(\tau-\tau')} & 0 \\ 0 & e^{Cr_2|v|(\tau-\tau')} \end{bmatrix}$$
$$\times \begin{bmatrix} -r_2r_3 & r_3\frac{1}{|v|} \\ r_1r_3 & -r_3\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} g(t-\tau')+\varepsilon \\ h(t-\tau')+\varepsilon \end{bmatrix} \mathrm{d}\tau'.$$

Directly, the RHS equals

$$\begin{bmatrix} r_3\left(r_1e^{Cr_2|v|(\tau-t)} - r_2e^{Cr_1|v|(\tau-t)}\right) & \frac{r_3}{|v|}\left(e^{Cr_1|v|(\tau-t)} - e^{Cr_2|v|(\tau-t)}\right) \\ -r_1r_2r_3|v|\left(e^{Cr_1|v|(\tau-t)} - e^{Cr_2|v|(\tau-t)}\right) & r_3\left(r_1e^{Cr_2|v|(\tau-t)} - r_2e^{Cr_1|v|(\tau-t)}\right) \end{bmatrix} \begin{bmatrix} A^\varepsilon(t) \\ B^\varepsilon(t) \end{bmatrix}$$
$$+\int_t^\tau \begin{bmatrix} r_3\left(r_1e^{Cr_2|v|(\tau-\tau')} - r_2e^{Cr_1|v|(\tau-\tau')}\right) & \frac{r_3}{|v|}\left(e^{Cr_1|v|(\tau-\tau')} - e^{Cr_2|v|(\tau-\tau')}\right) \\ -r_1r_2r_3|v|\left(e^{Cr_1|v|(\tau-\tau')} - e^{Cr_2|v|(\tau-\tau')}\right) & r_3\left(r_1e^{Cr_2|v|(\tau-\tau')} - r_2e^{Cr_1|v|(\tau-\tau')}\right) \end{bmatrix}$$
$$\times \begin{bmatrix} g(t-\tau')+\varepsilon \\ h(t-\tau')+\varepsilon \end{bmatrix} \mathrm{d}\tau'.$$

By expansion we have $|e^{Cr_1|v|(\tau-t)} - e^{Cr_2|v|(\tau-t)}| \lesssim_{C^\xi,\delta} |v||\tau-t|e^{C_{\xi,\delta}|v|(\tau-t)}$. Therefore we conclude (2.6).

Now we claim

$$a(\tau) \le A(\tau), \quad b(\tau) \le B(\tau), \quad \text{for all} \quad \tau \le t. \tag{2.7}$$

First we claim that $a(\tau) \le A^\varepsilon(\tau)$ and $b(\tau) \le B^\varepsilon(\tau)$ for all τ. Otherwise, we should have at least for some time τ_0 such that $a(\tau) \le A^\varepsilon(\tau)$ and $b(\tau) \le B^\varepsilon(\tau)$ for $\tau_0 \le \tau \le t$ but either $a(\tau) > A^\varepsilon(\tau)$ or $b(\tau) > B^\varepsilon(\tau)$ for a small neighborhood of $\tau > \tau_0$. Especially either $a(\tau_0) = A^\varepsilon(\tau_0)$ or $b(\tau_0) = B^\varepsilon(\tau_0)$.

But this is impossible. Since

$$\begin{bmatrix} A^\varepsilon(\tau)-a(\tau) \\ B^\varepsilon(\tau)-b(\tau) \end{bmatrix} \geq C \begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix} \begin{bmatrix} \int_\tau^t (A^\varepsilon(\tau')-a(\tau'))\mathrm{d}\tau' \\ \int_\tau^t (B^\varepsilon(\tau')-b(\tau'))\mathrm{d}\tau' \end{bmatrix} + \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix},$$

we have $\begin{bmatrix} A^\varepsilon(\tau)-a(\tau) \\ B^\varepsilon(\tau)-b(\tau) \end{bmatrix} \geq \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix} > 0$ as $\tau \to \tau_0^+$. Then we prove the inequalities (2.7) by letting $\varepsilon \to 0$.

Finally we prove the claim (2.4) from (2.6) and (2.7) and letting $\varepsilon \to 0$. □

Recall the expression of Γ_{gain} and ν in (1.7) and (1.8). Due to the Grad estimate in [4]

$$\Gamma_{\text{gain}}(\sqrt{\mu}, g) + \Gamma_{\text{gain}}(g, \sqrt{\mu}) = \int_{\mathbb{R}^3} \mathbf{k}_2(v,u)g(u)\mathrm{d}u,$$
$$\nu(\sqrt{\mu}g) = \int_{\mathbb{R}^3} \mathbf{k}_1(v,u)g(u)\mathrm{d}u, \tag{2.8}$$

where

$$\mathbf{k}_1(u,v) = |u-v|^\kappa e^{-\frac{|v|^2+|u|^2}{2}} \int_{\mathbb{S}^2} q_0(\frac{v-u}{|v-u|}\cdot\omega)\mathrm{d}\omega,$$
$$\mathbf{k}_2(u,v) = \frac{2}{|u-v|^2} e^{-\frac{1}{8}|u-v|^2-\frac{1}{8}\frac{(|u|^2-|v|^2)^2}{|u-v|^2}}$$
$$\times \int_{w\cdot(u-v)=0} q_0\left(\frac{u-v}{\sqrt{|u-v|^2+|w|^2}}\cdot\frac{u-v}{|u-v|}\right)e^{-|w+\varsigma|^2}$$
$$(|w|^2+|u-v|^2)^{\frac{\kappa}{2}}\mathrm{d}w, \tag{2.9}$$

and $\varsigma := \left(\frac{v+u}{2}\cdot\frac{w}{|w|}\right)\frac{w}{|w|}$. See page 315 of [8] for details.

Lemma 5 *For* $0 \leq \kappa \leq 1$,

$$|\mathbf{k}_1(u,v)| + |\mathbf{k}_2(u,v)| \lesssim \{|v-u|^\kappa + |v-u|^{-2+\kappa}\}e^{-\frac{1}{8}|v-u|^2-\frac{1}{8}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}}$$
$$\lesssim \frac{e^{-\frac{1}{10}|v-u|^2-\frac{1}{10}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}}}{|v-u|^{2-\kappa}}.$$

For $\varrho > 0$ *and* $-2\varrho < \theta < 2\varrho$ *and* $\zeta \in \mathbb{R}$, *we have for* $0 < \kappa \leq 1$,

$$\int_{\mathbb{R}^3} \{|v-u|^\kappa + |v-u|^{-2+\kappa}\}e^{-\varrho|v-u|^2-\varrho\frac{(|v|^2-|u|^2)^2}{|v-u|^2}} \frac{\langle v\rangle^\zeta e^{\theta|v|^2}}{\langle u\rangle^\zeta e^{\theta|u|^2}}\mathrm{d}u \lesssim \langle v\rangle^{-1}.$$

Proof The proof is based on [9]. Note that

$$\frac{\langle v\rangle^{\zeta}e^{\theta|v|^2}}{\langle u\rangle^{\zeta}e^{\theta|u|^2}} \lesssim [1+|v-u|^2]^{\frac{\zeta}{2}}e^{-\theta(|u|^2-|v|^2)}.$$

Set $v-u=\eta$ and $u=v-\eta$ in the integration. Now we compute the total exponent of the integrand as

$$\begin{aligned}
&-\varrho|\eta|^2-\varrho\frac{\||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\theta\{|v-\eta|^2-|v|^2\}\\
&\quad=-2\varrho|\eta|^2+4\varrho\{v\cdot\eta\}-4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2}-\theta\{|\eta|^2-2v\cdot\eta\}\\
&\quad=(-\theta-2\varrho)\,|\eta|^2+(4\varrho+2\theta)\,v\cdot\eta-4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2}.
\end{aligned}$$

Since $-2\varrho<\theta<2\varrho$, the discriminant of the above quadratic form of $|\eta|$ and $\frac{v\cdot\eta}{|\eta|}$ is negative: $(4\varrho+2\theta)^2+16\varrho(-\theta-2\varrho)=4\theta^2-16\varrho^2<0$. We thus have

$$-\varrho|\eta|^2-\varrho\frac{\||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\theta\{|v-\eta|^2-|v|^2\}\lesssim_{\varrho,\theta}-\left\{\frac{|\eta|^2}{2}+|v\cdot\eta|\right\}.$$

Hence, for $0\le\kappa\le1$ the integration is bounded by

$$\int_{\mathbb{R}^3}\left\{|\eta|^{\kappa}+|\eta|^{-2+\kappa}\right\}\langle\eta\rangle^{\zeta}e^{-C_{\varrho,\theta}|\eta|^2}\lesssim_{\varrho,\theta,\kappa}1.$$

Therefore in order to prove Lemma 5 it suffices to consider the case $|v|\ge1$. We make another change of variables $\eta_{\|}=\left\{\eta\cdot\frac{v}{|v|}\right\}\frac{v}{|v|}$ and $\eta_{\perp}=\eta-\eta_{\|}$, so that $|v\cdot\eta|=|v\|\eta_{\|}|$ and $|v-u|\ge|\eta_{\perp}|$. We can absorb $\langle\eta\rangle^{\zeta}$, $|\eta|^{\kappa}\langle\eta\rangle^{\zeta}$ by $e^{-C_{\varrho,\theta}|\eta|^2}$, and bound the integration by, for $0<\kappa\le1$,

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left\{1+|\eta|^{-2+\kappa}\right\}e^{-C_{\varrho,\theta}\left\{\frac{|\eta|^2}{2}+|v\cdot\eta|\right\}}\mathrm{d}\eta\\
&\le\int_{\mathbb{R}^3}\left\{1+|\eta|^{-2+\kappa}\right\}e^{-\frac{C_{\varrho,\theta}}{2}|\eta|^2}e^{-C_{\varrho,\theta}|v\cdot\eta|}\mathrm{d}\eta\\
&\le\int_{\mathbb{R}^2}\{1+|\eta_{\perp}|^{-2+\kappa}\}e^{-\frac{C_{\varrho,\theta}}{2}|\eta_{\perp}|^2}\left\{\int_{\mathbb{R}}e^{-C_{\varrho,\theta}|v|\times|\eta_{\|}|}\mathrm{d}|\eta_{\|}|\right\}\mathrm{d}\eta_{\perp}\\
&\lesssim\langle v\rangle^{-1}\int_{\mathbb{R}^2}\{1+|\eta_{\perp}|^{-2+\kappa}\}e^{-\frac{C_{\varrho,\theta}}{2}|\eta_{\perp}|^2}\left\{\int_0^{\infty}e^{-C_{\varrho,\theta}y}\mathrm{d}y\right\}\mathrm{d}\eta_{\perp}\\
&\lesssim\langle v\rangle^{-1},
\end{aligned}$$

where $y=|v||\eta_{\|}|$. □

We define

$$\Gamma_{\mathrm{gain},v}(g_1,g_2)(v) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)\nabla_v(\sqrt{\mu})(u)g_1(u')g_2(v')\mathrm{d}\omega\mathrm{d}u, \tag{2.10}$$

where $u' = u - [(u-v)\cdot\omega]\omega$ and $v' = v + [(u-v)\cdot\omega]\omega$.

Lemma 6 (i) *For* $0<\theta<\frac{1}{4}$ *and* $0\le\kappa\le 1$, *there exists* $C_\theta>0$ *such that, for* $(i,j)=(1,2)$ *or* $(i,j)=(2,1)$,

$$\left|\Gamma_{\mathrm{gain}}(g_1,g_2)(v)\right| \lesssim_\theta \|e^{\theta|v|^2}g_i\|_\infty \int_{\mathbb{R}^3} \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}|g_j(u)|\mathrm{d}u, \tag{2.11}$$

and

$$\begin{aligned}
\left|\Gamma_{\mathrm{gain}}(g_1,g_2)(v)\right| &\lesssim_\theta \langle v\rangle^\kappa e^{-\theta|v|^2}\|e^{\theta|v|^2}g_1\|_\infty\|e^{\theta|v|^2}g_2\|_\infty,\\
\left|\Gamma_{\mathrm{gain},v}(g_1,g_2)(v)\right| &\lesssim_\theta \langle v\rangle^\kappa e^{-\theta|v|^2}\|e^{\theta|v|^2}g_1\|_\infty\|e^{\theta|v|^2}g_2\|_\infty,\\
|\nu(\sqrt{\mu}g_1)g_2(v)| &\lesssim_\theta \|e^{\theta|v|^2}g_2\|_\infty \int_{\mathbb{R}^3}\frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}|g_1(u)|\mathrm{d}u.
\end{aligned}$$

(ii) *For* $p\in[1,\infty)$ *and* $0<\theta<\frac{1}{4}$, *and for* $(i,j)=(1,2)$ *or* $(i,j)=(2,1)$,

$$\begin{aligned}
\|\Gamma_{\mathrm{gain}}(g_1,g_2)\|_p &\lesssim_{\theta,p} \|e^{\theta|v|^2}g_i\|_\infty\|g_j\|_p,\\
\|\nu(\sqrt{\mu}g_1)g_2\|_p &\lesssim_{\theta,p} \|e^{\theta|v|^2}g_2\|_\infty\|g_1\|_p,\\
\left|\iint_{\Omega\times\mathbb{R}^3}\Gamma_{\mathrm{gain}}(g_1,g_2)g_3\mathrm{d}v\mathrm{d}x\right| &\lesssim_{\theta,p} \|e^{\theta|v|^2}g_i\|_\infty\|g_j\|_p\|g_3\|_q,\\
\left|\iint_{\Omega\times\mathbb{R}^3}\nu(\sqrt{\mu}g_1)g_2g_3\mathrm{d}v\mathrm{d}x\right| &\lesssim_{\theta,p} \|e^{\theta|v|^2}g_2\|_\infty\|g_1\|_p\|g_3\|_q.
\end{aligned}$$

(iii) *For* $p\in[1,\infty)$ *and* $0<\theta<\frac{1}{4}$, *and for* $(i,j)=(1,2)$ *or* $(i,j)=(2,1)$,

$$\begin{aligned}
\|\nabla_v[\Gamma_{\mathrm{gain}}(g_1,g_2)]\|_p &\lesssim_{\theta,p} \sum_{(i,j)}\|e^{\theta|v|^2}g_i\|_\infty\|\nabla_v g_j\|_p,\\
\|\nu(\sqrt{\mu}\nabla_v g_1)g_2\|_p &\lesssim_{\theta,p} \|e^{\theta|v|^2}g_2\|_\infty\|\nabla_v g_1\|_p,\\
\left|\iint_{\Omega\times\mathbb{R}^3}\nabla_v\Gamma_{\mathrm{gain}}(g_1,g_2)g_3\mathrm{d}v\mathrm{d}x\right| &\lesssim_{\theta,p} \sum_{(i,j)}\|e^{\theta|v|^2}g_i\|_\infty\|\nabla_v g_j\|_p\|g_3\|_q,\\
\left|\iint_{\Omega\times\mathbb{R}^3}\nu(\sqrt{\mu}\nabla_v g_1)g_2g_3\mathrm{d}v\mathrm{d}x\right| &\lesssim_{\theta,p} \|e^{\theta|v|^2}g_2\|_\infty\|\nabla_v g_1\|_p\|g_3\|_q.
\end{aligned}$$

(iv) *Let* $[Y, W] = [Y(x,v), W(x,v)] \in \Omega \times \mathbb{R}^3$. *For* $0 < \theta < \frac{1}{4}$ *and* $\partial_{\mathbf{e}} \in \{\partial_t, \nabla_x, \nabla_v\}$,

$$\begin{aligned}
&|\partial_{\mathbf{e}} \Gamma_{\text{gain}}(g,g)(Y,W)| \\
&\quad \lesssim |\partial_{\mathbf{e}} Y| \| e^{\theta|v|^2} g\|_\infty \int_{\mathbb{R}^3} \frac{e^{-C_\theta |u-W|^2}}{|u-W|^{2-\kappa}} |\nabla_x g(Y,u)| \mathrm{d}u \\
&\qquad + |\partial_{\mathbf{e}} W| \| e^{\theta|v|^2} g\|_\infty \int_{\mathbb{R}^3} \frac{e^{-C_\theta |u-W|^2}}{|u-W|^{2-\kappa}} |\nabla_v g(Y,u)| \mathrm{d}u \\
&\qquad + \langle v\rangle^\kappa e^{-\theta|v|^2} |\partial_{\mathbf{e}} W| \| e^{\theta|v|^2} g\|_\infty^2.
\end{aligned}$$

Proof (i) First we show (2.11) for $(i,j) = (1,2)$. Clearly

$$|\Gamma_{\text{gain}}(g_1, g_2)| \lesssim |\Gamma_{\text{gain}}(e^{-\theta|v|^2}, |g_2|)| \times \| e^{\theta|v|^2} g_1\|_\infty.$$

By the Grad estimate, (page 315, [8]) we bound $|\Gamma_{\text{gain}}(e^{-\theta|v|^2}, |g_2|)|$ by $\int_{\mathbb{R}^2} \mathbf{k}_2(v,u)|g_2(u)|\mathrm{d}u$ with different exponent of $\mathbf{k}_2(v,u)$. Then we use Lemma 5 to conclude (2.11).

For the second estimate we use (1.9)

$$\begin{aligned}
|\Gamma_{\text{gain}}(g_1,g_2)(v)| &\lesssim \Gamma_{\text{gain}}(e^{-\theta|v|^2}, e^{-\theta|v|^2}) \times \| e^{\theta|v|^2} g_1\|_\infty \| e^{\theta|v|^2} g_2\|_\infty \\
&= e^{-\theta|v|^2} \iint B(v-u,\omega)\sqrt{\mu(u)} e^{-\theta|u|^2} \mathrm{d}\omega \mathrm{d}u \\
&\quad \times \| e^{\theta|v|^2} g_1\|_\infty \| e^{\theta|v|^2} g_2\|_\infty \\
&\lesssim \langle v\rangle^\kappa e^{-\theta|v|^2} \| e^{\theta|v|^2} g_1\|_\infty \| e^{\theta|v|^2} g_2\|_\infty,
\end{aligned}$$

where we have used $|u'|^2 + |v'|^2 = |u|^2 + |v|^2$. The third estimate follows similarly with $\nabla_u(\sqrt{\mu})(u) \lesssim \mu(u)^{1/2-\delta}$ for any $\delta > 0$. The fourth estimate follows from

$$\begin{aligned}
e^{-\theta|v|^2} \nu(\sqrt{\mu} g_1)(v) &\lesssim \int_{\mathbb{R}^3} |v-u|^\kappa e^{-\theta|v|^2} \sqrt{\mu(u)} |g_1(u)| \mathrm{d}u \\
&\lesssim \int_{\mathbb{R}^3} \frac{e^{-C_\theta |v-u|^2}}{|v-u|^{2-\kappa}} |g_1(u)| \mathrm{d}u,
\end{aligned}$$

and $e^{-\theta|v|^2}|v-u|^\kappa \sqrt{\mu(u)} \lesssim \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}$.

(ii) The first two estimates are a direct consequence of *(i)*:

$$
\begin{aligned}
\|\Gamma_{\text{gain}}(g_1,g_2)\|_p &\lesssim \|e^{\theta|v|^2}g_i\|_\infty \left\|\left(\int_u \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}\right)^{1/q}\left(\int_u \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}|g_j(u)|^p\right)^{1/p}\right\|_{L^p_v}\\
&\lesssim \|e^{\theta|v|^2}g_i\|_\infty \left(\int_u \frac{e^{-C_\theta|u|^2}}{|u|^{2-\kappa}}\right)^{1/q}\left(\int_u |g_j(u)|^p\int_v \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}\right)^{1/p}\\
&\lesssim \|e^{\theta|v|^2}g_i\|_\infty \left(\int_u \frac{e^{-C_\theta|u|^2}}{|u|^{2-\kappa}}\right)\|g_j\|_p\\
&\lesssim \|e^{\theta|v|^2}g_i\|_\infty\|g_j\|_p.
\end{aligned}
$$

From the fourth estimate of (i), the same proof holds for $\|\nu(\sqrt{\mu}g_1)g_2\|_p \lesssim_{\theta,p} \|e^{\theta|v|^2}g_2\|_\infty\|g_1\|_p$.

For the third estimate we use (2.11) in order to bound it as

$$
\begin{aligned}
&\|e^{\theta|v|^2}g_i\|_\infty \iiint_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3}\frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}|g_j(x,u)||g_3(x,v)|\mathrm{d}u\mathrm{d}v\mathrm{d}x\\
&\lesssim \left(\iiint \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}|g_j(x,u)|^p\right)^{1/p}\left(\iiint \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}}|g_3(x,u)|^q\right)^{1/q}\\
&\lesssim \|g_j\|_p\|g_3\|_q.
\end{aligned}
$$

The same proof holds when exchanging i and j. Using the fourth estimate of (i), we conclude (ii).

(iii) We compute the velocity derivative of Γ_{gain} after the change of variable $u := v-u$:

$$
\begin{aligned}
&\nabla_v\Gamma_{\text{gain}}(g_1,g_2)\\
&\quad= \nabla_v\left[\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(u,\omega)\sqrt{\mu(u)}g_1(u-(u\cdot\omega)\omega)g_2(v+(u\cdot\omega)\omega)\mathrm{d}\omega\mathrm{d}u\right]\\
&\quad= \Gamma_{\text{gain}}(g_1,\nabla_v g_2)+\Gamma_{\text{gain}}(\nabla_v g_1,g_2)+\Gamma_{\text{gain},v}(g_1,g_2).
\end{aligned}
$$

The two first terms are estimated directly by *(ii)*. For $\Gamma_{\text{gain},v}$, we use the fact $|\nabla_v(\sqrt{\mu})(v-u)| \le C\mu(v-u)^{1/4}$ and then apply (ii). The other estimates are direct consequence of the previous estimates.

(iv) It suffices to show the following computation: For $0<\theta<\frac{1}{4}$ and $\partial_{\mathbf{e}} \in \{\partial_t,\nabla_x,\nabla_v\}$,

$$
\begin{aligned}
&|\partial_{\mathbf{e}}\Gamma_{\text{gain}}(g,g)(Y,W)|\\
&=\left|\partial_{\mathbf{e}}\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}|u|^{\kappa}q_0\Big(\frac{u}{|u|}\cdot\omega\Big)e^{-\frac{|u+W|^2}{4}}g(Y,W+[u\cdot\omega]\omega)\right.\\
&\qquad\qquad\qquad\left.\times g(Y,W+u-(u\cdot\omega)\omega)\mathrm{d}\omega\mathrm{d}u\right|\\
&=|\Gamma_{\text{gain}}(\partial_{\mathbf{e}}Y\cdot\nabla_x g,g)(Y,W)|+|\Gamma_{\text{gain}}(g,\partial_{\mathbf{e}}Y\cdot\nabla_x g)(Y,W)|\\
&\quad+|\Gamma_{\text{gain}}(\partial_{\mathbf{e}}W\cdot\nabla_v g,g)(Y,W)|+|\Gamma_{\text{gain}}(g,\partial_{\mathbf{e}}W\cdot\nabla_v g)(Y,W)|\\
&\quad+\left|\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}|u|^{\kappa}q_0\Big(\frac{u}{|u|}\cdot\omega\Big)(\frac{-1}{2})(u+W)\cdot\partial_{\mathbf{e}}W\sqrt{\mu(u+W)}\right.\\
&\qquad\qquad\left.\times g(Y,W+[u\cdot\omega]\omega)g(Y,W+u-(u\cdot\omega)\omega)\mathrm{d}\omega\mathrm{d}u\right|\\
&\le|\partial_{\mathbf{e}}Y|||e^{\theta|v|^2}g||_\infty\int_{\mathbb{R}^3}\frac{e^{-C_\theta|u-W|^2}}{|u-W|^{2-\kappa}}|\partial_x g(Y,u)|\mathrm{d}u\\
&\quad+|\partial_{\mathbf{e}}W|||e^{\theta|v|^2}g||_\infty\int_{\mathbb{R}^3}\frac{e^{-C_\theta|u-W|^2}}{|u-W|^{2-\kappa}}|\partial_v g(Y,u)|\mathrm{d}u\\
&\quad+|\partial_{\mathbf{e}}W|\langle v\rangle^{\kappa}e^{-\theta|v|^2}||e^{\theta|v|^2}g||_\infty^2,
\end{aligned}
\tag{2.12}
$$

where we have used the change of variables $u-V\mapsto u$. □

Lemma 7 (Local Existence) *For* $0\le\theta<1/4$, *if* $\|e^{\theta|v|^2}f_0\|_\infty<+\infty$ *then there exists* $T>0$ *depending on* $\|e^{\theta|v|^2}f_0\|_\infty$, *such that there exists a unique* $F=\mu+\sqrt{\mu}f$ *which solves the Boltzmann equation (1.1) in* $[0,T]$ *and satisfies the initial condition and boundary conditions* (1.10), (1.11), (1.12) *respectively. In addition* $F(t,x,v)\ge 0$ *on* $[0,T]\times\bar{\Omega}\times\mathbb{R}^3$ *and* f *satisfies, for some* $0<\theta'<\theta$,

$$
\sup_{0\le t\le T}\|e^{\theta'|v|^2}f(t)\|_\infty\lesssim P(\|e^{\theta|v|^2}f_0\|_\infty),
\tag{2.13}
$$

for some polynomial P. *If* f_0 *is continuous and satisfies the compatibility conditions* (1.14), (1.30), (1.37) *respectively, then* f *is continuous away from the grazing set* γ_0.

Moreover, for $0\le\bar{\theta}<\frac{1}{4}$, *if* $\|e^{\bar{\theta}|v|^2}\partial_t f_0\|_\infty\equiv\left\|e^{\bar{\theta}|v|^2}\frac{-v\cdot\nabla_x F_0+Q(F_0,F_0)}{\sqrt{\mu}}\right\|_\infty<+\infty$ *and compatibility conditions* (1.14), (1.30), (1.37) *hold respectively, then*

$$
\sup_{0\le t\le T^*}\left\|e^{\bar{\theta}|v|^2}\partial_t f(t)\right\|_\infty\lesssim P\left(\left\|e^{\bar{\theta}|v|^2}\partial_t f_0\right\|_\infty\right)+P\left(\left\|e^{\theta|v|^2}f_0\right\|_\infty\right),
\tag{2.14}
$$

for some polynomial P.

Furthermore for the diffuse and bounce-back boundary conditions, if $F_0 = \mu + \sqrt{\mu} g_0$ *with* $\|e^{\theta|v|^2} g_0\|_\infty \ll 1$ *for* $0 < \theta < \frac{1}{4}$ *then the results hold for all* $t \geq 0$. *For the specular reflection boundary condition, if* ξ *is real analytic, and if* $\|e^{\theta|v|^2} g_0\|_\infty \ll 1$ *for* $0 < \theta < \frac{1}{4}$ *then the results hold for all* $t \geq 0$.

Note that we only need the real analyticity assumption for the global-in-time solvability in the specular BC which is crucial to bootstrap $L^\infty-$bound using L^2-bound. ($L^2 - L^\infty$ interpolation argument in [9])

Proof We use the positive preserving iteration of [9,12]

$$\partial_t F^{m+1} + v \cdot \nabla_x F^{m+1} + \nu(F^m) F^{m+1} = Q_{\text{gain}}(F^m, F^m), \quad F^{m+1}|_{t=0} = F_0 \geq 0, \tag{2.15}$$

which is equivalent to the following equation when posing $F^m := \sqrt{\mu} f^m$,

$$\partial_t f^{m+1} + v \cdot \nabla_x f^{m+1} + \nu(F^m) f^{m+1} = \Gamma_{\text{gain}}(f^m, f^m), \quad f^{m+1}|_{t=0} = f_0. \tag{2.16}$$

We use the Duhamel formula (ignoring the boundary condition):

$$\begin{aligned}
f^{m+1}(t, x, v) &= e^{-\int_0^t \nu(\sqrt{\mu} f^m)(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)) \mathrm{d}s} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \\
&\quad + \int_0^t e^{-\int_s^t \nu(\sqrt{\mu} f^m)(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau)) \mathrm{d}\tau} \Gamma_{\text{gain}}(f^m, f^m)(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)) \mathrm{d}s.
\end{aligned}$$

The local existence theorem without boundary is standard:

$$\begin{aligned}
&|e^{(\theta - t)|v|^2} f^{m+1}(t, x, v)| \\
&\quad \lesssim |e^{(\theta - t)|v|^2} f_0| + \int_0^t |\Gamma_{\text{gain}}(f^m, f^m)(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))| \mathrm{d}s \\
&\quad \lesssim \|e^{\theta|v|^2} f_0\|_\infty + e^{(\theta - t)|v|^2} \int_0^t \iint_{\mathbb{R}^3 \times \mathbb{S}^2} B(V_{\mathbf{cl}}(s) - u, \omega) \sqrt{\mu(u)} | \\
&\qquad \times f^m(s, X_{\mathbf{cl}}(s), u')| |f^m(s, X_{\mathbf{cl}}(s), v')| \\
&\quad \lesssim \|e^{\theta|v|^2} f_0\|_\infty + \left(\sup_{0 \leq s \leq t} \|e^{(\theta - s)|v|^2} f^m(s)\|_\infty \right)^2 \\
&\qquad \times \int_0^t \iint B(v - u, \omega) \sqrt{\mu(u)} e^{(\theta - t)|v|^2} e^{-(\theta - s)|u'|^2} e^{-(\theta - s)|v'|^2}
\end{aligned}$$

$$\lesssim \|e^{\theta|v|^2} f_0\|_\infty + \left(\sup_{0 \le s \le t} \|e^{(\theta - s)|v|^2} f^m(s)\|_\infty \right)^2$$
$$\times \int_0^t e^{-(t-s)|v|^2} \int_u |v-u|^\kappa \sqrt{\mu(u)}$$
$$\lesssim \|e^{\theta|v|^2} f_0\|_\infty + \left(\sup_{0 \le s \le t} \|e^{(\theta - s)|v|^2} f^m(s)\|_\infty \right)^2$$
$$\times \int_0^t e^{-(t-s)|v|^2} \langle v \rangle \{\mathbf{1}_{|v|>N} + \mathbf{1}_{|v|\le N}\} \mathrm{d}s$$
$$\lesssim \|e^{\theta|v|^2} f_0\|_\infty + \left(\sup_{0 \le s \le t} \|e^{(\theta - s)|v|^2} f^m(s)\|_\infty \right)^2 \left\{ \frac{1}{N^2} + Nt \right\}.$$

Now we choose sufficiently large $N \gg 1$ and then small $0 < T \ll \theta$ to obtain the uniform-in-m estimate

$$\sup_{0 \le t \le T} \|e^{\theta'|v|^2} f^{m+1}(t)\|_\infty \lesssim \|e^{\theta|v|^2} f_0\|_\infty, \tag{2.17}$$

for some $0 < \theta' < \theta$.

With a boundary condition the Duhamel formula takes the form accordingly:

(i) Diffuse reflection boundary condition, on $(x, v) \in \gamma_-$,

$$f^{m+1}(t,x,v) = c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u > 0} f^m(t,x,u) \sqrt{\mu(u)} \{n(x) \cdot u\} \mathrm{d}u. \tag{2.18}$$

(ii) Specular reflection boundary condition, on $(x, v) \in \gamma_-$,

$$f^{m+1}(t,x,v) = f^m(t,x,R_x v), \tag{2.19}$$

where $R_x v = v - 2n(x)(n(x) \cdot v)$.

(iii) Bounce-back reflection boundary condition, on $(x, v) \in \gamma_-$,

$$f^{m+1}(t,x,v) = f^m(t,x,-v). \tag{2.20}$$

We then follow the proof of [9,12] to obtain the same estimates of (2.17). □

3 Traces and the in-flow problems

Recall the phase boundary in (1.3) and the almost grazing set γ_+^ε defined in (1.21). We first estimate the outgoing trace on $\gamma_+ \backslash \gamma_+^\varepsilon$. We remark that for the outgoing part, our estimate is global in time without the need to use cut-off functions, in contrast to the general trace theorem.

Lemma 8 *Assume that* $\varphi = \varphi(v)$ *is* $L^\infty_{loc}(\mathbb{R}^3)$. *For any* $\varepsilon > 0$, *there exists a constant* $C_{\varepsilon,T,\Omega} > 0$ *such that for any* h *in* $L^1([0,T], L^1(\Omega \times \mathbb{R}^3))$ *with* $\partial_t h + v \cdot \nabla_x h + \varphi h$ *is in* $L^1([0,T], L^1(\Omega \times \mathbb{R}^3))$, *we have for all* $0 \le t \le T$,

$$\int_0^t \int_{\gamma_+ \setminus \gamma_+^\varepsilon} |h| \mathrm{d}\gamma \mathrm{d}s \le C_{\varepsilon,T,\Omega} \left[\|h_0\|_1 + \int_0^t \left\{ \|h(s)\|_1 + |[\partial_t + v \cdot \nabla_x + \varphi] h(s)|_1 \right\} \mathrm{d}s \right].$$

Furthermore, for (s, x, v) *in* $[0,T] \times \Omega \times \mathbb{R}^3$, $h(s+s', x+s'v, v)$ *is absolutely continuous in* $s' \in [-\min\{t_{\mathbf{b}}(x,v), s\}, \min\{t_{\mathbf{b}}(x,-v), T-s\}]$.

Proof With a proper change of variables (e.g. Page 247 in [1]) we have

$$\begin{aligned} \int_0^T \iint_{\Omega \times \mathbb{R}^3} h(t,x,v) \mathrm{d}v \mathrm{d}x \mathrm{d}t &= \int_{-\min\{T, t_{\mathbf{b}}(x,v)\}}^0 \iint_{\Omega \times \mathbb{R}^3} h(T+s, x+sv, v) \mathrm{d}v \mathrm{d}x \mathrm{d}s \\ &\quad + \int_0^{\min\{T, t_{\mathbf{b}}(x,-v)\}} \iint_{\Omega \times \mathbb{R}^3} h(0+s, x+sv, v) \mathrm{d}v \mathrm{d}x \mathrm{d}s \\ &\quad + \int_0^T \int_{\gamma_+} \int_{-\min\{t, t_{\mathbf{b}}(x,v)\}}^0 h(t+s, x+sv, v) \mathrm{d}s \mathrm{d}\gamma \mathrm{d}t \\ &\quad + \int_0^T \int_{\gamma_-} \int_0^{\min\{T-t, t_{\mathbf{b}}(x,-v)\}} h(t+s, x+sv, v) \mathrm{d}s \mathrm{d}\gamma \mathrm{d}t. \end{aligned} \tag{3.1}$$

For $(t,x,v) \in [0,T] \times \gamma_+$ and $0 \le s \le \min\{t, t_{\mathbf{b}}(x,v)\}$,

$$\begin{aligned} h(t,x,v) &= h(t-s, x-sv, v) e^{-\varphi(v)s} \\ &\quad + \int_{-s}^0 e^{\varphi(v)\tau} [\partial_t h + v \cdot \nabla_x h + \varphi(v) h](t+\tau, x+\tau v, v) \mathrm{d}\tau. \end{aligned}$$

Now for $(t,x,v) \in [\varepsilon_1, T] \times \gamma_+ \setminus \gamma_+^\varepsilon$, we integrate over $\int_{\varepsilon_1}^T \int_{\gamma_+ \setminus \gamma_+^\varepsilon} \int_{\min\{t, t_{\mathbf{b}}(x,v)\}}^0$ to get

$$
\begin{aligned}
&\min\{\varepsilon_1, \varepsilon^3\} \times \int_{\varepsilon_1}^T \int_{\gamma_+ \setminus \gamma_+^\varepsilon} |h(t,x,v)| \mathrm{d}\gamma \mathrm{d}t \\
&\lesssim \min_{[\varepsilon_1, T]\times[\gamma_+ \setminus \gamma_+^\varepsilon]} \{t, t_{\mathbf{b}}(x,v)\} \times \int_{\varepsilon_1}^T \int_{\gamma_+ \setminus \gamma_+^\varepsilon} |h(t,x,v)| \mathrm{d}\gamma \mathrm{d}t \\
&\lesssim \int_0^T \int_{\gamma_+} \int_{-\min\{t, t_{\mathbf{b}}(x,v)\}}^0 |h(t+s, x+sv, v)| \mathrm{d}s \mathrm{d}\gamma \mathrm{d}t \\
&+T \int_0^T \int_{\gamma_+} \int_{-\min\{t, t_{\mathbf{b}}(x,v)\}}^0 |\partial_t h + v \cdot \nabla_x h + \varphi h|(t+\tau, x+\tau v, v) \mathrm{d}\tau \mathrm{d}\gamma \mathrm{d}t \\
&\lesssim \int_0^T \|h(t)\|_1 \mathrm{d}t + \int_0^T \|[\partial_t + v \cdot \nabla_x + \varphi] h(t)\|_1 \mathrm{d}t,
\end{aligned}
$$

where we have used the integration identity (3.1), and (40) of [9] to obtain $t_{\mathbf{b}}(x,v) \geq C_\Omega |n(x) \cdot v|/|v|^2 \geq C_\Omega \varepsilon^3$ for $(x,v) \in \gamma_+ \setminus \gamma_+^\varepsilon$. Now we choose $\varepsilon_1 = \varepsilon_1(\Omega, \varepsilon)$ as

$$
\varepsilon_1 \leq C_\Omega \varepsilon^3 \leq \inf_{(x,v)\in\gamma_+ \setminus \gamma_+^\varepsilon} t_{\mathbf{b}}(x,v).
$$

We only need to show, for $\varepsilon_1 \leq C_\Omega \varepsilon^3$, that

$$
\int_0^{\varepsilon_1} \int_{\gamma_+ \setminus \gamma_+^\varepsilon} |h(t,x,v)| \mathrm{d}\gamma \mathrm{d}t \lesssim_{\Omega, \varepsilon, \varepsilon_1} \|h_0\|_1 + \int_0^{\varepsilon_1} \|[\partial_t + v \cdot \nabla_x + \varphi] h(t)\|_1 \mathrm{d}t.
$$

Because of our choice ε and ε_1, $t_{\mathbf{b}}(x,v) > t$ for all $(t,x,v) \in [0, \varepsilon_1] \times \gamma_+ \setminus \gamma_+^\varepsilon$. Then

$$
|h(t,x,v)| \lesssim |h_0(x-tv, v)| + \int_0^t |[\partial_t + v \cdot \nabla_x + \varphi(v)] h(s, x-(t-s)v, v)| \,\mathrm{d}s,
$$

where the second contribution is bounded, from (3.1), by

$$
\begin{aligned}
&\int_0^{\varepsilon_1} \int_{\gamma_+ \setminus \gamma_+^\varepsilon} \int_0^t |[\partial_t + v \cdot \nabla_x + \varphi(v)] h(s, x-(t-s)v, v)| \,\mathrm{d}s \mathrm{d}\gamma \mathrm{d}t \\
&\qquad \lesssim \int_0^{\varepsilon_1} \|[\partial_t + v \cdot \nabla_x + \varphi(v)] h(t)\|_1 \mathrm{d}t.
\end{aligned}
$$

Consider the initial datum contribution $|h_0(x-tv, v)|$: We may assume $\partial_{x_3}\xi(x_0) \neq 0$. By the implicit function theorem, $\partial\Omega$ can be represented locally by the graph $\eta = \eta(x_1, x_2)$ satisfying $\xi(x_1, x_2, \eta(x_1, x_2)) = 0$ and $(\partial_{x_1}\eta(x_1, x_2), \partial_{x_2}\eta(x_1, x_2)) = (-\partial_{x_1}\xi/\partial_{x_3}\xi, -\partial_{x_2}\xi/\partial_{x_3}\xi)$ at $(x_1, x_2, \eta(x_1, x_2))$. We define the change of variables

$$(x,t) \in \{x \in \partial\Omega : |x - x_0| \ll 1\} \times [0, \varepsilon_1] \mapsto y = x - tv \in \bar{\Omega},$$

where $\left|\frac{\partial y}{\partial(x,t)}\right| = -v_1 \frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2 \frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} - v_3$.

Therefore

$$\begin{aligned}|n(x) \cdot v| \mathrm{d}S_x \mathrm{d}t &= (n(x) \cdot v) \left[1 + \left(\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi}\right)^2 + \left(\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi}\right)^2\right]^{1/2} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}t \\ &= \left[-v_1 \frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2 \frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} - v_3\right] \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}t = \mathrm{d}y,\end{aligned}$$

and $\int_0^{\varepsilon_1} \int_{\gamma_+ \backslash \gamma_+^{\varepsilon} \cap \{|x-x_0| \ll 1\}} |h_0(x-tv, v)| \mathrm{d}\gamma \mathrm{d}t \lesssim_{\varepsilon,\varepsilon_1,x_0} \iint_{\Omega\times\mathbb{R}^3} |h_0(y,v)| \mathrm{d}y \mathrm{d}v$. Since $\partial\Omega$ is compact we can choose finite covers of $\partial\Omega$ and repeat the same argument for each piece to conclude

$$\int_0^{\varepsilon_1} \int_{\gamma_+ \backslash \gamma_+^{\varepsilon}} |h_0(x - tv, v)| \mathrm{d}\gamma \mathrm{d}t \lesssim_{\Omega,\varepsilon,\varepsilon_1} \iint_{\Omega\times\mathbb{R}^3} |h_0(y,v)| \mathrm{d}y \mathrm{d}v.$$

□

Lemma 9 (Green's Identity) *For $p \in [1, \infty)$ assume $f, \partial_t f + v \cdot \nabla_x f \in L^p([0,T]; L^p(\Omega \times \mathbb{R}^3))$ and $f_{\gamma_-} \in L^p([0,T]; L^p(\gamma))$. Then $f \in C^0([0,T]; L^p(\Omega \times \mathbb{R}^3))$ and $f_{\gamma_+} \in L^p([0,T]; L^p(\gamma))$ and for almost every $t \in [0,T]$:*

$$\begin{aligned}\|f(t)\|_p^p + \int_0^t |f|_{\gamma_+,p}^p &= \|f(0)\|_p^p + \int_0^t |f|_{\gamma_-,p}^p \\ &\quad + \int_0^t \iint_{\Omega\times\mathbb{R}^3} \{\partial_t f + v \cdot \nabla_x f\} |f|^{p-2} f.\end{aligned}$$

See [9] for the proof. Now we state and prove following propositions for the in-flow problems:

$$\{\partial_t + v \cdot \nabla_x + \nu\} f = H, \quad f(0,x,v) = f_0(x,v), \quad f(t,x,v)|_{\gamma_-} = g(t,x,v), \tag{3.2}$$

where $\nu(t,x,v) \geq 0$. For notational simplicity, we define

$$\partial_t f_0 \equiv -v \cdot \nabla_x f_0 - \nu f_0 + H(0,x,v), \tag{3.3}$$

$$\nabla_x g \equiv \frac{n}{n \cdot v} \left\{-\partial_t g - \sum_{i=1}^2 (v \cdot \tau_i) \partial_{\tau_i} g - \nu g + H\right\} + \sum_{i=1}^2 \tau_i \partial_{\tau_i} g. \tag{3.4}$$

We remark that $\partial_t f_0$ is obtained from formally solving (3.2), and (3.3), which leads to the usual tangential derivatives of $\partial_{\tau_i} g$, and new 'normal derivative' $\partial_n g$ through the equation (3.2).

Proposition 1 *Assume the compatibility condition*

$$f_0(x, v) = g(0, x, v) \quad for \quad (x, v) \in \gamma_-. \tag{3.5}$$

Let $p \in [1, \infty)$ and $0 < \theta < 1/4$. Assume

$$\begin{aligned}
\nabla_x f_0, \nabla_v f_0, -v \cdot \nabla_x f_0 - \nu(0, \cdot, \cdot) f_0 + H(0, \cdot, \cdot) &\in L^p(\Omega \times \mathbb{R}^3), \\
\partial_t g, \nabla_v g, \partial_{\tau_i} g &\in L^p([0, T] \times \gamma_-), \\
\frac{1}{n(x) \cdot v}\{-\partial_t g - \sum_i (v \cdot \tau_i)\partial_{\tau_i} g - \nu g + H\} &\in L^p([0, T] \times \gamma_-), \\
\partial_t H,\ \nabla_x H,\ \nabla_v H, &\in L^p([0, T] \times \Omega \times \mathbb{R}^3), \\
e^{-\theta|v|^2}\partial_t \nu,\ e^{-\theta|v|^2}\nabla_x \nu,\ e^{-\theta|v|^2}\nabla_v \nu &\in L^p([0, T] \times \Omega \times \mathbb{R}^3), \\
e^{\theta|v|^2} f_0 \in L^\infty(\Omega \times \mathbb{R}^3),\ e^{\theta|v|^2} g &\in L^\infty([0, T] \times \gamma_-), \\
e^{\theta|v|^2} H &\in L^\infty([0, T] \times \Omega \times \mathbb{R}^3).
\end{aligned}$$

Then for sufficiently small $T > 0$, there exists a unique solution f to (3.2) such that $f, \partial_t f, \nabla_x f, \nabla_v f \in C^0([0, T]; L^p(\Omega \times \mathbb{R}^3))$ and their traces satisfy

$$\begin{aligned}
\partial_t f|_{\gamma_-} = \partial_t g,\ \nabla_v f|_{\gamma_-} = \nabla_v g,\ \nabla_x f|_{\gamma_-} &= \nabla_x g, \quad on \quad \gamma_-, \\
\nabla_x f(0, x, v) = \nabla_x f_0,\ \nabla_v f(0, x, v) &= \nabla_v f_0, \quad in \quad \Omega \times \mathbb{R}^3, \\
\partial_t f(0, x, v) &= \partial_t f_0, \quad in \quad \Omega \times \mathbb{R}^3,
\end{aligned} \tag{3.6}$$

where $\partial_t f_0$ and $\nabla_x g$ are given by (3.3) and (3.3). Moreover for $\partial_{\mathbf{e}} \in \{\partial_t, \partial_x, \partial_v\}$

$$\begin{aligned}
\|\partial_{\mathbf{e}} f(t)\|_p^p + \int_0^t |\partial_{\mathbf{e}} f|_{\gamma_+, p}^p &= \|\partial_{\mathbf{e}} f_0\|_p^p + \int_0^t |\partial_{\mathbf{e}} g|_{\gamma_-, p}^p \\
&+ p \int_0^t \iint_{\Omega \times \mathbb{R}^3} \{\partial_{\mathbf{e}} H - [\partial_{\mathbf{e}} v]\nabla_x f - [\partial_{\mathbf{e}} \nu] f\}|\partial_{\mathbf{e}} f|^{p-2}\partial_{\mathbf{e}} f.
\end{aligned} \tag{3.7}$$

Proof We apply the trace theorem to the derivatives of f by explicit computations. Denote $\nu(s) = \nu(s, x - (t - s)v, v)$. First we assume f_0, g and H have compact supports in $v \in \mathbb{R}^3$. We integrate the Eq. (3.2) along the backward trajectories. If the initial condition is reached before hitting the boundary (case $t < t_{\mathbf{b}}$), we have

$$f(t, x, v) = e^{-\int_0^t \nu} f_0(x - tv, v) + \int_0^t e^{-\int_0^s \nu} H(t - s, x - vs, v) \mathrm{d}s,$$

where we used the simplified notation $\int_0^t \nu = \int_0^t \nu(t-\tau, x-\tau v, v)\mathrm{d}\tau$. If the boundary has been reached before (case $t > t_{\mathbf{b}}$), we have

$$f(t,x,v) = e^{-\int_0^{t_{\mathbf{b}}} \nu} g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) + \int_0^{t_{\mathbf{b}}} e^{-\int_0^s \nu} H(t-s, x-vs, v)\mathrm{d}s,$$

where $\int_0^{t_{\mathbf{b}}} \nu = \int_0^{t_{\mathbf{b}}} \nu(t-\tau, x-\tau v, v)\mathrm{d}\tau$.

Let us rewrite it as

$$\begin{aligned} f(t,x,v) = \mathbf{1}_{\{t\le t_{\mathbf{b}}\}} e^{-\int_0^t \nu} f_0(x-tv, v) + \mathbf{1}_{\{t> t_{\mathbf{b}}\}} e^{-\int_0^{t_{\mathbf{b}}} \nu} g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) \\ + \int_0^{\min(t,t_{\mathbf{b}})} e^{-\int_0^s \nu} H(t-s, x-vs, v)\mathrm{d}s. \end{aligned} \tag{3.8}$$

We take the derivative of f with respect to time, space and velocity for $t \neq t_{\mathbf{b}}$. Recall the following derivatives of $x_{\mathbf{b}}$ and $t_{\mathbf{b}}$ (see lemma 2 in [9]):

$$\begin{aligned} &\nabla_x t_{\mathbf{b}} = \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \quad \nabla_v t_{\mathbf{b}} = -\frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \\ &\nabla_x x_{\mathbf{b}} = I - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \otimes v, \quad \nabla_v x_{\mathbf{b}} = -t_{\mathbf{b}} I + \frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \otimes v. \end{aligned} \tag{3.9}$$

Since g is defined on a surface, we cannot define its space gradient. Regarding $g(t-t_{\mathbf{b}}, x_{\mathbf{b}}(x,v), v)$ as function on $[0,T]\times\bar{\Omega}\times\mathbb{R}^3$, we obtain from (3.9)

$$\begin{aligned} \nabla_x[g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v)] &= -\nabla_x t_{\mathbf{b}} \partial_t g + \nabla_x x_{\mathbf{b}} \nabla_\tau g \\ &= -\frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \partial_t g + \left(I - \frac{n\otimes v}{n\cdot v} \right) \nabla_\tau g \\ &= \tau_1 \partial_{\tau_1} g + \tau_2 \partial_{\tau_2} g \\ &\quad - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \left\{ \partial_t g + v\cdot\tau_1 \partial_{\tau_1} g + v\cdot \tau_2 \partial_{\tau_2} g \right\}, \\ \nabla_v[g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v)] &= -t_{\mathbf{b}} \nabla_x [g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v)] + \nabla_v g, \end{aligned}$$

where $\tau_1(x)$ and $\tau_2(x)$ are unit vectors satisfying $\tau_1(x)\cdot n(x) = 0 = \tau_2(x)\cdot n(x)$ and $\tau_1(x)\times\tau_2(x) = n(x)$.

Therefore by direct computation for $t \neq t_{\mathbf{b}}$, we deduce

$$\begin{aligned} &\partial_t f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} \\ &\quad = -\mathbf{1}_{\{t<t_{\mathbf{b}}\}} e^{-\int_0^t \nu} \left[\nu f_0 + v\cdot\nabla_x f_0 - H_{|t=0} + \int_0^t \partial_t \nu \times f_0 \right](x-tv, v) \end{aligned}$$

$$
\begin{aligned}
&+\mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-\int_0^{t_{\mathbf{b}}}\nu}\left[\partial_t g-\int_0^{t_{\mathbf{b}}}\partial_t\nu\times g\right](t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
&+\int_0^{\min(t,t_{\mathbf{b}})}e^{-\int_0^s\nu}\left[\partial_t H-\int_0^s\partial_t\nu\times H\right](t-s,x-vs,v)\mathrm{d}s,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&\nabla_x f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\\
&=\mathbf{1}_{\{t<t_{\mathbf{b}}\}}e^{-\int_0^t\nu}\Big[\nabla_x f_0-\int_0^t\nabla_x\nu\times f_0\Big](x-tv,v)\\
&+\mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-\int_0^{t_{\mathbf{b}}}\nu}\Bigg\{\sum_{i=1}^2\tau_i\partial_{\tau_i}g-\int_0^{t_{\mathbf{b}}}\nabla_x\nu\times g\\
&-\frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\Big\{\partial_t g+\sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H\Big\}\Bigg\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
&+\int_0^{\min(t,t_{\mathbf{b}})}e^{-\int_0^s\nu}\Big[\nabla_x H-\int_0^s\nabla_x\nu\times H\Big](t-s,x-vs,v)\mathrm{d}s,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\nabla_v f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\\
&=\mathbf{1}_{\{t<t_{\mathbf{b}}\}}e^{-\int_0^t\nu}[-t\nabla_x f_0+\nabla_v f_0-\int_0^t(-\tau\nabla_x\nu+\nabla_v\nu)\times f_0](x-tv,v)\\
&-\mathbf{1}_{\{t>t_{\mathbf{b}}\}}t_{\mathbf{b}}e^{-\int_0^{t_{\mathbf{b}}}\nu}\Bigg\{\sum_{i=1}^2\tau_i\partial_{\tau_i}g\\
&\qquad-\frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\Big\{\partial_t g+\sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H\Big\}\Bigg\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
&+\mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-\int_0^{t_{\mathbf{b}}}\nu}\Bigg\{\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
&\qquad-\int_0^{t_{\mathbf{b}}}(-\tau\nabla_x\nu+\nabla_v\nu)\times g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\Bigg\}\\
&+\int_0^{\min(t,t_{\mathbf{b}})}e^{-\int_0^s\nu}\Bigg\{\nabla_v H-s\nabla_x H\\
&\qquad-(\int_0^s-\tau\nabla_x\nu+\nabla_v\nu)\times H\Bigg\}(t-s,x-vs,v)\mathrm{d}s.
\end{aligned}
\tag{3.12}
$$

Here we have abbreviated the notations as $\int_0^t\partial_{\mathbf{e}}\nu=\int_0^t\partial_{\mathbf{e}}\nu(t-\tau,x-\tau v,v)\mathrm{d}\tau$ and $\int_0^{t_{\mathbf{b}}}\partial_{\mathbf{e}}\nu=\int_0^{t_{\mathbf{b}}}\partial_{\mathbf{e}}\nu(t-\tau,x-\tau v,v)\mathrm{d}\tau$.

First we show that $\partial f\mathbf{1}_{\{t>t_{\mathbf{b}}\}}\in L^p$ and $\partial f\mathbf{1}_{\{t<t_{\mathbf{b}}\}}\in L^p$ separately. Then we take the L^p norms above with the changes of variables in Lemma 2.1 of

[7] and we use Jensen's inequality in $[0, t]$. More precisely, for $\phi \in L^1$ with $\phi \geq 0$,

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} \mathbf{1}_{\{x-tv\in\Omega\}}\phi(x-tv, v) \\
&\quad = \int_{\mathbb{R}^3}\left[\int_\Omega \mathbf{1}_{\{x-tv\in\Omega\}}\phi(x-tv, v)\mathrm{d}x\right]\mathrm{d}v \leq \iint_{\Omega\times\mathbb{R}^3}\phi(x, v), \\
&\iint_{\{\Omega\times\mathbb{R}^3\}\cap B((x_0,v_0);\delta)} \mathbf{1}_{\{t\geq t_{\mathbf{b}}\}}\phi(t-t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v) \\
&\quad \leq \int_0^t\int_{\partial\Omega\times\mathbb{R}^3}\phi(s, x, v)|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v\mathrm{d}s,
\end{aligned} \tag{3.13}$$

where for the second inequality we have used the change of variables for fixed t, v,

$$x \mapsto (t-t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v)). \tag{3.14}$$

In fact, without loss of generality we may assume $\partial_{x_3}\xi(x_{\mathbf{b}}(x, v)) \neq 0$ for $(x, v) \in B((x_0, v_0); \delta)$ so that $x_{\mathbf{b}}(x, v) = (x_{\mathbf{b},1}, x_{\mathbf{b},2}, \eta(x_{\mathbf{b},1}, x_{\mathbf{b},2}))$. Using (3.9), we compute the Jacobian

$$\det\begin{pmatrix} -\nabla_x t_{\mathbf{b}} \\ -\nabla_x x_{\mathbf{b},1} \\ -\nabla_x x_{\mathbf{b},2}\end{pmatrix} = \det\begin{pmatrix} -(v\cdot n)^{-1}n \\ -\nabla_x x_{\mathbf{b},1} \\ -\nabla_x x_{\mathbf{b},2}\end{pmatrix} = \left|-v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} + v_3\right|^{-1}.$$

Therefore $\mathrm{d}x\mathrm{d}v = \left|-v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} + v_3\right|\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}v\mathrm{d}t = |n\cdot v|\mathrm{d}S_x\mathrm{d}v\mathrm{d}t = \mathrm{d}\gamma\mathrm{d}t$. Using these changes of variables, we obtain

$$\|f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p \lesssim_t \|f_0\|_p + \left[\int_0^t\int_{\gamma_-}|g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + \left[\int_0^t\|H\|_p^p\mathrm{d}s\right]^{1/p},$$

and

$$\begin{aligned}
&\|\partial_t f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p \\
&\lesssim_t \|v\cdot\nabla_x f_0 + \nu f_0 - H(0,\cdot,\cdot)\|_p \\
&+ \left[\int_0^t\int_{\gamma_-}|\partial_t g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + \left[\int_0^t\|\partial_t H\|_p^p\right]^{1/p} \\
&+ \{\|e^{\theta|v|^2}f_0\|_\infty + \|e^{\theta|v|^2}H\|_\infty + |e^{\theta|v|^2}g|_\infty\}\left[\int_0^t\|e^{-\theta|v|^2}\partial_t\nu\|_p^p\right]^{1/p},
\end{aligned}$$

and

$$
\begin{aligned}
&\|\nabla_x f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p \\
&\lesssim_t \|\nabla_x f_0\|_p + \left[\int_0^t \|\nabla_x H\|_p^p\right]^{1/p} \\
&+ \{\|e^{\theta|v|^2} f_0\|_\infty + \|e^{\theta|v|^2} H\|_\infty + |e^{\theta|v|^2} g|_\infty\} \left[\int_0^t \|e^{-\theta|v|^2}\nabla_x \nu\|_p^p\right]^{1/p} \\
&+ \left[\int_0^t \int_{\gamma_-} \mathrm{d}\gamma \mathrm{d}s \Big| \sum_{i=1}^2 \tau_i \partial_{\tau_i} g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) \right.\\
&\quad \left. - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \Big\{\partial_t g + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H\Big\}(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v)\Big|^p\right]^{1/p},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\nabla_v f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p \\
&\lesssim_t \|\nabla_x f_0\|_p + \|\nabla_v f_0\|_p + C\|f_0\|_p + \left[\int_0^t \int_{\gamma_-} |\nabla_v g|^p \mathrm{d}\gamma \mathrm{d}s\right]^{1/p} \\
&+ \left[\int_0^t \|\nabla_v H\|_p^p + \|\nabla_x H\|_p^p \mathrm{d}s\right]^{1/p} \\
&+ \left[\int_0^t \int_{\gamma_-} \mathrm{d}\gamma \mathrm{d}s \Big| \sum_{i=1}^2 \tau_i \partial_{\tau_i} g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) \right.\\
&\quad \left. - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \Big\{\partial_t g + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H\Big\}(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v)\Big|^p\right]^{1/p} \\
&+ \{\|e^{\theta|v|^2} f_0\|_\infty + \|e^{\theta|v|^2} H\|_\infty + |e^{\theta|v|^2} g|_\infty\} \\
&\quad \times \left[\int_0^t \|e^{-\theta|v|^2}\nabla_x \nu\|_p^p + \|e^{-\theta|v|^2}\nabla_v \nu\|_p^p\right]^{1/p}.
\end{aligned}
$$

From our hypothesis and assumption on f_0, g and H with compact supports, these terms are bounded, therefore

$$
\partial f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} \equiv \left[\partial_t f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}, \nabla_x f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}, \nabla_v f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\right] \in L^\infty([0,T]; L^p(\Omega\times\mathbb{R}^3)).
$$

On the other hand, thanks to the compatibility condition, we need to show f has the same trace on the set

$$
\mathcal{M} \equiv \{t = t_{\mathbf{b}}(x,v)\} \equiv \{(t_{\mathbf{b}}(x,v), x, v) \in [0,T]\times\Omega\times\mathbb{R}^3\}. \tag{3.15}
$$

We claim the following fact: *Let* $\phi(t,x,v) \in C_c^\infty((0,T)\times\Omega\times\mathbb{R}^3)$ *then we have*

$$\int_0^T \iint_{\Omega\times\mathbb{R}^3} f\partial\phi = -\int_0^T \iint_{\Omega\times\mathbb{R}^3} \partial f \mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\phi, \tag{3.16}$$

so that $f \in W^{1,p}$ *with weak derivatives given by* $\partial f \mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}$.

Proof of claim. We first fix the test function $\phi(t,x,v)$. There exists $\delta = \delta_\phi > 0$ such that $\phi \equiv 0$ for $t \geq \frac{1}{\delta}$, or $\mathrm{dist}(x,\partial\Omega) < \delta$, or $|v| \geq \frac{1}{\delta}$. Let $\phi(t,x,v) \neq 0$ and $(t,x,v) \in \mathcal{M}$. By (3.15) and (1.27), $t = t_{\mathbf{b}}(x,v)$, $x_{\mathbf{b}} = x - t_{\mathbf{b}}v$, and $|x - x_{\mathbf{b}}| = t_{\mathbf{b}}|v|$, and

$$\mathrm{dist}(x,\Omega) \leq |x - x_{\mathbf{b}}| = t_{\mathbf{b}}|v|.$$

Since $t_{\mathbf{b}} \leq \frac{1}{\delta}$, this implies that $|v| \geq \frac{\delta}{t_{\mathbf{b}}} \geq \delta^2$.

Otherwise $\mathrm{dist}(x,\partial\Omega) \leq \delta$ so that $\phi(t,x,v) = 0$. Furthermore, by the Velocity lemma and this lower bound for $|v|$, we conclude that there exists $\delta'(\delta,\Omega) > 0$ such that

$$\begin{aligned}
|v\cdot n(x_{\mathbf{b}})|^2 &\gtrsim_\Omega |v\cdot\nabla_x\xi(x_{\mathbf{b}})|^2 = \alpha(t - t_{\mathbf{b}}; t,x,v)\\
&\geq e^{-C_\Omega\langle v\rangle t_{\mathbf{b}}}\alpha(t;t,x,v) \geq e^{-C_\Omega\langle v\rangle t_{\mathbf{b}}} C_\xi|v|^2|\xi(x)|\\
&\geq e^{-C_\Omega\delta^{-2}} C_\xi \delta^4 \min_{\mathrm{dist}(x,\partial\Omega)\geq\delta} |\xi(x)| = 2\delta'(\delta,\Omega) > 0.
\end{aligned}$$

In particular, this lower bound and a direct computation of (3.9) imply that $\{\phi \neq 0\}\cap\mathcal{M}$ is a smooth 6D hypersurface.

We next take a C^1 approximation of f_0^l, H^l, and g^l (by partition of unity and localization) such that

$$\begin{aligned}
&\|f_0^l - f_0\|_{W^{1,p}} \to 0, \quad \|g^l - g\|_{W^{1,p}([0,T]\times\gamma_-\backslash\gamma_-^{\delta'})} \to 0,\\
&\|H^l - H\|_{W^{1,p}([0,T]\times\Omega\times\mathbb{R}^3)} \to 0,
\end{aligned}$$

where $W^{1,p}([0,T]\times\gamma_-\backslash\gamma_-^{\delta'})$ is the standard Sobolev space in $[0,T]\times\gamma_-\backslash\gamma_-^{\delta'}$.

This implies, from the trace theorem, that

$$f_0^l(x,v) \to f_0(x,v) \quad \text{and} \quad g^l(0,x,v) \to g(0,x,v) \quad \text{in} \quad L^1(\gamma_-\backslash\gamma_-^{\delta'}).$$

We define accordingly, for $(t,x,v) \in [0,T]\times\Omega\times\mathbb{R}^3$,

$$\begin{aligned}
f^l(t,x,v) = \mathbf{1}_{\{t<t_{\mathbf{b}}\}} e^{-\int_0^t \nu} f_0^l(x - tv, v) + \mathbf{1}_{\{t>t_{\mathbf{b}}\}} e^{-\int_0^{t_{\mathbf{b}}} \nu} g^l(t - t_{\mathbf{b}}, x_{\mathbf{b}}, v)&\\
+ \int_0^{\min\{t,t_{\mathbf{b}}\}} e^{-\int_0^s \nu} H^l(t-s, x-sv, v)\mathrm{d}s,&
\end{aligned} \tag{3.17}$$

and $f^l_\pm(t,x,v) \equiv \mathbf{1}_{\{t \gtrless t_{\mathbf{b}}\}} f^l$. Therefore for all $(x,v) \in \gamma_-$,

$$f^l_+(s, x+sv, v) - f^l_-(s, x+sv, v) = e^{-\int_0^s \nu} g^l(0,x,v) - e^{-\int_0^s \nu} f^l_0(x,v).$$

Since $\{\phi \neq 0\} \cap \mathcal{M}$ is a smooth hypersurface, we apply the Gauss theorem to f^l to obtain

$$\begin{aligned} \iiint \partial_{\mathbf{e}} \phi f^l \mathrm{d}x\mathrm{d}v\mathrm{d}t &= \iint [f^l_+ - f^l_-] \phi \mathbf{e} \cdot \mathbf{n}_{\mathcal{M}} \mathrm{d}\mathcal{M} \\ &\quad - \left\{ \iiint_{t>t_{\mathbf{b}}} \phi \partial_{\mathbf{e}} f^l_+ \mathrm{d}x\mathrm{d}v\mathrm{d}t + \iiint_{t<t_{\mathbf{b}}} \phi \partial_{\mathbf{e}} f^l_- \mathrm{d}x\mathrm{d}v\mathrm{d}t \right\}, \end{aligned} \tag{3.18}$$

where $\partial_{\mathbf{e}} = [\partial_t, \nabla_x, \nabla_v] = [\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{v_1}, \partial_{v_2}, \partial_{v_3}]$ and

$$\mathbf{n}_{\mathcal{M}} = \frac{1}{\sqrt{1 + |\nabla_x t_{\mathbf{b}}|^2 + |\nabla_v t_{\mathbf{b}}|}} (1, -\nabla_x t_{\mathbf{b}}, -\nabla_v t_{\mathbf{b}}) \in \mathbb{R}^7.$$

We have used also $(s, x+sv, v)$ and $(x,v) \in \gamma_-$ as our parametrization for the manifold $\mathcal{M} \cap \{\phi \neq 0\}$, so that $n(x_{\mathbf{b}}(x,v)) \cdot v \geq 2\delta'$ is equivalent to $n(x) \cdot v \geq 2\delta'$. Therefore the above hypersurface integration over $\{t \neq t_{\mathbf{b}}\}$ is bounded by

$$\begin{aligned} &\lesssim_{\phi,\delta} \int_0^{\frac{1}{\delta}} \int_{n(x)\cdot v \geq 2\delta'} |f^l_+(s, x+sv, v) - f^l_-(s, x+sv, v)| \mathrm{d}S_x \mathrm{d}v\mathrm{d}s \\ &\lesssim_{\phi,\delta} \int_{n(x)\cdot v \geq 2\delta'} |g^l(0,x,v) - f^l_0(s,v)| \mathrm{d}S_x \mathrm{d}v \to 0, \quad \text{as } l \to \infty, \end{aligned}$$

due to the compatibility condition $f_0(x,v) = g(0,x,v)$ for $(x,v) \in \gamma_-$. Clearly, taking difference of (3.17) and (3.8), we deduce $f^l \to f$ strongly in $L^p(\{\phi \neq 0\})$ due to the first estimate of (3.15). Furthermore, due to (3.15), we have a uniform-in-l bound of $f^l_\pm$ in $W^{1,p}(\{t \gtrless t_{\mathbf{b}}, \phi \neq 0\})$ such that, up to a subsequence,

$$\partial_{\mathbf{e}} f^l_+ \rightharpoonup \partial_{\mathbf{e}} f \mathbf{1}_{\{t>t_{\mathbf{b}}\}}, \quad \partial_{\mathbf{e}} f^l_- \rightharpoonup \partial_{\mathbf{e}} f \mathbf{1}_{\{t<t_{\mathbf{b}}\}}, \quad \text{weakly in } L^p(\{\phi \neq 0\}).$$

Finally we conclude the claim (3.16) by letting $l \to \infty$ in (3.18).

Now, notice that from its explicit form (3.8), and since all the data are compactly supported in the velocity space, f is itself compactly supported in the velocity space. Recall $\partial = [\partial_t, \nabla_x, \nabla_v]$. From this and from the L^p bounds above, we conclude

$$\{\partial_t + v \cdot \nabla_x + \nu\} \partial f = \partial H - \partial v \cdot \nabla_x f - \partial \nu f \in L^p. \tag{3.19}$$

By the trace theorem (Lemma 8), the traces of $\partial_t f$, $\nabla_x f$, $\nabla_v f$ exist. To evaluate these traces, we take derivatives along characteristics. Letting $t \to t_{\mathbf{b}}$ and $t \to 0$, we deduce (3.6). From the Green's identity, Lemma 9, we have (3.7) and therefore we conclude $\partial f \in C^0([0,T]; L^p)$.

In order to remove the compact support assumption we employ the cut-off function χ used in (1.22). Define $f^m = \chi(|v|/m) f$ then f^m satisfies

$$\begin{aligned} \{\partial_t + v \cdot \nabla_x + \chi(|v|/m)\nu\} f^m &= \chi(|v|/m) H, \\ f^m(0,x,v) = \chi(|v|/m) f_0, \quad f^m|_{\gamma_-} &= \chi(|v|/m) g. \end{aligned} \tag{3.20}$$

Note that $\nabla_v[\chi(|v|/m)g] = \chi(|v|/m)\nabla_v g + g \nabla_v \chi(|v|/m)$ and $\chi(|v|/m) f_0(x,v) = \chi(|v|/m) g(0,x,v)$ for $(x,v) \in \gamma_-$. Apply the previous result to compute the traces of the derivatives of f^m. It is standard (using Green's identity) to show that $\partial_t f^m$, $\nabla_x f^m$ and $\nabla_v f^m$ are Cauchy sequences and we can pass to the desired limits. □

We now study the weighted $W^{1,p}$ estimate. Recall (1.22). We first define an effective collision frequency:

$$\nu_{\varpi,\beta}(t,x,v) = \nu(v) + \varpi \langle v \rangle - \beta \alpha^{-1}[v \cdot \nabla_x \alpha], \tag{3.21}$$

and

$$[\partial_t + v \cdot \nabla_x + \nu_{\varpi,\beta}](e^{-\varpi\langle v \rangle t} \alpha^\beta f) = e^{-\varpi \langle v \rangle t} \alpha^\beta [\partial_t f + v \cdot \nabla_x f + \nu f]. \tag{3.22}$$

Due to (2.1) and $\varpi \gg 1$, $\nu_{\varpi,\beta}(t,x,v) \lesssim \beta \langle v \rangle$.

