Ann. PDE (2018) 4:1
https://doi.org/10.1007/s40818-017-0037-5

Stationary Solutions to the Boltzmann Equation in the Hydrodynamic Limit

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Received: 27 August 2017 / Accepted: 12 October 2017 / Published online: 10 November 2017

Abstract Despite its conceptual and practical importance, a rigorous derivation of the steady incompressible Navier–Stokes–Fourier system from the Boltzmann theory has been an outstanding open problem for general domains in 3D. We settle this open question in the affirmative, in the presence of a small external field and a small boundary temperature variation for the diffuse boundary condition. We employ a recent quantitative L^2–L^∞ approach with new L^6 estimates for the hydrodynamic part $\mathbf{P} f$ of the distribution function. Our results also imply the validity of Fourier law in the hydrodynamical limit, and our method leads to an asymptotical stability of steady Boltzmann solutions as well as the derivation of the unsteady Navier–Stokes–Fourier system.

Keywords Boltzmann equation · Navier–Stokes equation · Hydrodynamic limit

Contents

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1 Introduction

1.1 Background

The hydrodynamic limit of the Boltzmann equation has been the subject of many studies since the pioneering work by Hilbert, who introduced his famous expansion in the Knudsen number ε in [36,37], realizing the first example of the program he proposed in the sixth of his famous questions [38]. Mathematical results on the closeness of the Hilbert expansion of the Bolzmann equation to the solutions of the compressible Euler equations for small Knudsen number ε, were obtained by Caflisch [12], and Lachowicz [45], while Nishida [49], Asano and Ukai [4] proved this by different methods. More recently, the convergence in the presence of singularities for the Euler equations have been obtained in [55] and [39]. The relativistic Euler limit has been studied in [52].

On a longer time scale ε^{-1}, where diffusion effects become significant, the problem can be faced only in the low Mach numbers regime (Mach number of order ε or smaller) due to the lack of scaling invariance of the compressible Navier–Stokes equations. Hence the Boltzmann solution has been proved to be close to the incompressible Navier–Stokes–Fourier system. Mathematical results were given, among the others, in [9,15,30,32,33] for smooth solutions. For weak solutions (renormalized solutions), after several partial steps [5–8,47,48], the full result for the convergence of the renormalized solutions has been obtained by Golse and Saint-Raymond [24]. For more references, see [50].

Much less is known about the steady solutions. It is worth to notice that, even for fixed Knudsen numbers, the analog of DiPerna-Lions' renormalized solutions

[16] is not available for the steady case, due to lack of L^1 and entropy estimates. In [26,27], steady solutions were constructed in convex domains near Maxwellians, and their positivity was left open. The only other results are for special, essentially one dimensional geometry (see [3] for results at fixed Knudsen numbers and [1,2,18,19] for results at small Knudsen numbers in certain special 2D geometry). In a recent paper [17], via a new L^2–L^∞ framework, we have constructed the steady solution to the Boltzmann equation close to Maxwellians, in 3D general domains, for a gas in contact with a boundary with a prescribed temperature profile modeled by the diffuse reflection boundary condition. The question about positivity of this steady solution was resolved as a consequence of their dynamical stability. As pointed in [22], despite the importance of steady Navier–Stokes–Fourier equations in applications, it has been an outstanding open problem to derive them from the steady Boltzmann theory. We list some recent developments along $L^2 - L^\infty$ framework [17,31–35,40,41,52,54].

The goal of our paper is to employ the L^2–L^∞ framework developed in [17] to study the hydrodynamical limit of the solutions to the steady Boltzmann equation, in the low Mach numbers regime, in a general domain with boundary where a temperature profile is specified. In order to perform the hydrodynamic limit, some uniform-in-ε estimates are required. To achieve this we need to show higher integrability than L^2 for the slow modes of the solution.

Let Ω be a bounded open region of $\mathbb{R}^d$ for either $d = 2$ or $d = 3$. We consider the Boltzmann equation for the distribution density $F(t, x, v)$ with $t \in \mathbb{R}_+ := [0, \infty)$, $x \in \Omega$, $v \in \mathbb{R}^3$. In the diffusive regime, the time evolution of the gas, subject to the action of a field $\vec{G}$, is described by the following *rescaled* Boltzmann equation:

$$\partial_t F + \varepsilon^{-1} v \cdot \nabla_x F + \vec{G} \cdot \nabla_v F = \varepsilon^{-2} Q(F, F), \tag{1.1}$$

where the Boltzmann collision operator is defined as

$$\begin{aligned} Q(F, H)(v) &:= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \omega)[F(v')H(u') - F(v)H(u)]\mathrm{d}\omega \mathrm{d}u \\ &:= Q_+(F, H)(v) - Q_-(F, H)(v), \end{aligned}$$

with $v' = v - [(v-u)\cdot\omega]\omega,\ u' = u + [(v-u)\cdot\omega]\omega$. Here, B is chosen as *the hard spheres cross section* throughout this paper,

$$B(V, \omega) = |V \cdot \omega|. \tag{1.2}$$

The interaction of the gas with the boundary $\partial\Omega$ is given by the diffuse reflection boundary condition, defined as follows: Let

$$M_{\rho,u,T} := \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \exp\Big[- \frac{|v-u|^2}{2T} \Big]$$

be the local Maxwellian with density ρ, mean velocity u, and temperature T. For a prescribed function T_w (wall temperature) on $\partial\Omega$, we define the wall Maxwellian

$$M_w = \sqrt{\frac{2\pi}{T_w}} M_{1,0,T_w}. \tag{1.3}$$

We impose *the diffuse boundary condition* as

$$F = \mathscr{P}^w_\gamma(F) \quad \text{on } \gamma_-, \tag{1.4}$$

where

$$\mathscr{P}^w_\gamma F(x,v) := M_w(x,v) \int_{n(x)\cdot u>0} F(x,u)\{n(x)\cdot u\}\mathrm{d}u. \tag{1.5}$$

Here, we denote by $n(x)$ the outward normal to $\partial\Omega$ at $x \in \partial\Omega$ and we decompose the phase boundary $\gamma := \partial\Omega \times \mathbb{R}^3$ as

$$\begin{aligned} \gamma_\pm &:= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 \colon n(x)\cdot v \gtrless 0\}, \\ \gamma_0 &:= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 \colon n(x)\cdot v = 0\}. \end{aligned} \tag{1.6}$$

We recall that the boundary condition (1.4) with (1.5) ensures the zero net mass flow at the boundary:

$$\int_{\mathbb{R}^3} F(x,v)\{n(x)\cdot v\}\mathrm{d}v = 0 \quad \text{for any } x \in \partial\Omega.$$

The rescaled Boltzmann equation (1.1) is studied under the assumption of low Mach numbers, meaning that the average velocity is small compared to the sound speed. This can be achieved by looking for solutions

$$F - \mu = \mathfrak{M}\sqrt{\mu} f, \tag{1.7}$$

with *the global Maxwellian*

$$\mu(v) = M_{1,0,1} = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}. \tag{1.8}$$

Here, the number $\mathfrak{M}$ is proportional to the Mach number. The case of $\mathfrak{M} = \varepsilon$ corresponds to the incompressible Navier–Stokes–Fourier limit (INSF) that will be discussed in this paper. The case of $\mathfrak{M} \ll \varepsilon$ corresponding to the incompressible Stokes–Fourier limit, is simpler and the results of this paper also cover this case which will not be discussed explicitly.

The condition (1.7), once assumed initially, needs to be checked at later times. By multiplying (1.1) by v and integrating on velocities, we see that the change of mean velocity is proportional to $\vec{G}$. Thus, we need to assume $\vec{G} = \mathfrak{M}\Phi$ with a bounded Φ. Moreover, to make (1.7) compatible with the boundary conditions, we need to assume that $T_w = 1 + \mathfrak{M}\vartheta_w$ where ϑ_w is a function defined on $\partial\Omega$ with size of order 1. In particular, for the INSF case, we have

$$\vec{G} = \varepsilon\Phi, \quad T_w = 1 + \varepsilon\vartheta_w. \tag{1.9}$$

1.2 Notation and Preliminary Definitions

Let Θ_w be any fixed smooth function on Ω such that $\Theta_w|_{\partial\Omega} = \vartheta_w$ and

$$\|\Theta_w\|_{W^{1,\infty}(\Omega)} \lesssim \|\vartheta_w\|_{W^{1,\infty}(\partial\Omega)}. \tag{1.10}$$

Let

$$f_w = \sqrt{\mu}\left[\Theta_w(|v|^2 - 3)/2 + \rho_w\right], \quad \rho_w = -\Theta_w + |\Omega|^{-1}\int_\Omega \Theta_w, \tag{1.11}$$

where μ is the standard Maxwellian in (1.8). The average of Θ_w is added so that $\iint_{\Omega\times\mathbb{R}^3} f_w = 0$. In order to enforce the incompressible regime we look for a solution which differs from the standard Maxwellian μ for terms of order ε. Moreover, to have only homogeneous Dirichlet boundary conditions in the limiting hydrodynamic equation we subtract the contribution f_w and seek for a solution in the form

$$F \;=\; \mu + \varepsilon\sqrt{\mu}\big(f_w + f\big). \tag{1.12}$$

Note that there is no loss of generality in assuming the zero-mass condition

$$\iint_{\Omega\times\mathbb{R}^3} f\sqrt{\mu} = 0, \tag{1.13}$$

so that $\iint_{\Omega\times\mathbb{R}^3} F = \iint_{\Omega\times\mathbb{R}^3} \mu = |\Omega|$.

Our aim is to show that we can construct f such that F solves (1.1) and (1.4) both in the steady and unsteady case. Moreover, as $\varepsilon \to 0$, f converges in some suitable sense to f_1 given by

$$f_1 := \left[\rho + u\cdot v + \frac{|v|^2 - 3}{2}\theta\right]\sqrt{\mu}, \tag{1.14}$$

where (ρ, u, θ) represents the density, velocity, and temperature fluctuations. The density and the temperature fluctuations satisfy the Boussinesq relation

$$\nabla_x(\rho + \theta) = 0, \tag{1.15}$$

and the velocity and the temperature fluctuations satisfy the Incompressible Navier–Stokes–Fourier System (INSF)

$$\begin{aligned} \partial_t u + u\cdot\nabla_x u + \nabla_x p = \mathfrak{v}\Delta u + \Phi,\ \nabla_x\cdot u = 0 \quad &\text{in } \Omega,\\ \partial_t\theta + u\cdot\nabla_x(\theta + \Theta_w) = \kappa\Delta(\theta + \Theta_w) \quad &\text{in } \Omega,\\ u(x,0) = u_0(x),\quad \theta(x,0) = \theta_0(x) \quad &\text{in } \Omega,\\ u(x) = 0,\quad \theta(x) = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.16}$$

where $\mathfrak{v}$ is the viscosity and κ is the heat conductivity and p the pressure. We have used the function Θ_w in our definition to impose the null boundary data for θ.

We recall the definition of the linearized collision operator:

$$Lf = -\frac{1}{\sqrt{\mu}}[Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)], \tag{1.17}$$

and the nonlinear collision operator:

$$\Gamma_{\pm}(f, g) = \frac{1}{\sqrt{\mu}}[Q_{\pm}(\sqrt{\mu}f, \sqrt{\mu}g) + Q_{\pm}(\sqrt{\mu}g, \sqrt{\mu}f)]. \tag{1.18}$$

The null space of L, Null L is a five-dimensional subspace of $L^2(\mathbb{R}^3)$ spanned by

$$\Big\{\sqrt{\mu},\ v\sqrt{\mu},\ \frac{|v|^2-3}{2}\sqrt{\mu}\Big\}.$$

We denote the orthogonal projection of f onto Null$L$ as

$$\mathbf{P}f \ = \ a\sqrt{\mu} + v\cdot b\sqrt{\mu} + c\frac{|v|^2-3}{2}\sqrt{\mu}, \tag{1.19}$$

where $a := (f, \sqrt{\mu})_2, b := (f, v\sqrt{\mu}), c = \frac{1}{3}(f, (|v|^2-3)\sqrt{\mu})_2$ with the $L^2(\mathbb{R}^3_v)$ inner product $(\cdot\,,\,\cdot)_2$, and $(\mathbf{I}-\mathbf{P})$ the projection on the orthogonal complement of Null$L$. The inverse operator L^{-1} is defined as follows: if $\mathbf{P}g = 0$, $L^{-1}g$ is the unique solution of $L(L^{-1}g) = g$, and $\mathbf{P}(L^{-1}g) = 0$. (see [14]) Note that the functions f_1 and f_w are in Null$L$.

It is well-known that, (see [14,29])

$$Lf = \nu f - Kf,$$

where the collision frequency is defined as

$$\nu(v) = \frac{1}{\sqrt{\mu}}Q_{-}(\sqrt{\mu}, \mu) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-u)\cdot\omega|\mu(u)\mathrm{d}\omega\mathrm{d}u.$$

For the hard sphere cross section (1.2), there are positive numbers C_0 and C_1 such that, for $\langle v\rangle := \sqrt{1+|v|^2}$,

$$C_0\langle v\rangle \le \nu(v) \le C_1\langle v\rangle. \tag{1.20}$$

Moreover the compact operator on $L^2(\mathbb{R}^3_v)$, K is defined as

$$\begin{aligned} Kf &= \frac{1}{\sqrt{\mu}}[Q_{+}(\mu, \sqrt{\mu}f) + Q_{+}(\sqrt{\mu}f, \mu) - Q_{-}(\mu, \sqrt{\mu}f)] \\ &= \int_{\mathbb{R}^3}[\mathbf{k}_1(v,u) - \mathbf{k}_2(v,u)]\, f(u)\mathrm{d}u. \end{aligned}$$

The operator L is symmetric on the dense subspace $D_L = \{f \in L^2(\mathbb{R}^3_v)\,|\,\nu^{\frac{1}{2}}f \in L^2(\mathbb{R}^3_v)\} : (f, Lg)_2 = (g, Lf)_2$.

The following spectral inequality holds for L:

$$(f, Lf)_2 \gtrsim \|\nu^{1/2}(\mathbf{I}-\mathbf{P})f\|^2_{L^2(\mathbb{R}^3_v)}. \tag{1.21}$$

For the proof see [14] or the proof of Lemma 1 of [29].

1.3 Boundary Conditions

From the definition of Θ_w, we have

$$M_{1+\varepsilon\rho_w,0,1+\varepsilon\Theta_w}\Big|_{\gamma_-} = \mathscr{P}^w_\gamma(M_{1+\varepsilon\rho_w,0,1+\varepsilon\Theta_w}).$$

Moreover, by expanding $M_{1+\varepsilon\rho_w,0,1+\varepsilon\Theta_w}$ in ε, we get

$$M_{1+\varepsilon\rho_w,0,1+\varepsilon\Theta_w} = \mu + \varepsilon f_w\sqrt{\mu} + \varepsilon^2\varphi_\varepsilon, \tag{1.22}$$

where

$$|\varphi_\varepsilon| \le O(\|\vartheta_w\|^2_{L^\infty(\partial\Omega)})\langle v\rangle^4\mu(v). \tag{1.23}$$

Therefore, on γ_-

$$\mu + \varepsilon f_w\sqrt{\mu} + \varepsilon^2\varphi_\varepsilon\sqrt{\mu} = \mathscr{P}^w_\gamma(\mu + \varepsilon f_w\sqrt{\mu} + \varepsilon^2\varphi_\varepsilon\sqrt{\mu}). \tag{1.24}$$

On the other hand, from (1.4) and (1.12), on γ_-,

$$\mathscr{P}^w_\gamma\left[\mu + \varepsilon(f_w+f)\sqrt{\mu}\right] \text{ to } \mathscr{P}^w_\gamma\left(\mu + \varepsilon(f_w+f)\sqrt{\mu}\right).$$

Subtracting above two equations, we obtain the boundary condition for f:

$$f|_{\gamma_-} = \sqrt{\mu}^{-1}\mathscr{P}^w_\gamma(\sqrt{\mu}f) + \varepsilon r,$$

with

$$r := \sqrt{\mu}^{-1}\mathscr{P}^w_\gamma\left(\sqrt{\mu}\varphi_\varepsilon\right) - \varphi_\varepsilon, \quad |r|_{L^\infty(\partial\Omega\times\mathbb{R}^3)} \lesssim \varepsilon|\vartheta_w|_{L^\infty(\partial\Omega)}. \tag{1.25}$$

Furthermore we can write

$$\sqrt{\mu}^{-1}\mathscr{P}^w_\gamma(\sqrt{\mu}f) = P_\gamma f + \varepsilon\mathscr{Q}f,$$

with

$$P_\gamma f(x,v) := \sqrt{2\pi}\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{1.26}$$

$$\mathscr{Q}f := \varepsilon^{-1}\left[\sqrt{\mu}^{-1}\mathscr{P}^w_\gamma(\sqrt{\mu}f) - P_\gamma f\right]. \tag{1.27}$$

Note that the boundary operator $\mathscr{Q}$ is bounded uniformly in ε and, for $0 \le \beta < \frac{1}{4}$,

$$|e^{\beta|v|^2}\mathscr{Q}f|_{L^\infty(\partial\Omega\times\mathbb{R}^3)} \lesssim |\vartheta_w|_{L^\infty(\partial\Omega)}. \tag{1.28}$$

This follows by expanding M_w in (1.3) with $T_w = 1 + \varepsilon\vartheta_w$ in ε to obtain

$$M_w(x,v) = \sqrt{2\pi}\mu(v) + \varepsilon\vartheta_w\sqrt{2\pi}\Big(\frac{|v|^2}{2} - 2\Big)\mu(v) + \varepsilon^2 O(|\vartheta_w|^2)\langle v\rangle^4\mu(v). \tag{1.29}$$

Hence the boundary condition for f becomes

$$f = P_\gamma f + \varepsilon[\mathscr{Q}f + r] \quad \text{on } \gamma_-, \tag{1.30}$$

with $\mathscr{Q}$ in (1.27) and r in (1.25).

From $\int_{n\cdot v<0}\mu\{n\cdot v\}\mathrm{d}v = -1 = \int_{n\cdot v<0} M_w\{n\cdot v\}\mathrm{d}v$ and (1.25) and (1.27), it follows that

$$\int_{n(x)\cdot v<0}\mathscr{Q}f\sqrt{\mu}\{n(x)\cdot v\}\mathrm{d}v = 0 = \int_{n(x)\cdot v<0} r\sqrt{\mu}\{n(x)\cdot v\}\mathrm{d}v \quad \text{for any } x\in\partial\Omega. \tag{1.31}$$

1.4 Main Results

We first focus on the steady case. The steady solution to the Boltzmann equation is obtained with the same procedure discussed before for the unsteady case:

$$F_s = \mu + \varepsilon\sqrt{\mu}[f_w + f_s], \tag{1.32}$$

where F_s and f_s do not depend on t.

The unknown f_s has to satisfy the following equation

$$v\cdot\nabla_x f_s + \varepsilon^2\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v\Big[\sqrt{\mu}f_s\Big] + \varepsilon^{-1}Lf_s = L_1 f_s + \Gamma(f_s,f_s) + A_s, \tag{1.33}$$

where

$$L_1 f_s = \frac{1}{\sqrt{\mu}}[Q(\sqrt{\mu}f_w, \sqrt{\mu}f_s) + Q(\sqrt{\mu}f_s, \sqrt{\mu}f_w)], \tag{1.34}$$

$$A_s = \varepsilon\Phi\cdot v\sqrt{\mu} - v\cdot\nabla_x f_w - \varepsilon^2\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v\Big[\sqrt{\mu}f_w\Big] + \Gamma(f_w,f_w), \tag{1.35}$$

with boundary conditions

$$f_s = P_\gamma f_s + \varepsilon[\mathscr{Q}f_s + r] \quad \text{on } \gamma_-, \tag{1.36}$$

with $\mathscr{Q}$ in (1.27) and r in (1.25).

Note that, by (1.11),

$$\mathbf{P}A_s = \varepsilon\Phi \cdot v\sqrt{\mu}, \quad \iint_{\Omega\times\mathbb{R}^3} A_s\sqrt{\mu} = 0. \tag{1.37}$$

Theorem 1.1 *Assume Ω is an open bounded subset of $\mathbb{R}^3$ with C^3 boundary $\partial\Omega$. We also assume the hard sphere cross section (1.2).*

If $\Phi = \Phi(x) \in C^1(\Omega)$, $\vartheta_w \in W^{1,\infty}(\partial\Omega)$ and

$$\|\vartheta_w\|_{H^{1/2}(\partial\Omega)} + \|\Phi\|_{L^2(\Omega)} \ll 1, \tag{1.38}$$

then, for $0 < \varepsilon \ll 1$, there is a unique positive solution $F_s \geq 0$, given by (1.12) with f_s satisfying (1.33) and the boundary condition (1.30).

Moreover,

$$\|f_s\|_2 + \|\mathbf{P}f_s\|_6 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_s\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f_s|_{2,+} + \varepsilon^{\frac{1}{2}}\|wf_s\|_\infty \ll 1, \tag{1.39}$$

where $w(v) = e^{\beta|v|^2}$ with $0 < \beta \ll 1$. Finally, as $\varepsilon \to 0$, f_s converges weakly in L^2 to $f_{1,s} = [u_s \cdot v + \theta_s(|v|^2-5)/2]\sqrt{\mu}$ with (p_s, u_s, θ_s) the unique solution to the steady INSF with Dirichlet boundary conditions and subject to the external field Φ:

$$\begin{aligned} u_s \cdot \nabla_x u_s + \nabla_x p_s = \mathfrak{v}\Delta u_s + \Phi, \quad \nabla_x \cdot u_s = 0 \text{ in } \Omega, \\ u_s \cdot \nabla_x(\theta_s + \Theta_w) = \kappa\Delta(\theta_s + \Theta_w) \text{ in } \Omega, \\ u_s(x) = 0, \quad \theta_s(x) = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.40}$$

Remark In particular Theorem 1.1 implies the existence of solutions to the stationary INSF boundary value problem for small force and boundary temperature in the sense of (1.38). The hard spheres cross section is used to control the term $\varepsilon v \cdot \Phi f$ coming from the external field.

Note that, if $\Phi = \nabla_x U$ is a potential field, $u_s \equiv 0$, $p_s \equiv U$ is solution to the above system. Therefore, in order to have a stationary solution with non vanishing velocity field, we may assume that Φ is not a potential field, such that $\nabla_x \cdot \Phi = 0$. (See [22])

We remark that the convergence results provided by Theorem 1.1 does not give any indication on the rate of convergence in ε of the solution to its limit. To discuss this we can be inspired by the Hilbert expansion.

Let us denote $F = \mu + \varepsilon\sqrt{\mu}g$ and $g_1 = \sqrt{\mu}(\rho_s + u_s \cdot v + (\theta_s + \Theta_w)(|v|^2-3)/2)$,

$$\begin{aligned} g_2 := \frac{1}{2}\sum_{i,j=1}^{3} \mathscr{A}_{ij}\left[\partial_{x_i}u_{j,s} + \partial_{x_j}u_{i,s}\right] + \sum_{i=1}^{3}\mathscr{B}_i\partial_{x_i}(\theta_s + \Theta_w) \\ - L^{-1}[\Gamma(f_1, f_1)] + \frac{|v|^2-3}{2}\theta_2\sqrt{\mu}, \end{aligned} \tag{1.41}$$

where $\mathscr{A}_{ij}$ and $\mathscr{B}_i$ are given by

$$\mathscr{A}_{ij} = L^{-1}\big(\sqrt{\mu}\big(v_i v_j - \frac{|v|^2}{3}\delta_{i,j}\big)\big), \quad \mathscr{B}_i = L^{-1}\big(\sqrt{\mu} v_i(|v|^2 - 5)\big),$$

and $\theta_2 = p_s - \fint p_s - (\theta_s + \Theta_w)\rho_s$. For the properties of $\mathscr{A}_{ij}$ and $\mathscr{B}_i$ see [11].

We define

$$R_s = \varepsilon^{-\frac{1}{2}}\{f_s + f_w - (g_1 + \varepsilon g_2)\},$$

so that

$$F_s = \mu + \varepsilon\sqrt{\mu}(g_1 + \varepsilon g_2 + \varepsilon^{\frac{1}{2}} R_s). \tag{1.42}$$

Then R_s satisfies the equation

$$v\cdot\nabla_x R_s + \varepsilon^2 \frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v\Big[\sqrt{\mu}R_s\Big] + \varepsilon^{-1} L R_s = L_1 R_s + \varepsilon^{1/2}\Gamma(R_s, R_s) + \varepsilon^{1/2}\overline{A}_s, \tag{1.43}$$

with the boundary condition

$$R_s \;=\; P_\gamma R_s + \varepsilon \mathscr{Q} R_s + \varepsilon^{\frac{1}{2}} r_s \quad \text{on } \gamma_-. \tag{1.44}$$

Here $\overline{A}_s$ is given by

$$\overline{A}_s \;= -(\mathbf{I}-\mathbf{P})[v\cdot\nabla_x g_2] - 2\Gamma(g_1, g_2) - \varepsilon\Big\{\Phi\cdot\frac{1}{\sqrt{\mu}}\nabla_v\big[\sqrt{\mu}(g_1+\varepsilon g_2)\big] - \Gamma(g_2, g_2)\Big\}. \tag{1.45}$$

The equation (1.43) is similar to the one for f_s with the extra factor $\sqrt{\varepsilon}$ in front of the nonlinear term $\Gamma(R_s, R_s)$. In fact, using the same arguments employed to prove Theorems 1.1 we can control the error between solutions of the Navier–Stokes–Fourier approximation and the Boltzmann equation:

Theorem 1.2 *If* $\Phi = \Phi(x) \in H^2(\Omega) \cap C^1(\Omega)$, $\vartheta_w \in H^{7/2}(\Omega)$ *and*

$$\|\vartheta_w\|_{H^{1+}(\partial\Omega)} + \|\Phi\|_{L^{\frac{3}{2}+}(\Omega)} \ll 1, \tag{1.46}$$

then, for $0 < \varepsilon \ll 1$, *there exist a unique* R_s *satisfying* (1.43) *and the boundary condition* (1.44).

Moreover,

$$\|R_s\|_2 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})R_s\|_\nu \ll 1, \quad \varepsilon^{\frac{1}{2}}\|wR_s\|_\infty \ll 1, \tag{1.47}$$

where $w(v) = e^{\beta|v|^2}$ *with* $0 < \beta \ll 1$.

Here, $\vartheta_w \in H^{7/2}(\Omega)$ means there exists $\bar{\vartheta}_w \in H^{7/2}(\Omega)$ such that $\bar{\vartheta}_w \equiv \vartheta_w$ on $\partial\Omega$. The convention is H^{1+} and $L^{\frac{3}{2}+}$ mean $H^{1+\delta}$ and $L^{\frac{3}{2}+\delta}$ for some $0 < \delta \ll 1$.

We remark that $\sqrt{\varepsilon}R_s$ is of higher order in L^p for $2 \leq p \leq 6$. On the other hand, $\sqrt{\varepsilon}R_s$ is small, but not infinitesimal in ε in L^∞, so that it is possible that $\{F_s - (\mu + \varepsilon\sqrt{\mu}g_1)\}/\varepsilon = O(1)$ in L^∞.