Proposition 2 *Let f be a solution of (3.2). Assume (3.5) and $\langle v \rangle g \in L^\infty([0,T] \times \gamma_-)$, and $\nu, \langle v \rangle H \in L^\infty([0,T] \times \Omega \times \mathbb{R}^3)$. For any fixed $p \in [2,\infty]$, $0 < \theta < 1/4$ and $\beta > 0$, assume*

$$\begin{aligned} &\alpha^\beta \nabla_x f_0, \alpha^\beta \nabla_v f_0, \alpha^\beta[-v \cdot \nabla_x f_0 - \nu(0,\cdot,\cdot) f_0 + H(0,\cdot,\cdot)] \in L^p(\Omega \times \mathbb{R}^3), \\ &e^{-\varpi \langle v \rangle t} \alpha^\beta \partial_t g, e^{-\varpi \langle v \rangle t} \alpha^\beta \nabla_v g, e^{-\varpi \langle v \rangle t} \alpha^\beta \partial_{\tau_i} g \in L^p([0,T] \times \gamma_-), \\ &\frac{e^{-\varpi \langle v \rangle t} \alpha^\beta}{n(x) \cdot v} \left\{ -\partial_t g - \sum_i (v \cdot \tau_i) \partial_{\tau_i} g - \nu g + H \right\} \in L^p([0,T] \times \gamma_-), \\ &e^{-\varpi \langle v \rangle t} \alpha^\beta \partial_t H, \ e^{-\varpi \langle v \rangle t} \alpha^\beta \nabla_x H, \ e^{-\varpi \langle v \rangle t} \alpha^\beta \nabla_v H \in L^p([0,T] \times \Omega \times \mathbb{R}^3), \\ &e^{-\theta|v|^2} e^{-\varpi \langle v \rangle t} \alpha^\beta \partial_t \nu, \ e^{-\theta|v|^2} e^{-\varpi \langle v \rangle t} \alpha^\beta \nabla_x \nu, \ e^{-\theta|v|^2} e^{-\varpi \langle v \rangle t} \alpha^\beta \nabla_v \nu \in L^p([0,T] \times \Omega \times \mathbb{R}^3), \\ &e^{\theta|v|^2} f_0 \in L^\infty(\Omega \times \mathbb{R}^3), \ e^{\theta|v|^2} g \in L^\infty([0,T] \times \gamma_-), \ e^{\theta|v|^2} H \in L^\infty([0,T] \times \Omega \times \mathbb{R}^3). \end{aligned}$$

Then $f(t,x,v)$ satisfies

$$\|f(t)\|_\infty \le \|f_0\|_\infty + \sup_{0 \le s \le t} \|g(s)\|_\infty + \left\| \int_0^t H(s) ds \right\|_\infty .$$

For $\partial_{\mathbf{e}} \in \{\partial_t, \nabla_x, \nabla_v\}$,

$$
\begin{aligned}
\{\partial_t+v\cdot\nabla_x+\nu_{\varpi,\beta}\}[e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f] &= e^{-\varpi\langle v\rangle t}\alpha^\beta\left[-\partial_{\mathbf{e}} v\cdot\nabla_x f-\partial_{\mathbf{e}}\nu f+\partial_{\mathbf{e}} H\right],\\
e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f|_{t=0} &= e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f_0,\\
e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f|_{\gamma_-} &= e^{-\varpi\langle v\rangle t}\alpha^\beta[\partial_{\mathbf{e}} g|_{\gamma_-}],
\end{aligned}
$$

where $[\partial_{\mathbf{e}} g|_{\gamma_-}]$ *is given in* (3.6). *Moreover, recalling* (3.3) *and* (3.4)*, we have for* $2 \le p < \infty$,

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f(t)|^p+\int_0^t\int_{\Omega\times\mathbb{R}^3}\nu_{\varpi,\beta}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f|^p\\
&\quad+\int_0^t\int_{\gamma_+}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f|^p\\
&\lesssim\int_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f_0|^p+\int_0^t\int_{\gamma_-}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} g|^p\\
&\quad+\int_0^t\int_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} H-e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} v\cdot\nabla_x f-\partial_{\mathbf{e}}\nu e^{-\varpi\langle v\rangle t}\alpha^\beta f|\\
&\qquad\qquad\times|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f|^{p-1},\\
&||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f(t)||_\infty\\
&\lesssim||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} f_0||_\infty+||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} g||_\infty\\
&+\int_0^t||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_{\mathbf{e}} H-\partial_{\mathbf{e}} v\cdot e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f-\partial_{\mathbf{e}}\nu e^{-\varpi\langle v\rangle t}\alpha^\beta f||_\infty.
\end{aligned}
\tag{3.23}
$$

Proof First we assume f_0, g and H have compact supports in $\{v\in\mathbb{R}^3 : |v| < m\}$. We estimate ∂f in the bulk. From Lemma 2, we have

$$
\sup_{t\le t_{\mathbf{b}}}\frac{e^{-\varpi\langle v\rangle t}\alpha^\beta(x,v)}{\alpha^\beta(x-tv,v)}\le e^{C_{m,\beta}t},\quad \sup_{t\ge t_{\mathbf{b}}}\frac{e^{-\varpi\langle v\rangle t}\alpha^\beta}{e^{-\varpi\langle v\rangle(t-t_{\mathbf{b}})}\alpha^\beta(x_{\mathbf{b}},v)}\le e^{C_{m,\beta}t_{\mathbf{b}}},
$$
$$
\sup_{\max\{t-t_{\mathbf{b}},0\}\le s\le t}\frac{e^{-\varpi\langle v\rangle t}\alpha^\beta}{e^{-\varpi\langle v\rangle(t-s)}\alpha(x-sv,v)^\beta}\le e^{C_{m,\beta}s}.
$$

Multiplying $e^{-\varpi\langle v\rangle t}\alpha^\beta$ to the direct computations (3.10), (3.11), and (3.12) and then using the change of variables (3.13) and (3.14), we get

$$
\begin{aligned}
\|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_t f(t)\|_p &\lesssim_{t,m,\beta}\|\alpha^\beta[v\cdot\nabla_x f_0+\nu f_0-H(0,\cdot,\cdot)]\|_p\\
&+\left[\int_0^t|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial_t g(s)|^p_{\gamma,p}+\|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial_t H(s)\|_p^p\mathrm{d}s\right]^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&+C'\left[\int_0^t \|e^{-\theta|v|^2}e^{-\varpi\langle v\rangle s}\alpha^\beta\partial_t\nu\|_p^p\right]^{1/p},\\
&\|e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f(t)\|_p \lesssim_{t,m,\beta} \|\alpha^\beta\nabla_x f_0\|_p\\
&+\left[\int_0^t \left|\frac{e^{-\varpi\langle v\rangle t}\alpha^\beta}{v\cdot n}\{\partial_t g+\sum(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H\}\right|_{\gamma,p}^p \mathrm{d}s\right]^{1/p}\\
&+\left[\int_0^t \sum_{i=1}^2 |e^{-\varpi\langle v\rangle s}\alpha^\beta\partial_{\tau_i}g(s)|_{\gamma,p}^p+\|e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_x H(s)\|_p^p\right]^{1/p}\\
&+C'\left[\int_0^t \|e^{-\theta|v|^2}e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_x\nu\|_p^p\right]^{1/p},\\
&\|e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_v f(t)\|_p \lesssim_{t,m,\beta} \|\alpha^\beta\nabla_{x,v} f_0\|_p\\
&+\left[\int_0^t \left|\frac{e^{-\varpi\langle v\rangle t}\alpha^\beta}{v\cdot n}\{\partial_t g+\sum(v\cdot\tau_i)\partial_{\tau_i}g+\nu g-H\}\right|_{\gamma,p}^p \mathrm{d}s\right]^{1/p}\\
&+\left[\int_0^t \sum_{i=1}^2 |e^{-\varpi\langle v\rangle s}\alpha^\beta\partial_{\tau_i}g(s)|_{\gamma,p}^p+|e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_v g(s)|_{\gamma,p}^p\right]^{1/p}\\
&+\left[\int_0^t \|e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_{x,v}H(s)\|_p^p\right]^{1/p}\\
&+C'\left[\int_0^t \|e^{-\theta|v|^2}e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_{x,v}\nu\|_p^p\right]^{1/p},
\end{aligned}
$$

where $C'=\{\|e^{\theta|v|^2}f_0\|_\infty+\|e^{\theta|v|^2}H\|_\infty+|e^{\theta|v|^2}g|_\infty\}$.

By the hypotheses of Proposition 2 and the assumption on f_0, g and H with compact supports, the right hand sides are bounded and hence $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_t f$, $e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f$, and $e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_v f$ are in $L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))$.

Since f_0, g and H are compactly supported inside $\{v\in\mathbb{R}^3:|v|\le m\}$, the derivatives $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_t f$, $e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f$ and $e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_v f$ are compactly supported inside $\{v\in\mathbb{R}^3:|v|\le m\}$ and hence from (3.22) and (3.19)

$$
\begin{aligned}
\{\partial_t+v\cdot\nabla_x+\nu_{\varpi,\beta}\}[e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f]&=e^{-\varpi\langle v\rangle t}\alpha^\beta\partial H\\
&-\partial v\cdot e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f-\partial\nu(v)e^{-\varpi\langle v\rangle t}\alpha^\beta f.
\end{aligned}
$$

Moreover, from the general definition of traces, by choosing a test function multiplied by $e^{-\varpi\langle v\rangle t}\alpha^\beta$, we deduce $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f$ has the same trace as $e^{-\varpi\langle v\rangle t}\alpha^\beta[\partial f|_\gamma]$.

Now we can apply Lemma 9 to establish (3.23) which does not depend on the velocity cut-off. Therefore, for the general case, we use (3.20) and pass to the limit to conclude the proof. □

4 Dynamical Non-local to Local Estimate

The main purpose of this section is to prove Lemma 1 and its variants (Lemma 11).

Lemma 10 *For $y \in \bar{\Omega}$, $\frac{1}{2} < \beta < \frac{3}{2}$, $0 < \kappa \leq 1$, and $\theta > 0$,*

$$\int_{\mathbb{R}^3} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}[\alpha(y,u)]^\beta} \mathrm{d}u \lesssim \frac{1}{|v|^{2\beta-1}|\xi(y)|^{\beta-\frac{1}{2}}}. \tag{4.1}$$

Proof Firstly, we consider the case of $|\xi(y)| \leq \delta_\Omega \ll 1$. From the assumption, we have $\nabla\xi(y) \neq 0$ and therefore there is a uniquely determined unit vector $n(y) = \frac{\nabla\xi(y)}{|\nabla\xi(y)|}$. We choose two unit vector τ_1 and τ_2 so that $\{\tau_1, \tau_2, n(y)\}$ is an orthonormal basis of $\mathbb{R}^3$.

We decompose the velocity variables $u \in \mathbb{R}^3$ as

$$u = u_n n(y) + u_\tau \cdot \tau = u_n n(y) + \sum_{i=1}^{2} u_{\tau,i}\tau_i.$$

We note that $u_\tau \in \mathbb{R}^2$ and $u_n \in \mathbb{R}$ are completely free coordinates. Therefore using the Fubini's theorem we can rearrange the order of integration freely. Now we split

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}} \frac{1}{[\alpha(y,u)]^\beta} \mathrm{d}u \\
&\lesssim \int_{\mathbb{R}^2}\int_{\mathbb{R}} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}\left[|u_n|^2 + |\xi(y)||u|^2\right]^\beta} \mathrm{d}u_n \mathrm{d}u_\tau \\
&= \int_{|u|\geq 5|v|} + \int_{|u|\leq\frac{|v|}{2}} + \int_{\frac{|v|}{2}\leq|u|\leq 5|v|} = \mathbf{(I)} + \mathbf{(II)} + \mathbf{(III)}.
\end{aligned}$$

For the first term **(I)** we use, for $|u| \geq 5|v|$ (therefore $|v| \leq \frac{|u|}{5}$),

$$\begin{aligned}
|u-v|^2 &= \frac{|u-v|^2}{2} + \frac{|u-v|^2}{2} \geq \frac{\frac{|u|^2}{2} - |v|^2}{2} + \frac{\frac{|u|^2}{2} - |v|^2}{2} \\
&\geq \frac{23}{4}|v|^2 + \frac{23}{100}|u|^2 \gtrsim |v|^2 + |u|^2,
\end{aligned}$$

and we use $\left[|u_n|^2 + |\xi||u|^2\right]^\beta \geq \left[|u_n|^2 + 25|\xi||v|^2\right]^\beta \gtrsim \left[|u_n|^2 + |\xi\|v|^2\right]^\beta$ for $|u| \geq 5|v|$ to have

$$(\mathbf{I}) \lesssim e^{-C|v|^2} \int_{\mathbb{R}^2} \mathrm{d}u_\tau \frac{e^{-C|u_\tau|^2}}{|v_\tau - u_\tau|^{2-\kappa}} \int_{\mathbb{R}} \mathrm{d}u_n \frac{e^{-C|u_n|^2}}{\left[|u_n|^2 + |\xi\|v|^2\right]^\beta}.$$

Since

$$\frac{1}{|v_\tau - u_\tau|^{2-\kappa}} \in L^1_{\mathrm{loc}}(\{u_\tau \in \mathbb{R}^2\}) \quad \text{for} \quad \kappa > 0, \tag{4.2}$$

the integral over u_τ is finite. Then

$$\begin{aligned}
(\mathbf{I}) &\lesssim e^{-C|v|^2} \int_{\mathbb{R}} \frac{e^{-C|u_n|^2}}{\left[|u_n|^2 + |\xi\|v|^2\right]^\beta} \mathrm{d}u_n \\
&\lesssim e^{-C|v|^2} \left\{ \int_{10}^{\infty} \frac{e^{-C|u_n|^2}}{|u_n|^{2\beta} \mathbf{1}_{\{|u_n| \geq 10\}}} \mathrm{d}|u_n| + \int_0^{10} \frac{\mathrm{d}|u_n|}{\left[|u_n|^2 + |\xi||v|^2\right]^\beta} \right\} \\
&\lesssim \left(1 + \int_0^{10} \frac{\mathrm{d}|u_n|}{\left[|u_n|^2 + |\xi\|v|^2\right]^\beta} \right) e^{-C|v|^2} \\
&\lesssim e^{-C|v|^2} \left(1 + \int_0^{10} \frac{\mathrm{d}[|\xi|^{\frac{1}{2}} |v| \tan\theta]}{|\xi|^\beta |v|^{2\beta} (1 + \tan^2\theta)^\beta} \right) \\
&\lesssim e^{-C|v|^2} \left(1 + \frac{1}{|v|^{2\beta-1}} \frac{1}{|\xi|^{\beta-1/2}} \int_0^{\pi/2} (\cos\theta)^{2\beta-2} \mathrm{d}\theta \right) \\
&\lesssim e^{-C|v|^2} \left(1 + \frac{1}{|v|^{2\beta-1}} \frac{1}{|\xi|^{\beta-1/2}} \right) \\
&\lesssim \frac{e^{-C_\theta|v|^2}}{|v|^{2\beta-1}} \frac{1}{|\xi(y)|^{\beta-1/2}},
\end{aligned}$$

where we have used a change of variables: $|u_n| = |\xi|^{\frac{1}{2}}|v|\tan\theta$ and $\mathrm{d}|u_n| = |\xi|^{\frac{1}{2}}|v|\sec^2\theta \mathrm{d}\theta$ and $(\cos\theta)^{2\beta-2} \in L^1_{\mathrm{loc}}(\{\theta \in [0, \frac{\pi}{2}]\})$ for $\beta > \frac{1}{2}$.

For the second term (**II**), we use $|v-u| \geq |v| - |u| \geq |v| - \frac{|v|}{2} \geq \frac{|v|}{2}$ from $|u| \leq \frac{|v|}{2}$, and apply the change of variables $u \mapsto |v|u$ to have

$$\begin{aligned}
(\mathbf{II}) &\lesssim \frac{1}{|v|^{2-\kappa}} \int_{|u_n|+|u_\tau| \leq \frac{|v|}{2}} \frac{e^{-C|v|^2} \mathrm{d}u_n \mathrm{d}u_\tau}{\left[|u_n|^2 + |\xi\|u_\tau|^2\right]^\beta} \\
&= \frac{1}{|v|^{2-\kappa}} \int_{|v|(|u_n|+|u_\tau|) \leq \frac{|v|}{2}} \frac{e^{-C|v|^2} |v| \mathrm{d}u_n |v|^2 \mathrm{d}u_\tau}{\left[|v|^2|u_n|^2 + |\xi||v|^2|u_\tau|^2\right]^\beta} \\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-\kappa-1}} \int_{|u_\tau| \leq \frac{1}{2}} \int_{|u_n| \leq \frac{1}{2}} \frac{1}{\left[|u_n|^2 + |\xi\|u_\tau|^2\right]^\beta} \mathrm{d}u_n \mathrm{d}u_\tau.
\end{aligned}$$

Now we apply the change of variables $|u_n| = |\xi|^{\frac{1}{2}}|u_\tau|\tan\theta$ for $\theta \in [0, \frac{\pi}{2}]$ with $\mathrm{d}u_n = |\xi|^{\frac{1}{2}}|u_\tau|\sec^2\theta\mathrm{d}\theta$ to have

$$\begin{aligned}
\textbf{(II)} &\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-\kappa-1}}\int_{|u_\tau|\le\frac{1}{2}}\mathrm{d}u_\tau\int_0^{\frac{\pi}{2}}\frac{|\xi|^{\frac{1}{2}}|u_\tau|\sec^2\theta\mathrm{d}\theta}{\left[|\xi||u_\tau|^2\tan^2\theta + |\xi||u_\tau|^2\right]^\beta}\\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-\kappa-1}|\xi|^{\beta-1/2}}\int_{|u_\tau|\le\frac{1}{2}}\frac{\mathrm{d}u_\tau}{|u_\tau|^{2\beta-1}}\int_0^{\pi/2}(\cos\theta)^{2\beta-2}\mathrm{d}\theta\\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-\kappa-1}|\xi|^{\beta-1/2}},
\end{aligned}$$

where we have used $\frac{1}{|u_\tau|^{2\beta-1}} \in L^1_{\mathrm{loc}}(\{u_\tau \in \mathbb{R}^2\})$ for $\beta < \frac{3}{2}$ and $(\cos\theta)^{2\beta-2} \in L^1_{\mathrm{loc}}(\{\theta \in [0, \frac{\pi}{2}]\})$ for $\beta > \frac{1}{2}$. We note $|v|^\kappa e^{-C|v|^2} \lesssim 1$, and (4.1) is valid in this case.

For the last term (**III**), we use the lower bound of $|u|$ ($|u| \ge \frac{|v|}{2}$) to have $\left[|u_n|^2 + |\xi\|u|^2\right]^\beta \ge \left[|u_n|^2 + |\xi|\frac{|v|^2}{4}\right]^\beta \gtrsim \left[|u_n|^2 + |\xi||v|^2\right]^\beta$ and

$$\begin{aligned}
\int_{\frac{|v|}{2}\le|u|\le5|v|} &\lesssim \int_{0\le|u_\tau|\le5|v|}\frac{e^{-\frac{C}{2}|v_\tau-u_\tau|^2}}{|v_\tau-u_\tau|^{2-\kappa}}\mathrm{d}u_\tau\int_0^{5|v|}\frac{1}{\left[|u_n|^2+|\xi\|v|^2\right]^\beta}\mathrm{d}u_n\\
&\lesssim \int_0^{5|v|}\frac{1}{\left[|u_n|^2+|\xi\|v|^2\right]^\beta}\mathrm{d}u_n,
\end{aligned}$$

where we have used $\frac{1}{|u_\tau|^{2-\kappa}} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for $\kappa > 0$. We apply a change of variables: $|u_n| = |\xi|^{1/2}|v|\tan\theta$ for $\theta \in [0, \pi/2]$ with $\mathrm{d}|u_n| = |\xi|^{\frac{1}{2}}|v|\sec^2\theta\mathrm{d}\theta$. Hence

$$\begin{aligned}
\textbf{(III)} &\lesssim \int_0^{5|v|}\frac{1}{\left[|u_n|^2+|\xi\|v|^2\right]^\beta}\mathrm{d}u_n = \int_0^{\frac{\pi}{2}}\frac{(\cos\theta)^{2\beta-2}}{|\xi|^{\beta-\frac{1}{2}}|v|^{2\beta-1}}\mathrm{d}\theta\\
&\lesssim \frac{1}{|\xi|^{\beta-\frac{1}{2}}}\frac{1}{|v|^{2\beta-1}},
\end{aligned}$$

where we used $(\cos\theta)^{2\beta-2} \in L^1_{\mathrm{loc}}(\{\theta \in [0, \frac{\pi}{2}]\})$ for $\beta > \frac{1}{2}$. Overall, we combine the estimates of (**I**), (**II**) and (**III**) to conclude (4.1).

Secondly, we consider the case of $|\xi(y)| > \delta_\Omega$. Then we can choose any orthonormal basis, for example the standard basis $\{\tau_1, \tau_2, n\} = (e_1, e_2, e_3)$, to decompose the velocity variables $u \in \mathbb{R}^3$ as $u = u_1e_1 + u_2e_2 + u_3e_3 :=$

$u_{\tau,1}e_1 + u_{\tau,2}e_2 + u_n e_3$. Then

$$\begin{aligned}\alpha(y,u) &= |u\cdot\nabla\xi(y)|^2 - 2\xi(y)\{u\cdot\nabla^2\xi(y)\cdot u\} \geq 2|\xi(y)|\{u\cdot\nabla^2\xi(y)\cdot u\}\\ &= \delta_\Omega C|u|^2 + |\xi(y)|\{u\cdot\nabla^2\xi(y)\cdot u\} \gtrsim |u_n|^2 + |\xi(y)\|u|^2.\end{aligned}$$

Then we follow all the proof with the same decomposition for $v := v_{\tau,1}e_1 + v_{\tau,2}e_2 + v_n e_3$ to conclude (4.1) for $|\xi(y)| > \delta_\Omega$. □

We now prove the dynamical non-local to local estimates:

Proof of (1) of Lemma 1 Since $\frac{\langle u\rangle^r}{\langle v\rangle^r} \lesssim \{1+|v-u|^2\}^{\frac{r}{2}}$, it suffices to consider the case $r=0$. We prove (1.28).

We first assume $v\cdot\nabla\xi(x)\geq 0$ and $x\in\partial\Omega$. There exist $\sigma_1,\sigma_2>0$ such that

$$\begin{aligned}&|v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| \gtrsim \sqrt{\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,v)}\\ &\quad\text{for all } s\in[0,\sigma_1]\cup[t_{\mathbf{b}}(x,v)-\sigma_2, t_{\mathbf{b}}(x,v)],\end{aligned} \tag{4.3}$$

and $|v|\sqrt{-\xi(x-(t_{\mathbf{b}}(x,v)-s)v)} \gtrsim \sqrt{\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,v)}$ for all $s\in[\sigma_1, t_{\mathbf{b}}(x,v)-\sigma_2]$. The mapping $s\mapsto\xi(x-(t_{\mathbf{b}}(x,v)-s)v)$ is one-to-one and onto on $s\in[0,\sigma_1]$ or on $s\in[t_{\mathbf{b}}(x,v)-\sigma_2, t_{\mathbf{b}}(x,v)]$. Moreover this mapping $s\mapsto\xi(x-(t_{\mathbf{b}}(x,v)-s)v)$ is a diffeomorphism and we have a change of variables on $s\in[0,\sigma_1]$ or $s\in[t_{\mathbf{b}}(x,v)-\sigma_2, t_{\mathbf{b}}(x,v)]$,

$$\mathrm{d}s = \frac{\mathrm{d}|\xi|}{|\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\cdot v|} \lesssim \frac{\mathrm{d}|\xi|}{\sqrt{\alpha(x-(t_{\mathbf{b}}(x,v)-s)v)}}. \tag{4.4}$$

Step 1 First we establish (4.3) and (4.4).

Firstly we prove (4.3). Recall the definition of α in Definition 1. It suffices to show

$$\begin{aligned}|v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| &\geq |v|\sqrt{-\xi(x-(t_{\mathbf{b}}(x,v)-s)v)},\\ &\qquad s\in[0,\sigma_1]\cup[t_{\mathbf{b}}(x,v)-\sigma_2, t_{\mathbf{b}}(x,v)],\\ |v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| &\leq |v|\sqrt{-\xi(x-(t_{\mathbf{b}}(x,v)-s)v)},\\ &\qquad s\in[\sigma_1, t_{\mathbf{b}}(x,v)-\sigma_2].\end{aligned}$$

If $v=0$ or $v\cdot\nabla\xi(x)=0$ then (4.3) holds clearly. Therefore we may assume $v\neq 0$ and $v\cdot\nabla\xi(x)>0$. Due to the Velocity lemma, $v\cdot\frac{\nabla\xi(x)}{|\nabla\xi(x)|}>0$ and $v\cdot\frac{\nabla\xi(x_{\mathbf{b}}(x,v))}{|\nabla\xi(x_{\mathbf{b}}(x,v))|}<0$. By the mean value theorem we choose $t^*\in(0,t_{\mathbf{b}}(x,v))$ solving $v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-t^*)v)=0$. Moreover due to the convexity of ξ we have

$$\frac{d}{ds}\left(v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\right)=v\cdot\nabla^2\xi(X_{\mathbf{cl}}(s))\cdot v\geq C_\xi|v|^2,$$

and therefore $t^*\in(0,t_{\mathbf{b}}(x,v))$ is uniquely determined. Clearly we have

$$\begin{aligned}v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)&\geq 0 \text{ for } s\in[t^*,t_{\mathbf{b}}(x,v)],\\ v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)&\leq 0 \text{ for } s\in[0,t^*].\end{aligned}$$

Define $\Phi(s)=\left\{|v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)|^2+|v|^2\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\right\}$. Since $2\left(v\cdot\nabla^2\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\cdot v\right)+|v|^2>0$ we have

$$\begin{aligned}\frac{d}{ds}\Phi(s)&=\left(v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\right)\\&\quad\times\left\{2\left(v\cdot\nabla^2\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\cdot v\right)+|v|^2\right\},\end{aligned}$$

is strictly negative for $s\in[0,t^*]$ and is strictly positive for $s\in[t^*,t_{\mathbf{b}}(x,v)]$. Note that $\Phi(0)>0$ and $\Phi(t_{\mathbf{b}}(x,v))>0$ from $v\cdot\frac{\nabla\xi(x)}{|\nabla\xi(x)|}>0$ and $v\cdot\frac{\nabla\xi(x_{\mathbf{b}}(x,v))}{|\nabla\xi(x_{\mathbf{b}}(x,v))|}<0$. Note that Φ is continuous function on the interval $[0,t_{\mathbf{b}}(x,v)]$ so that it has a minimum. If $\min_{[0,t_{\mathbf{b}}(x,v)]}\Phi(s)\leq 0$, there exist $\sigma_1,\sigma_2>0$ satisfying

$$\begin{aligned}\Phi(t_{\mathbf{b}}(x,v)+\sigma_1)&=\Phi(t_{\mathbf{b}}(x,v))+\int_0^{\sigma_1}\frac{d}{ds}\Phi(s)\mathrm{d}s=0,\\ \Phi(t_{\mathbf{b}}(x,v)-\sigma_2)&=\Phi(t_{\mathbf{b}}(x,v))-\int_{t_{\mathbf{b}}(x,v)-\sigma_2}^{t_{\mathbf{b}}(x,v)}\frac{d}{ds}\Phi(s)\mathrm{d}s=0,\end{aligned}$$

then $\sigma_1\leq t^*$ and $t_{\mathbf{b}}(x,v)-\sigma_2\geq t^*$ and there is no other $s\in[0,t_{\mathbf{b}}(x,v)]$ satisfying $\Phi(s)=0$. Moreover we have $\Phi(s)\leq 0$ for $s\in[\sigma_1,t_{\mathbf{b}}(x,v)-\sigma_2]$. If $\min_{[0,t_{\mathbf{b}}(x,v)]}\Phi(s)>0$, there do not exist such σ_1 and σ_2 then we let $\sigma_1=t^*$ and $\sigma_2=t_{\mathbf{b}}(x,v)-t^*$. This proves (4.3).

Secondly we prove (4.4). By the proof of (4.3), and the fact

$$\frac{d|\xi|}{ds}=-\frac{d}{ds}\xi(x-(t_{\mathbf{b}}(x,v)-s)v)=-v\cdot\nabla_x\xi(x-(t_{\mathbf{b}}(x,v)-s)v),$$

as well as the inverse function theorem we deduce (4.4).

Step 2 For small $0<\tilde{\delta}\ll 1$, we define

$$\tilde{\sigma}_1:=\min\left\{\sigma_1,\tilde{\delta}\frac{\sqrt{\alpha(x,v)}}{|v|^2}\right\},\quad \tilde{\sigma}_2:=\min\left\{\sigma_2,\tilde{\delta}\frac{\sqrt{\alpha(x,v)}}{|v|^2}\right\}. \tag{4.5}$$

Actually, thanks to the fact $\sigma_i \leq t_{\mathbf{b}} \lesssim \frac{\sqrt{\alpha(x,v)}}{|v|^2}$, by Lemma 3, we have $\tilde{\sigma}_i := \tilde{\delta}\sqrt{\alpha(x,v)}/|v|^2$.

Then both of (4.3) and (4.4) hold for $s \in [0, \tilde{\sigma}_1] \cup [t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2, t_{\mathbf{b}}(x,v)]$ without changing the constant. Moreover, if $s \in [0, \tilde{\sigma}_1] \cup [t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2, t_{\mathbf{b}}(x,v)]$ then by the Velocity lemma

$$\max\{|\xi|\} := \max_{s\in[0,\tilde{\sigma}_1]\cup[t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2,t_{\mathbf{b}}(x,v)]} |\xi(x-(t_{\mathbf{b}}-s)v)| \lesssim \tilde{\delta}\frac{\alpha(x,v)}{|v|^2}. \quad (4.6)$$

For $s \in [\tilde{\sigma}_1, t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2]$ we have the following estimate with $\tilde{\delta}-$dependent constant:

$$|v|\sqrt{-\xi(x-(t_{\mathbf{b}}(x,v)-s)v)} \gtrsim_{\xi,\tilde{\delta}} \sqrt{\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,v)}. \quad (4.7)$$

The proof of (4.6) is due to, for $s \in [0, \tilde{\sigma}_1]$,

$$\begin{aligned} |\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| &\leq \int_0^s |v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-\tau)v)|\mathrm{d}\tau \\ &\lesssim \sqrt{\alpha(x,v)}|s| \lesssim \min\left\{\sqrt{\alpha}t_Z, \frac{\tilde{\delta}\alpha}{|v|^2}\right\} =: B, \end{aligned} \quad (4.8)$$

where t_Z is defined in (1.28) and $\alpha(x-(t_{\mathbf{b}}-\tau)v,v) \lesssim_\xi \alpha(x,v)$ from the Velocity lemma (Lemma 2). The proof for $s \in [t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2, t_{\mathbf{b}}(x,v)]$ is exactly same.

Now we prove (4.7). Recall that for $t^* \in [0, t_{\mathbf{b}}(x,v)]$ in the previous step we proved: $v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-t^*)v) = 0$. Clearly $|\xi(x-(t_{\mathbf{b}}-s)v)|$ is an increasing function on $[0,t^*]$ and a decreasing function on $[t^*, t_{\mathbf{b}}(x,v)]$. This is due to the convexity of ξ:

$$\frac{d^2}{ds^2}[-\xi(s-(t_{\mathbf{b}}(x,v)-s)v)] = v\cdot\nabla^2\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\cdot v \gtrsim |v|^2,$$

and $v\cdot\nabla\xi(x) > 0$ and $v\cdot\nabla\xi(x_{\mathbf{b}}(x,v)) < 0$.

Therefore, from $\xi(x) = 0 = \xi(x_{\mathbf{b}})$,

$$\begin{aligned} -\xi(x-(t_{\mathbf{b}}(x,v)-s)v) &= -\xi(x) - \int_{t_{\mathbf{b}}(x,v)}^s v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-\tau)v)\mathrm{d}\tau \\ &= \int_s^{t_{\mathbf{b}}(x,v)} v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-\tau)v)\mathrm{d}\tau \\ &\geq (t_{\mathbf{b}}(x,v)-s)(v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)) \\ &\geq \tilde{\sigma}_2|v\cdot\nabla\xi(x-\tilde{\sigma}_2 v)| \quad \text{for } s\in[t^*, t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2], \end{aligned}$$

and

$$\begin{aligned} -\xi(x-(t_{\mathbf{b}}(x,v)-s)v) &= -\xi(x_{\mathbf{b}}(x,v)) \\ &\quad -\int_0^s v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-\tau)v)\mathrm{d}\tau \\ &\geq s|v\cdot\nabla\xi(x-(t_{\mathbf{b}}-s)v)| \\ &\geq \tilde{\sigma}_1|v\cdot\nabla\xi(x_{\mathbf{b}}(x,v)+\tilde{\sigma}_1 v)| \quad \text{for } s\in[0,\tilde{\sigma}_1]. \end{aligned}$$

Hence, for $s\in[\tilde{\sigma}_1, t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2]$,

$$\begin{aligned} |\xi(x-(t_{\mathbf{b}}-s)v)| &\geq \min\{|\xi(x-\tilde{\sigma}_2 v)|, |\xi(x_{\mathbf{b}}(x,v)+\tilde{\sigma}_1 v)|\} \\ &\geq \min\{\tilde{\sigma}_2|v\cdot\nabla\xi(x-\tilde{\sigma}_2 v)|, \tilde{\sigma}_1|v\cdot\nabla\xi(x_{\mathbf{b}}(x,v)+\tilde{\sigma}_1 v)|\}. \end{aligned}$$

From the definition of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ in (4.5) we have

$$\begin{aligned} |v|^2|\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| &\geq \tilde{\delta}\sqrt{\alpha(x,v)} \\ \times \min\{|v\cdot\nabla\xi(x-\tilde{\sigma}_2 v)|, &|v\cdot\nabla\xi(x_{\mathbf{b}}(x,v)+\tilde{\sigma}_1 v)|\}. \end{aligned}$$

Without loss of generality we may assume

$$|v\cdot\nabla\xi(x-\tilde{\sigma}_2 v)| = \min\{|v\cdot\nabla\xi(x-\tilde{\sigma}_2 v)|, |v\cdot\nabla\xi(x_{\mathbf{b}}(x,v)+\tilde{\sigma}_1 v)|\}.$$

Then by the Velocity lemma we have $\sqrt{\alpha(x,v)} \gtrsim_\xi |v||\xi(x-\tilde{\sigma}_2 v)|^{1/2}$. Then we choose $s = t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2$ to have $|v|^2|\xi(x-\tilde{\sigma}_2 v)| \geq \tilde{\delta}|v||\xi(x-\tilde{\sigma}_2 v)|^{1/2}\times |v\cdot\nabla\xi(x-\tilde{\sigma}_2 v)|$ and

$$|v||\xi(x-\tilde{\sigma}_2 v)|^{1/2} \gtrsim \tilde{\delta}\times|v\cdot\nabla\xi(x-\tilde{\sigma}_2 v)|.$$

The left hand side is the lower bound of $|v|^2|\xi(x-(t_{\mathbf{b}}(x,v)-s)v)|$ for $s\in[\tilde{\sigma}_1, t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2]$ and the right hand side is bounded below by the Velocity lemma: $e^{-\mathcal{C}|v|t_{\mathbf{b}}(x,v)}\alpha(x,v) \gtrsim_\xi \alpha(x,v)$. Therefore we conclude (4.7).

Step 3 We prove (1.28). From (4.1) with $y = x-(t_{\mathbf{b}}(x,v)-s)v$

$$\begin{aligned} &\int_0^{t_{\mathbf{b}}(x,v)}\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|v-u|^2}}{|v-u|^\kappa\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)}Z(s,x,v)\mathrm{d}u\mathrm{d}s \\ &\lesssim \int_0^{t_{\mathbf{b}}(x,v)} e^{-l\langle v\rangle(t-s)}\frac{1}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}}Z(s,x,v)\mathrm{d}s. \end{aligned}$$

According to (4.5) we split the time integration as

$$\int_0^{t_{\mathbf{b}}(x,v)} e^{-l\langle v\rangle(t-s)} \frac{1}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}} Z(s,x,v)\mathrm{d}s = \underbrace{\int_0^{\tilde{\sigma}_1} + \int_{t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2}^{t_{\mathbf{b}}(x,v)}}_{(\mathbf{IV})} + \underbrace{\int_{\tilde{\sigma}_1}^{t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2}}_{(\mathbf{V})}.$$

For the first two terms (**IV**), we use the mapping of (4.4)

$$s \in [0, \tilde{\sigma}_1] \cup [t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2, t_{\mathbf{b}}(x,v)] \mapsto |\xi(x - (t_{\mathbf{b}}(x,v) - s)v)| \in [0, B),$$

where the range of $|\xi|$ has been bounded in (4.6), and B is given by (4.8). By the change of variables of (4.4)

$$\begin{aligned}(\mathbf{IV}) &\lesssim \sup_{0\le s\le t_{\mathbf{b}}(x,v)} \{e^{-l\langle v\rangle(t-s)} Z(s,x,v)\} \frac{1}{|v|^{2\beta-1}} \int_0^{C\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}} \frac{1}{|\xi|^{\beta-1/2}} \frac{\mathrm{d}|\xi|}{\sqrt{\alpha(x,v)}} \\ &\lesssim \sup_{0\le s\le t_{\mathbf{b}}(x,v)} \{e^{-l\langle v\rangle(t-s)} Z(s,x,v)\} \frac{1}{|v|^{2\beta-1}} \frac{1}{\sqrt{\alpha(x,v)}} \left[|\xi|^{-\beta+\frac{3}{2}}\right]_{|\xi|=0}^{|\xi|=B},\end{aligned}$$

where we have used $\beta < \frac{3}{2}$. The lemma follows with B given by (4.8).

For (**V**) we use $\sqrt{\alpha(x-(t_{\mathbf{b}}-s)v)} \lesssim_{\xi,\tilde{\delta}} |v|\sqrt{-\xi(x-(t_{\mathbf{b}}-s)v)}$ for $s \in [\tilde{\sigma}_1, t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2]$, from (4.3), to have

$$\frac{1}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}} = \frac{1}{\left(|v|\sqrt{-\xi}\,\right)^{2(\beta-\frac{1}{2})}} \lesssim \frac{1}{[\alpha(x,v)]^{\beta-\frac{1}{2}}}.$$

Finally

$$\begin{aligned}(\mathbf{V}) &\lesssim \frac{1}{[\alpha(x,v)]^{\beta-1/2}} \int_0^{t_{\mathbf{b}}(x,v)} e^{-l\langle v\rangle(t-s)} Z(s,v)\mathrm{d}s \\ &\lesssim \frac{O(l^{-1})}{\langle v\rangle[\alpha(x,v)]^{\beta-1/2}} \sup_{0\le s\le t} \{e^{-l\langle v\rangle(t-s)} Z(s,x,v)\}.\end{aligned}$$

Now we assume $x \notin \partial\Omega$. We find $\bar{x} \in \partial\Omega$ and $\bar{t}_{\mathbf{b}}$ so that

$$x - (t_{\mathbf{b}}(x,v) - s)v = \bar{x} - (\bar{t}_{\mathbf{b}} - s)v.$$

Therefore, by *Step 1* and the fact $\bar{x} \in \partial\Omega$, we have

$$\int_0^{\bar{t}_{\mathbf{b}}} \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}\left[\alpha(\bar{x}-(\bar{t}_{\mathbf{b}}-s)v,u)\right]^{\beta}} Z(s,x,v)\mathrm{d}u\mathrm{d}s$$
$$\lesssim \int_0^{\bar{t}_{\mathbf{b}}} e^{-l\langle v\rangle(t-s)} \frac{e^{-C|v-u|^2}}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}} Z(s,x,v)\mathrm{d}s.$$

We then deduce our lemma since $\alpha(\bar{x},v) \simeq \alpha(x,v)$ via the Velocity Lemma and the fact $\bar{t}_{\mathbf{b}}|v| \lesssim_\Omega 1$. □

Proof of (2) Lemma 1 It suffices to consider $r = 0$. For the specular cycles and the bounce-back cycles it is important to control the *number of bounces*,

$$\ell_*(s) = \ell_*(s;t,x,v) \in \mathbb{N} \quad \text{such that} \quad t^{\ell_*+1} \leq s < t^{\ell_*}.$$

An important consequence of Velocity lemma is that for the specular cycles

$$\alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) \gtrsim e^{-\mathcal{C}|v||t-s|}\alpha(x,v),$$

and therefore for the specular cycles

$$\ell_*(s;t,x,v) \leq \frac{|t-s|}{\min_{0\leq\ell\leq\ell_*(s;t,x,v)}|t^\ell - t^{\ell+1}|} \lesssim \frac{|t-s|}{\min_{0\leq\ell\leq\ell_*(s;t,x,v)} \frac{\sqrt{\alpha(x^\ell,v^\ell)}}{|v^\ell|^2}}$$
$$\lesssim \frac{|t-s||v|^2}{\sqrt{\alpha(x,v)}} e^{\mathcal{C}|v|(t-s)}. \tag{4.9}$$

Remark that for the bounce-back cycles we do not have the growth factor $e^{\mathcal{C}|v|(t-s)}$. This is because of the fact $\alpha(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))$ is either $\alpha(x^1,v^1)$ or $\alpha(x^2,v^2)$, and the fact $|t-t^2| \leq 2|t^1-t^2| \lesssim \frac{C_\Omega}{|v|}$ for the bounded domain.

We consider the specular BC case first. For fixed (x,v) we use the following notation $\alpha(s) := \alpha(s;t,x,v) := \alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))$.

Firstly we consider the estimate (1.29) for $|v| < \delta$. Using (4.9),

$$\mathbf{1}_{\{|v|\leq\delta\}} \int_0^t \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} \frac{Z(s,x,v)}{[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)]^\beta}\mathrm{d}u\mathrm{d}s$$
$$\lesssim \sum_{\ell=0}^{\ell_*(0;t,x,v)} \int_{t^{\ell+1}}^{t^\ell} \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|v^\ell-u|^2}}{|v^\ell-u|^{2-\kappa}} \frac{Z(s,x,v)}{\left[\alpha(x^\ell-(t^\ell-s)v^\ell,u)\right]^\beta}\mathrm{d}u\mathrm{d}s$$

$$\lesssim \frac{t|v|^2 e^{\mathcal{C}t\delta}}{[\alpha(x,v)]^{1/2}} \sup_{\ell} \left\{ \frac{O(\tilde{\delta})e^{\mathcal{C}t\delta}}{|v|^2[\alpha(x,v)]^{\beta-1}} \sup_{t^{\ell+1}\leq s\leq t^{\ell}} \{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\} \right.$$
$$\left. + \frac{C_{\tilde{\delta}}e^{\mathcal{C}t\delta}}{[\alpha(x,v)]^{\beta-1/2}} \int_{t^{\ell+1}}^{t^{\ell}} e^{-l\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s \right\},$$

where we have used (1.28). By (2.2) and (2.3) and the Velocity lemma (Lemma 2), we have $|t^{\ell}-t^{\ell+1}| \lesssim_{\xi} \frac{\sqrt{\alpha(x,v)}}{|v|^2}e^{\mathcal{C}t|v|} \lesssim_{\xi,\delta} \frac{\sqrt{\alpha(x,v)}}{|v|^2}e^{\mathcal{C}t\delta}$, and hence we deduce (1.29) for $|v|<\delta$ by

$$\mathbf{1}_{\{|v|<\delta\}}\int\cdots \lesssim_{\xi} \frac{O(\tilde{\delta}+l^{-1})te^{2\mathcal{C}t\delta}}{[\alpha(x,v)]^{\beta-\frac{1}{2}}} \sup_{0\leq s\leq t}\left\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\right\}.$$

Now we consider $|v|\geq\delta$. We split the time interval as

$$[0,t] \subset [t-\frac{1}{|v|},t] \cup \bigcup_{j=1}^{[t|v|]+1}[t-(j+1)\frac{1}{|v|},t-j\frac{1}{|v|}]. \tag{4.10}$$

Consider the first time section $[t-\frac{1}{|v|},t]$. Due to (4.9), we bound the number of bounces within by

$$\sup_{s\in[t-\frac{1}{|v|},t]} \ell_*(s;t,x,v) \lesssim_{\xi} \frac{\frac{1}{|v|}|v|^2 e^{\mathcal{C}\frac{1}{|v|}|v|}}{[\alpha(x,v)]^{1/2}} \lesssim \frac{|v|e^{\mathcal{C}}}{[\alpha(x,v)]^{1/2}},$$

and for $s\in[t-\frac{1}{|v|},t]$, $e^{-\mathcal{C}}\alpha(x,v) \lesssim \alpha(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v)) \lesssim e^{\mathcal{C}}\alpha(x,v)$, and $|t^{\ell}-t^{\ell+1}| \lesssim \frac{[\alpha(x,v)]^{1/2}e^{\mathcal{C}}}{|v|^2}$ due to the Velocity lemma. Then we use (1.28) to have

$$\int_{t-1/|v|}^{t}\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} \frac{Z(s,x,v)}{[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)]^{\beta}}\mathrm{d}u\mathrm{d}s$$
$$\lesssim \sum_{\ell=0}^{\ell_*(0;t,x,v)} \int_{t^{\ell+1}}^{t^{\ell}}\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|v^{\ell}-u|^2}}{|v^{\ell}-u|^{2-\kappa}} \frac{Z(s,x,v)}{\left[\alpha(x^{\ell}-(t^{\ell}-s)v^{\ell},u)\right]^{\beta}}\mathrm{d}u\mathrm{d}s$$
$$\lesssim \frac{C_{\xi}|v|}{[\alpha(x,v)]^{1/2}} \sup_{\ell} \frac{O(\tilde{\delta})e^{C_{\xi}}}{|v|^2[\alpha(x,v)]^{\beta-1}} \sup_{t^{\ell+1}\leq s\leq t^{\ell}} \{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\}$$
$$+ \sum_{\ell=0}^{\ell_*(0,t,x,v)} \frac{C_{\tilde{\delta}}e^{C_{\xi}}}{[\alpha(x,v)]^{\beta-1/2}} \int_{t^{\ell+1}}^{t^{\ell}} e^{-l\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s$$

$$\lesssim \frac{O(\tilde{\delta})}{|v|[\alpha(x,v)]^{\beta-1/2}} \sup_{0\leq s\leq t}\{e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\}$$
$$+\frac{C_{\tilde{\delta},\xi}}{[\alpha(x,v)]^{\beta-1/2}}\int_0^t e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s$$
$$\lesssim \left(\frac{O(\tilde{\delta})}{|v|[\alpha(x,v)]^{\beta-1/2}}+\frac{C_{\tilde{\delta},\xi}}{l\langle v\rangle[\alpha(x,v)]^{\beta-1/2}}\right)$$
$$\times \sup_{0\leq s\leq t}\{e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\}.$$

Now we consider time sections $[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]$ for $j\geq 1$. Assume that

$$\left[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}\right]\subset [t^{\ell_{j+1}-1}, t^{\ell_{j+1}}]\cup\cdots\cup[t^{\ell_j+1}, t^{\ell_j}],$$

and $[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]\cap[t^{\ell_{j+1}-2}, t^{\ell_{j+1}-1}]=\emptyset$ and $[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]\cap[t^{\ell_j}, t^{\ell_j-1}]=\emptyset$.

Note that for all $s\in[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]$

$$e^{-C_\xi j}\alpha(t)\lesssim\alpha(s)\lesssim e^{C_\xi j}\alpha(t),$$

and

$$\ell_{j+1}-\ell_j\lesssim\frac{(j+1)\frac{1}{|v|}-j\frac{1}{|v|}}{\frac{\sqrt{\alpha(t-j\frac{1}{|v|})}}{|v|^2}}\lesssim\frac{|v|}{\sqrt{\alpha(t)}}e^{C_\xi j},$$

and for $\ell\in[\ell_{j+1}-1,\ell_j]$

$$|t^\ell-t^{\ell+1}|\lesssim\frac{\sqrt{\alpha(t-j\frac{1}{|v|})}}{|v|^2}\lesssim\frac{\sqrt{\alpha(t)}}{|v|^2}e^{C_\xi j}.$$

From (1.28), for all $\ell\in[\ell_{j+1}-1,\ell_j]$

$$\int_{t^\ell}^{t^{\ell+1}}\int_{\mathbb{R}^3}e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{Z(s,x,v)}{\left[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)\right]^\beta}\mathrm{d}u\mathrm{d}s$$
$$\lesssim\frac{\tilde{\delta}}{|v|^2\alpha(t-j/|v|)^{\beta-1}}\sup_{[t^\ell,t^{\ell+1}]}\{e^{-l\langle v\rangle(t-s)}Z\}$$
$$+\frac{C_{\tilde{\delta}}}{\alpha(t-j/|v|)^{\beta-1/2}}\int_{t^{\ell+1}}^{t^\ell}e^{-l\langle v\rangle(t-s)}Z,$$

is bounded by

$$\frac{\tilde{\delta}e^{C_\xi j}}{|v|^2\alpha(t)^{\beta-1}}e^{-\frac{l}{2}j}\sup_{[t^\ell,t^{\ell+1}]}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z\}+\frac{e^{C_\xi j}}{\alpha(t)^{\beta-1/2}}e^{-\frac{l}{2}j}\int_{t^{\ell+1}}^{t^\ell}e^{-\frac{l}{2}\langle v\rangle(t-s)}Z$$

$$\lesssim \frac{e^{-Clj}}{|v|^2\alpha(t)^{\beta-1}}\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s)\},$$

where we have used the fact $t-s\ge j\frac{1}{|v|}$ for $s\in[t-(j+1)\frac{1}{|v|},t-j\frac{1}{|v|}]$.

Therefore

$$\begin{aligned}&\int_{t-(j+1)\frac{1}{|v|}}^{t-j\frac{1}{|v|}}\int_{\mathbb{R}^3}e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{Z(s,x,v)}{[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)]^\beta}\mathrm{d}u\mathrm{d}s\\&\quad\lesssim|\ell_{j+1}-\ell_j|\sup_{\ell_{j+1}\le\ell\le\ell_j}\int_{t^\ell}^{t^{\ell+1}}\cdots\\&\quad\lesssim\frac{|v|}{\sqrt{\alpha(t)}}e^{C_\xi j}\times\frac{e^{-lj/4}}{|v|^2\alpha(t)^{\beta-1}}\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\}\\&\quad\lesssim\frac{e^{-lj/4}}{|v|\alpha(t)^{\beta-1/2}}\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\}.\end{aligned}$$

Now we sum up all contributions of $[t-(j+1)\frac{1}{|v|},t-j\frac{1}{|v|}]$ for $j\ge1$:

$$\begin{aligned}\sum_{j=1}^{t|v|}\int_{t-(j+1)/|v|}^{t-j/|v|}&\le\sum_{j=1}^{t|v|}\frac{e^{-lj/8}}{|v|\alpha(t)^{\beta-1/2}}\sup_{0\le s\le t}\left\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\right\}\\&\lesssim\frac{e^{-l/8}}{|v|[\alpha(x,v)]^{\beta-1/2}}\sup_{0\le s\le t}\left\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\right\},\end{aligned}$$

where we used $\sum_{j=1}^{t|v|}e^{-lj/8}=e^{-l/16}\sum_{j=2}^{t|v|}e^{-C'lj}\le e^{-l/16}$.

We conclude (1.29). For the bounce-back case we set $\mathcal{C}=0$ and we have the same conclusion. □

Lemma 11 *Let* $(t,x,v)\in[0,\infty)\times\bar{\Omega}\times\mathbb{R}^3$. *Let* $Z(s,x,v)\ge0$ *be an arbitrary non-negative function defined in the phase space.*

(1) *For* $0<\varepsilon\ll1$, $\frac{1}{2}<\beta<1$ *and* $0<\kappa\le1$, *we have*

$$\int_0^{t_{\mathbf{b}}(x,v)}\int_{\mathbb{R}^3}e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)]^\beta}\frac{|v|}{|u|}\frac{\langle u\rangle^r}{\langle v\rangle^r}\\ \times Z(s,x,v)\mathrm{d}u\mathrm{d}s$$

$$\lesssim_{\theta,r} \frac{O(\varepsilon)}{|v|^2[\alpha(x,v)]^{\beta-1}} \sup_{s\in[0,t_{\mathbf{b}}(x,v)]} \{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\}$$
$$+\frac{C_\varepsilon}{[\alpha(x,v)]^{\beta-1/2}}\int_0^{t_{\mathbf{b}}(x,v)} e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s. \tag{4.11}$$

(2) *Let* $[X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)]$ *be either the specular backward trajectory or the bounce-back backward trajectory in Definition 2. For* $0<\varepsilon\ll 1$, $\frac{1}{2}<\beta<1$ *and* $0<\kappa\leq 1$ *and* $r\in\mathbb{R}$, *there exists* $l\gg_\xi 1$ *and* $C=C_{l,\beta,\xi,r}>0$ *such that*

$$\int_0^t\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{|v|}{|u|}\frac{\langle u\rangle^r}{\langle v\rangle^r}\frac{Z(s,x,v)}{[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)]^\beta}\mathrm{d}u\mathrm{d}s$$
$$\lesssim_{\xi,r} \frac{O_{\beta,\kappa,r}(\varepsilon)}{\langle v\rangle\,[\alpha(x,v)]^{\beta-1/2}}\sup_{0\leq s\leq t}\left\{e^{-C_{l,\beta,\xi,r}\langle v\rangle(t-s)}Z(s,x,v)\right\}. \tag{4.12}$$

Proof of Lemma 11 We prove (4.11). Due to *Step* 2 and *Step* 3 in the proof of (1) of Lemma 1, it suffices to show

$$\int_{\mathbb{R}^3}\frac{|v|e^{-\theta|v-u|^2}\mathrm{d}u}{|v-u|^{2-\kappa}|u|\,[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)]^\beta}$$
$$\lesssim \frac{1}{|v|^{2\beta-1}|\xi(x-(t_{\mathbf{b}}(x,v)-s)v)|^{\beta-\frac{1}{2}}}. \tag{4.13}$$

As *Step* 1 in the proof of (1), for fixed s and $x-(t_{\mathbf{b}}(x,v)-s)v$, we decompose $u=u_{\tau,1}\tau_1+u_{\tau,2}\tau_2+u_n n$ where $\{\tau_1,\tau_2,n\}$ is the orthonormal basis that we chose in the proof of (1).

Now we split as

$$\int_{\mathbb{R}^3}\frac{|v|e^{-\theta|v-u|^2}\mathrm{d}u}{|v-u|^{2-\kappa}|u|[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)]^\beta}$$
$$\lesssim_\xi \int_{\mathbb{R}^2}\int_{\mathbb{R}}\frac{|v|e^{-\theta|v-u|^2}\mathrm{d}u_n\mathrm{d}u_\tau}{|v-u|^{2-\kappa}|u|\{|u_n|^2+|\xi(X_{\mathbf{cl}}(s))\||u|^2\}^\beta} = \int_{|u|\geq\frac{|v|}{5}}+\int_{|u|\leq\frac{|v|}{5}}.$$

For the first term, we have $\frac{|v|}{|u|}\leq 5$ so we reduce it to the previous case (4.1)

$$\int_{|u|\geq\frac{|v|}{5}}\lesssim\int_{\mathbb{R}^3}\frac{e^{-\theta|v-u|^2}\mathrm{d}u_n\mathrm{d}u_\tau}{|v-u|^{2-\kappa}\left\{|u_n|^2+|\xi(X_{\mathbf{cl}}(s))\||u|^2\right\}^\beta},$$

which is bounded by $\frac{1}{|v|^{2\beta-1}|\xi|^{\beta-1/2}}$.

Now we consider the case of $|u| \le \frac{|v|}{5}$. For fixed $0 < \kappa \le 1$

$$\frac{|v|}{|v-u|^{2-\kappa}} \lesssim \frac{|v|}{|v|^{2-\kappa}} \lesssim |v|^{-1+\kappa},$$

and we have, from $|v-u|^2 = \frac{|v-u|^2}{2} + \frac{|v-u|^2}{2} \ge \frac{4^2}{2\cdot 5^2}|v|^2 + \frac{4^2}{2}|u|^2$,

$$e^{-\theta|v-u|^2} \le e^{-C_\theta|v|^2} e^{-C_\theta|u|^2}.$$

We split $\int_{\mathbb{R}^3} \mathrm{d}u = \int_{|u_n| \ge |\xi|^{1/2}|u_\tau|} + \int_{|u_n| \le |\xi|^{1/2}|u_\tau|}$ to have $\left(\text{note}\frac{1}{2} < \beta < 1\right)$

$$\begin{aligned}
\int_{|u_n| \ge |\xi|^{1/2}|u_\tau|} &\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}} \int_{\mathbb{R}^2} \frac{e^{-C_\theta|u_\tau|^2}}{|u_\tau|} \int_{|\xi|^{1/2}|u_\tau|}^{|v|/5} |u_n|^{-2\beta} e^{-C_\theta|u_n|^2} \mathrm{d}|u_n| \mathrm{d}u_\tau \\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}} \int_{|u_\tau| \le \frac{|v|}{5}} \frac{e^{-C_\theta|u_\tau|^2}}{|u_\tau|} \int_{|\xi|^{1/2}|u_\tau|}^{\frac{|v|}{5}} \frac{\mathrm{d}u_n}{|u_n|^{2\beta}} \mathrm{d}u_\tau \\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}} \left\{ |v| + |v|^{-2\beta+1} \int_{|u_\tau| \le \frac{|v|}{5}} \frac{\mathrm{d}u_\tau}{|u_\tau|} \right. \\
&\qquad \left. + \frac{1}{|\xi|^{\beta-\frac{1}{2}}} \int_{|u_\tau| \le \frac{|v|}{5}} |u_\tau|^{-2\beta} \mathrm{d}u_\tau \right\} \\
&\lesssim e^{-C|v|^2}|v|^\kappa \left\{ 1 + \frac{1}{|v|^{2\beta-1}} \left(1 + \frac{1}{|\xi|^{\beta-\frac{1}{2}}}\right) \right\} \\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}}, \\
\int_{|u_n| \le |\xi|^{1/2}|u_\tau|} &\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}} \int_{|u_\tau| \le \frac{|v|}{5}} \frac{e^{-C_\theta|u_\tau|^2}}{|\xi|^\beta|u_\tau|^{2\beta}|u_\tau|} \int_{|u_n| \le |\xi|^{1/2}|u_\tau|} \mathrm{d}u_n \mathrm{d}u_\tau \\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}} \int_{|u_\tau| \le \frac{|v|}{5}} \frac{e^{-C_\theta|u_\tau|^2}}{|\xi|^{\beta-1/2}|u_\tau|^{2\beta}} \mathrm{d}u_\tau \\
&\lesssim \frac{|v|^{-2\beta+\kappa+1}e^{-C|v|^2}}{|\xi|^{\beta-1/2}} \lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}}.
\end{aligned}$$

Therefore, combining the cases of $|u| \le \frac{|v|}{5}$ and $|u| \ge \frac{|v|}{5}$, we conclude (4.13).

The proof of (4.12), (2) of Lemma 11 is a direct consequence of (4.13) and the proof of (1.29). □

5 Diffuse reflection BC

5.1 $W^{1,p}(1 < p < 2)$ Estimate

Consider the iteration (2.16) with (2.18), and the compatibility condition for the initial datum (1.14). Remark that the normalized Maxwellian $\mu(v) = e^{-\frac{|v|^2}{2}}$. From Lemma 7, we have a uniform bound (2.13) for $0 < T \ll 1$. We apply Proposition 1 for $m = 1, 2, \dots$ with

$$\begin{aligned} &\nu = \nu(\sqrt{\mu} f^m) \geq 0, \quad H = \Gamma_{\text{gain}}(f^m, f^m), \\ &g = c_\mu \sqrt{\mu(v)} \int_{n \cdot u > 0} f^m(t, x, u) \sqrt{\mu(u)} \{n(x) \cdot u\} \mathrm{d}u. \end{aligned}$$

For $\partial = [\partial_x, \partial_v]$, ∂f^m satisfies

$$\{\partial_t + v \cdot \nabla_x + \nu(\sqrt{\mu} f^m)\} \partial f^{m+1} = \mathcal{G}^m, \quad \partial f^{m+1}(0, x, v) = \partial f_0(x, v), \tag{5.1}$$

where

$$\begin{aligned} \mathcal{G}^m &= -[\partial v] \cdot \nabla_x f^{m+1} - \partial[\nu(\sqrt{\mu} f^m)] f^{m+1} + \partial[\Gamma_{\text{gain}}(f^m, f^m)], \\ |\mathcal{G}^m| &\lesssim |\nabla_x f^{m+1}| + e^{-\frac{\theta}{2}|v|^2} \| e^{\theta|v|^2} f_0\|_\infty^2 + P(\|e^{\theta|v|^2} f_0\|_\infty) \\ &\quad \times \int_{\mathbb{R}^3} \frac{e^{-C_\theta |v-u|^2}}{|v-u|^{2-\kappa}} |\partial f^m(u)| \mathrm{d}u, \end{aligned} \tag{5.2}$$

where we have used *(iv)* of Lemma 6 and (2.13) of Lemma 7.

We summarize boundary conditions (1.16), (1.17), (1.18) and (1.20) for f^{m+1} in (2.16). Now the boundary conditions for ∂f^{m+1} for $\partial = [\nabla_x, \nabla_v]$ is bounded by

$$\begin{aligned} &|\partial f^{m+1}(t, x, v)| \\ &\lesssim \sqrt{\mu(v)} \left(1 + \frac{\langle v \rangle}{|n(x) \cdot v|}\right) \int_{n(x) \cdot u > 0} |\partial f^m(t, x, u)| \mu^{1/4} \{n(x) \cdot u\} \mathrm{d}u \\ &\quad + \frac{e^{-\frac{\theta}{2}|v|^2}}{|n(x) \cdot v|} P(||e^{\theta|v|^2} f_0||_\infty). \end{aligned} \tag{5.3}$$

Now we are ready to prove Theorem 1:

Proof of Thoerem 1 We claim that for $1 \leq p < 2$, if $0 < T \ll 1$ (therefore, (2.13) and (2.14) hold for $0 < \theta < \frac{1}{4}$ from Lemma 7), and the compatibility condition (1.14), then uniformly-in-m,

$$\sup_{0\le t\le T_*}\|\partial f^m\|_p^p+\int_0^{T_*}|\partial f^m|_{\gamma,p}^p\lesssim_{\Omega,T_*}\|\partial f_0\|_p^p+P(\|e^{\theta|v|^2}f_0\|_\infty),\quad (5.4)$$

for $\partial=[\nabla_x,\nabla_v]$ and some polynomial P.

We remark that the sequence (2.16) is the one used in Lemma 7 and shown to be a Cauchy sequence in L^∞. Therefore the limit function f is a solution of the Boltzmann equation with the diffuse boundary condition. On the other hand, due to the weak lower semi-continuity for L^p in case of $p>1$, once we have (5.4), then we pass a limit $\partial f^m\rightharpoonup\partial f$ weakly in $\sup_{t\in[0,T_*]}\|\cdot\|_p^p$ and $\partial f^m|_\gamma\rightharpoonup\partial f|_\gamma$ in $\int_0^{T_*}|\cdot|_{\gamma,p}^p$ (up to a subsequence) to conclude that ∂f satisfies the same estimate of (5.4). Repeat the same procedure for $[T_*,2T_*],[2T_*,3T_*],\dots$, to conclude Theorem 1.

We prove the claim (5.4) by induction. From Proposition 1, ∂f^1 exists. Because of our choice ∂f^0 the estimate (5.4) is valid for $m=1$. Now assume that ∂f^i exists and (5.4) is valid for all $i=1,2,\dots,m$. Applying Proposition 1 to show that ∂f^{m+1} exists and to get (3.7), we have

$$\begin{aligned}&\sup_{0\le s\le t}\|\partial f^{m+1}(s)\|_p^p+\int_0^t|\partial f^{m+1}|_{\gamma_+,p}^p\\&\lesssim\|\partial f_0\|_p^p+\int_0^t|\partial f^{m+1}|_{\gamma_-,p}^p+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\mathcal{G}^m||\partial f^{m+1}|^{p-1}\\&\lesssim\|\partial f_0\|_p^p+\int_0^t|\partial f^{m+1}|_{\gamma_-,p}^p+P(\|e^{\theta|v|^2}f_0\|_\infty)\\&\quad\times\left\{\int_0^t\|\partial f^{m+1}(s)\|_p^p+\int_0^t\|\partial f^m(s)\|_p^p\right\},\end{aligned}\quad (5.5)$$

where we have used (5.2), Lemma 6 and the Hölder inequality.

Now we consider the boundary contributions. We use (5.3) to obtain

$$\begin{aligned}&\int_0^t\int_{\gamma_-}|\partial f^{m+1}(s)|^p\\&\lesssim_p\sup_{x\in\partial\Omega}\left(\int_{\gamma_-}\sqrt{\mu(v)}^p\left(|n\cdot v|+\frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\right)\mathrm{d}v\right)\\&\quad\times\int_0^t\int_{\partial\Omega}\left[\int_{u\cdot n(x)>0}|\partial f^m(s,x,u)|\mu^{1/4}(u)\{n\cdot u\}\mathrm{d}u\right]^p\mathrm{d}S_x\mathrm{d}s\\&\quad+\sup_{x\in\partial\Omega}\left(\int_{\gamma_-}\langle v\rangle^{-p\beta}|n\cdot v|^{1-p}\mathrm{d}v\right)\times t||e^{\theta|v|^2}f_0||_\infty^p\\&\lesssim_p\int_0^t\int_{\partial\Omega}\left[\int_{u\cdot n(x)>0}|\partial f^m(s,x,u)|\mu^{1/4}(u)\{n\cdot u\}\mathrm{d}u\right]^p\mathrm{d}S_x\mathrm{d}s\\&\quad+tP(||e^{\theta|v|^2}f_0||_\infty).\end{aligned}$$

Now we focus on $\int_0^t \int_{\partial\Omega} \left[\int_{u\cdot n(x)>0} |\partial f^m(s,x,u)| \mu^{1/4}(u)\{n\cdot u\} \mathrm{d}u \right]^p \mathrm{d}S_x \mathrm{d}s$. Recall (1.21). We split the $\{u \in \mathbb{R}^3 : n(x)\cdot u > 0\}$ as

$$
\begin{aligned}
&\int_0^t \int_{\partial\Omega} \left[\int_{n\cdot u>0} |\partial f^m| \mu^{1/4} \{n\cdot u\} \mathrm{d}u \right]^p \\
&\lesssim_p \int_0^t \int_{\partial\Omega} \left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon} \mathrm{d}u \right]^p + \int_0^t \int_{\partial\Omega} \left[\int_{(x,u)\in\gamma_+^\varepsilon} \mathrm{d}u \right]^p. \qquad (5.6)
\end{aligned}
$$

We use Hölder's inequality to bound

$$
\begin{aligned}
\left[\int_{(x,u)\in\gamma_+^\varepsilon} \mathrm{d}u \right]^p \le& \left[\int_{(x,u)\in\gamma_+^\varepsilon} \mu^{\frac{p}{4(p-1)}} \{n\cdot u\} \mathrm{d}u \right]^{p-1} \\
&\times \left[\int_{(x,u)\in\gamma_+^\varepsilon} |\partial f^m(s,x,u)|^p \{n(x)\cdot u\} \mathrm{d}u \right],
\end{aligned}
$$

and hence the second term of (5.6) can be bounded by

$$
\int_0^t \int_{\partial\Omega} \left[\int_{(x,u)\in\gamma_+^\varepsilon} \mathrm{d}u \right]^p \lesssim_p \varepsilon \int_0^t |\partial f^m(s)|_{\gamma_+,p}^p \mathrm{d}s. \qquad (5.7)
$$

For the first term (non-grazing part) of (5.6) we use the Hölder's inequality, Lemma 8, Lemma 5, and Lemma 6 for f^m to estimate

$$
\begin{aligned}
&\int_0^t \int_{\partial\Omega} \left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon} \mathrm{d}u \right]^p \\
&\lesssim_\varepsilon \|\partial f_0\|_p^p + \int_0^t \|\partial f^m(s)\|_p^p \mathrm{d}s + \int_0^t \iint_{\Omega\times\mathbb{R}^3} \left|\mathcal{G}^m\right| |\partial f^m|^{p-1} \\
&\lesssim_\varepsilon \|\partial f_0\|_p^p + P(\|e^{\theta|v|^2} f_0\|_\infty) \\
&\quad \times \sum_{i=m,m-1} \int_0^t \|\partial f^i(s)\|_p^p + tP(\|e^{\theta|v|^2} f_0\|_\infty^p). \qquad (5.8)
\end{aligned}
$$

Putting together estimates (5.5), (5.7), (5.8), and choosing sufficiently small $0<\varepsilon\ll 1, 0<T\ll 1$, we deduce that

$$\begin{aligned}&\sup_{0\le t\le T}\|\partial f^{m+1}(t)\|_p^p+\int_0^T|\partial f^{m+1}|_{\gamma_+,p}^p\\&\lesssim_{T,\Omega}\|\partial f_0\|_p^p+P(\|e^{\theta|v|^2}f_0\|_\infty)\\&\quad+\frac{1}{8}\max_{i=m,m-1}\left\{\sup_{0\le t\le T_*}\|\partial f^i(t)\|_p^p+\int_0^{T_*}|\partial f^i|_{\gamma_+,p}^p\right\}.\end{aligned}$$

To conclude the proof we use the following fact from [3]: Suppose $a_i\ge 0$, $D\ge 0$ and $A_i=\max\{a_i,a_{i-1},\dots,a_{i-(k-1)}\}$ for fixed $k\in\mathbb{N}$.

$$\text{If }\ a_{m+1}\le\frac{1}{8}A_m+D\ \text{ then }\ A_m\ \le\frac{1}{8}A_0+\left(\frac{8}{7}\right)^2D,\ \text{ for }\frac{m}{k}\gg 1.\tag{5.9}$$

Proof of (5.9): In fact, we can iterate for $m,m-1,\dots$ to get

$$\begin{aligned}a_m&\le\frac{1}{8}\max\left\{\frac{1}{8}A_{m-2}+D,\,A_{m-2}\right\}+D\le\frac{1}{8}A_{m-2}+\left(1+\frac{1}{8}\right)D\\&\le\frac{1}{8}\max\left\{\frac{1}{8}A_{m-3}+D,\,A_{m-3}\right\}+\left(1+\frac{1}{8}\right)D\le\frac{1}{8}A_{m-3}+\left(1+\frac{1}{8}+\frac{1}{8^2}\right)D\\&\le\frac{1}{8}A_{m-k}+\frac{8}{7}D.\end{aligned}$$

Similarly $a_{m-i}\le\frac{1}{8}A_{m-k}+\frac{8}{7}D$ for all $i=0,1,\dots,k-1$. Therefore if $1\ll m/k\in\mathbb{N}$,

$$\begin{aligned}A_m&=\max\{a_m,a_{m-1},\dots,a_{m-(k-1)}\}\le\frac{1}{8}A_{m-k}+\frac{8}{7}D\\&\le\frac{1}{8^2}A_{m-2k}+\frac{8}{7}\left(1+\frac{1}{8}\right)D\le\frac{1}{8^3}A_{m-3k}\\&\quad+\frac{8}{7}\left(1+\frac{1}{8}+\frac{1}{8^2}\right)D\\&\le\left(\frac{1}{8}\right)^{[\frac{m}{k}]}A_{m-[\frac{m}{k}]k}+\left(\frac{8}{7}\right)^2D\le\left(\frac{1}{8}\right)^{\frac{m}{k}}A_0+\left(\frac{8}{7}\right)^2D\le\frac{1}{8}A_0+\left(\frac{8}{7}\right)^2D.\end{aligned}$$

This completes the proof of (5.9).

Setting $k=2$ and

$$a_i=\sup_{0\le t\le T_*}\|\partial f^i(t)\|_p^p+\int_0^t|\partial f^i|_{\gamma_+,p}^p,\quad D=C_{T_*,\Omega}\left\{\|\partial f_0\|_p^p+P(\|\langle v\rangle^\beta f_0\|_\infty)\right\},$$

and applying (5.9), we complete the proof of the theorem. □

The following result indicates that Theorem 1 is optimal :

Lemma 12 *Let* $\Omega = B(0;1)$ *with* $B(0;1) = \{x \in \mathbb{R}^3 : |x| < 1\}$. *There exists a smooth initial datum with the support of contained* f_0 *in* $B(0;1) \times B(0;1)$ *so that the solution* f *to*

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f = 0, \quad f|_{t=0} = f_0, \\ &f(t,x,v)|_{\gamma_-} = c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \end{aligned} \tag{5.10}$$

satisfies

$$\int_0^1 \int_{\gamma_-} |\nabla_x f(s,x,v)|^2 \mathrm{d}\gamma \mathrm{d}s \ = \ +\infty,$$

hence the estimate (1.15) *of Theorem 1 fails for* $p = 2$.

Proof We can prove the existence and uniqueness from [9]. We prove this blow-up result by contradiction. Suppose $\int_0^1 \int_{\gamma_-} |\partial f(s,x,v)|^2 \mathrm{d}\gamma \mathrm{d}s < +\infty$. Then

$$\partial_n f(t,x,v) = \frac{1}{n\cdot v}\left\{-\partial_t f - (\tau_1\cdot v)\partial_{\tau_1} f - (\tau_2 \cdot v)\partial_{\tau_2} f\right\}, \quad \text{for } (x,v) \in \gamma_-.$$

We use the boundary condition to define:

$$\begin{aligned} \partial_t f(t,x,v)|_{\gamma_-} &= c_\mu\sqrt{\mu(v)}A(t,x) \equiv c_\mu\sqrt{\mu}\int_{n\cdot u>0} \partial_t f\sqrt{\mu}\{n\cdot u\}\mathrm{d}u, \\ \partial_{\tau_i} f(t,x,v)|_{\gamma_-} &= c_\mu\sqrt{\mu(v)}B_i(t,x) \\ &\equiv c_\mu\sqrt{\mu}\int_{n\cdot u>0}\partial_{\tau_i} f\sqrt{\mu}\{n\cdot u\}\mathrm{d}u \\ &\quad + c_\mu\sqrt{\mu}\int_{n\cdot u>0}\nabla_v f\frac{\partial \mathcal{T}}{\partial \tau_i}\mathcal{T}^{-1}u\sqrt{\mu}\{n\cdot u\}\mathrm{d}u. \end{aligned}$$

We make a change of variables $v_n = v\cdot n(x),\ v_{\tau_1} = v\cdot\tau_1(x),\ v_{\tau_2} = v\cdot\tau_2(x)$ to compute

$$\begin{aligned} &\int_{\partial\Omega}\mathrm{d}S_x\int_0^\infty \mathrm{d}v_n \iint_{\mathbb{R}^2}\mathrm{d}v_{\tau_1}\mathrm{d}v_{\tau_2} \\ &\quad\times\frac{\mu(v)}{v_\perp}\Big\{(A)^2 + (v_{\tau_1})^2(B_1)^2 + (v_{\tau_2})^2(B_2)^2 \end{aligned}$$

$$
\left.+2v_{\tau_1}AB_1 + 2v_{\tau_2}AB_2 + 2v_{\tau_1}v_{\tau_2}B_1B_2\right\}
$$

$$
= \int_0^\infty \mathrm{d}v_n \frac{e^{-\frac{|v_n|^2}{2}}}{v_n} \int_{\partial\Omega} \mathrm{d}S_x \left\{(A)^2 + 2\pi(B_1)^2 + 2\pi(B_2)^2\right\}.
$$

Note that the integration over $\partial\Omega$ is a function of t only (independent of v). Since $\int_0^\infty \frac{\mathrm{d}v_n}{v_n} = \infty$, we conclude that $A = B_1 = B_2 \equiv 0$ for $(t, x) \in [0, \infty) \times \partial\Omega$. In particular from $A(t, x) = 0$ we have for all $t \geq 0$

$$
\begin{aligned}
&\int_{n(x)\cdot u>0} f(t, x, u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u \\
&\quad = \int_{n(x)\cdot u>0} f(0, x, u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.
\end{aligned} \tag{5.11}
$$

We now choose an initial datum that vanishes near $\partial\Omega$:

$$
f_0(x, v) = \phi(|x|)\phi(|v|),
$$

where $\phi \in C^\infty([0, \infty))$ and $\phi \geq 0$ and supp$\phi \subset\subset [0, 1)$ and $\phi \equiv 1$ on $[0, \frac{1}{2}]$. Clearly

$$
c_\mu\sqrt{\mu(v)} \int_{n(x)\cdot u>0} f_0(x, u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = 0.
$$

Hence $f(t, x, v) \geq 0$ since $f_0 \geq 0$, and satisfies the zero inflow boundary condition from (5.11) and the above equality. Moreover following the backward trajectory to the initial plane for $t \in [\frac{1}{8}, \frac{1}{4}]$ and $(x, v) \in \gamma_+$ and $|v - \frac{x}{|x|}| < \frac{1}{64}$, and $|v| \in \left[\frac{1}{8}, \frac{1}{2}\right]$,

$$
f(t, x, v) = f_0(x - tv, v) = 1,
$$

which contradicts to $c_\mu\sqrt{\mu(v)} \int_{n\cdot u>0} f(t, x, u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = 0$ for $(t, x, v) \in [0, \infty) \times \gamma_-$ in (5.11). □

5.2 Weighted $W^{1,p}$ $(2 \leq p < \infty)$ Estimate

We now establish the weighted $W^{1,p}$ estimate for $2 \leq p < \infty$ with the same iteration (2.16). From Lemma 7 for $0 < \theta < \frac{1}{4}$, we have the uniform bounds (2.13) and (2.14). Recall the notation $\partial = [\nabla_x, \nabla_v]$. Then $e^{-\varpi\langle v\rangle t}[\alpha(x, v)]^\beta \partial f^m$ satisfies

$$
\begin{aligned}
&[\partial_t + v\cdot\nabla_x + \nu_{\varpi,\beta} + \nu(\sqrt{\mu} f^m)](e^{-\varpi\langle v\rangle t}\alpha(x,v)^\beta \partial f^{m+1}) \\
&\quad = e^{-\varpi\langle v\rangle t}\alpha(x,v)^\beta \mathcal{G}^m, \\
&\quad \alpha(x,v)^\beta \partial f^{m+1}(0,x,v) = \alpha(x,v)^\beta \partial f_0(x,v).
\end{aligned}
\tag{5.12}
$$

Here $\nu_{\varpi,\beta}$ is defined in (3.21) and $\mathcal{G}^m$ is defined in (5.2). Recall from (5.2)

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\alpha(x,v)^\beta |\mathcal{G}^m| \\
&\lesssim e^{-\varpi\langle v\rangle t}\alpha(x,v)^\beta \Big\{ |\nabla_x f^{m+1}| \\
&\qquad + P(||e^{\theta|v|^2} f_0||_\infty)\Big[e^{-\frac{\theta}{2}|v|^2} + \int_{\mathbb{R}^3} \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} |\partial f^m(u)| \mathrm{d}u\Big]\Big\}.
\end{aligned}
$$

For $(x,v)\in\gamma$, from (5.3), the boundary condition is bounded for $\beta < \frac{p-1}{2p}$ by

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta |\partial f^{m+1}(t,x,v)| \\
&\lesssim e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta \sqrt{\mu(v)}\Big(1 + \frac{\langle v\rangle}{|n(x)\cdot v|}\Big) \\
&\quad \times \int_{n\cdot u>0} |\partial f^m(t,x,u)|\langle u\rangle\sqrt{\mu}\{n\cdot u\}\mathrm{d}u \\
&\quad + \frac{e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta}{|n(x)\cdot v|} e^{-\frac{\theta}{4}|v|^2} P(||e^{\theta|v|^2} f_0||_\infty).
\end{aligned}
\tag{5.13}
$$

The main estimate is the following:

Proof of Theorem 2 Fix $p \geq 2$, $\frac{p-2}{2p} < \beta < \frac{p-1}{2p}$ and $\varpi \gg_\Omega 1$. We claim that there exists $0 < T_* \ll 1$ such that we have the following estimates uniformly-in-m ,

$$
\begin{aligned}
&\sup_{0\leq t\leq T_*} \|e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^m(t)\|_p^p + \int_0^{T_*} |e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m|_{\gamma,p}^p \lesssim_{\Omega,T_*} \\
&P(\|e^{\theta|v|^2} f_0\|_\infty) + \|\alpha^\beta \partial f_0\|_p^p,
\end{aligned}
\tag{5.14}
$$

for $\partial = [\nabla_x, \nabla_v]$ and some polynomial P.

Once we have (5.14) then we pass to the limit, $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f^m \rightharpoonup e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f$ weakly with norms $\sup_{t\in[0,T_*]}\|\cdot\|_p^p$ and $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f^m|_\gamma \rightharpoonup e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f|_\gamma$ in $\int_0^{T_*}|\cdot|_{\gamma,p}^p$ and $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f$ satisfies (5.14). Repeat the same procedure for $[T_*, 2T_*]$, $[2T_*, 3T_*]$, ..., up to the local existence time interval $[0, T]$ in Lemma 7 to conclude Theorem 2.

We prove (5.14) by induction. From Proposition 2, ∂f^1 exists. More precisely we construct $\partial_t f^1, \nabla_x f^1$ first and then $\nabla_v f^1$. Because of our choice of ∂f^0, the estimate (5.14) is valid for $m = 1$. Now assume that ∂f^i exists and (5.14) is valid for all $i = 1, 2, \ldots, m$. Applying the weighted inflow estimate (Proposition 2) we deduce that ∂f^{m+1} exists. From the Green's identity (Lemma 9) we have

$$\begin{aligned}
&\sup_{0\le s\le t} \|e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^{m+1}(s)\|_p^p + \int_0^t |e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^{m+1}|_{\gamma_+,p}^p \\
&\quad + \int_0^t \|\langle v\rangle^{1/p} e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^{m+1}\|_p^p \\
&\lesssim \|\alpha^\beta \partial f_0\|_p + tP(\|e^{\theta|v|^2} f_0\|_\infty) + \int_0^t |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^{m+1}|_{\gamma_-,p}^p \\
&\quad + t\sup_{0\le s\le t} \|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^{m+1}(s)\|_p^p \\
&\quad + \int_0^t \iint_{\Omega\times\mathbb{R}^3} [e^{-\varpi\langle v\rangle s}\alpha^\beta]^p |\mathcal{G}_m||\partial f^{m+1}|^{p-1} \\
&\lesssim \|\alpha^\beta \partial f_0\|_p + tP(\|e^{\theta|v|^2} f_0\|_\infty) + \int_0^t |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^{m+1}|_{\gamma_-,p}^p \\
&\quad + t\sup_{0\le s\le t} \|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^{m+1}(s)\|_p^p \\
&\quad + P(\|e^{\theta|v|^2} f_0\|_\infty)\int_0^t \iint_{\Omega\times\mathbb{R}^3} [e^{-\varpi\langle v\rangle s}\alpha^\beta]^p |\partial f^{m+1}|^{p-1} \\
&\quad \times \int_{\mathbb{R}^3} \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} |\partial f^m(u)|.
\end{aligned} \tag{5.15}$$

Step 1. Estimate for the nonlocal term: The key estimate is the following: For $0 < \beta < \frac{p-1}{2p}$, $0 < \theta < \frac{1}{4}$, and some $C_{\varpi,\beta,p} > 0$,

$$\sup_{x\in\Omega}\int_{\mathbb{R}^3} \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} \frac{[e^{-\frac{\varpi}{\beta}\langle v\rangle s}\alpha(x,v)]^{\frac{\beta p}{p-1}}}{[e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha(x,u)]^{\frac{\beta p}{p-1}}} du \lesssim_{\Omega,\theta} \langle v\rangle^{\frac{\beta p}{p-1}} e^{C_{\varpi,\beta,p}s^2}. \tag{5.16}$$

First we assume $|\xi(x)| < \delta_\Omega$ so that $n(x) := \frac{\nabla\xi(x)}{|\nabla\xi(x)|}$ is well-defined. We decompose $u_n = u\cdot n(x) = u\cdot\frac{\nabla\xi(x)}{|\nabla\xi(x)|}$ and $u_\tau = u - u_n n(x)$. We note $\alpha(x,v) \lesssim |v|$ and $\alpha(x,v) \ge |u\cdot\nabla\xi(x)|$. For $0\le\kappa\le 1$, we have the bound

$$
\begin{aligned}
&|v|^{\frac{\beta p}{p-1}} \int_{\mathbb{R}^3} \frac{1}{|v-u|^{2-\kappa}} e^{-C_\theta |v-u|^2} \frac{e^{-\frac{\varpi p}{p-1}\langle v\rangle t}}{e^{-\frac{\varpi p}{p-1}\langle u\rangle t}} \frac{1}{|u\cdot \nabla\xi(x)|^{\frac{2\beta p}{p-1}}} \mathrm{d}u \\
&\lesssim_\Omega \ |v|^{\frac{\beta p}{p-1}} \int_{\mathbb{R}^3} |v-u|^{-2+\kappa} e^{-\frac{C_\theta|v-u|^2}{2}} e^{\frac{\varpi p}{p-1} t |v-u|} |u_n|^{\frac{-2\beta p}{p-1}} \mathrm{d}u \\
&\lesssim_\Omega \ |v|^{\frac{\beta p}{p-1}} e^{C_{\varpi,\beta,p} t^2} \int_{\mathbb{R}^2} \mathrm{d}u_\tau \int_{\mathbb{R}} \mathrm{d}u_n |v-u|^{-2+\kappa} e^{-\frac{C_\theta |v-u|^2}{4}} |u_n|^{-\frac{2\beta p}{p-1}} \\
&\lesssim_\Omega \ C_\kappa |v|^{\frac{\beta p}{p-1}} e^{C_{\varpi,\beta,p} t^2}.
\end{aligned}
$$

where we have used

$$
e^{\frac{\varpi\beta p}{p-1} t|v-u|} \lesssim e^{C_{\varpi,\beta,p} t^2} \times e^{-\frac{C_\theta |v-u|^2}{4}}, \tag{5.17}
$$

for some $C_{\varpi,\beta,p} > 0$. Furthermore we split the last integration as $\int_{|u_n|/2 \le |v_n - u_n|} + \int_{|u_n|/2 \ge |v_n-u_n|}$. Both terms can be bounded together in this way

$$
C\left[\int \frac{e^{-\frac{C_\theta |u_n|^2}{8}}}{|u_n|^{\frac{\beta p}{p-1}}} \mathrm{d}u_n + \int \frac{e^{-\frac{C_\theta |v_n-u_n|^2}{8}}}{|v_n-u_n|^{\frac{\beta p}{p-1}}} \mathrm{d}u_n\right] \lesssim 1.
$$

If $|\xi(x)| \ge \delta_\Omega$ then

$$
\alpha(x,v) \ge 2|\xi(x)|\{v\cdot \nabla^2 \xi(x)\cdot v\} \gtrsim \delta_\Omega |v|^2 \gtrsim \delta_\Omega |v_3|^2,
$$

where $v = (v_1, v_2, v_3)$ is the standard Euclidian coordinate. We set $v_3 = v_n$ and $v_\tau = (v_1, v_2)$ and follow the exactly the same proof. Therefore we conclude (5.16).

Therefore

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle s} \alpha^\beta \left| \int_{\mathbb{R}^3} \frac{e^{-C_\theta |v-u|^2}}{|v-u|^{2-\kappa}} \partial f^m(u) \mathrm{d}u \right| \\
&\lesssim_\theta \left(\int_{\mathbb{R}^3} \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} \frac{[e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha]^{\beta q}}{[e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha]^{\beta q}} \mathrm{d}u \right)^{\frac{1}{q}} \\
&\quad \times \left(\int_{\mathbb{R}^3} \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} |[e^{-\varpi\langle u\rangle s}\alpha]^\beta \partial f^m(u)|^p \mathrm{d}u \right)^{\frac{1}{p}} \\
&\lesssim_\theta \langle v\rangle^\beta e^{Cs^2} \left(\int_{\mathbb{R}^3} \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} |e^{-\varpi\langle u\rangle s}\alpha^\beta \partial f^m(u)|^p \mathrm{d}u \right)^{\frac{1}{p}},
\end{aligned}
$$

where at the last line we used $\frac{p-2}{2p} < \beta < \frac{p-1}{2p}$ so that $\langle v\rangle^\beta \le \langle v\rangle$.

Finally we use the Hölder's estimate to bound the last term (nonlocal term) of (5.15) by

$$
\begin{aligned}
&Cte^{C_{\varpi,\beta,p}t^2}P(\|e^{\theta|v|^2}f_0\|_\infty)\sup_{0\le s\le t}\iint_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m|^p\\
&\quad+(\delta+\varepsilon)P(\|e^{\theta|v|^2}f_0\|_\infty)\max_{i=m,m+1}\int_0^t\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^i|^p.
\end{aligned}
\tag{5.18}
$$

Step 2. Boundary Estimate: Recall (1.21). We use (5.13) to estimate the contribution of γ_-

$$
\begin{aligned}
&\int_0^t\int_{\gamma_-}|e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta\partial f^{m+1}(s,x,v)|^p\\
&\lesssim_p\int_0^t\int_{\gamma_-}[e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p\sqrt{\mu}^p\left(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\right)^p\\
&\quad\times\left[\int_{n(x)\cdot u>0}|\partial f^m(s,x,u)|\mu^{1/4}\{n\cdot u\}\mathrm{d}u\right]^p\\
&\quad+P(\|e^{\theta|v|^2}f_0\|_\infty)\int_0^t\int_{\gamma_-}\frac{[e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p}{|n(x)\cdot v|^p}e^{-\frac{\theta p}{4}|v|^2}\mathrm{d}\gamma\mathrm{d}s.
\end{aligned}
\tag{5.19}
$$

By $e^{-\varpi\langle v\rangle s}\alpha(x,v)\le e^{-\frac{\varpi\langle v\rangle}{2}s}|\nabla_x\xi(x)\cdot v|^2$ for $x\in\partial\Omega$, the last term is bounded by

$$
\begin{aligned}
&C_\Omega P(\|e^{\theta|v|^2}f_0\|_\infty)\int_0^t\int_{\partial\Omega}\int_{\mathbb{R}^3}|n(x)\cdot v|^{2\beta p-p+1}e^{-\frac{\theta p}{4}|v|^2}\mathrm{d}v\mathrm{d}S_x\mathrm{d}s\\
&\lesssim_{\Omega,p,\zeta}tP(\|e^{\theta|v|^2}f_0\|_\infty),
\end{aligned}
$$

for $\beta>\frac{p-2}{2p}$ so that $2\beta p-p+1>-1$.

For the first term in (5.19) we split as

$$
\left[\int_{n(x)\cdot u>0}\cdots\mathrm{d}u\right]^p\lesssim_p\left[\int_{(x,u)\in\gamma_+^\varepsilon}\cdots\mathrm{d}u\right]^p+\left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon}\cdots\mathrm{d}u\right]^p.
$$

By the Hölder's inequality in u, the γ_+^ε contribution (grazing part) of (5.19) is bounded as

$$
\begin{aligned}
& C_p \int_0^t \int_{\gamma_-} [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p \sqrt{\mu}^p \left(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}} \right) \\
& \quad \times \left| \int_{(x,u)\in\gamma_+^\varepsilon} e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta \partial f^m \{n\cdot u\}^{1/p} \frac{\{n\cdot u\}^{1/q}\mu^{1/4}}{e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta} \mathrm{d}u \right|^p \mathrm{d}v\mathrm{d}S_x\mathrm{d}s \\
& \lesssim_{\Omega,p} \int_0^t \int_{\gamma_-} [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p \left(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}} \right) \sqrt{\mu}^p \\
& \quad \times \left[\int_{(x,u)\in\gamma_+} [e^{-\varpi\langle v\rangle s}\alpha(x,u)^\beta]^p |\partial f^m|^p \{n\cdot u\}\mathrm{d}u \right] \\
& \quad \times \left[\int_{(x,u)\in\gamma_+^\varepsilon} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^{-q} \mu^{q/4}\{n\cdot u\}\mathrm{d}u \right]^{p/q} \mathrm{d}v\mathrm{d}S_x\mathrm{d}s, \\
& \lesssim_{\Omega,p,\varpi,\beta} \varepsilon^a e^{C_{\varpi,\beta,p}t^2} \int_0^t |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m(s)|^p_{\gamma_+,p}\mathrm{d}s,
\end{aligned}
$$

where we used $[e^{-\varpi\langle v\rangle s}\alpha(x,v)] \leq |\nabla\xi(x)\cdot v|^2 \lesssim_\Omega |n(x)\cdot v|^2$ and, for $\beta > \frac{p-2}{2p}$ $(2\beta p - p + 1 > -1)$,

$$
\begin{aligned}
& [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p \left(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}} \right) \sqrt{\mu}^p \\
& \lesssim_\Omega \left(|n(x)\cdot v|^{1+2\beta p} + \langle v\rangle^p |n(x)\cdot v|^{2\beta p - p + 1} \right) \sqrt{\mu(v)}^p \in L^1(\{v\in\mathbb{R}^3\}),
\end{aligned}
$$

and, here, $a > 0$ is determined via , with $\frac{p-1}{p} = \frac{1}{q}$,

$$
\begin{aligned}
& \int_{\gamma_+^\varepsilon} [e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha(x,u)]^{-\frac{2\beta p}{p-1}} \mu^{\frac{p}{4(p-1)}} \{n\cdot u\}\mathrm{d}u \\
& \lesssim_\Omega \int_{\gamma_+^\varepsilon} \left[e^{-\frac{\frac{\varpi}{\beta}\langle u\rangle s}{2}} |u\cdot\nabla\xi(x)| \right]^{-\frac{2\beta p}{p-1}} e^{-\frac{p}{4(p-1)}|u|^2} |n\cdot u|\mathrm{d}u \\
& \lesssim_\Omega \int_{\gamma_+^\varepsilon} |u\cdot n|^{1-\frac{2\beta p}{p-1}} e^{\frac{\varpi p}{(p-1)}\langle u\rangle s} e^{-\frac{p}{4(p-1)}|u|^2}\mathrm{d}u \\
& \lesssim_\Omega e^{C_{\varpi,\beta,p}s^2} \int_{\gamma_+^\varepsilon} |u\cdot n|^{1-\frac{2\beta p}{p-1}} e^{-\frac{p}{8(p-1)}|u|^2}\mathrm{d}u \\
& \lesssim_{\Omega,p} \varepsilon^a e^{C_{\varpi,\beta,p}t^2},
\end{aligned}
$$

for some $a = 2(1 - \beta\frac{p}{p-1}) > 0$ since $1 - \frac{2\beta p}{p-1} > -1$.

On the other hand, for the non-grazing contribution $\gamma_+\backslash\gamma_+^\varepsilon$, we use a similar estimate to get

$$
\begin{aligned}
&\int_0^t \int_{\gamma_-} [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p \sqrt{\mu}^p \left(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\right)^p \\
&\quad\times \left[\int_{\gamma_+\backslash\gamma_+^\varepsilon} |\partial f^m(s,x,u)|\mu(u)^{1/4}\{n(x)\cdot u\}\mathrm{d}u\right]^p \mathrm{d}\gamma\mathrm{d}s \\
&\lesssim_\Omega \int_0^t \int_{\partial\Omega}\int_{\mathbb{R}^3} [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p \left(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\right)\sqrt{\mu}^p \\
&\quad\times\Bigg[\int_{\gamma_+\backslash\gamma_+^\varepsilon} e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta |\partial f^m(s,x,u)|\{n\cdot u\}^{1/p} \\
&\quad\times \frac{\{n\cdot u\}^{1/q}\mu(u)^{1/4}}{[e^{-\varpi\langle u\rangle s}\alpha(x,u)]^\beta}\mathrm{d}u\Bigg]^p \mathrm{d}v\mathrm{d}S_x\mathrm{d}s \\
&\lesssim_\Omega \int_0^t \int_{\gamma_-} [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p \left(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\right)\sqrt{\mu}^p \\
&\quad\times \left[\int_{\gamma_+\backslash\gamma_+^\varepsilon} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^p |\partial f^m|^p\{n\cdot u\}\mathrm{d}u\right] \\
&\quad\times \left[\int_{\gamma_+} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^{-q}\mu^{q/4}\{n\cdot u\}\mathrm{d}u\right]^{p/q} \mathrm{d}v\mathrm{d}S_x\mathrm{d}s \\
&\lesssim_\Omega e^{C_{\varpi,\beta,p}t^2}\int_0^t\int_{\gamma_+\backslash\gamma_+^\varepsilon} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^p |\partial f^m(s)|^p \mathrm{d}\gamma\mathrm{d}s,
\end{aligned}
$$

where we have used

$$
\begin{aligned}
&\int_{\gamma_+} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^{-q}\mu(u)^{q/4}\{n(x)\cdot u\}\mathrm{d}u \\
&= \int_{\gamma_+} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^{-\frac{p}{p-1}}\mu(u)^{\frac{p}{4(p-1)}}\{n\cdot u\}\mathrm{d}u \lesssim_{\Omega,p,\varepsilon} e^{C_{\varpi,\beta,p}t^2}.
\end{aligned}
$$

By Lemma 8, (5.12), and (5.18), the non-grazing part is further bounded by

$$
\begin{aligned}
\int_0^t\int_{\gamma_+\backslash\gamma_+^\varepsilon} &\lesssim_\varepsilon \int_0^t \|\alpha^\beta\partial f_0\|_p^p + \int_0^t \|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m\|_p^p \\
&\quad + \int_0^t \iint_{\Omega\times\mathbb{R}^3} |\mathcal{G}^m| [e^{-\varpi\langle v\rangle s}\alpha^\beta]^p |\partial f^m|^{p-1} \\
&\lesssim \int_0^t \|\alpha^\beta\partial f_0\|_p^p + \int_0^t \|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m\|_p^p
\end{aligned}
$$

$$
\begin{aligned}
&+t \sup_{0\le s\le t} \|e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m(s)\|_p^p + (1+t)P(\|e^{\theta|v|^2} f_0\|_\infty) \\
&+Cte^{C_{\varpi,\beta,p}t^2} P(\|e^{\theta|v|^2} f_0\|_\infty) \sup_{0\le s\le t} \iint_{\Omega\times\mathbb{R}^3} |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m|^p \\
&+(\delta+\varepsilon)P(\|e^{\theta|v|^2} f_0\|_\infty) \\
&\times \max_{i=m,m+1} \int_0^t \iint_{\Omega\times\mathbb{R}^3} \langle v\rangle |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^i|^p. \qquad (5.20)
\end{aligned}
$$

In summary, the boundary contribution of (5.15) is controlled by, for all $0 \le t \le T$,

$$
\begin{aligned}
&\int_0^t |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m(s)|_{\gamma_-,p}^p \mathrm{d}s \\
&\lesssim \int_0^T \|\alpha(v)^\beta \partial f_0\|_p^p + \varepsilon^a \int_0^T |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m|_{\gamma_+,p}^p \\
&\quad +T \max_{i=m-1,m} \sup_{0\le t\le T_*} \|e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^i(t)\|_p^p + P(\|e^{\theta|v|^2} f_0\|_\infty) \\
&\quad +Cte^{C_{\varpi,\beta,p}t^2} P(\|e^{\theta|v|^2} f_0\|_\infty) \sup_{0\le s\le t} \iint_{\Omega\times\mathbb{R}^3} |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m|^p \\
&\quad +(\delta+\varepsilon)P(\|e^{\theta|v|^2} f_0\|_\infty) \max_{i=m,m+1} \int_0^t \iint_{\Omega\times\mathbb{R}^3} \langle v\rangle |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^i|^p. \qquad (5.21)
\end{aligned}
$$

Finally we collect the terms to deduce

$$
\begin{aligned}
&\sup_{0\le t\le T} \|e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^{m+1}(t)\|_p^p + \int_0^T \|\langle v\rangle^{1/p} e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^{m+1}\|_p^p \\
&\quad + \int_0^T |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^{m+1}|_{\gamma_+,p}^p \mathrm{d}s \\
&\le C_{T,\Omega} \left\{ \|\alpha^\beta \partial f_0\|_p^p + P(\|e^{\theta|v|^2} f_0\|_\infty) \right\} \\
&\quad + \left\{ \varepsilon + \delta + Te^{C_{\varpi,\beta,p}(T)^2} \right\} P(\|e^{\theta|v|^2} f_0\|_\infty) \\
&\quad \times \max_{i=m,m-1} \left\{ \sup_{0\le t\le T} \|\alpha^\beta \partial f^i(t)\|_p^p + \int_0^T |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^i|_{\gamma_+,p}^p \right. \\
&\quad \left. + \int_0^T \|\langle v\rangle^{1/p} e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^i\|_p^p \right\}.
\end{aligned}
$$

Recall $C_{\varpi,\beta,p}$ from (5.16). Choose $0 < T \ll 1$, and $0 < \varepsilon \ll 1,\ 0 < \delta \ll 1$ and hence

$$\begin{aligned} &\sup_{0\le t\le T} \|e^{-\varpi\langle v\rangle t}\alpha^{\beta}\partial f^{m+1}(t)\|_p^p + \int_0^T |e^{-\varpi\langle v\rangle t}\alpha^{\beta}\partial f^{m+1}|_{\gamma_+,p}^p \\ &\quad\le C_{T,\Omega}\left\{\|\alpha^{\beta}\partial f_0\|_p^p + P(\|e^{\theta|v|^2}f_0\|_\infty)\right\} \\ &\qquad + \frac{1}{8}\max_{i=m,m-1}\left\{\sup_{0\le t\le T}\|e^{-\varpi\langle v\rangle t}\alpha^{\beta}\partial f^{i}(t)\|_p^p + \int_0^T |e^{-\varpi\langle v\rangle t}\alpha^{\beta}\partial f^{i}|_{\gamma_+,p}^p\right\}. \end{aligned}$$

Set

$$\begin{aligned} a_i &= \sup_{0\le t\le T_*}\|e^{-\varpi\langle v\rangle t}\alpha^{\beta}\partial f^{m+1}(t)\|_p^p + \int_0^T |e^{-\varpi\langle v\rangle t}\alpha^{\beta}\partial f^{m+1}|_{\gamma_+,p}^p, \\ D &= C_{T,\Omega}\left\{\|\alpha^{\beta}\partial f_0\|_p^p + P(\|e^{\theta|v|^2}f_0\|_\infty)\right\}. \end{aligned}$$

Applying (5.9) with $k = 2$, we complete the proof. $\square$

5.3 Weighted $\mathbf{C^1}$ estimate

We start with the same iterative sequences (5.12) with $\beta = \frac{1}{2}$. For $(x, v) \in \gamma$, note that $\sqrt{\alpha(x,v)} = |n(x)\cdot v|$. Recall $\mathcal{G}^m$ in (5.2). We define

$$\mathcal{N}^m(t,x,v) := e^{-\varpi\langle v\rangle t}\sqrt{\alpha(x,v)}\mathcal{G}^m(t,x,v). \tag{5.22}$$

From (5.13) with $\beta = \frac{1}{2}$, we have, for $(x, v) \in \gamma_-$,

$$\begin{aligned} &e^{-\varpi\langle v\rangle t}|\sqrt{\alpha(x,v)}\partial f^{m+1}(t,x,v)| \\ &\quad\lesssim \langle v\rangle c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} e^{-\varpi\langle u\rangle t}\sqrt{\alpha(x,u)}|\partial f^m(t,x,u)|e^{\varpi\langle u\rangle t}\langle u\rangle\sqrt{\mu(u)}\mathrm{d}u \\ &\qquad + e^{-\frac{\theta}{4}|v|^2}P(\|e^{\theta|v|^2}f_0\|_\infty), \end{aligned} \tag{5.23}$$

for $\partial = [\nabla_x, \nabla_v]$.

Let $(x, v) \notin \gamma_0$ and $(t^0, x^0, v^0) = (t, x, v)$. Define the stochastic (diffuse) cycles as $t^1 = t - t_{\mathbf{b}}(x,v)$, $x^1 = x - t_{\mathbf{b}}(x,v)v$, and $v^1 \in \mathbb{R}^3$ with $n(x^1)\cdot v^1 > 0$. For $\ell \ge 1$,

$$t^{\ell+1} = t^{\ell} - t_{\mathbf{b}}(x^{\ell}, v^{\ell}),\quad x^{\ell+1} = x_{\mathbf{b}}(x^{\ell}, v^{\ell}),$$
$$v^{\ell+1} \in \mathbb{R}^3 \quad \text{with } n(x^{\ell+1})\cdot v^{\ell+1} > 0.$$

Lemma 13 *If* $t^1 < 0$ *then*

$$|e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}\partial f^{m+1}(t,x,v)| \lesssim \|\alpha(x,v)^{1/2}\partial f_0\|_\infty + \int_0^t |\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s. \tag{5.24}$$

If $t^1 > 0$ *then*

$$\begin{aligned}
&|e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}\partial f^{m+1}(t,x,v)| \\
&\lesssim \int_{t^1}^t |\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s + e^{-\frac{\theta}{4}|v|^2}P(\|e^{\theta|v|^2}f_0\|_\infty) \\
&\quad + \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\sum_{i=1}^{\ell-1}1_{\{t^{\ell+1}<0<t^\ell\}}|\alpha^{1/2}\partial f^{m+1-i}(0,x^i-t^iv^i,v^i)|\,\mathrm{d}\Sigma_i^{\ell-1} \\
&\quad + \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\sum_{i=1}^{\ell-1}1_{\{t^{i+1}<0<t^i\}}\int_0^{t^i}|\mathcal{N}^{m-i}(s,x^i-(t^i-s)v^i,v^i)|\,\mathrm{d}s\,\mathrm{d}\Sigma_i^{\ell-1} \\
&\quad + \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\sum_{i=1}^{\ell-1}1_{\{t^{i+1}<0\}}\int_{t^{i+1}}^{t^i}|\mathcal{N}^{m-i}(s,x^i-(t^i-s)v^i,v^i)|\,\mathrm{d}s\,\mathrm{d}\Sigma_i^{\ell-1} \\
&\quad + \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\sum_{i=2}^{\ell-1}1_{\{t^{i-1}<0\}}e^{-\frac{\theta}{4}|v^{i-1}|^2}P(\|e^{\theta|v|^2}f_0\|_\infty)\,\mathrm{d}\Sigma_{i-1}^{\ell-1} \\
&\quad + \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\mathbf{1}_{\{t^\ell>0\}}|e^{-\varpi\langle v^{\ell-1}\rangle t^\ell}\alpha(x^\ell,v^{\ell-1})^{1/2}\partial f^{m+1-\ell}(t^\ell,x^\ell,v^{\ell-1})|\,\mathrm{d}\Sigma_{\ell-1}^{\ell-1},
\end{aligned} \tag{5.25}$$

where $\mathcal{V}_j = \{v^j \in \mathbb{R}^3 : n(x^j)\cdot v^j > 0\}$ *and*

$$w(v) = \frac{c_\mu}{\langle v\rangle\sqrt{\mu(v)}},$$

and

$$\begin{aligned}
\mathrm{d}\Sigma_i^{\ell-1} = &\left\{\Pi_{j=i+1}^{\ell-1}\mu(v^j)c_\mu|n(x^j)\cdot v^j|\mathrm{d}v^j\right\}\left\{w(v^i)e^{\varpi\langle v^i\rangle t^i}\langle v^i\rangle^2 c_\mu\mu(v^i)\mathrm{d}v^i\right\} \\
&\left\{\Pi_{j=1}^{i-1}e^{\varpi\langle v^j\rangle t^j}\langle v^j\rangle^2 c_\mu\mu(v^j)\mathrm{d}v^j\right\}.
\end{aligned}$$

Remark that $\mathrm{d}\Sigma_i^{\ell-1}$ is not a probability measure!

Proof For $t^1 < 0$ we use (5.12) with $\beta = 1$ to obtain

$$\begin{aligned}
e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}\partial f^{m+1}(t,x,v) &\lesssim \alpha(x-tv,v)^{1/2}\partial f_0(x-tv,v) \\
&\quad + \int_0^t e^{-\nu_{\varpi,1}(v)(t-s)}\mathcal{N}^m(s,x-(t-s)v,v)\mathrm{d}s.
\end{aligned}$$

Consider the case of $t^1 > 0$. We prove by the induction on ℓ, the number of iterations. First for $\ell = 1$, along the characteristics, for $t^1 > 0$, we have

$$\begin{aligned} &e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t,x,v) \\ &\quad\lesssim e^{-\nu_{\varpi,1}(v)(t-t_1)}e^{-\varpi\langle v\rangle t^1}\alpha^{1/2}\partial f^{m+1}(t^1,x^1,v) \\ &\qquad+\int_{t^1}^t e^{-\nu_{\varpi,1}(v)(t-s)}\mathcal{N}^m(s,x-(t-s)v,v)\mathrm{d}s. \end{aligned}$$

Now we apply (5.23) to the first term above to further estimate

$$\begin{aligned} &e^{-\varpi\langle v\rangle t}\alpha^{1/2}|\partial f^{m+1}(t,x,v)| \\ &\quad\lesssim\ e^{-\nu_{\varpi,1}(v)(t-t^1)}e^{-\frac{\theta}{4}|v|^2}P(\|e^{\theta|v|^2}f_0\|_\infty) \\ &\qquad+\int_{t^1}^t e^{-\nu_{\varpi,1}(v)(t-s)}|\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s \\ &\qquad+e^{-\nu_{\varpi,1}(v)(t-t^1)}\langle v\rangle c_\mu\sqrt{\mu(v)}\int_{\mathcal{V}_1}e^{-\varpi\langle v^1\rangle t^1}\alpha^{1/2}|\partial f^m(t^1,x^1,v^1)| \\ &\qquad\times e^{\varpi\langle v^1\rangle t^1}\langle v^1\rangle\sqrt{\mu(v^1)}\mathrm{d}v^1 \\ &\quad\lesssim\ e^{-\frac{\theta}{4}|v|^2}P(\|e^{\theta|v|^2}f_0\|_\infty)+\int_{t^1}^t|\mathcal{N}^m(s,x-(t-s)v,v)| \\ &\qquad+\frac{c_\mu}{w(v)}\int_{\mathcal{V}_1}e^{-\varpi\langle v^1\rangle t^1}\alpha^{1/2}|\partial f^m(t^1,x^1,v^1)|e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2\mu(v^1)\mathrm{d}v^1. \end{aligned} \tag{5.26}$$

Now we continue to express $\partial f^m(t^1,x^1,v^1)$ via backward trajectory to get

$$\begin{aligned} &e^{-\varpi\langle v^1\rangle t^1}\alpha(x^1,v^1)^{1/2}|\partial f^m(t^1,x^1,v^1)| \\ &\quad\le\ \mathbf{1}_{\{t^2<0<t^1\}}\Big\{\alpha^{1/2}|\partial f^m(0,x^1-t^1v^1,v^1)| \\ &\qquad\qquad+\int_0^{t^1}|\mathcal{N}^{m-1}(s,x^1-(t^1-s)v^1,v^1)|\mathrm{d}s\Big\} \\ &\qquad+\mathbf{1}_{\{t^2>0\}}\Big\{e^{-\varpi\langle v^1\rangle t^2}\alpha^{1/2}|\partial f^m(t^2,x^2,v^1)| \\ &\qquad\qquad+\int_{t^2}^{t^1}|\mathcal{N}^{m-1}(s,x^1-(t^1-s)v^1,v^1)|\mathrm{d}s\Big\}. \end{aligned}$$

Therefore we conclude from (5.26) that

$$\begin{aligned} &e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}|\partial f^{m+1}(t,x,v)| \\ &\lesssim\int_{t^1}^t|\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s+e^{-\frac{\theta}{4}|v|^2}P(||e^{\theta|v|^2}f_0||_\infty) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t^2<0<t^1\}}\alpha(x^1-t^1v^1,v^1)^{1/2}|\partial f_0(x^1-t^1v^1,v^1)\\
&\qquad\qquad\qquad\qquad \times|e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2c_\mu\mu(v^1)\mathrm{d}v^1\\
&+\frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t^2<0<t^1\}}\int_0^{t^1}|\mathcal{N}^{m-1}(s,x^1-(t^1-s)v^1,v^1)|\mathrm{d}s\\
&\qquad\qquad\qquad\qquad \times e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2c_\mu\mu(v^1)\mathrm{d}v^1\\
&+\frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t^2>0\}}\int_{t^2}^{t^1}|\mathcal{N}^{m-1}(s,x^1-(t^1-s)v^1,v^1)|\mathrm{d}s\\
&\qquad\qquad\qquad\qquad \times e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2c_\mu\mu(v^1)\mathrm{d}v^1\\
&+\frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t^2>0\}}e^{-\varpi\langle v^1\rangle t^2}\alpha(x^2,v^1)^{1/2}|\partial f^m(t^2,x^2,v^1)|\\
&\qquad\qquad\qquad\qquad \times e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2c_\mu\mu(v^1)\mathrm{d}v^1,
\end{aligned}
$$

and it equals (5.25) for $\ell=2$.

Assume (5.25) is valid for $\ell\in\mathbb{N}$. We use (5.23) and express the last term of (5.25) as

$$
\begin{aligned}
&\mathbf{1}_{\{t^\ell>0\}}e^{-\varpi\langle v^{\ell-1}\rangle t^\ell}\alpha(x^\ell,v^{\ell-1})|\partial f^{m+1-k}(t^\ell,x^\ell,v^{\ell-1})|\\
&\quad\lesssim\langle v^{\ell-1}\rangle c_\mu\sqrt{\mu(v^{\ell-1})}\int_{\mathcal{V}_\ell}\mathbf{1}_{\{t^\ell>0\}}e^{-\varpi\langle v^\ell\rangle t^\ell}\alpha^{1/2}|\partial f^{m+1-(k+1)}(t^\ell,x^\ell,v^\ell)|\\
&\qquad\times e^{\varpi\langle v^\ell\rangle t^\ell}\langle v^\ell\rangle\sqrt{\mu(v^\ell)}\mathrm{d}v^\ell+e^{-\frac{\theta}{4}|v_{\ell-1}|^2}P(\|e^{\theta|v|^2}f_0\|_\infty).
\end{aligned}\tag{5.27}
$$

Then we decompose $\mathbf{1}_{\{t^\ell>0\}}=\mathbf{1}_{\{t^{\ell+1}<0<t^\ell\}}+\mathbf{1}_{\{t^{\ell+1}>0\}}$, where the first part hits the initial plane as

$$
\begin{aligned}
&\mathbf{1}_{\{t^{\ell+1}<0<t^\ell\}}e^{-\varpi\langle v^\ell\rangle t^\ell}\alpha^{1/2}|\partial f^{m+1-(\ell+1)}(t^\ell,x^\ell,v^\ell)|\\
&\quad\lesssim\alpha^{1/2}|\partial f_0(x^\ell-t^\ell v^\ell,v^\ell)|\\
&\qquad+\int_0^{t^\ell}|\mathcal{N}^{m+1-(\ell+2)}(s,x^\ell-(t^\ell-s)v^\ell,v^\ell)|\mathrm{d}s,
\end{aligned}\tag{5.28}
$$

and the second part hits the boundary as

$$
\begin{aligned}
&\mathbf{1}_{\{t^{\ell+1}>0\}}e^{-\varpi\langle v\rangle t}\alpha^{1/2}|\partial f^{m+1-(\ell+1)}(t^\ell,x^\ell,v^\ell)|\\
&\quad\lesssim e^{-\varpi\langle v^\ell\rangle t^{\ell+1}}\alpha^{1/2}|\partial f^{m+1-(\ell+1)}(t^{\ell+1},x^{\ell+1},v^\ell)|\\
&\qquad+\int_{t^{\ell+1}}^{t^\ell}|\mathcal{N}^{m+1-(\ell+2)}(s,x^\ell-(t^\ell-s)v^\ell,v^\ell)|\mathrm{d}s.
\end{aligned}\tag{5.29}
$$

To summarize, from (5.27) upon integrating over $\prod_{j=1}^{\ell-1}\mathcal{V}_j$, we obtain a bound for the last term of (5.25) as

$$\begin{aligned}
&\frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\mathbf{1}_{\{t^\ell>0\}}|e^{-\varpi\langle v^{\ell-1}\rangle t^\ell}\alpha^{1/2}\partial f^{m+1-\ell}(t^\ell,x^\ell,v^{\ell-1})|\mathrm{d}\Sigma_{\ell-1}^{\ell-1}\\
&\lesssim P(\|e^{\theta|v|^2}f_0\|_\infty)\frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\mathbf{1}_{\{t^\ell>0\}}e^{-\frac{\theta}{4}|v^{\ell-1}|^2}\mathrm{d}\Sigma_{\ell-1}^{\ell-1}\\
&\quad+\frac{1}{w(v)}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\mathbf{1}_{\{t^\ell>0\}}e^{-\varpi\langle v^\ell\rangle t^\ell}\sqrt{\alpha}|\partial f^{m+1-(\ell+1)}(t^\ell,x^\ell,v^\ell)|\mathrm{d}\Sigma_\ell^\ell,
\end{aligned}$$

where by (5.28) and (5.29), the last term is bounded by

$$\begin{aligned}
&\frac{1}{w(v)}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\langle v^{\ell-1}\rangle c_\mu\sqrt{\mu(v^{\ell-1})}\sqrt{\mu(v^\ell)}\langle v^\ell\rangle e^{\varpi\langle v^\ell\rangle t^\ell}\mathrm{d}v^\ell\\
&\times\prod_{j=1}^{\ell-2}\left\{e^{\varpi\langle v^j\rangle t^j}\langle v^j\rangle^2 c_\mu\mu(v^j)\mathrm{d}v^j\right\}\\
&\times\left\{w(v^{\ell-1})e^{\varpi\langle v^{\ell-1}\rangle t^{\ell-1}}\langle v^{\ell-1}\rangle^2\mu(v^{\ell-1})\mathrm{d}v^{\ell-1}\right\}\\
&\times\Bigg\{\mathbf{1}_{\{t^{\ell+1}<0<t^\ell\}}\Big[\alpha^{1/2}|\partial f(0,x^\ell-t^\ell v^\ell,v^\ell)|\\
&\qquad+\int_0^{t^\ell}|\mathcal{N}^{m-\ell-2}(s,x^\ell-(t^\ell-s)v^\ell,v^\ell)|\mathrm{d}s\Big]\\
&\qquad+\mathbf{1}_{\{t^{\ell+1}>0\}}\Big[e^{-\varpi\langle v^\ell\rangle t^{\ell+1}}\alpha^{1/2}|\partial f^{m-\ell-1}(t^{\ell+1},x^{\ell+1},v^\ell)|\\
&\qquad+\int_{t^{\ell+1}}^{t^\ell}|\mathcal{N}^{m-\ell-2}(s,x^\ell-(t^\ell-s)v^\ell,v^\ell)|\mathrm{d}s\Big]\Bigg\}.
\end{aligned}$$

Now we use (5.26) to conclude Lemma 13. □

Lemma 14 *There exists $\ell_0(\varepsilon)>0$ such that for $\ell\geq\ell_0$ and for all $(t,x,v)\in[0,1]\times\bar{\Omega}\times\mathbb{R}^3$, we have*

$$\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\mathbf{1}_{\{t^\ell(t,x,v,v^1,\dots,v^{\ell-1})>0\}}\mathrm{d}\Sigma_{\ell-1}^{\ell-1}\lesssim_\Omega\left(\frac{1}{2}\right)^{-\ell/5}.$$

Proof The similar result with different weight is proven in Lemma 23 of [9]. We note that, for some fixed constant $C_0>0$,

$$
\begin{aligned}
d\Sigma_{\ell-1}^{\ell-1} &\le w(v^{\ell-1})e^{\varpi\langle v^{\ell-1}\rangle t^{\ell-1}}\langle v^{\ell-1}\rangle^2 c_\mu \mu(v^{\ell-1})\\
&\quad\times \Pi_{j=1}^{\ell-2} e^{\varpi\langle v^j\rangle t^j}\langle v^j\rangle^2 c_\mu\mu(v^j)\mathrm{d}v^1\ldots\mathrm{d}v^{\ell-1}\\
&\le \Pi_{j=1}^{\ell-1}\{C'e^{C't^2}\mu(v^j)^{\frac14}\}\,\mathrm{d}v^1\ldots\mathrm{d}v^{\ell-1}\le\{C_0\}^\ell\Pi_{j=1}^{\ell-1}\mu(v^j)^{\frac14}\mathrm{d}v^j.
\end{aligned}
$$

Choose a sufficiently small $\delta=\delta(C_0)>0$. Define

$$
\mathcal{V}_j^\delta\equiv\{v^j\in\mathcal{V}_j : v^j\cdot n(x^j)\ge\delta,\ |v^j|\le\delta^{-1}\},
$$

where we have $\int_{\mathcal{V}_j\backslash\mathcal{V}_j^\delta} C_0\mu(v^j)^{\frac14}\lesssim\delta$ for some $C_0>0$.

On the other hand if $v^j\in\mathcal{V}_j^\delta$ then by Lemma 6 of [9], $(t^j-t^{j+1})\ge\delta^3/C_\Omega$. Therefore if $t^\ell\ge 0$ then there can be at most $\left\{\left[\frac{C_\Omega}{\delta^3}\right]+1\right\}$ numbers of $v^m\in\mathcal{V}_m^\delta$ for $1\le m\le\ell-1$. Equivalently there are at least $\ell-2-\left[\frac{C_\Omega}{\delta^3}\right]$ numbers of $v^{m_i}\in\mathcal{V}_{m_i}\backslash\mathcal{V}_{m_i}^\delta$. Hence from $\{C_0\}^{\ell-1}=\{C_0\}^m\times\{C_0\}^{\ell-1-m}$, we have

$$
\begin{aligned}
&\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j} 1_{\{t^\ell(t,x,v,v^1,\ldots,v^{\ell-1})>0\}}d\Sigma_{\ell-1}^{\ell-1}\\
&\le\sum_{m=1}^{\left[\frac{C_\Omega}{\delta^3}\right]+1}\int_{\left\{\begin{array}{c}\text{there are exactly } m \text{ of } v_{m_i}\in\mathcal{V}_{m_i}^\delta\\ \text{and }\ \ell-1-m\ of\ v_{m_i}\in\mathcal{V}_{m_i}\backslash\mathcal{V}_{m_i}^\delta\end{array}\right\}}\prod_{j=1}^{\ell-1}C_0\mu(v^j)^{1/4}\mathrm{d}v^j\\
&\le\sum_{m=1}^{\left[\frac{C_\Omega}{\delta^3}\right]+1}\binom{\ell-1}{m}\left\{\int_{\mathcal{V}}C_0\mu(v)^{1/4}\mathrm{d}v\right\}^m\left\{\int_{\mathcal{V}\backslash\mathcal{V}^\delta}C_0\mu(v)^{1/4}\mathrm{d}v\right\}^{\ell-1-m}\\
&\le\left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right)\{\ell-1\}^{\left[\frac{C_\Omega}{\delta^3}\right]+1}\{\delta\}^{\ell-2-\left[\frac{C_\Omega}{\delta^3}\right]}\left\{\int_{\mathcal{V}}C_0\mu(v)^{1/4}dv\right\}^{\left[\frac{C_\Omega}{\delta^3}\right]+1}\\
&\lesssim\frac{\ell}{N}\{Ck\}^{\frac{\ell}{N}}\left(\frac{\ell}{N}\right)^{-\frac{N\ell}{10}}\\
&\le\{CN\}^{\frac{\ell}{N}}\left(\frac{\ell}{N}\right)^{\frac{\ell}{N}}\left(\frac{\ell}{N}\right)^{-\frac{\ell}{N}\frac{N^2}{20}}\le\left(\frac{\ell}{N}\right)^{\frac{\ell}{N}\left(-\frac{N^2}{20}+3\right)}\\
&\le\left(\frac{1}{\ell/N}\right)^{\frac{-\frac{N^2}{20}+3}{N}\ell}\le\left(\frac12\right)^{-\ell},
\end{aligned}
$$

where we have chosen $\ell=N\times\left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right)$ and $N=\left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right)\gg C>1$. □

Now we are ready to prove the weighted C^1 part of the main theorem:

Proof of weighted C^1 part in Theorem 2 First we show the $W^{1,\infty}$ estimate. Recall that we use the same sequences (5.12) with $\beta = \frac{1}{2}$ used for the weighted $W^{1,p}$ estimate ($2 \leq p < \infty$). We estimate along the stochastic cycles with (5.24) and (5.25). For $t^1 < 0$, the backward trajectory first hits $t = 0$. From Lemmas 13 and 6 for (5.22), we deduce, for $\partial = [\nabla_x, \nabla_v]$,

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|\mathbf{1}_{\{t_1<0\}} e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t)\|_\infty \\
&\lesssim \|\alpha^{1/2}\partial f_0\|_\infty + P(\|e^{\theta|v|^2} f_0\|_\infty) + T \sup_{0\leq t\leq T} \|e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t)\|_\infty \\
&\quad + \underbrace{\int_{t^1}^t \int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle(t-s)} \frac{e^{-C_\theta|v-u|^2}}{|v-u|^{2-\kappa}} \frac{\alpha(x-(t-s)v, v)^{1/2}}{\alpha(x-(t-s)v, u)^{1/2}} \mathrm{d}u\mathrm{d}s} \\
&\quad \times P(\|e^{\theta|v|^2} f_0\|_\infty) \max_m \sup_{0\leq t\leq T_*} \|e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t)\|_\infty,
\end{aligned}$$

where we have used an elementary fact (6.75) with bounded time. Note that, for any $\beta > \frac{1}{2}$,

$$\frac{1}{\alpha(x-(t-s)v, u)^{1/2}} \lesssim \frac{1}{\alpha(x-(t-s)v, u)^{\beta}} + 1 \tag{5.30}$$

We apply (1.28) to bound the underbrace term as, for $1 \geq \beta > \frac{1}{2}$,

$$\begin{aligned}
&\left\{ \mathbf{1}_{|v|\lesssim 1} \frac{\alpha(x,v)^{\frac{1}{2}+\frac{3}{4}-\frac{\beta}{2}} t_Z^{\frac{3}{2}-\beta}}{|v|^{2\beta-1}} + \mathbf{1}_{|v|\gtrsim 1} \frac{\varepsilon^{\frac{3}{2}-\beta}\alpha(x,v)^{\frac{1}{2}}}{|v|^2\alpha(x,v)^{\beta-1}} \right\} + \frac{\alpha(x,v)^{\frac{1}{2}}}{\varpi\langle v\rangle\varepsilon^2\alpha(x,v)^{\beta-\frac{1}{2}}} \\
&\lesssim t_Z^{\frac{3}{2}-\beta} + \varepsilon^{\frac{3}{2}-\beta} + \frac{1}{\varepsilon^2\varpi},
\end{aligned} \tag{5.31}$$

where we used $\alpha(x,v) \lesssim |v|^2$ and t_Z is define in (1.28).

If $t^1(t,x,v) \geq 0$, the backward trajectory first hits the boundary, then from (5.25) we have the following line-by-line estimate

$$\begin{aligned}
&|\mathbf{1}_{\{t_1>0\}} e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t,x,v)| \\
&\lesssim \underbrace{\int_{t^1}^t \int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle(t-s)} \frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2\kappa}} \frac{\alpha(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))^{\frac{1}{2}}}{\alpha(X_{\mathbf{cl}}(s), u)^{\frac{1}{2}}} \mathrm{d}u\mathrm{d}s} \\
&\quad \times \|e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^m(s)\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&+P(\|e^{\theta|v|^2}f_0\|_\infty)+\ell(Ce^{Ct^2})^\ell\max_{1\le i\le \ell-1}\|\alpha^{\frac12}\partial f_0^{m+1-i}\|_\infty\\
&+\ell(Ce^{Ct^2})^\ell\langle v\rangle\sqrt{\mu(v)}\\
&\underbrace{\times\max_i\int_0^{t^i}\int_{\mathbb{R}^3}e^{-\varpi\langle v^i\rangle(t-s)}\frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2\kappa}}\frac{\alpha(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))^{\frac12}}{\alpha(X_{\mathbf{cl}}(s),u)^{\frac12}}\mathrm{d}u\mathrm{d}s}\\
&\times\max_{1\le i\le\ell-1}\sup_{0\le s\le t}\|e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^{m+1-i}(s)\|_\infty\\
&+\ell(Ce^{Ct^2})^\ell P(\|e^{\theta|v|^2}f_0\|_\infty)+\left(\frac12\right)^{-\frac{\ell}{5}}\sup_{0\le s\le t}\|e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^{m+1-\ell}(s)\|_\infty,
\end{aligned}
$$

where we have used (5.12), Lemma 14, and Lemma 6 for (5.22) and (6.75). For the underbraced terms we apply (5.30) and (5.31). Therefore

$$
\begin{aligned}
&|\mathbf{1}_{\{t_1>0\}}e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t,x,v)|\\
&\quad\lesssim C_\ell C^{C\ell t^2}\left\{t^{\frac32-\beta}+\varepsilon^{\frac32-\beta}+\frac{1}{\varepsilon^2\varpi}\right\}\times\max_{0\le i\le m}\sup_{0\le s\le t}\|e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^i(s)\|_\infty\\
&\quad+C_\ell C^{C\ell t^2}\max_{0\le i\le m}\|\alpha^{1/2}\partial f_0^i\|_\infty+\left(\frac12\right)^{-\frac{\ell}{5}}\max_{0\le i\le m}\sup_{0\le s\le t}\|e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^i(s)\|_\infty.
\end{aligned}
$$

We choose a large ℓ then small t then small ε and then finally large ϖ to conclude

$$
\begin{aligned}
\sup_{0\le t\le T_*}\|e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t)\|_\infty\le\frac18\max_{m-\ell\le i\le m}\sup_{0\le t\le T_*}\|e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^i(t)\|_\infty\\
+\|\alpha^{1/2}\partial f_0\|_\infty+P(\|e^{\theta|v|^2}\partial f_0\|_\infty).
\end{aligned}
$$

Set $D=\|\alpha^{1/2}\partial f_0\|_\infty+P(\|e^{\theta|v|^2}\partial f_0\|_\infty)$,

$$
a_i=\sup_{0\le t\le T_*}\|e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^i(t)\|_\infty,\quad A_i=\max\{a_i,a_{i-1},\dots,a_{i-(\ell-1)}\},
$$

then we have $a_{m+1}\le\frac18 A_m+D$. Use (5.9) to conclude

$$
\sup_{0\le t\le T_*}\|e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f(t)\|_\infty\lesssim\|\alpha^{1/2}\partial f_0\|_\infty+P(\|e^{\theta|v|^2}f_0\|_\infty).
$$

The existence and uniqueness and the estimate in Theorem 2 are valid for a small time $T_*>0$. We follow the same procedure for $t\in[T_*,2T_*]$ to conclude

$$
\sup_{T_*\le t\le 2T_*}\|e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f(t)\|_\infty\lesssim_{\Omega,T_*}\|e^{-\varpi\langle v\rangle T_*}\partial f(T_*)\|_\infty+P(\|e^{\theta|v|^2}f_0\|_\infty).
$$

Then we conclude the weighted $W^{1,\infty}$ part of Theorem 2 following the same procedure for $[T_*, 2T_*]$, $[2T_*, 3T_*]$, ...

Now we consider the continuity of $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f$. Remark that for each step $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^m$ satisfies the condition of Proposition 2. Therefore we conclude from (1.26), (3.4) and [9] that $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^m \in C^0\big([0,T_*]\times(\bar{\Omega}\times\mathbb{R}^3)\setminus\gamma_0\big)$. Now we follow $W^{1,\infty}$ estimate part for $e^{-\varpi\langle v\rangle t}\alpha^{1/2}[\partial f^{m+1}-\partial f^m]$ to show that $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^m$ is a Cauchy sequence in L^∞. Then $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^m \rightarrow e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f$ strongly in L^∞ so that $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f \in C^0\big([0,T_*]\times(\bar{\Omega}\times\mathbb{R}^3)\setminus\gamma_0\big)$. $\square$

6 Specular reflection BC

We denote the standard spherical coordinate $\mathbf{x}_\parallel = \mathbf{x}_\parallel(\omega) = (\mathbf{x}_{\parallel,1}, \mathbf{x}_{\parallel,2})$ for $\omega \in \mathbb{S}^2$

$$\omega = (\cos\mathbf{x}_{\parallel,1}(\omega)\sin\mathbf{x}_{\parallel,2}(\omega), \sin\mathbf{x}_{\parallel,1}(\omega)\sin\mathbf{x}_{\parallel,2}(\omega), \cos\mathbf{x}_{\parallel,2}(\omega)),$$

where $\mathbf{x}_{\parallel,1}(\omega)\in[0,2\pi)$ is the azimuth and $\mathbf{x}_{\parallel,2}(\omega)\in[0,\pi)$ is the inclination.

We define an orthonormal basis of $\mathbb{R}^3$, $\{\hat{r}(\omega), \hat{\phi}(\omega), \hat{\theta}(\omega)\}$, with $\hat{r}(\omega) := \omega$ and

$$\begin{aligned}
\hat{\phi}(\omega) &:= (\cos\mathbf{x}_{\parallel,1}(\omega)\cos\mathbf{x}_{\parallel,2}(\omega), \sin\mathbf{x}_{\parallel,1}(\omega)\cos\mathbf{x}_{\parallel,2}(\omega), -\sin\mathbf{x}_{\parallel,2}(\omega)),\\
\hat{\theta}(\omega) &:= (-\sin\mathbf{x}_{\parallel,1}(\omega), \cos\mathbf{x}_{\parallel,1}(\omega), 0).
\end{aligned}$$

Moreover, $\hat{r}\times\hat{\phi}=\hat{\theta},\ \hat{\phi}\times\hat{\theta}=\hat{r},\ \hat{\theta}\times\hat{r}=\hat{\phi}$, and

$$\partial_{\mathbf{x}_{\parallel,1}}\hat{r} = \sin\mathbf{x}_{\parallel,2}\,\hat{\theta}, \quad \partial_{\mathbf{x}_{\parallel,2}}\hat{r} = \hat{\phi}, \tag{6.1}$$

where $\partial_{\mathbf{x}_{\parallel,1}}\hat{r}$ does not vanish (non-degenerate) away from $\mathbf{x}_{\parallel,2}=0$ or π.

Without loss of generality we assume $\mathbf{0}=(0,0,0)\in\Omega$. For

$$\mathbf{p} = (z,w)\in\partial\Omega\times\mathbb{S}^2 \ \text{ with } \ n(z)\cdot w = 0,$$

we define the north pole $\mathcal{N}_{\mathbf{p}}\in\partial\Omega$ and the south pole $\mathcal{S}_{\mathbf{p}}\in\partial\Omega$ as

$$\mathcal{N}_{\mathbf{p}} := |\mathcal{N}_{\mathbf{p}}|(n(z)\times w)\in\partial\Omega, \quad \mathcal{S}_{\mathbf{p}} := -|\mathcal{S}_{\mathbf{p}}|(n(z)\times w)\in\partial\Omega,$$

where $\partial_{\mathbf{x}_{\parallel,1}}\hat{r}$ is degenerate. We define the straight-line $\mathcal{L}_{\mathbf{p}}$ passing both poles

$$\mathcal{L}_{\mathbf{p}} := \{\tau\mathcal{N}_{\mathbf{p}} + (1-\tau)\mathcal{S}_{\mathbf{p}} : \tau\in\mathbb{R}\}.$$

Lemma 15 *Assume* Ω *is convex* (1.13). *Fix* $\mathbf{p}=(z,w)\in\partial\Omega\times\mathbb{S}^2$ *with* $n(z)\cdot w=0$.

(i) *There exists a smooth map (spherical-type coordinate)*

$$\begin{aligned} \eta_{\mathbf{p}} &: [0, 2\pi) \times (0, \pi) \to \partial\Omega \backslash \{\mathcal{N}_{\mathbf{p}}, \mathcal{S}_{\mathbf{p}}\}, \\ \mathbf{x}_{\parallel_{\mathbf{p}}} &:= (\mathbf{x}_{\parallel_{\mathbf{p}},1}, \mathbf{x}_{\parallel_{\mathbf{p}},2}) \mapsto \eta_{\mathbf{p}}(\mathbf{x}_{\parallel_{\mathbf{p}}}), \end{aligned} \tag{6.2}$$

which is one-to-one and onto. Here on $[0, 2\pi) \times (0, \pi)$ *we have* $\partial_i \eta_{\mathbf{p}} := \frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\parallel_{\mathbf{p}},i}} \neq 0$ *and*

$$\frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\parallel_{\mathbf{p}},1}}(\mathbf{x}_{\parallel_{\mathbf{p}}}) \times \frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\parallel_{\mathbf{p}},2}}(\mathbf{x}_{\parallel_{\mathbf{p}}}) \neq 0. \tag{6.3}$$

We define

$$\mathbf{n}_{\mathbf{p}} := n \circ \eta_{\mathbf{p}} : [0, 2\pi) \times (0, \pi) \to \mathbb{S}^2.$$

(ii) *We define the* $\mathbf{p}-$*spherical coordinate in the tubular neighborhood of the boundary:*
For $\delta > 0,\ \delta_1 > 0,\ C > 0,$ *we have a smooth one-to-one and onto map*

$$\begin{aligned} \Phi_{\mathbf{p}} : [0, C\delta) \times [0, 2\pi) \times (\delta_1, \pi - \delta_1) \times \mathbb{R} \times \mathbb{R}^2 &\to \{x \in \bar{\Omega} : |\xi(x)| < \delta\} \backslash B_{C\delta_1}(\mathcal{L}_{\mathbf{p}}) \times \mathbb{R}^3, \\ (\mathbf{x}_{\perp_{\mathbf{p}}}, \mathbf{x}_{\parallel_{\mathbf{p}},1}, \mathbf{x}_{\parallel_{\mathbf{p}},2}, \mathbf{v}_{\perp_{\mathbf{p}}}, \mathbf{v}_{\parallel_{\mathbf{p}},1}, \mathbf{v}_{\parallel_{\mathbf{p}},2}) &\mapsto \Phi_{\mathbf{p}}(\mathbf{x}_{\perp_{\mathbf{p}}}, \mathbf{x}_{\parallel_{\mathbf{p}},1}, \mathbf{x}_{\parallel_{\mathbf{p}},2}, \mathbf{v}_{\perp_{\mathbf{p}}}, \mathbf{v}_{\parallel_{\mathbf{p}},1}, \mathbf{v}_{\parallel_{\mathbf{p}},2}), \end{aligned}$$

where $B_{C\delta_1}(\mathcal{L}_{\mathbf{p}}) := \{x \in \mathbb{R}^3 : |x - y| < C\delta_1$ *for some* $y \in \mathcal{L}_{\mathbf{p}}\}$.
Explicitly,

$$\begin{aligned} &\Phi_{\mathbf{p}}(\mathbf{x}_{\perp_{\mathbf{p}}}, \mathbf{x}_{\parallel_{\mathbf{p}}}, \mathbf{v}_{\perp_{\mathbf{p}}}, \mathbf{v}_{\parallel_{\mathbf{p}}}) \\ &:= \begin{bmatrix} \mathbf{x}_{\perp_{\mathbf{p}}}[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\parallel_{\mathbf{p}}})] + \eta_{\mathbf{p}}(\mathbf{x}_{\parallel_{\mathbf{p}}}) \\ \mathbf{v}_{\perp_{\mathbf{p}}}[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\parallel_{\mathbf{p}}})] + \mathbf{v}_{\parallel_{\mathbf{p}}} \cdot \nabla \eta_{\mathbf{p}}(\mathbf{x}_{\parallel_{\mathbf{p}}}) + \mathbf{x}_{\perp_{\mathbf{p}}} \mathbf{v}_{\parallel_{\mathbf{p}}} \cdot \nabla[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\parallel_{\mathbf{p}}})] \end{bmatrix}, \end{aligned} \tag{6.4}$$

where $\nabla \eta_{\mathbf{p}} = \big(\partial_1 \eta_{\mathbf{p}}, \partial_2 \eta_{\mathbf{p}}\big) = \left(\frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\parallel_{\mathbf{p}},1}}, \frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\parallel_{\mathbf{p}},2}}\right)$ *and* $\nabla \mathbf{n}_{\mathbf{p}} = (\partial_1 \mathbf{n}_{\mathbf{p}}, \partial_2 \mathbf{n}_{\mathbf{p}}) = \left(\frac{\partial \mathbf{n}_{\mathbf{p}}}{\partial \mathbf{x}_{\parallel_{\mathbf{p}},1}}, \frac{\partial \mathbf{n}_{\mathbf{p}}}{\partial \mathbf{x}_{\parallel_{\mathbf{p}},2}}\right)$.
The Jacobian matrix is

$$\begin{aligned} &\frac{\partial \Phi(\mathbf{x}_{\perp}, \mathbf{x}_{\parallel}, \mathbf{v}_{\perp}, \mathbf{v}_{\parallel})}{\partial(\mathbf{x}_{\perp}, \mathbf{x}_{\parallel}, \mathbf{v}_{\perp}, \mathbf{v}_{\parallel})} \\ &= \left[\begin{array}{ccc|ccc} -n(\mathbf{x}_{\parallel}) & \begin{matrix} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_{\parallel}) \\ -\mathbf{x}_{\perp} \frac{\partial n}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_{\parallel}) \end{matrix} & \begin{matrix} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_{\parallel}) \\ -\mathbf{x}_{\perp} \frac{\partial n}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_{\parallel}) \end{matrix} & & \mathbf{0}_{3,3} & \\ \hline -\mathbf{v}_{\parallel} \cdot \nabla_{\mathbf{x}_{\parallel}} \mathbf{n}(\mathbf{x}_{\parallel}) & \begin{matrix} -\mathbf{v}_{\perp} \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_{\parallel}) \\ +\mathbf{v}_{\parallel} \cdot \nabla_{\mathbf{x}_{\parallel}} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_{\parallel}) \\ -\mathbf{x}_{\perp} \mathbf{v}_{\parallel} \cdot \nabla_{\mathbf{x}_{\parallel}} \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_{\parallel}) \end{matrix} & \begin{matrix} -\mathbf{v}_{\perp} \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_{\parallel}) \\ +\mathbf{v}_{\parallel} \cdot \nabla_{\mathbf{x}_{\parallel}} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_{\parallel}) \\ -\mathbf{x}_{\perp} \mathbf{v}_{\parallel} \cdot \nabla_{\mathbf{x}_{\parallel}} \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_{\parallel}) \end{matrix} & -n(\mathbf{x}_{\parallel}) & \begin{matrix} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_{\parallel}) \\ -\mathbf{x}_{\perp} \frac{\partial n}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_{\parallel}) \end{matrix} & \begin{matrix} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_{\parallel}) \\ -\mathbf{x}_{\perp} \frac{\partial n}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_{\parallel}) \end{matrix} \end{array}\right]. \end{aligned} \tag{6.5}$$

We fix an inverse map

$$\begin{aligned}\Phi_{\mathbf{p}}^{-1}:\{x\in\bar{\Omega}:|\xi(x)|<\delta\}\backslash B_{C\delta'}(\mathcal{L}_{\mathbf{p}})\times\mathbb{R}^3\to\ &[0,C\delta)\\ &\times[0,2\pi)\times(\delta_1,\pi-\delta_1)\times\mathbb{R}\times\mathbb{R}^2.\end{aligned}$$

In general this choice is not unique but once we fix the range as above then an inverse map is uniquely determined.
We denote, for $(x,v)\in\{x\in\bar{\Omega}:|\xi(x)|<\delta\}\backslash B_{C\delta'}(\mathcal{L}_{\mathbf{p}})\times\mathbb{R}^3$

$$(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}},1},\mathbf{v}_{\|_{\mathbf{p}},2})=\Phi_{\mathbf{p}}^{-1}(x,v).$$

(iii) *Let* $\mathbf{q}=(y,u)\in\partial\Omega\times\mathbb{S}^2$ *with* $n(y)\cdot u=0$ *and* $|\mathbf{p}-\mathbf{q}|\ll 1$ *and*

$$\Phi_{\mathbf{p}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}})=(x,v)=\Phi_{\mathbf{q}}(\mathbf{x}_{\perp_{\mathbf{q}}},\mathbf{x}_{\|_{\mathbf{q}}},\mathbf{v}_{\perp_{\mathbf{q}}},\mathbf{v}_{\|_{\mathbf{q}}}).$$

Then

$$\begin{aligned}\frac{\partial(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}})}{\partial(\mathbf{x}_{\perp_{\mathbf{q}}},\mathbf{x}_{\|_{\mathbf{q}}},\mathbf{v}_{\perp_{\mathbf{q}}},\mathbf{v}_{\|_{\mathbf{q}}})}&=\nabla\Phi_{\mathbf{q}}^{-1}\nabla\Phi_{\mathbf{p}}\\ &=\mathbf{Id}_{6,6}+O_{\xi}(|\mathbf{p}-\mathbf{q}|)\left[\begin{array}{ccc|ccc}0&0&0&&&\\0&1&1&&\mathbf{0}_{3,3}&\\0&1&1&&&\\\hline 0&0&0&0&0&0\\0&|v|&|v|&0&1&1\\0&|v|&|v|&0&1&1\end{array}\right].\end{aligned}\tag{6.6}$$

We remark that the purpose of change of chart (6.4) is designed such that the flow $\frac{dx}{dt}=v$ is preserved to $\frac{dX}{dt}=V$.

Proof of (i) in Lemma 15 We define the orthonormal matrix for $\mathbf{p}=(z,w)\in\partial\Omega\times\mathbb{S}^2$ which maps $\{e_1,e_3,e_3\}\mapsto\{n(z),w,n(z)\times w\}$:

$$\mathcal{O}_{\mathbf{p}}=\begin{bmatrix}n(z)&w&n(z)\times w\end{bmatrix}_{3\times3}^{T}.$$

For $x\in\partial\Omega$ with $x\neq\mathcal{N}_{\mathbf{p}}$ and $x\neq\mathcal{S}_{\mathbf{p}}$ we define

$$(\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2})\in[0,2\pi)\times(0,\pi),\quad\text{such that }\ \hat{r}(\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2})=\mathcal{O}_{\mathbf{p}}\frac{x}{|x|}.$$

Now we define $R_{\mathbf{p}}:[0,2\pi)\times[0,\pi)\to(0,\infty)$ such that

$$\xi(R_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2})\mathcal{O}_{\mathbf{p}}^{-1}\hat{r}(\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2}))=0.\tag{6.7}$$

We also define $\eta_{\mathbf{p}} : [0, 2\pi) \times [0, \pi) \to \partial\Omega$ such that

$$\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}) = R_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})\mathcal{O}_{\mathbf{p}}^{-1}\hat{r}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}).$$

Directly, from (6.1) and (6.7), with fixed $\mathbf{p} = (z, w)$,

$$\begin{aligned}
\frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},1}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}) &= \frac{-\sin(\mathbf{x}_{\|\mathbf{p},2})R_{\mathbf{p}}\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\boldsymbol{\theta}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})}{\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{r}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})} \\
&= \frac{-\sin(\mathbf{x}_{\|\mathbf{p},2})[R_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})]^2\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\boldsymbol{\theta}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})}{\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})}, \\
\frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},2}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}) &= \frac{-R_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\phi}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})}{\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{r}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})} \\
&= \frac{-[R_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})]^2\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\phi}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})}{\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2})}.
\end{aligned}$$

Here $\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}))\cdot\eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}) \neq 0$ due to the convexity.

And by (6.1)

$$\begin{aligned}
\frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},1}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}) &= \frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},1}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{r} + \sin(\mathbf{x}_{\|\mathbf{p},2})R_{\mathbf{p}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{\boldsymbol{\theta}}, \\
\frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},2}}(\mathbf{x}_{\|\mathbf{p},1}, \mathbf{x}_{\|\mathbf{p},2}) &= \frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},2}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{r} + R_{\mathbf{p}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{\boldsymbol{\phi}}.
\end{aligned}$$

Directly we check a non-degenerate condition (6.3)

$$\begin{aligned}
&\frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},1}}(\mathbf{x}_{\|\mathbf{p}}) \times \frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},2}}(\mathbf{x}_{\|\mathbf{p}}) \\
&= R_{\mathbf{p}}\frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},1}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{\boldsymbol{\theta}} + \sin(\mathbf{x}_{\|\mathbf{p},2})R_p\frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|\mathbf{p},2}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{\boldsymbol{\phi}} - \sin(\mathbf{x}_{\|\mathbf{p},2})R_{\mathbf{p}}^2\mathcal{O}_{\mathbf{p}}^{-1}\hat{r} \neq 0.
\end{aligned}$$

Proof of (ii) of Lemma 15. We fix $\mathbf{p} = (z, w)$ and drop $\mathbf{p}-$index (for the chart) in this step. Define

$$\Phi_1 : [0,\infty)\times[0,2\pi)\times(0,\pi) \to \bar{\Omega}\backslash\mathcal{L}_{\mathbf{p}}, \quad \Phi_1(\mathbf{x}_\perp, \mathbf{x}_\|) = \eta(\mathbf{x}_\|) + \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\|)]. \tag{6.8}$$

Note that this mapping is surjective: For any $x \in \bar{\Omega}\backslash\mathcal{L}_{\mathbf{p}}$, there exists (could be several) $y_0 \in \partial\Omega$ satisfying $|x - y_0| = \min_{y\in\partial\Omega}|x - y|$ ($\partial\Omega$ is compact). Now we choose $\tilde{\mathbf{p}} = (\tilde{z}, \tilde{w}) \in \partial\Omega\times\mathbb{S}^2$ such that $y_0 \notin \mathcal{L}_{\tilde{\mathbf{p}}}$. Note that this spherical-type coordinate $\eta_{\tilde{\mathbf{p}}} : (\mathbf{x}_{\|_{\tilde{\mathbf{p}}},1}, \mathbf{x}_{\|_{\tilde{\mathbf{p}}},2}) \mapsto \eta_{\tilde{\mathbf{p}}}(\mathbf{x}_{\|_{\tilde{\mathbf{p}}}}) \in \partial\Omega\backslash\{\mathcal{N}_{\tilde{\mathbf{p}}}, \mathcal{S}_{\tilde{\mathbf{p}}}\}$. Denote $\eta_{\tilde{\mathbf{p}}}(\mathbf{x}^*_{\|_{\tilde{\mathbf{p}}}}) = y_0$. Since y_0 is a minimizer of $|x-y|$ we have $(x - y_0)\cdot$

$\frac{\partial \eta_{\tilde{\mathbf{p}}}}{\partial \mathbf{x}_{\|,i}}(\mathbf{x}^*_{\|_{\tilde{\mathbf{p}}}}) = 0$ for $i = 1, 2$. Since $\nabla \eta_{\tilde{\mathbf{p}}}(\mathbf{x}_{\|_{\tilde{\mathbf{p}}}}) \neq 0$ from (6.3) and $\xi(\eta(\mathbf{x}_{\|})) = 0$, we have $0 \equiv \nabla \xi(\eta(\mathbf{x}^*_{\|_{\tilde{\mathbf{p}}}})) \cdot \frac{\partial \eta_{\tilde{\mathbf{p}}}}{\partial \mathbf{x}_{\|,i}}(\mathbf{x}^*_{\|_{\tilde{\mathbf{p}}}})$. Due to (6.3), we conclude $\frac{x - \eta_{\tilde{\mathbf{p}}}(\mathbf{x}^*_{\|_{\tilde{\mathbf{p}}}})}{|x - \eta_{\tilde{\mathbf{p}}}(\mathbf{x}^*_{\|_{\tilde{\mathbf{p}}}})|} = [-\mathbf{n}(y_0)]$. This implies $\eta_{\tilde{\mathbf{p}}}(\mathbf{x}^*_{\|_{\tilde{\mathbf{p}}}}) \notin \mathcal{L}_{\mathbf{p}}$ and hence we can choose $\tilde{\mathbf{p}} = \mathbf{p}$. Finally we write

$$x = \eta_{\mathbf{p}}(\mathbf{x}^*_{\|_{\mathbf{p}}}) + (x - \eta(\mathbf{x}^*_{\|_{\mathbf{p}}})) = \eta_{\mathbf{p}}(\mathbf{x}^*_{\|_{\mathbf{p}}}) + |x - \eta_{\mathbf{p}}(\mathbf{x}^*_{\|_{\mathbf{p}}})|[-\mathbf{n}(\mathbf{x}^*_{\|_{\mathbf{p}}})].$$

Since η and ξ (therefore n and $\mathbf{n}$) are smooth, the Φ_1 is smooth. Directly we compute the Jacobian matrix

$$\frac{\partial \Phi_1(\mathbf{x}_\perp, \mathbf{x}_\|)}{\partial(\mathbf{x}_\perp, \mathbf{x}_\|)} = \begin{bmatrix} -\mathbf{n}(\mathbf{x}_\|) & \begin{matrix} \frac{\partial \eta}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|) \\ +\mathbf{x}_\perp[-\frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|)] \end{matrix} & \begin{matrix} \frac{\partial \eta}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|) \\ +\mathbf{x}_\perp[-\frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|)] \end{matrix} \end{bmatrix}_{3\times 3}, \tag{6.9}$$

where $\left[-\mathbf{n}(\mathbf{x}_\|)\right]$, $\begin{bmatrix} \frac{\partial \eta}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|) \\ +\mathbf{x}_\perp[-\frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|)] \end{bmatrix}$, $\begin{bmatrix} \frac{\partial \eta}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|) \\ +\mathbf{x}_\perp[-\frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|)] \end{bmatrix}$ are column vectors in $\mathbb{R}^3$.