Next we investigate the stability properties of the stationary solution. To discuss this, we study the unsteady problem. The solution to (1.1) is written as

$$F(t) = \mu + \varepsilon\sqrt{\mu} f, \quad f = f_w + f_s + \tilde{f}. \tag{1.48}$$

The aim is to show that $\tilde{f} \to \tilde{f}_1 = \sqrt{\mu}(\tilde{\rho} + \tilde{u}\cdot v + \tilde{\vartheta}(\frac{|v|^2-3}{2}))$, with $\nabla_x\left[\tilde{\rho} + \tilde{\vartheta}\right] = 0$ and $(\tilde{u}, \tilde{\vartheta}, \tilde{p})$ satisfying

$$\begin{aligned}\partial_t\tilde{u} + \tilde{u}\cdot\nabla_x\tilde{u} + \tilde{u}\cdot\nabla_x u_s + u_s\cdot\nabla_x\tilde{u} + \nabla_x\tilde{p} = \mathfrak{v}\Delta\tilde{u}, \quad \nabla_x\cdot\tilde{u} = 0 \quad &\text{in } \Omega,\\ \partial_t\tilde{\vartheta} + \tilde{u}\cdot\nabla_x\tilde{\vartheta} + \tilde{u}\cdot\nabla_x\vartheta_s + u_s\cdot\nabla_x\tilde{\vartheta} = \kappa\Delta\tilde{\vartheta} \quad &\text{in } \Omega,\\ \tilde{u} = 0, \quad \tilde{\vartheta} = 0 \quad &\text{on } \partial\Omega.\end{aligned} \tag{1.49}$$

The equation of $\tilde{f}$ is given by

$$\partial_t\tilde{f} + \varepsilon^{-1}v\cdot\nabla_x\tilde{f} + \varepsilon\Phi\cdot\nabla_v\tilde{f} + \varepsilon^{-2}L\tilde{f} = \varepsilon^{-1}L_{f_w+f_s}\tilde{f} + \varepsilon^{-1}\Gamma(\tilde{f},\tilde{f}) + \varepsilon\frac{\Phi\cdot v}{2}\tilde{f}. \tag{1.50}$$

Here we have used the notation $L_\phi\psi := -[\Gamma(\phi,\psi) + \Gamma(\psi,\phi)]$. Note that, due to symmetry, for all $\psi_1, \psi_2 \in L^2$,

$$(L_\phi\psi_1, \psi_2) = (L_\phi\psi_1, (\mathbf{I}-\mathbf{P})\psi_2). \tag{1.51}$$

The boundary condition of $\tilde{f}$ is given by

$$\tilde{f}|_{\gamma_-} = P_\gamma\tilde{f} + \varepsilon\mathscr{Q}\tilde{f}. \tag{1.52}$$

We define the energy and the dissipation as

$$\mathscr{E}_\lambda[\tilde{f}](t) := \sup_{0\leq s\leq t}\|e^{\lambda s}\tilde{f}(s)\|_2^2 + \sup_{0\leq s\leq t}\|e^{\lambda s}\partial_t\tilde{f}(s)\|_2^2, \tag{1.53}$$

$$\begin{aligned}\mathscr{D}_\lambda[\tilde{f}](t) := &\frac{1}{\varepsilon^2}\int_0^t\|e^{\lambda s}(\mathbf{I}-\mathbf{P})\tilde{f}\|_v^2 + \frac{1}{\varepsilon^2}\int_0^t\|e^{\lambda s}(\mathbf{I}-\mathbf{P})\partial_t\tilde{f}\|_v^2\\ &+ \frac{1}{\varepsilon}\int_0^t|e^{\lambda s}(1-P_\gamma)\tilde{f}|_{2,\gamma}^2 + \frac{1}{\varepsilon}\int_0^t|e^{\lambda s}(1-P_\gamma)\tilde{f}_t|_{2,\gamma}^2\\ &+ \int_0^t|e^{\lambda s}\tilde{f}|_{2,\gamma}^2 + \int_0^t|e^{\lambda s}\partial_t\tilde{f}|_{2,\gamma}^2 + \int_0^t\|e^{\lambda s}\mathbf{P}\tilde{f}\|_2^2\\ &+ \int_0^t\|e^{\lambda s}\mathbf{P}\partial_t\tilde{f}\|_2^2.\end{aligned} \tag{1.54}$$

Theorem 1.3 *We assume the same hypotheses of Theorem 1.1 and let F_s be the steady solution of (1.12). Suppose the initial datum takes the form of $F_0 = F_s + \varepsilon\sqrt{\mu}\tilde{f}_0 \geq 0$ such that*

$$\mathscr{E}_0[\tilde{f}](0) + \varepsilon^{3/2}\|w\partial_t \tilde{f}_0\|_\infty + \Big\| \int_{\mathbb{R}^3} |\tilde{f}_0(x,v)|\langle v\rangle^2\sqrt{\mu}\mathrm{d}v\Big\|_{L^6(\Omega)} \ll 1, \tag{1.55}$$

where $w(v) = e^{\beta|v|^2}$ with $0 < \beta \ll 1$.

Then there exists a unique global solution $F \geq 0$ given by (1.48) with $\tilde{f}$ solving (1.50) and the boundary condition (1.4).

Moreover, for some $0 < \lambda \ll 1$,

$$\mathscr{E}_\lambda[\tilde{f}](\infty) + \mathscr{D}_\lambda[\tilde{f}](\infty) + \sup_{0\leq t\leq\infty} \varepsilon^{1/2}\|w\tilde{f}(t)\|_\infty + \sup_{0\leq t\leq\infty} \varepsilon^{3/2}\|w\partial_t\tilde{f}(t)\|_\infty \ll 1, \tag{1.56}$$

Finally, as $\varepsilon \to 0$, $\tilde{f}$ converges weakly to $f_1 = [\tilde{u}\cdot v + \tilde{\vartheta}(|v|^2 - 5)/2]\sqrt{\mu}$ with $(\tilde{p}, \tilde{u}, \tilde{\vartheta})$ is a solution to (1.49).

Remark 1.4 The initial assumption (1.55) in particular requires $\|\partial_t \tilde{f}(0)\|_{L^2_{x,v}} \ll 1$. This is a very sharp condition of $\tilde{f}_0$, because, from equation (1.50)

$$\begin{aligned}\partial_t \tilde{f}(0) = &-\varepsilon^{-1}v\cdot\nabla_x\tilde{f}_0 - \varepsilon\Phi\cdot\nabla_v\tilde{f}_0 - \varepsilon^{-2}L\tilde{f}_0 + \varepsilon^{-1}L_{f_w+f_s}\tilde{f}_0 \\ &+ \varepsilon^{-1}\Gamma(\tilde{f}_0,\tilde{f}_0) + \varepsilon\frac{\Phi\cdot v}{2}\tilde{f}_0.\end{aligned} \tag{1.57}$$

To compensate the diverging factors one has to choose $\tilde{f}_0$ properly. An example of such a choice is the following: let $\tilde{f}_{1,0} = (\tilde{\rho}_0 + \tilde{u}_0\cdot v + \tilde{\theta}_0(|v|^2-3)/2)\sqrt{\mu}$ and assume that $\nabla_x\cdot\tilde{u}_0 = 0$, $\nabla_x(\tilde{\rho}_0 + \tilde{\theta}_0) = 0$, so that $\mathbf{P}(v\cdot\nabla_x\tilde{f}_{1,0}) = 0$ and set

$$\tilde{f}_0 = \tilde{f}_{1,0} - \varepsilon L^{-1}[v\cdot\nabla_x\tilde{f}_{1,0} - L_{f_w+f_s}\tilde{f}_0 - \Gamma(\tilde{f}_0,\tilde{f}_0)] + \varepsilon^2 h,$$

for some $L^2_{x,v}$ function h. Then, clearly, the diverging factors are compensated and $\partial_t f(0)$ is bounded in $L^2_{x,v}$. Thus, the smallness condition is fulfilled by assuming $\tilde{\rho}, \tilde{u}, \tilde{\theta}$ and Lh sufficiently small. Note that the initial data for the hydrodynamic quantities are small but not depending on ε. Thus our result implies the exponential stability of the constructed solution to the INSF system.

We remark also that such an asymptotical stability implies non-negativity of steady solution F_s (Sect. 3.7).

As for the stationary case, Theorem 1.3 does not provide the rate of convergence. Proceeding as in the steady case we define:

$$\tilde{f}_2 := \frac{1}{2}\sum_{i,j=1}^{3}\mathscr{A}_{ij}\left[\partial_{x_i}\tilde{u}_j + \partial_{x_j}\tilde{u}_i\right] + \sum_{i=1}^{3}\mathscr{B}_i\partial_{x_i}\tilde{\theta} - L^{-1}[\Gamma(\tilde{f}_1,\tilde{f}_1)] + \frac{|v|^2-3}{2}\tilde{\theta}_2\sqrt{\mu},$$

with $\tilde{\theta}_2 = \tilde{p} - \fint \tilde{p} - \tilde{\theta}\tilde{\rho}$ and

$$\tilde{R} = \varepsilon^{-\frac{3}{2}}[F - F_s - \varepsilon \tilde{f}_1 - \varepsilon^2 \tilde{f}_2]. \tag{1.58}$$

Then $\tilde{R}$ has to solve

$$\begin{aligned}
&\partial_t \tilde{R} + \varepsilon^{-1} v \cdot \nabla_x \tilde{R} + \varepsilon \Phi \cdot \nabla_v \tilde{R} + \varepsilon^{-2} L \tilde{R} \\
&\quad = \varepsilon^{-1} L_1 \tilde{R} + \varepsilon^{-1} L_{\varepsilon^{1/2} R_s} \tilde{R} + \varepsilon^{-1} L_{R_s}(\tilde{f}_1 + \varepsilon \tilde{f}_2) + \varepsilon^{-1/2}\Gamma(\tilde{R}, \tilde{R}) \\
&\qquad + \varepsilon \frac{\Phi \cdot v}{2} \tilde{R} + \varepsilon^{-1/2} \tilde{A},
\end{aligned} \tag{1.59}$$

where, L_1 is defined in (1.34) and

$$\begin{aligned}
\tilde{A} = &-(\mathbf{I} - \mathbf{P})[v \cdot \nabla_x (f_2 + \tilde{f}_2)] - 2\Gamma(g_1 + \tilde{f}_1, g_2 + \tilde{f}_2) \\
&- \varepsilon \Big\{ \partial_t \tilde{f}_2 + \Phi \cdot \frac{1}{\sqrt{\mu}} \nabla_v \big[\sqrt{\mu}(g_1 + \tilde{f}_1 + \varepsilon (g_2 + \tilde{f}_2)) \big] \\
&- \Gamma(g_2 + \tilde{f}_2, g_2 + \tilde{f}_2) \Big\} - \overline{A}_s,
\end{aligned} \tag{1.60}$$

and has to satisfy the boundary condition

$$\tilde{R}|_{\gamma_-} = P_\gamma \tilde{R} + \varepsilon \mathscr{Q} \tilde{R} + \varepsilon^{1/2} \tilde{r}, \tag{1.61}$$

where $\tilde{r} := \varepsilon^{-1}[\mu^{-\frac{1}{2}} \mathscr{P}^w_\gamma(\tilde{f}_1 \sqrt{\mu}) - \tilde{f}_1] + [\mu^{-\frac{1}{2}} \mathscr{P}^w_\gamma(\tilde{f}_2 \sqrt{\mu}) - \tilde{f}_2]$.

We establish the error estimates between the solutions of the Navier–Stokes–Fourier approximation and the Boltzmann equation as follows:

Theorem 1.5 *Suppose $\tilde{R}_0(x, v)$ defined in $\Omega \times \mathbb{R}^3$ and satisfies*

$$\mathscr{E}_0[\tilde{R}](0) + \varepsilon^{3/2} \|w \partial_t \tilde{R}_0\|_\infty + \Big\| \int_{\mathbb{R}^3} |\tilde{R}_0(x, v)| \langle v \rangle^2 \sqrt{\mu} \mathrm{d}v \Big\|_{L^3(\Omega)} + \varepsilon^{\frac{1}{2}} \|w \tilde{R}_0\|_\infty \ll 1, \tag{1.62}$$

where $w(v) = e^{\beta |v|^2}$ with $0 < \beta \ll 1$.

Then for ε sufficiently small there exists a unique global solution $\tilde{R}$ solving (1.59) *and the boundary condition* (1.61).

Moreover, for some $0 < \lambda \ll 1$,

$$\mathscr{E}_\lambda[\tilde{R}](\infty) + \mathscr{D}_\lambda[\tilde{R}](\infty) + \sup_{0 \le t \le \infty} \varepsilon^{3/2} \|w \partial_t \tilde{R}(t)\|_\infty + \sup_{0 \le t \le \infty} \varepsilon^{\frac{1}{2}} \|w \tilde{R}(t)\|_\infty \ll 1, \tag{1.63}$$

where $\mathscr{E}_\lambda[\tilde{R}](T)$ and $\mathscr{D}_\lambda[\tilde{R}](T)$ are defined in (1.53) *and* (1.54) *with $\tilde{f}$ replaced by $\tilde{R}$.*

Since the nonlinearity for $\tilde{R}$ and $\tilde{R}_t$ is weaker with an extra power of $\varepsilon^{1/2}$, the proof of this theorem also follows along the same lines of Theorem 1.3 and it will omitted.

Theorem 1.6 *Assume that* $(u(t), \theta(t), p(t))$ *is a solution to the INSF initial boundary value problem for* $t \in [0, T]$, $T > 0$, *such that*

$$\sup_{0\le t\le T} \|u(t)\|_{H^4(\Omega)} + \sup_{0\le t\le T} \|\theta(t)\|_{H^4(\Omega)} < \infty.$$

Then there is $\varepsilon(T) > 0$ *such that for* $\varepsilon \le \varepsilon(T)$ *there exists a unique solution* $F(t) = \mu + \varepsilon[f_1 + \varepsilon f_2 + \varepsilon^{1/2}\tilde{R}(t)]\sqrt{\mu} \ge 0$ *on* $t \in [0, T]$ *such that*

$$\mathscr{E}_0[\tilde{R}](T) + \mathscr{D}_0[\tilde{R}](T) + \sup_{0\le t\le T} \varepsilon^{3/2}\|w\tilde{R}_t(t)\|_\infty + \sup_{0\le t\le T} \varepsilon^{\frac{1}{2}}\|w\tilde{R}(t)\|_\infty \lesssim 1,$$

where $\mathscr{E}_0[\tilde{R}](T)$ *and* $\mathscr{D}_0[\tilde{R}](T)$ *are defined in* (1.53) *and* (1.54) *with* $\tilde{f}$ *replaced by* $\tilde{R}$. *Moreover,* f_1 *and* f_2 *are defined in* (3.108).

The proof of this sharp local-in-time validity theorem is given in Sect. 3.9. We note that the interval of the validity is the same as the life-span of the classical solutions to the NSF system. In particular, if $\Omega \subset \mathbb{R}^2$, then T can be arbitrarily large.

1.5 Mathematical Difficulties and Their Resolutions

1.5.1 Theorem 1.1

It is important to note that the key difficulty in the study of the steady Boltzmann equation is that only the entropy production, not the mass nor entropy itself, is controlled a priori. At the linear level, this means only $(\mathbf{I} - \mathbf{P})f$ is bounded, and one needs to control f, or the missing $\mathbf{P}f$, in terms of $(\mathbf{I} - \mathbf{P})f$ via the steady Boltzmann equation. Such a so-called positivity estimate, to control $\mathbf{P}f$ in terms of $(\mathbf{I} - \mathbf{P})f$, is developed in [28] for the time dependent case in Sobolev norms. In [31] such an estimate is established in a bounded domain in L^2 setting, via an indirect method. The most important step forward comes in the paper [17], where a quantitative and direct positivity estimate is obtained

$$\|\mathbf{P}f\|_2 \lesssim \|(\mathbf{I} - \mathbf{P})f\|_2,$$

which leads to the construction of steady Boltzmann solutions with non-constant temperature for general domains.

We use the quantitative $L^2 - L^\infty$ approach developed in [17], in the presence of ε. We start with the energy estimates to get

$$\frac{1}{\varepsilon}\|(\mathbf{I} - \mathbf{P})f_s\|_\nu \lesssim \|\Gamma(f_s, f_s)\|_2 + 1.$$

The missing $\mathbf{P}f_s$ can be estimated by the coercivity estimates in [17], with carefully chosen proper test functions in the weak formulation, such that (Proposition 2.9):

$$\|\mathbf{P}f_s\|_2 \lesssim \frac{1}{\varepsilon}\|(\mathbf{I} - \mathbf{P})f_s\|_\nu + \|\Gamma(f_s, f_s)\|_2 + 1.$$

We split

$$|\Gamma(f_s, f_s)| \leq |\Gamma(\mathbf{P}f_s, f_s)| + |\Gamma((\mathbf{I}-\mathbf{P})f_s, f_s)| \leq |\Gamma(\mathbf{P}f_s, \mathbf{P}f_s)| \\ + |\Gamma(\mathbf{P}f_s, (\mathbf{I}-\mathbf{P})f_s)| + |\Gamma((\mathbf{I}-\mathbf{P})f_s, f_s)|.$$

Since we expect $\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_s\|_\nu \lesssim 1$, the last two parts of the nonlinear term are estimated as

$$\|\Gamma(\mathbf{P}f_s, (\mathbf{I}-\mathbf{P})f_s)\|_2 + \|\Gamma((\mathbf{I}-\mathbf{P})f_s, f_s)\|_2 \lesssim \left[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_s\|_2\right][\varepsilon\|f_s\|_\infty].$$

For the first main contribution, if we have

$$\|\mathbf{P}f_s\|_{L^6} \lesssim 1, \tag{1.64}$$

then we can close the energy estimate by

$$\|\Gamma(\mathbf{P}f_s, \mathbf{P}f_s)\|_2 \lesssim \|\mathbf{P}f_s\|_3\|\mathbf{P}f_s\|_6 \lesssim 1.$$

Thanks to this L^6 bound, we can control the other two terms by establishing Proposition 2.6

$$\|f_s\|_\infty \lesssim \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}f_s\|_{L^6} + \frac{1}{\sqrt{\varepsilon}}[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_s\|_2] \lesssim \frac{1}{\sqrt{\varepsilon}}.$$

We now sketch the main idea for establishing such a crucial L^6 estimate for $\mathbf{P}f$ as stated in Proposition 2.9. For simplicity, we consider a model problem of

$$v \cdot \nabla_x f = g \in L^2, \qquad \iint f \,\mathrm{d}x\mathrm{d}v = 0,$$

and look for estimate for $a(x) = \int_{\mathbf{R}^3} f\sqrt{\mu}dv$. In [17], we developed a quantitative method to estimate L^2 norm of $a(x)$, by carefully choosing a test function $\psi = (|v|^2 - \beta_a)v \cdot \nabla_x \varphi_a \sqrt{\mu}$ with $-\Delta_x \varphi_a = a(x)$, and the Neumann boundary condition $\frac{\partial \varphi_a}{\partial n} = 0$. This process is similar in spirit to the energy estimates in elliptic problems. The main contribution $\int -\Delta_x \varphi_a a = \|a\|_{L^2}^2$ results from the Green's identity of $\int v \cdot \nabla_x f \psi dx dv$, which leads to L^2 control of a. The new observation is that the key requirement for choosing φ_a is $\int\int \nabla_x \varphi_a \, g < +\infty$, or

$$\nabla_x \varphi_a \in L^2.$$

By Sobolev embedding in 3D, it suffices to require $-\Delta \varphi_a \in L^{\frac{6}{5}}$. Therefore, we may choose

$$-\Delta \varphi_a = a^5(x) - |\Omega|^{-1}\int_\Omega a^5,$$

to obtain a main contribution $\|a\|_{L^6}^6 = \int -\Delta_x \varphi_a a$. To close such L^6 estimate, we need L^6 bound of $(\mathbf{I}-\mathbf{P})f$, which is controled by interpolation between L^2 bound of

$\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f$ (from energy estimate) and L^∞ bound for $\varepsilon^{1/2}f$. We also need to control $L^{4/3}$ norm of $\nabla_x\varphi_a$ at the boundary, which is luckily bounded by $\|a\|_{L^6}^5$ exactly via the trace theorem. Such L^6 estimate seems natural in terms of scalings of Sobolev spaces in 3D and should lead to more applications in the future.

We remark that such L^6 estimate is very different from the celebrated Averaging Lemma for $v\cdot\nabla_x f = g \in L^2$, which states a gain of L^3 integrability ($H^{1/2}$ regularity) for *any* velocity average of f in the whole space without boundary. Our L^6 estimates are stronger than L^3, but only work with an additional assumption $(\mathbf{I}-\mathbf{P})f \in L^6$.

1.5.2 Theorem 1.2

Since the nonlinear interaction of R_s now is much weaker with an extra power of $\varepsilon^{1/2}$, the proof of Theorem 1.2 follows the same lines of the proof of Theorem 1.1 and hence the detailed proof will be omitted.

We note that the chosen power $\sqrt{\varepsilon}$ is forced from the fact that higher powers of ε make the boundary term too singular. In fact, if we use α instead of $\frac{1}{2}$ in the definition of R_s, the boundary condition for R becomes

$$R_s = P_\gamma R_s + \varepsilon \mathscr{Q} R_s + \varepsilon^{1-\alpha} r_s, \quad \text{on } \gamma_-, \tag{1.65}$$

with

$$r = \frac{1}{\sqrt{\mu}} \mathscr{P}_\gamma^w\big(\sqrt{\mu}[f_2-\varphi_\varepsilon]\big) - [f_2-\varphi_\varepsilon]. \tag{1.66}$$

Since in the energy inequality we need to compensate a factor ε^{-1} in front of $|\varepsilon^{1-\alpha} r|^2_{L^2(\gamma_-)}$, the choice $\alpha = \frac{1}{2}$ is the best we can do. In conclusion, the rate of convergence of the solution to its hydrodynamic limit is at least $O(\sqrt{\varepsilon})$ in L^2.

More accurate estimates of the errors would require the truncation of the expansion to higher order terms and boundary layer analysis, but there are serious difficulties in performing such a program. Although we do not follow this strategy, let us shortly indicate the main difficulties.

The usual approach is based on the representation of the solution by means of an Hilbert-like expansion in the bulk, suitably corrected at the boundary to satisfy the boundary conditions [1,2,18,19]:

$$F = \mu + \varepsilon\sqrt{\mu}\left[f_1 + \varepsilon f_2 + \cdots + \varepsilon^k f_{k+1} + \varepsilon f_1^B + \varepsilon^2 f_2^B + \cdots + \varepsilon^{k+1} f_{k+1}^B + \varepsilon^k R\right]. \tag{1.67}$$

Here, the functions f_k are corrections in the bulk, while f_k^B are boundary layer corrections which solve Milne-like problems, and $R = R^\varepsilon$ denotes the remainder.

This strategy has been used in [10] in the much simpler case of the neutron transport equations, but recently in [54] it has been proved that the result in [10] breaks down exactly because of the lack of regularity. Therefore, the geometric field, even if of small size, has to be included in the equation for the k-term of the expansion, as in [1,20] for the case of the gravity and [54] for the geometrical field in the neutron transport equation in a disk:

$$F = \mu + \varepsilon\sqrt{\mu}\Big[f_1 + \varepsilon f_2 + \cdots + \varepsilon^k f_{k+1} + \varepsilon f^B_{1,\varepsilon} + \varepsilon^2 f^B_{2,\varepsilon} + \cdots + \varepsilon^{k+1} f^B_{k+1,\varepsilon} + \varepsilon^k R\Big],$$

where $f^B_{k,\varepsilon}$ depends on ε. Unfortunately, this strategy fails even for a general 2D domain because the analysis of the derivatives' singularities presents severe difficulties (see [34,35] for the analysis at $\varepsilon \approx 1$). The only significant exceptions are the papers [53,54] where this expansion is completely proved in the cases of the neutron transport equation and the Boltzmann equation in a disk. See some formal expansions in [51].

1.5.3 Theorem 1.3

We start with the energy estimates, as the steady case, to get

$$\|\tilde f(t)\|_2^2 + \frac{1}{\varepsilon^2}\int_0^t \|(\mathbf{I}-\mathbf{P})\tilde f\|_\nu^2 \lesssim \int_0^t \|\Gamma(\tilde f, \tilde f)\|_2^2 + 1.$$

The missing $\mathbf{P}\tilde f$ can be estimated by the coercivity estimates in [17], with carefully chosen proper test functions in the weak formulation together with the local conservation laws (Lemma 3.9):

$$\int_0^t \|\mathbf{P}\tilde f\|_2^2 \lesssim \frac{1}{\varepsilon^2}\int_0^t \|(\mathbf{I}-\mathbf{P})\tilde f\|_\nu^2 + \int_0^t \|\Gamma(\tilde f, \tilde f)\|_2^2 + 1.$$

We estimate the main nonlinear contribution as

$$\|\Gamma(\mathbf{P}\tilde f, \mathbf{P}\tilde f)\|_{L^2_{t,x,v}} \lesssim \|\mathbf{P}\tilde f\|_{L^\infty_t L^6_{x,v}} \|\mathbf{P}\tilde f\|_{L^2_t L^3_{x,v}}.$$

The most important ingredient is to control $\|\mathbf{P}\tilde f\|_{L^\infty_t L^6_{x,v}}$ as in the steady case, in the presence of the term $\varepsilon\partial_t f$ (Proposition 3.13). For further control of $\varepsilon\partial_t \tilde f$, we repeat the energy estimate for $\partial_t \tilde f$ and estimate the nonlinear term

$$\|\Gamma(\mathbf{P}\tilde f, \mathbf{P}\partial_t \tilde f)\|_{L^2_{t,x,v}} \lesssim \|\mathbf{P}\tilde f\|_{L^\infty_t L^6_{x,v}} \|\mathbf{P}\partial_t \tilde f\|_{L^2_t L^3_{x,v}},$$

with the same norm $\|\mathbf{P}\tilde f\|_{L^\infty_t L^6_{x,v}}$.

To close the estimate, it suffices to control both $\|\mathbf{P}\tilde f\|_{L^2_t L^3_{x,v}}$ and $\|\mathbf{P}\partial_t \tilde f\|_{L^2_t L^3_{x,v}}$ by:

$$\|\mathbf{P}\tilde f\|_{L^2_t L^3_{x,v}} \lesssim \|\Gamma(\tilde f, \tilde f)\|_{L^2_{t,x,v}} + \|\Gamma(f_s, \tilde f)\|_{L^2_{t,x,v}} + 1,$$
$$\|\mathbf{P}\partial_t \tilde f\|_{L^2_t L^3_{x,v}} \lesssim \|\Gamma(\tilde f, \partial_t \tilde f)\|_{L^2_{t,x,v}} + \|\Gamma(\tilde f_t, \tilde f)\|_{L^2_{t,x,v}} + 1.$$

We now illustrate the estimate for $\|\mathbf{P}\tilde f\|_{L^2_t L^3_{x,v}}$. In the absence of the external field and the boundary, $\varepsilon^2\Phi \equiv 0$ and $\Omega = \mathbb{R}^3$, such gain of integrability is well-known from the Averaging Lemma [21,23] and the sharp Sobolev embedding $H^{1/2} \subset L^3$ (See also the case for a convex bounded domain with $\varepsilon^2\Phi \equiv 0$ in [23]). We need to extend this estimate properly to the case of the bounded domain Ω in the presence of an external field $\varepsilon^2\Phi \neq 0$. We first consider an extension of $\tilde{f}$ to the whole space, denoted by $\bar{f}$, such that $\bar{f} \in L^2$ and

$$\varepsilon\partial_t \bar{f} + v \cdot \nabla_x \bar{f} + \varepsilon^2\Phi \cdot \nabla_v \bar{f} \in L^2.$$

This would require that $\bar{f}$ is continuous along all exterior trajectories, matching with given incoming and outgoing data of f on the boundary. For a general domain Ω with $\varepsilon^2\Phi \neq 0$, the exterior trajectories can be complicated and they can connect the outgoing set γ_+ and incoming set γ_-, arbitrarily near the grazing set γ_0. It is not clear that an extension $\bar{f}$ would satisfy both $\bar{f} \in L^2$ and $\varepsilon\partial_t \bar{f} + v \cdot \nabla_x \bar{f} + \varepsilon^2\Phi \cdot \nabla_v \bar{f} \in L^2$, due to a possible discontinuity of $\bar{f}$ [40].

We circumvent this difficulty via an extension lemma, Lemma 3.6, which asserts that, for the function cutoff from the grazing set γ_0,

$$f_\delta \sim \mathbf{1}_{\{|v|<\frac{1}{\delta}\}}\mathbf{1}_{\{|n(x)\cdot v|>\delta \text{ or } \mathrm{dist}(x,\partial\Omega)>\delta\}}\tilde{f}, \quad \text{for } \delta \ll 1, \tag{1.68}$$

such an extension $\overline{f_\delta}$ is indeed possible. Here, $\mathrm{dist}(x, \partial\Omega) := \inf_{y\in\partial\Omega} |x - y|$. We note that $\mathbf{P} f_\delta \sim \mathbf{P}\tilde{f}$ thanks to the estimate $\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})\tilde{f}\|_2 \sim 1$. In the presence of an external field $\varepsilon^2\Phi \neq 0$, we modify the proof of averaging lemma for $\mathbf{P}\overline{f_\delta}$ (Lemma 3.7) via a careful splitting to show its effect is small for our purpose. Similarly to the steady case, as in [17], we may bootstrap such L^6 estimates to an improved L^∞ estimate for $\Omega \subset \mathbb{R}^3$

$$\|\tilde{f}\|_{L^\infty} \lesssim \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}\tilde{f}\|_{L^\infty_t L^6_{x,v}} + \frac{1}{\varepsilon^{3/2}}\|(\mathbf{I}-\mathbf{P})\tilde{f}\|_{L^\infty_t L^2_{x,v}} + 1 \lesssim \frac{1}{\sqrt{\varepsilon}}.$$

2 Steady Problems

2.1 Preliminary and the Linear Theorem

Assume $\partial\Omega$ is C^3. Then for any $x_0 \in \partial\Omega$, there exists $0 < r_0, r_1 \ll 1$ and C^3 function $\eta : \{x_\parallel = (x_{\parallel,1}, x_{\parallel,2}) \in \mathbb{R}^2 : |x_\parallel| < r_1\} \to \partial\Omega \cap B(x_0, r_0)$ such that if $x \in \partial\Omega \cap B(x_0, r_0)$ then there exists a unique $x_\parallel \in \mathbb{R}^2$ with $|x_\parallel| < r_1$ which satisfies $x = \eta(x_\parallel)$. Here, we have used the notation $B(x_0, r_0) := \{x \in \mathbb{R}^3 : |x - x_0| < r_0\}$. Without loss of generality we assume that $|\partial_{x_{\parallel,i}}\eta(x_\parallel)| \neq 0$ for $i = 1, 2$. Moreover we define the outward normal unit vector $n(x_\parallel)$ at the boundary with the same parametrization $x_\parallel = (x_{\parallel,1}, x_{\parallel,2})$.

Assume $\mathrm{dist}(x, \partial\Omega) \ll 1$ and $x_0 \in \partial\Omega$ such that $\mathrm{dist}(x, x_0) = \mathrm{dist}(x, \partial\Omega)$. Then there exists η which is a parametrization of $\partial\Omega$ around x_0. Clearly

$$\nabla_{x_\parallel}|\eta(x_\parallel) - x|^2 = (\partial_{x_{\parallel,1}}|\eta(x_\parallel) - x|^2, \partial_{x_{\parallel,2}}|\eta(x_\parallel) - x|^2) = 0, \quad \text{for some } x_\parallel. \tag{2.1}$$

On the other hand, if $|\eta(x_\parallel) - x| \ll 1$,

$$\partial^2_{x_{\parallel,i}} |\eta(x_\parallel) - x|^2 = \partial_{x_{\parallel,i}} \big[2\partial_i \eta(x_\parallel) \cdot (\eta(x_\parallel) - x)\big] = O(|\eta(x_\parallel) - x|) + 2|\partial_i \eta(x_\parallel)|^2 \neq 0,$$

where $\partial_i \eta := \partial_{x_{\parallel,i}} \eta$ for $i = 1, 2$.