By the basic linear algebra, the Jacobian (a determinant of the Jacobian matrix) equals

$$-\mathbf{n} \cdot \left(\frac{\partial \eta}{\partial \mathbf{x}_{\|,1}} \times \frac{\partial \eta}{\partial \mathbf{x}_{\|,2}} \right) + \mathbf{x}_\perp \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,1}} \times \frac{\partial \eta}{\partial \mathbf{x}_{\|,2}} \right) - \mathbf{x}_\perp \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,2}} \times \frac{\partial \eta}{\partial \mathbf{x}_{\|,1}} \right)$$
$$-|\mathbf{x}_\perp|^2 \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,1}} \times \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,2}} \right).$$

We use the facts $\nabla \eta(\mathbf{x}_\|) \neq 0$ and $\xi(\eta(\mathbf{x}_\|)) = 0$ and

$$0 \equiv \nabla \xi(\eta(\mathbf{x}_\|)) \cdot \frac{\partial \eta}{\partial \mathbf{x}_{\|,i}}(\mathbf{x}_\|) = |\nabla \xi(\eta(\mathbf{x}_\|))| \left(\mathbf{n}(\mathbf{x}_\|) \cdot \frac{\partial \eta}{\partial \mathbf{x}_{\|,i}}(\mathbf{x}_\|) \right).$$

Therefore

$$-\mathbf{n}(\mathbf{x}_\|) \cdot \left(\frac{\partial \eta}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|) \times \frac{\partial \eta}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|) \right) \neq 0, \quad \text{for all } \mathbf{x}_\| \in [0, 2\pi) \times (0, \pi).$$

We conclude that there exists a small $\delta > 0$ such that if $|\mathbf{x}_\perp| \leq \delta$ and $\mathbf{x}_\| \in [0, 2\pi) \times (0, \pi)$ then

$$\det \left(\frac{\partial \Phi_1(\mathbf{x}_\perp, \mathbf{x}_\|)}{\partial(\mathbf{x}_\perp, \mathbf{x}_\|)} \right) = -\mathbf{n}(\mathbf{x}_\|) \cdot \left(\frac{\partial \eta}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|) \times \frac{\partial \eta}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|) \right) + O_\xi(|\mathbf{x}_\perp|) \neq 0.$$

We use the inverse function theorem and we choose an inverse map

$$\Phi_1^{-1} : \bar{\Omega}\backslash\mathcal{L}_{\mathbf{p}} \to [0,\delta)\times[0,2\pi)\times(0,\pi).$$

If $x \in \Phi_1([0,\delta)\times[0,2\pi)\times(0,\pi))$ then

$$\Phi_1^{-1}(x) := (\mathbf{x}_\perp, \mathbf{x}_\parallel) \quad \text{and} \quad x = \eta(\mathbf{x}_\parallel) + \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)].$$

Since Φ_1 is surjective onto $\bar{\Omega}\backslash\mathcal{L}_{\mathbf{p}}$, for $x \in \bar{\Omega}\backslash\mathcal{L}_{\mathbf{p}}$ and $\mathbf{x}_\perp \geq 0$,

$$\begin{aligned}
\xi(x) &= \xi(\eta(\mathbf{x}_\parallel) + \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)]) \\
&= \xi(\eta(\mathbf{x}_\parallel)) + \int_0^{\mathbf{x}_\perp} \frac{d}{ds}\xi(\eta(\mathbf{x}_\parallel) + s[-\mathbf{n}(\mathbf{x}_\parallel)])\mathrm{d}s \\
&= \int_0^{\mathbf{x}_\perp} [-\mathbf{n}(\mathbf{x}_\parallel)]\cdot\nabla\xi(\eta(\mathbf{x}_\parallel) + s[-\mathbf{n}(\mathbf{x}_\parallel)])\mathrm{d}s \\
&= \int_0^{\mathbf{x}_\perp} \Big\{[-\mathbf{n}(\mathbf{x}_\parallel)]\cdot\nabla\xi(\eta(\mathbf{x}_\parallel)) + \int_0^s \mathbf{n}(\mathbf{x}_\parallel)\cdot\nabla^2\xi(\eta(\mathbf{x}_\parallel) \\
&\quad +\tau[-\mathbf{n}(\mathbf{x}_\parallel)])\cdot\mathbf{n}(\mathbf{x}_\parallel)\mathrm{d}\tau\Big\}\,\mathrm{d}s,
\end{aligned}$$

Then by the convexity of ξ in (1.13), we have the following equivalent relation:

Recall (6.4) and assume $\Phi_p(x_{\perp p}, x_{\parallel p}, v_{\perp p}, v_{\parallel p}) = (x,v)$. Assume $\mathrm{dist}(x,\partial\Omega) < \delta \ll 1$, then from the argument in the proof of (ii) of Lemma 15, we know that $(x_{\perp_\mathbf{p}}, x_{\parallel_\mathbf{p}})$ is uniquely determined. Now we focus on $v_{\perp_\mathbf{p}}, v_{\parallel_\mathbf{p}}$ in (6.4). Since both $\partial_{x_\parallel}\eta$ and $\partial_{x_\parallel}n$ are perpendicular to n, it follows that $v_{\perp p} = -(n(x_\parallel))\cdot v$ is uniquely defined. Since $\partial_{x_1}\eta$ and $\partial_{x_2}\eta$ are linearly independent for $\mathrm{dist}(x,\partial\Omega) < \delta \ll 1$, $v_{\parallel,i}$ are uniquely determined as well.

Proof of (iii) of Lemma 15. Let $\mathbf{q} = (y,u) \in \partial\Omega\times\mathbb{S}^2$ with $n(y)\cdot u = 0$ and $|\mathbf{p}-\mathbf{q}| \ll 1$. First we claim

$$\begin{aligned}
\mathbf{x}_{\perp_\mathbf{p}} &= \mathbf{x}_{\perp_\mathbf{q}}, \\
\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) &= \eta_\mathbf{q}(\mathbf{x}_{\parallel_\mathbf{q}}), \\
\mathbf{v}_{\perp_\mathbf{p}} &= \mathbf{v}_{\perp_\mathbf{q}}, \\
\mathbf{v}_{\parallel_\mathbf{p}}\cdot\nabla\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) - \mathbf{x}_{\perp_\mathbf{p}}\mathbf{v}_{\parallel_\mathbf{p}}\cdot\nabla\mathbf{n}_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) &= \mathbf{v}_{\parallel_\mathbf{q}}\cdot\nabla\eta_\mathbf{q}(\mathbf{x}_{\parallel_\mathbf{q}}) - \mathbf{x}_{\perp_\mathbf{q}}\mathbf{v}_{\parallel_\mathbf{q}}\cdot\nabla\mathbf{n}_\mathbf{q}(\mathbf{x}_{\parallel_\mathbf{q}}).
\end{aligned} \tag{6.10}$$

Once we show the first two equalities then the third and fourth equalities are clearly valid because $\mathbf{n}_\mathbf{p} \perp \mathbf{v}_\parallel\cdot\nabla_{\mathbf{x}_\parallel}\eta_\mathbf{p}$ and $\mathbf{n}_\mathbf{p} \perp \mathbf{v}_\parallel\cdot\nabla_{\mathbf{x}_\parallel}\mathbf{n}_\mathbf{p}$ for all $\mathbf{v}_\parallel \in \mathbb{R}^2$. (Since $\xi(\eta_\mathbf{p}) = 0$ we have $\mathbf{v}_{\parallel,i}\partial_{\mathbf{x}_{\parallel,i}}[\xi(\eta_\mathbf{p}(\mathbf{x}_{\parallel,1},\mathbf{x}_{\parallel,2}))] = \mathbf{v}_{\parallel,i}\partial_{\mathbf{x}_{\parallel,i}}\eta_\mathbf{p}\cdot\nabla_{\mathbf{x}_\parallel}\xi = 0$, and since $\mathbf{n}_\mathbf{p}\cdot\mathbf{n}_\mathbf{p} = 1$ we have $\mathbf{n}_\mathbf{p}\cdot[\mathbf{v}_\parallel\cdot\nabla_{\mathbf{x}_\parallel}\mathbf{n}_\mathbf{p}] = 0$).

Now we prove the first two equalities of (6.10) and it suffices to prove $\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) = \eta_{\mathbf{q}}(\mathbf{x}_{\|_{\mathbf{q}}})$. And it suffices to show that for $x \in \bar{\Omega}$ with $|\xi(x)| \ll 1$ there exists a unique $x^* \in \partial\Omega \cap B(x, \delta)$ for some $0 < \delta \ll 1$ such that

$$|x - x^*|^2 = \min_{y \in \partial\Omega, |y-x| \ll 1} |x - y|^2. \tag{6.11}$$

By the definition of (6.8) the uniqueness of such x^* in (6.11) implies $\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) = x^* = \eta_{\mathbf{q}}(\mathbf{x}_{\|_{\mathbf{q}}})$.

The existence of such $x^* \in \partial\Omega$ is clear from the compactness of $\partial\Omega$. Without loss of generality (up to rotation) we may assume $\partial_{x_3}\xi(y) \neq 0$ for $|y - x^*| \ll 1$ and $\partial_{x_1}\xi(x^*) = 0 = \partial_{x_2}\xi(x^*)$. Then we can find the graph $a : (x_1, x_2) \mapsto \mathbb{R}$ but $\xi(x_1, x_2, a(x_1, x_2)) = 0$ when $x^* = (x_1^*, x_2^*, a(x_1^*, x_2^*)) \in \partial\Omega$. By the implicit function theorem,

$$\partial_{x_1} a = -\partial_{x_1}\xi / \partial_{x_3}\xi, \quad \partial_{x_2} a = -\partial_{x_2}\xi / \partial_{x_3}\xi,$$

and $\partial_{x_1} a(x_1^*, x_2^*) = 0 = \partial_{x_2} a(x_1^*, x_2^*)$.

Clearly $x^* = (x_1^*, x_2^*, a(x_1^*, x_2^*))$ satisfies $\big|(x_1, x_2, x_3) - (x_1^*, x_2^*, a(x_1^*, x_2^*))\big| \ll 1$ and

$$\begin{aligned} &\frac{\partial}{\partial x_i^*} \big|(x_1, x_2, x_3) - (x_1^*, x_2^*, a(x_1^*, x_2^*))\big|^2 \\ &\quad = -\left\{ (x_i - x_i^*) + (x_3 - a(x_1^*, x_2^*)) \frac{\partial a}{\partial x_i^*}(x_1^*, x_2^*) \right\} = 0, \quad \text{for} \quad i = 1, 2, \end{aligned}$$

if and only if (6.11) holds. We take x_i^*−derivative to get

$$\begin{aligned} &1 + \left(\frac{\partial a}{\partial x_i^*}(x_1^*, x_2^*) \right)^2 - (x_3 - a(x_1^*, x_2^*)) \frac{\partial^2 a}{\partial x_i^* \partial x_i^*}(x_1^*, x_2^*) \\ &\quad = 1 - (x_3 - a(x_1^*, x_2^*)) \frac{\partial^2 a}{\partial x_i^* \partial x_i^*}(x_1^*, x_2^*) \neq 0, \end{aligned}$$

for $|x_3 - a(x_1^*, x_2^*)| \ll_\xi 1$. Using the inverse function theorem we have a uniquely determined $x^* : \{y \in \bar{\Omega} : |y - x| \ll 1\} \to \partial\Omega \cap B(x, \delta)$. This proves our claim (uniqueness of x^* in (6.11)) and therefore (6.10).

From the second equality of (6.10) and (6.2)

$$\hat{r}(\mathbf{x}_{\|_{\mathbf{q}},1}, \mathbf{x}_{\|_{\mathbf{q}},2}) = \mathcal{O}_{\mathbf{q}} \mathcal{O}_{\mathbf{p}}^{-1} \hat{r}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}).$$

Therefore for $i = 1, 2$,

$$\sum_{j=1,2} \frac{\partial \hat{r}}{\partial \mathbf{x}_{\|\mathbf{q}},j}(\mathbf{x}_{\|\mathbf{q}}) \frac{\partial \mathbf{x}_{\|\mathbf{q}},j}{\partial \mathbf{x}_{\|\mathbf{p}},i} = \mathcal{O}_{\mathbf{q}} \mathcal{O}_{\mathbf{p}}^{-1} \frac{\partial \hat{r}}{\partial \mathbf{x}_{\|\mathbf{p}},i}(\mathbf{x}_{\|\mathbf{p}}),$$

and from (6.1)

$$\begin{aligned} &\left[-\sin(\mathbf{x}_{\|\mathbf{q}},2)\hat{r}(\mathbf{x}_{\|\mathbf{q}}) \middle| \sin(\mathbf{x}_{\|\mathbf{q}},2)\hat{\boldsymbol{\theta}}(\mathbf{x}_{\|\mathbf{q}}) \middle| \hat{\boldsymbol{\phi}}(\mathbf{x}_{\|\mathbf{q}}) \right] \left[\begin{array}{c|c} 0 & \mathbf{0}_{1,2} \\ \hline \mathbf{0}_{2,1} & \frac{\partial \mathbf{x}_{\|\mathbf{q}}}{\partial \mathbf{x}_{\|\mathbf{p}}} \end{array} \right]_{3\times 3} \\ &= \mathcal{O}_{\mathbf{q}} \mathcal{O}_{\mathbf{p}}^{-1} \left[\mathbf{0}_{3,1} \middle| \sin(\mathbf{x}_{\|\mathbf{p}},2)\hat{\boldsymbol{\theta}}(\mathbf{x}_{\|\mathbf{p}}) \middle| \hat{\boldsymbol{\phi}}(\mathbf{x}_{\|\mathbf{p}}) \right], \end{aligned}$$

where we used $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = -\hat{r}$.

For $\mathbf{x}_{\|\mathbf{p}},2, \mathbf{x}_{\|\mathbf{q}},2 \notin \{0, \pi\}$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial \mathbf{x}_{\|\mathbf{q}},1}{\partial \mathbf{x}_{\|\mathbf{p}},1} & \frac{\partial \mathbf{x}_{\|\mathbf{q}},1}{\partial \mathbf{x}_{\|\mathbf{p}},2} \\ 0 & \frac{\partial \mathbf{x}_{\|\mathbf{q}},2}{\partial \mathbf{x}_{\|\mathbf{p}},1} & \frac{\partial \mathbf{x}_{\|\mathbf{q}},2}{\partial \mathbf{x}_{\|\mathbf{p}},2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sin(\mathbf{x}_{\|\mathbf{q}},2)} \hat{r}(\mathbf{x}_{\|\mathbf{q}})^T \\ \frac{1}{\sin(\mathbf{x}_{\|\mathbf{q}},2)} \hat{\boldsymbol{\theta}}(\mathbf{x}_{\|\mathbf{q}})^T \\ \hat{\boldsymbol{\phi}}(\mathbf{x}_{\|\mathbf{q}})^T \end{bmatrix} \mathcal{O}_{\mathbf{q}} \mathcal{O}_{\mathbf{p}}^{-1} \left[\mathbf{0}_{3,1} \middle| \sin(\mathbf{x}_{\|\mathbf{p}},2)\hat{\boldsymbol{\theta}}(\mathbf{x}_{\|\mathbf{p}}) \middle| \hat{\boldsymbol{\phi}}(\mathbf{x}_{\|\mathbf{p}}) \right].$$

Here $\mathcal{O}_{\mathbf{q}} = \mathcal{O}_{\mathbf{p}} + O_{\xi}(|\mathbf{p} - \mathbf{q}|)$, and $\sin(\mathbf{x}_{\|\mathbf{p}},2)\hat{\boldsymbol{\theta}}(\mathbf{x}_{\|\mathbf{p}}) = \sin(\mathbf{x}_{\|\mathbf{q}},2)\hat{\boldsymbol{\theta}}(\mathbf{x}_{\|\mathbf{q}}) + O_{\xi}(|\mathbf{p} - \mathbf{q}|)$ and $\hat{\boldsymbol{\phi}}(\mathbf{x}_{\|\mathbf{p}}) = \hat{\boldsymbol{\phi}}(\mathbf{x}_{\|\mathbf{q}}) + O_{\xi}(|\mathbf{p} - \mathbf{q}|)$.

Therefore for $\mathbf{x}_{\|\mathbf{p}},2, \mathbf{x}_{\|\mathbf{q}},2 \notin \{0, \pi\}$

$$\left[\frac{\partial \mathbf{x}_{\|\mathbf{q}}}{\partial \mathbf{x}_{\|\mathbf{p}}} \right]_{2\times 2} \lesssim \mathbf{Id}_{2,2} + O_{\xi}(|\mathbf{p} - \mathbf{q}|). \tag{6.12}$$

From the third equality of (6.10)

$$\left[-\mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \middle| \begin{array}{c} \partial_1 \eta_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \\ -\mathbf{x}_{\perp\mathbf{q}} \partial_1 \mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \end{array} \middle| \begin{array}{c} \partial_2 \eta_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \\ -\mathbf{x}_{\perp\mathbf{q}} \partial_2 \mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \end{array} \right] \left[\begin{array}{c|c} 0 & \mathbf{0}_{1,2} \\ \hline \mathbf{0}_{2,1} & \frac{\partial \mathbf{v}_{\|\mathbf{q}}}{\partial \mathbf{x}_{\|\mathbf{p}}} \end{array} \right] = \left[\mathbf{0}_{3,1} \middle| Z_1 \middle| Z_2 \right],$$

and

$$Z_i = \sum_{j=1}^{2} \mathbf{v}_{\|\mathbf{p}},j \sum_{m=1}^{2} \left(\partial_m \partial_j \eta_{\mathbf{p}} - \mathbf{x}_{\perp\mathbf{p}} \partial_m \partial_j \mathbf{n}_{\mathbf{p}} \right) \left(\delta_{mi} - \frac{\partial \mathbf{x}_{\|\mathbf{q}},m}{\partial \mathbf{x}_{\|\mathbf{p}},i} \right) \lesssim O_{\xi}(1)|v||\mathbf{p} - \mathbf{q}|,$$

where we have used (6.12).

Therefore

$$\left[\begin{array}{c|c} 0 & \mathbf{0}_{1,2} \\ \hline \mathbf{0}_{2,1} & \frac{\partial \mathbf{v}_{\|_\mathbf{q}}}{\partial \mathbf{x}_{\|_\mathbf{p}}} \end{array}\right] = \frac{1}{[-\mathbf{n}_\mathbf{q}]\cdot\big([\partial_1\eta_\mathbf{q} - \mathbf{x}_{\perp_\mathbf{q}}\partial_1\mathbf{n}_\mathbf{q}]\times[\partial_2\eta_\mathbf{q} - \mathbf{x}_{\perp_\mathbf{q}}\partial_2\mathbf{n}_\mathbf{q}]\big)}$$
$$\times\begin{bmatrix} (\partial_1\eta_\mathbf{q} - \mathbf{x}_{\perp_\mathbf{q}}\partial_1\mathbf{n}_\mathbf{q})\times(\partial_2\eta_\mathbf{q} - \mathbf{x}_{\perp_\mathbf{q}}\partial_2\mathbf{n}_\mathbf{q}) \\ (\partial_2\eta_\mathbf{q} - \mathbf{x}_{\perp_\mathbf{q}}\partial_2\mathbf{n}_\mathbf{q})\times(-\mathbf{n}_\mathbf{q}) \\ (-\mathbf{n}_\mathbf{q})\times(\partial_1\eta_\mathbf{q} - \mathbf{x}_{\perp_\mathbf{q}}\partial_1\mathbf{n}_\mathbf{q}) \end{bmatrix}\left[\mathbf{0}_{3,1}\middle|Z_1\middle|Z_2\right],$$

and hence from the above estimate of Z_i we have

$$\left[\frac{\partial \mathbf{v}_{\|_\mathbf{q}}}{\partial \mathbf{x}_{\|_\mathbf{p}}}\right]_{2\times 2} \lesssim_\xi |v||\mathbf{p}-\mathbf{q}|.$$

Again from the fourth equality of (6.10) and $\mathbf{n}_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}}) = \mathbf{n}_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})$,

$$\left[-\mathbf{n}_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\middle|\begin{smallmatrix}\partial_1\eta_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\\ -\mathbf{x}_{\perp_\mathbf{q}}\partial_1\mathbf{n}_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\end{smallmatrix}\middle|\begin{smallmatrix}\partial_2\eta_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\\ -\mathbf{x}_{\perp_\mathbf{q}}\partial_2\mathbf{n}_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\end{smallmatrix}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{\partial \mathbf{v}_{\|_\mathbf{q}}}{\partial \mathbf{v}_{\|_\mathbf{p}}}\end{array}\right]$$
$$= \left[-\mathbf{n}_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\middle|\begin{smallmatrix}\partial_1\eta_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\\ -\mathbf{x}_{\perp_\mathbf{p}}\partial_1\mathbf{n}_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\end{smallmatrix}\middle|\begin{smallmatrix}\partial_2\eta_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\\ -\mathbf{x}_{\perp_\mathbf{p}}\partial_2\mathbf{n}_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\end{smallmatrix}\right].$$

Since

$$\left[-\mathbf{n}_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\middle|\begin{smallmatrix}\partial_1\eta_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\\ -\mathbf{x}_{\perp_\mathbf{p}}\partial_1\mathbf{n}_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\end{smallmatrix}\middle|\begin{smallmatrix}\partial_2\eta_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\\ -\mathbf{x}_{\perp_\mathbf{p}}\partial_2\mathbf{n}_\mathbf{p}(\mathbf{x}_{\|_\mathbf{p}})\end{smallmatrix}\right]$$
$$= \left[-\mathbf{n}_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\middle|\begin{smallmatrix}\partial_1\eta_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\\ -\mathbf{x}_{\perp_\mathbf{q}}\partial_1\mathbf{n}_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\end{smallmatrix}\middle|\begin{smallmatrix}\partial_2\eta_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\\ -\mathbf{x}_{\perp_\mathbf{q}}\partial_2\mathbf{n}_\mathbf{q}(\mathbf{x}_{\|_\mathbf{q}})\end{smallmatrix}\right] + O_\xi(|\mathbf{p}-\mathbf{q}|),$$

we have

$$\begin{bmatrix} \frac{\partial \mathbf{v}_{\|_\mathbf{q},1}}{\partial \mathbf{v}_{\|_\mathbf{p},1}} & \frac{\partial \mathbf{v}_{\|_\mathbf{q},1}}{\partial \mathbf{v}_{\|_\mathbf{p},2}} \\ \frac{\partial \mathbf{v}_{\|_\mathbf{q},2}}{\partial \mathbf{v}_{\|_\mathbf{p},1}} & \frac{\partial \mathbf{v}_{\|_\mathbf{q},2}}{\partial \mathbf{v}_{\|_\mathbf{p},2}} \end{bmatrix} = \mathbf{Id}_{2,2} + O_\xi(|\mathbf{p}-\mathbf{q}|).$$

$\square$

Lemma 16 *Assume* $\frac{dx}{dt} = 0$, $\frac{dv}{dt} = 0$ *and let* $\Phi_\mathbf{p}(\mathbf{x}_{\perp_\mathbf{p}}, \mathbf{x}_{\|_\mathbf{p}}, \mathbf{v}_{\perp_\mathbf{p}}, \mathbf{v}_{\|_\mathbf{p}}) = (x, v)$.

(i) *For* $|\xi(X_{\mathbf{cl}}(s;t,x,v))| < \delta$ *and* $|X_{\mathbf{cl}}(s;t,x,v) - \mathcal{L}_\mathbf{p}| > C\delta_1$ *we define*

$$\begin{aligned}(\mathbf{X}_\mathbf{p}(s;t,x,v), \mathbf{V}_\mathbf{p}(s;t,x,v)) &:= \Phi_\mathbf{p}^{-1}(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) \\ &:= (\mathbf{x}_{\perp_\mathbf{p}}(s;t,x,v), \mathbf{x}_{\|_\mathbf{p}}(s;t,x,v), \\ &\qquad \mathbf{v}_{\perp_\mathbf{p}}(s;t,x,v), \mathbf{v}_{\|_\mathbf{p}}(s;t,x,v)).\end{aligned}$$

Then $|v| \simeq |\mathbf{V_p}|$ *and*

$$\begin{bmatrix} \dot{\mathbf{x}}_{\perp_\mathbf{p}} \\ \dot{\mathbf{x}}_{\parallel_\mathbf{p}} \\ \dot{\mathbf{v}}_{\perp_\mathbf{p}} \\ \dot{\mathbf{v}}_{\parallel_\mathbf{p}} \end{bmatrix} (s;t,x,v) = \begin{bmatrix} \mathbf{v}_{\perp_\mathbf{p}} \\ \mathbf{v}_{\parallel_\mathbf{p}} \\ F_{\perp_\mathbf{p}}(\mathbf{x_p},\mathbf{v_p}) \\ F_{\parallel_\mathbf{p}}(\mathbf{x_p},\mathbf{v_p}) \end{bmatrix} (s;t,x,v). \tag{6.13}$$

Here

$$\begin{aligned} F_{\perp_\mathbf{p}} &= F_{\perp_\mathbf{p}}(\mathbf{x}_{\perp_\mathbf{p}},\mathbf{x}_{\parallel_\mathbf{p}},\mathbf{v}_{\parallel_\mathbf{p}}) \\ &= \sum_{j,k=1}^{2} \mathbf{v}_{\parallel_\mathbf{p},k}\mathbf{v}_{\parallel_\mathbf{p},j}\ \partial_j\partial_k\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}})\cdot\mathbf{n_p}(\mathbf{x}_{\parallel_\mathbf{p}}) \\ &\quad -\mathbf{x}_{\perp_\mathbf{p}}\sum_{k=1}^{2}\mathbf{v}_{\parallel_\mathbf{p},k}(\mathbf{v}_{\parallel_\mathbf{p}}\cdot\nabla)\partial_k\mathbf{n_p}(\mathbf{x}_{\parallel_\mathbf{p}})\cdot\mathbf{n_p}(\mathbf{x}_{\parallel_\mathbf{p}}), \end{aligned} \tag{6.14}$$

where

$$\sum_{j,k=1}^{2} \mathbf{v}_{\parallel_\mathbf{p},k}\mathbf{v}_{\parallel_\mathbf{p},j}\ \partial_j\partial_k\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}})\cdot\mathbf{n_p}(\mathbf{x}_{\parallel_\mathbf{p}}) \lesssim_\xi -|\mathbf{v}_\parallel|^2,$$

and

$$\begin{aligned} F_{\parallel_\mathbf{p}} &= F_{\parallel_\mathbf{p}}(\mathbf{x}_{\perp_\mathbf{p}},\mathbf{x}_{\parallel_\mathbf{p}},\mathbf{v}_{\perp_\mathbf{p}},\mathbf{v}_{\parallel_\mathbf{p}}) \\ &= \sum_{i=1,2} G_{\mathbf{p},ij}(\mathbf{x}_{\perp_\mathbf{p}},\mathbf{x}_{\parallel_\mathbf{p}})\frac{(-1)^i}{\mathbf{n_p}(\mathbf{x}_{\parallel_\mathbf{p}})\cdot(\partial_1\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}})\times\partial_2\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}))} \\ &\quad \times\Big\{2\mathbf{v}_{\perp_\mathbf{p}}\mathbf{v}_{\parallel_\mathbf{p}}\cdot\nabla\mathbf{n_p}(\mathbf{x}_{\parallel_\mathbf{p}}) - \mathbf{v}_{\parallel_\mathbf{p}}\cdot\nabla^2\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}})\cdot\mathbf{v}_{\parallel_\mathbf{p}} \\ &\qquad +\mathbf{x}_{\perp_\mathbf{p}}\mathbf{v}_{\parallel_\mathbf{p}}\cdot\nabla^2\mathbf{n_p}(\mathbf{x}_{\parallel_\mathbf{p}})\cdot\mathbf{v}_{\parallel_\mathbf{p}}\Big\}\cdot\big\{\mathbf{n_p}(\mathbf{x}_{\parallel_\mathbf{p}})\times\partial_{i+1}\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}})\big\}, \end{aligned} \tag{6.15}$$

where a smooth bounded function $G_{\mathbf{p},ij}(\mathbf{x}_{\perp_\mathbf{p}},\mathbf{x}_{\parallel_\mathbf{p}})$ *is specified in* (6.22).

(ii) *For* $\tau \in (t^{\ell+1}, t^\ell)$, *if the* $\mathbf{p}^\ell-$*spherical coordinate is well-defined in* $[\tau, t^\ell)$ *then*

$$\begin{aligned} &[\mathbf{X}_\ell(\tau;t,x,v),\mathbf{V}_\ell(\tau;t,x,v)] \\ &\equiv [\mathbf{X}_\ell(\tau;t^\ell,0,\mathbf{x}^\ell_{\parallel_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\parallel_\ell}),\mathbf{V}_\ell(\tau;t^\ell,0,\mathbf{x}^\ell_{\parallel_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\parallel_\ell})] \end{aligned}$$

and, for $\partial_{\mathbf{v}^\ell_\ell} = [\partial_{\mathbf{v}^\ell_{\perp_\ell}}, \partial_{\mathbf{v}^\ell_{\parallel_\ell}}]$,

$$\begin{bmatrix} |\partial_{\mathbf{x}^\ell_{\parallel_\ell}}\mathbf{X}_\ell(\tau)| & |\partial_{\mathbf{v}^\ell_\ell}\mathbf{X}_\ell(\tau)| \\ |\partial_{\mathbf{x}^\ell_{\parallel_\ell}}\mathbf{V}_\ell(\tau)| & |\partial_{\mathbf{v}^\ell_\ell}\mathbf{V}_\ell(\tau)| \end{bmatrix} \lesssim \begin{bmatrix} 1 & |\tau-t^\ell| \\ |v|^2|\tau-t^\ell| & 1 \end{bmatrix}. \tag{6.16}$$

For $t^{\ell+1} < \tau < s < t^{\ell}$ *then*

$$\mathbf{X}_\ell(\tau;t,x,v) \equiv \mathbf{X}_\ell(\tau;s,\mathbf{X}_\ell(s;t,x,v),\mathbf{V}_\ell(s;t,x,v)),$$
$$\mathbf{V}_\ell(\tau;t,x,v) \equiv \mathbf{V}_\ell(\tau;s,\mathbf{X}_\ell(s;t,x,v),\mathbf{V}_\ell(s;t,x,v)),$$

and

$$\begin{bmatrix} |\partial_{\mathbf{X}_\ell(s)}\mathbf{X}_\ell(\tau)| & |\partial_{\mathbf{V}_\ell(s)}\mathbf{X}_\ell(\tau)| \\ |\partial_{\mathbf{X}_\ell(s)}\mathbf{V}_\ell(\tau)| & |\partial_{\mathbf{V}_\ell(s)}\mathbf{V}_\ell(\tau)| \end{bmatrix} \lesssim \begin{bmatrix} 1 & |\tau-s| \\ |v|^2|\tau-s| & 1 \end{bmatrix}. \tag{6.17}$$

Moreover, for either $[\partial_{\mathbf{X}},\partial_{\mathbf{V}}] = [\partial_{\mathbf{x}^\ell_{\|\ell}}, \partial_{\mathbf{v}^\ell_{\perp\ell}}, \partial_{\mathbf{v}^\ell_{\|\ell}}]$ *or* $[\partial_{\mathbf{X}},\partial_{\mathbf{V}}] = [\partial_{\mathbf{X}_\ell(s)},\partial_{\mathbf{V}_\ell(s)}]$

$$\begin{bmatrix} |\partial_{\mathbf{X}}F(\tau)| & |\partial_{\mathbf{V}}F(\tau)| \\ |\frac{d}{d\tau}\partial_{\mathbf{X}}F(\tau)| & |\frac{d}{d\tau}\partial_{\mathbf{V}}F(\tau)| \end{bmatrix} \lesssim \begin{bmatrix} |v|^2 & |v| \\ |v|^3 & |v|^2 \end{bmatrix}. \tag{6.18}$$

Proof From $\dot{v}=0$ and the second equation of (6.4) equals

$$\begin{aligned} 0 &= \dot{\mathbf{v}}_\perp(s)[-\mathbf{n}(\mathbf{x}_\|(s))] - 2\mathbf{v}_\perp(s)\mathbf{v}_\| \cdot \nabla\mathbf{n}(\mathbf{x}_\|(s)) + \dot{\mathbf{v}}_\|(s)\cdot\nabla\eta(\mathbf{x}_\|(s)) \\ &\quad + \mathbf{v}_\|\cdot\nabla^2\eta(\mathbf{x}_\|)\cdot\mathbf{v}_\| - \mathbf{x}_\perp\dot{\mathbf{v}}_\|\cdot\nabla\mathbf{n}(\mathbf{x}_\|) - \mathbf{x}_\perp\mathbf{v}_\|\cdot\nabla^2\mathbf{n}(\mathbf{x}_\|)\cdot\mathbf{v}_\|. \end{aligned} \tag{6.19}$$

We take the inner product with $\mathbf{n}(\mathbf{x}_\|(s))$ to the above equation to have

$$\begin{aligned} \dot{\mathbf{v}}_\perp(s) &= [\mathbf{v}_\|\cdot\nabla^2\eta(\mathbf{x}_\|)\cdot\mathbf{v}_\|]\cdot\mathbf{n}(\mathbf{x}_\|) - \mathbf{x}_\perp[\mathbf{v}_\|\cdot\nabla^2\mathbf{n}(\mathbf{x}_\|)\cdot\mathbf{v}_\|]\cdot\mathbf{n}(\mathbf{x}_\|) \\ &:= F_\perp(\mathbf{v}_\perp,\mathbf{v}_\|,\mathbf{x}_\|), \end{aligned} \tag{6.20}$$

where we have used the fact $\nabla\mathbf{n}\perp\mathbf{n}$ and $\nabla\eta\perp\mathbf{n}$.

Since $0=\xi(\eta(\mathbf{x}_\|))$ we take $\mathbf{x}_{\|,i}$ and $\mathbf{x}_{\|,j}$ derivatives to have

$$0 = \partial_{\mathbf{x}_{\|,j}}\left[\sum_k \partial_k\xi\,\partial_{\mathbf{x}_{\|,i}}\eta_k\right] = \sum_{k,m}\partial_k\partial_m\xi\,\partial_{\mathbf{x}_{\|,j}}\eta_m\partial_{\mathbf{x}_{\|,i}}\eta_k + \sum_k \partial_k\xi\,\partial_{\mathbf{x}_{\|,i}}\partial_{\mathbf{x}_{\|,j}}\eta_k,$$

and from the convexity (1.13) and $\mathbf{n}=\nabla\xi/|\nabla\xi|$,

$$\begin{aligned} \left[\mathbf{v}_\|\cdot\nabla^2\eta\cdot\mathbf{v}_\|\right]\cdot\mathbf{n} &= \sum_{i,j,k}\frac{\mathbf{v}_{\|,i}\partial_k\xi\,\partial_i\partial_j\eta_k\mathbf{v}_{\|,j}}{|\nabla\xi|} \\ &= -\sum_{i,j,k,m}\frac{\{\mathbf{v}_{\|,i}\partial_i\eta_m\}\partial_k\partial_m\xi\{\partial_j\eta_m\mathbf{v}_{\|,j}\}}{|\nabla\xi|} \lesssim_\xi -|\mathbf{v}_\||^2. \end{aligned}$$

Define $a_{ij}(\mathbf{x}_\parallel)$ via

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \partial_1\mathbf{n}\cdot\partial_1\mathbf{n} & \partial_1\mathbf{n}\cdot\partial_2\mathbf{n} \\ \partial_2\mathbf{n}\cdot\partial_1\mathbf{n} & \partial_2\mathbf{n}\cdot\partial_2\mathbf{n} \end{bmatrix} \begin{bmatrix} \partial_1\eta\cdot\partial_1\eta & \partial_1\eta\cdot\partial_2\eta \\ \partial_2\eta\cdot\partial_1\eta & \partial_2\eta\cdot\partial_2\eta \end{bmatrix}^{-1},$$

where $\det(\partial_i\eta\cdot\partial_j\eta) = |\partial_1\eta\times\partial_2\eta|^2 \neq 0$ due to (6.3). Then $\nabla\mathbf{n}$ is generated by $\nabla\eta$:

$$-\partial_i\mathbf{n}(\mathbf{x}_\parallel) = \sum_k a_{ik}(\mathbf{x}_\parallel)\partial_k\eta(\mathbf{x}_\parallel).$$

We take the inner product (6.19) with $(-1)^{i+1}(\mathbf{n}(\mathbf{x}_\parallel)\times\partial_i\mathbf{n}(\mathbf{x}_\parallel))$ to have

$$\begin{aligned}
&\sum_k(\delta_{ki}+\mathbf{x}_\perp a_{ki})\dot{\mathbf{v}}_{\parallel,k} \\
&\quad = \frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\parallel)\cdot(\partial_1\eta(\mathbf{x}_\parallel)\times\partial_2\eta(\mathbf{x}_\parallel))} \\
&\qquad \times\left\{-2\mathbf{v}_\perp\mathbf{v}_\parallel\cdot\nabla\mathbf{n}(\mathbf{x}_\parallel)+\mathbf{v}_\parallel\cdot\nabla^2\eta(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel-\mathbf{x}_\perp\mathbf{v}_\parallel\cdot\nabla^2\mathbf{n}(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel\right\} \\
&\qquad \cdot(-\mathbf{n}(\mathbf{x}_\parallel)\times\partial_{i+1}\eta(\mathbf{x}_\parallel)),
\end{aligned}$$

where we used the notational convention for $\partial_{i+1}\boldsymbol{\eta}$, the index $i+1 \bmod 2$. For $|\xi(x)|\ll 1$(and therefore $|\mathbf{x}_\perp|\ll 1$) the matrix $\delta_{ki}+\mathbf{x}_\perp a_{ki}$ is invertible: there exists the inverse matrix G_{ij} such that $\sum_i(\delta_{ki}+\mathbf{x}_\perp a_{ki}(\mathbf{x}_\parallel))G_{ij}(\mathbf{x}_\perp,\mathbf{x}_\parallel)=\delta_{kj}$. Therefore we have

$$\begin{aligned}
\dot{\mathbf{v}}_{\parallel,j} &= \sum_i G_{ij}(\mathbf{x}_\perp,\mathbf{x}_\parallel)\frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\parallel)\cdot(\partial_1\eta(\mathbf{x}_\parallel)\times\partial_2\eta(\mathbf{x}_\parallel))} \\
&\quad \times\left\{-2\mathbf{v}_\perp\mathbf{v}_\parallel\cdot\nabla\mathbf{n}(\mathbf{x}_\parallel)+\mathbf{v}_\parallel\cdot\nabla^2\eta(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel-\mathbf{x}_\perp\mathbf{v}_\parallel\cdot\nabla^2\mathbf{n}(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel\right\} \\
&\quad \cdot(-\mathbf{n}(\mathbf{x}_\parallel)\times\partial_{i+1}\eta(\mathbf{x}_\parallel)) \\
&:= F_{\parallel,j}(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\perp,\mathbf{v}_\parallel).
\end{aligned} \tag{6.21}$$

Here

$$\begin{aligned}
&\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \frac{1}{1+\mathbf{x}_\perp(a_{11}+a_{22})+(\mathbf{x}_\perp)^2(a_{11}a_{22}-a_{12}a_{21})}\begin{bmatrix} 1+\mathbf{x}_\perp a_{22} & -\mathbf{x}_\perp a_{12} \\ -\mathbf{x}_\perp a_{21} & 1+\mathbf{x}_\perp a_{11} \end{bmatrix}, \\
&\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{|\partial_1\eta|^2|\partial_2\eta|^2-(\partial_1\eta\cdot\partial_2\eta)^2} \\
&\quad\times\begin{bmatrix} |\partial_1\mathbf{n}|^2|\partial_2\eta|^2-(\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})(\partial_1\eta\cdot\partial_2\eta) & -|\partial_1\mathbf{n}|^2(\partial_1\eta\cdot\partial_2\eta)+(\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})|\partial_1\eta|^2 \\ (\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})|\partial_2\eta|^2-|\partial_2\mathbf{n}|^2(\partial_1\eta\cdot\partial_2\eta) & -(\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})(\partial_1\eta\cdot\partial_2\eta)+|\partial_2\mathbf{n}|^2|\partial_1\eta|^2 \end{bmatrix}.
\end{aligned} \tag{6.22}$$

To complete the proof of (6.13), from $\dot{x} = v$ and $\dot{v} = 0$, we have

$$\begin{aligned} v &= -\mathbf{v}_\perp \mathbf{n} + \mathbf{v}_\parallel \cdot \nabla \eta + \mathbf{x}_\perp[-\nabla n(\mathbf{x}_\parallel)]\dot{\mathbf{x}}_\parallel \\ &= \dot{\mathbf{x}}_\perp(-\mathbf{n}(\mathbf{x}_\parallel)) + \mathbf{x}_\perp[-\nabla \mathbf{n}(\mathbf{x}_\parallel)]\dot{\mathbf{x}}_\parallel + \nabla\eta\dot{\mathbf{x}}_\parallel \\ 0 &= \dot{\mathbf{v}}_\perp(-\mathbf{n}(\mathbf{x}_\parallel)) - \mathbf{v}_\perp \nabla\mathbf{n}\dot{\mathbf{x}}_\parallel + \dot{\mathbf{v}}_\parallel \nabla\eta + \mathbf{v}_\parallel \nabla^2\eta\dot{\mathbf{x}}_\parallel \\ &\quad + \dot{\mathbf{x}}_\perp \mathbf{v}_\parallel[-\nabla\mathbf{n}(\mathbf{x}_\parallel)] + \mathbf{x}_\perp\dot{\mathbf{v}}_\parallel[-\nabla\mathbf{n}(\mathbf{x}_\parallel)] + \mathbf{x}_\perp\mathbf{v}_\parallel[-\nabla^2 n]\dot{\mathbf{x}}_\parallel. \end{aligned}$$

We therefore conclude that $\dot{\mathbf{x}}_\perp = \mathbf{v}_\perp$, and $\dot{\mathbf{x}}_\parallel = \mathbf{v}_\parallel$ from $\Phi_{\mathbf{p}}^{-1}$. We then solve $\dot{\mathbf{v}}_\perp$ and $\dot{\mathbf{v}}_\parallel$ to obtain (6.13).

Now we prove (6.16) and (6.17). From (6.14) and (6.15), $\dot{\mathbf{x}}_{\parallel_\ell} = \mathbf{v}_{\parallel_\ell}$, $\dot{\mathbf{x}}_{\perp_\ell} = \mathbf{v}_{\perp_\ell}$ and $\dot{\mathbf{v}}_{\perp_\ell} = F_{\perp_\ell}$ and $\dot{\mathbf{v}}_{\parallel_\ell} = F_{\parallel_\ell}$. Denote $\partial = [\frac{\partial}{\partial \mathbf{x}^\ell_{\parallel_\ell}}, \frac{\partial}{\partial \mathbf{v}^\ell_{\perp_\ell}}, \frac{\partial}{\partial \mathbf{v}^\ell_{\parallel_\ell}}]$. From (6.14) and (6.15),

$$\begin{bmatrix} |\partial F_\perp| \\ |\partial F_\parallel| \end{bmatrix} \lesssim \begin{bmatrix} |v|^2\{|\partial \mathbf{x}_\perp| + |\partial \mathbf{x}_\parallel|\} + |v||\partial \mathbf{v}_\parallel| \\ |v|^2\{|\partial \mathbf{x}_\perp| + |\partial \mathbf{x}_\parallel|\} + |v|\{|\partial \mathbf{v}_\perp| + |\partial \mathbf{v}_\parallel|\} \end{bmatrix}. \tag{6.23}$$

Now we use a single (rough) bound of $|\partial F_\perp| + |\partial F_\parallel| \lesssim |v|^2\{|\partial \mathbf{x}_\perp| + |\partial \mathbf{x}_\parallel|\} + |v|\{|\partial \mathbf{v}_\perp| + |\partial \mathbf{v}_\parallel|\}$ to have

$$\begin{aligned} &\frac{d}{d\tau}\{|\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\parallel_\ell}(\tau)|\} \\ &\lesssim |\partial F_{\perp_\ell}(\tau)| + |\partial F_{\parallel_\ell}(\tau)| \\ &\lesssim |v|^2\big\{|\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\parallel_\ell}(\tau)|\big\} + |v|\big\{|\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\parallel_\ell}(\tau)|\big\}. \end{aligned}$$

Combining with $\frac{d}{d\tau}[\mathbf{x}_{\perp_\ell}(\tau), \mathbf{x}_{\parallel_\ell}(\tau)] = [\mathbf{v}_{\perp_\ell}(\tau), \mathbf{v}_{\parallel_\ell}(\tau)]$ yields

$$\frac{d}{d\tau}\begin{bmatrix} |\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\parallel_\ell}(\tau)| \\ |\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\parallel_\ell}(\tau)| \end{bmatrix} \lesssim_\xi \begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix} \begin{bmatrix} |\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\parallel_\ell}(\tau)| \\ |\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\parallel_\ell}(\tau)| \end{bmatrix}.$$

By Lemma 4 we prove our claim (6.16). The proof of (6.17) is exactly same but we use $\partial = [\partial_{\mathbf{X}_\ell}(s), \partial_{\mathbf{V}_\ell}(s)]$ to conclude the proof.

We prove the first row of (6.18) by (6.23). By taking the time derivative to (6.14), (6.15) and applying (6.13), we prove the second row of row of (6.18). □

We are ready to prove Theorem 5:

Proof of Theorem 5 First we consider the case of $t < t_{\mathbf{b}}(x, v)$. In this case

$$(X_{\mathbf{cl}}(s; t, x, v), V_{\mathbf{cl}}(s; t, x, v)) = (x - (t - s)v, v).$$

Directly

$$\frac{\partial(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s,t,x,v))}{\partial(t,x,v)}=\begin{bmatrix}-v & \mathbf{Id}_{3,3} & -(t-s)\mathbf{Id}_{3,3}\\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3}\end{bmatrix}_{6\times 7},$$

where $\mathbf{Id}_{m,m}$ is the m by m identity matrix and $\mathbf{0}_{m,n}$ is the m by n zero matrix.
Now we consider the case of $t\geq t_{\mathbf{b}}(x,v)$. We split our proof into 10 steps.

Step 1. Moving frames and grouping with respect to the scaling $t|v|=L_\xi$, *with fixed* $0<L_\xi\ll 1$.
Fix $(t,x,v)\in[0,\infty)\times\bar{\Omega}\times\mathbb{R}^3$. Also we fix small constant $\delta=\delta_\xi>0$ which depends on the domain. We define, at the boundary,

$$\mathbf{r}^\ell:=\frac{|\mathbf{v}^\ell_\perp|}{|v^\ell|}=\frac{|v\cdot n(x^\ell)|}{|v|}=\frac{|V_{\mathbf{cl}}(t^\ell;t,x,v)\cdot n(X_{\mathbf{cl}}(t^\ell;t,x,v))|}{|v|}. \tag{6.24}$$

Bounces ℓ(and (t^ℓ,x^ℓ,v^ℓ)) are categorized as *Type I* or *Type II*:

$$\begin{aligned}&\text{a bounce } \ell \text{ is } Type\ I \text{ (almost grazing) if and only if } \mathbf{r}^\ell\leq\sqrt{\delta},\\ &\text{a bounce } \ell \text{ is } Type\ II \text{ (non-grazing) if and only if } \mathbf{r}^\ell>\sqrt{\delta}.\end{aligned} \tag{6.25}$$

Let $s_*\in[t^{\ell+1},t^\ell]$ such that

$$|\xi(X_{\mathbf{cl}}(s_*;t^\ell,x^\ell,v^\ell))|=\max_{t^{\ell+1}\leq\tau\leq t^\ell}|\xi(X_{\mathbf{cl}}(\tau;t^\ell,x^\ell,v^\ell))|.$$

Since

$$\frac{d^2}{ds^2}\xi(X_{\mathbf{cl}}(s;t^\ell,x^\ell,v^\ell))=\frac{d^2}{ds^2}\xi(x^\ell-(t^\ell-s)v^\ell)=v^\ell\cdot\nabla_x^2\xi(x^\ell-(t^\ell-s)v^\ell)\cdot v^\ell>0$$

there exists a unique s_* solving

$$\frac{d}{ds}\xi(X_{\mathbf{cl}}(s_*;t^\ell,x^\ell,v^\ell))=v^\ell\cdot\nabla_x\xi(X_{\mathbf{cl}}(s_*;t^\ell,x^\ell,v^\ell))=0.$$

Note that $v^\ell \cdot \nabla\xi(x^\ell - (t^\ell - s)v^\ell)$ is monotone in either one of the interval $(t^{\ell+1}, s_*)$ or (s_*, t^ℓ). Without of generality we may assume $|t^\ell - s_*| \geq \frac{1}{2}|t^{\ell+1} - t^\ell|$. Then

$$
\begin{aligned}
|\xi(X_{\mathbf{cl}}(s_*; t^\ell, x^\ell, v^\ell))| &= \left| \int_{s_*}^{t^\ell} v^\ell \cdot \nabla\xi(x^\ell - (t^\ell - s)v^\ell, v^\ell) \mathrm{d}s \right| \\
&= \left| \int_{s_*}^{t^\ell} \int_s^{t^\ell} v^\ell \cdot \nabla^2\xi(x^\ell - (t^\ell - \tau)v^\ell, v^\ell) \cdot v^\ell \mathrm{d}\tau \mathrm{d}s \right| \\
&\simeq_\xi \frac{|v^\ell|^2 |t^\ell - s_*|^2}{2} \simeq_\xi \left(\sup_{s \in [t^{\ell+1}, t^\ell]} \frac{|v^\ell \cdot n(X_{\mathbf{cl}}(s))|}{|v^\ell|} \right)^2,
\end{aligned}
$$

where we used (2.2) and (2.3) and the Velocity lemma (Lemma 2).

Therefore if a bounce ℓ is *Type I* then $\max_{t^{\ell+1} \leq \tau \leq t^\ell} |\xi(X_{\mathbf{cl}}(\tau; t, x, v))| \leq C\delta$. If a bounce ℓ is *Type II* then $|\xi(X_{\mathbf{cl}}(\tau; t, x, v))| > C\delta$ for some $\tau \in [t^{\ell+1}, t^\ell]$. We always assume that $v \neq 0$.

Now we assign a coordinate chart for each bounce ℓ (moving frames). For *Type I* bounce ℓ in (6.25), we assign $\mathbf{p}^\ell \in \partial\Omega \times \mathbb{S}^2$ and $\mathbf{p}^\ell-$spherical coordinates in Lemma 15 and (6.4): we choose $\mathbf{p}^\ell := (z^\ell, w^\ell)$ on $\partial\Omega \times \mathbb{S}^2$ with $n(z^\ell) \cdot w^\ell = 0$

$$
z^\ell = x^\ell, \quad w^\ell = \frac{v^\ell - (v^\ell \cdot n(z^\ell))n(z^\ell)}{|v^\ell - (v^\ell \cdot n(z^\ell))n(z^\ell)|}. \tag{6.26}
$$

Note that, by the definition of *Type I bounce*, $|v^\ell - (v^\ell \cdot n(z^\ell)n(z^\ell))|^2 = |v|^2 - |\mathbf{v}_\perp^\ell|^2 \gtrsim |v|^2(1-\delta) \gtrsim_\delta |v|^2$ and hence w^ℓ is well-defined.

Moreover

$$
|X_{\mathbf{cl}}(s; t, x, v) - \mathcal{L}_{\mathbf{p}^\ell}| \gtrsim C_\delta > 0, \tag{6.27}
$$

for $|v||t^\ell - s| \leq \frac{1}{100} \min_{x \in \partial\Omega} |x|$. This is due to the fact that the projection of $V_{\mathbf{cl}}(s)$ on the plane passing z^ℓ and perpendicular to $n(z^\ell) \times w^\ell$ is at most $|v|$ magnitude but the distance from z^ℓ to the origin(the projection of poles $\mathcal{N}_{\mathbf{p}^\ell}$ and $\mathcal{S}_{\mathbf{p}^\ell}$) has lower bound $\frac{1}{10} \min_{x \in \partial\Omega} |x|$, $|s - t^\ell| \ll 1$.

For *Type II* bounce $\ell(t^\ell, x^\ell, v^\ell)$, we choose $\mathbf{p}^\ell = (z^\ell, w^\ell)$ with $|z^\ell - x^\ell| \leq \sqrt{\delta}$ but we choose arbitrary $w^\ell \in \mathbb{S}^2$ satisfying $n(z^\ell) \cdot w^\ell = 0$. We choose $\mathbf{p}^\ell-$spherical coordinate in Lemma 15 and (6.4) with this $\mathbf{p}^\ell$. Note that unlike *Type I*, this $\mathbf{p}^\ell-$spherical coordinate might not be defined for $s \in [t^{\ell+1}, t^\ell]$ but only defined near the boundary.

Whenever the moving frame is defined (for all $\tau \in (t^{\ell+1}, t^{\ell}]$ when ℓ is *Type I*, and $|\tau - t^{\ell}| \ll 1$ when ℓ is *Type II*) we denote, by (6.4) and (6.10),

$$(\mathbf{X}_{\ell}(\tau), \mathbf{V}_{\ell}(\tau)) = (\mathbf{x}_{\perp_{\ell}}(\tau), \mathbf{x}_{\|_{\ell}}(\tau), \mathbf{v}_{\perp_{\ell}}(\tau), \mathbf{v}_{\|_{\ell}}(\tau)) := \Phi^{-1}_{\mathbf{p}^{\ell}}(X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau)).$$

Especially at the boundary we denote

$$(\mathbf{x}^{\ell}_{\perp_{\ell}}, \mathbf{x}^{\ell}_{\|_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\|_{\ell}}) := \lim_{\tau \uparrow t^{\ell}} (\mathbf{X}_{\ell}(\tau), \mathbf{V}_{\ell}(\tau)), \quad \text{with } \mathbf{x}^{\ell}_{\perp_{\ell}} = 0, \ \mathbf{v}^{\ell}_{\perp_{\ell}} \geq 0.$$

Then we define

$$(\mathbf{x}^{\ell+1}_{\perp_{\ell}}, \mathbf{x}^{\ell+1}_{\|_{\ell}}, \mathbf{v}^{\ell+1}_{\|_{\ell}}) = \lim_{\tau \downarrow t^{\ell+1}} (\mathbf{x}_{\perp_{\ell}}(\tau), \mathbf{x}_{\|_{\ell}}(\tau), \mathbf{v}_{\|_{\ell}}(\tau)),$$

and

$$\mathbf{v}^{\ell+1}_{\perp_{\ell}} := - \lim_{\tau \downarrow t^{\ell+1}} \mathbf{v}_{\perp_{\ell}}(\tau). \tag{6.28}$$

Now we regroup the indices of the specular cycles, without order changing, as

$$\{0, 1, 2, \dots, \ell_* - 1, \ell_*\} = \{0\} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_{[\frac{|t-s\|v|}{L_\xi}]} \cup \mathcal{G}_{[\frac{|t-s\|v|}{L_\xi}]+1},$$

where $[a] \in \mathbb{N}$ is the greatest integer less than or equal to a. Each group is

$$\begin{aligned}
\mathcal{G}_1 &= \{1, \dots, \ell_1 - 1, \ell_1\}, \\
\mathcal{G}_2 &= \{\ell_1, \ell_1 + 1, \dots, \ell_2 - 1, \ell_2\}, \\
&\ \vdots \\
\mathcal{G}_{[\frac{|t-s\|v|}{L_\xi}]} &= \{\ell_{[\frac{|t-s\|v|}{L_\xi}]-1}, \ell_{[\frac{|t-s\|v|}{L_\xi}]-1} + 1, \dots, \ell_{[\frac{|t-s\|v|}{L_\xi}]} - 1, \ell_{[\frac{|t-s\|v|}{L_\xi}]}\}, \\
\mathcal{G}_{[\frac{|t-s\|v|}{L_\xi}]+1} &= \{\ell_{[\frac{|t-s\|v|}{L_\xi}]}, \ell_{[\frac{|t-s\|v|}{L_\xi}]} + 1, \dots, \ell_*\},
\end{aligned} \tag{6.29}$$

where $\ell_1 = \inf\{\ell \in \mathbb{N} : |v| \times |t^0 - t^{\ell_1}| \geq L_\xi\}$ and inductively

$$\ell_i = \inf\{\ell \in \mathbb{N} : |v| \times |t^{\ell_i} - t^{\ell_{i+1}}| \geq L_\xi\}, \tag{6.30}$$

and we have denoted $\ell_* = \ell_{[\frac{|t-s\|v|}{L_\xi}]+1}$.

Our analysis is carried out in each group G_i. We note that within each G_i, $|t^{\ell_i} - t^{\ell_{i+1}}\|v| < L_\xi$ by our design, so from the velocity lemma, r_{ℓ_i} is comparable to each other, so is $|v^{\ell}|$. We can also cover the entire G_i via a single chart in Section 8. By the chain rule, with the assigned $\mathbf{p}^{\ell}-$spherical coordinate (moving frame), we have for fixed $0 \leq s \leq t$ and $s \in (t^{\ell_*+1}, t^{\ell_*})$

$$
\begin{aligned}
&\frac{\partial(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)}\\
&=\underbrace{\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}}_{\text{from the last bounce to the } s\text{-plane}}\\
&\quad\times\underbrace{\prod_{i=1}^{[\frac{|t-s\|v|}{L_*}]}\underbrace{\frac{\partial(t^{\ell_{i+1}},\mathbf{x}^{\ell_{i+1}}_{\|_{\ell_{i+1}}},\mathbf{v}^{\ell_{i+1}}_{\perp_{\ell_{i+1}}},\mathbf{v}^{\ell_{i+1}}_{\|_{\ell_{i+1}}})}{\partial(t^{\ell_{i+1}-1},\mathbf{x}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}},\mathbf{v}^{\ell_{i+1}-1}_{\perp_{\ell_{i+1}-1}},\mathbf{v}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}})}\times\cdots\times\frac{\partial(t^{\ell_i+1},\mathbf{x}^{\ell_i+1}_{\|_{\ell_i+1}},\mathbf{v}^{\ell_i+1}_{\perp_{\ell_i+1}},\mathbf{v}^{\ell_i+1}_{\|_{\ell_i+1}})}{\partial(t^{\ell_i},\mathbf{x}^{\ell_i}_{\|_{\ell_i}},\mathbf{v}^{\ell_i}_{\perp_{\ell_i}},\mathbf{v}^{\ell_i}_{\|_{\ell_i}})}}_{i-\text{th intermediate group}}}_{\text{whole intermediate groups}}\\
&\quad\times\underbrace{\frac{\partial(t^{1},\mathbf{x}^{1}_{\|_{1}},\mathbf{v}^{1}_{\perp_{1}},\mathbf{v}^{1}_{\|_{1}})}{\partial(t,x,v)}}_{\text{from the } t\text{-plane to the first bounce}}.
\end{aligned}
\tag{6.31}
$$

Step 2. From the last bounce ℓ_* to the $s-$ plane

We choose $s^{\ell_*}\in(\frac{t^{\ell_*}+s}{2},t^{\ell_*})\subset(s,t^{\ell_*})$ such that $|v\|t^{\ell_*}-s^{\ell_*}|\ll 1$ and the ℓ_*-spherical coordinate $(\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))$ is well-defined regardless of types of ℓ_* in (6.25). Notice that s^{ℓ_*} is independent of t^{ℓ_*} and s so that $\frac{\partial s^{\ell_*}}{\partial t^{\ell_*}}=0=\frac{\partial s^{\ell_*}}{\partial s}$.

We first follow the flow in (x,v) co-ordinate to near the boundary at $t=s^{\ell_*}$, change to the chart to (X,V), then follow the flow in (X,V). Regarding s^{ℓ_*} as a free variable, by the chain rule,

$$
\begin{aligned}
&\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}\\
&=\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}\frac{\partial(s^{\ell_*},\mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\|_{\ell_*}}(s^{\ell_*}))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}\\
&=\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*},X_{\mathbf{cl}}(s^{\ell_*}),V_{\mathbf{cl}}(s^{\ell_*}))}\frac{\partial(s^{\ell_*},X_{\mathbf{cl}}(s^{\ell_*}),V_{\mathbf{cl}}(s^{\ell_*}))}{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}\\
&\quad\times\frac{\partial(s^{\ell_*},\mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\|}(s^{\ell_*}))}{\partial(t^{\ell_*},\mathbf{x}^{\ell_*}_{\|_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\|_{\ell_*}})}.
\end{aligned}
$$

Firstly, we claim

$$
\begin{aligned}
&\frac{\partial(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}\\
&\quad=\begin{bmatrix}-V_{\mathbf{cl}}(s^{\ell_*}) & O_\xi(1)(1+|v\|s^{\ell_*}-s|) & O_\xi(1)|s^{\ell_*}-s|\\ \mathbf{0}_{3,1} & O_\xi(1)|v| & O_\xi(1)\end{bmatrix}.
\end{aligned}
\tag{6.32}
$$

Since

$$X_{\mathbf{cl}}(s) = X_{\mathbf{cl}}(s^{\ell_*}) - (s^{\ell_*} - s)V_{\mathbf{cl}}(s^{\ell_*}), \quad V_{\mathbf{cl}}(s) = V_{\mathbf{cl}}(s^{\ell_*}),$$

and s^{ℓ_*} is independent of s, we have

$$\frac{\partial(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*}), V_{\mathbf{cl}}(s^{\ell_*}))} = \begin{bmatrix} -V_{\mathbf{cl}}(s^{\ell_*}) & \mathbf{Id}_{3,3} & -(s^{\ell_*} - s)\mathbf{Id}_{3,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3} \end{bmatrix}.$$

Furthermore due to Lemma 15, we conclude

$$\frac{\partial(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*}), V_{\mathbf{cl}}(s^{\ell_*}))}{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))}$$
$$= \left[\begin{array}{c|ccc|ccc} 1 & & \mathbf{0}_{1,3} & & & \mathbf{0}_{1,3} & \\ \hline \mathbf{0}_{3,1} & -\mathbf{n}_{\ell_*} & \partial_1\eta_{\ell_*} - \mathbf{x}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*} & \partial_2\eta_{\ell_*} - \mathbf{x}_{\perp_{\ell_*}}\partial_2\mathbf{n}_{\ell_*} & & \mathbf{0}_{3,3} & \\ \hline \mathbf{0}_{3,1} & -\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\mathbf{x}_{\|_{\ell_*}}\mathbf{n}_{\ell_*} & \begin{array}{c}\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\partial_1\eta_{\ell_*} \\ -\mathbf{v}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*} \\ -\mathbf{x}_{\perp_{\ell_*}}\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\partial_1\mathbf{n}_{\ell_*}\end{array} & \begin{array}{c}\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\partial_2\eta_{\ell_*} \\ -\mathbf{v}_{\perp_{\ell_*}}\partial_2\mathbf{n}\ell_* \\ -\mathbf{x}_{\perp_{\ell_*}}\mathbf{v}_{\|_{\ell_*}}\cdot\nabla\partial_2\mathbf{n}_{\ell_*}\end{array} & -\mathbf{n}_{\ell_*} & \partial_1\eta_{\ell_*} - \mathbf{x}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*} & \partial_2\eta_{\ell_*} - \mathbf{x}_{\perp_{\ell_*}}\partial_2\mathbf{n}_{\ell_*} \end{array}\right],$$

where all entries are evaluated at $(\mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))$. The multiplication of the above two matrices gives (6.32).

Secondly, we claim that whenever $\mathbf{p}^{\ell}-$spherical coordinate is defined for all $\tau \in [s^{\ell}, t^{\ell}]$, we have the following 7×6 matrix

$$\frac{\partial(s^{\ell}, \mathbf{x}_{\perp_{\ell}}(s^{\ell}), \mathbf{x}_{\|_{\ell}}(s^{\ell}), \mathbf{v}_{\perp_{\ell}}(s^{\ell}), \mathbf{v}_{\|_{\ell}}(s^{\ell}))}{\partial(t^{\ell}, \mathbf{x}^{\ell}_{\|_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\|_{\ell}})}$$
$$= \left[\begin{array}{c|c|c|c} 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline -\mathbf{v}_{\perp}(s^{\ell}) & O_{\xi}(1)|v|^2|t^{\ell} - s^{\ell}|^2 & O_{\xi}(1)|t^{\ell} - s^{\ell}| & O_{\xi}(1)|v||t^{\ell} - s^{\ell}|^2 \\ -\mathbf{v}_{\|}(s^{\ell}) & \mathbf{Id}_{2,2} + O_{\xi}(1)|v|^2|t^{\ell} - s^{\ell}|^2 & O_{\xi}(1)|v||t^{\ell} - s^{\ell}|^2 & O_{\xi}(1)|t^{\ell} - s^{\ell}|(\mathbf{Id}_{2,2} + |v||t^{\ell} - s^{\ell}|) \\ \hline O_{\xi}(1)|v|^2 & O_{\xi}(1)|v|^2|t^{\ell} - s^{\ell}| & 1 + O_{\xi}(1)|v||t^{\ell} - s^{\ell}| & O_{\xi}(1)|v||t^{\ell} - s^{\ell}| \\ O_{\xi}(1)|v|^2 & O_{\xi}(1)|v|^2|t^{\ell} - s^{\ell}| & O_{\xi}(1)|v||t^{\ell} - s^{\ell}| & \mathbf{Id}_{2,2} + O_{\xi}(1)|v||t^{\ell} - s^{\ell}| \end{array}\right]. \tag{6.33}$$

In this step we just need (6.33) for $\ell = \ell_*$ but we need (6.33) for general ℓ in Step 8.

Clearly the first raw is identically zero since s^{ℓ} is chosen to be independent of $(t^{\ell}, \mathbf{x}^{\ell}_{\|_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\|_{\ell}})$. The first column (temporal derivatives) holds due to the fact that the characteristics ODE (6.13) is autonomous. Moreover,

$$
\begin{aligned}
&\frac{\partial}{\partial t^{\ell}}(\mathbf{X}_{\ell}(s^{\ell};t^{\ell},x^{\ell},v^{\ell}),\mathbf{V}_{\ell}(s^{\ell};t^{\ell},x^{\ell},v^{\ell}))\\
&\quad=\frac{\partial}{\partial t^{\ell}}(\mathbf{X}_{\ell}(s^{\ell}-t^{\ell};0,x^{\ell},v^{\ell}),\mathbf{V}_{\ell}(s^{\ell}-t^{\ell};0,x^{\ell},v^{\ell}))\\
&\quad=-\frac{\partial}{\partial s^{\ell}}(\mathbf{X}_{\ell}(s^{\ell};t^{\ell},x^{\ell},v^{\ell}),\mathbf{V}_{\ell}(s^{\ell};t^{\ell},x^{\ell},v^{\ell}))\\
&\quad=-(\mathbf{V}_{\ell}(s^{\ell};t^{\ell},x^{\ell},v^{\ell}),F(\mathbf{X}_{\ell}(s^{\ell};t^{\ell},x^{\ell},v^{\ell}),\mathbf{V}_{\ell}(s^{\ell};t^{\ell},x^{\ell},v^{\ell}))\\
&\quad=(-\mathbf{v}_{\perp}(s^{\ell}),-\mathbf{v}_{\parallel}(s^{\ell}),O_{\xi}(1)|v|^{2},O_{\xi}(1)|v|^{2}).
\end{aligned}
$$

Now we turn to other entries in (6.33). From the characteristics ODE (6.13) in the $\mathbf{p}^{\ell}-$spherical coordinate (6.16), (6.17), and (6.18), we deduce (6.33) for $|v||s^{\ell}-t^{\ell}|\lesssim 1$.

Step 3. From $t-$ plane to the first bounce

We choose $s^{1}\in(t^{1},\frac{t^{1}+t}{2})\subset(t^{1},t)$ such that $|v||t^{1}-s^{1}|\ll 1$ and the polar coordinate $(\mathbf{X}_{1}(s^{1}),\mathbf{V}_{1}(s^{1}))$ is well-defined. More precisely we choose $0<\Delta$ such that $|v||t-\Delta-t^{1}|\ll 1$ and define

$$
s^{1}:=t-\Delta. \tag{6.34}
$$

We first follow the flow in the cartesian coordinate to near the boundary at s^{1}, change to the chart to $\mathbf{p}^{\ell}$-spherical coordinate, then follow the flow in that coordinate.

Then, by the chain rule,

$$
\begin{aligned}
&\frac{\partial(t^{1},\mathbf{x}^{1}_{\parallel_1},\mathbf{v}^{1}_{\perp_1},\mathbf{v}^{1}_{\parallel_1})}{\partial(t,x,v)}\\
&\quad=\frac{\partial(t^{1},\mathbf{x}^{1}_{\parallel_1},\mathbf{v}^{1}_{\perp_1},\mathbf{v}^{1}_{\parallel_1})}{\partial(s^{1},X_{\mathbf{cl}}(s^{1}),V_{\mathbf{cl}}(s^{1}))}\frac{\partial(s^{1},X_{\mathbf{cl}}(s^{1}),V_{\mathbf{cl}}(s^{1}))}{\partial(t,x,v)}\\
&\quad=\frac{\partial(t^{1},\mathbf{x}^{1}_{\parallel_1},\mathbf{v}^{1}_{\perp_1},\mathbf{v}^{1}_{\parallel_1})}{\partial(s^{1},\mathbf{x}_{\perp_1}(s^{1}),\mathbf{x}_{\parallel_1}(s^{1}),\mathbf{v}_{\perp_1}(s^{1}),\mathbf{v}_{\parallel_1}(s^{1}))}\frac{\partial(s^{1},\mathbf{X}_{1}(s^{1}),\mathbf{V}_{1}(s^{1}))}{\partial(s^{1},X_{\mathbf{cl}}(s^{1}),V_{\mathbf{cl}}(s^{1}))}\\
&\qquad\times\frac{\partial(s^{1},X_{\mathbf{cl}}(s^{1}),V_{\mathbf{cl}}(s^{1}))}{\partial(t,x,v)}.
\end{aligned}
$$

We fix $\mathbf{p}^{1}-$spherical coordinate and drop the index of the chart.

Firstly, we claim

$$\frac{\partial(t^1, \mathbf{x}_\parallel^1, \mathbf{v}_\perp^1, \mathbf{v}_\parallel^1)}{\partial(s^1, \mathbf{x}_\perp(s^1), \mathbf{x}_\parallel(s^1), \mathbf{v}_\perp(s^1), \mathbf{v}_\parallel(s^1))}$$
$$\lesssim_\Omega \left[\begin{array}{c|cc|cc} 1 & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{|v|^2|s^1-t^1|^2}{|\mathbf{v}_\perp^1|} & \frac{|s^1-t^1|}{|\mathbf{v}_\perp^1|} & \frac{|v||s^1-t^1|^2}{|\mathbf{v}_\perp^1|} \\ \hline \mathbf{0}_{2,1} & \frac{|v|}{|\mathbf{v}_\perp^1|} + |v|^2|s^1-t^1|^2 & \mathbf{Id}_{2,2} + |v||s^1-t^1| & \frac{|s^1-t^1||v|}{|\mathbf{v}_\perp^1|} + |s^1-t^1|^2|v| & |s^1-t^1| \\ \hline 0 & \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v|^2|s^1-t^1| & \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v|^2|s^1-t^1| & 1+|v||s^1-t^1| & |v||s^1-t^1| \\ \mathbf{0}_{2,1} & \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v|^2|s^1-t^1| & |v|^2|s^1-t^1| & 1+|v||s^1-t^1| & \mathbf{Id}_2 + |v||s^1-t^1| \end{array}\right]. \tag{6.35}$$

Note that since ODE is autonomous we have $\frac{\partial t^1}{\partial s} = 1$, $\frac{\partial(\mathbf{x}^1, \mathbf{v}^1)}{\partial s^1} = 0$.

The t^1 is determined via $\mathbf{x}_\perp(t^1) = 0$, i.e.

$$0 = \mathbf{x}_\perp(s^1) - \mathbf{v}_\perp(s^1)(s^1 - t^1) + \int_{t^1}^{s^1}\int_s^{s^1} F_\perp(\mathbf{X}(\tau), \mathbf{V}_{\mathbf{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s, \tag{6.36}$$

where $\mathbf{X}(\tau) = \mathbf{X}(\tau; s^1, \mathbf{X}(s^1; t, x, v), \mathbf{V}(s^1; t, x, v))$, $\mathbf{V}(\tau) = \mathbf{V}(\tau; s^1, \mathbf{X}(s^1; t, x, v), \mathbf{V}(s^1; t, x, v))$. For $\partial \in \{\partial_{\mathbf{x}_\perp(s^1)}, \partial_{\mathbf{x}_\parallel(s^1)}, \partial_{\mathbf{v}_\perp(s^1)}, \partial_{\mathbf{v}_\parallel(s^1)}\}$,

$$\begin{aligned} &\mathbf{v}_\perp(s^1)\partial t^1 - \partial t^1 \int_{t^1}^{s^1} F_\perp(\mathbf{X}(\tau), \mathbf{V}(\tau))\mathrm{d}\tau + \partial \mathbf{x}_\perp(s^1) - \partial \mathbf{v}_\perp(s^1)(s^1 - t^1) \\ &\quad + \int_{t^1}^{s^1}\int_s^{s^1} \{\partial \mathbf{X}(\tau)\cdot\nabla_{\mathbf{X}} F_\perp + \partial \mathbf{V}(\tau)\cdot\nabla_{\mathbf{V}} F_\perp\}(\mathbf{X}(\tau), \mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s = 0. \end{aligned} \tag{6.37}$$

But $\mathbf{v}_\perp^1 = -\lim_{s\downarrow t^1}\mathbf{v}_\perp(s) = -\mathbf{v}_\perp(s^1) + \int_{t^1}^{s^1} F_\perp(\mathbf{X}(\tau), \mathbf{V}(\tau))\mathrm{d}\tau$, we apply Lemma 16 and $|s^1 - t^1| \lesssim_\xi \min\{\frac{|\mathbf{v}_\perp^1|}{|v|^2}, t\}$ and (2.2) and (2.3), to obtain

$$\begin{bmatrix} \frac{\partial t^1}{\partial \mathbf{x}_\perp(s^1)} \\ \frac{\partial t^1}{\partial \mathbf{x}_\parallel(s^1)} \\ \frac{\partial t^1}{\partial \mathbf{v}_\perp(s^1)} \\ \frac{\partial t^1}{\partial \mathbf{v}_\perp(s^1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathbf{v}_\perp^1}\left\{1 + \int_{t^1}^{s^1}\int_s^{s^1} \frac{\partial}{\partial \mathbf{x}_\perp(s^1)} F_\perp(\mathbf{X}(\tau), \mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s\right\} \\ \frac{1}{\mathbf{v}_\perp^1}\int_{t^1}^{s^1}\int_s^{s^1} \frac{\partial}{\partial \mathbf{x}_\parallel(s^1)} F_\perp(\mathbf{X}(\tau), \mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s \\ \frac{1}{\mathbf{v}_\perp^1}\left\{(t^1 - s^1) + \int_{t^1}^{s^1}\int_s^{s^1} \frac{\partial}{\partial \mathbf{v}_\perp(s^1)} F_\perp(\mathbf{X}(\tau), \mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s\right\} \\ \frac{1}{\mathbf{v}_\perp^1}\int_{t^1}^{s_1}\int_s^{s_1} \frac{\partial}{\partial \mathbf{v}_\parallel(s_1)} F_\perp(\mathbf{X}(\tau), \mathbf{V}(\tau))\mathrm{d}\tau\mathrm{d}s \end{bmatrix}$$

$$\lesssim_{\xi,t} \begin{bmatrix} \frac{1}{|\mathbf{v}_\perp^1|} \\ \frac{|v|^2|s^1-t^1|^2}{|\mathbf{v}_\perp^1|} \\ \frac{|s^1-t^1|}{|\mathbf{v}_\perp^1|} \\ \frac{|v||s^1-t^1|^2}{|\mathbf{v}_\perp^1|} \end{bmatrix}.$$

Taking $(\mathbf{x}(s^1), \mathbf{v}(s^1))$ derivatives of the characteristic equations

$$\begin{aligned}
\mathbf{v}_\perp^1 &= -\lim_{s \downarrow t^1} \mathbf{v}_\perp(s) = -\mathbf{v}_\perp(s^1) + \int_{t^1}^{s^1} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau, \\
\mathbf{x}_\parallel^1 &= \mathbf{x}_\parallel(s^1) - \int_{t^1}^{s^1} \mathbf{v}_\parallel(s) \mathrm{d}s, \\
\mathbf{v}_\parallel^1 &= \mathbf{v}_\parallel(s^1) - \int_{t^1}^{s^1} F_\parallel(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau.
\end{aligned}$$

and using the above estimates, (6.37) and Lemma 16 yields

$$\begin{bmatrix} \frac{\partial \mathbf{x}_\parallel^1}{\partial \mathbf{x}_\perp(s^1)} \\ \frac{\partial \mathbf{x}_\parallel^1}{\partial \mathbf{x}_\parallel(s^1)} \\ \frac{\partial \mathbf{x}_\parallel^1}{\partial \mathbf{v}_\perp(s^1)} \\ \frac{\partial \mathbf{x}_\parallel^1}{\partial \mathbf{v}_\parallel(s^1)} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} \frac{|v|}{|\mathbf{v}_\perp^1|} + |v|^2|s^1 - t^1|^2 \\ \mathbf{Id}_{2,2} + |v||s^1 - t^1| \\ \frac{|s^1 - t^1||v|}{|\mathbf{v}_\perp^1|} + |s^1 - t^1|^2|v| \\ |s^1 - t^1| \end{bmatrix},$$

$$\begin{bmatrix} \frac{\partial \mathbf{v}_\perp^1}{\partial \mathbf{x}_\perp(s^1)} \\ \frac{\partial \mathbf{v}_\perp^1}{\partial \mathbf{x}_\parallel(s^1)} \\ \frac{\partial \mathbf{v}_\perp^1}{\partial \mathbf{v}_\perp(s^1)} \\ \frac{\partial \mathbf{v}_\perp^1}{\partial \mathbf{v}_\parallel(s^1)} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v|^2|s^1 - t^1| \\ \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v|^2|s^1 - t^1| \\ 1 + |v||s^1 - t^1| \\ |v||s^1 - t^1| \end{bmatrix},$$

and

$$\begin{bmatrix} \frac{\partial \mathbf{v}_\parallel^1}{\partial \mathbf{x}_\perp(s^1)} \\ \frac{\partial \mathbf{v}_\parallel^1}{\partial \mathbf{x}_\parallel(s^1)} \\ \frac{\partial \mathbf{v}_\parallel^1}{\partial \mathbf{v}_\perp(s^1)} \\ \frac{\partial \mathbf{v}_\parallel^1}{\partial \mathbf{v}_\parallel(s^1)} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v|^2|s^1 - t^1| \\ |v|^2|s^1 - t^1| \\ 1 + |v||s^1 - t^1| \\ \mathbf{Id}_{2,2} + |v||s^1 - t^1| \end{bmatrix}.$$

Secondly, we claim

$$\frac{\partial(\mathbf{X}_1(s^1), \mathbf{V}_1(s^1))}{\partial(t, x, v)} = \frac{\partial(\mathbf{X}_1(s^1), \mathbf{V}_1(s^1))}{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))} \frac{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(t, x, v)}$$

$$
= \left[\begin{array}{c|c|c}
\mathbf{0}_{3,1} & \frac{(\partial_1\eta\times\partial_2\eta)^T}{\mathbf{n}\cdot(\partial_1\eta\times\partial_2\eta)} + O_\xi(|v\|t^1-s^1|) & O_\xi(|t-s^1|) \\
 & \frac{(\partial_2\eta\times\mathbf{n})^T}{\mathbf{n}\cdot(\partial_1\eta\times\partial_2\eta)} + O_\xi(|v\|t^1-s^1|) & O_\xi(|t-s^1|) \\
 & \frac{(\mathbf{n}\times\partial_2\eta)^T}{\mathbf{n}\cdot(\partial_1\eta\times\partial_2\eta)} + O_\xi(|v\|t^1-s^1|) & O_\xi(|t-s^1|) \\ \hline
\mathbf{0}_{3,1} & O_\xi(|v|) & \frac{(\partial_1\eta\times\partial_2\eta)^T}{\mathbf{n}\cdot(\partial_1\eta\times\partial_2\eta)} + O_\xi(|v||t-s^1|) \\
 & O_\xi(|v|) & \frac{(\partial_2\eta\times\mathbf{n})^T}{\mathbf{n}\cdot(\partial_1\eta\times\partial_2\eta)} + O_\xi(|v||t-s^1|) \\
 & O_\xi(|v|) & \frac{(\mathbf{n}\times\partial_1\eta)^T}{\mathbf{n}\cdot(\partial_1\eta\times\partial_2\eta)} + O_\xi(|v||t-s^1|)
\end{array} \right]_{6\times 7}, \tag{6.38}
$$

where the entries are evaluated at $(\mathbf{X}_1(s^1), \mathbf{V}_1(s^1))$. Note that $|v\|t^1-s^1| \lesssim_\xi 1$.

From (6.5)

$$
\frac{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}(s^1), \mathbf{V}(s^1))} = \frac{\partial\Phi(\mathbf{X}(s^1), \mathbf{V}(s))}{\partial(\mathbf{X}(s^1), \mathbf{V}(s))} := \left[\begin{array}{c|c} A & \mathbf{0}_{3,3} \\ B & A \end{array}\right] + \mathbf{x}_\perp \left[\begin{array}{c|c} \mathbf{0}_{3,3} & \mathbf{0}_{3,3} \\ D & \mathbf{0}_{3,3} \end{array}\right].
$$

Note that, from (6.9) and (6.3),

$$
\begin{aligned}
\det(A) &= \det[\,[-\mathbf{n}(\mathbf{x}_\parallel)] \quad \partial_{\mathbf{x}_{\parallel,1}}\eta(\mathbf{x}_\parallel) \quad \partial_{\mathbf{x}_{\parallel,2}}\eta(\mathbf{x}_\parallel)\,] \\
&= [-\mathbf{n}(\mathbf{x}_\parallel)] \cdot (\partial_{\mathbf{x}_{\parallel,1}}\eta(\mathbf{x}_\parallel) \times \partial_{\mathbf{x}_{\parallel,2}}\eta(x_\parallel)) \neq 0, \\
A^{-1} &= \frac{1}{[-\mathbf{n}]\cdot(\partial_{\mathbf{x}_{\parallel,1}}\eta \times \partial_{\mathbf{x}_{\parallel,2}}\eta)} \\
&\quad \times [(\partial_{\mathbf{x}_{\parallel,1}}\eta \times \partial_{\mathbf{x}_{\parallel,2}}\eta)^T, (\partial_{\mathbf{x}_{\parallel,2}}\eta \times [-\mathbf{n}])^T, ([-\mathbf{n}] \times \partial_{\mathbf{x}_{\parallel,1}}\eta)^T].
\end{aligned}
$$

From basic linear algebra

$$
\begin{aligned}
\det\left(\frac{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1), \mathbf{V}_{\mathbf{cl}}(s^1))}\right) &= \det\left[\begin{array}{c|c} A & \mathbf{0}_{3,3} \\ \hline B + \mathbf{x}_\perp D & A \end{array}\right] \\
&= \{\det(A)\}^2 = \{[-\mathbf{n}]\cdot(\partial_1\eta \times \partial_2\eta)\}^2,
\end{aligned}
$$

and $\left(\frac{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1), \mathbf{V}_{\mathbf{cl}}(s^1))}\right)$ is invertible. From basic linear algebra

$$
\begin{aligned}
\frac{\partial(\mathbf{X}_{\mathbf{cl}}(s^1), \mathbf{V}_{\mathbf{cl}}(s^1))}{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))} &= \left[\frac{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1), \mathbf{V}_{\mathbf{cl}}(s^1))}\right]^{-1} = \left[\begin{array}{c|c} A & \mathbf{0}_{3,3} \\ \hline B + \mathbf{x}_\perp D & A \end{array}\right]^{-1} \\
&= \left[\begin{array}{c|c} A^{-1} & \mathbf{0}_{3,3} \\ \hline -A^{-1}(B + \mathbf{x}_\perp D)A^{-1} & A^{-1} \end{array}\right] = \left[\begin{array}{cc} A^{-1}(\mathbf{x}_\parallel) & \mathbf{0}_{3,3} \\ |v| + O_\xi(\mathbf{x}_\perp) & A^{-1}(\mathbf{x}_\parallel) \end{array}\right],
\end{aligned} \tag{6.39}
$$

and we obtain

$$\frac{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}=\left[\begin{array}{c|c} \begin{matrix}\frac{(\partial_1\eta\times\partial_2\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}\\ \frac{(\partial_2\eta\times[-\mathbf{n}])^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}\\ \frac{([-\mathbf{n}]\times\partial_1\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}\end{matrix} & \mathbf{0}_{3,3} \\ \hline \begin{matrix}O_\xi(1)(|v|)\\ O_\xi(1)(|v|)\\ O_\xi(1)(|v|)\end{matrix} & \begin{matrix}\frac{(\partial_1\eta\times\partial_2\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}\\ \frac{(\partial_2\eta\times[-\mathbf{n}])^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}\\ \frac{([-\mathbf{n}]\times\partial_1\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}\end{matrix}\end{array}\right].$$

From $X_{\mathbf{cl}}(s_1;t,x,v)=x-(t-s_1)v=x-\Delta\times v$ and $V_{\mathbf{cl}}(s_1;t,x,v)=v$,

$$\frac{\partial(X_{\mathbf{cl}}(s_1),V_{\mathbf{cl}}(s_1))}{\partial(t,x,v)}=\begin{bmatrix}\mathbf{0}_{3,1} & \mathbf{Id}_{3,3} & -(t-s^1)\mathbf{Id}_{3,3}\\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3}\end{bmatrix}.$$

Finally we multiply the above two matrices and use $|\mathbf{x}_\perp(s^1)|\lesssim|v||t^1-s^1|$ to conclude the second claim (6.38).

Step 4. Estimate of $\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})/\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})$

Recall $\mathbf{r}^\ell$ from (6.24). We show that there exists $M=M_{\xi,t}\gg 1$, which only depends on Ω, such that for all $\ell\in\mathbb{N}$ and $0\le t^{\ell+1}\le t^\ell\le t$ and $v\in\mathbb{R}^3$,

$$\begin{aligned} J^{\ell+1}_\ell &:= \frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})}\\ &\le \left[\begin{array}{c|ccc|cc} 1 & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|^2} & \frac{M}{|v|^2}\mathbf{r}^{\ell+1} & \frac{M}{|v|^2}\mathbf{r}^{\ell+1}\\ \hline 0 & 1+M\mathbf{r}^{\ell+1} & M\mathbf{r}^{\ell+1} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1}\\ 0 & M\mathbf{r}^{\ell+1} & 1+M\mathbf{r}^{\ell+1} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1}\\ \hline 0 & M|v|(\mathbf{r}^{\ell+1})^2 & M|v|(\mathbf{r}^{\ell+1})^2 & 1+M\mathbf{r}^{\ell+1} & M(\mathbf{r}^{\ell+1})^2 & M(\mathbf{r}^{\ell+1})^2\\ 0 & M|v|\mathbf{r}^{\ell+1} & M|v|\mathbf{r}^{\ell+1} & M & 1+M\mathbf{r}^{\ell+1} & M\mathbf{r}^{\ell+1}\\ 0 & M|v|\mathbf{r}^{\ell+1} & M|v|\mathbf{r}^{\ell+1} & M & M\mathbf{r}^{\ell+1} & 1+M\mathbf{r}^{\ell+1} \end{array}\right]\\ &:= \underbrace{J(\mathbf{r}^{\ell+1})}_{\text{Definition of } J(\mathbf{r}^{\ell+1})}. \end{aligned} \tag{6.40}$$

We also denote the Jacobian matrix within a single $\mathbf{p}^\ell-$ spherical coordinate:

$$\tilde{J}^{\ell+1}_\ell := \frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_\ell},\mathbf{v}^{\ell+1}_{\perp_\ell},\mathbf{v}^{\ell+1}_{\|_\ell})}{\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})}.$$

Note this bound (6.40) holds for both *Type I* and *Type II* in (6.25). We split the proof for each *Type*:

Proof of (6.40) *when* $\mathbf{r}^\ell < \sqrt{\delta}$ *and* $\mathbf{r}^{\ell+1} < \sqrt{\delta}$: Note that $\mathbf{p}^\ell-$spherical coordinate is well-defined of all $\tau \in [t^{\ell+1}, t^\ell]$. Due to the chart changing

$$\begin{aligned}
&\frac{\partial(t^{\ell+1}, \mathbf{x}^{\ell+1}_{\|_{\ell+1}}, \mathbf{v}^{\ell+1}_{\perp_{\ell+1}}, \mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^\ell, \mathbf{x}^\ell_{\|_\ell}, \mathbf{v}^\ell_{\perp_\ell}, \mathbf{v}^\ell_{\|_\ell})} \\
&\quad = \left[\begin{array}{c|c} 1 & \mathbf{0}_{1,5} \\ \hline \mathbf{0}_{5,1} & \frac{\partial(\mathbf{x}^{\ell+1}_{\|_{\ell+1}}, \mathbf{v}^{\ell+1}_{\perp_{\ell+1}}, \mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(\mathbf{x}^\ell_{\|_\ell}, \mathbf{v}^\ell_{\perp_\ell}, \mathbf{v}^\ell_{\|_\ell})} \end{array}\right] \underbrace{\frac{\partial(t^{\ell+1}, 0, \mathbf{x}^{\ell+1}_{\|_\ell}, \mathbf{v}^{\ell+1}_{\perp_\ell}, \mathbf{v}^{\ell+1}_{\|_\ell})}{\partial(t^\ell, 0, \mathbf{x}^\ell_{\|_\ell}, \mathbf{v}^\ell_{\perp_\ell}, \mathbf{v}^\ell_{\|_\ell})}}_{=\tilde{J}^{\ell+1}_\ell}.
\end{aligned}$$

where $\frac{\partial(\mathbf{x}^{\ell+1}_{\|_{\ell+1}}, \mathbf{v}^{\ell+1}_{\perp_{\ell+1}}, \mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(\mathbf{x}^\ell_{\|_\ell}, \mathbf{v}^\ell_{\perp_\ell}, \mathbf{v}^\ell_{\|_\ell})}$ is the 5×5 right lower submatrix of (6.6).

Note that $|\mathbf{p}^\ell - \mathbf{p}^{\ell+1}| \lesssim_\xi \mathbf{r}^\ell$ from $\mathbf{r}^\ell \leq C\sqrt{\delta}$, (6.26) and (2.2). In order to show (6.40) it suffices to show that $\tilde{J}^{\ell+1}_\ell$ is bounded as (6.40):

$$\tilde{J}^{\ell+1}_\ell \leq J(\mathbf{r}^{\ell+1}). \tag{6.41}$$

This is due to the following matrix multiplication

$$\begin{aligned}
&\left[\begin{array}{c|c} 1 & \mathbf{0}_{1,5} \\ \hline \mathbf{0}_{5,1} & \frac{\partial(\mathbf{x}^{\ell+1}_{\|_{\ell+1}}, \mathbf{v}^{\ell+1}_{\perp_{\ell+1}}, \mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(\mathbf{x}^\ell_{\|_\ell}, \mathbf{v}^\ell_{\perp_\ell}, \mathbf{v}^\ell_{\|_\ell})} \end{array}\right] \tilde{J}^{\ell+1}_\ell \\
&\leq \left[\begin{array}{c|cc|ccc} 1 & \multicolumn{2}{c|}{\mathbf{0}_{1,2}} & \multicolumn{3}{c}{\mathbf{0}_{1,3}} \\ \hline \mathbf{0}_{2,1} & 1 + C\mathbf{r}^{\ell+1} & C\mathbf{r}^{\ell+1} & \multicolumn{3}{c}{\mathbf{0}_{3,3}} \\ & C\mathbf{r}^{\ell+1} & 1 + C\mathbf{r}^{\ell+1} & & & \\ \hline & 0 & 0 & 1 & 0 & 0 \\ \mathbf{0}_{3,1} & C\mathbf{r}^{\ell+1}|v| & C\mathbf{r}^{\ell+1}|v| & 0 & 1 + C\mathbf{r}^{\ell+1} & C\mathbf{r}^{\ell+1} \\ & C\mathbf{r}^{\ell+1}|v| & C\mathbf{r}^{\ell+1}|v| & 0 & C\mathbf{r}^{\ell+1} & 1 + C\mathbf{r}^{\ell+1} \end{array}\right] J(\mathbf{r}^{\ell+1}) \\
&\leq J(C\mathbf{r}^{\ell+1}),
\end{aligned}$$

where we used (6.6) with an adjusted constant $C > 0$.

Now we prove the claim (6.41). We fix the $\mathbf{p}^\ell-$spherical coordinate and drop the index ℓ for the chart.

If $\mathbf{v}^\ell_\perp = 0$ then $t^{\ell+1} = t^\ell$. Otherwise if $\mathbf{v}^\ell_\perp \neq 0$ then $t^{\ell+1}$ is determined through

$$0 = \mathbf{v}^\ell_\perp (t^{\ell+1} - t^\ell) + \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} F_\perp(\mathbf{X}_\ell(\tau; t^\ell, x^\ell, v^\ell), \mathbf{V}_\ell(\tau; t^\ell, x^\ell, v^\ell)) \mathrm{d}\tau \mathrm{d}s. \tag{6.42}$$

Since the ODE for $[\mathbf{X}_\ell(\tau; t, x, v), \mathbf{V}_\ell(\tau; t, x, v)]$ is autonomous,

$$
\begin{aligned}
0 = \mathbf{v}_\perp^\ell (t^{\ell+1} - t^\ell)& \\
&+ \int_0^{t^\ell - t^{\ell+1}} \int_{t^{\ell+1} - t^\ell + s}^{0} F_\perp(\mathbf{X}_\ell(\tau; 0, x^\ell, v^\ell), \mathbf{V}_\ell(\tau; 0, x^\ell, v^\ell)) \mathrm{d}\tau \mathrm{d}s.
\end{aligned}
$$

We take t^ℓ-derivative to have

$$
\begin{aligned}
0 &= \frac{\partial (t^{\ell+1} - t^\ell)}{\partial t^\ell} \left\{ \mathbf{v}_\perp^\ell - \int_0^{t^\ell - t^{\ell+1}} F_\perp(\mathbf{X}_\ell, \mathbf{V}_\ell)(t^{\ell+1} - t^\ell + s; 0, x^\ell, v^\ell) \mathrm{d}s \right\}, \\
&= \frac{\partial (t^{\ell+1} - t^\ell)}{\partial t^\ell} \left\{ \mathbf{v}_\perp^\ell - \int_{t^{\ell+1}}^{t^\ell} F_\perp(\mathbf{X}_\ell(s; t^\ell, x^\ell, v^\ell), \mathbf{V}_\ell(s; t^\ell, x^\ell, v^\ell)) \mathrm{d}s \right\} \\
&= \frac{\partial (t^{\ell+1} - t^\ell)}{\partial t^\ell} (-\mathbf{v}_\perp^{\ell+1}),
\end{aligned}
$$

where we used the definition

$$
\mathbf{v}_\perp^{\ell+1} = - \lim_{s \downarrow t^{\ell+1}} \mathbf{v}_\perp(s) = -\mathbf{v}_\perp^\ell + \int_{t^{\ell+1}}^{t^\ell} F_\perp(\mathbf{X}(\tau; t, x, v), \mathbf{V}(\tau; t, x, v)) \mathrm{d}\tau. \tag{6.43}
$$

Therefore we conclude $\frac{\partial t^{\ell+1}}{\partial t^\ell} = 1$. Then combining with

$$
\mathbf{x}_\parallel^{\ell+1} = \mathbf{x}_\parallel^\ell + \int_0^{t^{\ell+1} - t^\ell} \mathbf{v}_\parallel(s; 0, x^\ell, v^\ell) \mathrm{d}s, \ \mathbf{v}^{\ell+1} = \mathbf{v}^\ell + \int_0^{t^{\ell+1} - t^\ell} F(s; 0, x^\ell, v^\ell) \mathrm{d}s, \tag{6.44}
$$

we conclude $\frac{\partial \mathbf{x}_\parallel^{\ell+1}}{\partial t^\ell} = \frac{\partial \mathbf{v}_\parallel^{\ell+1}}{\partial t^\ell} = \frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial t^\ell} = 0$.

Taking derivatives of (6.42) as before and using $|t^\ell - t^{\ell+1}| \lesssim_{\xi, t} \min\{\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|^2}, 1\}$, from (2.3), and Lemma 16, we obtain

$$
\begin{bmatrix} \frac{\partial t^{\ell+1}}{\partial \mathbf{x}_\parallel^\ell} \\ \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\perp^\ell} \\ \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\parallel^\ell} \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathbf{v}_\perp^{\ell+1}} \int_{t^\ell}^{t^{\ell+1}} \int_{t^\ell}^{s} \frac{\partial}{\partial \mathbf{x}_\parallel^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \mathrm{d}s \\ \frac{1}{\mathbf{v}_\perp^{\ell+1}} \left\{ (t^{\ell+1} - t^\ell) + \int_{t^\ell}^{t^{\ell+1}} \int_{t^\ell}^{s} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \mathrm{d}s \right\} \\ \frac{1}{\mathbf{v}_\perp^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_{s}^{t^\ell} \frac{\partial}{\partial \mathbf{v}_\parallel^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \mathrm{d}s \end{bmatrix}
$$

$$\lesssim_{\xi,t} \begin{bmatrix} \frac{1}{|v|}\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|} \\ \frac{1}{|v|^2} \\ \frac{1}{|v|^2}\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|} \end{bmatrix}. \tag{6.45}$$

Taking $(\mathbf{x}(t^\ell), \mathbf{v}(t^\ell))$ derivatives of the characteristic equations (6.44), by Lemma 16 and (6.45), we estimate directly

$$\begin{bmatrix} \frac{\partial \mathbf{x}_\parallel^{\ell+1}}{\partial \mathbf{x}_\parallel^\ell} \\ \frac{\partial \mathbf{x}_\parallel^{\ell+1}}{\partial \mathbf{v}_\perp^\ell} \\ \frac{\partial \mathbf{x}_\parallel^{\ell+1}}{\partial \mathbf{v}_\parallel^\ell} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} \mathbf{Id}_{2,2} + \frac{|\mathbf{v}_\perp^\ell|}{|v|} \\ \frac{1}{|v|} \\ \frac{1}{|v|}\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|} \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial \mathbf{v}_\parallel^{\ell+1}}{\partial \mathbf{x}_\parallel^\ell} \\ \frac{\partial \mathbf{v}_\parallel^{\ell+1}}{\partial \mathbf{v}_\perp^\ell} \\ \frac{\partial \mathbf{v}_\parallel^{\ell+1}}{\partial \mathbf{v}_\parallel^\ell} \end{bmatrix} \lesssim_{\xi,t} \begin{bmatrix} |\mathbf{v}_\perp^{\ell+1}| \\ 1 + \frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^\ell|} \\ \mathbf{Id}_{2,2} + \frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^\ell|} \end{bmatrix}.$$

Now we move to $D\mathbf{v}_\perp^{\ell+1}$ estimates. First we claim the crucial estimate of $t^\ell - t^{\ell+1}$:

$$(t^\ell - t^{\ell+1})F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell) = 2\mathbf{v}_\perp^{\ell+1} + O_\xi(1)|t^\ell - t^{\ell+1}|^2|v^\ell|^3. \tag{6.46}$$

As (6.42), we use the fact $\mathbf{x}_\perp^\ell = 0 = \mathbf{x}_\perp^{\ell+1}$ and the definition $\mathbf{v}_\perp^{\ell+1} = -\lim_{s\downarrow t^{\ell+1}} \mathbf{v}_\perp(s)$ and

$$\begin{aligned} \dot{\mathbf{v}}_\perp(s) &= F_\perp(\mathbf{X}_\ell(s; t^\ell, \mathbf{x}^\ell, \mathbf{v}^\ell), \mathbf{V}_\ell(s; t^\ell, \mathbf{x}^\ell, \mathbf{v}^\ell)) \\ &= F_\perp(\mathbf{X}_\ell(s; t^{\ell+1}, \mathbf{x}^{\ell+1}, \mathbf{v}^{\ell+1}), \mathbf{V}_\ell(s; t^{\ell+1}, \mathbf{x}^{\ell+1}, \mathbf{v}^{\ell+1})), \end{aligned}$$

to conclude the similar identity of (6.42)

$$\begin{aligned} 0 = &-\mathbf{v}_\perp^{\ell+1}(t^\ell - t^{\ell+1}) \\ &+ \int_{t^{\ell+1}}^{t^\ell}\int_{t^{\ell+1}}^{s} F_\perp(\mathbf{X}_\ell(\tau; t^{\ell+1}, \mathbf{x}^{\ell+1}, \mathbf{v}^{\ell+1}), \mathbf{V}_\ell(\tau; t^{\ell+1}, \mathbf{x}^{\ell+1}, \mathbf{v}^{\ell+1}))\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{6.47}$$

By Lemma 16, $F_\perp(\mathbf{X}_\ell(\tau; t^\ell, \mathbf{x}^\ell, \mathbf{v}^\ell), \mathbf{V}_\ell(\tau; t^\ell, \mathbf{x}^\ell, \mathbf{v}^\ell)) = F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell) + O_\xi(1)|t^{\ell+1} - t^\ell\|v|^3$. Plugging this into (6.47) we have

$$0 = -\mathbf{v}_\perp^{\ell+1}(t^\ell - t^{\ell+1}) + \frac{1}{2}(t^\ell - t^{\ell+1})^2 F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell) + O_\xi(1)|t^\ell - t^{\ell+1}|^3|v|^3,$$

and this proves our claim (6.46).

Taking derivatives in (6.43), from the extra cancellation in terms of order of $t^\ell - t^{\ell+1}$ in (6.46), by (6.45), we obtain

$$
\begin{aligned}
\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial \mathbf{x}_\parallel^\ell} &= \frac{-F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_s^{t^\ell} \frac{\partial}{\partial \mathbf{x}_\parallel^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \mathrm{d}s \\
&\quad + \int_{t^{\ell+1}}^{t^\ell} \frac{\partial}{\partial \mathbf{x}_\parallel^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \\
&= \left\{ \frac{(t^\ell - t^{\ell+1}) F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell)}{-2\mathbf{v}_\perp^{\ell+1}} + 1 \right\} (t^\ell - t^{\ell+1}) \frac{\partial}{\partial \mathbf{x}_\parallel^\ell} F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell) \\
&\quad + O_\xi(1) \left\{ |t^\ell - t^{\ell+1}|^2 |v^\ell|^3 + \frac{|t^\ell - t^{\ell+1}|^3 |v^\ell|^3}{|\mathbf{v}_\perp^{\ell+1}|} |\frac{\partial}{\partial \mathbf{x}_\parallel^\ell} F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell)| \right\} \\
&\lesssim_\xi \left\{ -1 + O_\xi(1) \frac{|t^\ell - t^{\ell+1}|^2 |v^\ell|^3}{|\mathbf{v}_\perp^{\ell+1}|} + 1 \right\} |t^\ell - t^{\ell+1}| |v^\ell|^2 \\
&\quad + |t^\ell - t^{\ell+1}|^2 |v^\ell|^3 \left\{ 1 + \frac{|t^\ell - t^{\ell+1}\| v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|} \right\} \\
&\lesssim_\xi |t^\ell - t^{\ell+1}|^2 |v^\ell|^3 \left(1 + \frac{|t^\ell - t^{\ell+1}\| v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|} \right) \lesssim_\xi |t^\ell - t^{\ell+1}|^2 |v^\ell|^3 \\
&\lesssim_{\xi, t} \frac{|\mathbf{v}_\perp^{\ell+1}|^2}{|v^\ell|},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial \mathbf{v}_\perp^\ell} &= -1 - \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell) + \int_{t^{\ell+1}}^{t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \\
&= -1 + \frac{F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} (t^\ell - t^{\ell+1}) - \frac{F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} \\
&\quad \times \int_{t^\ell}^{t^{\ell+1}} \int_{t^\ell}^{s} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \mathrm{d}s + \int_{t^{\ell+1}}^{t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) \mathrm{d}\tau \\
&= -1 + 2 + O_\xi(1) \frac{|t^\ell - t^{\ell+1}|^2 |v^\ell|^3}{\mathbf{v}_\perp^{\ell+1}} - \frac{F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} \frac{(t^\ell - t^{\ell+1})^2}{2} \\
&\quad \times \left\{ \lim_{s \uparrow t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) + O_\xi(1) |t^\ell - t^{\ell+1}\| v^\ell|^2 \right\} \\
&\quad + (t^\ell - t^{\ell+1}) \left\{ \lim_{s \uparrow t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) + O_\xi(1) |t^\ell - t^{\ell+1}\| v^\ell|^2 \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= 1 + O_\xi(1)\left\{\frac{|t^\ell - t^{\ell+1}|^2|v^\ell|^3}{|\mathbf{v}_\perp^{\ell+1}|} + |t^\ell - t^{\ell+1}|^2|v^\ell|^2\right.\\
&\qquad \left.+\frac{|t^\ell - t^{\ell+1}|^3}{|\mathbf{v}_\perp^{\ell+1}|}|v^\ell|^3\left|\lim_{s\uparrow t^\ell}\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\right|\right\}\\
&\lesssim 1 + |t^\ell - t^{\ell+1}|^2|v^\ell|^2\left\{1 + \frac{|v^\ell|}{|\mathbf{v}_\perp^{\ell+1}|} + \frac{|t^\ell - t^{\ell+1}||v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|}\right\} \lesssim_{\xi,t} 1 + \frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^\ell|},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial \mathbf{v}_\parallel^\ell} &= \frac{-F_\perp(\mathbf{x}^\ell, v^\ell)}{\mathbf{v}_\perp^{\ell+1}}\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\quad - \int_{t^{\ell+1}}^{t^\ell}\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{X}_\ell(\tau),\mathbf{V}_\ell(\tau))\mathrm{d}\tau\\
&= \left\{\frac{(t^\ell - t^{\ell+1})F_\perp(\mathbf{x}^\ell,\mathbf{v}^\ell)}{-2\mathbf{v}_\perp^{\ell+1}} + 1\right\}(t^\ell - t^{\ell+1})\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{x}^\ell,\mathbf{v}^\ell)\\
&\quad + O_\xi(1)|t^\ell - t^{\ell+1}|^2|v^\ell|^2\left\{\frac{|F_\perp(\mathbf{x}^\ell,\mathbf{v}^\ell)||t^\ell - t^{\ell+1}|}{|\mathbf{v}_\perp^{\ell+1}|} + 1\right\}\\
&\lesssim_\xi |t^\ell - t^{\ell+1}|^2|v^\ell|^2\left\{1 + \frac{|t^\ell - t^{\ell+1}||v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|}\right\} \lesssim_{\xi,t} \frac{|\mathbf{v}_\perp^{\ell+1}|^2}{|v^\ell|^2}.
\end{aligned}
\tag{6.48}
$$

These estimates complete the proof of the claims (6.40) and (6.41) when $\mathbf{r}^\ell < \sqrt{\delta}$ and $\mathbf{r}^{\ell+1} < \sqrt{\delta}$.

Proof of (6.40) *for either* $\mathbf{r}^\ell \geq \sqrt{\delta}$ *or* $\mathbf{r}^{\ell+1} \geq \sqrt{\delta}$: Without loss of generality we assume $\mathbf{r}^\ell > C\sqrt{\delta}$ in (6.25). Recall that we chose a $\mathbf{p}^\ell-$spherical coordinate as $\mathbf{p}^\ell = (z^\ell, w^\ell)$ with $|z^\ell - x^\ell| \leq \sqrt{\delta}$ and any $w^\ell \in \mathbb{S}^2$ with $n(z^\ell)\cdot w^\ell = 0$.

Fix ℓ. Let us choose fixed numbers $\Delta_1, \Delta_2 > 0$ such that $|v|\Delta_1 \ll 1$ as well as also $|v||t^{\ell+1} - (t^\ell - \Delta_1 - \Delta_2)| \ll 1$ so that

$$
s^\ell \equiv t^\ell - \Delta_1, \quad s^{\ell+1} \equiv s^\ell - \Delta_2 = t^\ell - \Delta_1 - \Delta_2,
$$

satisfying $|v||t^{\ell+1} - s^{\ell+1}| = |v||t^{\ell+1} - (t^\ell - \Delta_1 - \Delta_2)| \ll 1$ as well as also $|v||t^\ell - s^\ell| = |v||\Delta_1| \ll 1$ so that the spherical coordinates are well-defined for $s \in [t^{\ell+1}, s^{\ell+1}]$ and $s \in [s^\ell, t^\ell]$.

Notice that

$$\frac{\partial t^{\ell+1}}{\partial s^{\ell+1}}=\frac{\partial(s^{\ell+1}+\Delta_1+\Delta_2-t_{\mathbf{b}}(x^\ell,v^\ell))}{\partial s^{\ell+1}}=1,\quad \frac{\partial s^{\ell+1}}{\partial s^\ell}=\frac{\partial(s^\ell-\Delta_1)}{\partial s^\ell}=1,$$
$$\frac{\partial s^\ell}{\partial t^\ell}=\frac{\partial(t^\ell-\Delta_1)}{\partial t^\ell}=1.$$

We first follow the flow in $\mathbf{p}^\ell-$spherical coordinate, then change to the Euclidian coordinate to near the boundary s^ℓ, follow the flow until $s^{\ell+1}$, and then change to the chart to $\mathbf{p}^{\ell+1}-$spherical coordinate. By the chain rule,

$$\begin{aligned}
&\frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})}\\
&=\frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(s^{\ell+1},\mathbf{x}_{\perp_{\ell+1}}(s^{\ell+1}),\mathbf{x}_{\|_{\ell+1}}(s^{\ell+1}),\mathbf{v}_{\perp_{\ell+1}}(s^{\ell+1}),\mathbf{v}_{\|_{\ell+1}}(s^{\ell+1}))}\\
&\quad\times\frac{\partial(s^{\ell+1},\mathbf{X}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}),\mathbf{V}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}))}{\partial(s^{\ell+1},X_{\mathbf{cl}}(s^{\ell+1}),V_{\mathbf{cl}}(s^{\ell+1}))}\\
&\quad\times\frac{\partial(s^{\ell+1},X_{\mathbf{cl}}(s^{\ell+1}),V_{\mathbf{cl}}(s^{\ell+1}))}{\partial(s^\ell,X_{\mathbf{cl}}(s^\ell),V_{\mathbf{cl}}(s^\ell))}\\
&\quad\times\frac{\partial(s^\ell,X_{\mathbf{cl}}(s^\ell),V_{\mathbf{cl}}(s^\ell))}{\partial(s^\ell,\mathbf{X}_{\mathbf{p}^\ell}(s^\ell),\mathbf{V}_{\mathbf{p}^\ell}(s^\ell))}\frac{\partial(s^\ell,\mathbf{x}_{\perp_\ell}(s^\ell),\mathbf{x}_{\|_\ell}(s^\ell),\mathbf{v}_{\perp_\ell}(s^\ell),\mathbf{v}_{\|_\ell}(s^\ell))}{\partial(t^\ell,\mathbf{x}^\ell_{\|_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\|_\ell})}.
\end{aligned}$$

We can express that $t^{\ell+1}=t^\ell-t_{\mathbf{b}}(x^\ell,v^\ell)=s^{\ell+1}+\Delta_1+\Delta_2-t_{\mathbf{b}}(x^\ell,v^\ell)$. Let us regard $t^{\ell+1}$ as t^1 and $s^{\ell+1}$ as s^1 and $\Delta_1+\Delta_2$ as Δ in (6.34). Then we use (6.35) and (2.3) to have

$$\frac{\partial(t^{\ell+1},\mathbf{x}^{\ell+1}_{\|},\mathbf{v}^{\ell+1}_{\perp},\mathbf{v}^{\ell+1}_{\|})}{\partial(s^{\ell+1},\mathbf{x}_{\perp}(s^{\ell+1}),\mathbf{x}_{\|}(s^{\ell+1}),\mathbf{v}_{\perp}(s^{\ell+1}),\mathbf{v}_{\|}(s^{\ell+1}))}\leq\left[\begin{array}{c|c|c}1 & O_{\delta,\xi}(1)\frac{1}{|v|} & O_{\delta,\xi}(1)\frac{1}{|v|^2}\\ \hline \mathbf{0}_{2,1} & O_{\delta,\xi}(1) & O_{\delta,\xi}(1)\frac{1}{|v|}\\ \hline \mathbf{0}_{3,1} & O_{\delta,\xi}(1)|v| & O_{\delta,\xi}(1)\end{array}\right].$$

From (6.39)

$$\frac{\partial(s^{\ell+1},\mathbf{X}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}),\mathbf{V}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}))}{\partial(s^{\ell+1},X_{\mathbf{cl}}(s^{\ell+1}),V_{\mathbf{cl}}(s^{\ell+1}))}\lesssim_\xi\left[\begin{array}{c|c|c}1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3}\\ \hline \mathbf{0}_{3,1} & O_\xi(1) & \mathbf{0}_{3,3}\\ \hline \mathbf{0}_{3,1} & O_\xi(1)|v| & O_\xi(1)\end{array}\right],$$

and from $s^{\ell+1} = s^{\ell} - \Delta_2,\ X_{\mathbf{cl}}(s^{\ell+1}) = X_{\mathbf{cl}}(s^{\ell}) - (s^{\ell+1} - s^{\ell})V_{\mathbf{cl}}(s^{\ell}),\ V_{\mathbf{cl}}(s^{\ell+1}) = V_{\mathbf{cl}}(s^{\ell})$,

$$\frac{\partial(s^{\ell+1}, X_{\mathbf{cl}}(s^{\ell+1}), V_{\mathbf{cl}}(s^{\ell+1}))}{\partial(s^{\ell}, X_{\mathbf{cl}}(s^{\ell}), V_{\mathbf{cl}}(s^{\ell}))} \lesssim_{\xi} \begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & \mathbf{Id}_{3,3} & |s_1 - s_2|\mathbf{Id}_{3,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3} \end{bmatrix},$$

and from (6.5)

$$\frac{\partial(s^{\ell}, X_{\mathbf{cl}}(s^{\ell}), V_{\mathbf{cl}}(s^{\ell}))}{\partial(s^{\ell}, \mathbf{X}_{\mathbf{p}^{\ell}}(s^{\ell}), \mathbf{V}_{\mathbf{p}^{\ell}}(s^{\ell}))} \lesssim_{\xi} \begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & O_{\xi}(1) & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,1} & |v| & O_{\xi}(1) \end{bmatrix}.$$

Recalling (6.33), we have

$$\frac{\partial(s^{\ell}, \mathbf{x}_{\perp_{\ell}}(s^{\ell}), \mathbf{x}_{\parallel_{\ell}}(s^{\ell}), \mathbf{v}_{\perp_{\ell}}(s^{\ell}), \mathbf{v}_{\parallel_{\ell}}(s^{\ell}))}{\partial(t^{\ell}, \mathbf{x}^{\ell}_{\parallel_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\parallel_{\ell}})} \lesssim_{\xi} \left[\begin{array}{c|c|c} 1 & \mathbf{0}_{1,2} & \mathbf{0}_{1,3} \\ \hline O_{\xi}(1)|v| & O_{\xi}(1) & O_{\xi}(1)|t^{\ell} - s_1| \\ \hline O_{\xi}(1)|v|^2 & O_{\xi}(1)|v| & O_{\xi}(1) \end{array}\right].$$

By a direct matrix multiplication

$$\frac{\partial(t^{\ell+1}, \mathbf{x}^{\ell+1}_{\parallel_{\ell+1}}, \mathbf{v}^{\ell+1}_{\perp_{\ell+1}}, \mathbf{v}^{\ell+1}_{\parallel_{\ell+1}})}{\partial(t^{\ell}, \mathbf{x}^{\ell}_{\parallel_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\parallel_{\ell}})} \lesssim_{t,\xi} \left[\begin{array}{c|c|c} 1 & \frac{1}{|v|} & \frac{1}{|v|^2} \\ \hline \mathbf{0}_{2,1} & 1 & \frac{1}{|v|} \\ \hline \mathbf{0}_{3,1} & |v| & 1 \end{array}\right].$$

Note that for *Type II* we have $\mathbf{r}^{\ell+1} \gtrsim \sqrt{\delta}$ so that from (6.40)

$$\begin{aligned} J(\mathbf{r}^{\ell+1}) \quad &\gtrsim \quad \left[\begin{array}{c|c|c} 1 & \frac{M}{|v|}\sqrt{\delta} & \frac{M}{|v|^2}\min\{1, \sqrt{\delta}\} \\ \hline \mathbf{0}_{2,1} & M\sqrt{\delta} & \frac{M}{|v|}\min\{1, \sqrt{\delta}\} \\ \hline \mathbf{0}_{3,1} & M|v|\min\{\delta, \sqrt{\delta}\} & M\min\{\delta, \sqrt{\delta}\} \end{array}\right] \\ &\gtrsim_{\delta,t,\xi} \frac{\partial(t^{\ell+1}, \mathbf{x}^{\ell+1}_{\parallel_{\ell+1}}, \mathbf{v}^{\ell+1}_{\perp_{\ell+1}}, \mathbf{v}^{\ell+1}_{\parallel_{\ell+1}})}{\partial(t^{\ell}, \mathbf{x}^{\ell}_{\parallel_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\parallel_{\ell}})}. \end{aligned}$$

This proves our claim (6.40) for *Type II*.

Step 5. Eigenvalues and diagonalization of (6.40)

By a basic linear algebra (row and column operations), the characteristic polynomial of (6.40) equals, with $\mathbf{r} = \mathbf{r}^{\ell+1}$,

$$\det \begin{bmatrix} 1-\lambda & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|^2} & \frac{M}{|v|^2}\mathbf{r} & \frac{M}{|v|^2}\mathbf{r} \\ 0 & 1+M\mathbf{r}-\lambda & M\mathbf{r} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} \\ 0 & M\mathbf{r} & 1+M\mathbf{r}-\lambda & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} \\ 0 & M|v|\mathbf{r}^2 & M|v|\mathbf{r}^2 & 1+M\mathbf{r}-\lambda & M\mathbf{r}^2 & M\mathbf{r}^2 \\ 0 & M|v|\mathbf{r} & M|v|\mathbf{r} & M & 1+M\mathbf{r}-\lambda & M\mathbf{r} \\ 0 & M|v|\mathbf{r} & M|v|\mathbf{r} & M & M\mathbf{r} & 1+M\mathbf{r}-\lambda \end{bmatrix}$$
$$= (\lambda-1)^5[\lambda-(1+5M\mathbf{r})].$$

Therefore eigenvalues are

$$\begin{aligned} &\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1, \\ &\lambda_6 = 1 + 5M\mathbf{r}^{\ell+1} = 1 + 5M\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^{\ell+1}|}. \end{aligned} \tag{6.49}$$

Corresponding eigenvectors are

$$\begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\-1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\-|v|\mathbf{r}\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\-|v|\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\0\\-|v| \end{pmatrix}, \begin{pmatrix} 1\\|v|\\|v|\\|v|^2\mathbf{r}\\|v|^2\\|v|^2 \end{pmatrix}.$$

Write $P = P(\mathbf{r}^\ell)$ as a block matrix of above column eigenvectors. Then

$$\mathcal{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & |v| \\ 0 & -1 & 0 & 0 & 0 & |v| \\ 0 & 0 & -|v|\mathbf{r} & 0 & 0 & |v|^2\mathbf{r} \\ 0 & 0 & 0 & -|v| & 0 & |v|^2 \\ 0 & 0 & 0 & 0 & -|v| & |v|^2 \end{bmatrix},$$

$$\mathcal{P}^{-1} = \begin{bmatrix} 1 & \frac{-1}{5|v|} & \frac{-1}{5|v|} & \frac{-1}{5|v|^2\mathbf{r}} & \frac{-1}{5|v|^2} & \frac{-1}{5|v|^2} \\ 0 & \frac{1}{5} & \frac{-4}{5} & \frac{1}{5|v|\mathbf{r}} & \frac{1}{5|v|} & \frac{1}{5|v|} \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{-4}{5|v|\mathbf{r}} & \frac{1}{5|v|} & \frac{1}{5|v|} \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5|v|\mathbf{r}} & \frac{-4}{5|v|} & \frac{1}{5|v|} \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5|v|\mathbf{r}} & \frac{1}{5|v|} & \frac{-4}{5|v|} \\ 0 & \frac{1}{5|v|} & \frac{1}{5|v|} & \frac{1}{5|v|^2\mathbf{r}} & \frac{1}{5|v|^2} & \frac{1}{5|v|^2} \end{bmatrix}. \tag{6.50}$$

Therefore

$$J(\mathbf{r}) = \mathcal{P}(\mathbf{r})\Lambda(\mathbf{r})\mathcal{P}^{-1}(\mathbf{r}),$$

and

$$\Lambda(\mathbf{r}) := \operatorname{diag}[1, 1, 1, 1, 1, 1 + 5M\mathbf{r}],$$

where the notation diag$[a_1, \ldots, a_m]$ is a $m \times m-$matrix with $a_{ii} = a_i$ and $a_{ij} = 0$ for all $i \neq j$.

Step 6. The i-th intermediate group

We claim that, for $i = 1, 2, \ldots, [\frac{|t-s||v|}{L_\xi}]$,

$$\begin{aligned} & J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \\ & = \frac{\partial(t^{\ell_{i+1}}, \mathbf{x}^{\ell_{i+1}}_{\|_{\ell_{i+1}}}, \mathbf{v}^{\ell_{i+1}}_{\perp_{\ell_{i+1}}}, \mathbf{v}^{\ell_{i+1}}_{\|_{\ell_{i+1}}})}{\partial(t^{\ell_{i+1}-1}, \mathbf{x}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}}, \mathbf{v}^{\ell_{i+1}-1}_{\perp_{\ell_{i+1}-1}}, \mathbf{v}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}})} \times \cdots \times \frac{\partial(t^{\ell_i+1}, \mathbf{x}^{\ell_i+1}_{\|_{\ell_i+1}}, \mathbf{v}^{\ell_i+1}_{\perp_{\ell_i+1}}, \mathbf{v}^{\ell_i+1}_{\|_{\ell_i+1}})}{\partial(t^{\ell_i}, \mathbf{x}^{\ell_i}_{\|_{\ell_i}}, \mathbf{v}^{\ell_i}_{\perp_{\ell_i}}, \mathbf{v}^{\ell_i}_{\|_{\ell_i}})} \\ & \leq \mathcal{P}(\mathbf{r}_i)(\Lambda(\mathbf{r}_i))^{\frac{C_\xi}{\mathbf{r}_i}} \mathcal{P}^{-1}(\mathbf{r}_i). \end{aligned} \tag{6.51}$$

By the definition of the group, $L_\xi \leq |v||t^{\ell_i} - t^{\ell_{i+1}}| \leq C_1 < +\infty$ for all i. By the Velocity lemma (Lemma 2),

$$\begin{aligned} & \frac{1}{\mathcal{C}_1} e^{-\frac{\mathcal{C}}{2} C_1} \mathbf{r}^{\ell_i} \leq \mathbf{r}^{\ell_{i+1}} \equiv \frac{|\mathbf{v}^{\ell_{i+1}}_\perp|}{|v|}, \quad \mathbf{r}^{\ell_{i+1}-1} \equiv \frac{|\mathbf{v}^{\ell_{i+1}-1}_\perp|}{|v|}, \ldots, \mathbf{r}^{\ell_i+1} \equiv \frac{|\mathbf{v}^{\ell_i+1}_\perp|}{|v|}, \\ & \quad \mathbf{r}^{\ell_i} \equiv \frac{|\mathbf{v}^{\ell_i}_\perp|}{|v|} \leq \mathcal{C}_1 e^{\frac{\mathcal{C}}{2} C_1} \mathbf{r}^{\ell_i}, \end{aligned}$$

and define

$$\mathbf{r}_i \equiv \mathcal{C}_1 e^{\frac{\mathcal{C}}{2} C_1} \mathbf{r}^{\ell_i}.$$

Then we have

$$\frac{1}{(\mathcal{C}_1)^2} e^{-\mathcal{C} C_1} \mathbf{r}_i \;\leq\; \mathbf{r}^j \;\leq\; \mathbf{r}_i \quad \text{for all} \quad \ell_{i+1} \leq j \leq \ell_i. \tag{6.52}$$

From (6.40), we have a uniform bound for all $\ell_{i+1} \leq j \leq \ell_i$

$$J^{j+1}_j \lesssim J(\mathbf{r}_i) = \mathcal{P}(\mathbf{r}_i)\Lambda(\mathbf{r}_i)\mathcal{P}^{-1}(\mathbf{r}_i).$$

Therefore

$$J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \leq \mathcal{P}(\mathbf{r}_i)[\Lambda(\mathbf{r}_i)]^{|\ell_{i+1}-\ell_i|} \mathcal{P}^{-1}(\mathbf{r}_i).$$

Now we only left to prove $|\ell_{i+1}-\ell_i| \lesssim_\Omega \frac{1}{\mathbf{r}_i}$: For any $\ell_{i+1} \le j \le \ell_i$, we have $\xi(x^j)=0=\xi(x^{j+1})=\xi(x^j-(t^j-t^{j+1})v^j)$. We expand $\xi(x^j-(t^j-t^{j+1})v^j)$ in time to have

$$\begin{aligned}\xi(x^{j+1}) &= \xi(x^j)+\int_{t^j}^{t^{j+1}}\frac{d}{ds}\xi(X_{\mathbf{cl}}(s))\mathrm{d}s\\ &= \xi(x^j)+(v^j\cdot\nabla\xi(x^j))(t^{j+1}-t^j)+\int_{t^j}^{t^{j+1}}\int_{t^j}^{s}\frac{d^2}{d\tau^2}\xi(X_{\mathbf{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}0 &= (v^j\cdot\nabla\xi(x^j))(t^{j+1}-t^j)+\frac{(t^j-t^{j+1})^2}{2}(v^j\cdot\nabla^2\xi(X_{\mathbf{cl}}(\tau_*))\cdot v^j),\\ &\quad \text{for some } \tau_*\in[t^{j+1},t^j].\end{aligned}$$

Therefore

$$\frac{v^j\cdot\nabla\xi(x^j)}{|v|}=(t^j-t^{j+1})|v|\frac{v^j\cdot\nabla^2\xi(X_{\mathbf{cl}}(\tau_*))\cdot v^j}{2|v|^2}.$$

From the convexity (1.13), there exists $C_2 \gg 1$

$$\frac{1}{C_2}|t^j-t^{j+1}\|v| \le |\mathbf{r}^j| = \frac{|\mathbf{v}_\perp^j|}{|v|} = \frac{|v^j\cdot\nabla\xi(x^j)|}{|v|} \le C_2|t^j-t^{j+1}\|v|. \quad (6.53)$$

Therefore we have a lower bound of

$$|v\|t^j-t^{j+1}| : |v\|t^j-t^{j+1}| \ge \frac{1}{C_2}|\mathbf{r}^j| \ge \frac{1}{(\mathcal{C}_1)^2C_2}e^{-\mathcal{C}C_1}\mathbf{r}_i,$$

where we have used (6.52). Finally, using the definition of one group($1 \le |v\|t^{\ell_i}-t^{\ell_{i+1}}| \le C_1$), we have the following upper bound of the number of bounces in this one group($i-$th intermediate group)

$$|\ell_i-\ell_{i+1}| \le \frac{|v\|t^{\ell_i}-t^{\ell_{i+1}}|}{\min_{\ell_i\le j\le\ell_{i+1}}|v\|t^j-t^{j+1}|} \le \frac{C_1}{\frac{1}{(\mathcal{C}_1)^2C_2}e^{-\mathcal{C}C_1}\mathbf{r}_i} \lesssim_\xi \frac{1}{\mathbf{r}_i},$$

and this completes our claim (6.51).

Step 7. Whole intermediate groups

Recall $\mathcal{P}$ and $\mathcal{P}^{-1}$ from (6.50). We claim that, there exists $C_3 > 0$ such that

$$\prod_{i=1}^{[\frac{|t-s||v|}{L_\xi}]} J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \le (C_3)^{|t-s||v|} \mathcal{P}\left(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}\right) \mathcal{P}^{-1}(\mathbf{r}_1). \tag{6.54}$$

From the one group estimate (6.51),

$$\begin{aligned} &\prod_{i=1}^{[\frac{|t-s||v|}{L_\xi}]} J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \lesssim \mathcal{P}(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}) \\ &\times \prod_{i=1}^{[\frac{|t-s||v|}{L_\xi}]-1} \left[(\Lambda(\mathbf{r}_{i+1}))^{\frac{C_\xi}{\mathbf{r}_{i+1}}} \underbrace{\mathcal{P}^{-1}(\mathbf{r}_{i+1}) \times \mathcal{P}(\mathbf{r}_i)} \right] \times (\Lambda(\mathbf{r}_1))^{\frac{C_\xi}{\mathbf{r}_1}} \mathcal{P}^{-1}(\mathbf{r}_1). \end{aligned}$$

Now we focus on the underbraced matrix multiplication. Directly

$$\mathcal{P}^{-1}(\mathbf{r}_{i+1})\mathcal{P}(\mathbf{r}_i) = \begin{bmatrix} 1 & 0 & \frac{-1+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5|v|} & 0 & 0 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\ 0 & 1 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} & 0 & 0 & |v|\frac{-1+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\ 0 & 0 & \frac{1+4\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} & 0 & 0 & 4|v|\frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\ 0 & 0 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} & 1 & 0 & |v|\frac{-1+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\ 0 & 0 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} & 0 & 1 & |v|\frac{-1+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\ 0 & 0 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5|v|} & 0 & 0 & \frac{4+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \end{bmatrix}.$$

Due to the choice of $\mathbf{r}_i \equiv \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}C_1}\mathbf{r}^{\ell_i}$ in (6.52) we have $\left|\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}\right| \lesssim \left|\frac{\mathbf{r}^{\ell_i}}{\mathbf{r}^{\ell_{i+1}}}\right| \le C_\xi$, where we have used the Velocity lemma and (2.2) and (2.3): $\frac{1}{\mathcal{C}_1}e^{-\frac{\mathcal{C}}{2}C_1}\mathbf{r}^{\ell_{i+1}} \le \frac{1}{\mathcal{C}_1}e^{-\frac{\mathcal{C}}{2}|t^{\ell_i}-t^{\ell_{i+1}}|}\mathbf{r}^{\ell_{i+1}} \le \mathbf{r}^{\ell_i} \le \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}|t^{\ell_i}-t^{\ell_{i+1}}|}\mathbf{r}^{\ell_{i+1}} \le \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}C_1}\mathbf{r}^{\ell_{i+1}}$.

Therefore for sufficiently large $C_\xi > 0$, for all i

$$\mathcal{P}^{-1}\widetilde{(\mathbf{r}_{i+1})\mathcal{P}}(\mathbf{r}_i) \;\le\; \mathcal{Q} := \begin{bmatrix} 1 & 0 & \frac{C_\xi}{|v|} & 0 & 0 & C_\xi \\ 0 & 1 & C_\xi & 0 & 0 & C_\xi|v| \\ 0 & 0 & C_\xi & 0 & 0 & C_\xi|v| \\ 0 & 0 & C_\xi & 1 & 0 & C_\xi|v| \\ 0 & 0 & C_\xi & 0 & 1 & C_\xi|v| \\ 0 & 0 & \frac{C_\xi}{|v|} & 0 & 0 & C_\xi \end{bmatrix}, \tag{6.55}$$

where we use a notation: For a matrix A, the entries of a matrix $\widetilde{A}$ are absolute values of the entries of A, i.e. $(\widetilde{A})_{ij} = |(A)_{ij}|$. In particular, the entries of $\mathcal{P}^{-1}\widetilde{(\mathbf{r}_{i+1})\mathcal{P}}(\mathbf{r}_i)$ are absolute values of the entries of the matrix multiplication $\mathcal{P}^{-1}(\mathbf{r}_{i+1})\mathcal{P}(\mathbf{r}_i)$.

Again we diagonalize $\mathcal{Q}$ as

$$\mathcal{Q} = \mathcal{F}\mathcal{A}\mathcal{F}^{-1}$$

$$:= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{2C_\xi}{2C_\xi - 1} \\ 0 & 1 & 0 & 0 & 0 & \frac{2C_\xi|v|}{2C_\xi - 1} \\ 0 & 0 & 0 & 0 & -|v| & |v| \\ 0 & 0 & 1 & 0 & 0 & \frac{2C_\xi|v|}{2C_\xi - 1} \\ 0 & 0 & 0 & 1 & 0 & \frac{2C_\xi|v|}{2C_\xi - 1} \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 0 & \\ & & & & & 2C_\xi \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & \frac{-C_\xi}{2C_\xi - 1}\frac{1}{|v|} & 0 & 0 & \frac{-C_\xi}{2C_\xi - 1} \\ 0 & 1 & \frac{-C_\xi}{2C_\xi - 1} & 0 & 0 & \frac{-C_\xi|v|}{2C_\xi - 1} \\ 0 & 0 & \frac{-C_\xi}{2C_\xi - 1} & 1 & 0 & \frac{-C_\xi|v|}{2C_\xi - 1} \\ 0 & 0 & \frac{-C_\xi}{2C_\xi - 1} & 0 & 1 & \frac{-C_\xi|v|}{2C_\xi - 1} \\ 0 & 0 & \frac{-1}{2|v|} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2|v|} & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

and directly

$$
\begin{aligned}
\mathcal{Q}^{[\frac{|t-s||v|}{L_\xi}]} &= \mathcal{F}\mathcal{A}^{[\frac{|t-s||v|}{L_\xi}]}\mathcal{F}^{-1} \\
&= \mathcal{F}\ \mathrm{diag}\left[1,1,1,1,0,(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}\right]\mathcal{F}^{-1} \\
&= \left[\begin{array}{cccccc}
1 & 0 & \frac{1}{|v|}\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) & 0 & 0 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) \\
0 & 1 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) & 0 & 0 & |v|\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) \\
0 & 0 & \frac{(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}}{2} & 0 & 0 & |v|\frac{(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}}{2} \\
0 & 0 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) & 1 & 0 & |v|\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) \\
0 & 0 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) & 0 & 1 & |v|\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) \\
0 & 0 & \frac{1}{|v|}\frac{(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}}{2} & 0 & 0 & \frac{(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}}{2}
\end{array}\right].
\end{aligned}
\tag{6.56}
$$

Notice that from (6.49)

$$
[\Lambda(\mathbf{r}_i)]^{\frac{C_\xi}{\mathbf{r}_i}} \le (1+5M\mathbf{r}_i)^{\frac{C_\xi}{\mathbf{r}_i}}\ \mathbf{Id}_{6,6} \le C_\xi'\ \mathbf{Id}_{6,6}.
$$

Now we use (6.51) and take the absolute value of the entries and then use (6.55) and (6.56), for $\tilde{t} := t-s$,

$$
\begin{aligned}
\prod_{i=1}^{[\frac{\tilde{t}|v|}{L_\xi}]} J^{\ell_{i+1}}_{\ell_{i+1}-1}\times\cdots\times J^{\ell_i+1}_{\ell_i} &\le \widetilde{\mathcal{P}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})}\prod_{i=1}^{[\frac{\tilde{t}|v|}{L_\xi}]}\left[(1+5M\mathbf{r}_i)^{\frac{C_\xi}{\mathbf{r}_i}}\mathcal{Q}\right]\mathcal{Q}^{-1}\widetilde{\mathcal{P}^{-1}(\mathbf{r}_1)} \\
&\le (C_\xi')^{[\frac{\tilde{t}|v|}{L_\xi}]}\times\widetilde{\mathcal{P}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})}\mathcal{F}\mathcal{A}^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathcal{F}^{-1}\widetilde{\mathcal{P}^{-1}(\mathbf{r}_1)}.
\end{aligned}
$$

Now we use the explicit form of (6.56) to bound

$$
C^{C\tilde{t}|v|}\left[\begin{array}{c|cc|c|cc}
1 & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|^2}\frac{1}{|\mathbf{r}_1|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|^2} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|^2} \\
\hline
0 & (C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|}\frac{1}{|\mathbf{r}_1|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} \\
0 & (C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|}\frac{1}{|\mathbf{r}_1|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} \\
\hline
0 & |v|(C_\xi)^{\tilde{t}|v|}\left|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}\right| & |v|(C_\xi)^{\tilde{t}|v|}\left|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}\right| & (C_\xi)^{\tilde{t}|v|}\frac{\left|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}\right|}{|\mathbf{r}_1|} & (C_\xi)^{\tilde{t}|v|}\left|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}\right| & (C_\xi)^{\tilde{t}|v|}\left|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}\right| \\
\hline
0 & |v|(C_\xi)^{\tilde{t}|v|} & |v|(C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|}\frac{1}{|\mathbf{r}_1|} & (C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|} \\
0 & |v|(C_\xi)^{\tilde{t}|v|} & |v|(C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|}\frac{1}{|\mathbf{r}_1|} & (C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|}
\end{array}\right]
$$

$$\lesssim C^{C|t-s\|v|} \left[\begin{array}{c|c|c|c} 1 & \frac{1}{|v|} & \frac{1}{|v||\mathbf{v}_\perp^1|} & \frac{1}{|v|^2} \\ \hline \mathbf{0}_{2,1} & O_\xi(1) & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline 0 & |\mathbf{v}_\perp^1| & O_\xi(1) & \frac{|\mathbf{v}_\perp^1|}{|v|} \\ \hline \mathbf{0}_{2,1} & |v| & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1) \end{array}\right]_{6\times 6}, \tag{6.57}$$

where we have used (6.53) and the Velocity lemma (Lemma 2) and (2.2), (2.3) and

$$\mathbf{r}_i = \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}C_1}\mathbf{r}^i \lesssim e^{C|t-s\|v|}\frac{|\mathbf{v}_\perp^1|}{|v|}, \quad \text{and} \quad \frac{\mathbf{r}_{[\frac{|t-s\|v|}{L_\xi}]}}{\mathbf{r}_1}$$

$$= \frac{\mathbf{r}^{[\frac{|t-s\|v|}{L_\xi}]}}{\mathbf{r}^1} = \frac{\left|\mathbf{v}_\perp^{[\frac{|t-s\|v|}{L_\xi}]}\right|}{|\mathbf{v}_\perp^1|} \leq \mathcal{C}_1 e^{\frac{\mathcal{C}}{2}|v\|t-s|}.$$

Step 8. Intermediate summary for the matrix method and the final estimate for Type II

Recall from (6.31) and (6.33), (6.57), (6.35),

$$\frac{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))} \equiv \frac{\partial(s^{\ell_*}, \mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}), \mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\|_{\ell_*}}(s^{\ell_*}))}{\partial(s^1, \mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\|_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\|_1}(s^1))}$$

$$= \frac{\partial(s^{\ell_*}, \mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}), \mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\|_{\ell_*}}(s^{\ell_*}))}{\partial(t^{\ell_*}, \mathbf{x}_{\|_{\ell_*}}^{\ell_*}, \mathbf{v}_{\perp_{\ell_*}}^{\ell_*}, \mathbf{v}_{\|_{\ell_*}}^{\ell_*})}$$

$$\times \prod_{i=1}^{[\frac{|t-s\|v|}{L_\xi}]} \frac{\partial(t^{\ell_{i+1}}, \mathbf{x}_{\|_{\ell_{i+1}}}^{\ell_{i+1}}, \mathbf{v}_{\perp_{\ell_{i+1}}}^{\ell_{i+1}}, \mathbf{v}_{\|_{\ell_{i+1}}}^{\ell_{i+1}})}{\partial(t^{\ell_{i+1}-1}, \mathbf{x}_{\|_{\ell_{i+1}-1}}^{\ell_{i+1}-1}, \mathbf{v}_{\perp_{\ell_{i+1}-1}}^{\ell_{i+1}-1}, \mathbf{v}_{\|_{\ell_{i+1}-1}}^{\ell_{i+1}-1})}$$

$$\times \cdots \times \frac{\partial(t^{\ell_i+1}, \mathbf{x}_{\|_{\ell_i+1}}^{\ell_i+1}, \mathbf{v}_{\perp_{\ell_i+1}}^{\ell_i+1}, \mathbf{v}_{\|_{\ell_i+1}}^{\ell_i+1})}{\partial(t^{\ell_i}, \mathbf{x}_{\|_{\ell_i}}^{\ell_i}, \mathbf{v}_{\perp_{\ell_i}}^{\ell_i}, \mathbf{v}_{\|_{\ell_i}}^{\ell_i})}$$

$$\times \frac{\partial(t^1, \mathbf{x}_{\|_1}^1, \mathbf{v}_{\perp_1}^1, \mathbf{v}_{\|_1}^1)}{\partial(s^1, \mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\|_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\|_1}(s^1))}$$

$$\leq (6.33) \times (6.57) \times (6.35).$$

Then directly we bound it by

$$
\begin{aligned}
&\leq \ (6.33) \times C^{C|t-s\|v|} \\
&\times \left[\begin{array}{cc|cc}
\frac{1}{|\mathbf{v}_\perp^1|} + \frac{|v|}{|\mathbf{v}_\perp^1|^2} + |t^1 - s^1| & \frac{1}{|v|} + \frac{|v|}{|\mathbf{v}_\perp^1|^2} + |s^1 - t^1| & \frac{1}{|v\|\mathbf{v}_\perp^1|} + |s^1 - t^1|^2 & \frac{1}{|v|^2} + \frac{|s^1-t^1|}{|v|} \\
\hline
\frac{|v|^2}{|\mathbf{v}_\perp^1|^2} + \frac{|v|}{|\mathbf{v}_\perp^1|} + |v\|s^1 - t^1| & 1 + \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|} + |s^1 - t^1| & \frac{1}{|v|} \\
\frac{|v|^2}{|\mathbf{v}_\perp^1|^2} + \frac{|v|}{|\mathbf{v}_\perp^1|} + |v\|s^1 - t^1| & 1 + \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|} + |s^1 - t^1| & \frac{1}{|v|} \\
\hline
\frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v| & |\mathbf{v}_\perp^1| + \frac{|v|^2}{|\mathbf{v}_\perp^1|} & O_\xi(1) & \frac{|\mathbf{v}_\perp^1|}{|v|} \\
\frac{|v|^3}{|\mathbf{v}_\perp^1|^2} + \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v|^2|s_1 - t^1| & |v| + \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} + |v\|s_1 - t^1| & O_\xi(1)
\end{array}\right],
\end{aligned}
\tag{6.58}
$$

where we have used the Velocity lemma (Lemma 2) and (6.53), (2.2), (2.3) and

$$
\begin{aligned}
&|v\|t^1 - s^1| \leq \min\{|v|(t_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x, -v)), (t-s)|v|\} \\
&\lesssim_\Omega \min\left\{\frac{|\mathbf{v}_\perp^1|}{|v|}, (t-s)|v|\right\} \lesssim_\Omega C^{C|t-s\|v|} \min\left\{\frac{|\mathbf{v}_\perp^1|}{|v|}, 1\right\}.
\end{aligned}
$$

Again we use the velocity lemma (Lemma 2) and (6.53), (2.2), (2.3) and

$$
\begin{aligned}
&|v\|t^{\ell_*} - s^{\ell_*}| \leq \min\{|v\|t^{\ell_*} - t^{\ell_*+1}|, |t-s||v|\} \\
&\lesssim_\Omega \min\left\{\frac{|\mathbf{v}_\perp^{\ell_*}|}{|v|}, |t-s\|v|\right\} \lesssim_\Omega C^{C|t-s\|v|} \min\left\{\frac{|\mathbf{v}_\perp^1|}{|v|}, 1\right\},
\end{aligned}
$$

and $|\mathbf{v}_\perp(s^{\ell_*})| \lesssim_\Omega C^{C|v|(t-s)}|\mathbf{v}_\perp^1|$ to have, from (6.58)

$$
\frac{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))} \lesssim C^{C|t-s\|v|} \left[\begin{array}{c|c|cc}
0 & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,2} \\
\hline
|\mathbf{v}_\perp^1| & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} \\
|v| & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\
\hline
|v|^2 & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1)
\end{array}\right]_{7\times 7}.
\tag{6.59}
$$

We consider the following case:

$$
\text{There exists } \ell \in [\ell_*(s; t, x, v), 0] \text{ such that } \mathbf{r}^\ell \geq \sqrt{\delta}. \tag{6.60}
$$

Therefore ℓ is *Type II* in (6.25). Equivalently $\tau \in [t^{\ell+1}, t^{\ell}]$ for some $\ell_* \leq \ell \leq 0$ and $|\xi(X_{\mathbf{cl}}(\tau; t, x, v))| \geq C\delta$. By the Velocity lemma (Lemma 2), for all $1 \leq i \leq \ell_*(s; t, x, v)$,

$$|\mathbf{r}^i| = \frac{|\mathbf{v}_{\perp}^i|}{|v|} \gtrsim_{\xi} e^{-C_{\xi}|v||t^i - t^{\ell}|}|\mathbf{r}^{\ell}| \gtrsim_{\xi} e^{-C_{\xi}|v|(t-s)}\sqrt{\delta}.$$

Especially, for all $1 \leq i \leq \ell_*(s; t, x, v)$,

$$|\mathbf{r}^1| \gtrsim_{\xi} e^{-C_{\xi}|v|(t-s)}\sqrt{\delta}, \quad \frac{1}{|\mathbf{r}^i|} = \frac{|v|}{|\mathbf{v}_{\perp}^i|} \lesssim_{\xi} \frac{e^{C_{\xi}|v|(t-s)}}{\sqrt{\delta}}.$$

Note that $\ell_*(s; t, x, v) \lesssim \max_i \frac{|v||t-s|}{\mathbf{r}^i} \lesssim_{\delta} C^{C|v||t-s|}$.