Then, by the implicit function theorem, there exists a unique $x_\parallel(x) \in C^2$ satisfying (2.1). Moreover,

$$\begin{bmatrix} \partial_{x_i} x_{\parallel,1} \\ \partial_{x_i} x_{\parallel,2} \end{bmatrix} = \begin{bmatrix} |\partial_1 \eta|^2 + \partial_1^2 \eta \cdot (\eta - x) & \partial_1 \eta \cdot \partial_2 \eta + \partial_1 \partial_2 \eta \cdot (\eta - x) \\ \partial_1 \eta \cdot \partial_2 \eta + \partial_1 \partial_2 \eta \cdot (\eta - x) & |\partial_2 \eta| + \partial_2^2 \eta \cdot (\eta - x) \end{bmatrix}^{-1} \begin{bmatrix} -\partial_1 \eta_i \\ -\partial_2 \eta_i \end{bmatrix},$$

where $\eta = \eta(x_\parallel)$. Then we define $x_\perp \in C^2$ for $\mathrm{dist}(x, \partial\Omega) \ll 1$,

$$x_\perp(x) := [x - \eta(x_\parallel(x))] \cdot n(x_\parallel(x)). \tag{2.2}$$

Note that $\mathrm{dist}(x, \partial\Omega) = |x_\perp(x)|$ if $\mathrm{dist}(x, \partial\Omega) \ll 1$.

By the compactness of $\partial\Omega$, we conclude that if $\mathrm{dist}(x, \partial\Omega) < 4r$ for some $0 < r \ll_\Omega 1$ then there exists $(x_\parallel(x), x_\perp(x)) \in C^2$ such that $x = \eta(x_\parallel(x)) + x_\perp(x) n(x_\parallel(x))$.

Finally we define the C^2 function $\xi : \mathbb{R}^3 \to \mathbb{R}$ as

$$\xi(x) := x_\perp(x) \chi\left(\frac{|\mathrm{dist}(x, \Omega)|^2}{4r^2}\right) + r\big[1 - \chi\big(\frac{|\mathrm{dist}(x, \Omega)|^2}{r^2}\big)\big], \tag{2.3}$$

where

$$\chi \in C_c^\infty(\mathbb{R}) \text{ such that } 0 \le \chi \le 1,\ \chi'(x) \ge -4 \times \mathbf{1}_{\frac{1}{2} \le |x| \le 1} \text{ and } \chi(x) = \begin{cases} 1 & \text{if } |x| \le \frac{1}{2}, \\ 0 & \text{if } |x| \ge 1. \end{cases} \tag{2.4}$$

Then $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$. If $|\xi(x)| \ll 1$ then $\xi(x) = x_\perp(x)$.

Moreover $n(x) \equiv \frac{\nabla \xi(x)}{|\nabla \xi(x)|}$ at the boundary $x \in \partial\Omega$. From now we define

$$n(x) := \nabla \xi(x)/|\nabla \xi(x)| \quad \text{for } x \in \mathbb{R}^3. \tag{2.5}$$

We use this new coordinate (2.2) to extend Φ on the whole space, and denote this extension by $\bar{\Phi}$, with $\|\bar{\Phi}\|_\infty \le \|\Phi\|_\infty$: For $0 < \delta \ll 1$,

$$\bar{\Phi}(x) := \Phi(x) \mathbf{1}_{x \in \bar{\Omega}} + \Phi(\eta(x_\parallel(x))) \chi\left(\frac{|\xi(x)|}{\delta}\right) \mathbf{1}_{x \in \mathbb{R}^3 \setminus \bar{\Omega}}.$$

Therefore without loss of generality we assume that Φ is defined on the whole space $\mathbb{R}^3$.

Definition 2.1 Assume $\Phi = \Phi(x) \in C^1$. Consider the steady linear transport equation

$$v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_x f = g. \tag{2.6}$$

The equations of the characteristics for (2.6) are

$$\dot{X} = V, \quad \dot{V} = \varepsilon^2 \Phi(X), \quad X(t; t, x, v) = x, \quad V(t; t, x, v) = v. \tag{2.7}$$

If $X(\tau; t, x, v) \in \Omega$ for all τ between s and t, then

$$\begin{aligned} X(s; t, x, v) &= x + v(s-t) + \varepsilon^2 \int_t^s \int_t^\tau \Phi(X(\tau'; t, x, v)) \mathrm{d}\tau' \mathrm{d}\tau, \\ V(s; t, x, v) &= v + \varepsilon^2 \int_t^s \Phi(X(\tau; s, x, v)) \mathrm{d}\tau. \end{aligned} \tag{2.8}$$

Note that the ODE (2.7) is autonomous since Φ is time-independent.

Define

$$\begin{aligned} t_{\mathbf{b}}(x, v) &:= \inf\{t \geq 0 : X(-t; 0, x, v) \notin \Omega\}, \\ x_{\mathbf{b}}(x, v) &:= X(-t_{\mathbf{b}}(x, v); 0, x, v), \quad v_{\mathbf{b}}(x, v) := V(-t_{\mathbf{b}}(x, v); 0, x, v), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} t_{\mathbf{f}}(x, v) &:= \inf\{t \geq 0 : X(t; 0; x, v) \notin \Omega\}, \\ x_{\mathbf{f}}(x, v) &:= X(t_{\mathbf{f}}(x, v); 0; x, v), \quad v_{\mathbf{f}}(x, v) := V(t_{\mathbf{f}}(x, v); 0; x, v). \end{aligned} \tag{2.10}$$

Clearly $(x_{\mathbf{b}}(x, v), v_{\mathbf{b}}(x, v)) \in \gamma_-$ and $(x_{\mathbf{f}}(x, v), v_{\mathbf{f}}(x, v)) \in \gamma_+$.

Lemma 2.2 *For any open subset $\Omega \subset \mathbb{R}^3$, $B \subset \partial\Omega$, and $f \in L^1(\Omega \times \mathbb{R}^3)$,*

$$\begin{aligned} &\iint_{\Omega \times \mathbb{R}^3} |f(x, v)| \mathbf{1}_{x_{\mathbf{b}}(x,v) \in B} \mathbf{1}_{t_{\mathbf{b}}(x,v) \leq \frac{1}{m} \ln \frac{1}{\varepsilon}} \mathrm{d}x \mathrm{d}v \\ &\quad = \int_B \int_{n(y) \cdot u < 0} \int_0^{\min\{t_{\mathbf{f}}(y,u), \frac{1}{m} \ln \frac{1}{\varepsilon}\}} |f(X(s; 0, y, u), V(s; 0, y, u))| \\ &\quad \times \{|n(y) \cdot u| + O(\varepsilon)(1 + |u|)s\} \mathrm{d}s \mathrm{d}u \mathrm{d}S_y, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\iint_{\Omega \times \mathbb{R}^3} |f(x, v)| \mathbf{1}_{x_{\mathbf{f}}(x,v) \in B} \mathbf{1}_{t_{\mathbf{f}}(x,v) \leq \frac{1}{m} \ln \frac{1}{\varepsilon}} \mathrm{d}x \mathrm{d}v \\ &\quad = \int_B \int_{n(y) \cdot u > 0} \int_{-\min\{t_{\mathbf{b}}(y,u), \frac{1}{m} \ln \frac{1}{\varepsilon}\}}^0 |f(X(s; 0, y, u), V(s; 0, y, u))| \\ &\quad \times \{|n(y) \cdot u| + O(\varepsilon)(1 + |u|)|s|\} \mathrm{d}s \mathrm{d}u \mathrm{d}S_y. \end{aligned} \tag{2.12}$$

Proof *Step 1*. From (2.7), for $\nabla \in \{\nabla_x, \nabla_v\}$,

$$\frac{d}{ds}\begin{pmatrix} \nabla X \\ \nabla V \end{pmatrix} = \mathbb{A} \begin{pmatrix} \nabla X \\ \nabla V \end{pmatrix}, \quad \mathbb{A} = \left(\begin{array}{c|c} 0_{3,3} & I_{3,3} \\ \hline \varepsilon^2 \nabla_x \Phi & 0_{3,3} \end{array} \right). \tag{2.13}$$

Note $\binom{\nabla X}{\nabla V}|_{s=t} = Id$. Since the matrix $\mathbb{A}$ is bounded, there exists $C_\Phi > 0$ such that

$$\begin{aligned}
&|\partial_{x_j} X_i(s;t,x,v)| \leq C_\Phi e^{C_\Phi|t-s|}, && |\partial_{v_j} X_i(s;t,x,v)| \leq C_\Phi |t-s| e^{C_\Phi|t-s|},\\
&|\partial_{x_j} V_i(s;t,x,v)| \leq C_\Phi \varepsilon^2 |t-s| e^{C_\Phi|t-s|}, && |\partial_{v_j} V_i(s;t,x,v)| \leq C_\Phi e^{C_\Phi|t-s|}.
\end{aligned} \tag{2.14}$$

Step 2. Assume $n(x_{\mathbf{b}}(y,u)) \cdot \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \neq 0$ so that the boundary $\partial\Omega$ is locally a graph of $\eta(y_1, y_2)$: $(y_1, y_2, y_3) \in \partial\Omega$ iff $y_3 = \eta(y_1, y_2)$. By the definitions,

$$\begin{aligned}
X &:= X(s;0,y,u) = \begin{pmatrix} y_2 \\ y_2 \\ \eta(y_1,y_2) \end{pmatrix} + us + \varepsilon^2 \int_0^s \int_0^\tau \Phi(X(\tau';0,y,u)) \mathrm{d}\tau' \mathrm{d}\tau,\\
V &:= V(s;0,y,u) = u + \varepsilon^2 \int_0^s \Phi(X(\tau;0,y,u)) \mathrm{d}\tau.
\end{aligned}$$

From (2.14),

$$\begin{aligned}
\frac{\partial X}{\partial(y_1,y_2)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1\eta(y_1,y_2) & \partial_2\eta(y_1,y_2) \end{pmatrix}\\
&\quad + \varepsilon^2 \int_0^s \int_0^\tau \nabla_x \Phi(X(\tau';0,y,u)) \cdot \nabla_x X(\tau';0,y,u) \mathrm{d}\tau' \mathrm{d}\tau\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1\eta(y_1,y_2) & \partial_2\eta(y_1,y_2) \end{pmatrix} + O(\varepsilon^2) s^2 e^{C_\Phi s},\\
\frac{\partial X}{\partial s} &= V = u + O(\varepsilon^2)s, \quad \frac{\partial V}{\partial s} = \varepsilon^2 \Phi(X),\\
\frac{\partial X}{\partial v} &= sI_{3,3} + \varepsilon^2 \int_0^s \int_0^\tau \nabla_x \Phi(X(\tau';0,y,u)) \cdot \nabla_v X(\tau';0,y,u) \mathrm{d}\tau' \mathrm{d}\tau\\
&= sI_{3,3} + O(\varepsilon^2) s^3 e^{C_\Phi s},\\
\frac{\partial V}{\partial(y_1,y_2)} &= \varepsilon^2 \int_0^s \nabla_x \Phi(X(\tau;0,y,u)) \cdot \nabla_x X(\tau;0,y,u) \mathrm{d}\tau = O(\varepsilon^2) s e^{C_\Phi s},\\
\frac{\partial V}{\partial v} &= I_{3,3} + \varepsilon^2 \int_0^s \nabla_x \Phi(X(\tau;0,y,u)) \cdot \nabla_v X(\tau;0,y,u) \mathrm{d}\tau\\
&= I_{3,3} + O(\varepsilon^2) s^2 e^{C_\Phi s}.
\end{aligned}$$

We consider

$$\det\left(\frac{\partial(X,V)}{\partial(y_1,y_2,s,v)}\right) = \det\left(\begin{array}{c|c|c} I_{2,2} + O(\varepsilon^2)s^2 e^{C_\Phi s} & v + O(\varepsilon^2)s & sI_{3,3} + O(\varepsilon^2)s^3 e^{C_\Phi s} \\ \nabla\eta + O(\varepsilon^2)s^2 e^{C_\Phi s} & & \\ \hline O(\varepsilon^2)s e^{C_\Phi s} & \varepsilon^2\Phi(X) & I_{3,3} + O(\varepsilon^2)s^2 e^{C_\Phi s} \end{array}\right).$$

Recall the formula for the block matrix when a submatrix D is invertible

$$\det\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \det(D)\det(A - BD^{-1}C).$$

For $s \leq \frac{1}{m}\ln\frac{1}{\varepsilon}$ for $m \gg 1$, the submatrix $\frac{\partial V}{\partial v}$ is invertible and

$$\det\left(\frac{\partial V}{\partial v}\right) = 1 + O(\varepsilon^2)s^2 e^{C_\Phi s} \neq 0, \quad \left(\frac{\partial V}{\partial v}\right)^{-1} = I_{3,3} - \frac{O(\varepsilon^2)s^2 e^{C_\Phi s}}{1 + 3O(\varepsilon^2)s^2 e^{C_\Phi s}}.$$

Furthermore, for $s \leq \frac{1}{m}\ln\frac{1}{\varepsilon}$ for $m \gg 1$, we have $s^k e^{C_\Phi s} \lesssim \varepsilon^{0+}$ and therefore

$$\begin{aligned}
&\det\left(\frac{\partial V}{\partial v}\right)\det\left(\frac{\partial X}{\partial(x,s)} - \frac{\partial X}{\partial v}\left(\frac{\partial V}{\partial v}\right)^{-1}\frac{\partial V}{\partial(x,s)}\right) \\
&= \{1 + O(\varepsilon)s\}\det\left(\begin{pmatrix} I_{2,2} + O(\varepsilon^2)s^2 e^{C_\Phi s} & u + O(\varepsilon^2)s \\ \nabla\eta + O(\varepsilon^2)s^2 e^{C_\Phi s} & \end{pmatrix}\right. \\
&\quad \left. - \left(sI_{3,3} + O(\varepsilon^2)s^2 e^{C_\Phi s}\right)\left(I_{3,3} - \frac{O(\varepsilon^2)s^2 e^{C_\Phi s}}{1 + 3O(\varepsilon^2)s^2 e^{C_\Phi s}}\right)\left(O(\varepsilon^2)s^2 e^{C_\Phi s}\right)\right) \\
&= \{1 + O(\varepsilon)s\}\det\left(\begin{pmatrix} I_{2,2} + O(\varepsilon^2)s^2 e^{C_\Phi s} & u + O(\varepsilon^2)s \\ \nabla\eta + O(\varepsilon^2)s^2 e^{C_\Phi s} & \end{pmatrix} + \left(O(\varepsilon^2)s^3 e^{C_\Phi s}\right)\right) \\
&= \{1 + O(\varepsilon)s\}\det\begin{pmatrix} I_{2,2} + O(\varepsilon)s & u + O(\varepsilon)s \\ \nabla\eta + O(\varepsilon)s & \end{pmatrix} \\
&= u \cdot \big(-\partial_1\eta(y_1, y_2), -\partial_2\eta(y_1, y_2), 1\big) + O(\varepsilon)s(1 + |u|) \\
&= -u \cdot n(y)\sqrt{1 + (\partial_1\eta(y_1, y_2))^2 + (\partial_2\eta(y_1, y_2))^2} + O(\varepsilon)s(1 + |u|).
\end{aligned}$$

Therefore,

$$\det\left(\frac{\partial(X, V)}{\partial(s, y, u)}\right) = O(1)n(y) \cdot u + O(\varepsilon)(1 + |u|)s.$$

These prove (2.11). For (2.12), note that if $s \leq \frac{1}{m}\ln\frac{1}{\varepsilon}$, $n(y) \cdot u > 0$ then $t_{\mathbf{b}}(X(s; 0, y, u), V(s; 0, y, u)) = s \leq \frac{1}{m}\ln\frac{1}{\varepsilon}$, and if $s \leq \frac{1}{m}\ln\frac{1}{\varepsilon}$, $n(y) \cdot u < 0$ then $t_{\mathbf{f}}(X(s; 0, y, u), V(s; 0, y, u)) = s \leq \frac{1}{m}\ln\frac{1}{\varepsilon}$. This confirms (2.12). $\square$

Next lemma extends the Ukai's Lemma ([14]) to the case with external fields.

Lemma 2.3 *Assume Ω is an open bounded subset of $\mathbb{R}^3$ with $\partial\Omega$ is C^3. We define*

$$\gamma_\pm^\delta := \left\{(x, v) \in \gamma_\pm : |n(x) \cdot v| > \delta, \ \ \delta \leq |v| \leq \frac{1}{\delta}\right\}. \tag{2.15}$$

If $f \in L^1(\Omega \times \mathbb{R}^3)$ *and* $v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f \in L^1(\Omega \times \mathbb{R}^3)$ *then* f *has a trace on* $\gamma_\pm^\delta$ *and satisfies*

$$|f \mathbf{1}_{\gamma_\pm^\delta}|_1 \lesssim_{\delta,\Omega} \|f\|_1 + \|v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f\|_1.$$

Proof Let f solve (2.6) in the sense of distributions. Then along the trajectory for $(x, v) \in \gamma_+$, with $X(s) \equiv X(s; t; x, v)$ and $V(s) \equiv V(s; t; x, v)$,

$$|f(x, v)| \lesssim |f(X(s), V(s))| + \int_s^t |g(X(\tau), V(\tau))| \mathrm{d}\tau.$$

Integrating over $s \in [t - t_{\mathbf{b}}(x, v), t]$, we obtain

$$\begin{aligned} t_{\mathbf{b}}(x, v)|f(x, v)| \lesssim & \int_{t-t_{\mathbf{b}}(x,v)}^t |f(X(s), V(s))| \mathrm{d}s \\ & + t_{\mathbf{b}}(x, v) \int_{t-t_{\mathbf{b}}(x,v)}^t |g(X(\tau), V(\tau))| \mathrm{d}\tau. \end{aligned} \tag{2.16}$$

On the other hand, for $(x, v) \in \gamma_+^\delta$, from $x_{\mathbf{b}}(x, v) = x - t_{\mathbf{b}}(x, v)v + O(\varepsilon^2)(t_{\mathbf{b}}(x, v))^2$,

$$t_{\mathbf{b}}(x, v) = |v|^{-1}|x_{\mathbf{b}}(x, v) - x| + O(\varepsilon^2)|v|^{-1}(t_{\mathbf{b}}(x, v))^2. \tag{2.17}$$

We claim, for $(x, v) \in \gamma_+^\delta$,

$$t_{\mathbf{b}}(x, v) \gtrsim_\delta 1. \tag{2.18}$$

For large $C \gg 1$, we only need to consider $(x, v) \in \gamma_+^\delta$ such that $t_{\mathbf{b}}(x, v) \leq C$. From (2.17), $t_{\mathbf{b}}(x, v) = \frac{|x_{\mathbf{b}}(x,v)-x|}{|v|} - t_{\mathbf{b}}(x, v)O(\varepsilon^2)\frac{t_{\mathbf{b}}(x,v)}{|v|} \geq \frac{|x_{\mathbf{b}}(x,v)-x|}{|v|} - t_{\mathbf{b}}(x, v)O(\varepsilon^2)\frac{C}{\delta}$ so that

$$t_{\mathbf{b}}(x, v) \geq \left[1 + O(\varepsilon^2)\frac{C}{\delta}\right]^{-1} |v|^{-1}|x_{\mathbf{b}}(x, v) - x| \gtrsim |v|^{-1}|x_{\mathbf{b}}(x, v) - x|.$$

From $|x_{\mathbf{b}} - x| \gtrsim |n(x) \cdot \frac{x - x_{\mathbf{b}}}{|x - x_{\mathbf{b}}|}|$ for $x_{\mathbf{b}}, x \in \partial\Omega$ ([31]),

$$t_{\mathbf{b}}(x, v) \gtrsim |n(x) \cdot (x - x_{\mathbf{b}}(x, v))| / [|v||x - x_{\mathbf{b}}(x, v)|].$$

On the other hand, for $(x, v) \in \gamma_+^\delta$ and $\varepsilon \ll 1$,

$$\begin{aligned} |n(x) \cdot (x_{\mathbf{b}} - x)| = |n(x) \cdot [t_{\mathbf{b}} v + O(\varepsilon^2)(t_{\mathbf{b}})^2]| = t_{\mathbf{b}}|n(x) \cdot v| \\ + O(\varepsilon^2)(t_{\mathbf{b}})^2 \gtrsim t_{\mathbf{b}}|n(x) \cdot v|, \end{aligned}$$

and $|x - x_{\mathbf{b}}| \leq t_{\mathbf{b}}|v| + O(\varepsilon^2)(t_{\mathbf{b}})^2 \lesssim t_{\mathbf{b}}|v|$. Therefore, we conclude our claim (2.18) by

$$t_{\mathbf{b}} \gtrsim t_{\mathbf{b}}|n(x) \cdot v| / [t_{\mathbf{b}}|v|] \gtrsim |n(x) \cdot v||v|^{-1} \gtrsim_\delta 1.$$

From (2.16) and (2.18),

$$\mathbf{1}_{(x,v)\in\gamma_+^\delta}|f(x,v)| \lesssim_\delta \int_{t-t_{\mathbf{b}}(x,v)}^t |f(X(s),V(s))|\mathrm{d}s + \int_{t-t_{\mathbf{b}}(x,v)}^t |g(X(s),V(s))|\mathrm{d}s.$$

Integrating the above over γ_+^δ, we deduce

$$\begin{aligned}
&\int_{\gamma_+^\delta} |f(x,v)||n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v \\
&\lesssim \int_{\gamma_+^\delta}\int_{t-t_{\mathbf{b}}(x,v)}^t |f(X(s;t,x,v),V(s;t,x,v))||n(x)\cdot v|\mathrm{d}s\mathrm{d}S_x\mathrm{d}v \\
&\quad + \int_{\gamma_+^\delta}\int_{t-t_{\mathbf{b}}(x,v)}^t |g(X(s;t,x,v),V(s;t,x,v))||n(x)\cdot v|\mathrm{d}s\mathrm{d}S_x\mathrm{d}v.
\end{aligned}$$

We check that there exist $\varepsilon_0>0$, $m\gg 1$, and $\delta>0$ such that, for all $0<\varepsilon<\varepsilon_0$,

$$\gamma_+^\delta \subset \left\{(x,v)\in\gamma_+ : t_{\mathbf{b}}(x,\pm v)\le m\ln\frac{1}{\varepsilon} \text{ and } |v|\ge m\varepsilon^2\ln\frac{1}{\varepsilon}\right\}.$$

Clearly, $|v|>\delta\ge m\varepsilon_0^2\ln\frac{1}{\varepsilon_0}\ge m\varepsilon^2\ln\frac{1}{\varepsilon}$. Since Ω is bounded, we have $|v|t_{\mathbf{b}}(x,\pm v)\lesssim_\Omega 1$ and $t_{\mathbf{b}}(x,\pm v)\lesssim\frac{1}{|v|}\le\frac{1}{\delta}\le m\ln\frac{1}{\varepsilon_0}\le m\ln\frac{1}{\varepsilon}$. Then

$$O(\varepsilon)(1+|v|)s \lesssim O(\varepsilon)\left(1+\frac{1}{\delta}\right)m\ln\frac{1}{\varepsilon} \lesssim_\delta o(1)|n(x)\cdot v|.$$

From (2.12), we conclude that

$$\int_{\gamma_+^\delta}|f(x,v)||n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v \lesssim \|f\|_1+\|g\|_1.$$

The same arguments can be applied to bound $|f\mathbf{1}_{\gamma_-^\delta}|$. □

Lemma 2.4 *Let $\Phi\in C^1$. Assume that $f(x,v)$, $g(x,v)\in L^2(\Omega\times\mathbb{R}^3)$, $\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}f$, $\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}g\in L^2(\Omega\times\mathbb{R}^3)$ and $f_\gamma, g_\gamma\in L^2(\partial\Omega\times\mathbb{R}^3)$. Then*

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3}\{v\cdot\nabla_x f+\varepsilon^2\Phi\cdot\nabla_v f\}g+\{v\cdot\nabla_x g+\varepsilon^2\Phi\cdot\nabla_v g\}f \\
&=\int_{\gamma_+}fg-\int_{\gamma_-}fg.
\end{aligned}\tag{2.19}$$

Proof It is easy to check that the proof in Chapter 9 of [14], equation (2.18), still holds in the presence of C^1 field. □

In the following sections, Sects. 2.2 and 2.3, we prove the next linear estimate.

Theorem 2.5 *For the steady case, we define a norm*

$$[\![f]\!] := \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{\nu}+\varepsilon^{-1/2}|(1-P_{\gamma})f|_2+|f|_2+\|\mathbf{P}f\|_6+\varepsilon^{1/2}\|wf\|_{\infty}. \tag{2.20}$$

Suppose $\Phi \in L^{\infty}$, $g \in L^2(\Omega \times \mathbb{R}^3)$, *and* $r \in L^2(\gamma_-)$ *such that*

$$\iint_{\Omega\times\mathbb{R}^3} g(x,v)\sqrt{\mu}\mathrm{d}x\mathrm{d}v \;=\; 0 \;=\; \int_{\gamma_-} r(x,v)\sqrt{\mu}\mathrm{d}\gamma. \tag{2.21}$$

Then, for sufficiently small $\varepsilon > 0$, *there exists a unique solution to*

$$v\cdot\nabla_x f \;+\; \varepsilon^2\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v\left[\sqrt{\mu}f\right]+\varepsilon^{-1}Lf = g, \quad f|_{\gamma_-} = P_{\gamma}f + r, \tag{2.22}$$

such that

$$\iint_{\Omega\times\mathbb{R}^3} f(x,v)\sqrt{\mu}\,\mathrm{d}x\mathrm{d}v = 0, \tag{2.23}$$

and

$$[\![f]\!] \lesssim \varepsilon^{-1/2}|r|_2 + \|\nu^{-1/2}(\mathbf{I}-\mathbf{P})g\|_2 + \varepsilon^{-1}\|\mathbf{P}g\|_2 + \varepsilon^{\frac{1}{2}}|wr|_{\infty} + \varepsilon^{\frac{3}{2}}\|\langle v\rangle^{-1}wg\|_{\infty}. \tag{2.24}$$

The proof is in the end of Sect. 2.3.

2.2 L^{∞} Estimate

The main goal of this section is to prove the following proposition.

Proposition 2.6 *Let* f *satisfy,*

$$\begin{aligned}\left[v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v+\varepsilon^{-1}C_0\langle v\rangle+\lambda\right]|f| &\le \varepsilon^{-1}K_{\beta}|f|+|g|,\\ \left|f|_{\gamma_-}\right| &\le P_{\gamma}|f|+|r|,\end{aligned} \tag{2.25}$$

where $\lambda \ge 0$, *for* $0<\beta<\frac{1}{4}$, $K_{\beta}|f| = \int_{\mathbb{R}^3}\mathbf{k}_{\beta}(v,u)|f(u)|\mathrm{d}u$ *and*

$$\mathbf{k}_{\beta}(v,u) := \left\{|v-u|+|v-u|^{-1}\right\}\exp\left[-\beta|v-u|^2-\beta\frac{[|v|^2-|u|^2]^2}{|v-u|^2}\right]. \tag{2.26}$$

If $\mathbf{P}f \in L^6(\Omega\times\mathbb{R}^3)$ *and* $(\mathbf{I}-\mathbf{P})f \in L^2(\Omega\times\mathbb{R}^3)$, *then, for* $w(v) = e^{\beta'|v|^2}$ *with* $0<\beta'\ll\beta$,

$$\begin{aligned}\varepsilon^{\frac{1}{2}}\|wf\|_{\infty} \lesssim\, &\varepsilon^{\frac{1}{2}}|wr|_{\infty}+\varepsilon^{\frac{3}{2}}\|\langle v\rangle^{-1}wg\|_{\infty}\\ &+\|\mathbf{P}f\|_{L^6(\Omega\times\mathbb{R}^3)}+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)}.\end{aligned} \tag{2.27}$$

We define the stochastic cycles for the steady case.

Definition 2.7 Define, for free variables $v_k \in \mathbb{R}^3$, from (2.9)

$$
\begin{aligned}
t_1(t,x,v) = t - t_{\mathbf{b}}(x,v), \quad x_1 &= X(t_1;t,x,v) = x_{\mathbf{b}}(x,v),\\
t_2(t,x,v,v_1) = t_1 - t_{\mathbf{b}}(x_1,v_1), \quad x_2 &= X(t_2;t_1,x_1,v_1) = x_{\mathbf{b}}(x_1,v_1),\\
&\vdots\\
t_{k+1}(t,x,v,\ldots,v_k) = t_k - t_{\mathbf{b}}(x_k,v_k), \quad x_{k+1} &= X(t_{k+1};t_k,x_k,v_k) = x_{\mathbf{b}}(x_k,v_k).
\end{aligned}
$$

Set

$$
\begin{aligned}
X_{\mathbf{cl}}(s;t,x,v) &:= \sum_k \mathbf{1}_{[t_{k+1},t_k)}(s) X(s;t_k,x_k,v_k),\\
V_{\mathbf{cl}}(s;t,x,v) &:= \sum_k \mathbf{1}_{[t_{k+1},t_k)}(s) V(s;t_k,x_k,v_k).
\end{aligned}
$$

For $x \in \partial\Omega$, we define

$$
\mathcal{V}(x) := \{v \in \mathbb{R}^3 : n(x)\cdot v > 0\}, \quad \mathrm{d}\sigma(x,v) := \sqrt{2\pi}\mu(v)\{n(x)\cdot v\}\mathrm{d}v. \tag{2.28}
$$

For $j \in \mathbb{N}$, we denote

$$
\mathcal{V}_j := \{v_j \in \mathbb{R}^3 : n(x_j)\cdot v_j > 0\}, \quad \mathrm{d}\sigma_j := \sqrt{2\pi}\mu(v_j)\{n(x_j)\cdot v_j\}\mathrm{d}v_j. \tag{2.29}
$$

The following lemma is a generalized version of Lemma 23 of [31].