Therefore in the case of (6.60), from (6.59),

$$\frac{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))} \lesssim C^{C(t-s)|v|} \left[\begin{array}{c|cc|cc} 0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline |\mathbf{v}_{\perp}^1| & \frac{1}{\sqrt{\delta}} & \frac{1}{\sqrt{\delta}} & \frac{1}{|v|} & \frac{1}{|v|} \\ |v| & \frac{1}{\delta} & \frac{1}{\delta} & \frac{1}{|v|}\frac{1}{\sqrt{\delta}} & \frac{1}{|v|} \\ \hline |v|^2 & |v|\frac{1}{\delta} & |v|\frac{1}{\delta} & \frac{1}{\sqrt{\delta}} & 1 \end{array}\right]$$

$$\lesssim_{\delta} C^{C|v|(t-s)} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline |v| & 1 & \frac{1}{|v|} \\ \hline |v|^2 & |v| & 1 \end{array}\right].$$

From (6.32) and (6.38) we conclude

$$\begin{aligned} &\frac{\partial(X_{\mathbf{cl}}(s; t, x, v), V_{\mathbf{cl}}(s; t, x, v))}{\partial(t, x, v)} \\ &\lesssim_{\delta,\xi} C^{C|v|(t-s)} \frac{\partial(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline |v| & 1 & \frac{1}{|v|} \\ \hline |v|^2 & |v| & 1 \end{array}\right] \\ &\quad \times \frac{\partial(s^1, \mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\parallel_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\parallel_1}(s^1))}{\partial(t, x, v)} \\ &\lesssim_{\delta,\xi} C^{C|v|(t-s)} \begin{bmatrix} |v| & 1 & |s^{\ell_*} - s| \\ \mathbf{0}_{3,1} & |v| & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ |v| & 1 & \frac{1}{|v|} \\ |v|^2 & |v| & 1 \end{bmatrix} \left[\begin{array}{c|c|c} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & 1 & |t - s^1| \\ \mathbf{0}_{3,1} & |v| & 1 \end{array}\right] \\ &\lesssim_{\delta,\xi} C^{C|v|(t-s)} \begin{bmatrix} |v| & 1 & \frac{1}{|v|} \\ |v|^2 & |v| & 1 \end{bmatrix}_{6\times 7}. \end{aligned} \tag{6.61}$$

We remark $\partial \mathbf{x}_{\perp_{\ell_*}}$ and $\partial \mathbf{v}_{\perp_{\ell_*}}$ have desired bounds but $\partial \mathbf{x}_{\|_{\ell_*}}$ and $\partial \mathbf{v}_{\|_{\ell_*}}$ still have undesired bounds in (6.59). We only need to consider the remaining case of (6.60), i.e.

$$\text{For all } \ell \in [\ell_*(s;t,x,v),0], \quad \text{we have } \mathbf{r}_\ell \leq \sqrt{\delta}. \tag{6.62}$$

Note that in this case the moving frame ($\mathbf{p}^\ell-$spherical coordinate) is well-defined for all $\tau \in [s,t]$. In next two step we use the ODE method to refine the estimates for submatrix of (6.59):

$$\begin{aligned}
&\frac{\partial(\mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\|_{\ell_*}}(s^{\ell_*}))}{\partial(\mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\|_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\|_1}(s^1))}\\
&\quad = \begin{bmatrix}
\frac{\partial \mathbf{x}_{\|_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{x}_{\perp_1}(s^1)} & \frac{\partial \mathbf{x}_{\|_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{x}_{\|_1}(s^1)} & \frac{\partial \mathbf{x}_{\|_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{v}_{\perp_1}(s^1)} & \frac{\partial \mathbf{x}_{\|_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{v}_{\|_1}(s^1)} \\
\frac{\partial \mathbf{v}_{\|_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{x}_{\perp_1}(s^1)} & \frac{\partial \mathbf{v}_{\|_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{x}_{\|_1}(s^1)} & \frac{\partial \mathbf{v}_{\|_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{v}_{\perp_1}(s^1)} & \frac{\partial \mathbf{v}_{\|_{\ell_*}}(s^{\ell_*})}{\partial \mathbf{v}_{\|_1}(s^1)}
\end{bmatrix}_{4\times 6}.
\end{aligned}$$

Step 9. ODE method within the time scale $|t-s\|v| \simeq L_\xi$

Recall the end points (time) of intermediate groups from (6.29):

$$\begin{aligned}
s &< \underbrace{t^{\ell_*} < t^{\ell_{[\frac{|t-s\|v|}{L_\xi}]}+1}}_{[\frac{|t-s\|v|}{L_\xi}]+1} < \underbrace{t^{\ell_{[\frac{|t-s\|v|}{L_\xi}]}} < t^{\ell_{[\frac{|t-s\|v|}{L_\xi}]-1}+1}}_{[\frac{|t-s\|v|}{L_\xi}]} < \cdots < \underbrace{t^{\ell_i} < t^{\ell_{i-1}+1}}_{i} < \cdots \\
&< \underbrace{t^{\ell_2} < t^{\ell_1+1}}_{2} < \underbrace{t^{\ell_1} < t^1}_{1} < t,
\end{aligned}$$

where the underbraced numbering indicates the index of the intermediate group. We further choose points independently on (t,x,v) for all $i = 1,2,\ldots,[\frac{|t-s\|v|}{L_\xi}]$:

$$\begin{gathered}
t^{\ell_1+1} < s^2 < t^{\ell_1}, \\
t^{\ell_2+1} < s^3 < t^{\ell_2}, \\
\vdots \\
t^{\ell_i+1} < s^{i+1} < \underbrace{t^{\ell_i} < \cdots < t^{\ell_{i-1}+1}}_{i-\text{intermediate group}} < s^i < t^{\ell_{i-1}}, \\
\vdots \\
t^{\ell_{[\frac{|t-s\|v|}{L_\xi}]}+1} < s^{\ell_{[\frac{|t-s\|v|}{L_\xi}]}+1} < t^{\ell_{[\frac{|t-s\|v|}{L_\xi}]}}.
\end{gathered}$$

We claim the following estimate at s^{i+1} via s^i. Within the i-th intermediate group, we fix $\mathbf{p}^{\ell_i}$-spherical coordinate in *Step 9*. The goal is to estimate derivatives with respect to initial $(\mathbf{x}_1, \mathbf{v}_1)$ at s^{i+1} in terms of s^i. This is a bit different from previous steps.

$$\begin{aligned}
&\begin{bmatrix} \left|\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\perp_1}(s^1)}\right| & \left|\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\|_1}(s^1)}\right| \\ \left|\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\perp_1}(s^1)}\right| & \left|\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\|_1}(s^1)}\right| \end{bmatrix} \lesssim_{\delta,\xi} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} \left|\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\perp_1}(s^1)}\right| & \left|\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\|_1}(s^1)}\right| \\ \left|\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\perp_1}(s^1)}\right| & \left|\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\|_1}(s^1)}\right| \end{bmatrix} \\
&\quad + e^{C|v||t-s^i|} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ |v|\left(1+\frac{|v|}{|\mathbf{v}_\perp^1|}\right) & |v|\left(1+\frac{|v|}{|\mathbf{v}_\perp^1|}\right) \end{bmatrix}, \\
&\begin{bmatrix} \left|\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\perp_1}(s^1)}\right| & \left|\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\|_1}(s^1)}\right| \\ \left|\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\perp_1}(s^1)}\right| & \left|\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\|_1}(s^1)}\right| \end{bmatrix} \\
&\lesssim_{\delta,\xi} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} \left|\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\perp_1}(s^1)}\right| & \left|\frac{\partial \mathbf{x}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\|_1}(s^1)}\right| \\ \left|\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\perp_1}(s^1)}\right| & \left|\frac{\partial \mathbf{v}_{\|_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\|_1}(s^1)}\right| \end{bmatrix} + e^{C|v||t-s^i|} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.
\end{aligned} \tag{6.63}$$

For the sake of simplicity we drop the index ℓ_i.

Denote, from (6.15),

$$F_\|(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|) := D(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\|) + E(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\|)\mathbf{v}_\perp, \tag{6.64}$$

where D is a $\mathbf{r}^3$-vector-valued function and E is a 3×3 matrix-valued function:

$$\begin{aligned}
D(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\|) &= \sum_i G_{ij}(\mathbf{x}_\perp, \mathbf{x}_\|) \frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\|)\cdot(\partial_1\eta(\mathbf{x}_\|)\times\partial_2\eta(\mathbf{x}_\|))} \\
&\quad \times \left\{\mathbf{v}_\| \cdot \nabla^2\eta(\mathbf{x}_\|)\cdot\mathbf{v}_\| - \mathbf{x}_\perp\mathbf{v}_\|\cdot\nabla^2\mathbf{n}(\mathbf{x}_\|)\cdot\mathbf{v}_\|\right\}\cdot(-\mathbf{n}(\mathbf{x}_\|)\times\partial_{i+1}\eta(\mathbf{x}_\|)),
\end{aligned}$$

and

$$\begin{aligned}
E(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\|) &= \sum_i G_{ij}(\mathbf{x}_\perp, \mathbf{x}_\|) \frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\|)\cdot(\partial_1\eta(\mathbf{x}_\|)\times\partial_2\eta(\mathbf{x}_\|))} 2\mathbf{v}_\|\cdot \\
&\quad \times \nabla\mathbf{n}(\mathbf{x}_\|)\cdot(-\mathbf{n}(\mathbf{x}_\|)\times\partial_{i+1}\eta(\mathbf{x}_\|)).
\end{aligned}$$

Note that E is linear in $\mathbf{v}_\|$. Here $G_{ij}(\cdot,\cdot)$ is a smooth bounded function defined in (6.22) and we used the notational convention $i \equiv i \bmod 2$.

From Lemma 15 we take the time integration of (6.13) along the characteristics to have

$$\begin{aligned}
\mathbf{x}_{\|}(s^{i+1}) &= \mathbf{x}_{\|}(s^i) - \int_{s^{i+1}}^{s^i} \mathbf{v}_{\|}(\tau)\mathrm{d}\tau, \\
\mathbf{v}_{\|}(s^{i+1}) &= \mathbf{v}_{\|}(s^i) - \int_{s^{i+1}}^{s^i} \big\{E(\mathbf{x}_{\perp}(\tau), \mathbf{x}_{\|}(\tau), \mathbf{v}_{\|}(\tau))\mathbf{v}_{\perp}(\tau) \\
&\qquad + D(\mathbf{x}_{\perp}(\tau), \mathbf{x}_{\|}(\tau), \mathbf{v}_{\|}(\tau))\big\}\,\mathrm{d}\tau.
\end{aligned}$$

Note that $\mathbf{v}_{\perp}(\tau)$ is not continuous with respect to the time τ. Using (6.13) we rewrite this time integration as

$$\int_{s^{i+1}}^{s^i} E(\mathbf{x}_{\perp}(\tau), \mathbf{x}_{\|}(\tau), \mathbf{v}_{\|}(\tau))\mathbf{v}_{\perp}(\tau)\mathrm{d}\tau = \int_{t^{\ell_{i-1}+1}}^{s^i} + \sum_{\ell=\ell_i-1}^{\ell_{i-1}+1} \int_{t^{\ell+1}}^{t^\ell} + \int_{s^{i+1}}^{t^{\ell_i}},$$

then we use $\mathbf{v}_{\perp}(\tau) = \dot{\mathbf{x}}_{\perp}(\tau)$ and the integration by parts to have

$$\begin{aligned}
&\int_{t^{\ell_{i-1}+1}}^{s^i} E(\mathbf{x}_{\perp}(\tau), \mathbf{x}_{\|}(\tau), \mathbf{v}_{\|}(\tau))\dot{\mathbf{x}}_{\perp}(\tau)\mathrm{d}\tau \\
&\quad + \sum_{\ell=\ell_i-1}^{\ell_{i-1}+1} \int_{t^{\ell+1}}^{t^\ell} E(\mathbf{x}_{\perp}(\tau), \mathbf{x}_{\|}(\tau), \mathbf{v}_{\|}(\tau))\dot{\mathbf{x}}_{\perp}(\tau)\mathrm{d}\tau \\
&\quad + \int_{s^{i+1}}^{t^{\ell_i}} E(\mathbf{x}_{\perp}(\tau), \mathbf{x}_{\|}(\tau), \mathbf{v}_{\|}(\tau))\dot{\mathbf{x}}_{\perp}(\tau)\mathrm{d}\tau \\
&= E(s^i)\mathbf{x}_{\perp}(s^i) - E(t^{\ell_{i-1}+1})\underbrace{\mathbf{x}_{\perp}(t^{\ell_{i-1}+1})}_{=0} \\
&\quad - \int_{t^{\ell_{i-1}+1}}^{s^i} \big[\mathbf{v}_{\perp}(\tau), \mathbf{v}_{\|}(\tau), F_{\|}(\tau)\big] \cdot \nabla E(\tau)\mathbf{x}_{\perp}(\tau)\mathrm{d}\tau \\
&\quad + \sum_{\ell=\ell_i-1}^{\ell_{i-1}+1} \Bigg\{ E(t^\ell)\underbrace{\mathbf{x}_{\perp}(t^\ell)}_{=0} - E(t^{\ell+1})\underbrace{\mathbf{x}_{\perp}(t^{\ell+1})}_{=0} \\
&\quad - \int_{t^{\ell+1}}^{t^\ell} \big[\mathbf{v}_{\perp}(\tau), \mathbf{v}_{\|}(\tau), F_{\|}(\tau)\big] \cdot \nabla E(\tau)\mathbf{x}_{\perp}(\tau)\mathrm{d}\tau \Bigg\} \\
&\quad + E(t^{\ell_i})\underbrace{\mathbf{x}_{\perp}(t^{\ell_i})}_{=0} - E(s^{i+1})\mathbf{x}_{\perp}(s^{i+1}) \\
&\quad - \int_{s^{i+1}}^{t^{\ell_i}} \big[\mathbf{v}_{\perp}(\tau), \mathbf{v}_{\|}(\tau), F_{\|}(\tau)\big] \cdot \nabla E(\tau)\mathbf{x}_{\perp}(\tau)\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&= E(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\|)(s^i)\mathbf{x}_\perp(s^i) - E(s^{i+1})\mathbf{x}_\perp(s^{i+1}) \\
&\quad - \int_{s^i}^{s^{i+1}} \left[\mathbf{v}_\perp(\tau), \mathbf{v}_\|(\tau), F_\|(\tau)\right] \cdot \nabla E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau,
\end{aligned}
$$

where we have used the fact $X_{\mathbf{cl}}\ (t^\ell) \in \partial\Omega$ (therefore $\mathbf{x}_\perp(t^\ell) = 0$) and the notation $E(\tau) = E(\mathbf{x}_\perp(\tau), \mathbf{x}_\|(\tau), \mathbf{v}_\|(\tau))$, $D(\tau) = D(\mathbf{x}_\perp(\tau), \mathbf{x}_\|(\tau), \mathbf{v}_\|(\tau))$, $F_\|(\tau) = F_\|(\mathbf{x}_\perp(\tau), \mathbf{x}_\|(\tau), \mathbf{v}_\perp(\tau), \mathbf{v}_\|(\tau))$.

Overall we have

$$
\begin{aligned}
\mathbf{x}_\|(s^{i+1}) &= \mathbf{x}_\|(s^i) - \int_{s^{i+1}}^{s^i} \mathbf{v}_\|(\tau)\mathrm{d}\tau, \\
\mathbf{v}_\|(s^{i+1}) &= \mathbf{v}_\|(s^i) - E(s^i)\mathbf{x}_\perp(s^i) + E(s^{i+1})\mathbf{x}_\perp(s^{i+1}) \\
&\quad + \int_{s^{i+1}}^{s^i} \left[\mathbf{v}_\perp(\tau), \mathbf{v}_\|(\tau), F_\|(\tau)\right] \cdot \nabla E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau - \int_{s^{i+1}}^{s^i} D(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{6.65}
$$

Denote

$$
\begin{aligned}
\partial &= \left[\partial_{\mathbf{x}_\perp(s^1)}, \partial_{\mathbf{x}_\|(s^1)}, \partial_{\mathbf{v}_\perp(s^1)}, \partial_{\mathbf{v}_\|(s^1)}\right] \\
&= \left[\frac{\partial}{\partial \mathbf{x}_\perp(s^1)}, \frac{\partial}{\partial \mathbf{x}_\|(s^1)}, \frac{\partial}{\partial \mathbf{v}_\perp(s^1)}, \frac{\partial}{\partial \mathbf{v}_\|(s^1)}\right].
\end{aligned}
$$

We claim that, in a sense of distribution on $(s^1, \mathbf{x}_\perp(s^1), \mathbf{x}_\|(s^1), \mathbf{v}_\perp(s^1), \mathbf{v}_\|(s^1)) \in [0, \infty) \times (0, C_\xi) \times (0, 2\pi] \times (\delta, \pi - \delta) \times \mathbb{R} \times \mathbb{R}^2$,

$$
\begin{aligned}
&\Big[\partial \mathbf{x}_\perp(s^{i+1}; s^1, \mathbf{x}(s^1), \mathbf{v}(s^1)), \partial \mathbf{x}_\|(s^{i+1}; s^1, \mathbf{x}(s^1), \mathbf{v}(s^1)), \\
&\qquad \partial \mathbf{v}_\|(s^{i+1}; s^1, \mathbf{x}(s^1), \mathbf{v}(s^1))\Big] \\
&= \sum_\ell \mathbf{1}_{[t^{\ell+1}, t^\ell)}(s^{i+1}) \left[\partial \mathbf{x}_\perp, \partial \mathbf{x}_\|, \partial \mathbf{v}_\|\right], \\
&\partial \left[\mathbf{v}_\perp(s^{i+1}; s^1, \mathbf{x}(s^1), \mathbf{v}(s^1))\mathbf{x}_\perp(s^{i+1}; s^1, \mathbf{x}(s^1), \mathbf{v}(s^1))\right] \\
&= \sum_\ell \mathbf{1}_{[t^{\ell+1}, t^\ell)}(s^{i+1}) \left\{\partial \mathbf{v}_\perp \mathbf{x}_\perp + \mathbf{v}_\perp \partial \mathbf{x}_\perp\right\},
\end{aligned}
\tag{6.66}
$$

i.e. the distributional derivatives of $[\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\|]$ and $\mathbf{v}_\perp \mathbf{x}_\perp$ equal the piecewise derivatives.

Proof of (6.66). Let $\phi(\tau', \mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|) \in C_c^\infty([0, \infty) \times (0, C_\xi) \times \mathbb{S}^2 \times \mathbb{R} \times \mathbb{R}^2)$. Therefore $\phi \equiv 0$ when $\mathbf{x}_\perp < \delta$. For $\mathbf{x}_\perp \geq \delta$ we use the proof of Lemma 15: For $x = \eta(\mathbf{x}_\|) + \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\|)]$,

$$|\mathbf{x}_\perp| \lesssim_\xi \xi(x) = \xi(\eta(\mathbf{x}_\|) + \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\|)]) \lesssim_\xi |\mathbf{x}_\perp|,$$

and therefore $\xi(x) \gtrsim_\xi \delta$ and $\alpha(x, v) \gtrsim_\xi |\xi(x)||v|^2 \gtrsim_\xi |v|^2\delta$. Since we are considering the case $t - s > t_{\mathbf{b}}(x, v)$, from $|v|t_{\mathbf{b}}(x, v) \gtrsim \mathbf{x}_\perp \geq \delta$ we have $|v| \gtrsim_\xi \frac{\delta}{t-s}$ and finally we obtain the lower bound $\alpha(x, v) \gtrsim_\xi \frac{\delta^3}{|t-s|^2} > 0$. By the Velocity lemma, for $(x, v) \in \mathrm{supp}(\phi)$

$$\alpha(x^\ell, v^\ell) \gtrsim_\xi e^{-C|v||t^1 - t^\ell|}\alpha(x, v) \gtrsim_\xi e^{-C|v|(t-s)}\frac{\delta^3}{|t-s|^2} \gtrsim_{\xi,|t-s|,\delta,\phi} 1 > 0,$$

where we used the fact that ϕ vanishes away from a compact subset $\mathrm{supp}(\phi)$. Therefore $t^\ell(t, x, v) = t^\ell(t, \mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|)$ is smooth with respect to $\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|$ locally on $\mathrm{supp}(\phi)$ and therefore $\mathcal{M} = \{(\tau', \mathbf{x}, \mathbf{v}) \in \mathrm{supp}(\phi) : \tau' = t^\ell(t, \mathbf{x}, \mathbf{v})\}$ is a smooth manifold.

It suffices to consider the case $|\tau' - t^\ell(t, x, v)| \ll 1$. Denote $\partial_{\mathbf{e}} \in \{\partial_{\mathbf{x}_\perp}, \partial_{\mathbf{x}_{\|,1}}, \partial_{\mathbf{x}_{\|,2}}, \partial_{\mathbf{v}_\perp}, \partial_{\mathbf{v}_{\|,1}}, \partial_{\mathbf{v}_{\|,2}}\}$ and $n_{\mathcal{M}} = e_1$ to have

$$\begin{aligned}
&\int_{\{(\tau',\mathbf{x},\mathbf{v})\in\mathrm{supp}(\phi)\}} [\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau'; t, \mathbf{x}, \mathbf{v}), \partial_{\mathbf{e}}\mathbf{x}_\|(\tau'; t, \mathbf{x}, \mathbf{v}), \partial_{\mathbf{e}}\mathbf{v}_\|(\tau'; t, \mathbf{x}, \mathbf{v})] \\
&\quad \times\phi(\tau', \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\tau' \\
&= \int_{\tau'<t^\ell} + \int_{\tau'\geq t^\ell} \\
&= \int_{\mathcal{M}} \left(\lim_{\tau'\uparrow t^\ell}[\mathbf{x}_\perp(\tau'), \mathbf{x}_\|(\tau'), \mathbf{v}_\|(\tau')] - \lim_{\tau'\downarrow t^\ell}[\mathbf{x}_\perp(\tau'), \mathbf{x}_\|(\tau'), \mathbf{v}_\|(\tau')] \right) \\
&\qquad\qquad\qquad \times \phi(\tau', \mathbf{x}, \mathbf{v})\{\mathbf{e}\cdot n_{\mathcal{M}}\}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&\quad - \int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau'), \mathbf{x}_\|(\tau'), \mathbf{v}_\|(\tau')]\partial_{\mathbf{e}}\phi(\tau', \mathbf{x}, \mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&= -\int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau'), \mathbf{x}_\|(\tau'), \mathbf{v}_\|(\tau')]\partial_{\mathbf{e}}\phi(\tau', \mathbf{x}, \mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x},
\end{aligned}$$

where we used the continuity of $[\mathbf{x}_\perp(\tau'; t, \mathbf{x}, \mathbf{v}), \mathbf{x}_\|(\tau'; t, \mathbf{x}, \mathbf{v}), \mathbf{v}_\|(\tau'; t, \mathbf{x}, \mathbf{v})]$ in terms of τ' near $t^\ell(t, \mathbf{x}, \mathbf{v})$.

Note that $\mathbf{v}_\perp(\tau'; t, \mathbf{x}, \mathbf{v})$ is discontinuous around $|\tau' - t^\ell| \ll 1$ ($\lim_{\tau'\downarrow t^\ell} \mathbf{v}_\perp(\tau') = -\lim_{\tau'\uparrow t^\ell} \mathbf{v}_\perp(\tau')$). However with crucial $\mathbf{x}_\perp(\tau')-$multiplication, we have $\mathbf{x}_\perp(t^\ell)\mathbf{v}_\perp(t^\ell) = 0$ and therefore

$$\begin{aligned}
&\int_{\{(\tau',\mathbf{x},\mathbf{v})\in\mathrm{supp}(\phi)\}} \partial_{\mathbf{e}}[\mathbf{x}_\perp(\tau'; t, \mathbf{x}, \mathbf{v})\mathbf{v}_\perp(\tau'; t, \mathbf{x}, \mathbf{v})]\phi(\tau', \mathbf{x}, \mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\tau' \\
&= \int_{\tau'<t^\ell} + \int_{\tau'\geq t^\ell}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{M}} \left(\lim_{\tau' \uparrow t^{\ell}} [\mathbf{x}_{\perp}(\tau')\mathbf{v}_{\perp}(\tau')] - \lim_{\tau' \downarrow t^{\ell}} [\mathbf{x}_{\perp}(\tau')\mathbf{v}_{\perp}(\tau')] \right) \\
&\quad \times \phi(\tau', \mathbf{x}, \mathbf{v}) \{\mathbf{e} \cdot n_{\mathcal{M}}\} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&\quad - \int_{\{\tau' \neq t^{\ell}(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_{\perp}(\tau')\mathbf{v}_{\perp}(\tau')] \partial_{\mathbf{e}} \phi(\tau', \mathbf{x}, \mathbf{v}) \mathrm{d}\tau' \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&= - \int_{\{\tau' \neq t^{\ell}(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_{\perp}(\tau'; t, \mathbf{x}, \mathbf{v})\mathbf{v}_{\perp}(\tau'; t, \mathbf{x}, \mathbf{v})] \partial_{\mathbf{e}} \phi(\tau', \mathbf{x}, \mathbf{v}) \mathrm{d}\tau' \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}.
\end{aligned}
$$

This completes the proof of (6.66).

Since $\mathbf{v}_{\perp}$ always is multiplied with $\mathbf{x}_{\perp}$ in (6.65), we may apply (6.66) and take derivative inside each $\int_{s^{i+1}}^{s^i}$ of (6.65), separating the main terms with $\partial_{\mathbf{e}}\mathbf{x}_{\parallel}$ and $\partial_{\mathbf{e}}\mathbf{v}_{\parallel}$, and treating the rest (underbraced terms) as forcing terms to obtain, for $\partial_{\mathbf{e}} \in \{\partial_{\mathbf{x}_{\perp}}, \partial_{\mathbf{x}_{\parallel,1}}, \partial_{\mathbf{x}_{\parallel,2}}, \partial_{\mathbf{v}_{\perp}}, \partial_{\mathbf{v}_{\parallel,1}}, \partial_{\mathbf{v}_{\parallel,2}}\}$,

$$
\begin{aligned}
&\partial_{\mathbf{e}}\mathbf{x}_{\parallel}(s^{i+1}) = \partial_{\mathbf{e}}\mathbf{x}_{\parallel}(s^{i}) - \int_{s^{i+1}}^{s^i} \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(\tau)\mathrm{d}\tau, \\
&\partial\mathbf{v}_{\parallel}(s^{i+1}) = \partial_{\mathbf{e}}E(s^{i+1})\mathbf{x}_{\perp}(s^{i+1}) + E(s^{i+1})\underbrace{\partial_{\mathbf{e}}\mathbf{x}_{\perp}(s^{i+1})} + \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(s^i) \\
&\quad -\partial_{\mathbf{e}}[E(\mathbf{x}_{\perp}, \mathbf{x}_{\parallel}, \mathbf{v}_{\parallel})\mathbf{x}_{\perp}](s^{i+1}) \\
&\quad + \int_{s^{i+1}}^{s^i} \underbrace{\partial_{\mathbf{e}}\mathbf{v}_{\perp}(\tau)}\partial_{\mathbf{x}_{\perp}}E(\tau)\mathbf{x}_{\perp}(\tau) + \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(\tau)\cdot\nabla_{\mathbf{x}_{\parallel}}E(\tau)\mathbf{x}_{\perp}(\tau)\mathrm{d}\tau \\
&\quad + \int_{s^{i+1}}^{s^i} \Bigg\{ \Bigg[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_{\perp}(\tau)}\partial_{\mathbf{x}_{\perp}}E(\tau) + \partial_{\mathbf{e}}\mathbf{x}_{\parallel}(\tau)\cdot\nabla_{\mathbf{x}_{\parallel}}E(\tau) + \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(\tau)\cdot\nabla_{\mathbf{v}_{\parallel}}E(\tau)\Bigg]\mathbf{v}_{\perp}(\tau) \\
&\quad + E(\tau)\underbrace{\partial_{\mathbf{e}}\mathbf{v}_{\perp}(\tau)} + \underbrace{\partial_{\mathbf{e}}\mathbf{x}_{\perp}(\tau)}\partial_{\mathbf{x}_{\perp}}D(\tau) + \partial_{\mathbf{e}}\mathbf{x}_{\parallel}(\tau)\cdot\nabla_{\mathbf{x}_{\parallel}}D(\tau) + \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(\tau)\nabla_{\mathbf{v}_{\parallel}}D(\tau)\Bigg\} \\
&\quad \cdot\nabla_{\mathbf{v}_{\parallel}}E(\tau)\mathbf{x}_{\perp}(\tau)\mathrm{d}\tau \\
&\quad + \int_{s^{i+1}}^{s^i} \Big\{ \mathbf{v}_{\perp}(\tau)[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_{\perp}(\tau)}, \partial_{\mathbf{e}}\mathbf{x}_{\parallel}(\tau), \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(\tau)]\cdot\nabla\partial_{\mathbf{x}_{\perp}}E(\tau) \\
&\quad + \mathbf{v}_{\parallel}(\tau)\cdot[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_{\perp}(\tau)}, \partial_{\mathbf{e}}\mathbf{x}_{\parallel}(\tau), \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(\tau)]\cdot\nabla\nabla_{\mathbf{x}_{\parallel}}E(\tau) \\
&\quad + F_{\parallel}(\tau)\cdot[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_{\perp}(\tau)}, \partial_{\mathbf{e}}\mathbf{x}_{\parallel}(\tau), \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(\tau)]\cdot\nabla\nabla_{\mathbf{v}_{\parallel}}E(\tau)\Big\}\mathbf{x}_{\perp}(\tau)\mathrm{d}\tau \\
&\quad + \int_{s^{i+1}}^{s^i} \{\mathbf{v}_{\perp}(\tau)\partial_{\mathbf{x}_{\perp}}E(\tau) + \mathbf{v}_{\parallel}(\tau)\cdot\nabla_{\mathbf{x}_{\parallel}}E(\tau) + F_{\parallel}(\tau)\cdot\nabla_{\mathbf{v}_{\parallel}}E(\tau)\}\underbrace{\partial_{\mathbf{e}}\mathbf{x}_{\perp}(\tau)}\mathrm{d}\tau \\
&\quad - \int_{s^{i+1}}^{s^i} \Big[\underbrace{\partial_{\mathbf{e}}\mathbf{x}_{\perp}(\tau)}, \partial_{\mathbf{e}}\mathbf{x}_{\parallel}(\tau), \partial_{\mathbf{e}}\mathbf{v}_{\parallel}(\tau)\Big]\cdot\nabla D(\tau)\mathrm{d}\tau. \qquad (6.67)
\end{aligned}
$$

Now we use (6.59) to control the underbraced term of (6.67). Notice that we cannot directly use (6.59) since now we fix the chart for whole i-th intermediate group but the estimate (6.59) is for the moving frame. (For clarity, we write the index for the chart for this part.) Note the time of bounces within the i-th intermediate group ($|t^{\ell_{i-1}} - t^{\ell_i}\|v| \simeq L_\xi$) are

$$t^{\ell_i+1} < s^{i+1} < t^{\ell_i} < t^{\ell_i-1} < \cdots < t^{\ell_{i-1}+2} < t^{\ell_{i-1}+1} < s^i < t^{\ell_{i-1}}.$$

Now we apply (6.6) and (6.59) to bound, for $\tau \in (s^{i+1}, s^i)$ and $\ell \in \{\ell_i, \ell_i - 1, \ldots, \ell_{i-1}+2, \ell_{i-1}+1, \ell_{i-1}\}$

$$\begin{aligned}
&\frac{\partial(\mathbf{x}_{\perp_\ell}(\tau), \mathbf{x}_{\|_\ell}(\tau), \mathbf{v}_{\perp_\ell}(\tau), \mathbf{v}_{\|_\ell}(\tau))}{\partial(\mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\|_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\|_1}(s^1))} \\
&= \frac{\partial(\mathbf{x}_{\perp_\ell}(\tau), \mathbf{x}_{\|_\ell}(\tau), \mathbf{v}_{\perp_\ell}(\tau), \mathbf{v}_{\|_\ell}(\tau))}{\partial(\mathbf{x}_{\perp_{\ell_i}}(\tau), \mathbf{x}_{\|_{\ell_i}}(\tau), \mathbf{v}_{\perp_{\ell_i}}(\tau), \mathbf{v}_{\|_{\ell_i}}(\tau))} \frac{\partial(\mathbf{x}_{\perp_{\ell_i}}(\tau), \mathbf{x}_{\|_{\ell_i}}(\tau), \mathbf{v}_{\perp_{\ell_i}}(\tau), \mathbf{v}_{\|_{\ell_i}}(\tau))}{\partial(\mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\|_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\|_1}(s^1))} \\
&\lesssim e^{C|t-s\|v|} \left\{ \mathbf{Id}_{6,6} + O_\xi(|\mathbf{p}^\ell - \mathbf{p}^{\ell_i}|) \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & & & \\ 0 & 1 & 1 & & \mathbf{0}_{3,3} & \\ 0 & 1 & 1 & & & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |v| & |v| & 0 & 1 & 1 \\ 0 & |v| & |v| & 0 & 1 & 1 \end{array}\right] \right\} \\
&\times \begin{bmatrix} \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} & \frac{1}{|v|} \\ \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} \\ \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} \\ \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1) & O_\xi(1) \\ \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1) & O_\xi(1) \\ \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1) & O_\xi(1) \end{bmatrix} \\
&\lesssim e^{C|t-s\|v|} \begin{bmatrix} \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} & \frac{1}{|v|} \\ \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} \\ \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^2}{|\mathbf{v}_\perp^1|^2} & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} \\ \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1) & O_\xi(1) \\ \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1) & O_\xi(1) \\ \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1) & O_\xi(1) \end{bmatrix},
\end{aligned} \tag{6.68}$$

where we have used $|\mathbf{p}^\ell - \mathbf{p}^{\ell_i}| \lesssim 1$.

We plug in (6.67) with (6.68) respectively, collecting terms with tedious but straightforward bounds (E is linear in $\mathbf{v}_\|$). We summarize the results as: for $s \in [s^{i+1}, s^i]$

$$
\begin{aligned}
\begin{bmatrix} |\frac{\partial \mathbf{x}_{\|}(s)}{\partial \mathbf{x}_{\perp}}| \\ |\frac{\partial \mathbf{v}_{\|}(s)}{\partial \mathbf{x}_{\perp}}| \end{bmatrix} &\lesssim_{\xi} \begin{bmatrix} |\frac{\partial \mathbf{x}_{\|}(s^i)}{\partial \mathbf{x}_{\perp}}| \\ |\frac{\partial \mathbf{v}_{\|}(s^i)}{\partial \mathbf{x}_{\perp}}| + |v||\frac{\partial \mathbf{x}_{\|}(s^i)}{\partial \mathbf{x}_{\perp}}| \end{bmatrix} \\
&\quad + \begin{bmatrix} \int_s^{s^i} |\frac{\partial \mathbf{v}_{\|}}{\partial \mathbf{x}_{\perp}}| \\ \int_s^{s^i} |v|^2 |\frac{\partial \mathbf{x}_{\|}}{\partial \mathbf{x}_{\perp}}| + |v||\frac{\partial \mathbf{v}_{\|}}{\partial \mathbf{x}_{\perp}}| \end{bmatrix} + \begin{bmatrix} 0 \\ e^{C|v||t-s|} \frac{|v|^2}{|\mathbf{v}_{\perp}^1|} \end{bmatrix}, \\
\begin{bmatrix} |\frac{\partial \mathbf{x}_{\|}(s)}{\partial \mathbf{x}_{\|}}| \\ |\frac{\partial \mathbf{v}_{\|}(s)}{\partial \mathbf{x}_{\|}}| \end{bmatrix} &\lesssim_{\xi} \begin{bmatrix} |\frac{\partial \mathbf{x}_{\|}(s^i)}{\partial \mathbf{x}_{\|}}| \\ |\frac{\partial \mathbf{v}_{\|}(s^i)}{\partial \mathbf{x}_{\|}}| + |v||\frac{\partial \mathbf{x}_{\|}(s^i)}{\partial \mathbf{x}_{\|}}| \end{bmatrix} \\
&\quad + \begin{bmatrix} \int_s^{s^i} |\frac{\partial \mathbf{v}_{\|}}{\partial \mathbf{x}_{\|}}| \\ \int_s^{s^i} |v|^2 |\frac{\partial \mathbf{x}_{\|}}{\partial \mathbf{x}_{\|}}| + |v||\frac{\partial \mathbf{v}_{\|}}{\partial \mathbf{x}_{\|}}| \end{bmatrix} + \begin{bmatrix} 0 \\ e^{C|v||t-s|} \frac{|v|^2}{|\mathbf{v}_{\perp}^1|} \end{bmatrix}, \\
\begin{bmatrix} |\frac{\partial \mathbf{x}_{\|}(s)}{\partial \mathbf{v}_{\perp}}| \\ |\frac{\partial \mathbf{v}_{\|}(s)}{\partial \mathbf{v}_{\perp}}| \end{bmatrix} &\lesssim_{\xi} \begin{bmatrix} |\frac{\partial \mathbf{x}_{\|}(s^i)}{\partial \mathbf{v}_{\perp}}| \\ |\frac{\partial \mathbf{v}_{\|}(s^i)}{\partial \mathbf{v}_{\perp}}| + |v||\frac{\partial \mathbf{x}_{\|}(s^i)}{\partial \mathbf{v}_{\perp}}| \end{bmatrix} \\
&\quad + \begin{bmatrix} \int_s^{s^i} |\frac{\partial \mathbf{v}_{\|}}{\partial \mathbf{v}_{\perp}}| \\ \int_s^{s^i} |v|^2 |\frac{\partial \mathbf{x}_{\|}}{\partial \mathbf{v}_{\perp}}| + |v||\frac{\partial \mathbf{v}_{\|}}{\partial \mathbf{v}_{\perp}}| \end{bmatrix} + \begin{bmatrix} 0 \\ e^{C|v||t-s|}|v| \end{bmatrix}, \\
\begin{bmatrix} |\frac{\partial \mathbf{x}_{\|}(s)}{\partial \mathbf{v}_{\|}}| \\ |\frac{\partial \mathbf{v}_{\|}(s)}{\partial \mathbf{v}_{\|}}| \end{bmatrix} &\lesssim_{\xi} \begin{bmatrix} |\frac{\partial \mathbf{x}_{\|}(s^i)}{\partial \mathbf{v}_{\|}}| \\ |\frac{\partial \mathbf{v}_{\|}(s^i)}{\partial \mathbf{v}_{\|}}| + |v||\frac{\partial \mathbf{x}_{\|}(s^i)}{\partial \mathbf{v}_{\|}}| \end{bmatrix} \\
&\quad + \begin{bmatrix} \int_s^{s^i} |\frac{\partial \mathbf{v}_{\|}}{\partial \mathbf{v}_{\|}}| \\ \int_s^{s^i} |v|^2 |\frac{\partial \mathbf{x}_{\|}}{\partial \mathbf{v}_{\|}}| + |v||\frac{\partial \mathbf{v}_{\|}}{\partial \mathbf{v}_{\|}}| \end{bmatrix} + \begin{bmatrix} 0 \\ e^{C|v||t-s|}|v| \end{bmatrix}. \qquad (6.69)
\end{aligned}
$$

We apply Lemma 4 to (6.69) and we prove the claim (6.63).

Step 10. ODE method within the time scale $|t-s| \simeq 1$: *Refinement of the estimate* (6.59)

We claim that

$$
\begin{bmatrix} |\frac{\partial \mathbf{x}_{\|_{\tilde{\ell}}}(s)}{\partial \mathbf{x}_{\perp_1}}| & |\frac{\partial \mathbf{x}_{\|_{\tilde{\ell}}}(s)}{\partial \mathbf{x}_{\|_1}}| & |\frac{\partial \mathbf{x}_{\|_{\tilde{\ell}}}(s)}{\partial \mathbf{v}_{\perp_1}}| & |\frac{\partial \mathbf{x}_{\|_{\tilde{\ell}}}(s)}{\partial \mathbf{v}_{\|_1}}| \\ |\frac{\partial \mathbf{v}_{\|_{\tilde{\ell}}}(s)}{\partial \mathbf{x}_{\perp_1}}| & |\frac{\partial \mathbf{v}_{\|_{\tilde{\ell}}}(s)}{\partial \mathbf{x}_{\|_1}}| & |\frac{\partial \mathbf{v}_{\|_{\tilde{\ell}}}(s)}{\partial \mathbf{v}_{\perp_1}}| & |\frac{\partial \mathbf{v}_{\|_{\tilde{\ell}}}(s)}{\partial \mathbf{v}_{\|_1}}| \end{bmatrix} \lesssim C^{C|v||t-s|} \begin{bmatrix} \frac{|v|}{|\mathbf{v}_{\perp}^1|} & \frac{|v|}{|\mathbf{v}_{\perp}^1|} & \frac{1}{|v|} & \frac{1}{|v|} \\ \frac{|v|^2}{|\mathbf{v}_{\perp}^1|} & \frac{|v|^2}{|\mathbf{v}_{\perp}^1|} & 1 & 1 \end{bmatrix}, \qquad (6.70)
$$

where $\tilde{\ell} = \left[\frac{|t-s||v|}{L_{\xi}}\right]$.

Proof of the claim (6.70). By the chain rule

$$
\begin{bmatrix} D_{\mathbf{x}}\mathbf{x}_{\|_i} & D_{\mathbf{v}}\mathbf{x}_{\|_i} \\ D_{\mathbf{x}}\mathbf{v}_{\|_i} & D_{\mathbf{v}}\mathbf{v}_{\|_i} \end{bmatrix} = \frac{\partial(\mathbf{x}_{\|_i}, \mathbf{v}_{\|_i})}{\partial(\mathbf{x}_{\|_{i-1}}, \mathbf{v}_{\|_{i-1}})} \begin{bmatrix} D_{\mathbf{x}}\mathbf{x}_{\|_{i-1}} & D_{\mathbf{v}}\mathbf{x}_{\|_{i-1}} \\ D_{\mathbf{x}}\mathbf{v}_{\|_{i-1}} & D_{\mathbf{v}}\mathbf{v}_{\|_{i-1}} \end{bmatrix}.
$$

Note, from (6.6)

$$\frac{\partial(\mathbf{x}_{\|_i}, \mathbf{v}_{\|_i})}{\partial(\mathbf{x}_{\|_{i-1}}, \mathbf{v}_{\|_{i-1}})} \leq C \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline |v| & 1 \end{array}\right] \leq C \left[\begin{array}{c|c} 1 & \frac{1}{|v|} \\ \hline |v| & 1 \end{array}\right] := C\mathbf{B}.$$

Denote

$$\mathbf{D}_i(s) = \begin{bmatrix} |D_{\mathbf{x}}\mathbf{x}_{\|_i}(s)| & |D_{\mathbf{v}}\mathbf{x}_{\|_i}(s)| \\ |D_{\mathbf{x}}\mathbf{v}_{\|_i}(s)| & |D_{\mathbf{v}}\mathbf{v}_{\|_i}(s)| \end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \frac{|v|^2}{|\mathbf{v}_\perp^1|} & 1 \end{bmatrix}.$$

Note that from (6.63)

$$\mathbf{D}_i(s^{i+1}) \leq C\mathbf{B}\mathbf{D}_i(s^i) + C\mathbf{B}\mathbf{G}.$$

Therefore, by induction,

$$\begin{aligned} \mathbf{D}_{[\frac{|t-s\|v|}{L_\xi}]}(s) &\leq C\mathbf{D}_{[\frac{|t-s\|v|}{L_\xi}]}(s^{[\frac{|t-s\|v|}{L_\xi}]}) + C\mathbf{B}\mathbf{G} \\ &\leq C^2\mathbf{B}\mathbf{D}_{[\frac{|t-s\|v|}{L_\xi}]-1}(s^{[\frac{|t-s\|v|}{L_\xi}]}) + C\mathbf{B}\mathbf{G} \\ &\leq C^2\mathbf{B}\mathbf{D}_{[\frac{|t-s\|v|}{L_\xi}]-1}(s^{[\frac{|t-s\|v|}{L_\xi}]-1}) + C^3\mathbf{B}\mathbf{G} + C\mathbf{B}\mathbf{G} \\ &\leq C^3\mathbf{B}^2\mathbf{D}_{[\frac{|t-s\|v|}{L_\xi}]-1}(s^{[\frac{|t-s\|v|}{L_\xi}]-2}) + \{C^2\mathbf{B} + \mathbf{Id}\}C\mathbf{B}\mathbf{G} \\ &\leq C^4\mathbf{B}^3\mathbf{D}_{[\frac{|t-s\|v|}{L_\xi}]-2}(s^{[\frac{|t-s\|v|}{L_\xi}]-2}) + \{C^3\mathbf{B}^2 + C^2\mathbf{B} + \mathbf{Id}\}C\mathbf{B}\mathbf{G} \\ &\ \ \vdots \\ &\lesssim C^{C|t-s\|v|}\mathbf{B}^{C[|t-s\|v|]}\mathbf{D}_1(s^1) + \sum_{i=0}^{C[|t-s\|v|]} C^{i+1}\mathbf{B}^i\mathbf{B}\mathbf{G}. \end{aligned}$$

By a direct computation yields $\mathbf{B}^j \leq 2^j\mathbf{B}$. Therefore

$$\mathbf{D}_{[\frac{|t-s\|v|}{L_\xi}]}(s) \lesssim C^{C|t-s\|v|}\mathbf{B}\{\mathbf{D}_1(s^1) + \mathbf{B}\mathbf{G}\}.$$

From (6.16) we have $\mathbf{D}_1(s^1) \lesssim \left[\begin{array}{c|c} 1 & \frac{1}{|v|} \\ \hline |v| & 1 \end{array}\right]$ and we conclude our claim (6.70).

With these estimates, we refine (6.59) to give a final estimate for the case that $|\xi(X_{\mathbf{cl}}(\tau; t, x, v))| < \delta$ for all $\tau \in [s, t]$:

$$\frac{\partial(s^{\ell_*}, \mathbf{x}_\perp(s^{\ell_*}), \mathbf{x}_\parallel(s^{\ell_*}), \mathbf{v}_\perp(s^{\ell_*}), \mathbf{v}_\parallel(s^{\ell_*}))}{\partial(s^1, \mathbf{x}_\perp(s^1), \mathbf{x}_\parallel(s^1), \mathbf{v}_\perp(s^1), \mathbf{v}_\parallel(s^1))} \lesssim C^{C|v|(t-s)} \left[\begin{array}{c|cc|cc} 0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline |\mathbf{v}_\perp^1| & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} \\ |v| & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{|v|}{|\mathbf{v}_\perp^1|} & |t-s| & |t-s| \\ \hline |v|^2 & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1) \\ |v|^2 & \frac{|v|^2}{|\mathbf{v}_\perp^1|} & \frac{|v|^2}{|\mathbf{v}_\perp^1|} & O_\xi(1) & O_\xi(1) \end{array}\right], \tag{6.71}$$

and from (6.32) and (6.38)

$$\begin{aligned} &\frac{\partial(X_{\mathbf{cl}}(s; t, x, v), V_{\mathbf{cl}}(s; t, x, v))}{\partial(t, x, v)} \\ &\lesssim C^{C|v|(t-s)} \frac{\partial(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, \mathbf{X}_{\mathbf{cl}}(s^{\ell_*}), \mathbf{V}_{\mathbf{cl}}(s^{\ell_*}))} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ |v| & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline |v|^2 & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right] \\ &\times \frac{\partial(s^1, \mathbf{x}_\perp(s^1), \mathbf{x}_\parallel(s^1), \mathbf{v}_\perp(s^1), \mathbf{v}_\parallel(s^1))}{\partial(t, x, v)} \\ &\lesssim C^{C|v|(t-s)} \begin{bmatrix} |v| & 1 & |s^{\ell_*} - s| \\ \mathbf{0}_{3,1} & |v| & 1 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline |v| & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline |v|^2 & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right] \\ &\times \begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & 1 & |t - s^1| \\ \mathbf{0}_{3,1} & |v| & 1 \end{bmatrix} \\ &\lesssim C^{C|v|(t-s)} \left[\begin{array}{c|c|c} |v| & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline |v|^2 & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right]. \end{aligned} \tag{6.72}$$

Finally from (6.61) and (6.72), we conclude, for all $\tau \in [s, t]$

$$\frac{\partial(X_{\mathbf{cl}}(s; t, x, v), V_{\mathbf{cl}}(s; t, x, v))}{\partial(t, x, v)} \leq Ce^{C|v|(t-s)} \left[\begin{array}{c|c|c} |v| & \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline |v|^2 & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & \frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right]_{6\times 7}.$$

From the Velocity lemma (Lemma 2),

$$
\begin{aligned}
|\mathbf{v}^1_\perp| &= |v^1 \cdot [-n(x^1)]| = |V_{\mathbf{cl}}(t^1; t, x, v) \cdot n(X_{\mathbf{cl}}(t^1; t, x, v))| \\
&= \sqrt{\alpha(X_{\mathbf{cl}}(t^1), V_{\mathbf{cl}}(t^1))} \geq e^{\mathcal{C}|v||t-t^1|}\sqrt{\alpha(x, v)} \gtrsim \sqrt{\alpha(x, v)},
\end{aligned}
$$

and this completes the proof for the case (6.60). □

Proof of Theorem 3 We use the approximation sequence (2.16) with (2.19). Due to Lemma 7 we have $\sup_m \sup_{0\leq t\leq T} \|e^{\theta|v|^2} f^m(t)\|_\infty \lesssim_{\xi,T} P(\|e^{\theta'|v|^2} f_0\|_\infty)$.

Now we claim that the distributional derivatives coincide with the piecewise derivatives. This is due to Proposition 1 and Proposition 2 together with an invariant property of $\Gamma(f, f) = \Gamma_{\text{gain}}(f, f) - \nu(\sqrt{\mu} f) f$: *Assume* $f^m(v) = f^{m-1}(\mathcal{O}v)$ *holds for some orthonormal matrix. Then*

$$
\Gamma(f^m, f^m)(v) = \Gamma(f^{m-1}, f^{m-1})(\mathcal{O}v). \tag{6.73}
$$

Using (6.73), we apply Proposition 1 to have

$$
\begin{aligned}
&f^m(t, x, v) \\
&= e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1}, t^\ell)}(s)\nu(\sqrt{\mu} f^{m-\ell})(s)\mathrm{d}s} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \\
&\quad + \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1}, t^\ell)}(s) e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1}, t^j)}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \\
&\quad \times \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\mathrm{d}s.
\end{aligned}
$$

Now we consider the spatial and velocity derivatives. In the sense of distributions, we have for $\partial_{\mathbf{e}} \in \{\nabla_x, \nabla_v\}$

$$
\partial_{\mathbf{e}} f^m(t, x, v) = \mathrm{I}_{\mathbf{e}} + \mathrm{II}_{\mathbf{e}} + \mathrm{III}_{\mathbf{e}}. \tag{6.74}
$$

Here

$$
\mathrm{I}_{\mathbf{e}} = e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1}, t^\ell)}(s)\nu(F^{m-\ell})(s)\mathrm{d}s} \partial_{\mathbf{e}}[X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)] \cdot \nabla_{x,v} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)),
$$

and

$$
\begin{aligned}
\mathrm{II}_{\mathbf{e}} = &\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1}, t^\ell)}(s) e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1}, t^j)}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \\
&\times \partial_{\mathbf{e}}\left[\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\right] \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) e^{-\int_s^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \\
&\quad \times \int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\partial_{\mathbf{e}}[\nu(F^{m-j})(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))]\mathrm{d}\tau \\
&\quad \times \Gamma_{\mathrm{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\mathrm{d}s \\
&-e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\nu(F^{m-\ell})(s)\mathrm{d}s}\ f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \\
&\quad \times \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\partial_{\mathbf{e}}\left[\nu(F^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\right]\mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{III}_{\mathbf{e}} = &\sum_{\ell=0}^{\ell_*(0)} \Big[-\partial_{\mathbf{e}} t^\ell \lim_{s\uparrow t^\ell} \nu(\sqrt{\mu} f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)) \\
&\qquad + \partial_{\mathbf{e}} t^{\ell+1} \lim_{s\downarrow t^{\ell+1}} \mu(\sqrt{\mu} f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\Big] \\
&\qquad \times e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\nu(\sqrt{\mu} f^{m-\ell})(s)} \\
&+\sum_{\ell=0}^{\ell_*(0)} \Big[\lim_{s\uparrow t^\ell} e^{-\int_s^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \\
&\qquad \times \Gamma_{\mathrm{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)) \\
&\quad - \lim_{s\downarrow t^{\ell+1}} e^{-\int_s^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \\
&\qquad \times \Gamma_{\mathrm{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)) \\
&+\int_0^t \sum_\ell \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) \sum_{j=0}^{\ell_*(s)} \Big[-\lim_{\tau\downarrow t^j} \nu(F^{m-j})(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau)) \\
&\qquad + \lim_{\tau\uparrow t^{j+1}} \nu(F^{m-j})(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))\Big] \\
&\qquad \times e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \\
&\qquad \times \Gamma_{\mathrm{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)).
\end{aligned}
$$

For $\mathbf{III_e}$ we rearrange the summation, use (6.28) and apply (6.73) to get

$$
\begin{aligned}
\mathbf{III_e} = & \sum_{\ell=0}^{\ell_*(0)} \Big[-\nu(\sqrt{\mu} f^{m-\ell})(t^\ell, x^\ell, v^\ell) + \nu(\sqrt{\mu} f^{m-\ell+1})(t^\ell, x^\ell, R_{x^\ell} v^\ell) \Big] \\
& \times \partial_{\mathbf{e}} t^\ell e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1}, t^\ell)}(s) \nu(\sqrt{\mu} f^{m-\ell})(s)} \\
& + \sum_{\ell=0}^{\ell_*(0)} e^{-\int_{t^\ell}^t \sum_j \mathbf{1}_{[t^{j+1}, t^j)}(\tau) \nu(\sqrt{\mu} f^{m-j})(\tau) \mathrm{d}\tau} \Big[\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(t^\ell, x^\ell, v^\ell) \\
& - \Gamma_{\text{gain}}(f^{m-\ell+1}, f^{m-\ell+1})(t^\ell, x^\ell, R_{x^\ell} v^\ell) \Big] \\
& + \int_0^t \sum_\ell \mathbf{1}_{[t^{\ell+1}, t^\ell)}(s)^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1}, t^j)}(\tau) \nu(F^{m-j})(\tau) \mathrm{d}\tau} \\
& \times \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)) \\
& \times \sum_{\ell=0}^{\ell_*(s)} \Big[-\nu(\sqrt{\mu} f^{m-\ell})(t^\ell, x^\ell, v^\ell) + \nu(\sqrt{\mu} f^{m-\ell+1})(t^\ell, x^\ell, R_{x^\ell} v^\ell) \Big] \\
= & \ 0.
\end{aligned}
$$

Proof of (6.73). The proof is due to the change of variables

$$
\tilde{u} = \mathcal{O} u, \quad \tilde{\omega} = \mathcal{O} \omega, \quad \mathrm{d}\tilde{u} = \mathrm{d}u, \quad \mathrm{d}\tilde{\omega} = \mathrm{d}\omega.
$$

Note

$$
\begin{aligned}
& \Gamma(f^m, f^m)(v) \\
& = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v-u|^\kappa q_0 \left(\frac{v-u}{|v-u|} \cdot \omega \right) \sqrt{\mu(u)} \\
& \times \left\{ f^m(u - [(u-v)\cdot\omega]\omega) f^m(v + [(u-v)\cdot\omega]\omega) - f^m(u) f^m(v) \right\} \mathrm{d}\omega \mathrm{d}u \\
& = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |\mathcal{O}v - \mathcal{O}u|^\kappa q_0 \left(\frac{\mathcal{O}v - \mathcal{O}u}{|\mathcal{O}v - \mathcal{O}u|} \cdot \mathcal{O}\omega \right) \sqrt{\mu(\mathcal{O}u)} \\
& \times \Big\{ f^{m-1}(\mathcal{O}u - [(\mathcal{O}u - \mathcal{O}v)\cdot \mathcal{O}\omega]\mathcal{O}\omega) f^{m-1}(\mathcal{O}v + [(\mathcal{O}u - \mathcal{O}v)\cdot\mathcal{O}\omega]\mathcal{O}\omega) \\
& \quad - f^{m-1}(\mathcal{O}u) f^{m-1}(\mathcal{O}v) \Big\} \mathrm{d}\omega \mathrm{d}u \\
& = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |\mathcal{O}v - \tilde{u}|^\kappa q_0 \left(\frac{\mathcal{O}v - \tilde{u}}{|\mathcal{O}v - \tilde{u}|} \cdot \tilde{\omega} \right) \sqrt{\mu(\tilde{u})} \\
& \quad \times \left\{ f^{m-1}(\tilde{u} - [(\tilde{u} - \mathcal{O}v)\cdot\tilde{\omega}]\tilde{\omega}) f^{m-1}(\mathcal{O}v + [(\tilde{u} - \mathcal{O}v)]\cdot\tilde{\omega})\tilde{\omega} \right. \\
& \quad \left. - f^{m-1}(\tilde{u}) f^{m-1}(\mathcal{O}v) \right\} \mathrm{d}\tilde{\omega} \mathrm{d}\tilde{u} \\
& = \Gamma(f^{m-1}, f^{m-1})(\mathcal{O}v).
\end{aligned}
$$

This proves (6.73). Especially we can apply (6.73) for the specular reflection BC (2.19) with $\mathcal{O}v = R_x v$ as well as the bounce-back reflection BC (2.20) with $\mathcal{O}v = -v$.

Using Lemma 6 and (2.12), we obtain for $\partial_{\mathbf{e}} \in \{\nabla_x, \nabla_v\}$

$$
\begin{aligned}
\mathrm{II}_{\mathbf{e}} \lesssim\ & P(\|e^{\theta|v|^2} f_0\|_\infty) \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) |\partial_{\mathbf{e}} X_{\mathbf{cl}}(s)| \\
& \times \int_{\mathbb{R}^3} \frac{e^{-C_\theta |V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} |\nabla_x f^{m-\ell}(s, X_{\mathbf{cl}}(s), u)| \mathrm{d}u \mathrm{d}s \\
& + P(\|e^{\theta|v|^2} f_0\|_\infty) \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) |\partial_{\mathbf{e}} V_{\mathbf{cl}}(s)| \\
& \times \int_{\mathbb{R}^3} \frac{e^{-C_\theta |V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} |\nabla_v f^{m-\ell}(s, X_{\mathbf{cl}}(s), u)| \mathrm{d}u \mathrm{d}s \\
& + t P(\|e^{\theta|v|^2} f_0\|_\infty) \langle v\rangle^\kappa e^{-\theta|v|^2} \sup_{0\le s\le t} |\partial_{\mathbf{e}} V(s;t,x,v)|.
\end{aligned}
$$

We shall estimate the following:

$$
e^{-\varpi\langle v\rangle t} \frac{[\alpha(x,v)]^\beta}{\langle v\rangle^{b+1}} |\partial_x f(t,x,v)|, \quad e^{-\varpi\langle v\rangle t} \frac{|v|[\alpha(x,v)]^{\beta-\frac{1}{2}}}{\langle v\rangle^{b}} |\partial_v f(t,x,v)|.
$$

From (1.35), the Velocity lemma (Lemma 2), Lemma 7, and $F^m \ge 0$ for all m, with $\varpi \gg 1$

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t} \frac{1}{\langle v\rangle^{b+1}} [\alpha(x,v)]^\beta\ \mathrm{I}_{\mathbf{x}} \\
& \lesssim_{\xi,t} e^{-\varpi\langle v\rangle t} \frac{1}{\langle v\rangle^{b+1}} [\alpha(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))]^\beta e^{2C|v|t} \\
& \quad \times \left\{ \frac{|v|}{\sqrt{\alpha(x,v)}} |\partial_x f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))| + \frac{|v|^3}{\alpha(x,v)} |\partial_v f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))| \right\} \\
& \lesssim_{\xi,t} \left\| \frac{|v|}{\langle v\rangle^{b+1}} \alpha^{\beta-\frac{1}{2}} \partial_x f_0 \right\|_\infty + \left\| \frac{|v|^3}{\langle v\rangle^{b+1}} \alpha^{\beta-1} \partial_v f_0 \right\|_\infty \\
& \lesssim_{\xi,t} \left\| \frac{\alpha^{\beta-\frac{1}{2}}}{\langle v\rangle^{b}} \partial_x f_0 \right\|_\infty + \left\| \frac{|v|^2\alpha^{\beta-1}}{\langle v\rangle^{b}} \partial_v f_0 \right\|_\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^b}[\alpha(x,v)]^{\beta-\frac12}\ \mathrm{I}_{\mathbf{v}} \\
&\lesssim_{\xi,t}\ e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^b}[\alpha(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))]^{\beta-\frac12}e^{2\mathcal{C}|v|t} \\
&\quad\times\left\{\frac{1}{|v|}|\partial_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|+\frac{|v|}{\sqrt{\alpha(x,v)}}|\partial_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|\right\} \\
&\lesssim_{\xi,t}\ \left\|\frac{\alpha^{\beta-\frac12}}{\langle v\rangle^b}\partial_x f_0\right\|_\infty+\left\|\frac{|v|^2}{\langle v\rangle^b}\alpha^{\beta-1}\partial_v f_0\right\|_\infty,
\end{aligned}
$$

where we have used $\alpha(x,v)\lesssim_\xi |v|^2$ and the choice of $\varpi\gg 1$.

From Lemma 5, Lemma 6, and Lemma 7,

$$
\begin{aligned}
\mathrm{II}_{\mathbf{e}} \lesssim_t\ & P(\|e^{\theta|v|^2}f_0\|_\infty)\int_0^t \mathrm{d}s\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\int_{\mathbb{R}^3}\mathrm{d}u\,\frac{e^{-C_\theta|u-V_{\mathbf{cl}}(s)|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}} \\
&\times\Big\{|\partial_{\mathbf{e}}X_{\mathbf{cl}}(s)||\partial_x f^{m-j}(s,X_{\mathbf{cl}}(s),u)|+|\partial_{\mathbf{e}}V_{\mathbf{cl}}(s)| \\
&\times\Big(1+|\partial_v f^{m-j}(s,X_{\mathbf{cl}}(s),u)|\Big)\Big\}.
\end{aligned}
$$

Now we use (1.35) to have

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{[\alpha(x,v)]^\beta}{\langle v\rangle^{b+1}}\ \mathrm{II}_{\mathbf{x}}\lesssim_{t,\xi}\ P(\|e^{\theta|v|^2}f_0\|_\infty) \\
&\times\Bigg\{\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v|\|t-s|}\frac{|v|[\alpha(x,v)]^{\beta-\frac12}}{[\alpha(X_{\mathbf{cl}}(s),u)]^\beta}\frac{\langle u\rangle^{b+1}}{\langle v\rangle^{b+1}}\mathrm{d}u\mathrm{d}s \\
&\times\sup_m\sup_{0\le s\le t}\left\|e^{-\varpi\langle u\rangle s}\frac{[\alpha(X_{\mathbf{cl}}(s),u)]^\beta}{\langle u\rangle^{b+1}}\partial_x f^{m-j}(s,X_{\mathbf{cl}}(s),u)\right\|_\infty \\
&+\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v\|t-s|}\frac{\langle u\rangle^b}{\langle v\rangle^b}\frac{|v|^2[\alpha(x,v)]^{\beta-1}}{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac12}} \\
&\times\sup_m\sup_{0\le s\le t}\left\|e^{-\varpi\langle u\rangle s}\frac{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac12}}{\langle u\rangle^b}\partial_v f^{m-j}(s,X_{\mathbf{cl}}(s),u)\right\|_\infty\Bigg\}.
\end{aligned}
$$

We first claim that

$$
e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v|(t-s)}e^{-C'|v-u|^2}\lesssim e^{-\frac{\varpi\langle v\rangle}{2}(t-s)}e^{C''(s+s^2)}e^{-C''|v-u|^2}. \tag{6.75}
$$

Using $\langle u\rangle \leq 1+|u| \leq 1+|v|+|u-v| \leq 1+\langle v\rangle+|v-u|$, we bound the first three exponents as

$$\begin{aligned}&-(\varpi-C)\langle v\rangle(t-s)-\varpi(\langle v\rangle-\langle u\rangle)s\\&\quad\leq -(\varpi-C)\langle v\rangle(t-s)+\varpi|v-u|s+\varpi s.\end{aligned}$$

Then we have, for $0<\sigma\ll 1$

$$\varpi|v-u|s=\frac{\sigma\varpi^2}{2}|v-u|^2+\frac{s^2}{2\sigma}-\frac{1}{2\sigma}\,[s-\sigma\varpi|v-u|]^2\leq \frac{\sigma\varpi^2}{2}|v-u|^2+\frac{s^2}{2\sigma},$$

to bound the whole exponents of (6.75) by

$$\begin{aligned}&-(\varpi-C)\langle v\rangle(t-s)+\varpi|v-u|s-C'|v-u|^2+\varpi s\\&\leq -(\varpi-C)\langle v\rangle(t-s)-(C-\frac{\sigma\varpi^2}{2})|v-u|^2+\frac{s^2}{2\sigma}+\varpi s\\&\leq -(\varpi-C)\langle v\rangle(t-s)-C_{\sigma,\varpi}|v-u|^2+C'_{\sigma,\varpi}\{s^2+s\}.\end{aligned}$$

Hence we prove the claim (6.75) for $\varpi\gg 1$.

Now we use (6.75) to bound

$$\begin{aligned}&e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b+1}}[\alpha(x,v)]^{\beta}\ \mathrm{II}_{\mathbf{x}}\\&\lesssim_{t,\xi}\ P(\|e^{\theta|v|^2}f_0\|_\infty)\times\\&\quad\times\Bigg\{\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi\langle v\rangle}{2}(t-s)}\frac{e^{-C'_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{\langle u\rangle^{b+1}}{\langle v\rangle^{b+1}}\frac{\langle v\rangle[\alpha(x,v)]^{\beta-\frac12}}{[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta}}\mathrm{d}u\mathrm{d}s}_{(\mathbf{A})}\\&\qquad\times\sup_m\sup_{0\leq s\leq t}\left\|e^{-\varpi\langle v\rangle s}\frac{\alpha^{\beta}}{\langle v\rangle^{b+1}}\partial_x f^m(s)\right\|_\infty\\&\quad+\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi\langle v\rangle}{2}(t-s)}\frac{e^{-C'_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{\langle u\rangle^{b}}{\langle v\rangle^{b}}\frac{|v|^2[\alpha(x,v)]^{\beta-1}}{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac12}}\mathrm{d}u\mathrm{d}s}_{(\mathbf{B})}\\&\qquad\times\sup_m\sup_{0\leq s\leq t}\left\|e^{-\varpi\langle v\rangle s}\frac{|v|\alpha^{\beta-\frac12}}{\langle v\rangle^{b}}\partial_v f^m(s)\right\|_\infty\Bigg\}.\end{aligned}\tag{6.76}$$

For (**A**) we use (1.29) with $Z=\langle v\rangle[\alpha(x,v)]^{\beta-\frac12}$ and $l=\frac{\varpi}{2}$ and $r=b+1$. For (**B**) we use (4.12) with $\beta\mapsto\beta-\frac12$ and $Z=\langle v\rangle[\alpha(x,v)]^{\beta-1}$ and $l=\frac{\varpi}{2}$ and $r=b$. Then

$$(\mathbf{A}),\ (\mathbf{B}) \ll 1.$$

Similarly, but with a different weight $e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^b}[\alpha(x,v)]^{\beta-\frac{1}{2}}$, we use (1.35) to have

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^b}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ \mathrm{II}_{\mathbf{v}}\\
&\quad\lesssim_{t,\xi} P(\|e^{\theta|v|^2}f_0\|_\infty)\\
&\quad\times\left\{\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v\|t-s|}\frac{\langle v\rangle[\alpha(x,v)]^{\beta-\frac{1}{2}}}{[\alpha(X_{\mathbf{cl}}(s),u)]^\beta}\frac{\langle u\rangle^{b+1}}{\langle v\rangle^{b+1}}\mathrm{d}u\mathrm{d}s\right.\\
&\quad\times\sup_m\sup_{0\le s\le t}\left\|e^{-\varpi\langle u\rangle s}\frac{[\alpha(X_{\mathbf{cl}}(s),u)]^\beta}{\langle u\rangle^{b+1}}\partial_x f^m(s,X_{\mathbf{cl}}(s),u)\right\|_\infty\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|u-V_{\mathbf{cl}}(s)|^{2-\kappa}}e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v\|t-s|}\frac{\langle u\rangle^b}{\langle v\rangle^b}\frac{|v|^2[\alpha(x,v)]^{\beta-1}}{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac{1}{2}}}\\
&\quad\left.\times\sup_m\sup_{0\le s\le t}\left\|e^{-\varpi\langle u\rangle s}\frac{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac{1}{2}}}{\langle u\rangle^b}\partial_v f^m(s,X_{\mathbf{cl}}(s),u)\right\|_\infty\right\}.
\end{aligned}$$

Again we use (6.75) and (1.29) and (4.12) exactly as (6.76). Therefore for $0<\delta=\delta(\|e^{\theta|v|^2}f_0\|_\infty)\ll 1$

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{b+1}}[\alpha(x,v)]^\beta\mathrm{II}_{\mathbf{x}}+e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^b}[\alpha(x,v)]^{\beta-\frac{1}{2}}\mathrm{II}_{\mathbf{v}}\\
&\quad\lesssim\ \delta\left\{\sup_m\sup_{0\le s\le t}\left\|e^{-\varpi\langle v\rangle s}\frac{\alpha^\beta}{\langle v\rangle^{b+1}}\partial_x f^m(s)\right\|_\infty\right.\\
&\qquad\left.+\sup_m\sup_{0\le s\le t}\left\|e^{-\varpi\langle v\rangle s}\frac{|v|\alpha^{\beta-\frac{1}{2}}}{\langle v\rangle^b}\partial_v f^m(s)\right\|_\infty\right\}.
\end{aligned}$$

Collecting all the terms, for $1<\beta<\frac{3}{2}$ and $b\in\mathbb{R}$ with $\varpi\gg 1$ and $0<\delta\ll 1$, we get

$$\begin{aligned}
&\sup_m\sup_{0\le s\le t}\left\|e^{-\varpi\langle v\rangle t}\frac{\alpha^\beta}{\langle v\rangle^{b+1}}\partial_x f^m(t)\right\|_\infty+\sup_m\sup_{0\le s\le t}\left\|e^{-\varpi\langle v\rangle t}\frac{|v|\alpha^{\beta-\frac{1}{2}}}{\langle v\rangle^b}\partial_v f^m(t)\right\|_\infty\\
&\lesssim\left\|\frac{\alpha^{\beta-\frac{1}{2}}}{\langle v\rangle^b}\partial_x f_0\right\|_\infty+\left\|\frac{|v|^2\alpha^{\beta-1}}{\langle v\rangle^b}\partial_v f_0\right\|_\infty+P(\left\|e^{\theta|v|^2}f_0\right\|_\infty).
\end{aligned}$$

We remark that this sequence f^m is Cauchy in $L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}^3)$ for $0<T\ll 1$. Therefore the limit function f is a solution of the Boltzmann

equation satisfying the specular reflection BC. On the other hand, due to the weak lower semi-continuity of L^p, $p > 1$, we pass a limit $\partial f^m \rightharpoonup \partial f$ weakly in the weighted $L^\infty-$norm.

Now we consider the continuity of $e^{-\varpi \langle v\rangle t} \langle v\rangle^{-1} \alpha^\beta \partial_x f$ and $e^{-\varpi \langle v\rangle t} |v| \alpha^{\beta - \frac{1}{2}} \partial_v f$. Remark that $e^{-\varpi \langle v\rangle t} \langle v\rangle^{-1} \alpha^\beta \partial_x f^m$ and $e^{-\varpi \langle v\rangle t} |v| \alpha^{\beta-\frac{1}{2}} \partial_v f^m$ satisfy all the conditions of Proposition 2. Therefore we conclude

$$
\begin{aligned}
e^{-\varpi \langle v\rangle t} \langle v\rangle^{-1} \alpha^\beta \partial_x f^m &\in C^0([0, T] \times (\bar{\Omega} \times \mathbb{R}^3 \setminus \gamma_0)),\\
e^{-\varpi \langle v\rangle t} |v| \alpha^{\beta-\frac{1}{2}} \partial_v f^m &\in C^0([0, T] \times (\bar{\Omega} \times \mathbb{R}^3 \setminus \gamma_0)).
\end{aligned}
$$

Now we follow $W^{1,\infty}$ estimate proof for $e^{-\varpi \langle v\rangle t} \langle v\rangle^{-1} \alpha^\beta [\partial_x f^{m+1} - \partial_x f^m]$ and $e^{-\varpi \langle v\rangle t} |v| \alpha^{\beta-\frac{1}{2}} [\partial_v f^{m+1} - \partial f^m]$ to show that $e^{-\varpi \langle v\rangle t} \langle v\rangle^{-1} \alpha^\beta \partial_x f^m$ and $e^{-\varpi \langle v\rangle t} |v| \alpha^{\beta-\frac{1}{2}} \partial_v f^m$ are Cauchy in L^∞. Then we pass a limit $e^{-\varpi \langle v\rangle t} \langle v\rangle^{-1} \alpha^\beta \partial_x f^m \to e^{-\varpi \langle v\rangle t} \langle v\rangle^{-1} \alpha^\beta \partial_x f$ and $e^{-\varpi \langle v\rangle t} |v| \alpha^{\beta-\frac{1}{2}} \partial_v f^m \to e^{-\varpi \langle v\rangle t} |v| \alpha^{\beta-\frac{1}{2}} \partial_v f$ strongly in L^∞ so that $e^{-\varpi \langle v\rangle t} \langle v\rangle^{-1} \alpha^\beta \partial_x f \in C^0([0, T^*] \times (\bar{\Omega} \times \mathbb{R}^3 \setminus \gamma_0))$ and $e^{-\varpi \langle v\rangle t} |v| \alpha^{\beta-\frac{1}{2}} \partial_v f \in C^0([0, T] \times (\bar{\Omega} \times \mathbb{R}^3 \setminus \gamma_0))$. □

7 Bounce-back reflection BC

We recall the bounce-back cycles from (iv) of Definition 2: $(t^0, x^0, v^0) = (t, x, v)$ and for $\ell \geq 1$,

$$
\begin{aligned}
t^\ell &= t^1 - (\ell - 1) t_{\mathbf{b}}(x^1, v^1),\\
x^\ell &= \frac{1 - (-1)^\ell}{2} x^1 + \frac{1 + (-1)^\ell}{2} x^2,\\
v^{\ell+1} &= (-1)^{\ell+1} v,
\end{aligned}
$$

where $t_{\mathbf{b}}(x, v)$ is defined in (1.27).

Lemma 17 *For all* $0 \leq s \leq t$,

$$
\begin{aligned}
\min\{\alpha(x^1, v^1), \alpha(x^2, v^2)\} &\lesssim_\Omega \alpha(X_{\mathbf{cl}}(s; t, x, v), V_{\mathbf{cl}}(s; t, x, v))\\
&\lesssim_\Omega \max\{\alpha(x^1, v^1), \alpha(x^2, v^2)\}.
\end{aligned}
$$

For $\ell_*(s; t, x, v) \in \mathbb{N}$ *(therefore* $t^{\ell_*+1}(t, x, v) \leq s \leq t^{\ell_*}(t, x, v)$*)*

$$
\ell_*(s; t, x, v) \ \leq \ \frac{|t - s|}{t_{\mathbf{b}}(x_1, v_1)} \ \lesssim_\Omega \ \frac{|t - s||v|^2}{\sqrt{\alpha(x, v)}}.
$$

For all $0 \leq s \leq t$ *uniformly*

$$
\begin{aligned}
|\partial_{x_i} t^{\ell}(t,x,v)| &= \left| -\ell \frac{\partial_{x_i}\xi(x^1)}{v\cdot\nabla\xi(x^1)} - (\ell-1)\frac{\partial_{x_i}\xi(x^2)}{-v\cdot\nabla\xi(x^2)} \right| \lesssim_{\Omega} \frac{t|v|^2}{\alpha(x,v)},\\
|\partial_{v_i} t^{\ell}(t,x,v)| &= \left| \ell t_{\mathbf{b}}(x,v) \frac{\partial_{x_i}\xi(x^1)}{v\cdot\nabla\xi(x^1)} + (\ell-1) t_{\mathbf{b}}(x,-v)\frac{\partial_{x_i}\xi(x^2)}{-v\cdot\nabla\xi(x^2)} \right| \\
&\lesssim_{\Omega} \frac{t}{\sqrt{\alpha(x,v)}},\\
|\partial_{x_i} x^{\ell}_j(x,v)| &= \left| \frac{1-(-1)^{\ell}}{2}\left\{ \delta_{ij} - \frac{v_j \partial_{x_i}\xi(x_1)}{v\cdot\nabla\xi(x_1)} \right\} + \frac{1+(-1)^{\ell}}{2}\left\{ \delta_{ij} - \frac{v_j \partial_{x_i}\xi(x_2)}{v\cdot\nabla\xi(x_2)} \right\} \right| \\
&\lesssim_{\Omega} 1 + \frac{|v|}{\sqrt{\alpha(x,v)}},\\
|\partial_{v_i} x^{\ell}_j(x,v)| &= \left| \frac{1-(-1)^{\ell}}{2}(-t_{\mathbf{b}}(x,v))\left\{ \delta_{ij} - \frac{v_j \partial_{x_i}\xi(x_1)}{v\cdot\nabla\xi(x_1)} \right\} \right. \\
&\quad \left. + \frac{1+(-1)^{\ell}}{2}(-t_{\mathbf{b}}(x,-v))\left\{ \delta_{ij} - \frac{v_j \partial_{x_i}\xi(x_2)}{v\cdot\nabla\xi(x_2)} \right\} \right|,\\
&\lesssim_{\Omega} \frac{1}{|v|},\\
\partial_{x_i} v^{\ell} &= 0, \quad |\partial_{v_i} v^{\ell}_j| = |(-1)^{\ell}\delta_{ij}| \lesssim_{\Omega} 1,\\
|\partial_{x_i}(t^{\ell}-t^{\ell+1})| &= \left| \frac{\partial_{x_i}\xi(x_1)}{v\cdot\nabla\xi(x_1)} + \frac{\partial_{x_i}\xi(x_2)}{-v\cdot\nabla\xi(x_2)} \right| \lesssim_{\Omega} \frac{1}{\sqrt{\alpha(x,v)}},\\
|\partial_{v_i}(t^{\ell}-t^{\ell+1})| &= \left| t_{\mathbf{b}}(x,v)\frac{-\partial_{x_i}\xi(x_1)}{v\cdot\nabla\xi(x_1)} + t_{\mathbf{b}}(x,-v)\frac{\partial_{x_i}\xi(x_2)}{v\cdot\nabla\xi(x_2)} \right| \lesssim_{\Omega} \frac{1}{|v|^2}.
\end{aligned}
$$

Proof These are direct consequence of (3.9) and the Velocity lemma (Lemma 2). □

Now we state the key ingredient in the case of the bounce-back BC: In the sense of distribution,

$$
\begin{aligned}
&\partial_{\mathbf{e}}\left[\sum_{\ell=0}^{\ell_*(s)} \int_{\max\{s,t^{j+1}\}}^{t^j} A^{m-j}(\tau, x^j-(t^j-\tau)v^j, v^j)\mathrm{d}\tau \right]\\
&= \sum_{j=0}^{\ell_*(s)} \int_{\max\{s,t^{j+1}\}}^{t^j} \left[\partial_{\mathbf{e}} t^j, \partial_{\mathbf{e}} x^j + \tau\partial_{\mathbf{e}} v^j, \partial_{\mathbf{e}} v^j \right]\\
&\quad \cdot\nabla_{t,x,v} A^{m-j}(\tau, x^j-(t^j-\tau)v^j, v^j)\mathrm{d}\tau\\
&\quad + \sum_{j=0}^{\ell_*(s)-1} \partial_{\mathbf{e}}[t^j - t^{j+1}] \lim_{\tau\downarrow -(t^j-t^{j+1})} A^{m-j}(\tau+t^j, x^j+\tau v^j, v^j)\\
&\quad + \partial_{\mathbf{e}} t^{\ell_*(s)} \lim_{\tau\downarrow -(t^{\ell_*(s)}-s)} A^{m-\ell_*(s)}(\tau+t^{\ell_*(s)}, x^{\ell_*(s)}+\tau v^{\ell_*(s)}, v^{\ell_*(s)}). \quad (7.1)
\end{aligned}
$$

Proof of (7.1) For each time intervals $[t^{j+1}, t^j]$, we apply the change of variables

$$\begin{aligned} &x^j - (t^j - \tau)v^j,\ \tau \in [t^{j+1}, t^j] \ \mapsto \ x^j + \tau v^j,\ \tau \in [-(t^j - t^{j+1}), 0], \\ &\quad \text{for } j = 0, 1, \ldots, \ell_*(s) - 1, \\ &x^{\ell_*(s)} - (t^{\ell_*(s)} - \tau)v^{\ell_*(s)},\ \tau \in [s, t^{\ell_*(s)}] \ \mapsto \ x^{\ell_*(s)} \\ &\quad +\tau v^{\ell_*(s)},\ \tau \in [-(t^{\ell_*(s)} - s), 0]. \end{aligned} \tag{7.2}$$

From (3.16) the piecewise derivatives equal distributional derivatives almost everywhere. Moreover

$$\begin{aligned} &\partial_{\mathbf{e}} \left[\sum_{j=0}^{\ell_*(s)} \int_{\max\{s, t^{j+1}\}}^{t^j} A^{m-j}(\tau, x^j - (t^j - \tau)v^j, v^j) \mathrm{d}\tau \right] \\ &= \partial_{\mathbf{e}} \left[\sum_{j=0}^{\ell_*(s)-1} \int_{t^{j+1}}^{t^j} \cdots \right] + \partial_{\mathbf{e}} \left[\int_s^{t^{\ell_*(s)}} \cdots \right] \\ &= \partial_{\mathbf{e}} \left[\sum_{j=0}^{\ell_*(s)-1} \int_{-(t^j - t^{j+1})}^{0} A^{m-j}(\tau + t^j, x^j + \tau v^j, v^j) \mathrm{d}\tau \right] \\ &\quad + \partial_{\mathbf{e}} \left[\int_{-(t^{\ell_*(s)} - s)}^{0} A^{m-\ell_*(s)}(\tau + t^{\ell_*(s)}, x^{\ell_*(s)} + \tau v^{\ell_*(s)}, v^{\ell_*(s)}) \right] \\ &= \sum_{j=0}^{\ell_*(s)-1} \int_{-(t^j - t^{j+1})}^{0} \partial_{\mathbf{e}} \left[A^{m-j}(\tau + t^j, x^j + \tau v^j, v^j) \right] \mathrm{d}\tau \\ &\quad + \sum_{j=0}^{\ell_*(s)-1} \partial_{\mathbf{e}}[t^j - t^{j+1}] \lim_{\tau \downarrow -(t^j - t^{j+1})} A^{m-j}(\tau + t^j, x^j + \tau v^j, v^j) \\ &\quad + \int_{-(t^{\ell_*(s)} - s)}^{0} \partial_{\mathbf{e}} \left[A^{m-\ell_*(s)}(\tau + t^{\ell_*(s)}, x^{\ell_*(s)} + \tau v^{\ell_*(s)}, v^{\ell_*(s)}) \right] \\ &\quad + \partial_{\mathbf{e}} t^{\ell_*(s)} \lim_{\tau \downarrow -(t^{\ell_*(s)} - s)} A^{m-\ell_*(s)}(\tau + t^{\ell_*(s)}, x^{\ell_*(s)} + \tau v^{\ell_*(s)}, v^{\ell_*(s)}). \end{aligned}$$

Directly we have

$$\begin{aligned} \partial_{\mathbf{e}} \left[A^{m-j}(\tau + t^j, x^j + \tau v^j, v^j) \right] = &\left[\partial_{\mathbf{e}} t^j, \partial_{\mathbf{e}} x^j + \tau \partial_{\mathbf{e}} v^j, \partial_{\mathbf{e}} v^j \right] \\ &\cdot \nabla_{t,x,v} A^{m-j}(\tau + t^j, x^j + \tau v^j, v^j). \end{aligned}$$

Then we apply the inverse of the change of variables in (7.2) to the time integration terms:

$$\sum_{j=0}^{\ell_*(s)} \int_{\max\{s,t^{j+1}\}}^{t^j} \left[\partial_{\mathbf{e}} t^j, \partial_{\mathbf{e}} x^j + \tau \partial_{\mathbf{e}} v^j, \partial_{\mathbf{e}} v^j\right] \cdot \nabla_{t,x,v} A^{m-j}(\tau, x^j - (t^j - \tau)v^j, v^j) \mathrm{d}\tau.$$

We collect the terms and conclude (7.1). □

Now we are ready to proof the main theorem:

Proof of Theorem 6 We use the approximation sequence (2.16) with (2.20). Due to Lemma 7 we have (2.13) and (2.14).

Now we consider the spatial and velocity derivatives. From the iteration (2.15) and (2.20), for $\ell_*(0; t, x, v) = \ell_*$ with $t^{\ell_*+1} \leq 0 < t^{\ell_*}$,

$$\begin{aligned} &f^{m+1}(t, x, v) \\ &= e^{-\sum_{j=0}^{\ell_*(0)} \int_{\max\{0,t^{j+1}\}}^{t^j} \nu(F^{m-j})(\tau)\mathrm{d}\tau} f_0(x^{\ell_*(0)} - t^{\ell_*(0)} v^{\ell_*(0)}, v^{\ell_*(0)}) \\ &\quad + \sum_{\ell=0}^{\ell_*(0)} \int_{\max\{0,t^{\ell+1}\}}^{t^\ell} e^{-\sum_{j=0}^{\ell_*(s)} \int_{\max\{0,t^{j+1}\}}^{t^j} \nu(F^{m-j})(\tau)\mathrm{d}\tau} \\ &\quad \times \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, x^\ell - (t^\ell - s)v^\ell, v^\ell)\mathrm{d}s, \end{aligned}$$

where $\nu(F^{m-j})(\tau) = \mu(\sqrt{\mu} f^{m-j})(\tau) = \nu(\sqrt{\mu} f^{m-j})(\tau, x^j - (t^j - \tau)v^j, v^j)$.

From (7.1) and (6.73), in the sense of distribution, for $\partial_{\mathbf{e}} \in \{\partial_x, \partial_v\}$,

$$\begin{aligned} &\partial_{\mathbf{e}} f^m(t, x, v) = \mathbf{I_e} + \mathbf{II_e} = e^{-\int_0^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))\mathrm{d}\tau} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \\ &\times \Bigg\{ \underbrace{-\sum_{j=0}^{\ell_*(0)} \int_{\max\{0,t^{j+1}\}}^{t^j} \left[\partial_{\mathbf{e}} t^j, \partial_{\mathbf{e}} x^j + \tau \partial_{\mathbf{e}} v^j, \partial_{\mathbf{e}} v^j\right] \cdot \nabla_{t,x,v} \nu(F^{m-j})(\tau, x^j - (t^j - \tau)v^j, v^j)\mathrm{d}\tau}_{\mathbf{II_e}} \\ &- \sum_{j=0}^{\ell_*(0)-1} \underbrace{\partial_{\mathbf{e}}[t^j - t^{j+1}]\nu(F^{m-j})(t^{j+1}, x^{j+1}, v^j)}_{\mathbf{I_e}} - \underbrace{\partial_{\mathbf{e}} t^{\ell_*(0)} \nu(F^{m-\ell_*(0)})(0, x^j - t^j v^j, v^j)}_{\mathbf{I_e}} \Bigg\} \\ &+ \underbrace{e^{-\int_0^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))\mathrm{d}\tau} \partial_{\mathbf{e}} \left[x^{\ell_*(0)} - t^{\ell_*(0)} v^{\ell_*(0)}, v^{\ell_*(0)}\right] \cdot \nabla_{x,v} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))}_{\mathbf{I_e}} \\ &+ \sum_{\ell=0}^{\ell_*(0)-1} \partial_{\mathbf{e}}[t^\ell - t^{\ell+1}] e^{-\sum_{j=0}^{\ell_*(t^\ell - t^{\ell+1})} \int_{\max\{t^\ell - t^{\ell+1} - t^j, -(t^j - t^{j+1})\}}^0 \nu(F^{m-j})(\tau + t^j, x^j + \tau v^j, v^j)\mathrm{d}\tau} \\ &\times \underbrace{\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(t^{\ell+1}, x^{\ell+1}, v^\ell)}_{\mathbf{I_e}} \\ &+ \underbrace{\partial_{\mathbf{e}} t^{\ell_*(0)} e^{-\int_0^t \mathbf{1}_{[t^{j+1},t^j)}(s)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \Gamma_{\text{gain}}(f^{m-\ell_*(0)}, f^{m-\ell_*(0)})(0, x^{\ell_*(0)} - t^{\ell_*(0)} v^{\ell_*(0)}, v^{\ell_*(0)})}_{\mathbf{I_e}} \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^{j})}(s)\nu(F^{m-j})(\tau)\mathrm{d}\tau}\\
&\underbrace{\times[\partial_{\mathbf{e}}t^\ell, \partial_{\mathbf{e}}x^\ell + s\partial_{\mathbf{e}}v^\ell, \partial_{\mathbf{e}}v^\ell]\cdot\nabla_{t,x,v}\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})](s, x^\ell-(t^\ell-s)v^\ell, v^\ell)\mathrm{d}s}_{\mathbf{II_e}}\\
&+\int_0^t \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\mathrm{d}s\\
&\times\Bigg\{-\sum_{j=0}^{\ell_*(s)-1}\underbrace{\partial_{\mathbf{e}}[t^j-t^{j+1}]\nu(F^{m-j})(t^{j+1}, x^{j+1}, v^j)}_{\mathbf{I_e}}-\underbrace{\partial_{\mathbf{e}}t^{\ell_*(s)}\nu(F^{m-\ell_*(s)})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}_{\mathbf{I_e}}\\
&-\sum_{j=0}^{\ell_*(s)}\underbrace{\int_{\max\{s,t^{j+1}\}}^{t^j}[\partial_{\mathbf{e}}t^j, \partial_{\mathbf{e}}x^j+\tau\partial_{\mathbf{e}}v^j, \partial_{\mathbf{e}}v^j]\cdot\nabla_{t,x,v}\nu(F^{m-j})(\tau, x^j-(t^j-\tau)v^j, v^j)\mathrm{d}\tau}_{\mathbf{II_e}}\Bigg\}.
\end{aligned}
\tag{7.3}
$$

We shall estimate the followings:

$$
e^{-\varpi\langle v\rangle t}\frac{\alpha(x,v)}{\langle v\rangle^2}\partial_x f(t,x,v), \qquad e^{-\varpi\langle v\rangle t}\frac{|v|\alpha(x,v)^{1/2}}{\langle v\rangle^2}\partial_v f(t,x,v).
$$

Firstly, we estimate $\mathbf{I_e}$. Using Lemma 17 and Lemma 6 and $F^m\geq 0$ from (2.15) and Lemma 7, for some polynomial P,

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\langle v\rangle^{-2}\alpha(x,v)\mathbf{I_x}\\
&\lesssim e^{-\varpi\langle v\rangle t}\langle v\rangle^{-2}\alpha(x,v)P(\|e^{\theta|v|^2}f\|_\infty)\\
&\quad\times\Bigg\{e^{-\theta|v|^2}\frac{t|v|^2}{\alpha(x,v)}\langle v\rangle^\kappa+\Bigg[\Bigg(1+\frac{|v|}{\alpha(x,v)}\Bigg)+\frac{t|v|^3}{\alpha(x,v)}\Bigg]|\partial_x f_0|\\
&\quad+\frac{t|v|^2}{\alpha(x,v)}e^{-\frac{\theta}{2}|v|^2}+te^{-\frac{\theta}{2}|v|^2}\langle v\rangle^\kappa\frac{t|v|^2}{\alpha(x,v)}\Bigg\}\\
&\lesssim\|\langle v\rangle^{-2}\alpha(1+\frac{|v|+|v|^3}{\alpha(x,v)})\partial_x f_0\|_\infty+\langle v\rangle^{-2}e^{-C_\theta|v|^2}P(\|e^{\theta|v|^2}f_0\|_\infty)\\
&\lesssim(1+\|\langle v\rangle\partial_x f_0\|_\infty)\times P(\|e^{\theta|v|^2}f_0\|_\infty).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}|v|\langle v\rangle^{-2}\alpha^{1/2}\mathbf{I_v}\\
&\lesssim e^{-\varpi\langle v\rangle t}|v|\langle v\rangle^{-2}[\alpha(x,v)]^{1/2}P(\|e^{\theta|v|^2}f\|_\infty)\\
&\quad\times\Bigg\{e^{-\frac{\theta}{2}|v|^2}\frac{t}{\alpha(x,v)^{1/2}}\langle v\rangle^\kappa+|\nabla_v f_0|\\
&\quad+\Bigg[\Bigg(\frac{1}{|v|}+\frac{\alpha(x,v)^{1/2}}{|v|^2}\Bigg)+\frac{t|v|}{\alpha(x,v)^{1/2}}+t\Bigg]|\partial_x f_0|
\end{aligned}
$$

$$+ \frac{te^{-\frac{\theta}{2}|v|^2}}{\alpha(x,v)^{1/2}} + te^{-\frac{\theta}{2}|v|^2}\langle v\rangle^{\kappa}\frac{t}{\alpha(x,v)^{1/2}}\Bigg\}$$
$$\lesssim (1+\|\langle v\rangle\partial_x f_0\|_\infty + \|\partial_v f_0\|_\infty) P(\|e^{\theta|v|^2} f_0\|_\infty).$$

Secondly, we estimate $\mathbf{II_e}$. Let $\phi_{\mathbf{e}} \in \{\phi_{\mathbf{x}}, \phi_{\mathbf{v}}\}$ with $\phi_{\mathbf{x}} = e^{-\varpi\langle v\rangle t}\frac{\alpha(x,v)}{\langle v\rangle^2}$ and $\phi_{\mathbf{v}} = e^{-\varpi\langle v\rangle t}\frac{|v|\alpha(x,v)^{1/2}}{\langle v\rangle^2}$. We have

$$e^{-\varpi\langle v\rangle t}\phi_{\mathbf{e}}(v)[\alpha(x,v)]^{\beta_{\mathbf{e}}}\mathbf{II_e}$$
$$\lesssim e^{-\varpi\langle v\rangle t}\phi_{\mathbf{e}}(v)[\alpha(x,v)]^{\beta_{\mathbf{e}}}\left\{1+(1+t)e^{-\frac{\theta}{2}|v|^2}\left\|e^{\theta|v|^2}f\right\|_\infty\right\}$$
$$\times\Bigg\{\int_0^t\sum_{j=0}^{\ell_*(0)}\mathbf{1}_{[t^{j+1},t^j)}(s)|\partial_{\mathbf{e}}t^j|\langle v\rangle^\kappa \mathrm{d}s\times\left\|e^{\theta|v|^2}\partial_t f\right\|_\infty \tag{7.4}$$
$$+\int_0^t\sum_{j=0}^{\ell_*(0)}\mathbf{1}_{[t^{j+1},t^j)}(s)\{|\partial_{\mathbf{e}}x^\ell|+t|\partial_{\mathbf{e}}v^\ell|\}\nu(\sqrt{\mu}\partial_x f^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\mathrm{d}s \tag{7.5}$$
$$+\int_0^t\sum_{j=0}^{\ell_*(0)}\mathbf{1}_{[t^{j+1},t^j)}(s)|\partial_{\mathbf{e}}v^\ell|\int_{\mathbb{R}^3}|V_{\mathbf{cl}}(s)-u|^{\kappa-1}\sqrt{\mu(u)}f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)\mathrm{d}u\mathrm{d}s \tag{7.6}$$
$$+\int_0^t\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)|\partial_{\mathbf{e}}t^\ell|$$
$$\times\left[\left|\Gamma_{\mathrm{gain}}(\partial_t f^{m-\ell},f^{m-\ell})\right|+\left|\Gamma_{\mathrm{gain}}(f^{m-\ell},\partial_t f^{m-\ell})\right|\right]\mathrm{d}s \tag{7.7}$$
$$+\int_0^t\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\left\{|\partial_{\mathbf{e}}x^\ell|+t|\partial_{\mathbf{e}}v^\ell|\right\}$$
$$\times\left[\left|\Gamma_{\mathrm{gain}}(\partial_x f^{m-\ell},f^{m-\ell})\right|+\left|\Gamma_{\mathrm{gain}}(f^{m-\ell},\partial_x f^{m-\ell})\right|\right]\mathrm{d}s \tag{7.8}$$
$$+\int_0^t\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)|\partial_{\mathbf{e}}v^\ell|\left[\left|\Gamma_{\mathrm{gain},v}(f^{m-\ell},f^{m-\ell})\right|\right.$$
$$\left.+\left|\Gamma_{\mathrm{gain}}(f^{m-\ell},\partial_v f^{m-\ell})\right|+\left|\Gamma_{\mathrm{gain}}(\partial_v f^{m-\ell},f^{m-\ell})\right|\right]\mathrm{d}s\Bigg\}. \tag{7.9}$$

Firstly, we consider $\partial_{\mathbf{e}}t^j$–contribution. Then from Lemma 17 and (2) of Lemma 6

$$e^{-\varpi\langle v\rangle t}\frac{\alpha(x,v)}{\langle v\rangle^2}\{(7.4)_{\mathbf{x}}+(7.7)_{\mathbf{x}}\}$$
$$\lesssim e^{-\varpi\langle v\rangle t}\frac{\alpha(x,v)}{\langle v\rangle^2}e^{-\frac{\theta}{2}|v|^2}t\frac{t|v|^2}{\alpha(x,v)}\langle v\rangle\|e^{\theta|v|^2}f\|_\infty\|e^{\theta|v|^2}\partial_t f\|_\infty$$

$$
\begin{aligned}
&+e^{-\varpi\langle v\rangle t}\frac{\alpha(x,v)}{\langle v\rangle^2}(1+t)\|e^{\theta|v|^2}f\|_\infty t\frac{t|v|^2}{\alpha(x,v)}e^{-\frac{\theta}{2}|v|^2}\|e^{\theta|v|^2}\partial_t f\|_\infty\\
&\lesssim_t 1+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)+P(\|e^{\theta|v|^2}f_0\|_\infty).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{|v|\alpha(x,v)^{1/2}}{\langle v\rangle^2}\{(7.4)_{\mathbf{v}}+(7.7)_{\mathbf{x}}\}\\
&\lesssim_t \frac{|v|}{\langle v\rangle^2}e^{-C_\theta|v|^2}\left[P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)+P(\|e^{\theta|v|^2}f_0\|_\infty)\right]\\
&\lesssim_t 1+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)+P(\|e^{\theta|v|^2}f_0\|_\infty).
\end{aligned}
$$

Secondly, we consider the terms (7.5) and (7.8), which include $|\partial_{\mathbf{e}}x^\ell|+t|\partial_{\mathbf{e}}v^\ell|$. We use (2) of Lemmas 6 and 17, $|\partial_x x^\ell|+t|\partial_x v^\ell|\lesssim\frac{|v|}{\sqrt{\alpha(x,v)}}$, and (6.75) to estimate

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{\alpha(x,v)}{\langle v\rangle^2}\{(7.5)_{\mathbf{x}}+(7.7)_{\mathbf{x}}\}\\
&\lesssim_t \left[1+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)+P(\|e^{\theta|v|^2}f_0\|_\infty)\right]\\
&\quad\times\sum_{\ell=0}^{\ell_*(0;t,x,v)}\int_{t^{\ell+1}}^{t^\ell}\int_{\mathbb{R}^3}e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^2}\alpha(x,v)^{1/2}\\
&\quad\times\frac{e^{-C_\theta|u-v^\ell|^2}}{|u-v^\ell|^{2-\kappa}}|\partial_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\mathrm{d}u\mathrm{d}s,\\
&\lesssim_t \left[1+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)+P(\|e^{\theta|v|^2}f_0\|_\infty)\right]\\
&\quad\times\max_{0\le\ell\le m}\sup_{0\le s\le t}\|e^{-\varpi\langle v\rangle s}\frac{\alpha}{\langle v\rangle^2}\partial_x f^{m-\ell}(s)\|_\infty\\
&\quad\times\int_0^t\sum_{\ell=0}^{\ell_*(0;t,x,v)}\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\int_{\mathbb{R}^3}e^{-\frac{\varpi}{2}\langle v\rangle(t-s)}\frac{\langle u\rangle^2}{\langle v\rangle^2}\\
&\quad\times\frac{|v|\alpha(x,v)^{\frac12}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}\alpha(X_{\mathbf{cl}}(s),u)}e^{-C_\theta|v-u|^2}\mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

We use $\frac{\langle u\rangle^2}{\langle v\rangle^2}\lesssim\langle v-u\rangle^2$ and (1.29) to bound it by

$$
\begin{aligned}
&\lesssim_t \left[1+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)+P(\|e^{\theta|v|^2}f_0\|_\infty)\right]\\
&\quad\times\max_{0\le\ell\le m}\sup_{0\le s\le t}\|e^{-\varpi\langle v\rangle s}\frac{\alpha}{\langle v\rangle^2}\partial_x f^{m-\ell}(s)\|_\infty
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{O(\delta)}{\langle v\rangle \alpha(x,v)^{1/2}} |v| \alpha(x,v)^{1/2} \\
&\lesssim O(\delta)\Big[1 + P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty) + P(\|e^{\theta|v|^2} f_0\|_\infty)\Big] \\
&\times \max_{0\le \ell\le m} \sup_{0\le s\le t} \|e^{-\varpi\langle v\rangle s} \frac{\alpha}{\langle v\rangle^2} \partial_x f^{m-\ell}(s)\|_\infty .
\end{aligned}
$$

Similarly we further use $|\partial_v x^\ell| + t|\partial_v v^\ell| \lesssim \frac{1}{|v|}$ from Lemma 17 to obtain

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t} \frac{|v|\alpha(x,v)^{1/2}}{\langle v\rangle^2} \{(7.5)_{\mathbf{v}} + (7.7)_{\mathbf{v}}\} \\
&\lesssim_t \Big[1 + P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty) + P(\|e^{\theta|v|^2} f_0\|_\infty)\Big] \\
&\times \sum_{\ell=0}^{\ell_*(0;t,x,v)} \int_{t^{\ell+1}}^{t^\ell} \int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle t} \frac{|v|}{\langle v\rangle^2} (\frac{1}{|v|} + 1) \alpha(x,v)^{1/2} \\
&\times \frac{e^{-C_\theta|u-v^\ell|^2}}{|u-v^\ell|^{2-\kappa}} |\partial_x f^{m-\ell}(s, X_{\mathbf{cl}}(s), u)| \mathrm{d}u\mathrm{d}s, \\
&\lesssim_t \Big[1 + P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty) + P(\|e^{\theta|v|^2} f_0\|_\infty)\Big] \\
&\times \max_{0\le \ell\le m} \sup_{0\le s\le t} \|e^{-\varpi\langle v\rangle t} \frac{|u|\alpha}{\langle u\rangle^2} \partial_x f^{m-\ell}(s, X_{\mathbf{cl}}(s), u)\|_\infty \\
&\times \sum_{\ell=0}^{\ell_*(0;t,x,v)} \int_{t^{\ell+1}}^{t^\ell} \int_{\mathbb{R}^3} \\
&\qquad \times e^{-\frac{\varpi}{2}\langle v\rangle(t-s)} \frac{|v|\langle u\rangle^2}{\langle v\rangle^2} \frac{(\frac{1}{|v|}+1)\alpha(x,v)^{1/2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}\alpha(x,u)} \mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

From $\frac{\langle u\rangle^2}{\langle v\rangle^2} \lesssim \langle v-u\rangle^2$, the last integration is bounded by

$$
\sum_{\ell=0}^{\ell_*(0;t,x,v)} \int_{t^{\ell+1}}^{t^\ell} \int_{\mathbb{R}^3} e^{-\frac{\varpi}{2}\langle v\rangle(t-s)} \langle v-u\rangle^2 \frac{\langle v\rangle \alpha(x,v)^{1/2}}{\alpha(X_{\mathbf{cl}}(s),u)} \frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} \mathrm{d}u\mathrm{d}s.
$$

By the dynamical non-local to local estimate (1.29), this is bounded by

$$
\begin{aligned}
&O(\delta)\Big[1 + P(\|e^{\theta|v|^2}\partial_t f\|_\infty) + P(\|e^{\theta|v|^2} f\|_\infty)\Big] \\
&\times \max_{0\le \ell\le m} \sup_{0\le s\le t} \|e^{-\varpi\langle v\rangle s} \frac{\alpha(x,v)}{\langle v\rangle^2} \partial_x f^{m-\ell}(s)\|_\infty .
\end{aligned}
$$

Thirdly, we consider $\partial_{\mathbf{e}} v^{\ell}-$contribution, (7.6) and (7.9). Note that $(7.6)_{\mathbf{x}} = 0 = (7.9)_{\mathbf{x}}$ since $\partial_x v^j \equiv 0$. From Lemma 17 and *(3)* of Lemma 6,

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{|v|\alpha(x,v)^{1/2}}{\langle v\rangle^2}\{(7.6)_{\mathbf{v}}+(7.9)_{\mathbf{v}}\} \\
&\quad \lesssim \left[1+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)+P(\|e^{\theta|v|^2} f_0\|_\infty)\right] \\
&\qquad \times \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) e^{-\varpi\langle v\rangle t}\frac{|v|\alpha(x,v)^{1/2}}{\langle v\rangle^2} e^{-C|v|^2} \\
&\qquad \times \int_{\mathbb{R}^3} \frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} |\partial_v f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)| \mathrm{d}u\mathrm{d}s \\
&\quad \lesssim \left[1+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)+P(\|e^{\theta|v|^2} f_0\|_\infty)\right] \\
&\qquad \times \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) \int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle t} e^{-\varpi\langle u\rangle s} e^{-C|v|^2} \frac{|v|\langle u\rangle^2\alpha(x,v)^{1/2}}{|u|\langle v\rangle^2\alpha(X_{\mathbf{cl}}(s),u)^{1/2}} \\
&\qquad \times \frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} \mathrm{d}u\mathrm{d}s \\
&\qquad \times \sup_{0\le s\le t}\max_{0\le\ell\le m} \|e^{-\varpi\langle u\rangle s}\frac{|u|\alpha(x,u)^{1/2}}{\langle u\rangle^2}\partial_v f^{m-\ell}(s,x,u)\|_\infty.
\end{aligned}
$$

Now we choose $\beta' \in (\frac{1}{2},1)$ and use $\alpha(x,u) \lesssim |u|^2$ to have

$$
\frac{1}{[\alpha(X_{\mathbf{cl}}(s),u)]^{1/2}} \lesssim \frac{|u|^{2(\beta'-\frac{1}{2})}}{[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta'}}.
$$

Now we use (6.75) to bound the integration by

$$
\begin{aligned}
& \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) \int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle(t-s)} \frac{|v|}{|u|} \frac{|u|^{2\beta'-1}\alpha(x,v)^{1/2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}\alpha(X_{\mathbf{cl}}(s),u)^{\beta'}} \\
& \quad \times e^{-C|v|^2} e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2} \mathrm{d}u\mathrm{d}s
\end{aligned}
$$

Now we use $|u|^{2\beta'-1} \le \langle v\rangle^{2\beta'-1}\langle u-v\rangle^{2\beta'-1}$ and we apply (4.12) to bound this integration by

$$
O(\delta)\langle v\rangle^{-2+2\beta'}\alpha(x,v)^{1-\beta'} \lesssim O(\delta),
$$

Hence

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{|v|\alpha(x,v)^{1/2}}{\langle v\rangle^2}\{(7.6)_{\mathbf{v}}+(7.9)_{\mathbf{v}}\}\\
&\lesssim\Big[1+P(\|e^{\theta|v|^2}f_0\|_\infty)+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)\Big]\\
&\quad\times\sup_{0\le s\le t}\max_{0\le\ell\le m}\|e^{-\varpi\langle v\rangle s}\frac{|u|\alpha(x,u)^{1/2}}{\langle u\rangle^2}\partial_v f^{m-\ell}(s,x,u)\|_\infty\\
&\quad\times\left\{\frac{O(\delta)}{\langle v\rangle[\alpha(x,v)]^{\beta'-1/2}}\alpha(x,v)^{1/2}e^{-C_\theta|v|^2}+O(\delta)\right\}\\
&\lesssim\Big[1+P(\|e^{\theta|v|^2}f_0\|_\infty)+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)\Big]\\
&\quad+O(\delta)\Big[P(\|e^{\theta|v|^2}f_0\|_\infty)+P(\|e^{\theta|v|^2}\partial_t f_0\|_\infty)\Big]\\
&\quad\times\sup_{0\le s\le t}\max_{0\le\ell\le m}\|e^{-\varpi\langle v\rangle s}\frac{|u|\alpha(x,u)^{1/2}}{\langle u\rangle^2}\partial_v f^{m-\ell}(s,x,u)\|_\infty.
\end{aligned}
$$

Now we gather all the estimates with small $0<\delta\ll 1$ to close the estimate. Then we follow the exactly same argument as the specular case and this complete the proof of Theorem 6. $\square$

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Appendix: Non-existence of second derivatives

In the previous theorem, we consider the *first-order derivative* of the Boltzmann solution with several boundary conditions. Now we show that some second order spatial derivative does not exist up to the boundary in general so that our result is quite optimal.

Assume that all the second order spatial derivatives exist away from the grazing set $\gamma_0=\{(x,v)\in\partial\Omega\times\mathbb{R}^3:n(x)\cdot v=0\}$ but up to some boundary $\partial\Omega\times\mathbb{R}^3$. Taking the normal derivative $\partial_n=n(x)\cdot\nabla_x=\frac{\nabla_x\xi(x)}{|\nabla_x\xi(x)|}\cdot\nabla_x$ to the

Boltzmann equation directly yields

$$v \cdot \partial_n \nabla_x f = -\partial_n \partial_t f - \nu(\sqrt{\mu} f)\partial_n f + \underbrace{\partial_n \Gamma_{\text{gain}}(f, f) - \partial_n \nu(\sqrt{\mu} f) f}.$$

In this section we show that the underbraced term blows up at the boundary with any velocity for symmetric domains.

Assume $f_0 \lesssim (\sqrt{\mu})^{1-\delta}$ for some $0 < \delta \ll 1$. Then there exists $\mathbf{k}_{f_0}(v, u)$ such that

$$\Gamma_{\text{gain}}(f, f_0) + \Gamma_{\text{gain}}(f_0, f) - \nu(\sqrt{\mu} f) f_0 := \int_{\mathbb{R}^3} \mathbf{k}_{f_0}(v, u) f(u) \mathrm{d}u.$$

First consider the diffuse reflection boundary condition. Theorem 2 plays an important role in our proof.

Proposition 3 (Diffuse BC) *Assume* $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ *and* $\xi(x) = |x|^2 - 1$. *Assume the initial datum* f_0 *satisfies, for some* $x_0 \in \partial\Omega$,

$$\lim_{n(x_0)\cdot u = 0} \int \mathbf{k}_{f_0}(v, u) u \cdot n(x_0) \partial_n f_0(x_0, u) \mathrm{d}u_\tau > C > 0, \tag{8.1}$$

where $u_\tau = u - (u \cdot n(x_0)) n(x_0)$. *Then for any* $t > 0$ *such that for all* $v \in \mathbb{R}^3$,

$$\partial_n \Gamma_{\text{gain}}(f, f)(t, x_0, v) - \partial_n \nu(\sqrt{\mu} f) f(t, x_0, v) = \infty. \tag{8.2}$$

We remark that for $0 < \theta < \frac{1}{4}$ we have $\sup_t \|e^{\theta|v|^2} f(t)\|_\infty \lesssim \|e^{\theta|v|^2} f_0\|_\infty$ due to Lemma 7 or [3,9] and $\|\alpha^{1/2} \partial f(t)\|_\infty \lesssim 1$ for $\partial = [\partial_t, \nabla_x, \nabla_v]$ due to Theorem 2. The blow-up of (8.2) is due to the interaction between expected singular behavior $\frac{1}{\sqrt{\alpha}}$ with the non-local collision operator. For the diffuse reflection, indeed this blow-up happens for any time $0 < t \ll 1$. Our proof assumes crucially the initial condition is non-zero at the boundary. However, for initial data vanish near the boundary, it is strongly believed that solutions will become non-zero at a short time later at the boundary and such a blow-up will happen generically.

Proof We denote the different quotient

$$\triangle_\varepsilon f(t, x, v) := \frac{f(t, x + \varepsilon[-n(x)], v) - f(t, x, v)}{\varepsilon}.$$

Then

$$\begin{aligned}\triangle_\varepsilon \{\Gamma_{\text{gain}}(f, f)\} - \nu(\sqrt{\mu}\, \triangle_\varepsilon f) f = \Gamma_{\text{gain}}&(\triangle_\varepsilon f, f) \\ &+ \Gamma_{\text{gain}}(f, \triangle_\varepsilon f) - \nu(\sqrt{\mu}\, \triangle_\varepsilon f) f.\end{aligned}$$

Assuming $f \simeq f_0 \lesssim (\sqrt{\mu})^{1-\delta}$ for $0 < \delta \ll 1$, we have

$$\begin{aligned}&\Gamma_{\text{gain}}(\triangle_\varepsilon f, f) + \Gamma_{\text{gain}}(f, \triangle_\varepsilon f) - \nu(\sqrt{\mu}\,\triangle_\varepsilon f) f \\ &\simeq \int_{\mathbb{R}^3} \mathbf{k}_{f_0}(v,u)\,\triangle_\varepsilon f(x,u)\mathrm{d}u \simeq \int_{\mathbb{R}^3} \mathbf{k}_{f_0}(v,u)\frac{f(x-\varepsilon n(x),u)-f(x,u)}{\varepsilon}\mathrm{d}u,\end{aligned} \tag{8.3}$$

where $\mathbf{k}_{f_0}(v,u) \simeq \mathbf{k}(v,u)$ in (2.9) with slightly different exponents. For simplicity let us assume $\mathbf{k}_{f_0}(v,u)$ is bounded. We split as with $0 < \Lambda \ll 1$

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathbf{k}_{f_0}(v,u)\frac{f(t,x-\varepsilon n(x),u) - f(t,x,u)}{\varepsilon}\mathrm{d}u \\ &= \underbrace{\int_{|n(x)\cdot u|\le\varepsilon}}_{\mathbf{I}} + \underbrace{\int_{\varepsilon\le|n(x)\cdot u|\le\Lambda}}_{\mathbf{II}} + \underbrace{\int_{\Lambda\le|n(x)\cdot u|}}_{\mathbf{III}}.\end{aligned} \tag{8.4}$$

The first term is bounded as $\mathbf{I} \lesssim O(1)\|e^{\theta|v|^2} f\|_\infty$. The last term is bounded due to Theorem 2. Since $\xi(x) = |x|^2 - 1$, for all $0 < r < \varepsilon \ll 1$,

$$\begin{aligned}\nabla\xi(x - rn(x))\cdot u &= \nabla\xi(x)\cdot u - \int_0^r \left\{\nabla\xi(x)\cdot\nabla^2\xi(x - r'n(x))\cdot u\right\}\mathrm{d}r' \\ &= \nabla\xi(x)\cdot u - 2\int_0^r \nabla\xi(x)\cdot u\mathrm{d}r' \\ &= \nabla\xi(x)\cdot u + O(\varepsilon)|\nabla\xi(x)\cdot u| \\ &= \nabla\xi(x)\cdot u.\end{aligned} \tag{8.5}$$

Therefore $\Lambda \le |n(x)\cdot u|$ implies $\Lambda \lesssim \sqrt{\alpha(x,u)}$ and

$$\begin{aligned}\mathbf{III} &\lesssim \|e^{-\varpi\langle v\rangle t}\sqrt{\alpha}\nabla_x f(t)\|_\infty \int_{\Lambda\lesssim\sqrt{\alpha}} \frac{e^{\varpi\langle u\rangle t}}{\sqrt{\alpha}}\mathbf{k}_{f_0}(v,u)\mathrm{d}u \\ &= \int_{\Lambda\le\sqrt{\alpha},|u|\le N} + \int_{\Lambda\le\sqrt{\alpha},|u|\ge N} \lesssim \frac{O(1)+e^{CNt}}{\Lambda}.\end{aligned}$$

For the second term of (8.4) we use (8.5) to conclude, for $0 \le r \le \varepsilon$,

$$\varepsilon \lesssim |n(x - rn(x))\cdot u| \lesssim \Lambda.$$

Therefore $f(t, x - \varepsilon n(x), u)$ is differentiable so that

$$\frac{f(t,x-\varepsilon n(x),u) - f(t,x,u)}{\varepsilon} = \int_0^1 \partial_n f(t, x - \varepsilon r n(x), u)\mathrm{d}r. \tag{8.6}$$

We further split **II** as

$$\mathbf{II} = \underbrace{\int_{\substack{\varepsilon \le |n(x)\cdot u| \le \Lambda \\ \frac{1}{N} \le |u| \le N}}}_{\mathbf{II}_a} + \underbrace{\int_{\substack{\varepsilon \le |n(x)\cdot u| \le \Lambda \\ |u| \le \frac{1}{N}, |u| \ge N}}}_{\mathbf{II}_b} .$$

For the second term we use Theorem 2 to have

$$\begin{aligned}
\mathbf{II}_b &\lesssim e^{-N} \int_0^1 \mathrm{d}r \int_{\varepsilon \lesssim |u_n| \lesssim \Lambda} \mathrm{d}u_n \int_{|u_\tau| \gtrsim N} \mathrm{d}u_\tau \, \mathbf{k}_{f_0}(v,u) \partial_n f(t, x - \varepsilon r n(x), u) \\
&\lesssim e^{-N} \int_0^1 \mathrm{d}r \int_{\varepsilon \lesssim |u_n| \lesssim \Lambda} \mathrm{d}u_n \int_{|u_\tau| \gtrsim N} \mathrm{d}u_\tau \\
&\qquad \times \frac{\mathbf{k}_{f_0}(v,u)}{\sqrt{|u_n|^2 + Cr\varepsilon N^2}},
\end{aligned} \tag{8.7}$$

where we used

$$\xi(x - \varepsilon r n(x)) = \xi(x) + C\varepsilon r = C\varepsilon r.$$

The main term is $\mathbf{II}_a$:

$$\mathbf{II}_a = \int_0^1 \mathrm{d}r \iint_{\substack{\varepsilon \lesssim |u_n| \lesssim \Lambda \\ \frac{1}{N} \le |u| \le N}} \mathrm{d}u_\tau \mathrm{d}u_n \, \mathbf{k}_{f_0}(v,u) \partial_n f(t, x - \varepsilon r n(x), u).$$

From (2.3), for $\varepsilon \lesssim |u_n| \lesssim \Lambda$ and $\frac{1}{N} \le |u| \le N$,

$$t_{\mathbf{b}}(x - \varepsilon r n(x), u) \lesssim \frac{\sqrt{\alpha(x - \varepsilon r n(x), u)}}{|u|^2} \lesssim \frac{\sqrt{\Lambda^2 + \varepsilon r N^2}}{\frac{1}{N^2}} \lesssim N^2 \sqrt{\Lambda^2 + \varepsilon N^2}.$$

Let $x(r) = x - \varepsilon r n(x)$. For $\varepsilon \lesssim |u_n| \lesssim \Lambda$ and $\frac{1}{N} \le |u| \le N$ and $t \gtrsim N^2 \sqrt{\Lambda^2 + \varepsilon N^2}$,

$$\begin{aligned}
&\partial_n f(t, x(r), u) \\
&\quad = n(x(r)) \cdot \nabla_x \Big\{ f(t - t_{\mathbf{b}}, x_{\mathbf{b}}, u) \\
&\qquad + \int_0^{t_{\mathbf{b}}} [\Gamma_{\text{gain}}(f,f) - \nu(F) f](t - s, x(r) - su, u) \mathrm{d}s \Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} n(x(r)) \cdot \tau_i(x_{\mathbf{b}}) \partial_{\tau_i} f(t - t_{\mathbf{b}}, x_{\mathbf{b}}, u) \\
&\quad + \frac{n(x(r)) \cdot n(x_{\mathbf{b}})}{n(x_{\mathbf{b}}) \cdot u} \underline{u \cdot n(x_{\mathbf{b}}) \partial_n f(t - t_{\mathbf{b}}, x_{\mathbf{b}}, u)} \\
&\quad + \int_0^{t_{\mathbf{b}}} n(x(r)) \cdot \{\Gamma_{\text{gain}}(\nabla_x f, f) + \Gamma_{\text{gain}}(f, \nabla_x f) \\
&\quad - \nu(\sqrt{\mu}\nabla_x f) f - \nu(\sqrt{\mu} f)\nabla_x f\}(t - s, x(r) - su, u)\mathrm{d}s.
\end{aligned}
$$

Now we expand in time for the underlined term and choose $0 < t \ll 1$ ($N^2\sqrt{\Lambda^2 + \varepsilon N^2} \ll 1$) so that

$$
\begin{aligned}
& u \cdot n(x_{\mathbf{b}}) \partial_n f(t - t_{\mathbf{b}}, x_{\mathbf{b}}, u) \\
&= u \cdot n(x_{\mathbf{b}}) \partial_n f_0(x_{\mathbf{b}}, u) + \int_0^{t - t_{\mathbf{b}}} \{u \cdot n(x_{\mathbf{b}})\} \partial_t \partial_n f(s, x_{\mathbf{b}}, u)\mathrm{d}s \\
&= u \cdot n(x_{\mathbf{b}}) \partial_n f_0(x_{\mathbf{b}}, u) + O(1) t e^{\varpi N t} \| e^{-\varpi \langle v \rangle t} \sqrt{\alpha} \partial_t \partial_n f(t) \|_\infty.
\end{aligned}
$$

We remark that $\partial_t f$ can satisfy the same estimate as in Theorem 2. The tangential derivative term is bounded by

$$
\begin{aligned}
& |n(x(r)) \cdot \tau_i(x_{\mathbf{b}}(x(r), u))| |\partial_{\tau_i} f(t - t_{\mathbf{b}}, x_{\mathbf{b}}, u)| \\
&\quad \lesssim |n(x_{\mathbf{b}}) \cdot \tau_i(x_{\mathbf{b}}) + O(t_{\mathbf{b}}(x(r), u)) u \cdot \nabla_x n(x(r))| |\partial_{\tau_i} f(t - t_{\mathbf{b}}, x_{\mathbf{b}}, u)| \\
&\quad \lesssim \frac{\sqrt{\alpha(x_{\mathbf{b}}, u)}}{|u|} |\nabla_x f(t - t_{\mathbf{b}}, x_{\mathbf{b}}, u)| \\
&\quad \lesssim N e^{\varpi N t} \| e^{-\varpi \langle v \rangle t} \sqrt{\alpha} \nabla_x f(t, x, v) \|_\infty,
\end{aligned}
$$

and the time integration terms are bounded by, from $t_{\mathbf{b}}(x(r), u) \lesssim \frac{\sqrt{\alpha(x(r),u)}}{|u|^2}$,

$$
\begin{aligned}
& \| e^{\theta|v|^2} f \|_\infty \int_0^{t_{\mathbf{b}}} \int_{\mathbb{R}^3} \frac{e^{-C|u-u'|^2}}{|u - u'|^{2-\kappa}} |\partial_x f(t - s, x(r) - su, u')| \mathrm{d}u' \mathrm{d}s \\
&\quad + N e^{\varpi N t} \| e^{\theta|v|^2} f \|_\infty \| e^{-\varpi \langle v \rangle t} \sqrt{\alpha} \nabla_x f(t) \|_\infty \\
&\lesssim \| e^{\theta|v|^2} f \|_\infty \| e^{-\varpi \langle v \rangle t} \sqrt{\alpha} \nabla_x f(t) \|_\infty e^{\varpi N t} \\
&\quad \times \left\{ \int_0^{t_{\mathbf{b}}} \int_{\mathbb{R}^3} e^{-\varpi \langle u \rangle (t-s)} \frac{e^{-C|u-u'|^2}}{|u - u'|^{2-\kappa}} \frac{|u'|^\delta}{\alpha(x(r) - su, u')^{\frac{1+\delta}{2}}} \mathrm{d}u' \mathrm{d}s \right\} \\
&\lesssim \| e^{\theta|v|^2} f \|_\infty \| e^{-\varpi \langle v \rangle t} \sqrt{\alpha} \nabla_x f(t) \|_\infty \frac{e^{\varpi N t} C_N}{[\alpha(x(r), u)]^{\delta/2}}.
\end{aligned}
$$

Now we plug these estimates into $\mathbf{II}_a$ to have

$$\begin{aligned}
&\mathbf{II}_a + \mathbf{II}_b \\
&\gtrsim \int_0^1 \int_{\substack{\varepsilon \lesssim |u_n| \lesssim \Lambda \\ \frac{1}{N} \lesssim |u| \lesssim N}} \frac{1}{\sqrt{\alpha(x_0 - \varepsilon r n(x_0), u)}} \\
&\quad \times \left[\int_{\cdot n(x_0)\cdot u_\tau = 0} \mathbf{k}_{f_0}(v,u) u \cdot n(x_0) \partial_n f_0(x_0, u) \mathrm{d}u_\tau \right]_{u \cdot n(x_0) = 0} \\
&- \left\{ O(t) e^{-\varpi N t} \| e^{-\varpi \langle v \rangle t} \sqrt{\alpha} \partial_t \partial_n f(t) \|_\infty + e^{-N} \right\} \\
&\quad \times \int_0^1 \iint_{\substack{\varepsilon \lesssim |u_n| \lesssim \Lambda \\ \frac{1}{N} \lesssim |u| \lesssim N}} \frac{1}{\sqrt{\alpha(x_0 - \varepsilon r n(x_0), u)}} \\
&- O(1) N e^{\varpi N t} \| e^{-\varpi \langle v \rangle t} \sqrt{\alpha} \nabla_x f(t) \|_\infty \\
&- O_N(1) e^{\varpi N t} \| e^{\theta |v|^2} f_0 \|_\infty \| e^{-\varpi \langle v \rangle t} \sqrt{\alpha} \nabla_x f(t) \|_\infty \\
&\quad \times \int_0^1 \iint_{\substack{\varepsilon \lesssim |u_n| \lesssim \Lambda \\ \frac{1}{N} \lesssim |u| \lesssim N}} \frac{1}{[\alpha(x_0 - \varepsilon r n(x_0), u)]^{\delta/2}}.
\end{aligned}$$

Due to (8.1), if $\delta < 1$, for $N \gg 1$ and $t \ll 1$ with $N^2 \sqrt{\Lambda^2 + \varepsilon N^2} \ll 1$

$$\begin{aligned}
\mathbf{II} &\gtrsim \int_0^1 \int_{\substack{\varepsilon \lesssim |u_n| \lesssim \Lambda \\ \frac{1}{N} \lesssim |u| \lesssim N}} \frac{1}{\sqrt{|u_n|^2 + C \varepsilon r |u_\tau|^2}} \mathrm{d}u_n \mathrm{d}u_\tau \mathrm{d}r \\
&\gtrsim \int_{\varepsilon \leq |u_n| \leq \Lambda} \frac{N^2}{|u_n| + \sqrt{\varepsilon} N} \mathrm{d}u_\tau \mathrm{d}u_n \\
&\gtrsim N^2 \ln \frac{1}{\varepsilon + \sqrt{\varepsilon} N} - O_{N,\Lambda}(1) \\
&\gtrsim \frac{N^2}{2} \ln \frac{1}{\varepsilon} - o(1) \ln \frac{1}{\varepsilon} - O_{N,\Lambda}(1) \to \infty.
\end{aligned}$$

□

For the bounce-back and specular cases we consider small solutions

$$f = \sigma g \quad \text{for } 0 < \sigma \ll 1. \tag{8.8}$$

where f solves (1.7). Then g solves

$$\partial_t g + v \cdot \nabla_x g + \sigma \nu(\sqrt{\mu} g) g = \sigma \Gamma_{\text{gain}}(g, g).$$

Note that for either the bounce-back or specular cases we can express g along the trajectory

$$g(t,x,v) = e^{-\sigma\int_0^t \nu(\sqrt{\mu}g)(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\mathrm{d}s} g_0(X_{\mathbf{cl}}(0;t,x,v),V_{\mathbf{cl}}(0;t,x,v))$$
$$+\sigma\int_0^t e^{-\sigma\int_s^t \nu(\sqrt{\mu}g)(\tau,X_{\mathbf{cl}}(\tau),V_{\mathbf{cl}}(\tau))\mathrm{d}\tau}$$
$$\times\Gamma_{\mathrm{gain}}(g,g)(s,X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))\mathrm{d}s, \tag{8.9}$$

where the backward trajectories are defined in Definition 2 respectively. We remark that for $0<\sigma\ll 1$, g is uniformly bounded in C^1 away from the grazing set γ_0.

Proposition 4 (Bounce-Back BC) *Assume* $\Omega=\{x\in\mathbb{R}^3 : |x|<1\}$ *and* $\xi(x)=|x|^2-1$. *Assume the initial datum* $f_0=\sigma g_0\geq 0$ *supported in* $\frac{1}{2}\leq|v|\leq 2$ *and*

$$\lim_{n(x_0)\cdot u=0}\int \mathbf{k}_{g_0}(v,u)u\cdot\nabla_x g_0(x_0,u)\mathrm{d}u_\tau > C>0, \tag{8.10}$$

for some $x_0\in\partial\Omega$ *and some* $v\in\mathbb{R}^3$ *where* $u_\tau=u-[u\cdot n(x_0)]n(x_0)$. *Then there exists* $0<t=C\sigma$ *for some* $C>0$ *such that*

$$\partial_n\Gamma_{\mathrm{gain}}(g,g)(t,x_0,v)-\partial_n\nu(\sqrt{\mu}g)g(t,x_0,v)=\infty. \tag{8.11}$$

We remark that $u\cdot\nabla_x g_0(x_0,u)$ is rather arbitrary for $u\cdot n(x_0)=0$.

Proof Assuming $g\simeq g_0\lesssim(\sqrt{\mu})^{1-\delta}$ for $0<\delta\ll 1$, we have

$$\Gamma_{\mathrm{gain}}(\triangle_\varepsilon g,\sqrt{\mu}+g)+\Gamma_{\mathrm{gain}}(\sqrt{\mu}+g,\triangle_\varepsilon g)-\nu(\sqrt{\mu}\triangle_\varepsilon g)(\sqrt{\mu}+g)$$
$$\simeq\int_{\mathbb{R}^3}\mathbf{k}_{g_0}(v,u)\triangle_\varepsilon g(x,u)\mathrm{d}u\simeq\int_{\mathbb{R}^3}\mathbf{k}_{g_0}(v,u)\frac{g(x-\varepsilon n(x),u)-g(x,u)}{\varepsilon}\mathrm{d}u, \tag{8.12}$$

where $\mathbf{k}_{g_0}(v,u)\simeq\mathbf{k}(v,u)$ in (2.9) with slightly different exponents. For simplicity let us assume $\mathbf{k}_{g_0}(v,u)$ is bounded.

We choose $(x,v)\in\bar{\Omega}\times\mathbb{R}^3$ so that $|x^\ell-x_0|\ll 1$ and $|v^\ell-\pm v_0|\ll 1$ for all $\ell\in\mathbb{N}$ (even or odd). Then

$$\int_{\mathbb{R}^3}\mathbf{k}_{g_0}(v,u)\frac{g(t,x-\varepsilon n(x),u)-g(t,x,u)}{\varepsilon}\mathrm{d}u=\int_{|n(x)\cdot u|\leq\varepsilon}+\int_{\varepsilon\leq|n(x)\cdot u|}.$$

The first term is bounded. Due to (8.5) we have $|n(x)\cdot u|\geq\varepsilon$ implies $|n(x-rn(x))\cdot u|\gtrsim\varepsilon$ for all $0\leq r\leq\varepsilon$. Then by Theorem 6, the function g is differentiable and the second term equals

$$
\begin{aligned}
&\int_{|n(x)\cdot u|\geq\varepsilon}\mathbf{k}_{g_0}(v,u)\frac{g(t,x-\varepsilon n(x),u)-g(t,x,u)}{\varepsilon}\mathrm{d}u\\
&\quad=\int_0^1\int_{|n(x)\cdot u|\geq\varepsilon}\mathbf{k}_{g_0}(v,u)\partial_n g(t,x-\varepsilon rn(x),u)\mathrm{d}u\mathrm{d}r\\
&\quad=\int_0^1\int_{\varepsilon\leq|n(x)\cdot u|\leq 1,|\tau(x)\cdot u|\leq N}+\int_0^1\int_{\varepsilon\leq|n(x)\cdot u|\leq 1,|\tau(x)\cdot u|\geq N}\\
&\qquad+\int_0^1\int_{|n(x)\cdot u|\geq 1}.
\end{aligned}
\tag{8.13}
$$

For the third term of (8.13) we use Theorem 6 to have

$$
|\partial_n g(t,x-\varepsilon rn(x),u)|\lesssim\frac{\langle u\rangle^2 e^{\varpi\langle u\rangle t}}{\alpha(x-\varepsilon rn(x),u)}\lesssim\langle u\rangle^2 e^{\varpi\langle u\rangle t},
$$

and therefore the third term of (8.13) is bounded. For the second term of (8.13) we use Theorem 6 to bound

$$
\begin{aligned}
&\|e^{-\varpi\langle v\rangle t}\frac{\alpha}{\langle v\rangle^2}\nabla_x g(t)\|_\infty\times\int_0^1\mathrm{d}r\int_\varepsilon^1\mathrm{d}u_n\frac{e^{\varpi\delta}N^2}{|u_n|^2+Cr\varepsilon N^2}\int_{\mathbb{R}^2}\mathrm{d}u_\tau\mathbf{k}_{g_0}(v,u)\\
&\lesssim e^{-N}\times\int_0^1\mathrm{d}r\int_\varepsilon^1\mathrm{d}u_n\frac{1}{|u_n|^2+Cr\varepsilon N^2}.
\end{aligned}
\tag{8.14}
$$

Now we focus on the first term of (8.13). Using (8.9), (7.3), Theorem 6, and Lemma 17, we have

$$
\begin{aligned}
&\partial_n g(t,y,u)\\
&=e^{-\sigma\int_0^t\nu(\sqrt{\mu}g)(\tau)\mathrm{d}\tau}\Big\{[\partial_n x^{\ell_*(0)}-\partial_n t^{\ell_*(0)}v^{\ell_*(0)}]\cdot\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\\
&\quad+v^{\ell_*(0)}\cdot\nabla_v g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\Big\}\\
&\quad+\sigma\|e^{\theta|v|^2}g_0\|_\infty\Big\{\frac{t^2|u|^2}{\alpha(y,u)}e^{-\frac{\theta}{4}|u|^2}\|e^{\theta|v|^2}\partial_t g_0\|_\infty+\frac{t(1+t)|u|^2}{\alpha(y,u)}e^{-\frac{\theta}{4}|u|^2}\\
&\quad+\sup_{0\leq s\leq t}\|e^{-\varpi\langle v\rangle t}\alpha\partial_x g(t)\|_\infty\int_0^t\int_{\mathbb{R}^3}\\
&\quad\sum_\ell\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\frac{e^{-\frac{\theta}{8}|v^\ell-u'|^2}}{|v^\ell-u'|^{2-\kappa}}\frac{|\partial_n x^\ell|}{e^{-\varpi\langle u'\rangle s}\alpha(X_{\mathbf{cl}}(s),u')}\mathrm{d}u'\mathrm{d}s\Big\}.
\end{aligned}
$$

Using Lemma 1.29, Lemma 17, and (6.75), we bound the last integration by

$$
\begin{aligned}
& e^{\varpi\langle u\rangle t}\frac{|u|}{\sqrt{\alpha(y,u)}}\int_0^t\int_{\mathbb{R}^3} e^{-\varpi\langle u\rangle(t-s)}\frac{e^{-\frac{\theta}{8}|V_{\mathbf{cl}}(s)-u'|^2}}{|V_{\mathbf{cl}}(s)-u'|^{2-\kappa}}\frac{1}{\alpha(X_{\mathbf{cl}}(s),u')}\mathrm{d}u'\mathrm{d}s \\
&\quad \lesssim_\varepsilon e^{\varpi\langle u\rangle t}\frac{|u|}{\langle u\rangle}\frac{1}{\alpha(y,u)}.
\end{aligned}
$$

Now by the explicit computations in Lemma 17

$$
\begin{aligned}
& e^{-\sigma\int_0^t \nu(\mu+\sqrt{\mu}g)(\tau)\mathrm{d}\tau}\Big\{[\partial_n x^{\ell_*(0)}-\partial_n t^{\ell_*(0)}v^{\ell_*(0)}]\cdot\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) \\
&\quad +v^{\ell_*(0)}\cdot\nabla_v g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\Big\} \\
&\geq e^{-\sigma t\langle u\rangle\|e^{\theta|v|^2}g_0\|_\infty}\underbrace{\left\{\ell_*(0)\frac{n(y)\cdot\nabla\xi(x^1)}{v\cdot\nabla\xi(x^1)}+(\ell_*(0)-1)\frac{n(y)\cdot\nabla\xi(x^2)}{-v\cdot\nabla\xi(x^2)}\right\}}V_{\mathbf{cl}}(0) \\
&\qquad \cdot\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) \\
&\quad -C_\xi\frac{|u|}{\sqrt{\alpha(y,u)}}|\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|-C_\xi|u||\nabla_v g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))| \\
&\geq e^{-t\sigma\langle u\rangle\|e^{\theta|v|^2}g_0\|_\infty}O_\xi(1)\frac{t|u|^2}{\alpha(y,u)}V_{\mathbf{cl}}(0)\cdot\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) \\
&\quad -\frac{O_\xi(1+t|u|)}{\sqrt{\alpha(y,u)}}|u\|\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|-C_\xi|u||\nabla_v g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|,
\end{aligned}
$$

where for the underbraced term we used

$$
\begin{aligned}
& n(y)\cdot\left\{\frac{n(x_{\mathbf{b}})}{n(x_{\mathbf{b}})\cdot v}-\frac{n(y)}{n(y)\cdot v}\right\} \\
&= \frac{n(y)\cdot\nabla\xi(y-t_{\mathbf{b}}v)\,(\nabla\xi(y)\cdot v)-n(y)\cdot\nabla\xi(y)\,(\nabla\xi(y-t_{\mathbf{b}}v)\cdot v)}{(\nabla\xi(y)\cdot v)\,(\nabla\xi(y-t_{\mathbf{b}}v)\cdot v)} \\
&= \frac{t_{\mathbf{b}}\left\{(n(y)\cdot[v\cdot\nabla]\nabla\xi(y-\tilde{\tau}v))\,(\nabla\xi(y)\cdot v)-n(y)\cdot\nabla\xi(y)\left(v\cdot\nabla^2\xi(y-\tilde{\tau}v)\cdot v\right)\right\}}{(\nabla\xi(y)\cdot v)\,(-\nabla\xi(y-t_{\mathbf{b}}v)\cdot v)} \\
&= -\frac{t_{\mathbf{b}}}{(-\nabla\xi(x_{\mathbf{b}})\cdot v)}\frac{\left(v\cdot\nabla^2\xi(y-\tilde{\tau}v)\cdot v\right)}{n(y)\cdot v}+\frac{t_{\mathbf{b}}}{(-\nabla\xi(x_{\mathbf{b}})\cdot v)}(n(y)\cdot[v\cdot\nabla]\nabla\xi(y-\tilde{\tau}v)) \\
&:= -\frac{A(y,v)}{n(y)\cdot v}+B(y,v),
\end{aligned}
\tag{8.15}
$$

where for some $\tilde{\tau}\in[0,t_{\mathbf{b}}]$, and from (2.2), (2.3), and the Velocity lemma (Lemma 2) we have $A\geq 0$ and

$$
A(y,v)\geq C_\xi\frac{v}{|v|}\cdot\nabla^2\xi(y-\tilde{\tau}v)\cdot\frac{v}{|v|}\gtrsim_\Omega 1,\quad B(y,v)\simeq_\Omega\frac{1}{|v|}.
$$

Therefore finally the underbraced term has the following explicit lower bound:

$$
\begin{aligned}
&\ell_*(0)\left[\frac{n(y)\cdot\nabla\xi(x^1)}{v\cdot\nabla\xi(x^1)}-\frac{n(y)\cdot\nabla\xi(x^2)}{v\cdot\nabla\xi(x^2)}\right]+\frac{n(y)\cdot\nabla\xi(x^2)}{v\cdot\nabla\xi(x^2)}\\
&=\ell_*(0)\frac{A(y,v)}{n(x^1)\cdot v}+\ell_*(0)B(y,v)+O\left(\frac{1}{n(x^1)\cdot v}\right)\\
&=O_\xi(1)\frac{t|v|^2}{\alpha(y,v)}+\frac{O_\xi(1+t|v|)}{\sqrt{\alpha(y,v)}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\partial_n g(t,y,u)\\
&\geq e^{-\sigma t\langle u\rangle}\|e^{\theta|v|^2}g_0\|_\infty O_\xi(1)\frac{t|u|^2}{\alpha(y,u)}V_{\mathbf{cl}}(0)\cdot\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\\
&-\frac{O_\xi(1+t|u|)}{\sqrt{\alpha(y,u)}}|u\|\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|-C_\xi|u||\nabla_v g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|\\
&-\sigma\|e^{\theta|v|^2}g_0\|_\infty\left\{\frac{t^2|u|^2}{\alpha(y,u)}e^{-\frac{\theta}{4}|u|^2}\|e^{\theta|v|^2}\partial_t g_0\|_\infty-\frac{t(1+t)|u|^2}{\alpha(y,u)}e^{-\frac{\theta}{4}|v|^2}\right.\\
&\qquad\left.-\sup_{0\leq s\leq t}\|e^{-\varpi\langle v\rangle t}\alpha\partial_x g(t)\|_\infty\times e^{\varpi\langle u\rangle t}\frac{|u|}{\langle u\rangle}\frac{1}{\alpha(y,u)}\right\}.
\end{aligned}
\tag{8.16}
$$

Choose $y=x-\varepsilon rn(x)$, $t=C\sigma\ for\ C\gg 1$, $0<\delta\ll 1$ and $\alpha(y,u)\ll 1$. Then the first contribution of (8.13) is bounded below by

$$
\begin{aligned}
&\int_0^1\int_{\varepsilon\leq|n(x)\cdot u|\leq\delta}\int_{|\tau(y)\cdot u|\leq N}\\
&\gtrsim\sigma\int_0^1\mathrm{d}r\int_{\varepsilon\leq|u_n|\leq\delta}\mathrm{d}u_n\int_{|u_\tau|\leq N}\mathrm{d}u_\tau\frac{1}{|u_n|^2+Cr\varepsilon N^2}\\
&\quad\times\mathbf{k}_{g_0}(v,u)V_{\mathbf{cl}}(0)\cdot\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\\
&\simeq\sigma\int_0^1\mathrm{d}r\int_{\varepsilon\leq|u_n|\leq\delta}\frac{\mathrm{d}u_n}{|u_n|^2+Cr\varepsilon N^2}\int_{|u_\tau|\leq N}\mathrm{d}u_\tau\mathbf{k}_{g_0}(v,u)u\cdot\nabla_x g_0(x_0,u).
\end{aligned}
\tag{8.17}
$$

Now we use the condition (8.10) for $\varepsilon\ll 1$ and $u_n\ll 1$

$$
\begin{aligned}
&\int_{|u_\tau|\geq N}\mathbf{k}_{g_0}(v,u)V_{\mathbf{cl}}(0)\cdot\nabla_x g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\mathrm{d}u_\tau\\
&\quad\simeq\int_{n(x_0)\cdot u_\tau=0}\mathbf{k}_{g_0}(v,u)u\cdot\nabla_x g_0(x_0,u)\mathrm{d}u_\tau=C\neq 0.
\end{aligned}
$$

We combine this term with the second term of (8.13) to conclude

$$\begin{aligned}(8.17)-(8.14) &\gtrsim \{C\sigma - e^{-N} - o(1)\}\int_{\varepsilon}^{\delta} \mathrm{d}u_n \int_0^1 \frac{\mathrm{d}r}{|u_n|^2 + Cr\varepsilon N^2}\\ &\gtrsim \int_{\varepsilon}^{1} \frac{1}{C\varepsilon N^2}\ln\left(1+\frac{C\varepsilon N^2}{|u_n|^2}\right)\mathrm{d}u_n \gtrsim \frac{1}{N^2}\frac{1}{\varepsilon}.\end{aligned} \tag{8.18}$$

Now all the other terms of (8.16) except the first term are bounded by

$$\int_{\varepsilon}^{1} \mathrm{d}u_n \frac{1}{|u_n|} + \delta \int_{\varepsilon}^{1} \mathrm{d}u_n \frac{1}{|u_n|^2} \lesssim |\ln\varepsilon| + \delta \times \frac{1}{\varepsilon}.$$

Therefore we conclude (8.2) as $\varepsilon \to 0$. □

In order to show the non-existence of $\nabla^2 f$ up to the boundary for the specular reflection BC (Proposition 5) we first obtain the explicit lower bound of (1.35) with a lower dimensional symmetric domain, 2D disk.