Lemma 2.8 ([31]) *Assume $\Phi = \Phi(x) \in C^1$. For sufficiently large $T_0 > 0$, there exist constant $C_1, C_2 > 0$, independent of T_0, such that for $k = C_1 T_0^{5/4}$,*

$$
\sup_{(t,x,v)\in[0,T_0]\times\bar{\Omega}\times\mathbb{R}^3} \int_{\prod_{\ell=1}^{k-1}\mathcal{V}_\ell} \mathbf{1}_{t_k(t,x,v,v_1,v_2,\ldots,v_{k-1})>0} \Pi_{\ell=1}^{k-1}\mathrm{d}\sigma_\ell < \left\{\frac{1}{2}\right\}^{C_2 T_0^{5/4}}. \tag{2.30}
$$

Proof For $0 < \delta \ll 1$, we define

$$
\mathcal{V}_\ell^\delta := \left\{v_\ell \in \mathcal{V}_\ell : |v_\ell \cdot n(x_\ell)| > \delta \text{ and } \delta < |v_\ell| < \frac{1}{\delta}\right\}.
$$

Clearly, $\int_{\mathcal{V}_\ell \setminus \mathcal{V}_\ell^\delta} \mathrm{d}\sigma_\ell \le C\delta$, where C is independent of ℓ. We claim that

$$
|t_\ell - t_{\ell+1}| \ \ge\ \delta^3/C_\Omega, \quad \text{for} \quad v_\ell \in \mathcal{V}_\ell^\delta. \tag{2.31}
$$

It suffices to prove, for $(x,v) \in \gamma_-^\delta$ and $0 < \varepsilon \ll 1$,

$$
t_{\mathbf{b}}(x,v) \ \gtrsim\ |v|^{-2}|n(x)\cdot v|.
$$

Note that $\frac{|n(x)\cdot v|}{|v|^2} \le \delta^2$. Therefore we only need to consider the case of $t_{\mathbf{b}}(x,v) < \delta^2$.

From $|v| > \delta$ and $x_{\mathbf{b}} = x + t_{\mathbf{b}} v + O(\varepsilon^2)(t_{\mathbf{b}})^2$,

$$t_{\mathbf{b}} = |x_{\mathbf{b}} - x||v|^{-1} + O(\varepsilon^2)(t_{\mathbf{b}})^2|v|^{-1} = |x_{\mathbf{b}} - x||v|^{-1} + t_{\mathbf{b}}\ O(\varepsilon^2)\delta.$$

For fixed $\delta > 0$ and $\varepsilon < \varepsilon_0 \ll_\delta 1$,

$$t_{\mathbf{b}}(x, v) \gtrsim |x_{\mathbf{b}}(x, v) - x||v|^{-1}.$$

From the fact $|x_{\mathbf{b}} - x| \gtrsim |n(x) \cdot \frac{x - x_{\mathbf{b}}}{|x - x_{\mathbf{b}}|}|$ for $x_{\mathbf{b}}, x \in \partial\Omega$ from [31], we have

$$t_{\mathbf{b}}(x, v) \gtrsim \big|n(x) \cdot [x - x_{\mathbf{b}}(x, v)]\big|^{1/2}|v|^{-1}.$$

On the other hand, for $(x, v) \in \gamma_-^\delta$ and $\varepsilon \ll 1$

$$|n(x)\cdot(x_{\mathbf{b}} - x)| = \big|n(x)\cdot[t_{\mathbf{b}} v + O(\varepsilon^2)(t_{\mathbf{b}})^2]\big| = t_{\mathbf{b}}|n(x)\cdot v| + O(\varepsilon^2)(t_{\mathbf{b}})^2 \gtrsim t_{\mathbf{b}}|n(x)\cdot v|.$$

Therefore we prove our claim. The rest of the proof of (2.30) is identical to the proof of Lemma 23 on [31]. □

Now we are ready to prove the main result of this section:

Proof of Proposition 2.6 Define, for $w(v) = e^{\beta'|v|^2}$,

$$h(t, x, v) := w(v) f(t, x, v). \tag{2.32}$$

From Lemma 3 of [31], there exists $\tilde{\beta} = \tilde{\beta}(\beta, \beta') > 0$ such that $\mathbf{k}_\beta(v, u)\frac{w(v)}{w(u)} \lesssim \mathbf{k}_{\tilde{\beta}}(v, u)$.

Then, from (2.25),

$$\begin{aligned} &\big[v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v + \varepsilon^{-1} C_0\langle v\rangle + \frac{\varepsilon^2 \Phi \cdot \nabla_v w}{w}\big]|\varepsilon^{\frac12} h| \\ &\quad \le \varepsilon^{-1} \int_{\mathbb{R}^3} \mathbf{k}_{\tilde{\beta}}(v, u)|\varepsilon^{\frac12} h(u)| \mathrm{d}u + \varepsilon^{\frac12}|wg|. \end{aligned} \tag{2.33}$$

Clearly $\varepsilon^{-1} C_0\langle v\rangle + \frac{\varepsilon^2 \Phi\cdot\nabla_v w}{w} \sim \varepsilon^{-1} C_0\langle v\rangle$.

From (2.25), on $(x, v) \in \gamma_-$,

$$\begin{aligned} \varepsilon^{\frac12}|h(x, v)| &\le \sqrt{2\pi}\, w(v)\sqrt{\mu(v)} \int_{n(x)\cdot u>0} \varepsilon^{\frac12}|h(x, u)|\frac{\sqrt{\mu(u)}}{w(u)}\{n(x)\cdot u\}\mathrm{d}u \\ &\quad + \varepsilon^{\frac12} w(v)|r(x, v)|. \\ &\lesssim \frac{1}{\tilde{w}(v)} \int_{n(x)\cdot u>0} \varepsilon^{\frac12}|h(x, u)|\tilde{w}(u)\mathrm{d}\sigma + \varepsilon^{\frac12} w(v)|r(x, v)|, \end{aligned} \tag{2.34}$$

where we define

$$\tilde{w}(v) := \frac{1}{w(v)\sqrt{\mu(v)}}. \tag{2.35}$$

We claim, for $t = T_0$ defined as in Lemma 2.8 (not depending on ε),

$$
\begin{aligned}
|\varepsilon^{\frac{1}{2}} h(x, v)| \le\ & \Big[C T_0^{5/4} \Big\{\frac{4}{5}\Big\}^{C_2 T_0^{5/4}} + o(1) C_{T_0} \Big] \|\varepsilon^{\frac{1}{2}} h\|_\infty + C_{T_0} \varepsilon^{\frac{1}{2}} \|wr\|_\infty \\
& + C_{T_0} \varepsilon^{\frac{3}{2}} \|\langle v\rangle^{-1} wg\|_\infty \\
& + C_{T_0} \Big[\|\mathbf{P} f\|_{L^6(\Omega)} + \frac{1}{\varepsilon} \|(\mathbf{I}-\mathbf{P}) f\|_{L^2(\Omega\times\mathbb{R}^3)} \Big],
\end{aligned}
\tag{2.36}
$$

where $o(1)$ denotes any number that can be chosen arbitrarily small independently of ε.

Once (2.36) holds, Proposition 2.6 is a direct consequence.

We first prove (2.36). From (2.33), for $t_1(t, x, v) < s \le t$,

$$
\begin{aligned}
& \frac{d}{ds}\Big[e^{-\int_s^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau} \varepsilon h(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) \Big] \\
& \le e^{-\int_s^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau} \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \mathbf{k}_{\tilde\beta}(V_{\mathbf{cl}}(s;t,x,v), v') |\varepsilon h(X_{\mathbf{cl}}(s;t,x,v), v')| \mathrm{d}v' \\
& \quad + e^{-\int_s^t \frac{C_0}{\varepsilon}\langle V(\tau;t,x,v)\rangle \mathrm{d}\tau} |\varepsilon wg(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))|.
\end{aligned}
$$

Along the stochastic cycles, for $k = C_1 T_0^{5/4}$, following the arguments presented in Section 4 of [17], which we do not repeat here for brevity, we deduce the following estimate:

$$
\begin{aligned}
& |\varepsilon^{\frac{1}{2}} h(x, v)| \\
& \le \mathbf{1}_{\{t_1(t,x,v)<0\}} e^{-\int_0^t \frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau} |\varepsilon^{\frac{1}{2}} h(X_{\mathbf{cl}}(0;t,x,v), V_{\mathbf{cl}}(0;t,x,v))| && (2.37) \\
& \quad + \int_{\max\{0,t_1(t,x,v)\}}^t \mathrm{d}s\, \frac{e^{-\int_s^t \frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau}}{\varepsilon} \\
& \quad \times \int_{\mathbb{R}^3} \mathrm{d}v'\, \mathbf{k}_{\tilde\beta}(V_{\mathbf{cl}}(s;t,x,v), v') \Big|\varepsilon^{\frac{1}{2}} h(X_{\mathbf{cl}}(s;t,x,v), v')\Big| && (2.38) \\
& \quad + \int_{\max\{0,t_1(t,x,v)\}}^t \mathrm{d}s\, \frac{e^{-\int_s^t \frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau}}{\varepsilon} |\varepsilon^{\frac{3}{2}} wg(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))| && (2.39) \\
& \quad + \mathbf{1}_{\{t_1(t,x,v)\ge 0\}} e^{-\int_{t_1(t,x,v)}^t \frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau} |\varepsilon^{\frac{1}{2}} wr(x_1(x,v), v_1(x,v))| \\
& \quad + \mathbf{1}_{\{t_1(t,x,v)\ge 0\}} \frac{e^{-\int_{t_1(t,x,v)}^t \frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon} \mathrm{d}\tau}}{\tilde w(v_1)} \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} H, && (2.40)
\end{aligned}
$$

where $X_{\mathbf{cl}}(s) := X_{\mathbf{cl}}(s;t,x,v)$ and $V_{\mathbf{cl}}(s) := V_{\mathbf{cl}}(s) := V_{\mathbf{cl}}(s;t,x,v)$, and H is given by

$$\sum_{l=1}^{k-1} \mathbf{1}_{\tilde{t}_{l+1}\le 0<\tilde{t}_l} \big|\varepsilon^{\frac{1}{2}} h(0, X_{\mathbf{cl}}(\tilde{t}_l - \frac{\tilde{t}_l}{\varepsilon}), V_{\mathbf{cl}}(\tilde{t}_l - \frac{\tilde{t}_l}{\varepsilon}))\big|$$

$$\times \Pi_{m=1}^{l-1} \frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1} - \frac{\tilde{t}_{m+1}}{\varepsilon}; v_m))} \mathrm{d}\Sigma_l(0) \tag{2.41}$$

$$+ \sum_{l=1}^{k-1} \int_{\max\{0,\tilde{t}_{l+1}\}}^{\tilde{t}_l} \mathrm{d}\tau \mathbf{1}_{\tilde{t}_l>0} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \mathbf{k}_{\tilde{\beta}}(V_{\mathbf{cl}}(\tilde{t}_l - \frac{\tilde{t}_l - \tau}{\varepsilon}), u)$$

$$\times \big|\varepsilon^{\frac{1}{2}} h(\tau, X_{\mathbf{cl}}(\tilde{t}_l - \frac{\tilde{t}_l - \tau}{\varepsilon}), u)\big| \Pi_{m=1}^{l-1} \frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1} - \frac{\tilde{t}_{m+1}}{\varepsilon}; v_m))} \mathrm{d}u \mathrm{d}\Sigma_l(\tau) \tag{2.42}$$

$$+ \sum_{l=1}^{k-1} \int_{\max\{0,\tilde{t}_{l+1}\}}^{\tilde{t}_l} \mathbf{1}_{\tilde{t}_l>0} \Pi_{m=1}^{l-1} \frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1} - \frac{\tilde{t}_{m+1}}{\varepsilon}; v_m))}$$

$$\times \varepsilon^{-\frac{1}{2}} \big|wg(\tau, X_{\mathbf{cl}}(\tilde{t}_l - \frac{\tilde{t}_l - \tau}{\varepsilon}), V_{\mathbf{cl}}(\tilde{t}_l - \frac{\tilde{t}_l - \tau}{\varepsilon}))\big| \mathrm{d}\Sigma_l(\tau) \mathrm{d}\tau \tag{2.43}$$

$$+ \sum_{l=1}^{k-1} \mathbf{1}_{\tilde{t}_l>0} \varepsilon^{\frac{1}{2}} w(v_l) |r(\tilde{t}_l, x_{l+1}, v_l)| \Pi_{m=1}^{l-1} \frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1} - \frac{\tilde{t}_{m+1}}{\varepsilon}; v_m))} \mathrm{d}\Sigma_l(\tilde{t}_{l+1}) \tag{2.44}$$

$$+ \mathbf{1}_{\tilde{t}_k>0} |\varepsilon^{\frac{1}{2}} h(\tilde{t}_k, x_k, v_{k-1})| \Pi_{m=1}^{k-2} \frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1} - \frac{\tilde{t}_{m+1}}{\varepsilon}; v_m))} \mathrm{d}\Sigma_{k-1}(\tilde{t}_k), \tag{2.45}$$

where $V_{\mathbf{cl}}(t_{m+1}; v_m) := V_{\mathbf{cl}}(t_{m+1}; t_m, x_m, v_m)$ and $\mathrm{d}\Sigma_{k-1}(t_k)$ is evaluated at $s = t_k$ of

$$\mathrm{d}\Sigma_l(s) := \left\{ \prod_{j=l+1}^{k-1} \mathrm{d}\sigma_j \right\} \left\{ e^{-\int_s^{t_l} \frac{C_0 \langle V_{\mathbf{cl}}(\tau; t_l, x_l, v_l)\rangle}{\varepsilon} \mathrm{d}\tau} w(v_l) \mathrm{d}\sigma_l \right\}$$

$$\times \prod_{j=1}^{l-1} \left\{ e^{-\int_{t_{j+1}}^{t_j} \frac{C_0 \langle V_{\mathbf{cl}}(\tau; t_j, x_j, v_j)\rangle}{\varepsilon} \mathrm{d}\tau} \mathrm{d}\sigma_j \right\}. \tag{2.46}$$

Note that if $t_{\mathbf{b}}(x, v) = \infty$ only (2.37), (2.38), (2.39) remain. Then it is bounded directly by (2.49).

Since

$$|V_{\mathbf{cl}}(t_{m+1}; t_m, x_m, v_m) - v_m| \le \varepsilon^2 |t_{m+1} - t_m| \|\Phi\|_\infty \lesssim \varepsilon^2 T_0,$$

then for $\varepsilon^2 T_0 \le 1$, we have

$$\frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(t_{m+1}; t_m, x_m, v_m))} \le 1 + O(\varepsilon). \tag{2.47}$$

Directly, from our choice $k = C_1 T_0^{5/4}$

$$(2.37) + (2.41) \lesssim_{T_0} e^{-\frac{C_0}{\varepsilon} t} \|\varepsilon^{\frac{1}{2}} h\|_\infty, \quad (2.40) + (2.44) \lesssim_{T_0} \|\varepsilon^{\frac{1}{2}} wr\|_\infty,$$

and

$$
\begin{aligned}
&(2.39) + (2.43)\\
&\lesssim_{T_0} \|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1} wg\|_\infty \times \left\{ \int_0^t \frac{\langle V_{\mathbf{cl}}(s;t,x,v)\rangle}{\varepsilon} e^{-\int_s^t \frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon}\mathrm{d}\tau}\mathrm{d}s \right.\\
&\quad \left. + \; C_1 T_0^{5/4} \sup_l \int_0^{t_l} \frac{\langle V_{\mathbf{cl}}(\tau;t_l,x_l,v_l)\rangle}{\varepsilon} e^{-\int_s^{t_l} \frac{C_0\langle V_{\mathbf{cl}}(\tau;t_l,x_l,v_l)\rangle}{\varepsilon}\mathrm{d}\tau}\mathrm{d}\tau \right\}\\
&\lesssim_{T_0} \|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1} wg\|_\infty \times \int_0^t \frac{d}{ds} e^{-\int_s^t \frac{C_0\langle V_{\mathbf{cl}}(\tau;t,x,v)\rangle}{\varepsilon}\mathrm{d}\tau}\mathrm{d}s \;\lesssim_{T_0}\; \|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1} wg\|_\infty,
\end{aligned}
$$

where we have used the fact that $\mathrm{d}\sigma_j$ is a probability measure of $\mathscr{V}_j$.

Now we focus on (2.38) and (2.42). For $N > 1$, we can choose $m = m(N) \gg 1$ such that

$$
\mathbf{k}_m(v,u) := \mathbf{1}_{|v-u|\geq \frac{1}{m}} \mathbf{1}_{|u|\leq m}\mathbf{1}_{|v|\leq m}\mathbf{k}_{\tilde{\beta}}(v,u), \quad \sup_v \int_{\mathbb{R}^3} |\mathbf{k}_m(v,u) - \mathbf{k}_{\tilde{\beta}}(v,u)|\mathrm{d}u \leq \frac{1}{N}. \tag{2.48}
$$

We split $\mathbf{k}_{\tilde{\beta}}(v,u) = [\mathbf{k}_{\tilde{\beta}}(v,u) - \mathbf{k}_m(v,u)] + \mathbf{k}_m(v,u)$, and the first difference would lead to a small contribution in (2.38) and (2.42) as, for $N \gg_{T_0} 1$,

$$
\frac{k}{N}\|\varepsilon^{\frac{1}{2}} h\|_\infty = \frac{C_1 T_0^{5/4}}{N}\|\varepsilon^{\frac{1}{2}} h\|_\infty.
$$

We further split the time integrations in (2.38) and (2.42) as $[t_l - \kappa\varepsilon, t_l]$ and $[\max\{0, t_{l+1}\}, t_l - \kappa\varepsilon]$:

$$
(2.38) = \underbrace{\int_{t-\kappa\varepsilon}^t} + \int_{\max\{0,t_1\}}^{t-\kappa\varepsilon}, \qquad (2.42) = \mathbf{1}_{\{t_1\geq 0\}} \int_{\prod_{j=1}^{k-1}\mathscr{V}_j} \sum_{l=1}^{k-1} \left\{ \underbrace{\int_{t_l-\kappa\varepsilon}^{t_l}} + \int_{\max\{0,t_{l+1}\}}^{t_l-\kappa\varepsilon} \right\}.
$$

The small-in-time contributions of both (2.38) and (2.42), underbraced terms, are bounded by

$$
\begin{aligned}
&\kappa\varepsilon \frac{1}{\varepsilon} \sup_v \int_{|v'|\leq N} \mathbf{k}_m(v,v')\mathrm{d}v' \|\varepsilon^{\frac{1}{2}} h\|_\infty \lesssim \kappa \|\varepsilon^{\frac{1}{2}} h\|_\infty,\\
&C_1 T_0^{5/4} \kappa\varepsilon \frac{1}{\varepsilon} \sup_v \int_{|v'|\leq N} \mathbf{k}_m(v,v')\mathrm{d}v' \|\varepsilon^{\frac{1}{2}} h\|_\infty \lesssim \kappa C_1 T_0^{5/4} \|\varepsilon^{\frac{1}{2}} h\|_\infty.
\end{aligned}
$$

For (2.45), by Lemma 2.8 and (2.47),

$$
\begin{aligned}
(2.45) &\lesssim \{1 + O(\varepsilon)\}^{C_1 T_0^{5/4}} \quad \|\varepsilon^{\frac{1}{2}} h\|_\infty\\
&\quad \times \sup_{(t,x,v)\in[0,T_0]\times\bar{\Omega}\times\mathbb{R}^3} \int_{\prod_{j=1}^{k-1}\mathscr{V}_j} \mathbf{1}_{t_k(t,x,v,v_1,v_2,\cdots,v_{k-1})>0} \Pi_{j=1}^{k-1}\mathrm{d}\sigma_j\\
&\quad \lesssim \left\{\frac{4}{5}\right\}^{C_2 T_0^{5/4}} \|\varepsilon h\|_\infty.
\end{aligned}
$$

Overall, for $(t, x, v) \in [0, T_0] \times \bar{\Omega} \times \mathbb{R}^3$,

$$
\begin{aligned}
&|\varepsilon^{\frac{1}{2}} h(x, v)| \\
&\lesssim \int_{\max\{0, t_1(t,x,v)\}}^{t-\kappa\varepsilon} \frac{e^{-\frac{C_0}{\varepsilon}(t-s)}}{\varepsilon} \int_{|v'|\le m} \underbrace{\left|\varepsilon^{\frac{1}{2}} h(X_{\mathbf{cl}}(s; t, x, v), v')\right|} \mathrm{d}v' \mathrm{d}s \\
&\quad + \mathbf{1}_{\{t_1 \ge 0\}} \frac{e^{-\frac{C_0}{\varepsilon}(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \sum_{\ell=1}^{k-1} \int_{\max\{0, t_{\ell+1}\}}^{t_\ell - \kappa\varepsilon} \frac{\mathbf{1}_{t_\ell > 0}}{\varepsilon} \\
&\quad \times \int_{|v''|\le m} \underbrace{\left|\varepsilon^{\frac{1}{2}} h(X_{\mathbf{cl}}(\tau; t_\ell, x_\ell, v_\ell), v'')\right|} \mathrm{d}v'' \mathrm{d}\Sigma_\ell(\tau) \mathrm{d}\tau \\
&\quad + C T_0^{5/4} \left\{ e^{-\frac{C_0}{\varepsilon} t} \|\varepsilon^{\frac{1}{2}} h\|_\infty + C_{T_0} \|\varepsilon^{\frac{1}{2}} w r(s)\|_\infty + C_{T_0} \|\varepsilon^{\frac{3}{2}} \langle v\rangle^{-1} w g\|_\infty \right\} \\
&\quad + o(1) C T_0^{5/4} \|\varepsilon^{\frac{1}{2}} h\|_\infty + \left\{\frac{1}{2}\right\}^{C_2 T_0^{5/4}} \|\varepsilon^{\frac{1}{2}} h\|_\infty.
\end{aligned} \tag{2.49}
$$

Note that the same estimate holds for the underbraced terms in (2.49). We plug these estimates into the underbraced terms of (2.49) to have a bound as

$$
|\varepsilon^{\frac{1}{2}} h^{\ell+1}(t, x, v)| \le \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.
$$

Here, using $w(u) \lesssim_m 1$ for $|u| \le m$,

$$
\begin{aligned}
\mathbf{I}_1 \lesssim_m &\int_{\max\{0, t_1\}}^{t-\kappa\varepsilon} \mathrm{d}s \, \frac{e^{-\frac{C_0}{\varepsilon}(t-s)}}{\varepsilon} \int_{|v'|\le m} \mathrm{d}v' \int_{\max\{0, t_1'\}}^{s-\kappa\varepsilon} \mathrm{d}s' \frac{e^{-\frac{C_0(s-s')}{\varepsilon}}}{\varepsilon} \\
&\times \int_{|u|\le m} \mathrm{d}u \left|\varepsilon^{\frac{1}{2}} h(X_{\mathbf{cl}}(s'; s, X_{\mathbf{cl}}(s; t, x, v), v'), u)\right| \\
&+ \int_{\max\{0, t_1\}}^{t-\kappa\varepsilon} \mathrm{d}s \, \frac{e^{-\frac{C_0}{\varepsilon}(t-s)}}{\varepsilon} \int_{|v'|\le m} \mathrm{d}v' \, \mathbf{1}_{\{t_1' \ge 0\}} \frac{e^{-\frac{C_0}{\varepsilon}(s-t_1')}}{\tilde{w}(v)} \\
&\times \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j'} \sum_{\ell'=1}^{k-1} \int_{\max\{0, t_{\ell'+1}'\}}^{t_{\ell'}'-\kappa\varepsilon} \mathbf{1}_{t_{\ell'}' > 0} \frac{1}{\varepsilon} \left|\varepsilon^{\frac{1}{2}} h(\tau, X_{\mathbf{cl}}(\tau; t_{\ell'}', x_{\ell'}', v_{\ell'}'), u)\right| \mathrm{d}u \mathrm{d}\Sigma_{\ell'}(\tau) \mathrm{d}\tau,
\end{aligned}
$$

where $t_{\ell'} := \tilde{t}_{\ell'}(s, X_{\mathbf{cl}}(s; t, x, v), v')$, $x_{\ell'}' := x_{\ell'}(X_{\mathbf{cl}}(s; t, x, v), v')$, $v_{\ell'}' := v_{\ell'}(X_{\mathbf{cl}}(s; t, x, v), v')$. Moreover

$$
\begin{aligned}
\mathbf{I}_2 \lesssim_m \; &\mathbf{1}_{\{t_1 \ge 0\}} \frac{e^{-\frac{C_0}{\varepsilon}(t-t_1)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1} \mathcal{V}_j} \sum_{\ell=1}^{k-1} \int_{\max\{0, t_{\ell+1}\}}^{t_\ell - \kappa\varepsilon} \mathrm{d}\Sigma_\ell(\tau) \mathrm{d}\tau \, \mathbf{1}_{t_\ell > 0} \frac{1}{\varepsilon} \int_{|v''|\le m} \mathrm{d}v'' \\
&\times \int_{\max\{0, t_1''\}}^{\tau-\kappa\varepsilon} \mathrm{d}s'' \, \frac{e^{-\frac{C_0}{\varepsilon^2}(\tau-s'')}}{\varepsilon} \int_{|u|\le m} \mathrm{d}u \; \left|\varepsilon^{\frac{1}{2}} h\big(X_{\mathbf{cl}}(s''; \tau, X_{\mathbf{cl}}(\tau; t_\ell, x_\ell, v_\ell), v''), u\big)\right|
\end{aligned}
$$

$$+\mathbf{1}_{\{t_1\geq 0\}}\frac{e^{-\frac{C_0}{\varepsilon}(t-t_1)}}{\tilde{w}(v)}\int_{\Pi_{j=1}^{k-1}\mathcal{V}_j}\sum_{\ell=1}^{k-1}\int_{\max\{0,t_{\ell+1}\}}^{t_\ell-\kappa\varepsilon}\mathrm{d}\Sigma_\ell(\tau)\mathrm{d}\tau\ \mathbf{1}_{t_\ell>0}\frac{1}{\varepsilon}\int_{|v''|\leq m}\mathrm{d}v''$$
$$\times\mathbf{1}_{t_1''\geq 0}\frac{e^{-\frac{C_0}{\varepsilon}(\tau-t_1'')}}{\tilde{w}(v'')}\int_{\prod_{j=1}^{k-1}\mathcal{V}_j''}\sum_{\ell''=1}^{k-1}\int_{\max\{0,t_{\ell''+1}''\}}^{t_{\ell''}''-\kappa\varepsilon}\mathbf{1}_{t_{\ell''}''>0}\frac{1}{\varepsilon}$$
$$\times\int_{|u|\leq m}\left|\varepsilon^{\frac{1}{2}}h\big(\tau'',X_{\mathbf{cl}}(\tau'';t_{\ell''}'',x_{\ell''}'',v_{\ell''}''),u\big)\right|\mathrm{d}u\mathrm{d}\Sigma_{\ell''}''(\tau'')\mathrm{d}\tau'',$$

where $t_{\ell''}'':=t_{\ell''}(\tau,X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v'')$, $x_{\ell''}'':=x_{\ell''}(X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v'')$, $v_{\ell''}'':=v_{\ell''}(X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v'')$. Furthermore

$$\mathbf{I}_3\lesssim CT_0^{5/2}\left\{e^{-\frac{C_0}{\varepsilon}t}\|\varepsilon^{\frac{1}{2}}h\|_\infty+C_{T_0}\|\varepsilon^{\frac{1}{2}}wr\|_\infty+C_{T_0}\|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1}wg\|_\infty\right\}$$
$$+o(1)CT_0^{5/2}\|\varepsilon^{\frac{1}{2}}h\|_\infty+T_0^{5/4}\left\{\frac{4}{5}\right\}^{C_2T_0^{5/4}}\|\varepsilon^{\frac{1}{2}}h\|_\infty.$$

This bound of $\mathbf{I}_3$ is already included in the RHS of (2.36).

Now we focus on $\mathbf{I}_1$ and $\mathbf{I}_2$. Consider the change of variables

$$v'\ \mapsto\ y:=X(s';s,X_{\mathbf{cl}}(s;t,x,v),v'). \tag{2.50}$$

By a direct computation and (2.14), for $\max\{0,t_1'\}\leq s'\leq s-\kappa\varepsilon\leq T_0$,

$$\begin{aligned}\frac{\partial X_i(s';s)}{\partial v_j'}&=-(s-s')\delta_{ij}+\int_s^{s'}\mathrm{d}\tau'\int_s^{\tau'}\mathrm{d}\tau''\varepsilon^2\sum_m\partial_m\Phi_i(X(\tau'';s))\frac{\partial X_m}{\partial v_j'}(\tau'';s)\\&=-(s-s')\big[\delta_{ij}+O(\varepsilon^2)\|\Phi\|_{C^1}T_0^2e^{C_\Phi T_0}\big].\end{aligned}$$

By the lower bound of $|s-s'|\geq\kappa\varepsilon$,

$$\det\nabla_{v'}X(s';s)=|s-s'|^3\det\big(\delta_{ij}+O(\varepsilon^2)\|\Phi\|_{C^1}T_0^2e^{C_\Phi T_0}\big)\gtrsim\kappa^3\varepsilon^3.$$

Now integrating over time first

$$\int_{\max\{0,t_1\}}^{t-\kappa\varepsilon}\mathrm{d}s\,\frac{e^{-\frac{C_0}{\varepsilon}(t-s)}}{\varepsilon}\int_{|v'|\leq m}\mathrm{d}v'\int_{\max\{0,t_1'\}}^{s-\kappa\varepsilon}\mathrm{d}s'\frac{e^{-\frac{C_0(s-s')}{\varepsilon}}}{\varepsilon}$$
$$\times\int_{|u|\leq m}\mathrm{d}u\left|\varepsilon^{1/2}h(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u)\right|$$
$$\lesssim\sup_{0\leq s'\leq s-\kappa\varepsilon\leq s\leq t-\kappa\varepsilon}\int_{|v'|\leq m}\mathrm{d}v'\int_{|u|\leq m}\mathrm{d}u\ |\varepsilon^{1/2}h(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u)|.$$

From $|h(u)|=w(u)|f(u)|\lesssim_m|f(u)|$ for $|u|\leq m$

$$\lesssim\sup_{0\leq s'\leq s-\kappa\varepsilon\leq s\leq t-\kappa\varepsilon}\varepsilon^{\frac{1}{2}}\int_{|v'|\leq m}\int_{|u|\leq m}|f(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u)|\mathrm{d}u\mathrm{d}v'$$

$$\lesssim \sup_{0\le s'\le s-\kappa\varepsilon\le s\le t-\kappa\varepsilon} \varepsilon^{\frac{1}{2}} \int_{|v'|\le m}\int_{|u|\le m} \mathbf{P}f(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'))|\langle u\rangle^2\sqrt{\mu(u)}\mathrm{d}u\mathrm{d}v'$$
$$+\sup_{0\le s'\le s-\kappa\varepsilon\le s\le t-\kappa\varepsilon} \varepsilon^{\frac{1}{2}} \int_{|v'|\le m}\int_{|u|\le m} |(\mathbf{I}-\mathbf{P})f(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'))|\mathrm{d}u\mathrm{d}v'.$$

For $\mathbf{P}f-$contribution,

$$\begin{aligned}
&\varepsilon^{\frac{1}{2}}\int_{v'}\int_u \left|\mathbf{P}f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v')\big)\langle u\rangle^2\sqrt{\mu(u)}\right|\mathrm{d}u\mathrm{d}v'\\
&\lesssim_m \varepsilon^{\frac{1}{2}}\Big[\int_{v'}\left|\mathbf{P}f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v')\big)\right|^6\mathrm{d}v'\Big]^{1/6}\\
&\lesssim_m \varepsilon^{\frac{1}{2}}\Big[\int_{\Omega\times\mathbb{R}^3}\left|\mathbf{P}f\big(y,v'\big)\right|^6\frac{1}{\kappa^3\varepsilon^3}\mathrm{d}y\mathrm{d}v'\Big]^{1/6}\\
&\lesssim_m \|\mathbf{P}f\|_{L^6(\Omega\times\mathbb{R}^3)}.
\end{aligned}$$

For $(\mathbf{I}-\mathbf{P})f$ contribution,

$$\begin{aligned}
&\varepsilon^{\frac{1}{2}}\int_{v'}\int_u |(\mathbf{I}-\mathbf{P})f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u\big)|\mathrm{d}u\mathrm{d}v'\\
&\lesssim_m \varepsilon^{\frac{1}{2}}\Big[\iint\left|(\mathbf{I}-\mathbf{P})f\big(X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),v'),u\big)\right|^2\mathrm{d}v'\mathrm{d}u\Big]^{1/2}\\
&\lesssim_m \varepsilon^{\frac{1}{2}}\Big[\iint_{\Omega\times\mathbb{R}^3}\left|(\mathbf{I}-\mathbf{P})f(y,u)\right|^2\frac{1}{\kappa^3\varepsilon^3}\mathrm{d}y\mathrm{d}u\Big]^{1/2}\\
&\lesssim_m \frac{1}{\varepsilon}\|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)}.
\end{aligned}$$

We have the similar change of variables for $v'_{\ell'}\mapsto X_{\mathbf{cl}}(\tau;t'_{\ell'},x'_{\ell'},v'_{\ell'})$, and $v''_{\ell''}\mapsto X_{\mathbf{cl}}(\tau'';t''_{\ell''},x''_{\ell''},v''_{\ell''})$, and $v''\mapsto X_{\mathbf{cl}}(s'';\tau,X_{\mathbf{cl}}(\tau;t_\ell,x_\ell,v_\ell),v'')$.

Following the same proof, we conclude

$$\mathbf{I}_1+\mathbf{I}_2 \lesssim T_0^{5/2}\big(\|\mathbf{P}f\|_{L^6(\Omega\times\mathbb{R}^3)}+\frac{1}{\varepsilon}\|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)}\big). \tag{2.51}$$

All together we prove our claims (2.36). □

2.3 Steady L^2-Coercivity and L^6 Bound

The main purpose of this section is to prove the following:

Proposition 2.9 *Suppose all the assumptions of Theorem 2.5 hold. Then, for sufficiently small $\varepsilon>0$, there exists a unique solution to (2.22). Moreover,*

$$\begin{aligned}
&\|\mathbf{P}f\|_2+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu+\varepsilon^{-1/2}|(1-P_\gamma)f|_{2,+}+|f|_2\\
&\quad\lesssim \|\nu^{-\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_2+\varepsilon^{-1}\|\mathbf{P}g\|_2+\varepsilon^{-1/2}|r|_{2,-}.
\end{aligned} \tag{2.52}$$

Furthermore

$$\begin{aligned}\|\mathbf{P}f\|_6 \lesssim & \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f|_{2,+} + |r|_{2,-} \\ & + \|\frac{g}{\sqrt{\nu}}\|_2 + o(1)\varepsilon^{\frac{1}{2}}\|wf\|_\infty \\ & + |\varepsilon^{\frac{1}{2}}wr|_\infty + \|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1}wg\|_\infty.\end{aligned} \tag{2.53}$$

As the first step of the proof of Proposition 2.9, we consider the following penalized problem:

$$\begin{aligned}\mathscr{L}f &:= (\lambda + \varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v)f + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f = g \quad \text{in } \Omega\times\mathbb{R}^3, \\ f &= P_\gamma f + r \quad \text{on } \gamma_-.\end{aligned} \tag{2.54}$$

Lemma 2.10 *Assume that $g \in L^2(\Omega\times\mathbb{R}^3)$ and $r \in L^2(\gamma_-)$ and satisfy (2.21). Moreover, let $\Phi \in L^\infty(\Omega)$ and $\lambda > 0$. Then, if $\varepsilon > 0$ is sufficiently small, the solution to (2.54) exists and is unique. Moreover it satisfies*

$$\varepsilon^{-1}\|f\|_\nu^2 + |(1-P_\gamma)f|_{2,+}^2 \lesssim \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_{2,-}^2. \tag{2.55}$$

We remark that Lemma 2.10 implies that, for ε sufficiently small, the operator $\mathscr{L}^{-1}$ is well-defined and bounded as a map from L^2 to L^2.

Proof Step 1. Denote $\varpi := \lambda + \varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v$. Since $\nu \geq \nu_0\langle v\rangle$, with $\nu_0 > 0$, if Φ satisfies $\frac{1}{2}\varepsilon^2\|\Phi\|_\infty|v| \leq \frac{1}{2}\varepsilon^{-1}\nu$, then we have $\varpi \geq \frac{1}{2}\varepsilon^{-1}\nu_0\langle v\rangle$.

For the existence, we first consider the following problem:

$$\varpi f + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f = g \quad \text{in } \Omega\times\mathbb{R}^3, \quad f\big|_{\gamma_-} = r, \tag{2.56}$$

with a prescribed positive function $\varpi(x,v)$ and prescribed g, r.

From (2.7), for $-t_{\mathbf{b}}(x,v) < t < t_{\mathbf{f}}(x,v)$,

$$\begin{aligned}&\frac{d}{dt}\Big[f(X(t;0,x,v),V(t;0,x,v))e^{\int_0^t \varpi(X(\tau;0,x,v),V(\tau;0,x,v))\mathrm{d}\tau}\Big] \\ &\quad = g(X(t;0,x,v),V(t;0,x,v))e^{\int_0^t \varpi(X(\tau;0,x,v),V(\tau;0,x,v))\mathrm{d}\tau}.\end{aligned}$$

Then, for $(X(t),V(t)) := (X(t;0,x,v),V(t;0,x,v))$ and $\varpi(\tau) := \varpi(X(\tau;0,x,v), V(\tau;0,x,v))$,

$$f(x,v) = r(x_{\mathbf{b}}(x,v),v_{\mathbf{b}}(x,v))e^{-\int_{-t_{\mathbf{b}}(x,v)}^0 \varpi(\tau)} + \int_{-t_{\mathbf{b}}(x,v)}^0 g(X(s),V(s))e^{-\int_s^0 \varpi(\tau)}\mathrm{d}s. \tag{2.57}$$

This proves the existence.

Combining with $\int_{-\infty}^0 \varpi(s) e^{-\int_s^0 \varpi(\tau)\mathrm{d}\tau} \mathrm{d}s = \int_{-\infty}^0 \frac{d}{ds} e^{-\int_s^0 \varpi(\tau)\mathrm{d}\tau} \mathrm{d}s = 1 - e^{-\int_{-\infty}^0 \varpi(\tau)\mathrm{d}\tau} \lesssim 1$ and $\varpi(s) \gtrsim \varepsilon^{-1}\langle V(s)\rangle$,

$$\|f\|_\infty + |f|_\infty \lesssim \|\frac{g}{\nu}\|_\infty + |r|_\infty.$$

Similarly, we can prove

$$\|e^{\beta|v|^2} f\|_\infty + |e^{\beta|v|^2} f|_\infty \lesssim \|e^{\beta|v|^2} \frac{g}{\nu}\|_\infty + |e^{\beta|v|^2} r|_\infty.$$

Step 2. Next we consider the diffuse reflection boundary conditions. This is done by introducing the sequence f^ℓ solving

$$\varpi f^{\ell+1} + v \cdot \nabla_x f^{\ell+1} + \varepsilon^2 \Phi \cdot \nabla_v f^{\ell+1} = g, \quad f_-^{\ell+1} = \vartheta P_\gamma f^\ell + r,$$

with $f^0 = 0$, $\ell \geq 0$ integer and $\vartheta \in [0, 1)$.

By multiplying by $f^{\ell+1}$ and using the Green identity we obtain

$$\frac{1}{2} \iint_{\Omega\times\mathbb{R}^3} \varpi |f^{\ell+1}|^2 + \frac{1}{2}|f^{\ell+1}|_{2,+}^2 = (f^{\ell+1}, g) + \frac{1}{2}\int_{\gamma_-} |\vartheta P_\gamma f^\ell + \frac{1}{2} r|^2.$$

From (2.21) and the definition of P_γ, we have $\int_{\gamma_-} r P_\gamma f^\ell = 0$ and hence

$$\frac{1}{2}\int_{\gamma_-} |\vartheta P_\gamma f^\ell + r|^2 \leq \frac{1}{2}\vartheta^2 |P_\gamma f^\ell|_{2,-}^2 + |r|_{2,-}^2.$$

Therefore, from $\varpi \geq \frac{\nu_0}{\varepsilon}\langle v\rangle$, $|P_\gamma f^\ell|_{2,-} \leq |f^\ell|_{2,+}$ and

$$|(f^{\ell+1}, g)| = \big|\big(\frac{1}{\sqrt{\varepsilon}}\sqrt{\nu} f^{\ell+1}, \sqrt{\varepsilon}\frac{g}{\sqrt{\nu}}\big)\big| \lesssim o(1)\varepsilon^{-1}\|f^{\ell+1}\|_\nu^2 + \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2,$$

we find

$$\frac{1}{8\varepsilon}\|f^{\ell+1}\|_\nu^2 + \frac{1}{2}|f^{\ell+1}|_{2,+}^2 \leq \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + \frac{1}{2}|r|_{2,-}^2 + \frac{1}{2}\vartheta^2|f^\ell|_{2,+}^2. \tag{2.58}$$

By iteration, since $\vartheta < 1$, we conclude that

$$\varepsilon^{-1}\|f^{\ell+1}\|_\nu^2 + |f^{\ell+1}|_{2,+}^2 \lesssim_\vartheta \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_{2,-}^2.$$

Let us look now, for $\ell \geq 1$ at the difference $f^{\ell+1} - f^\ell$. By the Green's identity, we obtain

$$\frac{1}{4\varepsilon}\|f^{\ell+1} - f^\ell\|_\nu^2 + \frac{1}{2}|f^{\ell+1} - f^\ell|_{2,+}^2 \lesssim \frac{1}{2}\vartheta^2|f^\ell - f^{\ell-1}|_{2,+}^2.$$

Again by iteration, we obtain that the sequence $\{f^\ell\}$ is a Cauchy sequence and has a limit f_ϑ depending on ϑ. Moreover, taking the limit $\ell \to \infty$in (2.58), we have

$$\frac{1}{8\varepsilon}\|f_\vartheta\|_\nu^2 + (1-\vartheta^2)|f_\vartheta|_{2,+}^2 \lesssim \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_2^2,$$

where we used the trace theorem, Lemma 2.3, for the boundary integration. Then we see that f_ϑ satisfies the uniform-in-ϑbounds $\frac{1}{8\varepsilon}\|f_\vartheta\|_\nu^2 \lesssim \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_2^2$.

Thus we can take the weak L^2-limit as $\vartheta \to 1$ to obtain the existence of the solution $f$to the first line of (2.54). For the boundary condition we use Lemma 2.3 to show the second line of (2.54).

Then the difference $f_\vartheta - f$satisfies the bound

$$\frac{1}{8\varepsilon}\|f - f_\vartheta\|_\nu^2 + \frac{1}{2}|f - f_\vartheta|_{2,+}^2 \le (1-\vartheta)|f|_{2,+}^2 \to 0 \quad \text{as } \vartheta \to 1.$$

Hence the convergence is strong.
Step 3. We can prove (2.55) by applying the Green's identity to (2.54): We establish an important positivity property of $\mathscr{L}$. Using Lemma 2.4 and the boundary condition for f, we get

$$(f, \mathscr{L}f) = \iint_{\Omega\times\mathbb{R}^3} (\lambda + \varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v)f^2 + \frac{1}{2}\int_{\gamma_+} f^2 = (f, g) + \frac{1}{2}\int_{\gamma_-}(P_\gamma f + r)^2.$$

Following *Step 2*, we derive

$$\iint_{\Omega\times\mathbb{R}^3} (\lambda + \varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v)f^2 + \frac{1}{2}\int_{\gamma_+} |(1-P_\gamma)f|^2 \lesssim o(1)\varepsilon^{-1}\|f\|_\nu^2 + \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_{2,-}^2. \tag{2.59}$$

If $\varepsilon \ll 1$ then $\frac{1}{2}\varepsilon^2\|\Phi\|_\infty \le \frac{\nu_0}{4}$, and

$$\lambda\|f\|_2^2 + \frac{\varepsilon^{-1}}{2}\|f\|_\nu^2 + \frac{1}{2}|(1-P_\gamma)f|_2^2 \lesssim \varepsilon\|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_{2,-}^2. \tag{2.60}$$

The inequality (2.55) follows immediately from (2.60). The uniqueness follows from (2.55) because, if there are two solutions, their difference satisfies (2.54) with $g = 0$ and $r = 0$. Hence it must vanish. □

Lemma 2.11 *For any $\lambda, \varepsilon > 0$, the operator $K\mathscr{L}^{-1}$ is compact in L^2. Explicitly, if $g^n \in L^2$ and $\sup_n \|g^n\|_2 < \infty$ then there exists a subsequence n_k such that $Kf^{n_k} \to Kf$ in L^2, where f^n solve*

$$\lambda f^n + v\cdot\nabla_x f^n + \frac{1}{\varepsilon}\nu f^n + \varepsilon^2\Phi\cdot\nabla_v f^n - \frac{1}{2}\varepsilon^2\Phi\cdot v f^n = g^n, \quad f^n|_{\gamma_-} = P_\gamma f^n + r.$$

This lemma is proved in Appendix A.2.

Next we prove the essential bound for $\mathbf{P}f$, where, for $\mathfrak{r} \in [0,1]$, f solves

$$\Big[\lambda + (1-\mathfrak{r})\varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v)\Big] f + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f + \varepsilon^{-1}\mathfrak{r}Lf = g, \quad \text{in } \Omega\times\mathbb{R}^3$$
$$f_- = P_\gamma f + r, \ \text{ on } \gamma_-. \tag{2.61}$$

Note that the total mass of f in general does not vanish. But since this condition is essential for our proof, we first establish, in Lemma 2.12, the estimates for $\mathring{f}$ defined as

$$\mathring{f} \ := \ f - <f> \sqrt{\mu}, \quad <f> := \ \Big(\iint_{\Omega\times\mathbb{R}^3} f\sqrt{\mu}\mathrm{d}x\mathrm{d}v\Big)\Big/\Big(\iint_{\Omega\times\mathbb{R}^3}\mu\mathrm{d}x\mathrm{d}v\Big). \tag{2.62}$$

Lemma 2.12 *Assume* (2.21). *Let f be a solution to* (2.61) *in the sense of distribution. Then, for all $\lambda \geq 0$ sufficiently small and all $\mathfrak{r}\in[0,1]$ sufficiently close to 1,*

$$\|\mathbf{P}\mathring{f}\|_2^2 \lesssim \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_\nu^2 + |(1-P_\gamma)f|_{2,+}^2 + |r|_{2,-}^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + \varepsilon^2\|\Phi\|_\infty| <f> |^2, \tag{2.63}$$

and

$$\lambda| <f> | \lesssim (1-\mathfrak{r})\varepsilon^{-1}\|f\|_2. \tag{2.64}$$

Moreover, for $0<\eta\ll 1$

$$\begin{aligned}\|\mathbf{P}\mathring{f}\|_6 \lesssim\ & \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f|_{2,+} + |r|_{2,-} + \|\frac{g}{\sqrt{\nu}}\|_2 \\ & + |\varepsilon^{\frac{1}{2}}wr|_\infty + \|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1}wg\|_\infty + \eta\{| <f> | + \varepsilon^{\frac{1}{2}}\|wf\|_\infty\},\end{aligned} \tag{2.65}$$

and, in particular, for $\lambda = 0$ and $\mathfrak{r}=1$, (2.53) *is verified.*

Proof Set $\varpi_{\mathfrak{r}} = \lambda + (1-\mathfrak{r})\varepsilon^{-1}\nu - \frac{1}{2}\varepsilon^2\Phi\cdot v$. By the Green's identity (2.19) and (2.61),

$$\begin{aligned}&\iint_{\Omega\times\mathbb{R}^3} \varpi_{\mathfrak{r}} f\psi - v\cdot\nabla_x\psi f - \varepsilon^2 f\Phi\cdot\nabla_v\psi + \int_{\gamma_+}\psi f - \int_{\gamma_-}\psi f \\ &= -\varepsilon^{-1}\mathfrak{r}\iint_{\Omega\times\mathbb{R}^3}\psi L(\mathbf{I}-\mathbf{P})f + \iint_{\Omega\times\mathbb{R}^3}\psi g.\end{aligned} \tag{2.66}$$

Step 1. Proof of (2.64). From (2.66) with $\psi = \sqrt{\mu}$,

$$<f> \Big[\lambda + (1-\mathfrak{r})\varepsilon^{-1}\iint_{\Omega\times\mathbb{R}^3}\nu\sqrt{\mu}\Big] + (1-\mathfrak{r})\varepsilon^{-1}\iint_{\Omega\times\mathbb{R}^3}\nu\mathring{f}\sqrt{\mu} = 0, \tag{2.67}$$

where we have used (2.21), $\iint_{\Omega\times\mathbb{R}^3}\mathring{f}\sqrt{\mu}=0$, and

$$\int_{\mathbb{R}^3}\sqrt{\mu}\left[\Phi\cdot\nabla_v f-\frac{1}{2}\Phi\cdot vf\right]\mathrm{d}v=\int_{\mathbb{R}^3}\sqrt{\mu}\Phi\cdot\frac{\nabla_v(\sqrt{\mu}f)}{\sqrt{\mu}}\mathrm{d}v=0,$$
$$\int_{\gamma_-}P_\gamma f\sqrt{\mu}\mathrm{d}\gamma-\int_{\gamma_+}f\sqrt{\mu}\mathrm{d}\gamma=0.$$

Clearly, $\iint\nu\mathring{f}\sqrt{\mu}\lesssim\|\mathring{f}\|_2\le\|f\|_2$ and these prove (2.64).

Now we prove (2.63). Denote $\mathring{a}=:a-<f>$ so that

$$\mathbf{P}\mathring{f}=\left\{\mathring{a}+v\cdot b+c\left[\frac{|v|^2}{2}-\frac{3}{2}\right]\right\}\sqrt{\mu}.$$

Step 2. Estimate of c. We claim that, for sufficiently small $\varepsilon>0$,

$$\|c\|_2\lesssim o(1)\|\mathbf{P}f\|_2+|(1-P_\gamma)f|_{2,+}+\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_2+\|\frac{g}{\sqrt{\nu}}\|_2+|r|_{2,-},\tag{2.68}$$

$$\begin{aligned}\|c\|_6\lesssim\ & o(1)\{\|\mathbf{P}f\|_6+\varepsilon^{1/2}\|wf\|_\infty\}+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu+\|(\mathbf{I}-\mathbf{P})f\|_6\\ &+\varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f|_{2,+}+\|\frac{g}{\sqrt{\nu}}\|_2+|r|_{2,-}+|\varepsilon^{\frac{1}{2}}wr|_\infty+\|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1}wg\|_\infty.\end{aligned}\tag{2.69}$$

For $k=2,6$ we choose the test functions

$$\psi=\psi_{c,k}\equiv(|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\varphi_{c,k}(x),\ \text{where}\ -\Delta_x\varphi_{c,k}(x)=c^{k-1}(x),\ \varphi_{c,k}|_{\partial\Omega}=0,\tag{2.70}$$

and β_c is a constant to be determined.

From the standard elliptic estimate, we have

$$\|\varphi_{c,2}\|_{H^2}\lesssim\|c\|_2.\tag{2.71}$$

With the choice (2.70), the right hand side of (2.66) is bounded by

$$\text{r.h.s.}(2.66)\lesssim\|c\|_2\big\{\varepsilon^{-1}\mathfrak{r}\|(\mathbf{I}-\mathbf{P})f\|_2+\|\frac{g}{\sqrt{\nu}}\|_2\big\}.\tag{2.72}$$

For $k=6$ we use the Sobolev–Gagliardo–Nirenberg inequality: for $1\le p\le N$ and a bounded C^1 domain $\Omega\subset\mathbb{R}^N$, and $u\in W^{1,p}(\Omega)$,

$$\left(\int_{\Omega} |u|^{p^*}\right)^{\frac{1}{p^*}} \leq C(N, p, \Omega)\|u\|_{W^{1,p}(\Omega)}, \quad \text{for any } p \leq p^* = \frac{Np}{N-p}, \tag{2.73}$$

and $W^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$ (see [46], page 312).

Here $N = 3$ and we are interested in $p^* = 2$ which means $p = \frac{6}{5}$. Thus for any $q \in [\frac{6}{5}, 2]$, we have

$$\|\nabla\varphi_{c,6}\|_q \lesssim \|\varphi_{c,6}\|_{W^{2,\frac{6}{5}}}.$$

Hence, by a standard elliptic estimate ([44]),

$$\|\nabla\phi_{c,6}\|_2 \lesssim \|\phi_{c,6}\|_{W^{2,\frac{6}{5}}} \lesssim \|c^5\|_{\frac{6}{5}} = \|c\|_6^5. \tag{2.74}$$

Therefore, the right hand side of (2.66), for $k = 6$ is bounded by

$$\begin{aligned} \text{r.h.s.}(2.66) &\lesssim \|\nabla\varphi_{c,6}\|_2 \left(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \|g\|_2\right) \\ &\leq \|c\|_6^5 \left(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \|\frac{g}{\sqrt{\nu}}\|_2\right). \end{aligned} \tag{2.75}$$

Thus, by Young inequality ($|xy| \leq \eta|x|^p + C_{\eta,p,q}|y|^q$, $p^{-1} + q^{-1} = 1$), we have

$$\text{r.h.s.}(2.66) \lesssim \eta\|c\|_6^6 + C_\eta\left(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \|\frac{g}{\sqrt{\nu}}\|_2\right)^6. \tag{2.76}$$

We have $v\cdot\nabla_x\psi_{c,k} = \sum_{i,j=1}^d(|v|^2-\beta_c)\sqrt{\mu}v_iv_j\partial_{ij}\varphi_{c,k}(x)$, and

$$\begin{aligned} \varepsilon^2\left[\Phi\cdot\nabla_v\psi_{c,k} - \frac{1}{2}v\cdot\Phi\right]f &= \varepsilon^2\sqrt{\mu}f\Phi\cdot\nabla_v\left(\frac{\psi_{c,k}}{\sqrt{\mu}}\right) \\ &= \varepsilon^2\sqrt{\mu}f\sum_{i,j}\Phi_i[\delta_{ij}(|v|^2-\beta_c)+2v_iv_j]\partial_j\varphi_{c,k}. \end{aligned}$$

Then the left hand side of (2.66) takes the form, for $i = 1, \ldots, d$,

$$\iint_{\partial\Omega\times\mathbb{R}^3}(n(x)\cdot v)(|v|^2-\beta_c)\sqrt{\mu}\sum_{i=1}^d v_i\partial_i\varphi_{c,k}f\,\mathrm{d}S_x\mathrm{d}v \tag{2.77}$$

$$-\iint_{\Omega\times\mathbb{R}^3}[(\lambda+(1-\mathfrak{r})\varepsilon^{-1}\nu]f(|v|^2-\beta_c)\sqrt{\mu}\sum_i v_i\partial_i\varphi_{c,k}\mathrm{d}x\mathrm{d}v \tag{2.78}$$

$$-\iint_{\Omega\times\mathbb{R}^3}(|v|^2-\beta_c)\sqrt{\mu}\Big\{\sum_{i,j=1}^d v_iv_j\partial_{ij}\varphi_{c,k}\Big\}f\,\mathrm{d}x\mathrm{d}v \tag{2.79}$$

$$+\varepsilon^2\sqrt{\mu}\sum_{i,j}\iint_{\Omega\times\mathbb{R}^3}\Phi_i[\delta_{ij}(|v|^2-\beta_c)+2v_iv_j]\partial_j\varphi_{c,k}f\,\mathrm{d}x\mathrm{d}v. \tag{2.80}$$

We decompose

$$f = \left\{ a + v\cdot b + c\left[\frac{|v|^2}{2} - \frac{3}{2}\right]\right\}\sqrt{\mu} + (\mathbf{I}-\mathbf{P})f, \quad \text{on } \Omega\times\mathbb{R}^3, \tag{2.81}$$

$$f_\gamma = P_\gamma f + \mathbf{1}_{\gamma_+}(1-P_\gamma)f + \mathbf{1}_{\gamma_-}r, \quad \text{on } \gamma, \tag{2.82}$$

and substitute (2.81), (2.82) into (2.77)–(2.80). Note that the off-diagonal parts ($v_i v_j$ with $i \neq j$) and b term vanish by oddness in v. Now we choose $\beta_c = 5$ so that,

$$\int (|v|^2 - \beta_c) v_i^2 \mu(v) dv = 0, \text{ for } i = 1, 2, 3. \tag{2.83}$$

Note that, thanks to the choice of $\beta_c = 5$, we eliminate the a contribution in the bulk. Then (2.79) becomes

$$(2.79) = -\sum_{i=1}^{d} \int_{\mathbb{R}^3} (|v|^2 - \beta_c) v_i^2 \Big(\frac{|v|^2}{2} - \frac{3}{2}\Big)\mu(v)\mathrm{d}v \int_\Omega \partial_{ii}\varphi_{c,k}(x)c(x)\mathrm{d}x \tag{2.84}$$

$$-\sum_{i=1}^{d} \iint_{\Omega\times\mathbb{R}^3} (|v|^2 - \beta_c) v_i \sqrt{\mu}(v\cdot\nabla_x)\partial_i\varphi_{c,k}(\mathbf{I}-\mathbf{P})f. \tag{2.85}$$

From $\int_{\mathbb{R}^3}(|v|^2 - \beta_c)v_i^2(\frac{|v|^2}{2} - \frac{3}{2})\mu(v)dv = 5$ and $-\Delta_x\varphi_{c,k} = c^{k-1}$ for $k = 2, 6$,

$$(2.84) = -5\int_\Omega \Delta_x\varphi_{c,k} c = 5\|c\|_k^k. \tag{2.86}$$

Moreover, for $k = 2$,

$$(2.85) \le \|\nabla^2\varphi_{c,2}\|_2 \|(\mathbf{I}-\mathbf{P})f\|_2 \le \eta\|c\|_2^2 + C_\eta\|(\mathbf{I}-\mathbf{P})f\|_2, \tag{2.87}$$

and, for $k = 6$,

$$(2.85) \le \|\nabla^2\varphi_{c,2}\|_{\frac{6}{5}} \|(\mathbf{I}-\mathbf{P})f\|_6 \le \eta\|c\|_6^6 + C_\eta\|(\mathbf{I}-\mathbf{P})f\|_6^6. \tag{2.88}$$

Consider (2.77). Because of the choice of β_c in (2.83), there is no $P_\gamma f$ contribution at the boundary in (2.77). Then for $k = 2$ we have

$$(2.77) \lesssim \|c\|_2\{|(1-P_\gamma)f|_{2,+} + |r|_{2,-}\}, \tag{2.89}$$

where we used $|\nabla_x\varphi_c|_2 \lesssim \|\varphi_c\|_{H^2} \lesssim \|c\|_2$ from an elliptic estimate and the trace estimate.