Lemma 18 *Let* $\Omega = \{\bar{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1|^2 + |x_2|^2 < 1\}$. *Define*

$$r := \sqrt{x_1^2 + x_2^2} \in [0,1], \quad \theta \in [0, 2\pi) \ \textit{such that} \ (\cos\theta, \sin\theta) = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1, x_2),$$

$$\bar{v}_n := v_1\cos\theta + v_2\sin\theta, \quad \bar{v}_\theta := -v_1\sin\theta + v_2\cos\theta.$$

We claim that as $\alpha \to 0$ (*therefore* $|r-1| \ll 1$, $\bar{v}_n \ll 1$) *asymptotically*

$$\begin{aligned}|\partial_n \bar{X}_{\mathbf{cl}}(s;t,x,v)\cdot \bar{n}(\bar{X}_{\mathbf{cl}}(s;t,x,v))| &\simeq \frac{|t-s||\bar{v}_\theta|^2}{\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} \simeq \frac{|t-s||\bar{v}|^2}{\sqrt{\alpha(x,v)}},\\ |\partial_n \bar{V}_{\mathbf{cl}}(s;t,x,v)\cdot \bar{n}(\bar{X}_{\mathbf{cl}}(s;t,x,v))| &\simeq \frac{|t-s||\bar{v}|^4}{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2} \simeq \frac{|t-s||\bar{v}|^4}{\alpha(x,v)}.\end{aligned} \tag{8.19}$$

Proof Explicitly $x = (r\cos\theta, r\sin\theta, x_3)$, and $v = (\bar{v}_n\cos\theta - \bar{v}_\theta\sin\theta, \bar{v}_n \sin\theta + \bar{v}_\theta\cos\theta, v_3)$, and

$$
\begin{aligned}
x^\ell &= (\cos\theta^\ell, \sin\theta^\ell, x_3 - (t - t^\ell)v_3),\\
v^\ell &= \left(\sqrt{\bar{v}_n^2 + \bar{v}_\theta^2}\cos\psi^\ell, \sqrt{\bar{v}_n^2 + \bar{v}_\theta^2}\sin\psi^\ell, v_3\right),\\
t^1 &= t - \frac{r|\bar{v}_n| + \sqrt{(1-r^2)\bar{v}_\theta^2 + \bar{v}_n^2}}{\bar{v}_n^2 + \bar{v}_\theta^2},\\
t^\ell &= t - \frac{r|\bar{v}_n| + (2\ell - 1)\sqrt{(1-r^2)\bar{v}_\theta^2 + \bar{v}_n^2}}{\bar{v}_n^2 + \bar{v}_\theta^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\ell_*(s;t,x,v) \le \frac{(t-s)|\bar{v}|^2}{2\sqrt{(1-r^2)\bar{v}_\theta^2 + \bar{v}_n^2}} - \frac{r|\bar{v}_n|}{2\sqrt{(1-r^2)\bar{v}_\theta^2 + \bar{v}_n^2}} + \frac{1}{2}\\
< \ell_*(s;t,x,v) + 1,
\end{aligned}
$$

where, for $\ell \ge 1$

$$
\begin{aligned}
&\theta^0 = \theta, \quad \theta^\ell = \theta - \cos^{-1}\left(\frac{\bar{v}_\theta}{\sqrt{\bar{v}_\theta^2 + \bar{v}_n^2}}\right) - (2\ell - 1)\cos^{-1}\left(\frac{r\bar{v}_\theta}{\sqrt{\bar{v}_n^2 + \bar{v}_\theta^2}}\right),\\
&\psi^0 = \cos^{-1}\left(\frac{\bar{v}_n\cos\theta - \bar{v}_\theta\sin\theta}{\sqrt{\bar{v}_n^2 + \bar{v}_\theta^2}}\right), \quad \psi^\ell = \psi^0 - 2\ell\cos^{-1}\left(\frac{r\bar{v}_\theta}{\sqrt{\bar{v}_n^2 + \bar{v}_\theta^2}}\right).
\end{aligned}
$$

Therefore, for $t^{\ell+1} < s < t^\ell$, we have $X_{\mathbf{cl}}(s) = x^\ell - (t^\ell - s)v^\ell$, $V_{\mathbf{cl}}(s) = v^\ell$, and

$$
\begin{aligned}
&r(s) = |\bar{X}_{\mathbf{cl}}(s)| = |\bar{x}^\ell - (t^\ell - s)\bar{v}^\ell|,\\
&\bar{v}_n(s) = \bar{V}_{\mathbf{cl}}(s)\cdot\frac{\bar{X}_{\mathbf{cl}}(s)}{|\bar{X}_{\mathbf{cl}}(s)|}, \quad \bar{v}_\theta(s) = \bar{V}_{\mathbf{cl}}(s)\cdot\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{\bar{X}_{\mathbf{cl}}(s)}{|\bar{X}_{\mathbf{cl}}(s)|}, \quad v_3(s) = v_3.
\end{aligned}
$$

Directly

$$
\begin{aligned}
&\partial_\theta\bar{v}_n = v_\theta, \quad \partial_\theta\bar{v}_\theta = -\bar{v}_n,\\
&\partial_n\cos^{-1}\left(\frac{r\bar{v}_\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right) = \frac{-\bar{v}_\theta}{\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}},
\end{aligned}
$$

$$\partial_\theta \cos^{-1}\left(\frac{\bar{v}_\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=1, \quad \partial_\theta \cos^{-1}\left(\frac{r\bar{v}_\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=\frac{r\bar{v}_n}{\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}},$$

$$\partial_{\bar{v}_n} \cos^{-1}\left(\frac{\bar{v}_\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=\frac{\bar{v}_\theta}{\bar{v}_n^2+\bar{v}_\theta^2}, \quad \partial_{\bar{v}_n} \cos^{-1}\left(\frac{r\bar{v}_\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=\frac{r\bar{v}_\theta}{\bar{v}_n^2+\bar{v}_\theta^2},$$

$$\partial_{\bar{v}_\theta} \cos^{-1}\left(\frac{\bar{v}_\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=\frac{-\bar{v}_n}{\bar{v}_n^2+\bar{v}_\theta^2}, \quad \partial_{\bar{v}_\theta} \cos^{-1}\left(\frac{r\bar{v}_\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=\frac{-r\bar{v}_n}{\bar{v}_n^2+\bar{v}_\theta^2},$$

$$\partial_{\bar{v}_n} \cos^{-1}\left(\frac{\bar{v}_n\cos\theta-\bar{v}_\theta\sin\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=\frac{\bar{v}_\theta}{\bar{v}_n^2+\bar{v}_\theta^2},$$

$$\partial_{\bar{v}_\theta} \cos^{-1}\left(\frac{\bar{v}_n\cos\theta-\bar{v}_\theta\sin\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=\frac{\bar{v}_n}{\bar{v}_n^2+\bar{v}_\theta^2},$$

$$\partial_\theta \cos^{-1}\left(\frac{\bar{v}_n\cos\theta-\bar{v}_\theta\sin\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)=0=\partial_n \cos^{-1}\left(\frac{\bar{v}_n\cos\theta-\bar{v}_\theta\sin\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right),$$

and

$$\partial_{\bar{v}_\theta}\theta^\ell=\frac{|\bar{v}_n|}{|\bar{v}|^2}+|t-s|, \quad \partial_{\bar{v}_r}\theta^\ell=-\frac{\bar{v}_\theta}{|\bar{v}|^2}-(2\ell-1)\frac{r\bar{v}_\theta}{|\bar{v}|^2},$$
$$\partial_\theta\theta^\ell\lesssim\frac{|t-s\|\bar{v}|^2|\bar{v}_n|}{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}, \ \partial_n\theta^\ell=\frac{(2\ell-1)\bar{v}_\theta}{\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}},$$

and

$$\partial_\theta\psi^\ell\lesssim\frac{|t-s\|\bar{v}|^2|\bar{v}_n|}{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}, \quad \partial_n\psi^\ell=\frac{2\ell\bar{v}_\theta}{\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}},$$
$$\partial_{\bar{v}_\theta}\psi^\ell\lesssim\frac{|t-s||\bar{v}_n|}{\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}}\lesssim|t-s|, \quad \partial_{\bar{v}_n}\psi^\ell=-2\ell\frac{r\bar{v}_\theta}{\bar{v}_n^2+\bar{v}_\theta^2}+O_\xi(1)\frac{1}{|\bar{v}|},$$

and

$$t^{\ell} - t^{\ell+1} \leq \frac{2\sqrt{(1-r^2)\bar{v}_\theta^2 + \bar{v}_n^2}}{\bar{v}_n^2 + \bar{v}_\theta^2},$$

$$\ell_*(s) \leq \frac{|t-s||\bar{v}|^2}{2\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} - \frac{r|\bar{v}_n|}{2\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} + \frac{1}{2} \leq \ell_*(s) + 1,$$

and

$$\partial_r t^{\ell} = \frac{-|\bar{v}_n|}{\bar{v}_n^2 + \bar{v}_\theta^2} + (2\ell - 1)\frac{r\bar{v}_\theta^2}{|\bar{v}|^2\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}},$$

$$\partial_\theta t^{\ell} = \frac{-(2\ell-1)\bar{v}_n\bar{v}_\theta r^2}{|\bar{v}|^2\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} \lesssim \frac{|t-s\|\bar{v}_\theta|r^2}{\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}},$$

$$\partial_{\bar{v}_n} t^{\ell} = -(2\ell-1)\frac{\bar{v}_n}{|\bar{v}|^2\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} + O_\xi(1)\frac{1+|\bar{v}\|t-s|}{|\bar{v}|^2},$$

$$\partial_{\bar{v}_\theta} t^{\ell} \leq (2\ell-1)\frac{(1-r^2)|\bar{v}_\theta|}{|\bar{v}|^2\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} + 2(2\ell-1)\frac{|\bar{v}_\theta|\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}}{|\bar{v}|^4}$$
$$\lesssim |t-s|\frac{(1-r^2)|\bar{v}_\theta|}{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2} + |t-s|\frac{|\bar{v}_\theta|}{|\bar{v}|^2}.$$

If $r < \frac{1}{2}$ then $(1-r^2)\bar{v}_\theta^2 + \bar{v}_n^2 \geq \frac{3}{4}|\bar{v}|^2$ and

$$\partial_{\bar{v}_\theta} t^{\ell} \lesssim |t-s|\frac{4|\bar{v}_\theta|}{3|\bar{v}|^2} + |t-s|\frac{|\bar{v}_\theta|}{|\bar{v}|^2} \lesssim |t-s|\frac{|\bar{v}_\theta|}{|\bar{v}|^2}.$$

If $r \geq \frac{1}{2}$ and $|\bar{v}_\theta| \leq |\bar{v}_n|$ then

$$\partial_{\bar{v}_\theta} t^{\ell} \lesssim \frac{|t-s||\bar{v}_\theta|}{\frac{\bar{v}_n^2}{2} + \frac{\bar{v}_n^2}{2}} + |t-s|\frac{|\bar{v}_\theta|}{|\bar{v}|^2} \lesssim |t-s|\frac{|\bar{v}_\theta|}{|\bar{v}|^2}.$$

If $r \geq \frac{1}{2}$ and $|\bar{v}_\theta| \geq |\bar{v}_n|$ then

$$\partial_{\bar{v}_\theta} t^{\ell} \lesssim |t-s|\frac{(1-r^2)|\bar{v}_\theta|}{(1-r^2)|\bar{v}_\theta|(\frac{|\bar{v}_\theta|}{2} + \frac{|\bar{v}_\theta|}{2})} + |t-s|\frac{|\bar{v}_\theta|}{|\bar{v}|^2} \lesssim \frac{|t-s|}{|\bar{v}|}.$$

Therefore

$$\partial_{\bar{v}_\theta} t^\ell \lesssim \frac{|t-s|}{|\bar{v}|}.$$

Directly

$$\begin{aligned}
\partial_n \bar{X}_{\mathbf{cl}}(s) = \partial_n \theta^\ell \begin{pmatrix} -\sin\theta^\ell \\ \cos\theta^\ell \end{pmatrix} &- \frac{\partial t^\ell}{\partial n}|\bar{v}| \begin{pmatrix} \cos\psi^\ell \\ \sin\psi^\ell \end{pmatrix} \\
&- (t^\ell - s)|\bar{v}|\frac{\partial \psi^\ell}{\partial n}\begin{pmatrix} -\sin\psi^\ell \\ \cos\psi^\ell \end{pmatrix} \\
&= \frac{(2\ell-1)\bar{v}_\theta^2}{|\bar{v}|\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} \begin{pmatrix} -\sin\theta^\ell - \cos\psi^\ell \\ \cos\theta^\ell - \sin\psi^\ell \end{pmatrix} \\
&+ O_\xi(1)\left\{ \frac{(2\ell-1)|\bar{v}_\theta||\bar{v}_n|}{|\bar{v}|\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} + \frac{(2\ell-1)(1-r)|\bar{v}_\theta|^2}{|\bar{v}|\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} \right. \\
&\left. + \frac{|\bar{v}_n|}{|\bar{v}|} + \ell|t^\ell - t^{\ell+1}| \right\},
\end{aligned}$$

where $O_\xi(1)$-remainder is bounded by

$$\begin{aligned}
&\lesssim \frac{|t-s||\bar{v}|^2}{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}\left\{ \frac{|\bar{v}_\theta||\bar{v}_n|}{|\bar{v}|} + \frac{|1-r||\bar{v}_\theta^2|}{|\bar{v}|} + \frac{\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}}{|\bar{v}|} \right\} + \frac{|\bar{v}_n|}{|\bar{v}|} \\
&\lesssim \frac{|t-s||\bar{v}|(1+|\bar{v}|)}{\sqrt{\bar{v}_n^2 + (1-r^2)\bar{v}_\theta^2}} + |t-s||\bar{v}| + \frac{|\bar{v}_n|}{|\bar{v}|}.
\end{aligned}$$

Now we focus on the first term (most singular term) of $\partial_n \bar{X}_{\mathbf{cl}}(s)$. We note that

$$\begin{aligned}
\psi^\ell &= \psi^0 - 2\ell\cos^{-1}\left(\frac{r\bar{v}_\theta}{\sqrt{\bar{v}_n^2 + \bar{v}_\theta^2}}\right) \\
&= \theta^\ell + \left\{ \cos^{-1}\left(\frac{\bar{v}_n\cos\theta - \bar{v}_\theta\sin\theta}{\sqrt{\bar{v}_n^2 + \bar{v}_\theta^2}}\right) - \theta + \cos^{-1}\left(\frac{\bar{v}_\theta}{\sqrt{\bar{v}_n^2 + \bar{v}_\theta^2}}\right) \right.
\end{aligned}$$

$$-\cos^{-1}\left(\frac{r\bar{v}_\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)\Bigg\}$$

In the limit of $r \to 1$, we have $\frac{\bar{v}_n\cos\theta-\bar{v}_\theta\sin\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}} \simeq -\mathrm{sgn}\{\bar{v}_\theta\}\cos\theta$. Without the loss of generality we assume $\bar{v}_\theta > 0$. Then for $\cos\theta \neq 0$, $\cos^{-1}$ is decreasing, and

$$\cos^{-1}\left(\frac{\bar{v}_n\cos\theta-\bar{v}_\theta\sin\theta}{\sqrt{\bar{v}_n^2+\bar{v}_\theta^2}}\right)-\theta=\theta+\frac{\pi}{2}-a\bar{v}_n\cos\theta-\theta+o(\bar{v}_n)\simeq\frac{\pi}{2}-a\bar{v}_n\cos\theta.$$

for some constant $a > 0$. In particular, at $\theta = 0$, $a = 1$. Therefore, we have for the leading order term of $\bar{v}_n$ in $\partial_n X_{\mathbf{cl}}$ is

$$\begin{aligned}\begin{pmatrix}-\sin\theta^\ell-\cos\psi^\ell\\ \cos\theta^\ell-\sin\psi^\ell\end{pmatrix}&=\begin{pmatrix}-\sin\theta^\ell-\cos(\theta^\ell+\frac{\pi}{2}-a\cos\theta\bar{v}_n)\\ \cos\theta^\ell-\sin(\theta^\ell+\frac{\pi}{2}-a\cos\theta\bar{v}_n)\end{pmatrix}\\ &=\begin{pmatrix}-\sin\theta^\ell+\sin(\theta^\ell-a\cos\theta\bar{v}_n)\\ \cos\theta^\ell-\cos(\theta^\ell-a\cos\theta\bar{v}_n)\end{pmatrix}\simeq-a\cos\theta\bar{v}_n\begin{pmatrix}\cos\theta^\ell\\ \sin\theta^\ell\end{pmatrix}\end{aligned}$$

which is parallel to $n(\bar{X}_{\mathbf{cl}}(s))$. Without loss of generality we now fix $\theta = 0$ and $|v| = 1$, $r = 1$,

$$\begin{aligned}&\partial_n\bar{X}_{\mathbf{cl}}(s)\cdot\bar{n}(\bar{X}_{\mathbf{cl}}(s;t,x,v))\\ &=\lim_{r\downarrow 0}\left\{\frac{(2\ell-1)\bar{v}_\theta^2}{|\bar{v}|\sqrt{\bar{v}_n^2+(1-r)\bar{v}_\theta^2}}\bar{v}_n+\frac{|t-s\|\bar{v}|(1+|\bar{v}|)}{\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}}+|t-s||\bar{v}|+\frac{|\bar{v}_n|}{|\bar{v}|}\right\}.\end{aligned}$$

This prove the first estimate of (8.19).

Using the same estimates leads to

$$\begin{aligned}\partial_{\bar{v}_n}\bar{X}_{\mathbf{cl}}(s)&=(2\ell-1)\left\{\frac{-r\bar{v}_\theta}{|\bar{v}|^2}\begin{pmatrix}-\sin\theta^\ell\\ \cos\theta^\ell\end{pmatrix}+\frac{\bar{v}_n}{|\bar{v}|\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}}\begin{pmatrix}\cos\psi^\ell\\ \sin\psi^\ell\end{pmatrix}\right\}\\ &\quad+O_\xi(1)\frac{1+|\bar{v}\|t-s|}{|\bar{v}|}\\ &=\frac{2\ell-1}{|\bar{v}|}\begin{pmatrix}\sin\theta^\ell+\cos\psi^\ell\\ -\cos\theta^\ell+\sin\psi^\ell\end{pmatrix}+O_\xi(1)\frac{1+|\bar{v}\|t-s|}{|\bar{v}|}\lesssim\frac{1}{|\bar{v}|}.\end{aligned}$$

Since $t^{\ell_*+1} < 0 < t^{\ell_*}$,

$$\partial_n v^\ell=\partial_n\psi^\ell(-|\bar{v}|\sin\psi^\ell,|\bar{v}|\cos\psi^\ell,0)$$

$$= \frac{|t-s\|\bar{v}|^2|\bar{v}_\theta|}{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}(-|\bar{v}|\sin\psi^\ell, |\bar{v}|\cos\psi^\ell, 0).$$

Therefore we conclude our claim for $\partial_n \bar{V}_{\mathbf{cl}}(0)$. We can easily check

$$\begin{aligned}
|\partial_\theta \bar{X}_{\mathbf{cl}}(s)| &\lesssim \frac{|\bar{v}|}{\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}}|\bar{v}\|t-s|, \\
|\partial_\theta \bar{V}_{\mathbf{cl}}(s)| &\lesssim \frac{|\bar{v}|^2}{\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}}|\bar{v}\|t-s|, \\
|\partial_{\bar{v}_n} \bar{V}_{\mathbf{cl}}(s)| &\lesssim 1+\frac{|\bar{v}|^2|t-s|}{\sqrt{\bar{v}_n^2+(1-r^2)\bar{v}_\theta^2}}, \\
|\partial_{\bar{v}_\theta} \bar{X}_{\mathbf{cl}}(s)| &\lesssim \frac{1}{|\bar{v}|}, \quad |\partial_{\bar{v}_\theta} \bar{V}_{\mathbf{cl}}(s)| \lesssim 1+|\bar{v}\|t-s|.
\end{aligned} \tag{8.20}$$

□

Based on Example 1, we naturally consider the 2D specular problem. We consider the 2D specular problem for $f(t,x_1,x_2,v_1,v_2,v_3)$ solving (1.33) where v_3 is a parameter. Here $(x_1,x_2)\in\Omega=\{x\in\mathbb{R}^2:\xi(x)>0\}$ and the convexity (1.13) is valid for all $\zeta\in\mathbb{R}^2$. We study (1.33) with specular boundary condition (1.11). Denote $v:=(\bar{v},v_3)=(v_1,v_2;v_3)\in\mathbb{R}^3$. We define

$$\alpha(x,\bar{v}) = |\bar{v}\cdot\nabla\xi(x)|^2 - 2\{\bar{v}\cdot\nabla^2\xi(x)\cdot\bar{v}\}\xi(x).$$

Note that $\nabla\xi(x)=(\partial_{x_1}\xi(x),\partial_{x_2}\xi(x),0)$.

The following estimate is crucial to establish the weighted C^1 estimate (Theorem 4) and non-existence of $\nabla^2 f$ up to the boundary (Proposition 5).

Lemma 19 *For $\theta>0$ and for $i=1,2$,*

$$\begin{aligned}
&e^{-\varpi\langle v\rangle s}|\partial_{v_i}\Gamma_{\text{gain}}(f,f)| \\
&\lesssim \|e^{\theta|v|^2}f\|_\infty\left\{\|e^{\theta|v|^2}f\|_\infty + \|\partial_{v_3}f\|_\infty + \left\|e^{-\varpi\langle v\rangle s}\frac{|\bar{v}|}{\langle\bar{v}\rangle}\alpha^{1/2}\nabla_{\bar{v}}f\right\|_\infty\right\},
\end{aligned} \tag{8.21}$$

where $v=(\bar{v},v_3)=(v_1,v_2,v_3)$.

Proof The key is to split $u_{\|,3}$ with respect to the size of $|\bar{v}+\bar{u}_\perp|\sqrt{\alpha(\bar{v}+\bar{u}_\perp)}$.

Recall from [8] that the gain term of the nonlinear Boltzmann operator in (1.9) equals

$$\Gamma_{\text{gain}}(g_1,g_2)(v)$$

$$
\begin{aligned}
&= C \int_{\mathbb{R}^3} \mathrm{d}u \int_{u\cdot w=0} \mathrm{d}w\ g_1(v+w)g_2(v+u)q_0^*\left(\frac{|u|}{|u+w|}\right) \\
&\quad \frac{|u+w|^{\kappa-1}}{|u|} e^{-\frac{|u+v+w|^2}{4}}, \\
&= C \int_{\mathbb{R}^3} \mathrm{d}u \int_{u\cdot w=0} \mathrm{d}w\ g_2(v+w)g_1(v+u)q_0^*\left(\frac{|u|}{|u+w|}\right) \\
&\quad \frac{|u+w|^{\kappa-1}}{|u|} e^{-\frac{|u+v+w|^2}{4}}, \\
&= C \int_{\mathbb{R}^3} \mathrm{d}u \int_{(u-v)\cdot w=0} \mathrm{d}w\ g_1(v+w)g_2(u)q_0^*\left(\frac{|u-v|}{|u-v+w|}\right) \\
&\quad \frac{\left|u-v+w\right|^{\kappa-1}}{|u-v|} e^{-\frac{|u+w|^2}{4}}, \\
&= C \int_{\mathbb{R}^3} \mathrm{d}u \int_{(u-v)\cdot w=0} \mathrm{d}w\ g_2(v+w)g_1(u)q_0^*\left(\frac{|u-v|}{|u-v+w|}\right) \\
&\quad \frac{|u-v+w|^{\kappa-1}}{|u-v|} e^{-\frac{|u+w|^2}{4}},
\end{aligned}
\tag{8.22}
$$

where $q_0^*(\cos\theta) = \frac{q_0(\cos\theta)}{|\cos\theta|}$. This is due to two change of variables (37),(38) and page 316 of [8]. Then

$$
\begin{aligned}
&\partial_{v_i}\Gamma_{\mathrm{gain}}(f,f) = 2\Gamma_{\mathrm{gain}}(\partial_{v_i} f, f) \\
&\quad + C\int_{\mathbb{R}^3} \mathrm{d}u_\parallel \int_{u_\parallel\cdot u_\perp=0} \mathrm{d}w f(v+u_\perp) f(v+u_\parallel) \\
&\quad \times q_0^*\left(\frac{|u_\parallel|}{|u_\parallel+u_\perp|}\right)\frac{|u_\parallel+u_\perp|^{\kappa-1}}{|u_\parallel|} e^{-\frac{|u_\parallel+v+u_\perp|^2}{4}}\left(-\frac{\mathbf{e}_i}{2}\right)\cdot(u_\parallel+v+u_\perp) \\
&\quad = 2\Gamma_{\mathrm{gain}}(\partial_{v_i} f, f) + O_\xi(1)e^{-C|v|^2}\|e^{\theta|v|^2} f\|_\infty^2.
\end{aligned}
$$

Denote the standard cutoff function $\chi \geq 0$: $\chi \equiv 1$ on $[0, 1]$ and $\chi \equiv 0$ for $[2, \infty)$. We have

$$
\begin{aligned}
\Gamma_{\mathrm{gain}}(\partial_{v_i} f, f) &= \int_{\mathbb{R}^3} \mathrm{d}u_\parallel f(v+u_\parallel) \int_{u_\parallel\cdot u_\perp=0} \mathrm{d}u_\perp \partial_{v_i} f(v+u_\perp) \\
&\quad \times e^{-\frac{|u_\parallel+u_\perp+v|^2}{4}} q_0^*\left(\frac{|u_\parallel|}{|u_\parallel+u_\perp|}\right)\frac{|u_\parallel+u_\perp|^{\kappa-1}}{|u_\parallel|}.
\end{aligned}
$$

We further split it into, for $0 < \varepsilon \ll 1$,

$$\int_{\mathbb{R}^3} \mathrm{d}u_\| f(v+u_\|) \int_{u_\| \cdot u_\perp = 0} \chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}{u_{\|,3}}\right)$$
$$\times \partial_{v_i} f(v+u_\perp) e^{-\frac{|u_\|+v+u_\perp|^2}{4}} q_0^*\left(\frac{|u_\||}{|u_\|+u_\perp|}\right) \frac{|u_\|+u_\perp|^{\kappa-1}}{|u_\||} \mathrm{d}u_\perp$$
$$+ \int_{\mathbb{R}^3} du_\| f(v+u_\|) \int_{u_\| \cdot u_\perp = 0} \left\{1 - \chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}{u_{\|,3}}\right)\right\}$$
$$\times \partial_{v_i} f(v+u_\perp) e^{-\frac{|u_\|+v+u_\perp|^2}{4}} q_0^*\left(\frac{|u_\||}{|u_\|+u_\perp|}\right) \frac{|u_\|+u_\perp|^{\kappa-1}}{|u_\||} \mathrm{d}u_\perp.$$

For the first part, $|u_{\|,3}| \geq |\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}$, and we parametrize $u_\perp$, on a plain perpendicular to $u_\| \in \mathbb{R}^3$, as $u_{\perp,3} = -\frac{\bar{u}_\| \cdot \bar{u}_\perp}{u_{\|,3}}$ so that

$$\mathrm{d}u_\perp = \frac{|u_\||}{|u_{\|,3}|} \mathrm{d}\bar{u}_\perp := \frac{|u_\||}{|u_{\|,3}|} \mathrm{d}u_{\perp,1}\mathrm{d}u_{\perp,2}, \tag{8.23}$$

and the first part equals

$$\int_{\mathbb{R}^3} \mathrm{d}u_\| f(v+u_\|) \int_{\mathbb{R}^2} \mathrm{d}\bar{u}_\perp \partial_{v_i} f\left(v_1+u_{\perp,1}, v_2+u_{\perp,2}, v_3 - \frac{\bar{u}_\| \cdot \bar{u}_\perp}{u_{\|,3}}\right)$$
$$\times \chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}{u_{\|,3}}\right) e^{-\frac{|u_\|+v+u_\perp|^2}{4}} \frac{q_0^*(\frac{|u_\||}{|u_\|+u_\perp|})}{|u_{\|,3}||u_\|+u_\perp|^{1-\kappa}}.$$

We now integrate by part in $u_{\perp,i}$ for $i = 1, 2$ to get

$$-\int_{\mathbb{R}^3} \mathrm{d}u_\| f(v+u_\|) \int_{\mathbb{R}^2} \partial_{v_3} f\left(\bar{v}+\bar{u}_\perp, v_3 - \frac{\bar{u}_\| \cdot \bar{u}_\perp}{u_{\|,3}}\right)$$
$$\times \chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}{u_{\|,3}}\right) e^{-\frac{|u_\|+v+u_\perp|^2}{4}} \frac{q_0^*\left(\frac{|u_\||}{|u_\|+u_\perp|}\right) u_{\|,i}\mathrm{d}\bar{u}_\perp}{|u_{\|,3}|^2|u_\|+u_\perp|^{1-\kappa}}$$
$$-\int_{\mathbb{R}^3} \mathrm{d}u_\| f(v+u_\|) \int_{\mathbb{R}^2} f\left(\bar{v}+\bar{u}_\perp, v_3 - \frac{\bar{u}_\| \cdot \bar{u}_\perp}{u_{\|,3}}\right)$$
$$\times \partial_{u_{\perp,i}} \left\{\chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}{u_{\|,3}}\right) e^{-\frac{|u_\|+v+u_\perp|^2}{4}} \frac{q_0^*\left(\frac{|u_\||}{|u_\|+u_\perp|}\right) \bar{u}_{\|,i}}{|u_{\|,3}||u_\|+u_\perp|^{1-\kappa}}\right\} \mathrm{d}\bar{u}_\perp.$$

Directly $|\partial_{u_{\perp,i}}\alpha(\bar v+\bar u_\perp)|\lesssim \alpha(\bar v+\bar u_\perp)^{1/2}$ and $\left|\frac{du_{\perp,3}}{du_{\perp,i}}\right|\le \frac{|\bar u_\parallel|}{|u_{\parallel,3}|}$, we conclude

$$
\begin{aligned}
|\partial_{u_{\perp,i}}\{\ \}| \simeq\ & \chi'\frac{|\bar v+\bar u_\perp|^{-\varepsilon}\alpha^{1/2-\varepsilon}+|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{-\varepsilon}}{u_{\parallel,3}}e^{-\frac{|u_\parallel+v+u_\perp|^2}{4}}\frac{\|q_0^*\|_\infty|\bar u_\parallel|}{|u_{\parallel,3}\|u_\parallel+u_\perp|^{1-\kappa}}\\
&+\chi e^{-C|u_\parallel+u_\perp+v|^2}\left\{\frac{\|q_0^*\|_\infty|\bar u_\parallel|^2}{|u_{\parallel,3}|^2|u_\parallel+u_\perp|^{1-\kappa}}+\frac{\|q_0^*\|_{C^1}|\bar u_\parallel|^2}{|u_{\parallel,3}|^2|u_\parallel+u_\perp|^{3-\kappa}}+\frac{\|q_0^*\|_\infty|\bar u_\parallel|(1+\frac{|\bar u_\parallel|}{|u_{\parallel,3}|})}{|u_{\parallel,3}\|u_\parallel+u_\perp|^{2-\kappa}}\right\}\\
\lesssim_{q_0^*}\ & \mathbf{1}_{\{u_{\parallel,3}\simeq|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}\}}\left\{\frac{1}{|\bar v+\bar u_\perp|}+\frac{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{-\varepsilon}}{|u_{\parallel,3}|}\right\}\frac{e^{-\frac{|u_\parallel+v+u_\perp|^2}{4}}|\bar u_\parallel|}{|u_{\parallel,3}\|u_\parallel+u_\perp|^{1-\kappa}}\\
&+\mathbf{1}_{\{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}\le u_{\parallel,3}\}}\frac{|\bar u_\parallel|(1+|\bar u_\parallel|)}{|u_{\parallel,3}|^2|u_\parallel+u_\perp|^{1-\kappa}}\left(1+\frac{1}{|u_\parallel+u_\perp|^2}\right)e^{-C|u_\parallel+v+u_\perp|^2}.
\end{aligned}
$$

Note that $|f(v+u_\parallel)|\lesssim e^{-C|v+u_\parallel|^2}\|e^{\theta|v|^2}f\|_\infty$ and

$$
\left|f\left(\bar v+\bar u_\perp, v_3-\frac{\bar u_\parallel\cdot\bar u_\perp}{u_{\parallel,3}}\right)\right|\lesssim e^{-C|\bar v+\bar u_\perp|^2-C|v_3+u_{\perp,3}(u_\parallel,\bar u_\perp)|^2}\|e^{\theta|v|^2}f\|_\infty, \tag{8.24}
$$

and

$$
\begin{aligned}
&e^{-|u_\parallel+u_\perp+v|^2}e^{-C|v+u_\parallel|^2}e^{-C|\bar v+\bar u_\perp|^2-C|v_3+u_{\perp,3}(u_\parallel,\bar u_\perp)|^2}\\
&\lesssim e^{-C'|v|^2}e^{-C'|u_\perp|^2}e^{-C|v+u_\parallel|^2},
\end{aligned}
$$

where $v:=v_\parallel+v_\perp$ with $v_\parallel:=v\cdot\frac{u_\parallel}{|u_\parallel|}$ and

$$
|v+u_\parallel|^2+|v+u_\perp|^2=|v_\parallel+u_\parallel|^2+|v_\perp|^2+|v_\perp+u_\perp|^2+|v_\parallel|^2\ge|v|^2.
$$

The $\partial_{u_{\perp,i}}\{\ \}$ −contribution are bounded by the following three estimates: For the first term

$$
\begin{aligned}
&e^{-|v|^2}\|e^{\theta|v|^2}f\|_\infty^2\int_{\mathbb{R}^2}\frac{e^{-|u_\perp|^2}\mathrm{d}\bar u_\perp}{|\bar v+\bar u_\perp|}\int_{\mathbb{R}^2}|\bar u_\parallel|^\kappa e^{-|\bar v+\bar u_\parallel|^2}\mathrm{d}\bar u_\parallel\\
&\times\int_{|u_{\parallel,3}-|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{\frac12-\varepsilon}|\ll1}\frac{\mathrm{d}u_{\parallel,3}}{|u_{\parallel,3}|}\lesssim e^{-C'|v|^2}.
\end{aligned}
$$

For the second term we use

$$
f(\bar v+\bar u_\perp, v_3-\frac{\bar u_\parallel\cdot\bar u_\perp}{u_{\parallel,3}})\lesssim e^{-C|v+u_\perp|^2}\|e^{\theta|v|^2}f\|_\infty\frac{1}{1+\left(v_3-\frac{\bar u_\parallel\cdot\bar u_\perp}{|u_{\parallel,3}|}\right)^\varepsilon}
$$

such that

$$
\begin{aligned}
& f(v+u_\parallel)\frac{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{-\varepsilon}}{|u_{\parallel,3}|^2} f\left(v_3-\frac{\bar u_\parallel\cdot\bar u_\perp}{u_{\parallel,3}}\right)\\
&\quad\times\frac{e^{-\frac{|u_\parallel+u_\perp+v|^2}{4}}|\bar u_\parallel|}{|u_\parallel+u_\perp|^{1-\kappa}}\\
&\quad\lesssim e^{-|v|^2}e^{-|v+u_\parallel|^2}e^{-|u_\perp|^2}\|e^{\theta|v|^2}f\|_\infty^2\frac{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{-\varepsilon}|\bar u_\parallel|}{|u_\parallel+u_\perp|^{1-\kappa}}\\
&\quad\times\frac{1}{|u_{\parallel,3}|^{2-\varepsilon}}\frac{1}{|u_{\parallel,3}|^{\varepsilon}+[v_3u_{\parallel,3}-\bar u_\parallel\cdot\bar u_\perp]^{\varepsilon}}\\
&\quad\lesssim e^{-|v|^2}e^{-|v+u_\parallel|^2}e^{-|u_\perp|^2}\|e^{\theta|v|^2}f\|_\infty^2\frac{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{-\varepsilon}\langle v\rangle}{|u_\parallel+u_\perp|^{1-\kappa}}\frac{1}{|u_{\parallel,3}|^{2-\varepsilon}|\bar u_\perp|^{\varepsilon}}\\
&\quad\times\frac{1}{\left[\frac{v_3u_{\parallel,3}}{|\bar u_\perp|}-\bar u_\parallel\cdot\frac{\bar u_\perp}{|\bar u_\perp|}\right]^{\varepsilon}},
\end{aligned}
$$

where we have used

$$
e^{-|v+u_\parallel|^2}|u_\parallel|\lesssim\{|u_\parallel+v|+|v|\}e^{-|v+u_\parallel|^2}\lesssim(1+|v|)e^{-C|v+u_\parallel|^2}.
$$

Now we decompose $\bar u_\parallel=\bar u_{\parallel,a}+\bar u_{\parallel,b}:=\bar u_\parallel\cdot\frac{\bar u_\perp}{|\bar u_\perp|}+(\bar u_\parallel-\bar u_\parallel\cdot\frac{\bar u_\perp}{|\bar u_\perp|})$ and bound $e^{-|v|^2}\|e^{\theta|v|^2}f\|_\infty^2\times$

$$
\begin{aligned}
&\int_{|u_{\parallel,3}-|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{\frac12-\varepsilon}|\ll1}\mathrm{d}u_{\parallel,3}\int_{\mathbb{R}^2}\mathrm{d}\bar u_\perp e^{-|\bar u_\perp|^2}\frac{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{-\varepsilon}}{|\bar u_\perp|^{\varepsilon}|u_{\parallel,3}|^{2-\varepsilon}}\\
&\quad\times\int_{\mathbb{R}}\frac{e^{-|\bar u_{\parallel,b}+(v-v\cdot\frac{\bar u_\perp}{|\bar u_\perp|})|^2}\mathrm{d}\bar u_{\parallel,b}}{|\bar u_{\parallel,b}|^{1-\kappa}}\int_{\mathbb{R}}\frac{\mathrm{d}\bar u_{\parallel,a}e^{-|\bar u_{\parallel,a}|^2}}{[\bar u_{\parallel,a}-\frac{v_3u_{\parallel,3}}{|\bar u_\perp|}]^{\varepsilon}}\\
&\quad\lesssim\int_{\mathbb{R}^2}\mathrm{d}\bar u_\perp e^{-|\bar u_\perp|^2}\int_{|u_{\parallel,3}-|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{\frac12-\varepsilon}|\ll1}\mathrm{d}u_{\parallel,3}\frac{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha(\bar v+\bar u_\perp)^{-\varepsilon}}{|\bar u_\perp|^{\varepsilon}|u_{\parallel,3}|^{2-\varepsilon}},
\end{aligned}
$$

where we have first integrated $u_\parallel$ 3D integral and then changed variables $\bar u_{\parallel,a}\mapsto\bar u_{\parallel,a}-v\cdot\frac{\bar u_\perp}{|\bar u_\perp|}$. The $u_{\parallel,3}$-integration yields

$$
\begin{aligned}
&\lesssim\int_{\mathbb{R}^2}\mathrm{d}\bar u_\perp e^{-|\bar u_\perp|^2}\frac{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha(\bar v+\bar u_\perp)^{-\varepsilon}}{|\bar u_\perp|^{\varepsilon}}\int_{|u_{\parallel,3}-|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha^{\frac12-\varepsilon}|\ll1}\frac{\mathrm{d}u_{\parallel,3}}{|u_{\parallel,3}|^{2-\varepsilon}}\\
&\lesssim\int_{\mathbb{R}^2}\mathrm{d}\bar u_\perp e^{-|\bar u_\perp|^2}\frac{|\bar v+\bar u_\perp|^{1-\varepsilon}\alpha(\bar v+\bar u_\perp)^{-\varepsilon}}{|\bar u_\perp|^{\varepsilon}}
\end{aligned}
$$

$$\times \frac{1}{|\bar{v}+\bar{u}_\perp|^{(1-\varepsilon)(1-\varepsilon)}} \frac{1}{\alpha(\bar{v}+\bar{u}_\perp)^{(\frac{1}{2}-\varepsilon)(1-\varepsilon)}}$$
$$\lesssim \int_{\mathbb{R}^2} \mathrm{d}\bar{u}_\perp e^{-|\bar{u}_\perp|^2} \frac{|\bar{v}+\bar{u}_\perp|^{\varepsilon(1-\varepsilon)}}{|\bar{u}_\perp|^\varepsilon} \frac{1}{\alpha(\bar{v}+\bar{u}_\perp)^{\frac{1}{2}-(1-\varepsilon)\varepsilon}}.$$

Note that $\alpha(\bar{v}+\bar{u}_\perp)^{\frac{1}{2}-\varepsilon(1-\varepsilon)} \gtrsim [n(x)\cdot(\bar{v}+\bar{u}_\perp)]^{1-\varepsilon(1-\varepsilon)}$ and $|\bar{u}_\perp|^\varepsilon \gtrsim [n^\perp\cdot\bar{u}_\perp]^\varepsilon$ and we bound the above by

$$\lesssim \langle v\rangle \int_{\mathbb{R}} \frac{e^{-|n\cdot\bar{u}_\perp|^2}}{[n\cdot\bar{u}_\perp+n\cdot\bar{v}]^{1-2\varepsilon(1-\varepsilon)}} \mathrm{d}[n\cdot\bar{u}_\perp] \int_{\mathbb{R}} \frac{e^{-|n^\perp\cdot\bar{u}_\perp|^2}}{|n^\perp\cdot\bar{u}_\perp|^\varepsilon} \mathrm{d}[n^\perp\cdot\bar{u}_\perp] \lesssim \langle v\rangle.$$

For the third term in $\partial_{u_{\perp,i}}\{\ \}$ we bound it by $\|e^{\theta|v|^2} f\|_\infty^2 \times$

$$\int_{\mathbb{R}^3} \mathrm{d}\bar{u}_\parallel \int_{\mathbb{R}^2} \mathrm{d}\bar{u}_\perp \left\{ \int_{|u_{\parallel,3}|\geq|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}} \frac{e^{-|u_{\parallel,3}-v_3|^2}}{|u_{\parallel,3}|^2} du_{\parallel,3} \right\}$$
$$\times \frac{\langle u_\parallel\rangle^2 e^{-C\{|u_\parallel+v|^2-|u_\perp|^2\}}}{|u+w|^{1-\kappa}} \left(1+\frac{1}{|u_\parallel+u_\perp|^2}\right)$$
$$\lesssim \int_{\mathbb{R}^3} \mathrm{d}\bar{u}_\parallel \left\{ \int_{\mathbb{R}^2} \frac{e^{-C|u_\perp|^2}\mathrm{d}\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}} \right\} \frac{\langle \bar{u}_\parallel\rangle^2 e^{-C\{|u_\parallel+v|^2\}}}{|u_\parallel+u_\perp|^{1-\kappa}} \left(1+\frac{1}{|u_\parallel+u_\perp|^2}\right).$$

We note that, by separating $|\xi| \geq \delta$ or $|\xi| \leq \delta$, we can write $\alpha^{1/2-(1-\varepsilon)(1+\varepsilon^2)} \geq \{n\cdot[\bar{v}+\bar{u}_\perp]\}^{1-(1-\varepsilon)(1+\varepsilon^2)}$ and $|\bar{v}+\bar{u}_\perp|^{1-(1-\varepsilon)(1+\varepsilon^2)} \geq \{n^\perp\cdot[\bar{v}+\bar{u}_\perp]\}^{1-(1-\varepsilon)(1+\varepsilon^2)}$, where $n^\perp = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} n$, so that the inner 2D integral are two convergent 1D one

$$\int_{|(\bar{v}+\bar{u}_\perp)\cdot n^\perp|\geq 1} \frac{e^{-C|u_\perp|^2} d\bar{u}_\perp}{\alpha^{1/2-\varepsilon}} + \int_{|\bar{v}+\bar{u}_\perp|\leq 1,|n^\perp\{\bar{v}+\bar{u}_\perp\}|\leq 1} \frac{d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}$$
$$\leq 1 + \int_{|n^\perp\{\bar{v}+\bar{u}_\perp\}|\leq 1} \frac{e^{-C|u_\perp|^2} d\bar{u}_\perp}{\alpha^{1/2-\varepsilon}} + \int_{|n^\perp\{\bar{v}+\bar{u}_\perp\}|\leq 1} \frac{d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}$$
$$+ \int_{|n^\perp\{\bar{v}+\bar{u}_\perp\}|\leq 1} \frac{e^{-C|u_\perp|^2} d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}}$$
$$< +\infty.$$

Similarly, the first term is bounded by

$$\|\langle v\rangle^{\zeta}e^{\theta|v|^2}f\|_\infty\|\partial_{v_3}f\|\int_{\mathbb{R}^3}\mathrm{d}u_\parallel\left\{\int_{|u_{\parallel,3}|\geq|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}\frac{e^{-|v_3-u_{\parallel,3}|^2}}{|u_{\parallel,3}|^2}\right\}$$
$$\times\frac{q_0^*(\frac{|u|}{|v+w|})u_{\parallel,i}\,\mathrm{d}\bar{u}_\perp}{|u_\parallel+u_\perp|^{\kappa-1}}e^{-\frac{|u+v+w|^2}{4}},$$

and the same argument yields the same bound.

We now turn to

$$e^{-\varpi\langle v\rangle s}\int_{\mathbb{R}^3}\mathrm{d}u_\parallel f(v+u_\parallel)\int_{u_\perp\cdot u_\parallel=0}\left\{1-\chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}{u_{\parallel,3}}\right)\right\}\mathrm{d}u_\perp$$
$$\partial_{v_i}f(v+u_\perp)e^{-\frac{|u+v+w|^2}{4}}\frac{q_0^*(\frac{|u|}{|v+w|})}{|u_\parallel\|u+w|^{1-\kappa}}.$$

In this case,

$$|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}\geq|u_{\parallel,3}|.$$

We now parametrize $\mathrm{d}u_\perp$ in two different ways.

We decompose

$$\bar{u}_\parallel=\bar{u}_{\parallel,n}+\bar{u}_{\parallel,n^\perp}:=\bar{u}_\parallel\cdot n+\bar{u}_\parallel\cdot n^\perp. \tag{8.25}$$

If $|u_{\parallel,3}|\geq|\bar{u}_{\parallel,n^\perp}|$, then we use the same parametrization as in (8.23) and absorb the factor $(\bar{v}+\bar{u}_\perp)\alpha^{1/2}$ in L^∞ norm, to get

$$e^{-\varpi\langle v\rangle s}e^{\varpi\langle v+u_\perp\rangle s}\int_{\mathbb{R}^3}\mathrm{d}u_\parallel f(v+u_\parallel)$$
$$\times\int_{\mathbb{R}^2}\mathrm{d}\bar{u}_\perp e^{-\varpi\langle v+u_\perp\rangle s}\partial_{v_i}f\left(\bar{v}+\bar{u}_\perp,\,v_3-\frac{\bar{u}_\parallel\cdot\bar{u}_\perp}{u_{\parallel,3}}\right)$$
$$\times[1-\chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}{u_{\parallel,3}}\right)]e^{-\frac{|u_\parallel+v+u_\perp|^2}{4}}\frac{q_0^*\left(\frac{|u_\parallel|}{|u_\parallel+u_\perp|}\right)}{|u_{\parallel,3}\|u_\parallel+u_\perp|^{1-\kappa}}$$
$$\lesssim_s\|e^{\theta|v|^2}f\|_\infty\|e^{-\varpi\langle v\rangle s}\frac{\bar{v}\alpha^{1/2}}{\langle\bar{v}\rangle}\partial_{v_i}f(s)\|_\infty$$
$$\times\int_{\mathbb{R}^3}\mathrm{d}u_\parallel\int_{|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}\geq u_{\parallel,3}}\frac{\mathrm{d}\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|\alpha^{1/2}}\frac{e^{-C|u_\perp|^2}e^{-C|v+u_\parallel|^2}}{|u_{\parallel,3}||u_\parallel+u_\perp|^{1-\kappa}}.$$

First we integrate $\bar{u}_{\|,n}$ to drop $\frac{1}{|u_\| + u_\perp|^{1-\kappa}}$ singular term for $0 < \kappa \leq 1$

$$\int \frac{e^{-|v_n + u_{\|,n}|^2}}{|u_\| - u_\perp|^{1-\kappa}} \mathrm{d}u_{\|,n} \leq \int \frac{e^{-|v_n + u_{\|,n}|^2}}{|u_{\|,n} - u_{\perp,n}|^{1-\kappa}} \mathrm{d}u_{\|,n} < \infty,$$

so that we only need to bound, for $|\bar{v} + \bar{u}_\perp|^{1-\varepsilon} \alpha^{1/2-\varepsilon} \geq u_{\|,3}$,

$$\begin{aligned} &\int \mathrm{d}\bar{u}_{\|,n^\perp} \int \frac{e^{-|\bar{u}_\perp|^2 - |v + u_\||^2} \mathrm{d}\bar{u}_\perp}{|\bar{v} + \bar{u}_\perp|^{2\varepsilon} \alpha^{2\varepsilon} |\bar{v} + \bar{u}_\perp|^{1-2\varepsilon} \alpha^{1/2-2\varepsilon}} \frac{1}{|u_{\|,3}|} \\ &\quad \leq \int \mathrm{d}\bar{u}_{\|,n^\perp} \left\{ \int \frac{e^{-|\bar{u}_\perp|^2} \mathrm{d}\bar{u}_\perp}{|\bar{v} + \bar{u}_\perp|^{2\varepsilon} \alpha^{2\varepsilon}} \right\} \frac{e^{-|v + u_\||^2}}{|u_{\|,3}|^{2-\varepsilon/4}}. \end{aligned}$$

The inner integral is finite, since $\alpha \geq |n \cdot \{\bar{v} + \bar{u}_\perp\}| = |\bar{v} \cdot n + \bar{u}_{\perp,n}|$, and the integral is a 1D integral:

$$\int_{\mathbb{R}} \frac{e^{-|\bar{u}_{\perp,n}|^2} \mathrm{d}\bar{u}_{\perp,n}}{|n \cdot \bar{v} + \bar{u}_{\perp,n}|^{4\varepsilon}} < +\infty.$$

Moreover, from $|u_{\|,3}| \geq |\bar{u}_{\|,n^\perp}|$, the outer integral takes the form

$$\begin{aligned} &\int_{\mathbb{R}} \frac{e^{-|u_{\|,3} + v_3|^2} e^{-|v_{\|,n^\perp} + u_{\|,n^\perp}|^2} \mathrm{d}u_{\|,3} \mathrm{d}u_{\|,n^\perp}}{|u_{\|,3}|^{2-\varepsilon/4}} \\ &\quad \leq \int \frac{e^{-|u_{\|,3} + v_3|^2} e^{-|v_{\|,n^\perp} + u_{\|,n^\perp}|^2} \mathrm{d}u_{\|,n^\perp} \mathrm{d}u_{\|,3}}{\{|u_{\|,n^\perp}| + |u_{\|,3}|\}^{2-\varepsilon/4}} < \infty. \end{aligned}$$

We are done in this case.

We now consider the case $|u_{\|,3}| \leq |u_{\|,n^\perp}|$. We now choose a different parametrization. We define

$$u_{\perp,n} := u_\perp \cdot n, \quad u_{\perp,n^\perp} := u_\perp \cdot n^\perp.$$

Now we choose $u_{\perp,n}$ and $u_{\perp,3}$ as parameters so that $u_{\perp,n^\perp} = -\frac{u_{\perp,n} u_{\|,n} + u_{\perp,3} u_{\|,3}}{u_{\|,n^\perp}}$ and

$$\mathrm{d}u_\perp = \frac{|u_\||}{|u_{\|,n^\perp}|} \mathrm{d}u_{\perp,n} \mathrm{d}u_{\perp,3},$$

so that we need to bound

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle s}\int_{\mathbb{R}^3}\mathrm{d}u_{\|}\times f(v+u_{\|})\int_{\mathbb{R}^2}\mathrm{d}u_{\perp,n}\mathrm{d}u_{\perp,3}\\
& \partial_{v_i} f\left(v_n+u_{\perp,n},\, v_{n^\perp}-\frac{u_{\perp,n}u_{\|,n}+u_{\perp,3}u_{\|,3}}{u_{\|,n^\perp}},\, v_3+u_{\perp,3}\right)\frac{|u_{\|}|}{|u_{\|,n^\perp}|}\\
& \times\left\{1-\chi\left(\frac{|\bar v+\bar u_{\perp}|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}{u_{\|,3}}\right)\right\}e^{-\frac{|u_{\|}+v+u_{\perp}|^2}{4}}\frac{q_0^*\left(\frac{|u_{\|}|}{|u_{\|}+u_{\perp}|}\right)}{|u_{\|}||u_{\|}+u_{\perp}|^{1-\kappa}}.
\end{aligned}
$$

Directly this is bounded by $\|e^{\theta|v|^2}f\|_\infty\|e^{-\varpi\langle v\rangle s}\frac{|\bar v|\alpha^{1/2}\partial_{v_i}f}{\langle\bar v\rangle}\|_\infty\times$

$$
\begin{aligned}
& \int_{\mathbb{R}^3}\mathrm{d}u_{\|}\int_{|\bar v+\bar u_{\perp}|^{1-\varepsilon}\alpha^{1/2-\varepsilon}\geq|u_{\|,3}|}\frac{\langle\bar v+\bar u_{\perp}\rangle e^{-|\bar u_{\perp}|^2-|v+u_{\|}|^2}\mathrm{d}u_{\perp,n}\mathrm{d}u_{\perp,3}}{|\bar v+\bar u_{\perp}|\alpha^{1/2}}\frac{\mathrm{d}u_{\|}}{|u_{\|,n^\perp}||u_{\|}+u_{\perp}|^{1-\kappa}}\\
& \lesssim_s\int_{\mathbb{R}^3}\mathrm{d}u_{\|}\int_{|\bar v+\bar u_{\perp}|^{1-\varepsilon}\alpha^{1/2-\varepsilon}\geq|u_{\|,3}|}\frac{\langle\bar v+\bar u_{\perp}\rangle e^{-|\bar u_{\perp,n}|^2-|v+u_{\|}|^2}}{|\bar v+\bar u_{\perp}|^{2\varepsilon}\alpha^{2\varepsilon}|\bar v+\bar u_{\perp}|^{1-2\varepsilon}\alpha^{1/2-2\varepsilon}}\frac{1}{|u_{\|,n^\perp}|}\mathrm{d}\bar u_{\perp,n}.
\end{aligned}
$$

where we integrate $u_{\perp,3}$ first to drop $\int_{\mathbb{R}}\frac{e^{-C|u_{\perp,3}|^2}\mathrm{d}u_{\perp,3}}{|u_{\|}+u_{\perp}|^{1-\kappa}}\lesssim\int_{\mathbb{R}}\frac{e^{-C|u_{\perp,3}|^2}\mathrm{d}u_{\perp,3}}{|u_{\|,3}+u_{\perp,3}|^{1-\kappa}}<+\infty$.

In the case of $|\bar v+\bar u_{\perp}|\leq 1$, this is bounded by

$$
\begin{aligned}
& \lesssim_s\int_{\mathbb{R}^3}\mathrm{d}u_{\|}\int\frac{e^{-|u_{\perp}|^2-|v+u_{\|}|^2}\mathrm{d}u_{\perp,n}}{|\bar v+\bar u_{\perp}|^{2\varepsilon}\alpha^{2\varepsilon}}\frac{1}{|u_{\|,n^\perp}||u_{\|,3}|^{1-\varepsilon}}\\
& \lesssim_s\int_{\mathbb{R}^3}du_{\|}\int\frac{e^{-|u_{\perp}|^2-|v+u_{\|}|^2}\mathrm{d}u_{\perp,n}}{|\bar u_{\perp,n}+\bar v_n|^{4\varepsilon}}\frac{1}{|u_{\|,n^\perp}||u_{\|,3}|^{1-\varepsilon}}\\
& \lesssim_s\int_{\mathbb{R}^3}\mathrm{d}u_{\|}\left\{\int\frac{e^{-|\bar u_{\perp,n}|^2}\mathrm{d}u_{\perp,n}}{|\bar u_{\perp,n}+\bar v_n|^{4\varepsilon}}\right\}\frac{e^{-|v+u_{\|}|^2}}{|u_{\|,n^\perp}||u_{\|,3}|^{1-\varepsilon}},
\end{aligned}
$$

where the inner integral is 1D which is finite and bounded. On the other hand, from the assumption $|u_{\|,3}|\leq|\bar u_{\|,n^\perp}|$, the outer integral is

$$
\begin{aligned}
& \int\frac{e^{-|v+u_{\|}|^2}\mathrm{d}\bar u_{\|,n^\perp}\mathrm{d}\bar u_{\|,n}\mathrm{d}u_{\|,3}}{|\bar u_{\|,n^\perp}||u_{\|,3}|^{1-\varepsilon}}\\
& \quad\leq\int\left\{\int_0^{|\bar u_{\|,n^\perp}|}\frac{\mathrm{d}u_{\|,3}}{|u_{\|,3}|^{1-\varepsilon}}\right\}\frac{e^{-|\bar v_n+\bar u_{\|,n}|^2}e^{-|\bar v_{n^\perp}+\bar u_{\|,n^\perp}|^2}}{|\bar u_{\|,n^\perp}|}\mathrm{d}\bar u_{\|,n^\perp}\mathrm{d}\bar u_{\|,n}\\
& \quad\leq\iint_{\mathbb{R}^2}\frac{e^{-|\bar v_n+\bar u_{\|,n}|^2}e^{-|\bar v_{n^\perp}+\bar u_{\|,n^\perp}|^2}}{|\bar u_{\|,n^\perp}|^{1-\varepsilon}}\mathrm{d}\bar u_{\|,n^\perp}\mathrm{d}\bar u_{\|,n}<\infty.
\end{aligned}
$$

In the case of $|\bar{v}+\bar{u}_\perp|\geq 1$ we bound the integration as

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{\langle\bar{v}+\bar{u}_\perp\rangle^{\varepsilon}\langle\bar{v}+\bar{u}_\perp\rangle^{1-\varepsilon}}{|u_{\|,3}|^{\frac{2\varepsilon}{1-2\varepsilon}}|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{\frac{1}{2}-\varepsilon}}\frac{e^{-|\bar{u}_{\perp,n}|^2}e^{-|v+u_\||^2}}{|u_{\|,n^\perp}||u_\|+u_\perp|^{1-\kappa}}\mathrm{d}\bar{u}_{\perp,n}\mathrm{d}u_\|\\ &\lesssim\int_{\mathbb{R}^3}\mathrm{d}u_\|\int\frac{\langle\bar{v}_n+\bar{u}_{\perp,n}\rangle^{\varepsilon}\langle\bar{v}_{n^\perp}\rangle^{\varepsilon}e^{-|\bar{u}_{\perp,n}|^2}e^{-|v+u_\||^2}}{|u_{\|,3}|^{\frac{2\varepsilon}{1-2\varepsilon}}[\bar{u}_{\perp,n}+\bar{v}_n]^{2(\frac{1}{2}-\varepsilon)}|\bar{u}_{\|,n^\perp}|}\mathrm{d}\bar{u}_{\perp,n}.\end{aligned}$$

Again $\int_0^{|\bar{u}_{\|,n^\perp}|}\frac{\mathrm{d}u_{\|,3}}{|u_{\|,3}|^{\frac{2\varepsilon}{1-2\varepsilon}}}\lesssim|\bar{u}_{\|,n^\perp}|^{1-\frac{2\varepsilon}{1-2\varepsilon}}$ and hence the integration is bounded by

$$\begin{aligned}&\langle\bar{v}_n\rangle^{\varepsilon}\langle\bar{v}_{n^\perp}\rangle^{\varepsilon}\iint\frac{e^{-|\bar{u}_{\perp,n}|^2}e^{-|\bar{v}+\bar{u}_\||^2}}{|\bar{u}_{\|,n^\perp}|^{\frac{2\varepsilon}{1-2\varepsilon}}|\bar{u}_{\perp,n}+\bar{v}_n|^{1-2\varepsilon}}\\ &\leq\langle\bar{v}_n\rangle^{\varepsilon}\langle\bar{v}_{n^\perp}\rangle^{\varepsilon}\langle\bar{v}_{n^\perp}\rangle^{-\frac{2\varepsilon}{1-2\varepsilon}}\langle\bar{v}_n\rangle^{-(1-2\varepsilon)}\lesssim 1.\end{aligned}$$

□

Now we prove the main result for 2D specular case.

Proof of Theorem 4 We repeat our proof in 3D for the simpler 2D case, and we only point out the differences. Lemma 7 is valid with easy adaptations. The new $\partial_{v_3}f(t)$ estimate follows from taking the v_3 derivative

$$\begin{aligned}&\{\partial_t+v_1\partial_{x_1}+v_2\partial_{x_2}\}\partial_{v_3}f+\nu(F)\partial_{v_3}f\\ &=\Gamma_{\mathrm{gain},v_3}(f,f)+\Gamma_{\mathrm{gain}}(\partial_{v_3}f,f)\\ &\quad+\Gamma_{\mathrm{gain}}(f,\partial_{v_3}f)-\nu_{v_3}(\sqrt{\mu}f)f-\nu(\sqrt{\mu}\partial_{v_3}f)f.\end{aligned}$$

Since

$$\begin{aligned}&|\nu_{v_3}(\sqrt{\mu}f)f|+|\Gamma_{\mathrm{gain},v_3}(f,f)|\lesssim P(\|e^{\theta|v|^2}f\|_\infty),\\ &|\nu(\sqrt{\mu}\partial_{v_3}f)f|+|\Gamma_{\mathrm{gain}}(\partial_{v_3}f,f)|\lesssim P(\|e^{\theta|v|^2}f\|_\infty)\int\frac{e^{-C|v-u|^2}}{|v-u|^{2-\kappa}}|\partial_{v_3}f(u)|\mathrm{d}u,\end{aligned}$$

and $\partial_{v_3}f(t,x,v)=\partial_{v_3}f(t,x,R_xv)$ for $(x,v)\in\gamma_-$ then we follow the proof of Lemma 7 (similar to $\partial_t f$ proof) to conclude $\|\partial_{v_3}f(t)\|_\infty\lesssim\|\partial_{v_3}f_0\|_\infty+P(\|e^{\theta|v|^2}f\|_\infty)$.

The Velocity lemma (Lemma 2) is valid with changing v to $\bar{v}$. The non-local to local estimates (1.28) and (1.29) are valid for $0<\kappa\leq 1$ for $\bar{v}=(v_1,v_2)$: In the proof of (1.28) in Lemma 1, Step 1, the claim (4.1) is valid. In Step 2, (4.3) and (4.4) are valid with $\alpha(x,\bar{v})$. In Step 3 we define $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ by changing

v to $\bar{v}$. Then (4.6), and (4.8) hold by changing v to $\bar{v}$. We follow the same proof of Step 4 to bound $\int_0^{t_{\mathbf{b}}(x,\bar{v})} \frac{e^{-l\langle \bar{v}\rangle(t-s)}}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}} Z(s,v)\mathrm{d}s$. We use $\frac{1}{|v|} \leq \frac{1}{|\bar{v}|}$ to conclude (1.28). For the proof of (1.29) in Lemma 1, we use the same time splitting of (4.10) with changing $|v|$ to $|\bar{v}|$. Then all the proofs are followed and we conclude the proof using $\frac{1}{|v|} \leq \frac{1}{|\bar{v}|}$.

The fundamental Theorem 5 is valid with a simpler proof with changing all v to $\bar{v}$. In fact, due to topological advantage, we can use a global chart $x_{\parallel} = \theta$ in $\mathbb{R}^1$ (such as the polar co-ordinates) for the boundary as

$$\eta(x_{\parallel}) = [R(x_{\parallel})\cos x_{\parallel}, R(x_{\parallel})\sin x_{\parallel}],$$

(vector-valued function) with a global ODE for in the polar co-ordinate system near the boundary! The proof of Theorem 5 follows step by step of the 3D case but with simpler arguments without changes of charts. The estimate of $e^{-\varpi\langle v\rangle t}\frac{\alpha}{1+|v|^2}\nabla_x f(t)$ exactly as in 3D case, valid for α. The most delicate part is to estimate $\nabla_{\bar{v}}\Gamma_{\text{gain}}(f,f)$, where a weight stronger than $\sqrt{\alpha}$, due to $\beta > 1/2$ in (1.29). It is important to know, that we are unable to establish (1.29) in the 2D case with $\beta = 1/2$. However, we are able to close the estimate by using additional bounds on $\partial_{v_3} f$.

Basically we follow the *Proof of Theorem* 3 but we use Lemma 19 when derivatives act on $\bar{V}_{\mathbf{cl}}(s)$ argument of $\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), \bar{V}_{\mathbf{cl}}(s), v_3)$.

More precisely we use Lemma 6 for $\mathbf{e} \in \{x_1, x_2, v_1, v_2\}$

$$\begin{aligned}
&\mathrm{II}_{\mathbf{e}} \text{ of } (6.74)\\
&= \int_0^t \mathrm{d}s \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^{j})}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau}\\
&\quad\times\Big\{\partial_{\mathbf{e}}\bar{X}_{\mathbf{cl}}(s)\cdot\Big[\Gamma_{\text{gain}}(\nabla_{\bar{x}} f^{m-\ell}, f^{m-\ell}) + \Gamma_{\text{gain}}(f^{m-\ell}, \nabla_{\bar{x}} f^{m-\ell})\Big]\\
&\quad\times(s, \bar{X}_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\\
&\quad+\partial_{\mathbf{e}}\bar{V}_{\mathbf{cl}}(s)\cdot\nabla_{\bar{v}}\Big[\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})\Big](s, \bar{X}_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\Big\}
\end{aligned}$$

$$\begin{aligned}
&-\int_0^t \mathrm{d}s \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) e^{-\int_s^t \sum_j \mathbf{1}_{[t^{j+1},t^{j})}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau}\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, \bar{X}_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\\
&\times\int_s^t \mathrm{d}\tau \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^{j})}(\tau)\Big\{\partial_{\mathbf{e}}\bar{X}_{\mathbf{cl}}(s)\cdot\nu(\sqrt{\mu}\nabla_{\bar{x}} f^{m-j})(\tau, \bar{X}_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))\\
&+\int_{\mathbb{R}^3}\partial_{\mathbf{e}}\bar{V}_{\mathbf{cl}}(s)\cdot\nabla_{\bar{v}} B(v-u,\omega)\sqrt{\mu(u)} f^{m-j}(\tau, \bar{X}_{\mathbf{cl}}(\tau), u)\mathrm{d}u\Big\}
\end{aligned}$$

$$
\begin{aligned}
&-e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\nu(F^{m-\ell})(s)\mathrm{d}s} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^{\ell})}(s) \\
&\times \Big\{ \partial_{\mathbf{e}} \bar{X}_{\mathbf{cl}}(s) \cdot \nu(\sqrt{\mu}\nabla_{\bar{x}} f^{m-j})(\tau, \bar{X}_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau)) \\
&+ \int_{\mathbb{R}^3} \partial_{\mathbf{e}} \bar{V}_{\mathbf{cl}}(s) \cdot \nabla_{\bar{v}} B(v-u,\omega)\sqrt{\mu(u)} f^{m-j}(\tau, \bar{X}_{\mathbf{cl}}(\tau), u)\mathrm{d}u \Big\}.
\end{aligned}
$$

We repeat the estimates of x and v derivatives with weight $\frac{\alpha}{\langle \bar{v}\rangle^2}$, $\frac{|\bar{v}|}{\langle \bar{v}\rangle}\sqrt{\alpha}$ respectively. From (1.35),

$$
\begin{aligned}
&\int_0^t e^{-\varpi\langle v\rangle t} \frac{\alpha(x,\bar{v})}{\langle \bar{v}\rangle^2} |\partial_{\bar{x}} \bar{V}_{\mathbf{cl}}(s)||\partial_{\bar{v}} \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})|\mathrm{d}s \\
&\lesssim \int_0^t e^{-\varpi\langle v\rangle (t-s)} \frac{|\bar{v}|^3 e^{C|\bar{v}\|t-s|}}{\langle \bar{v}\rangle^2} |e^{\varpi\langle v\rangle s} \partial_{\bar{v}} \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})|\mathrm{d}s \\
&\lesssim \int_0^t e^{-\varpi\langle \bar{v}\rangle (t-s)} |\bar{v}|\mathrm{d}s \times \{ \text{ RHS of (8.21)}\} \\
&\lesssim \frac{1}{\varpi} P(\|e^{\theta|v|^2} f_0\|_\infty)\{1 + \|\partial_{v_3} f_0\|_\infty + \|e^{-\varpi\langle v\rangle s} \frac{|\bar{v}|}{\langle \bar{v}\rangle} \alpha^{1/2} \partial_{\bar{v}} f\|_\infty\}.
\end{aligned}
$$

From (1.35),

$$
\begin{aligned}
&\int_0^t e^{-\varpi\langle v\rangle t} \frac{|\bar{v}|\sqrt{\alpha(x,\bar{v})}}{\langle \bar{v}\rangle} |\partial_{\bar{v}} \bar{V}_{\mathbf{cl}}(s)||\partial_{\bar{v}} \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})|\mathrm{d}s \\
&\lesssim \int_0^t e^{-\varpi\langle v\rangle (t-s)} \frac{|\bar{v}|^2 e^{C|\bar{v}\|t-s|}}{\langle \bar{v}\rangle} |e^{\varpi\langle v\rangle s} \partial_{\bar{v}} \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})|\mathrm{d}s \\
&\lesssim \int_0^t e^{-\varpi\langle \bar{v}\rangle (t-s)} |\bar{v}|\mathrm{d}s \times \{ \text{ RHS of (8.21)}\} \\
&\lesssim \frac{1}{\varpi} P(\|e^{\theta|v|^2} f_0\|_\infty)\{1 + \|\partial_{v_3} f_0\|_\infty + \left\| e^{-\varpi\langle v\rangle s} \frac{|\bar{v}|}{\langle \bar{v}\rangle} \alpha^{1/2} \partial_{\bar{v}} f \right\|_\infty\}.
\end{aligned}
$$

For the term containing $\partial_x \bar{V}_{\mathbf{cl}}(s) \cdot \nabla_{\bar{v}} B(v-u,\omega)$ we use (1.34). The estimate for the other terms are same as the proof of Theorem 3. □

Proposition 5 (Specular BC) *Recall (8.8). Let* $\Omega := \{\bar{x} \in \mathbb{R}^2 : |\bar{x}| < 1\}$ *be 2D disk and* $\xi(\bar{x}) = |\bar{x}|^2 - 1$. *Assume the initial datum* $f_0 = \sigma g_0(|x|, v) \geq 0$ *supported in* $\frac{1}{2} \leq |v| \leq 2$ *and*

$$
\lim_{n(x_0)\cdot u=0} \int \mathbf{k}_{g_0}(v,u) u_n \partial_n g_0(x_0,u)\mathrm{d}u_\tau > C > 0, \tag{8.26}
$$

for some $x_0 \in \partial\Omega$ *and some* $v \in \mathbb{R}^3$. *Then there exists* $0 < t = C\sigma$ *for some* $C > 0$ *such that if* $|X_{\mathbf{cl}}(0; t, x, v) - x_0| \ll 1$ *then we have a blow-up* (8.11).

Remark. We note that the condition (8.26) is compatible with the compatibility condition of the specular BC (1.32). The specular reflection BC entails, at $x \in \partial\Omega$, $g(t,x,v) = g(t,x,v-2(n(x)\cdot v)n(x))$. Now we take a spatial derivative parallel to the boundary, ∂_τ

$$\begin{aligned}\partial_\tau g(x,v) &= \partial_\tau g(x, v-2(n\cdot v)n) - [2(n\cdot v)\partial_\tau n + 2(\partial_\tau n\cdot v)n]\\ &\quad \cdot\nabla_v g(x, v-2(n\cdot v)n)\\ &= \partial_\tau g(x, R(x)v) - 2(n\cdot v)\partial_{v_\tau} g(x, R(x)v)\\ &\quad -2(\partial_\tau n\cdot v)\partial_{v_n} g(x, R(x)v),\end{aligned}$$

and take ∂_v

$$\partial_v g(x,v) = \{\mathbf{I} - n\otimes n\}\partial_v g(x, v-2(n\cdot v)n).$$

The compatibility condition reads $v\cdot\nabla_x g_0(x,v) = R(x)v\cdot\nabla_x g_0(x,R(x)v)$ and hence

$$\begin{aligned}&v_n\partial_n g_0(x,v) + v_\tau\partial_\tau g_0(x,v)\\ &\quad = -v_n\partial_n g_0(x,R(x)v) + v_\tau\partial_\tau g_0(x,R(x)v)\\ &\quad = -v_n\partial_n g_0(x,R(x)v) + v_\tau\partial_\tau g_0(x,v) + 2v_n v_\tau\partial_{v_\tau} g_0(x,R(x)v)\\ &\qquad +2v_\tau(\partial_\tau n\cdot v)\partial_{v_n} g_0(x,R(x)v).\end{aligned}$$

Now from the specular BC we can set $\partial_{v_n} g(x,v) = 0$ (oddness) and $\partial_{v_\tau} g_0(x,v)$ bounded at $x\in\partial\Omega$. Hence in the limit as $n(x)\cdot v\to 0$ with $n(x)\cdot v < 0$ and $x\in\partial\Omega$, the compatibility condition reduces to *evenness* for the function $v_n\partial_n g(x,v)$ at the grazing set

$$\begin{aligned}\lim_{v_n\to 0} v_n\partial_n g(x,v) &= \lim_{v_n\to 0}(R(x)v)_n\partial_n g(x,R(x)v)\\ &= \lim_{v_n\to 0}(-v_n)\partial_n g(x,R(x)v). \qquad (8.27)\end{aligned}$$

By setting $\lim_{v_n\to 0} v_n\partial_n g(x,v) = C > 0$ we can have (8.26).

Proof of Proposition 5 The crucial ingredients of the proof are a 2D borderline estimate of Theorem 4 (due to Lemma 19) and the explicit lower bound of (8.19) in Example 1.

We decompose

$$\int_{\mathbb{R}^3} \mathbf{k}_{g_0}(v,u)\frac{g(t,x-\varepsilon n(x),u)-g(t,x,u)}{\varepsilon} = \int_{|n(x)\cdot u|\leq \varepsilon} + \underbrace{\int_{\varepsilon\leq|n(x)\cdot u|\leq 1}}_{\mathbf{II}} + \int_{1\leq|n(x)\cdot u|}.$$

By Lemma 7, the first term is bounded by

$$\int_{|n(x)\cdot u|\leq\varepsilon} \lesssim O(1)\|g\|_\infty.$$

Due to (8.5), $1 \leq |n(x)\cdot u|$ implies $1 \lesssim |n(x-\varepsilon n(x))\cdot u|$ for $0\leq \varepsilon \ll 1$. Therefore we use Theorem 4 to bound the third term as

$$\begin{aligned}\int_{1\leq|n(x)\cdot u|} &\lesssim \int e^{\varpi\langle\bar{u}\rangle t}\frac{1+|\bar{u}|^2}{1+\varepsilon|\bar{u}|^2}\mathbf{k}_{g_0}(v,u)\mathrm{d}u \times \|e^{-\varpi\langle\bar{v}\rangle t}\frac{\alpha}{1+|\bar{v}|^2}\nabla_{\bar{x}} f(t)\|_\infty \\ &\lesssim O_{N,t}(1)\|e^{-\varpi\langle\bar{v}\rangle t}\frac{\alpha}{1+|\bar{v}|^2}\nabla_{\bar{x}} f(t)\|_\infty.\end{aligned}$$

Now we focus on the second term **II**. Due to (8.5), $g(t,x-\varepsilon rn(x),u)$ is differentiable for all $0\leq r\leq 1$ and we have (8.6). We further decompose

$$\mathbf{II} = \int_{\varepsilon\leq|n(x)\cdot u|\leq 1}\int_0^1 \mathbf{k}_{g_0}(v,u)\partial_n g(t,x-\varepsilon rn(x),u)\mathrm{d}r\mathrm{d}u = \int_{\substack{\varepsilon\leq|u_n|\leq 1\\ |u_\tau|\leq N}} + \int_{\substack{\varepsilon\leq|u_n|\leq 1\\ |u_\tau|\geq N}}.$$

Set $x(r) := x-\varepsilon rn(x)$. We apply Theorem 4 as (8.14) to bound the second term as

$$\int_0^1\int_{\varepsilon\leq|u_n|\leq 1}\frac{e^{-N}}{|u_n|^2+C\varepsilon rN^2}\mathrm{d}u_n\mathrm{d}r \lesssim \frac{e^{-N}}{\varepsilon}.$$

From (8.9), (6.74), and Theorem 4,

$$\begin{aligned}\mathbf{II} \gtrsim& \int_{\substack{\varepsilon\leq|u_n|\leq 1\\ |u_\tau|\leq N}}\mathrm{d}u\int_0^1 \mathrm{d}r\ \mathbf{k}_{g_0}(v,u)\big\{\partial_n\bar{X}_{\mathbf{cl}}(0)\cdot\nabla_{\bar{x}} g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) \\ &+\partial_n\bar{V}_{\mathbf{cl}}(0)\cdot\nabla_{\bar{v}} g_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\big\} \\ &+\sigma P(\|e^{\theta|v|^2}f_0\|_\infty)\int_{\substack{\varepsilon\leq|u_n|\leq 1\\ |u_\tau|\leq N}}\mathrm{d}u\int_0^1\mathrm{d}r\ \mathbf{k}_{g_0}(v,u)\end{aligned}$$

$$\times\left\{\int_0^t |\partial_n \bar{X}_{\mathbf{cl}}(s)| \int_{\mathbb{R}^3} \frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} |\nabla_{\bar{x}} g(s)| \mathrm{d}u \mathrm{d}s \right.$$
$$+\int_0^t |\partial_n \bar{V}_{\mathbf{cl}}(s)| \int_{\mathbb{R}^3} \frac{e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} |\nabla_{\bar{v}} g(s)| \mathrm{d}u \mathrm{d}s$$
$$\left. +\langle u\rangle^\kappa e^{-\theta|u|^2} \sup_{0\le s\le t} |\partial_n \bar{V}_{\mathbf{cl}}(s;t,x,v)| \right\}.$$

From (8.19),

$$\partial_n \bar{X}_{\mathbf{cl}}(0)\cdot\nabla_{\bar{x}} g_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \simeq \frac{|t||\bar{u}|^2}{\alpha(x(r),u)} V_{\mathbf{cl}}(0)$$
$$\cdot n(X_{\mathbf{cl}}(0))\partial_n g_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)).$$

If $|X_{\mathbf{cl}}(0)-x_0| \ll 1$ in (8.26), for $0<\delta\ll 1$

$$t\int_{\substack{\varepsilon\le|u_n|\le\delta \\ |u_\tau|\le N}} \mathrm{d}u \int_0^1 \mathrm{d}r\, \mathbf{k}_{g_0}(v,u)\partial_n \bar{X}_{\mathbf{cl}}(0)\cdot\nabla_{\bar{x}} g_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))$$
$$\simeq t\int_0^1 \mathrm{d}r \int_{\varepsilon\le|u_n|\le\delta} \mathrm{d}u_n \frac{1}{|u_n|^2+C\varepsilon r N^2}.$$

From our assumption ($\nabla_{\bar{v}} g_0 \equiv 0$ for $|u|\le N$) $\partial_n \bar{V}_{\mathbf{cl}}(0)\cdot\nabla_{\bar{v}} g_0$ contribution vanishes. From Theorem 4, (8.19) and (1.29) all the other contributions with time integration are bounded by

$$\sigma \times \int_0^1 \mathrm{d}r \int_{\varepsilon\le|u_n|\le\delta} \mathrm{d}u_n \frac{1}{|u_n|^2+C\varepsilon r N^2}.$$

Collecting the terms and using (8.18) yields for $t=C\sigma$ with $C\gg 1$

$$\mathbf{II} \gtrsim (t-\sigma-e^{-N}) \int_0^1 \mathrm{d}r \int_{\varepsilon\le|u_n|\le\delta} \mathrm{d}u_n \frac{1}{|u_n|^2+C\varepsilon r N^2} \gtrsim_N \frac{1}{\varepsilon} \to +\infty,$$

as $\varepsilon\to 0$ and this proves the proposition. □

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