Now we consider $k = 6$ case. By the assumption that Ω is a C^1 domain in $\mathbb{R}^N$ with $N = 3$, we can use the following trace estimate (see [46], page 466):

$$\left(\int_{\partial\Omega} dS(x)|u|^{\frac{p(N-1)}{N-p}}\right)^{\frac{N-p}{p(N-1)}} \leq C(N,P)\left(\int_\Omega dx|u|^p + \int_\Omega dx|\nabla u|^p\right)^{\frac{1}{p}}. \quad (2.90)$$

This is a consequence of the trace theorem $W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$, and the Sobolev embedding in $N-1$ dimensional sub-manifold ($W^{1-\frac{1}{p},p}(\partial\Omega) \subset L^{\frac{p(N-1)}{N-p}}(\partial\Omega)$ for $\frac{N-p}{p(N-1)} = \frac{1}{p} - \frac{1-\frac{1}{p}}{N-1}$). In particular, with $p = \frac{6}{5}$ and $N = 3$ we have $\frac{p(N-1)}{N-p} = \frac{4}{3}$. With $u = \nabla\varphi_{c,6}$, we have from (2.47)

$$\|\nabla_x\varphi_c\|_{L^{4/3}(\partial\Omega)} \lesssim \|c\|^5_{L^6(\Omega)}. \quad (2.91)$$

On the other hand, by Holder inequality

$$\begin{aligned}
|\mu^{1/4}(1-P_\gamma)f|_{4,+} &\leq \varepsilon^{1/4}\big[\varepsilon^{-1/2}|(1-P_\gamma)f|_{2,+}\big]^{1/2}|\mu^{1/2}(1-P_\gamma)f|^{1/2}_{\infty,+} \\
&\lesssim \big[\varepsilon^{-1/2}|(1-P_\gamma)f|_{2,+}\big]^{1/2}\big[\varepsilon^{1/2}\|wf\|_\infty\big]^{1/2}.
\end{aligned}$$

Therefore, by the Young inequality, we conclude

$$\begin{aligned}
(2.77) &\lesssim \big\{|\mu^{1/4}(1-P_\gamma)f|_{4,+} + |\mu^{1/4}r|_{4,-}\big\}|\nabla_x\varphi_c|_{4/3,+} \\
&\lesssim \Big\{\big[\varepsilon^{-1/2}|(1-P_\gamma)f|_{2,+}\big]^{1/2}\big[\varepsilon^{1/2}\|wf\|_\infty\big]^{1/2} + |\mu^{1/4}r|_{4,-}\Big\}\|c\|^5_{L^6} \\
&\leq \eta\|c\|_6^6 + \eta'\big[\varepsilon^{\frac{1}{2}}\|wf\|_\infty + |\mu^{1/4}r|_{4,-}\big]^6 + C_{\eta,\eta'}\big[\varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f|_{2,+}\big]^6.
\end{aligned} \quad (2.92)$$

Now we consider (2.78). Using (2.71) for $k = 2$ and (2.74) for $k = 6$ respectively, we conclude that

$$\begin{aligned}
(2.78) &\lesssim [(\lambda + (1-\mathfrak{r})\varepsilon^{-1}] \times \big\{\|\mathbf{P}f\|_k + \|(\mathbf{I}-\mathbf{P})f\|_2\big\}\|c\|_k^{k-1} \\
&\lesssim [(\lambda + (1-\mathfrak{r})\varepsilon^{-1}] \times \big\{\|\mathbf{P}f\|_k^k + \|(\mathbf{I}-\mathbf{P})f\|_2^k\big\} \\
&\lesssim (\lambda + o(1))\|\mathbf{P}f\|_k^k + \|(\mathbf{I}-\mathbf{P})f\|_2^k,
\end{aligned} \quad (2.93)$$

where we have used $\mathfrak{r} = 1 + o(1)\varepsilon$.

Moreover, since $\int_{\mathbb{R}^3}\mu[(|v|^2-\beta_c) + 2v_i^2]\left[\frac{|v|^2}{2} - \frac{3}{2}\right] = 2\sqrt{2\pi}$, and $\int_{\mathbb{R}^3}\mu[(|v|^2 - \beta_c) + 2v_i^2] = 3 - \beta_c + 2 = 0$, the term (2.80) becomes

$$2\varepsilon^2\int_\Omega c\Phi\cdot\nabla_x\varphi_{c,k} + \varepsilon^2\sqrt{\mu}\sum_{i,j}\iint_{\Omega\times\mathbb{R}^3}\Phi_i\left[\left(\delta_{i,j} - \frac{1}{2}v_iv_j\right)(|v|^2-\beta_c) + 2v_iv_j\right]$$
$$\times\,\partial_j\varphi_{c,k}(\mathbf{I}-\mathbf{P})f.$$

Using $\big|\int_\Omega c\Phi\cdot\nabla_x\varphi_{c,k}\big|\le\|c\|_k^k\|\Phi\|_\infty$, (2.80) is bounded by

$$(2.80)\lesssim\varepsilon^2\big[\|c\|_k^k+\|(\mathbf{I}-\mathbf{P})f\|_k^k\big]\|\Phi\|_\infty. \tag{2.94}$$

By collecting the estimates (2.72), (2.86), (2.87), (2.89), (2.93), (2.94), for sufficiently small $\varepsilon>0$ we prove (2.68).

Similarly, collecting the estimates (2.76), (2.86), (2.88), (2.92), (2.93), (2.94), for ε sufficiently small we obtain (2.69).

Step 3. Estimate of b. We claim that, for sufficiently small $\varepsilon>0$,

$$\begin{aligned}\|b\|_2^2 &\lesssim o(1)\|\mathbf{P}f\|_2^2+\,|(1-P_\gamma)f|_{2,+}^2+\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_2^2\\ &\quad+\,\|\frac{g}{\sqrt{\nu}}\|_2^2+|r|_{2,-}^2.\end{aligned} \tag{2.95}$$

$$\begin{aligned}\|b\|_6^6 &\lesssim o(1)\|\mathbf{P}f\|_6^6+\Big(\varepsilon^{-\frac12}|(1-P_\gamma)f|_{2,+}+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu+\|(\mathbf{I}-\mathbf{P})f\|_6\\ &\quad+\,\|\frac{g}{\sqrt{\nu}}\|_2+|r|_2+|\varepsilon^{\frac12}wr|_\infty+\|\varepsilon^{\frac32}\langle v\rangle^{-1}wg\|_\infty\Big)^6.\end{aligned} \tag{2.96}$$

For $k=2,6$ we shall establish the estimate of b by estimating $(\partial_i\partial_j\Delta^{-1}b_j^{k-1})b_i$ for all $i,j=1,\dots,d$, and $(\partial_j\partial_j\Delta^{-1}b_i^{k-1})b_i$ for $i\neq j$.

We fix i,j. To estimate $\partial_i\partial_j\Delta^{-1}b_j^{k-1}b_i$ we choose a test function in (2.66) as, for $k=2,6$

$$\psi=\psi_{b,k}^{i,j}\equiv(v_i^2-\beta_b)\sqrt{\mu}\partial_j\varphi_{b,k}^j,\quad i,j=1,\dots,d, \tag{2.97}$$

where β_b is a constant to be determined, and

$$-\Delta_x\varphi_{b,k}^j(x)=b_j^{k-1}(x),\quad \varphi_{b,k}^j|_{\partial\Omega}=0. \tag{2.98}$$

For $k=2$, from the standard elliptic estimate $\|\varphi_b^j\|_{H^2}\lesssim\|b\|_2$. Hence, for $k=2$ the right hand side of (2.66) is now bounded by

$$\text{r.h.s}(2.66)\le\|b\|_2\big\{\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_2+\|\frac{g}{\sqrt{\nu}}\|_2\big\}. \tag{2.99}$$

With the same argument as before, the right hand side of (2.66) for $k=6$ is bounded by

$$\text{r.h.s.}(2.66)\lesssim\eta\|b\|_6^6+C_\eta\left(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu+\|g\|_2\right)^6. \tag{2.100}$$

Now we substitute (2.82) and (2.81) into the left hand side of (2.66). Note that $(v_i^2-\beta_b)\{n(x)\cdot v\}\mu$ is odd in v, therefore $P_\gamma f$ contribution to (2.66) vanishes. Moreover, by (2.81), the a,c contributions to (2.66) also vanish by oddness. Finally, in the field term only the $\mathbf{P}f$ part survives because $\nabla_v\frac{\psi}{\sqrt{\mu}}=2v_i\partial_j\varphi_b^j$ and, by oddness, the a and c contributions disappear. Therefore the left hand side of (2.66) takes the form

$$\iint_{\partial\Omega\times\mathbb{R}^3}(n(x)\cdot v)(v_i^2-\beta_b)\sqrt{\mu}\partial_j\varphi_{b,k}^j[(1-P_\gamma)f+r]\mathbf{1}_{\gamma_+} \tag{2.101}$$

$$+\sum_i \iint_{\Omega\times\mathbb{R}^3} [(\lambda + (1-\mathfrak{r})\varepsilon^{-1}\nu] f (v_i^2 - \beta_b)\sqrt{\mu} v_i \partial_j \varphi_{b,k}^j \tag{2.102}$$

$$-\iint_{\Omega\times\mathbb{R}^3} (v_i^2 - \beta_b)\sqrt{\mu} \left\{ \sum_l v_l \partial_{lj} \varphi_{b,k}^j \right\} f \tag{2.103}$$

$$-\varepsilon^2 \sum_k \int_{\mathbb{R}^3} 2 v_i v_k \int_\Omega \Phi_i b_k \partial_j \varphi_{b,k}^j. \tag{2.104}$$

By (2.98) and the trace estimate, for $k = 2$, $|\partial_j \varphi_b^j|_2 \le \|\varphi_b^j\|_{H^2} \le \|b\|_2$, for any $\eta > 0$, the term (2.101) is bounded by

$$(2.101) \le \frac{1}{4\eta}\left(|(1-P_\gamma) f|_{2,+}^2 + |r|_2^2 \right) + \eta \|b\|_2^2. \tag{2.105}$$

For $k = 6$, by the same argument as before, the term (2.101) is bounded by

$$\begin{aligned}(2.101) &\le \eta\|b\|_6^6 + \eta' \|\mathbf{P} f\|_{L^6(\Omega\times\mathbb{R}^3)} + C_{\eta,\eta'}(\varepsilon^{-\frac{1}{2}} |(1-P_\gamma) f|_2)^6 \\ &\quad + \eta'(|\varepsilon^{\frac{1}{2}} w r|_\infty + \varepsilon^{\frac{3}{2}} \|\langle v\rangle^{-1} w g\|_\infty + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P}) f\|_{L^2(\Omega\times\mathbb{R}^3)})^6. \end{aligned} \tag{2.106}$$

The term (2.102) is bounded, as in (2.93).
The term (2.103) equals

$$-\sum_l \int (v_i^2 - \beta_b) v_l^2 \mu \partial_{lj} \varphi_{b,k}^j (x) b_l - \int (v_i^2 - \beta_b) v_l \sqrt{\mu} \partial_{lj} \varphi_{b,k}^j (x) (\mathbf{I} - \mathbf{P}) f. \tag{2.107}$$

We can choose $\beta_b > 0$ such that for all i,

$$\int_{\mathbb{R}^3} [(v_i)^2 - \beta_b] \mu(v) \mathrm{d}v = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} [v_1^2 - \beta_b] e^{-\frac{|v_1|^2}{2}} \mathrm{d}v_1 = 0. \tag{2.108}$$

Note that for such chosen β_b, and for $i \neq k$, by an explicit computation

$$\begin{aligned} &\int (v_i^2 - \beta_b) v_k^2 \mu \mathrm{d}v = \int (v_1^2 - \beta_b) v_2^2 \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v_1|^2}{2}} e^{-\frac{|v_2|^2}{2}} e^{-\frac{|v_3|^2}{2}} \mathrm{d}v = 0, \\ &\int (v_i^2 - \beta_b) v_i^2 \mu \mathrm{d}v = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} [v_1^4 - \beta_b v_1^2] e^{-\frac{|v_1|^2}{2}} \mathrm{d}v_1 \neq 0. \end{aligned}$$

The first term in (2.107) becomes

$$-\iint_{\Omega\times\mathbb{R}^3} (v_i^2 - \beta_b) v_i^2 \mu \mathrm{d}v \partial_{ij} \varphi_{b,k}^j (x) b_i + \sum_{k\neq i} \underbrace{\int_{\mathbb{R}^3} (v_i^2 - \beta_b) v_k^2 \mu}_{=0} \int_\Omega \partial_{kj} \varphi_{b,k}^j (x) b_k$$

$$= 2\int_\Omega (\partial_i\partial_j\Delta^{-1}b_j^{k-1})b_i. \tag{2.109}$$

The second term in (2.107), for any $\eta > 0$ is bounded by

$$\text{second term in } (2.107) \lesssim \eta\|b\|_k^k + \frac{1}{4\eta}\|(\mathbf{I}-\mathbf{P})f\|_2^2. \tag{2.110}$$

The term (2.104) is bounded by

$$(2.104) \lesssim \varepsilon^2\|\Phi\|_\infty\|b\|_k^k. \tag{2.111}$$

Collecting the bounds (2.99), (2.105), (2.109), (2.110), (2.111), we have the following estimate for all i, j:

$$\begin{aligned}&\left|\int_\Omega \partial_i\partial_j\Delta^{-1}b_jb_i\right|\\ &\lesssim \left[(\varepsilon^{-2}+\varepsilon^2\|\Phi\|_\infty)\|(\mathbf{I}-\mathbf{P})f\|_\nu^2 + |(1-P_\gamma)f|_{2,+}^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_2^2\right]\\ &\quad + (\eta+\varepsilon^2\|\Phi\|_\infty)\|b\|_2^2 + o(1)\{\|\mathbf{P}f\|_2^2 + \|(\mathbf{I}-\mathbf{P})f\|_\nu^2\}.\end{aligned} \tag{2.112}$$

Collecting the estimates (2.100), (2.106), (2.109), (2.110), (2.111), for ε sufficiently small we obtain

$$\begin{aligned}&\left|\int_\Omega \partial_i\partial_j\Delta^{-1}b_j^5b_i\right| \lesssim \eta(\|b\|_6^6 + \|\mathbf{P}f\|_6^6)\\ &+ C_{\eta.\eta'}\Big[(\varepsilon^{-2}+\varepsilon^2\|\Phi\|_\infty)\|(\mathbf{I}-\mathbf{P})f\|_\nu^2 + \varepsilon^{-1}|(1-P_\gamma)f|_{2,+}^2\\ &+\left|\frac{g}{\sqrt{\nu}}\right|_2^2 + |r|_2^2\Big]^3 + \eta'(|\varepsilon^{\frac12}wr|_\infty + \varepsilon^{\frac32}\|\langle v\rangle^{-1}wg\|_\infty + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2(\Omega\times\mathbb{R}^3)})^6.\end{aligned} \tag{2.113}$$

To estimate $\partial_j(\partial_j\Delta^{-1}b_i^{k-1})b_i$ for $i\neq j$, we choose a test function in (2.66) as

$$\psi = |v|^2v_iv_j\sqrt{\mu}\partial_j\varphi_{b,k}^i(x), \quad i\neq j, \tag{2.114}$$

where $\varphi_{b,k}^i$ is given by (2.98). Clearly, the right hand side of (2.66) is again bounded by (2.99) for $k=2$ and and (2.100) for $k=6$. We substitute again (2.82) and (2.81) into the left hand side of (2.66). The $P_\gamma f$ contribution and a, c contributions vanish again due to oddness. With this choice of ψ, we have

$$\int_{\mathbb{R}^3}\sqrt{\mu}|v|^2v_iv_j\sqrt{\mu}\left(a+b\cdot v+c\frac{|v|^2-3}{2}\right)=0, \quad \text{if } i\neq j.$$

The contribution from the field is

$$\varepsilon^2\sum_\ell\iint_{\Omega\times\mathbb{R}^3}\partial_j\varphi_{b,k}^i\Phi_\ell\sqrt{\mu}[2v_\ell v_iv_j + |v|^2(v_i\delta_{j,\ell}+v_j\delta_{i,\ell})]f$$

$$\lesssim (\varepsilon^2\|\Phi\|_\infty + \eta)\|b\|_k^k + \varepsilon^2\|\Phi\|_\infty \frac{1}{4\eta}\|(\mathbf{I}-\mathbf{P})f\|_\nu^2.$$

The contribution from the term containing $\lambda + \varepsilon^{-1}(1-\mathfrak{r})\nu$ is bounded again as (2.93). The boundary terms is bounded by (2.106).

Finally, the bulk term becomes

$$\begin{aligned}
&-\iint_{\Omega\times\mathbb{R}^3} |v|^2 v_i v_j \sqrt{\mu}\{\sum_\ell v_\ell \partial_{\ell j}\varphi_b^i\} f \\
&= -\iint_{\Omega\times\mathbb{R}^3} |v|^2 v_i^2 v_j^2 \mu \left[\partial_{ij}\varphi_{b,k}^i b_j + \partial_{jj}\varphi_{b,k}^i(x) b_i\right] \\
&\quad - \iint_{\Omega\times\mathbb{R}^3} |v|^2 v_i v_j v_\ell \sqrt{\mu}\partial_{\ell j}\varphi_b^i(x)[\mathbf{I}-\mathbf{P}]f.
\end{aligned} \tag{2.115}$$

Note that the first term in (2.115) is evaluated as $\int_\Omega\{(\partial_i\partial_j\Delta^{-1}b_i^{k-1})b_j + (\partial_j\partial_j\Delta^{-1}b_i^{k-1})b_i\}$, thus collecting the above bounds we get a bound for $(\partial_j\partial_j\Delta^{-1} b_i^{k-1})b_i$ which, combined with (2.112) for $k=2$, and with (2.113) for $k=6$, gives (2.95) and (2.96).

Step 4. Estimate of a. We claim that, for ε sufficiently small,

$$\begin{aligned}
&\|\mathring{a}\|_2^2 \lesssim \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_2^2 + |(1-P_\gamma)f|_{2,+}^2 + |r|_2^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 \\
&\quad + \varepsilon^2\|\Phi\|_\infty(\|\mathbf{P}\mathring{f}\|_2^2 + | < f > |^2). &(2.116)\\
&\|\mathring{a}\|_6^6 \lesssim \eta(\|a\|_6^6 + \|\mathbf{P}f\|_6^6) + C_{\eta,\eta'}\Big(\|(\mathbf{I}-\mathbf{P})f\|_6 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_2 \\
&\quad + \varepsilon^{-\frac12}|(1-P_\gamma)f|_{2,+} + |r|_2 + \|\frac{g}{\sqrt{\nu}}\|_2\Big)^6 \\
&\quad + \eta'\Big(|\varepsilon^{\frac12} wr|_\infty + \varepsilon^{\frac32}\|\langle v\rangle^{-1} wg\|_\infty\Big)^6. &(2.117)
\end{aligned}$$

We choose a test function

$$\psi = \psi_{a,k} \equiv (|v|^2-\beta_a)v\cdot\nabla_x\varphi_a\sqrt{\mu} = \sum_{i=1}^d (|v|^2-\beta_a)v_i\partial_i\varphi_{a,k}\sqrt{\mu}, \tag{2.118}$$

where

$$-\Delta_x\varphi_{a,k}(x) = \mathring{a}^{k-1}(x) - \fint \mathring{a}^{k-1}, \quad \frac{\partial}{\partial n}\varphi_{a,k}|_{\partial\Omega} = 0, \quad \fint \varphi_{a,k} = 0.$$

For $k=2$ it follows from the elliptic estimate that $\|\varphi_{a,k}\|_{H^2} \lesssim \|\mathring{a}\|_2$. Since $\int_{\mathbb{R}^3}(\frac{|v|^2}{2} - \frac32)(v_i)^2\mu(v)\mathrm{d}v \neq 0$, we choose $\beta_a = 10 > 0$ so that, for all i,

$$\int_{\mathbb{R}^3}(|v|^2-\beta_a)\left(\frac{|v|^2}{2}-\frac32\right)(v_i)^2\mu(v) = 0. \tag{2.119}$$

Plugging ψ_a into (2.66), we bound its right hand side by

$$\text{r.h.s}(2.66) \lesssim \|\mathring{a}\|_2 \big\{\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_2 + \|g\|_2\big\}. \tag{2.120}$$

For $k=6$ we have the bound

$$\text{r.h.s.}(2.66) \lesssim \eta \|\mathring{a}\|_6^6 + C_\eta \left(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \|g\|_2\right)^6. \tag{2.121}$$

By (2.82) and (2.81), since the c contribution vanishes in (2.66) due to our choice of β_a and the b contribution vanishes in (2.66) due to the oddness, the left hand side of (2.66) takes the form of

$$\sum_{i=1}^d \int_\gamma \{n\cdot v\}(|v|^2-\beta_a)v_i\sqrt{\mu}\partial_i\varphi_{a,k}(x)[P_\gamma f + (I-P_\gamma)f\mathbf{1}_{\gamma_+} + r\mathbf{1}_{\gamma_+}] \tag{2.122}$$

$$-\iint_{\Omega\times\mathbb{R}^3} [(\lambda + (1-\mathfrak{r})\varepsilon^{-1}\nu]f(|v|^2-\beta_a)\sqrt{\mu}\sum_i v_i\partial_i\varphi_{a,k} \tag{2.123}$$

$$-\sum_{i,\ell=1}^d \iint_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_a)v_iv_\ell\partial_{i\ell}\varphi_{a,k}(x)a(x)\mu(v) \tag{2.124}$$

$$-\varepsilon^2\sum_{i,\ell}\iint_{\Omega\times\mathbb{R}^3} \Phi_\ell[2v_iv_\ell + (|v|^2-\beta_a)\delta_{i,\ell}]\left\{a + c\frac{|v|^2-3}{2}\right\}\mu\partial_i\varphi_a \tag{2.125}$$

$$-\sum_{i,\ell=1}^d \iint_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_a)v_iv_\ell\partial_{i\ell}\varphi_{a,k}(x)(\mathbf{I}-\mathbf{P})f \tag{2.126}$$

We make an orthogonal decomposition at the boundary, $v_i = (v\cdot n)n_i + (v_\perp)_i = v_n n_i + (v_\perp)_i$. The contribution of $P_\gamma f = z_\gamma(x)\sqrt{\mu}$ in (2.122) is

$$\int_\gamma (|v|^2-\beta_a)v\cdot\nabla_x\varphi_{a,k}v_n\mu z_\gamma = \int_\gamma (|v|^2-\beta_a)v_n\frac{\partial\varphi_{a,k}}{\partial n}v_n\mu z_\gamma + \int_\gamma (|v|^2-\beta_a)v_\perp\cdot\nabla_x\varphi_{a,k}v_n\mu z_\gamma.$$

The first term vanishes by the Neumann boundary condition, while the second term also vanishes due to the oddness of $(v_\perp)_i v_n$ for all i. Therefore, for $k=2$, (2.122) and (2.126) are bounded by $\|\mathring{a}\|_2\big\{\|(\mathbf{I}-\mathbf{P})f\|_2 + |(1-P_\gamma)f|_{2,+} + |r|_2\big\}$. The term (2.123) is bounded, as before, by (2.93).

The term (2.124), for $\ell\neq i$ vanishes due to the oddness. Hence we only have the $\ell=i$ contribution:

$$\sum_{i=1}^d \iint_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_a)(v_i)^2\mu(\partial_{ii}\varphi_a)a$$

$$
\begin{aligned}
&= \sum_{i=1}^{d} \iint_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_a)(v_i)^2\mu(\partial_{ii}\varphi_{a,2})\mathring{a} \\
&= -5\|\mathring{a}\|_2^2,
\end{aligned}
$$

because $\int(|v|^2-10)v_i^2\mu \neq 0$ and $\sum_i \int_\Omega dx\partial_{ii}\varphi_{a,2} = \int_{\partial\Omega}\partial_n\varphi_{a,2} = 0$. Finally, the term (2.125) is bounded by

$$
\varepsilon^2\|\Phi\|_\infty\|\mathring{a}\|_2(\|\mathring{a}\|_2 + | < f > | + \|c\|_2).
$$

Using $-\Delta_x\varphi_a = \mathring{a}$, (2.67), and (2.68) we obtain

$$
\begin{aligned}
\|\mathring{a}\|_2^2 \;\lesssim\; & \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_2^2 + |(1-P_\gamma)f|_{2,+}^2 + |r|_2^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 \\
& + o(1)\{\|\mathbf{P}f\|_2^2 + \|(\mathbf{I}-\mathbf{P})f\|_\nu^2\}.
\end{aligned}
$$

Since $\|\mathbf{P}f\|_2^2 \leq \|\mathbf{P}\mathring{f}\|_2^2 + | < f > |^2$, we conclude (2.116). Finally we conclude (2.63). The case $k=6$ is handled in a similar way using the same estimates as for b and c. The only term we have to check is

$$
\begin{aligned}
&\sum_{i=1}^{d}\iint_{\Omega\times\mathbb{R}^3}(|v|^2-\beta_a)(v_i)^2\mu(\partial_{ii}\varphi_{a,6})a = \sum_{i=1}^{d}\iint_{\Omega\times\mathbb{R}^3}(|v|^2-\beta_a)(v_i)^2\mu(\partial_{ii}\varphi_{a,6})\mathring{a} \\
&= \sum_{i=1}^{d}\iint_{\Omega\times\mathbb{R}^3}(|v|^2-\beta_a)(v_i)^2\mu\Big(\mathring{a}^5 - \fint \mathring{a}^5\Big)\mathring{a} = -5\|\mathring{a}\|_6^6,
\end{aligned}
$$

where the first equality is due to again to $\sum_i \int_\Omega dx\partial_{ii}\varphi_{a,6} = \int_{\partial\Omega}\partial_n\varphi_{a,6} = 0$ and the second to $\fint \mathring{a} = 0$. Since

$$
\begin{aligned}
\|(\mathbf{I}-\mathbf{P})f\|_6^6 \leq [\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_2^2][\varepsilon^2\|(\mathbf{I}-\mathbf{P})f\|_\infty^4] \leq C_\eta(\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_2)^6 \\
+ \eta(\varepsilon^{\frac{1}{2}}\|wf\|_\infty)^6,
\end{aligned}
\tag{2.127}
$$

we obtain (2.65). □

Now we are ready to prove the main result of this section:

Proof of Proposition 2.9 Step 1. We claim that for any $\lambda > 0$ and $0 < \varepsilon \ll 1$, there exists a (unique) solution to

$$
\lambda f^\lambda + v\cdot\nabla_x f^\lambda + \varepsilon^2\Phi\cdot\nabla_v f^\lambda - \frac{1}{2}\varepsilon^2\Phi\cdot v f^\lambda + \varepsilon^{-1}Lf^\lambda = g \quad \text{in } \Omega\times\mathbb{R}^3,\;\; f^\lambda\big|_{\gamma_-} = P_\gamma f^\lambda + r.
\tag{2.128}
$$

Moreover,

$$
\begin{aligned}
< f^\lambda > = 0,\;\; & \|\mathbf{P}f^\lambda\|_2^2 + \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f^\lambda\|_\nu^2 + \varepsilon^{-1}|(1-P_\gamma)f^\lambda|_2^2 \\
& \lesssim \|\nu^{-\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_2^2 + \varepsilon^{-2}\|\mathbf{P}g\|_2^2 + |r|_2^2.
\end{aligned}
\tag{2.129}
$$

From (2.54), a solution to (2.128) is a fixed point of the map

$$f^\lambda \;\mapsto\; \mathscr{L}^{-1}\big[\varepsilon^{-1}Kf^\lambda + g\big]. \tag{2.130}$$

Note that from Lemma 2.10, the operator $\mathscr{L}^{-1}$ is well-defined and bounded. Hence for any $f^\lambda \in L^2$ there is $h \in L^2$ such that $f^\lambda = \mathscr{L}^{-1}h$. Thus (2.130), the fixed point problem for f^λ, is equivalent to the fixed point problem for h:

$$h \;\mapsto\; \varepsilon^{-1}K\mathscr{L}^{-1}h + g. \tag{2.131}$$

In view of the application of the Schaefer's fixed point Theorem ([25], page 504), to show the existence of the fixed point, we need to show that $K\mathscr{L}^{-1}$ is compact, which is proven in Lemma 2.11, and the following *a priori* uniform bound: if $h^\mathfrak{r}$ solves

$$h^\mathfrak{r} \;=\; \mathfrak{r}\varepsilon^{-1}K\mathscr{L}^{-1}h^\mathfrak{r} + g, \quad \text{for some } \mathfrak{r} \in [1^-, 1], \tag{2.132}$$

then $\|h^\mathfrak{r}\|_2$ is bounded uniformly in $\mathfrak{r}$. Since $\mathscr{L}^{-1}$ is bounded, it suffices to show a uniform bound of $f^\mathfrak{r} = \mathscr{L}^{-1}h^\mathfrak{r}$ solving (2.61) with $f = f^\mathfrak{r}$.

By the Green's identity,

$$\begin{aligned}
&\lambda\|f^\mathfrak{r}\|_2^2 + \mathfrak{r}\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f^\mathfrak{r}\|_\nu^2 - o(1)\|\mathbf{P}f^\mathfrak{r}\|_2^2 + |(1-P_\gamma)f^\mathfrak{r}|_{2,+}^2\\
&\quad\lesssim o(1)\|f^\mathfrak{r}\|_\nu^2 + \varepsilon\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2^2 + \varepsilon^{-1}\|\mathbf{P}g\|_2^2 + |r|_{2,-}^2 + \varepsilon^2\|\Phi\|_\infty\|f^\mathfrak{r}\|_\nu^2.
\end{aligned}$$

From $\|f^\mathfrak{r}\|_\nu \lesssim \|\mathbf{P}f^\mathfrak{r}\|_\nu + \|(\mathbf{I}-\mathbf{P})f^\mathfrak{r}\|_\nu \lesssim \|f^\mathfrak{r}\|_2 + \|(\mathbf{I}-\mathbf{P})f^\mathfrak{r}\|_\nu$, we have, for $\mathfrak{r}\sim 1$ and $\varepsilon \ll 1$,

$$\begin{aligned}
&\lambda\|f^\mathfrak{r}\|_2^2 + \mathfrak{r}\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f^\mathfrak{r}\|_\nu^2 + |(1-P_\gamma)f^\mathfrak{r}|_{2,+}^2 - o(1)\|\mathbf{P}f^\mathfrak{r}\|_2^2\\
&\quad\lesssim \varepsilon\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2^2 + \varepsilon^{-1}\|\mathbf{P}g\|_2^2 + |r|_{2,-}^2.
\end{aligned} \tag{2.133}$$

Therefore we obtain a uniform-in-$\mathfrak{r}$ bound on $\|f^\mathfrak{r}\|_2$. Since $f^\mathfrak{r} = \mathscr{L}^{-1}h^\mathfrak{r}$, from (2.132), we have

$$h^\mathfrak{r} = \mathfrak{r}\varepsilon^{-1}Kf^\mathfrak{r} + g, \tag{2.134}$$

so, $\|h^\mathfrak{r}\|_2$ is also bounded uniformly in $\mathfrak{r}$. Note that in this argument ε is fixed. Therefore, by the Schaefer's fixed point theorem there is a fixed point h^λ for (2.131) and in consequence, a fixed point $f^\lambda = \mathscr{L}^{-1}h^\lambda$ for (2.130). Thus, we conclude the existence of a solution f^λ to (2.128).

Now we prove the first identity of (2.129). Estimating $[f^\lambda - f^\mathfrak{r}]$ using the Green's identity,

$$\begin{aligned}
&\lambda\|f^\lambda - f^\mathfrak{r}\|_2^2 + \mathfrak{r}\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})[f^\lambda - f^\mathfrak{r}]\|_\nu^2 + |(1-P_\gamma)[f^\lambda - f^\mathfrak{r}]|_{2,+}^2\\
&\quad\lesssim O(1-\mathfrak{r})\varepsilon^{-1}\big\{\|f^\mathfrak{r}\|_2^2 + \|f^\lambda - f^\mathfrak{r}\|_2^2 + \|(\mathbf{I}-\mathbf{P})f^\mathfrak{r}\|_\nu^2\\
&\qquad + \|(\mathbf{I}-\mathbf{P})[f^\lambda - f^\mathfrak{r}]\|_\nu^2\big\} \;\to\; 0, \quad \mathfrak{r}\uparrow 1.
\end{aligned}$$

From the above estimate and (2.64), for fixed $\lambda > 0$,

$$|< f^{\lambda} >| = \lim_{\mathfrak{r}\to 1} |< f^{\mathfrak{r}} >| \lesssim_{\lambda} \lim_{\mathfrak{r}\to 1} (1-\mathfrak{r})\varepsilon^{-1}\Big\{\|\frac{g}{\sqrt{\nu}}\|_2 + |r|_{2,-}\Big\} = 0.$$

By Lemma 2.12 applied to (2.128) and from the first identity of (2.129),

$$\|\mathbf{P}f^{\lambda}\|_2^2 \lesssim \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f^{\lambda}\|_2^2 + |(1-P_{\gamma})f^{\lambda}|_{2,+}^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + |r|_{2,-}^2. \tag{2.135}$$

The second estimate of (2.129) is a direct consequence of (2.135) and (2.133) with $\mathfrak{r} \uparrow 1$.
Step 2. To show the existence of the solution to (2.22) we take the limit as $\lambda \to 0$ for f^{λ} solving (2.128). Using (2.64), the uniform-in-λ estimate, we have $f^{\lambda} \rightharpoonup f$ weakly in L^2 where f solves the linear problem (2.22) with the estimate (2.53).

Moreover, since $< f^{\lambda} >= 0$, then also $< f >= 0$ and we conclude (2.23).

The difference $[f - f^{\lambda}]$ satisfies

$$\begin{aligned}
&\lambda[f-f^{\lambda}] + v\cdot\nabla_x[f-f^{\lambda}] + \varepsilon^2\Phi\cdot\nabla_v[f-f^{\lambda}] - \frac{1}{2}(\varepsilon^2\Phi\cdot v)[f-f^{\lambda}] \\
&\quad + \varepsilon^{-1}L[f-f^{\lambda}] = \lambda f, \\
&\quad\ [f-f^{\lambda}]\big|_{\gamma_-} = P_{\gamma}[f-f^{\lambda}].
\end{aligned}$$

By the Green's identity and Lemma 2.12,

$$\|f-f^{\lambda}\|_2^2 \lesssim \lambda\|f\|_2^2.$$

Therefore f^{λ} converges strongly to f. The uniqueness follows using the same argument with $\lambda = 0$.
Step 3. By Lemma 2.3,

$$\begin{aligned}
|P_{\gamma}f|_{2,+}^2 &\lesssim |\mathbf{1}_{\gamma_+^{\delta}}P_{\gamma}f|_{2,+}^2 \lesssim |\mathbf{1}_{\gamma_+^{\delta}}f|_{2,+}^2 + |(1-P_{\gamma})f|_{2,+}^2 \\
&\lesssim \|f\|_2^2 + \varepsilon^{-1}\|L(\mathbf{I}-\mathbf{P})ff\|_1 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + |(1-P_{\gamma})f|_{2,+}^2 \\
&\lesssim \|\mathbf{P}f\|_2^2 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{\nu}^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + |(1-P_{\gamma})f|_{2,+}^2.
\end{aligned}$$

For the incoming part, using the boundary condition,

$$\begin{aligned}
|f|_{2,-}^2 \lesssim\ |P_{\gamma}f|_{2,+}^2 + |r|_{2,-}^2 &\lesssim \|\mathbf{P}f\|_2^2 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{\nu}^2 \\
&+ \left|\frac{g}{\sqrt{\nu}}\right|_2^2 + |(1-P_{\gamma})f|_{2,+}^2 + |r|_{2,-}^2.
\end{aligned}$$

Combing (2.129), (2.135), and the above estimates, we conclude (2.53). □

Proof of Theorem 2.5 We only need to prove (2.24). Using (2.27) in Proposition 2.6 to bound $\varepsilon^{\frac{1}{2}}\|wf\|_\infty$ in (2.53), we conclude, for ε sufficiently small,

$$\|\mathbf{P}f\|_6 \lesssim \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f|_{2,+} + \|\frac{g}{\sqrt{\nu}}\|_2 + \|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1}wg\|_\infty \\ + |\varepsilon^{\frac{1}{2}}wr|_\infty + |r|_{2,-}. \tag{2.136}$$

From (2.27), (2.52), and (2.136) we conclude (2.24). □

2.4 Validity of the Steady Problem

The main purpose of this section is to prove Theorem 1.1. We need several estimates before proving the main theorem.

Lemma 2.13 *Recall the expression of* $\mathbf{P}f$ *in (1.19). Then, for* $w = e^{\beta|v|^2}$, $0<\beta\ll 1$,

$$\begin{aligned}&\|\nu^{-\frac{1}{2}}\Gamma_\pm(f,g)\|_{L^2_{x,v}}\\ &\lesssim \ \varepsilon^{1/2}\big\{\varepsilon^{1/2}\|wg\|_\infty\big[\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})f\|_{L^2_{x,v}}\big]\\ &\quad+\varepsilon^{1/2}\|wf\|_\infty\big[\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_{L^2_{x,v}}\big]\big\}\\ &\quad+\|\mathbf{P}f\|_{L^6_{x,v}}\|\mathbf{P}g\|_{L^3_{x,v}}.\end{aligned} \tag{2.137}$$

Proof By the decomposition

$$\begin{aligned}&\|\nu^{-\frac{1}{2}}\Gamma_\pm(f,g)\|_{L^2_{x,v}}\\ &\lesssim \|\nu^{-\frac{1}{2}}\Gamma_\pm(|(\mathbf{I}-\mathbf{P})f|,|g|)\|_{L^2_{x,v}}+\|\nu^{-\frac{1}{2}}\Gamma_\pm(|f|,|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{x,v}}\\ &\quad+\|\nu^{-\frac{1}{2}}\Gamma_\pm(|\mathbf{P}f|,|\mathbf{P}g|)\|_{L^2_{x,v}}.\end{aligned} \tag{2.138}$$

Note that terms like $\|\nu^{-\frac{1}{2}}\Gamma_\pm(\mathbf{P}f,(\mathbf{I}-\mathbf{P})g)\|_{L^2_{x,v}}$ and $\|\nu^{-\frac{1}{2}}\Gamma_\pm((\mathbf{I}-\mathbf{P})f,\mathbf{P}g)\|_{L^2_{x,v}}$ are bound by $\|\nu^{-\frac{1}{2}}\Gamma_\pm(|f|,|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{x,v}}$ and $\|\nu^{-\frac{1}{2}}\Gamma_\pm((\mathbf{I}-\mathbf{P})f|,|g|)\|_{L^2_{x,v}}$ respectively.

The first two terms of the RHS of (2.138) are bounded by

$$\varepsilon\|wg\|_{L^\infty_{x,v}}\|\nu^{-1/2}\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|,w^{-1})\|_{L^2_{x,v}}\\ +\varepsilon\|wf\|_{L^\infty_{x,v}}\|\nu^{-1/2}\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|,w^{-1})\|_{L^2_{x,v}}.$$

From $|v|^2+|u|^2=|v'|^2+|u'|^2$ and $\nu^{-1/2}|(v-u)\cdot\omega|\sqrt{\mu(u)}\lesssim\nu^{-1/2}[|v|+|u|]\sqrt{\mu(u)}\lesssim[1+|v|+|u|]^{\frac{1}{2}}\mu(u)^{\frac{1}{2}-}$,

$$\int_{\mathbb{R}^3}\nu^{-1}|\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|,w^{-1})(v)|^2\mathrm{d}v$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[1+|v'|+|u'|]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(v')|^2 w(u')^{-2}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\\
&\quad+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[1+|v'|+|u'|]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(u')|^2 w(v')^{-2}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\\
&\quad+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[1+|v|+|u|]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(v)|^2 w(u)^{-2}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\\
&\quad+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}[1+|v|+|u|]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(u)|^2 w(v)^{-2}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v. \qquad (2.139)
\end{aligned}
$$

Now by the change of variables $(v,u)\leftrightarrow(v',u')$ for the first term, $(v,u)\leftrightarrow(u',v')$ for the second term and $(v,u)\leftrightarrow(u,v)$ for the last term, we bound all the above terms as

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\nu^{-1}|\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|,w^{-1})|^2\\
&\lesssim \int_{\mathbb{R}^3}\Big[\iint_{\mathbb{R}^3\times\mathbb{S}^2}[1+|v|+|u|]w(u)^{-1}\mathrm{d}\omega\mathrm{d}u\Big]|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(v)|^2\mathrm{d}v\\
&\lesssim \int_{\mathbb{R}^3}\nu|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(v)|^2\mathrm{d}v. \qquad (2.140)
\end{aligned}
$$

Similarly,

$$
\int_{\mathbb{R}^3}\nu^{-1}|\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|,w^{-1})(v)|^2\mathrm{d}v\lesssim\int_{\mathbb{R}^3}\nu|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})g(v)|^2\mathrm{d}v. \qquad (2.141)
$$

Therefore, the first two terms of the RHS of (2.138) are bounded by

$$
\varepsilon\|wg\|_\infty\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})f\|_\nu+\varepsilon\|wf\|_\infty\|\varepsilon^{-1}(\mathbf{I}-\mathbf{P})g\|_\nu.
$$

Due to the strong decay in v of $\mathbf{P}f$, we have $\big\|\frac{1}{\mu^{0+}}|\mathbf{P}f(x,v)|\big\|_{L^\infty_v}\lesssim\|\mathbf{P}f(x)\|_{L^p_v}$ for any $1\le p\le\infty$. Recall that μ^{0+} stands μ^δ for some $0<\delta\ll 1$. The last term of (2.138) is bounded as, for fixed v, by $\|\nu^{-1/2}\Gamma(\mu^{0+},\mu^{0+})\|_{L^2_v}<\infty$,

$$
\begin{aligned}
\|\nu^{-\frac12}\Gamma_\pm(\mathbf{P}f,\mathbf{P}g)\|_{L^2_{x,v}}&\lesssim\|\nu^{-1/2}\Gamma(\mu^{0+},\mu^{0+})\|_{L^2_v}\Big\|\|\mathbf{P}f(\cdot)\|_{L^6_v}\|\mathbf{P}g(\cdot)\|_{L^3_v}\Big\|_{L^2_x}\\
&\lesssim\|\mathbf{P}f\|_{L^6_{x,v}}\|\mathbf{P}g\|_{L^3_{x,v}}.
\end{aligned}
$$

All together we prove (2.137). ◻

Lemma 2.14 *Recall r_s, f_w, A_s, $\mathcal{Q}$ in (1.25), (1.11), (1.35), (1.27). We have*

$$
\begin{aligned}
&|r_s|_{2,-}+|wr_s|_{\infty,-}\lesssim|\vartheta_w|_\infty,\\
&\|f_w\|_{L^6_xL^2_v}+\|wf_w\|_\infty\lesssim|\vartheta_w|_\infty,\\
&\|(\mathbf{I}-\mathbf{P})A_s\|_{L^2(\Omega\times\mathbb{R}^3)}\lesssim|\vartheta_w|_{W^{1,\infty}(\partial\Omega)}+\varepsilon^2\|\Phi\|_\infty|\vartheta_w|_2,
\end{aligned}
$$

$$\|\mathbf{P}A_s\|_{L^2(\Omega\times\mathbb{R}^3)} \le \varepsilon\|\Phi\|_2,$$
$$\|wA_s\|_{L^\infty(\Omega\times\mathbb{R}^3)} \lesssim |\vartheta_w|_{W^{1,\infty}(\partial\Omega)} + \varepsilon^2\|\Phi\|_\infty|\vartheta_w|_\infty + \varepsilon\|\Phi\|_\infty + |\vartheta_w|^2_\infty,$$
$$|\mathscr{Q}f|_{2,-} \lesssim \|\vartheta_w\|_{L^\infty(\partial\Omega)}\big[1+\varepsilon\|\vartheta_w\|_{L^\infty(\partial\Omega)}\big]\|\sqrt{\mu}f\|_{L^2(\gamma_+)},$$
$$\|w\mathscr{Q}f\|_\infty \lesssim |\vartheta_w|_\infty\big[1+\varepsilon|\vartheta_w|_\infty\big]\|\sqrt{\mu}f\|_{L^\infty(\bar\Omega\times\mathbb{R}^3)}.$$

Proof From (1.25) and (1.23) we have the first estimate. From (1.11) and (1.10) we have the second estimate. From the (1.35) we have the third, fourth, and fifth estimates. Finally, from (1.13), (1.27) and (1.29) we have

$$\mathscr{Q}f(x,v) = \sqrt{2\pi}\Big(\frac{|v|^2}{2}-2\Big)\sqrt{\mu(v)}\vartheta_w(x)\int_{n(x)\cdot u>0} f(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u$$
$$+\varepsilon O(|\vartheta_w|^2)\langle v\rangle^4\sqrt{\mu(v)}\int_{n(x)\cdot u>0} R(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u,$$

which leads the last two estimates. □

Now we are ready to prove the main theorem for the steady case:

Proof of Theorem 1.1 We prove Theorem 1.1 by considering a sequence f^ℓ, for $\ell \ge 0$,

$$v\cdot\nabla_x f^{\ell+1} + \varepsilon^2\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v\Big[\sqrt{\mu}f^{\ell+1}\Big] + \frac{1}{\varepsilon}Lf^{\ell+1} = \Gamma(f^\ell,f^\ell) + L_1f^\ell + A_s,$$
$$f^{\ell+1}|_{\gamma_-} = P_\gamma f^{\ell+1} + \varepsilon\mathscr{Q}f^\ell + \varepsilon r_s, \quad R^0\equiv 0, \tag{2.142}$$

where L_1 and A_s are defined at (1.34) and (1.35), $\mathscr{Q}$ at (1.27), and r_s at (1.25). Note that Theorem 2.5, with (1.37), (1.31), guarantees the solvability of such a linear problem (2.142).

Step 1. We claim $[\![f^{l+1}]\!]^2 < \eta_0$. For $0 < \eta_0 \ll 1$, we assume that,

$$\|\vartheta_w\|^2_{H^{1/2}(\partial\Omega)} + \|\Phi\|_2^2 + \varepsilon(\|\vartheta_w\|^2_{W^{1,\infty}(\partial\Omega)} + \|\Phi\|^2_\infty) < c_0\eta_0, \tag{2.143}$$

for $0 < c_0 \ll 1$, and the induction hypothesis

$$\sup_{0\le j\le\ell} [\![f^j]\!]^2 < \eta_0, \tag{2.144}$$

where the norm $[\![\cdot]\!]$ is defined in (2.20).

We apply Theorem 2.5 for

$$f = f^{\ell+1},\ g = \Gamma(f^\ell,f^\ell) + L_1f^\ell + A_s,\ r = \varepsilon\mathscr{Q}f^\ell + \varepsilon r_s,$$

to achieve the same upper bound as in (2.144) for $[\![f^{\ell+1}]\!]^2$ in the next two steps.

We estimate the right hand side of (2.24) for our case. From (1.34), $\mathbf{P}L_1f = 0$ and

$$|L_1f^\ell(x,v)| \lesssim |\Theta_w(x)|\big\{|\Gamma_\pm(\langle v\rangle^2\sqrt{\mu},\mathbf{P}f^\ell)| + |\Gamma_\pm(\langle v\rangle^2\sqrt{\mu},(\mathbf{I}-\mathbf{P})f^\ell)|\big\}.$$

Then

$$\|\nu^{-1/2}(\mathbf{I}-\mathbf{P})L_1 f^\ell\|_2 \lesssim \|\Theta_w\|_3\|\mathbf{P}f^\ell\|_6 + \varepsilon\|\Theta_w\|_\infty[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f^\ell\|_\nu]. \quad (2.145)$$

Using (2.145), (2.137), (1.34), (1.11) and Lemma 2.14, we obtain

$$\begin{aligned}
\|\nu^{-\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_2 &\lesssim [\![f^\ell]\!]^2 + (\|\Theta_w\|_3 + \varepsilon\|\Theta_w\|_\infty)[\![f^\ell]\!] \\
&\quad + \|\nabla_x\Theta_w\|_2 + \varepsilon^2\|\Phi\Theta_w\|_2 + \||\Theta_w|^2\|_2, \\
\varepsilon^{-1}\|\mathbf{P}g\|_2 &\lesssim \|\Phi\|_2, \\
\varepsilon^{3/2}\|\langle v\rangle^{-1}wg\|_\infty &\lesssim \varepsilon^{1/2}[\![f^\ell]\!]^2 + \varepsilon\|\Theta_w\|_\infty[\![f^\ell]\!] + \varepsilon^{5/2}\|\Phi\|_\infty + \varepsilon^{3/2}\|\nabla_x\Theta_w\|_\infty \\
&\quad + \varepsilon^{7/2}\|\Phi\|_\infty\|\Theta_w\|_\infty + \varepsilon^{3/2}\|\Theta_w\|_\infty^2.
\end{aligned}$$

From Lemma 2.14

$$\begin{aligned}
\varepsilon^{-1/2}|r|_2 &\lesssim \varepsilon^{1/2}|\vartheta_w|_\infty[\![f^\ell]\!] + \varepsilon^{1/2}|\vartheta_w|_\infty, \\
\varepsilon^{1/2}|wr|_\infty &\lesssim \varepsilon^{3/2}|\vartheta_w|_\infty[\![f^\ell]\!] + \varepsilon^{3/2}|\vartheta_w|_\infty.
\end{aligned}$$

Finally applying Theorem 2.5, we conclude that

$$[\![f^{\ell+1}]\!]^2 \lesssim \{(1+\varepsilon)[\![f^\ell]\!]^2 + (\|\Theta_w\|_3 + \varepsilon\|\Theta_w\|_\infty)^2 + (\varepsilon^2+\varepsilon^3)|\vartheta_w|_\infty^2\}[\![f^\ell]\!]^2 + c_0\eta_0. \quad (2.146)$$

By $\|\Theta_w\|_3 \lesssim \|\Theta_w\|_{H^1(\Omega)} \lesssim |\vartheta_w|_{H^{\frac{1}{2}}(\partial\Omega)} < c_0\eta_0 \ll 1$ we prove that $[\![f^{\ell+1}]\!]^2 < \eta_0$.
Step 2. We repeat *Step 1* for $f^{\ell+1} - f^\ell$ to show that f^ℓ is Cauchy sequence in $L^\infty \cap L^2$ for fixed ε. Now it is standard to conclude that the limiting $f^\ell \to f$ solves the equation. The uniqueness is standard. (See [17] for the details)
Step 3. To prove the weak convergence of f^ε, we use the argument of [5] where it is proved, in the unsteady case and without boundary, that, if f^ε converges weakly to a limit, then the limit has to be in the null space of L and its components have to solve the INSF system. Adding a force field is straightforward. The boundary condition issue requires a little more care. Let $g^\varepsilon = f_w + f_s^\varepsilon$. The equation for g^ε is

$$v\cdot\nabla_x g^\varepsilon + \varepsilon^2\frac{\Phi\cdot\nabla_v(g^\varepsilon\sqrt{\mu})}{\sqrt{\mu}} + \frac{1}{\varepsilon}Lg^\varepsilon = \Gamma(g^\varepsilon, g^\varepsilon) + \varepsilon\Phi\cdot v\sqrt{\mu}. \quad (2.147)$$

From the previous results we know that $\|(\mathbf{I}-\mathbf{P})g^\varepsilon\|_\nu \to 0$ as $\varepsilon \to 0$. Moreover $\mathbf{P}g^\varepsilon$ is bounded in L^6_x and hence weakly compact and $\langle v\rangle^{-1}\Gamma(g^\varepsilon, g^\varepsilon)$ is bounded in $L^2_{x,v}$. Therefore

$$v\cdot\nabla_x(g^\varepsilon\langle v\rangle^{-1}) + \varepsilon^2\langle v\rangle^{-1}\mu^{-\frac{1}{2}}\Phi\cdot\nabla_v(g^\varepsilon\sqrt{\mu}) \in L^2_{x,v}.$$

Passing to the (weak) limit as $\varepsilon \to 0$, up to subsequences, $g^\varepsilon \to g_1$ weakly and $\varepsilon^2\langle v\rangle^{-1}\mu^{-\frac{1}{2}}\Phi\cdot\nabla_v(g^\varepsilon\sqrt{\mu}) \to 0$ in the sense of distribution, so that, as distribution

$$v\cdot\nabla_x(g^\varepsilon\langle v\rangle^{-1}) + \varepsilon^2\langle v\rangle^{-1}\mu^{-\frac{1}{2}}\Phi\cdot\nabla_v(g^\varepsilon\sqrt{\mu}) \to v\cdot\nabla_x(g_1\langle v\rangle^{-1}).$$

But $v \cdot \nabla_x (g^\varepsilon \langle v\rangle^{-1}) + \varepsilon^2 \langle v\rangle^{-1} \mu^{-\frac{1}{2}} \Phi \cdot \nabla_v (g^\varepsilon \sqrt{\mu})$ has a weak limit in $L^2_{x,v}$. By the uniqueness of the distribution limit, we deduce that the limit $g_1 = \mathbf{P} g_1$ is such that

$$v \cdot \nabla_x (g_1 \langle v\rangle^{-1}) \in L^2_{x,v}, \quad \|g_1\|_{L^6} \ll 1.$$

But $g_1 = \{\rho + u \cdot v + \theta(|v|^2 - 3)/2\}\sqrt{\mu}$, and, from the linear independence of $\frac{v}{\sqrt{v}}\{1, v, v \otimes v, |v|^2, v|v|^2\}\sqrt{\mu}$, we deduce that $\rho, u, \theta \in H^1_x$. The equation for the hydrodynamic fields are deduced as in [5] as follows: we apply $\mathbf{P}$ to equation (2.147) and take the weak limit to obtain that

$$\mathbf{P}(v \cdot \nabla_x g_1) = 0,$$

which is equivalent to

$$\nabla_x (\rho + \theta) = 0, \quad \nabla_x \cdot u = 0.$$

Then we multiply equation (2.147) by $\varepsilon^{-1} v \sqrt{\mu}$, integrate on velocity and take the weak limit. We obtain

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \nabla_x \cdot (v\sqrt{\mu}, v \cdot \nabla_x g^\varepsilon)_{L^2_v} = \Phi.$$

To compute the above limit we write

$$\varepsilon^{-1} \nabla_x \cdot (v \otimes v \sqrt{\mu}, g^\varepsilon)_{L^2_v} = \left(L^{-1} \left(v \otimes v - \frac{|v|^2}{3} \mathbb{I} \right), \varepsilon^{-1} L g^e \right)_{L^2_v} + \nabla_x p^e,$$

where $p^\varepsilon = \varepsilon^{-1} (\frac{1}{3}|v|^2 \sqrt{\mu}, g^\varepsilon)_{L^2_v}$. In order to compute the limit of the first term, we note that, from the equation, the weak limit

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} L g^\varepsilon = \Gamma(g_1, g_1) - v \cdot \nabla_x g_1.$$

Hence we obtain

$$\Phi = \nabla_x \left(\left(v \otimes v - \frac{|v|^2}{3} \mathbb{I} \right), L^{-1} (\Gamma(g_1, g_1) - v \cdot \nabla_x g_1)_{L^2_v} + \nabla_x p, \right.$$

where $p = \lim_\varepsilon p_\varepsilon$. It is standard to compute that

$$\nabla_x \cdot \left(v \otimes v - \frac{|v|^2}{3} \mathbb{I}), L^{-1} (\Gamma(g_1, g_1) - v \cdot \nabla_x g_1 \right)_{L^2_v} = u \otimes u - \mathfrak{v} \Delta u,$$

and hence u is a weak solution to the incompressible Navier–Stokes equation $(1.40)_1$. Similar arguments can be used to obtain $(1.40)_2$. The conditions $\nabla_x (\rho + \theta)$ and $\nabla_x \cdot u$ have been already obtained.

We only need to check the boundary conditions. We return to $f^\varepsilon_s = g^\varepsilon - f_w$. By the smoothness of f_w,

$$v\cdot\nabla_x(f_s^\varepsilon\langle v\rangle^{-1})+\varepsilon^2\langle v\rangle^{-1}\mu^{-\frac{1}{2}}\Phi\cdot\nabla_v(f_s^\varepsilon\sqrt{\mu})\to v\cdot\nabla_x(f_1\langle v\rangle^{-1})\in L^2_{x,v}\text{ weakly.}$$

By Lemma 2.3, $f_s^\varepsilon\langle v\rangle^{-1}$ has local trace which weakly converges to $\langle v\rangle^{-1}P_\gamma f_1$ because $(1-P_\gamma)f_s^\varepsilon\langle v\rangle^{-1}\to 0$. So f_1 has a local trace $P_\gamma f_1$ on γ. But $f_1\in H_x^1C_v^\infty$, so that, for each $v\in\mathbb{R}^3$ the components $\rho-\rho_w$, u and $\theta-\Theta_w$ of $\mathbf{P}f_1$ have trace on the boundary. Hence $\mathbf{P}f_1=P_\gamma f_1$. Thus $u=0$ and $\theta=\vartheta_w$ on $\partial\Omega$. Recall $||g_1||_{L^6}\ll 1$, so is $||u||_{L^6}+||\theta||_{L^6}$, hence all the weak limit points must coincide with the unique solution to the steady Navier–Stokes–Fourier solution.

The positivity $F_s\geq 0$ is left for the unsteady case in Sect. 3.7. □

3 Unsteady Problems

3.1 Preliminary and the Linear Theorem

Definition 3.1 Assume $\Phi=\Phi(x)\in C^1$. Consider an unsteady linear transport equation

$$\varepsilon\partial_t f+v\cdot\nabla_x f+\varepsilon^2\Phi\cdot\nabla_v f\ =\ g. \tag{3.1}$$

The equations of the characteristics for (3.1) are

$$\dot{Y}=\varepsilon^{-1}W,\quad \dot{W}=\varepsilon\Phi(Y),\quad Y(t;t,x,v)=x,\ \ W(t;t,x,v)=v. \tag{3.2}$$

By the uniqueness of ODE

$$\begin{aligned}&[Y(s;t,x,v),W(s;t,x,v)]\\&=\left[X(t-\frac{t-s}{\varepsilon};t,x,v),V(t-\frac{t-s}{\varepsilon};t,x,v)\right]\\&=[X(\varepsilon^{-1}s;0,x,v),V(\varepsilon^{-1}s;0,x,v)],\end{aligned} \tag{3.3}$$

where (X,V) is defined in (2.7).

Define

$$\begin{aligned}\tilde{t}_{\mathbf{b}}(x,v)&:=\ \sup\{t>0:Y(-s;0,x,v)\in\Omega\text{ for all }0<s<t\}\\&=\ \varepsilon\sup\left\{\frac{t}{\varepsilon}>0:X\left(-\frac{s}{\varepsilon};0,x,v\right)\in\Omega\ \text{ for all }0<\frac{s}{\varepsilon}<\frac{t}{\varepsilon}\right\}\ =\ \varepsilon t_{\mathbf{b}}(x,v),\\\tilde{t}_{\mathbf{f}}(x,v)&:=\ \sup\{t>0:Y(s;0,x,v)\in\Omega\text{ for all }0<s<t\}\\&=\ \varepsilon\sup\left\{\frac{t}{\varepsilon}>0:X\left(\frac{s}{\varepsilon};0,x,v\right)\in\Omega\ \text{ for all }0<\frac{s}{\varepsilon}<\frac{t}{\varepsilon}\right\}\ =\ \varepsilon t_{\mathbf{f}}(x,v).\end{aligned} \tag{3.4}$$

Moreover

$$\begin{aligned}\tilde{x}_{\mathbf{b}}(x,v)&=Y(-\tilde{t}_{\mathbf{b}}(x,v);0,x,v)=X\left(-\frac{\tilde{t}_{\mathbf{b}}(x,v)}{\varepsilon};0,x,v\right)\\&=X(-t_{\mathbf{b}}(x,v);0,x,v)=x_{\mathbf{b}}(x,v),\end{aligned}$$

$$\begin{aligned}
\tilde{x}_{\mathbf{f}}(x,v) &= Y(-\tilde{t}_{\mathbf{f}}(x,v);0,x,v) = X\left(-\frac{\tilde{t}_{\mathbf{f}}(x,v)}{\varepsilon};0,x,v\right)\\
&= X(-t_{\mathbf{f}}(x,v);0,x,v) = x_{\mathbf{f}}(x,v),\\
\tilde{v}_{\mathbf{b}}(x,v) &= W(-\tilde{t}_{\mathbf{b}}(x,v);0,x,v) = V\left(-\frac{\tilde{t}_{\mathbf{b}}(x,v)}{\varepsilon};0,x,v\right)\\
&= V(-t_{\mathbf{b}}(x,v);0,x,v) = v_{\mathbf{b}}(x,v),\\
\tilde{v}_{\mathbf{f}}(x,v) &= W(-\tilde{t}_{\mathbf{f}}(x,v);0,x,v) = V\left(-\frac{\tilde{t}_{\mathbf{f}}(x,v)}{\varepsilon};0,x,v\right)\\
&= V(-t_{\mathbf{f}}(x,v);0,x,v) = v_{\mathbf{f}}(x,v).
\end{aligned}\tag{3.5}$$

Lemma 3.2 *For* $f \in L^1([0,T]\times\Omega\times\mathbb{R}^3)$,

$$\begin{aligned}
&\int_0^T \int_{\gamma_+^\delta} |f(t,x,v)|\mathrm{d}\gamma\mathrm{d}t\\
&\lesssim \varepsilon \iint_{\Omega\times\mathbb{R}^3} |f(0,x,v)|\mathrm{d}v\mathrm{d}x + \varepsilon\int_0^T \iint_{\Omega\times\mathbb{R}^3} |f(t,x,v)|\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\quad+ \int_0^T \iint_{\Omega\times\mathbb{R}^3} \big|[\varepsilon\partial_t f + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f](t,x,v)\big|\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.6}$$

Proof First we claim

$$\begin{aligned}
&\int_{\gamma_+^\delta} \mathrm{d}S_x\mathrm{d}v|n(x)\cdot v| \int_0^T \mathrm{d}t \int_{\max\{-\varepsilon t_{\mathbf{b}}(x,v),-t\}}^0 \mathrm{d}s\\
&\quad\times|f(t+s, X(\varepsilon^{-1}s;0,x,v), V(\varepsilon^{-1}s;0,x,v))|\\
&\quad\lesssim \varepsilon\int_0^T \iint_{\Omega\times\mathbb{R}^3} |f(t,x,v)|\mathrm{d}x\mathrm{d}v\mathrm{d}t.
\end{aligned}\tag{3.7}$$

By the Fubini theorem and the change of variables $\tilde{s} = \varepsilon^{-1}s$,

$$\begin{aligned}
&\int_{\gamma_+^\delta} \mathrm{d}S_x\mathrm{d}v|n(x)\cdot v| \int_0^T \mathrm{d}t \int_{\max\{-\varepsilon t_{\mathbf{b}}(x,v),-t\}}^0 \mathrm{d}s\cdots\\
&= \int_{\gamma_+^\delta} \mathrm{d}S_x\mathrm{d}v|n(x)\cdot v| \int_{\max\{-\varepsilon t_{\mathbf{b}}(x,v),-T\}}^0 \mathrm{d}s \int_{-s}^T \mathrm{d}t\\
&\quad\times|f(t+s, X(\varepsilon^{-1}s;0,x,v), V(\varepsilon^{-1}s;0,x,v))|\\
&\le \varepsilon\int_{\gamma_+^\delta} \mathrm{d}S_x\mathrm{d}v|n(x)\cdot v| \int_{\max\{-t_{\mathbf{b}}(x,v),-\varepsilon^{-1}T\}}^0 \mathrm{d}\tilde{s}\int_0^T \mathrm{d}t\\
&\quad\times|f(t, X(\tilde{s};0,x,v), V(\tilde{s};0,x,v))|\\
&\lesssim \varepsilon\iint_{\Omega\times\mathbb{R}^3}\int_0^T f(t,x,v)\mathrm{d}t\mathrm{d}x\mathrm{d}v,
\end{aligned}$$

where we have used Lemma 2.2 and $t_{\mathbf{b}}(x,v) \lesssim_\delta 1$ for $(x,v)\in\gamma_+^\delta$.

Now we recall that

$$\begin{aligned}&\frac{d}{ds}|f(t+s, X(\varepsilon^{-1}\, s; 0, x, v), V(\varepsilon^{-1}\, s; 0, x, v))|\\&\lesssim \big|\big[\partial_t f+\varepsilon^{-1} V\cdot\nabla_x f+\varepsilon\Phi(X)\cdot\nabla_v f\big](t+s, X(\varepsilon^{-1}\, s), V(\varepsilon^{-1}\, s))\big|,\end{aligned}$$

and

$$\begin{aligned}|f(t,x,v)| &\lesssim |f(t+s, X(\varepsilon^{-1}s), V(\varepsilon^{-1}s))|\\&\quad+\int_s^t \big|[\partial_t f+\varepsilon^{-1}V\cdot\nabla_x f+\varepsilon\Phi(X)\cdot\nabla_v f](t+\tau, X(\varepsilon^{-1}\tau), V(\varepsilon^{-1}\tau))\big|\mathrm{d}\tau.\end{aligned}$$

For $(y,u)\in\gamma_+$ and for $s\in[\max\{-\varepsilon t_{\mathbf{b}}(y,-u),-t\},0]$

$$\begin{aligned}&\min\{\varepsilon t_{\mathbf{b}}(y,-u),t\}\times|f(t,y,u)|\\&=\int_{\max\{-\varepsilon t_{\mathbf{b}}(y,-u),-t\}}^0 |f(t+s, X(\varepsilon^{-1}s;0,y,u), V(\varepsilon^{-1}s;0,y,u))|\mathrm{d}s\\&\quad+\int_{\max\{-\varepsilon t_{\mathbf{b}}(y,-u),-t\}}^0\int_s^t \Big|\Big[\partial_t f+\varepsilon^{-1}V\cdot\nabla_x f+\varepsilon\Phi\cdot\nabla_v f\Big]\\&\quad\times(t+\tau, X(\varepsilon^{-1}\tau;0,y,u), V(\varepsilon^{-1}\tau;0,y,u))\Big|\mathrm{d}\tau\mathrm{d}s.\end{aligned}$$

If $(t,y,u)\in[\varepsilon\delta_1,T]\times\gamma_+\backslash\gamma_+^\delta$ then $t_{\mathbf{b}}(y,-u)\gtrsim_\Omega |n(y)\cdot u|/|u|^2\gtrsim\delta^3$. We use (3.7) to bound

$$\begin{aligned}&\min\{\varepsilon\delta^3,\varepsilon\delta_1\}\times\int_{\gamma_+^\delta}\mathrm{d}S_x\mathrm{d}v|n(y)\cdot u|\int_{\varepsilon\delta_1}^T \mathrm{d}t\ |f(t,y,u)|\\&\le\int_{\gamma_+^\delta}\mathrm{d}S_x\mathrm{d}v|n(y)\cdot u|\int_{\varepsilon\delta_1}^T\mathrm{d}t\int_{\max\{-\varepsilon t_{\mathbf{b}}(y,-u),-t\}}^0 |f(t+s,X(\varepsilon^{-1}s),V(\varepsilon^{-1}s))|\mathrm{d}s\\&\quad+\int_{\gamma_+^\delta}\mathrm{d}S_x\mathrm{d}v|n(y)\cdot u|\int_{\varepsilon\delta_1}^T\mathrm{d}t\int_{\max\{-\varepsilon t_{\mathbf{b}}(y,-u),-t\}}^0\mathrm{d}s\\&\quad\times\int_s^t\big|[\partial_t f+\varepsilon^{-1}V\cdot\nabla_x f+\varepsilon\Phi\cdot\nabla_v f](t+\tau,X(\varepsilon^{-1}\tau),V(\varepsilon^{-1}\tau))\big|\mathrm{d}\tau\\&\lesssim\varepsilon\int_0^T\iint_{\Omega\times\mathbb{R}^3}|f(t,x,v)|\mathrm{d}v\mathrm{d}x\mathrm{d}t\\&\quad+\int_0^T\iint_{\Omega\times\mathbb{R}^3}\big|[\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v]f(t,x,v)\big|\mathrm{d}v\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $(X(\varepsilon^{-1}\tau),V(\varepsilon^{-1}\tau))=(X(\varepsilon^{-1}\tau;0,y,u),V(\varepsilon^{-1}\tau;0,y,u))$. We choose $\varepsilon\delta_1\ll\varepsilon\delta^3\lesssim\varepsilon\times\inf_{\gamma_+\backslash\gamma_+^\delta}t_{\mathbf{b}}(y,-u)$. Then $\varepsilon t_{\mathbf{b}}(y,-u)>t$ for all $(t,y,u)\in[0,\varepsilon\delta_1]\times\gamma_+\backslash\gamma_+^\delta$ so that the backward trajectory hits the initial plan first:

$$
\begin{aligned}
|f(t,y,u)| \le\, & |f(0, X(\varepsilon^{-1}t), V(\varepsilon^{-1}t))| \\
& + \int_{-t}^{0} \big|[\partial_t f + \varepsilon^{-1} V\cdot\nabla_x f + \varepsilon\Phi(X)\cdot\nabla_v f](t+s, X(\varepsilon^{-1}s), V(\varepsilon^{-1}s))\big|.
\end{aligned}
$$

Then from (3.7) and the change of variables

$$
\begin{aligned}
&\int_0^{\varepsilon\delta_1}\int_{\gamma_+} |f(t,y,u)| \\
&\le \int_0^{\varepsilon\delta_1} \mathrm{d}t \int_{\gamma_+} \mathrm{d}S_y \mathrm{d}v\, |n(y)\cdot u||f(0, X(\varepsilon^{-1}t;0,y,u), V(\varepsilon^{-1}t;0,y,u))| \\
&+ \int_0^{\varepsilon\delta_1} \mathrm{d}t \int_{\gamma_+} \mathrm{d}S_y \mathrm{d}v\, |n(y)\cdot u| \int_{-t}^{0} \\
&\quad \times \big|[\partial_t f + \varepsilon^{-1} V\cdot\nabla_x f + \varepsilon\Phi(X)\cdot\nabla_v f](t+s, X(\varepsilon^{-1}s), V(\varepsilon^{-1}s))\big|\mathrm{d}s \\
&\le \varepsilon \int_0^{\delta_1} \mathrm{d}\tilde{t} \int_{\gamma_+} \mathrm{d}S_y \mathrm{d}v\, |n(y)\cdot u||f(0, X(\tilde{t}), V(\tilde{t}))| \qquad (\varepsilon^{-1}t = \tilde{t}) \\
&+ \int_0^T \iint_{\Omega\times\mathbb{R}^3} \big|[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(t,x,v)\big|\mathrm{d}v\mathrm{d}x\mathrm{d}t \qquad (\varepsilon^{-1}s = \tilde{s}) \\
&\lesssim \varepsilon\delta_1 \iint_{\Omega\times\mathbb{R}^3} |f_0(x,v)|\mathrm{d}v\mathrm{d}x \\
&\quad + \int_0^T \iint_{\Omega\times\mathbb{R}^3} \big|[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(t,x,v)\big|\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

□

Lemma 3.3 *Assume* $\Phi \in C^1$. *Assume that* $f(t,x,v),\ g(t,x,v) \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$, $\{\partial_t + \varepsilon^{-1} v\cdot\nabla_x + \varepsilon\Phi\cdot\nabla_v\} f$, $\{\partial_t + \varepsilon^{-1} v\cdot\nabla_x + \varepsilon\Phi\cdot\nabla_v\} g \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$ *and* $f_\gamma, g_\gamma \in L^2(\mathbb{R}_+ \times \gamma)$. *Then*

$$
\begin{aligned}
&\int_s^t \iint_{\Omega\times\mathbb{R}^3} \{\varepsilon\partial_t + v\cdot\nabla_x f + \varepsilon^2\Phi\cdot\nabla_v f\} g + \{\varepsilon\partial_t + v\cdot\nabla_x g + \varepsilon^2\Phi\cdot\nabla_v g\} f\ \mathrm{d}v\mathrm{d}x\mathrm{d}\tau \\
&= \varepsilon \iint_{\Omega\times\mathbb{R}^3} f(s,x,v)g(s,x,v)\mathrm{d}v\mathrm{d}x - \varepsilon \iint_{\Omega\times\mathbb{R}^3} f(t,x,v)g(t,x,v)\mathrm{d}v\mathrm{d}x \\
&\quad + \int_s^t \left[\int_{\gamma_+} fg\mathrm{d}\gamma - \int_{\gamma_-} fg\mathrm{d}\gamma\right]\mathrm{d}\tau.
\end{aligned}
$$

Proof The proof is from Chapter 9 of [14] with the same modification as Lemma 2.4. □

3.2 Gain of Integrability: $L^2_t L^3_x$ Estimate

The main goal of this section is the following:

Proposition 3.4 *Assume* $g \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$, $f_0 \in L^2(\Omega \times \mathbb{R}^3)$, *and* $f_\gamma \in L^2(\mathbb{R}_+ \times \gamma)$. *Let* $f \in L^\infty(\mathbb{R}_+; L^2(\Omega \times \mathbb{R}^3))$ *solve (3.1) in the sense of distribution and satisfy* $f(t,x,v) = f_\gamma(t,x,v)$ *on* $\mathbb{R}_+ \times \gamma$ *and* $f(0,x,v) = f_0(x,v)$ *on* $\Omega \times \mathbb{R}^3$. *Recall* $\mathbf{P}f$ *in* (1.19).

Then there exist $\mathbf{S}_1 f(t,x)$, $\mathbf{S}_2 f(t,x)$, *and* $\mathbf{S}_3 f(t,x)$ *satisfying*

$$|a(t,x)| + |b(t,x)| + |c(t,x)| \leq \mathbf{S}_1 f(t,x) + \mathbf{S}_2 f(t,x) + \mathbf{S}_3 f(t,x), \tag{3.8}$$

where the precise form of $\mathbf{S}_i f$ *is defined in* (3.40).

Moreover,

$$\begin{aligned} &\|\mathbf{S}_1 f\|_{L^2_t L^3_x} + \varepsilon^{-\frac{1}{2}} \|\mathbf{S}_2 f\|_{L^2_t L_x^{\frac{12}{5}}} \\ &\lesssim \|g\|_{L^2_{t,x,v}} + \|f_0\|_{L^2_{x,v}} + \|[v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v] f_0\|_{L^2_{x,v}} + \|f_0\|_{L^2(\gamma)} \\ &\quad + \|f_\gamma\|_{L^2(\mathbb{R}_+ \times \gamma)}, \end{aligned} \tag{3.9}$$

and

$$\|\mathbf{S}_3 f\|_{L^2_{t,x}} \lesssim \|(\mathbf{I} - \mathbf{P}) f\|_{L^2_{t,x,v}}. \tag{3.10}$$

We need several lemmas to prove Proposition 3.4. First we define f_δ which represents either the interior or the non-grazing parts of f near the boundary.

Definition 3.5 We define, for $(t,x,v) \in \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^3$ and for $0 < \delta \ll 1$,

$$\begin{aligned} &f_\delta(t,x,v) \\ &:= \left[1 - \chi\left(\frac{n(x)\cdot v}{\delta}\right)\chi\left(\frac{\xi(x)}{\delta}\right)\right]\chi(\delta|v|)\big\{\mathbf{1}_{t\in[0,\infty)} f(t,x,v) \\ &\quad + \mathbf{1}_{t\in(-\infty,0]}\chi(t) f_0(x,v)\big\}. \end{aligned} \tag{3.11}$$

Here $n(x)$ and χ are defined in (2.5) and (2.4) respectively.

Here we extend f_δ to the negative time so that we are able to take the time-derivative. Clearly,

$$\begin{aligned} \|f_\delta\|_{L^2(\mathbb{R}\times\Omega\times\mathbb{R}^3)} &\lesssim \|f\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)} + \|f_0\|_{L^2(\Omega\times\mathbb{R}^3)}, \\ \|f_\delta\|_{L^2(\mathbb{R}\times\gamma)} &\lesssim \|f_\gamma\|_{L^2(\mathbb{R}_+\times\gamma)} + \|f_0\|_{L^2(\gamma)}. \end{aligned}$$

Note that, at the boundary $(x,v) \in \gamma := \partial\Omega \times \mathbb{R}^3$,

$$f_\delta(t,x,v)|_\gamma \equiv 0, \quad \text{for } |n(x)\cdot v| \leq \delta \ \text{ or } \ |v| \geq \frac{1}{\delta}. \tag{3.12}$$

Lemma 3.6 *Assume the same hypothesis of Proposition* 3.4. *Then there exists* $\bar{f}(t,x,v) \in L^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$, *an extension of* f_δ *in* (3.11), *such that*

$$\bar{f}|_{\Omega\times\mathbb{R}^3} \equiv f_\delta \ \ \textit{and} \ \ \bar{f}|_\gamma \equiv f_\delta|_\gamma \ \ \textit{and} \ \ \bar{f}|_{t=0} = f_\delta|_{t=0}.$$

Moreover, in the sense of distributions on $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$\{\varepsilon \partial_t + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v\} \bar{f} = h_1 + h_2 + h_3 + h_4, \tag{3.13}$$

where

$$\begin{aligned}
h_1(t,x,v) &= \mathbf{1}_{(x,v)\in\Omega\times\mathbb{R}^3} \left[1 - \chi\left(\frac{n(x)\cdot v}{\delta}\right)\chi\big(\frac{\xi(x)}{\delta}\big)\right]\chi(\delta|v|) \\
&\quad \times \Big[\mathbf{1}_{t\in[0,\infty)} g(t,x,v) + \mathbf{1}_{t\in(-\infty,0]}\chi(t)\{\varepsilon\frac{\chi'(t)}{\chi(t)} \\
&\quad + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\} f_0(x,v)\Big], && (3.14)\\
h_2(t,x,v) &= \mathbf{1}_{(x,v)\in\Omega\times\mathbb{R}^3}\big[\mathbf{1}_{t\in[0,\infty)} f(t,x,v) + \mathbf{1}_{t\in(-\infty,0]}\chi(t) f_0(x,v)\big] \\
&\quad \times\{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\}\Big(\big[1 - \chi\big(\frac{n(x)\cdot v}{\delta}\big)\chi\big(\frac{\xi(x)}{\delta}\big)\big]\chi(\delta|v|)\Big), && (3.15)\\
h_3(t,x,v) &= \mathbf{1}_{\ (x,v)\in\left[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}\right]\times\mathbb{R}^3}\ \frac{1}{\tilde{C}\delta^4} v\cdot\nabla_x\xi(x)\chi'\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big) \\
&\quad \times\big[f_\delta(t - \varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\
&\quad + f_\delta(t + \varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big], && (3.16)\\
h_4(t,x,v) &= \mathbf{1}_{\ (x,v)\in\left[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}\right]\times\mathbb{R}^3} f_\delta(t - \varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) \\
&\quad \times\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big)\chi'(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\
&\quad +\mathbf{1}_{\ (x,v)\in\left[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}\right]\times\mathbb{R}^3} f_\delta(t + \varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v)) \\
&\quad \times\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big)\chi'(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}, && (3.17)
\end{aligned}$$

where

$$\Omega_{\tilde{C}\delta^4} := \big\{x\in\mathbb{R}^3 : \xi(x) < \tilde{C}\delta^4\big\}, \tag{3.18}$$

and, for $(x,v)\in\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}$, *with* $\bar{\Omega} = \Omega\cup\partial\Omega$,

$$\begin{aligned}
t^*_{\mathbf{b}}(x,v) &:= \inf\{s>0 : 0 < \xi(X(s;0,x,v)) < \tilde{C}\delta^4 \ \textit{for all}\ 0<\tau<s\},\\
t^*_{\mathbf{f}}(x,v) &:= t^*_{\mathbf{b}}(x,-v),\\
(x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) &:= (X(-t^*_{\mathbf{b}}(x,v);0,x,v), V(-t^*_{\mathbf{b}}(x,v);0,x,v)),\\
(x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v)) &:= (X(t^*_{\mathbf{f}}(x,v);0,x,v), V(t^*_{\mathbf{f}}(x,v);0,x,v)). && (3.19)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|h_1\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)} \lesssim \|g\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)} + \varepsilon\|f_0\|_{L^2(\Omega\times\mathbb{R}^3)}\\
&\quad + \|[v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v]f_0\|_{L^2(\Omega\times\mathbb{R}^3)},\\
&\|h_2\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)} \lesssim_\delta \|f\|_{L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)} + \|f_0\|_{L^2(\Omega\times\mathbb{R}^3)},\\
&\|h_3\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)} + \|h_4\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)} \lesssim_\delta \|f_\gamma\|_{L^2(\mathbb{R}_+\times\gamma)} + \|f_0\|_{L^2(\gamma)}. && (3.20)
\end{aligned}$$

Proof The proof of this lemma is given in Appendix A.1. □

The next step is to prove a version of the velocity averaging lemma in the presence of a small external force. The presence of the external force requires significant modifications to the original argument in [23,50].

Lemma 3.7 *Let $f \in L^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$ be a function solving the transport equation*

$$\varepsilon \partial_t f + v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f + f = q, \tag{3.21}$$

in the sense of distributions with $q \in L^2_{t,x,v}$. Let $\psi(v)$ be a smooth function, which vanishes as $e^{C|v|}|\psi(v)| \to 0$ as $|v| \to \infty$ for some $C > 0$.

Then we can decompose f as

$$f = f_{la} + f_{sm}, \tag{3.22}$$

such that

$$\left\| \int_{\mathbb{R}^3} \psi f_{la} \mathrm{d}v \right\|_{L^2_t L^3_x} \lesssim_\psi \|w^{-1} q\|_{L^2_{t,x,v}}, \tag{3.23}$$

for $w = e^{\beta|v|^2}$ with $0 < \beta \ll 1$, and

$$\left\| \int_{\mathbb{R}^3} \psi f_{sm} \mathrm{d}v \right\|_{L^2_t L^{\frac{12}{5}}_x} \lesssim_{\psi, \|\Phi\|_\infty} \varepsilon^{\frac{1}{2}} \|w^{-1} q\|_{L^2_{t,x,v}}. \tag{3.24}$$

Proof We only prove the case $\beta = 0$, because the growing factor w can be absorbed in ψ by redefining f_{la}.

Step 1. We define a "large" part as

$$f_{la}(t, x, v) := \int_0^\infty e^{-\tau} q(t - \varepsilon\tau, x - \tau v, v) \mathrm{d}\tau. \tag{3.25}$$

Then, clearly f_{la} solves the transport equation without an external force, in the sense of distributions,

$$\varepsilon \partial_t f_{la} + v \cdot \nabla_x f_{la} + f_{la} = q. \tag{3.26}$$

Via the change of variables $(t - \varepsilon\tau, x - \tau v, v) \mapsto (t, x, v)$ for fixed τ and the Minkowski inequality

$$\begin{aligned} \|f_{la}\|_{L^2_{t,x,v}} &\le \int_0^\infty e^{-\tau} \|q(t - \varepsilon\tau, x - \tau v, v)\|_{L^2_{t,x,v}} \mathrm{d}\tau \\ &= \|q\|_{L^2_{t,x,v}} \int_0^\infty e^{-\tau} \mathrm{d}\tau \lesssim \|q\|_{L^2_{t,x,v}}. \end{aligned} \tag{3.27}$$

Therefore, f_{la} is in L^2. By the standard velocity averaging lemma [23,50], $\int_{\mathbb{R}^3} \psi f_{la} \mathrm{d}v$ is in $L^2_t H^{\frac{1}{2}}_x$ and satisfies the bound (3.23) by the Sobolev embedding $L^2(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}^3)) \subset L^2(\mathbb{R}; L^3(\mathbb{R}^3))$.

Step 2. Now we define a "small" part as

$$f_{sm}(t,x,v) := f(t,x,v) - f_{la}(t,x,v). \tag{3.28}$$

Clearly,

$$\varepsilon\partial_t f_{sm} + v\cdot\nabla_x f_{sm} + f_{sm} = -\varepsilon^2\Phi\cdot\nabla_v f. \tag{3.29}$$

We use the shorthand notation

$$\zeta = -\varepsilon^2\Phi f, \tag{3.30}$$

so that (3.29) becomes

$$\varepsilon\partial_t f_{sm} + v\cdot\nabla_x f_{sm} + f_{sm} = \nabla_v\zeta. \tag{3.31}$$

By Fourier transforming,

$$\hat f_{sm} = \frac{\nabla_v\hat\zeta}{i(\varepsilon\tau + v\cdot\xi)+1}, \quad |\hat f_{sm}| \lesssim \frac{|\nabla_v\hat\zeta|}{|\varepsilon\tau + v\cdot\xi|+1}. \tag{3.32}$$

Recall χ in (2.4). Then, for $\alpha>0$,

$$\begin{aligned}\int_{\mathbb{R}^3}\hat f_{sm}\psi &= \int_{\mathbb{R}^3}\hat f_{sm}\psi\chi\left(\frac{\varepsilon\tau+\xi\cdot v}{\alpha}\right) + \int_{\mathbb{R}^3}\hat f_{sm}\psi\left[1-\chi\left(\frac{\varepsilon\tau+\xi\cdot v}{\alpha}\right)\right]\\ &:= \mathscr{T}_1 + \mathscr{T}_2.\end{aligned}$$

Since in $\mathscr{T}_1$ we have only the contribution for $|\varepsilon\tau+\xi\cdot v|\le\alpha$, by integrating first on the velocity $v_{\xi^\perp}$ orthogonal to ξ and then on the one dimensional variable $v_\xi = \frac{v\cdot\xi}{|\xi|}$, with the change of variables $\frac{|\xi|v_\xi}{\alpha}$ we estimate,

$$|\mathscr{T}_1| \le \frac{\alpha^{\frac12}}{|\xi|^{\frac12}}\|\hat f_{sm}\|_{L^2_v}.$$

As for $\mathscr{T}_2$, we use (3.32) to obtain

$$\begin{aligned}\mathscr{T}_2 &= \int_{\mathbb{R}^3} dv\frac{\nabla_v\hat\zeta}{i(\varepsilon\tau+v\cdot\xi)+1}\psi\left[1-\chi\left(\frac{\varepsilon\tau+\xi\cdot v}{\alpha}\right)\right]\\ &= i\int_{\mathbb{R}^3} dv\frac{\xi\hat\zeta}{[i(\varepsilon\tau+v\cdot\xi)+1]^2}\psi\left[1-\chi\left(\frac{\varepsilon\tau+\xi\cdot v}{\alpha}\right)\right]\\ &\quad+\int_{\mathbb{R}^3} dv\frac{\xi\hat\zeta}{\alpha[i(\varepsilon\tau+v\cdot\xi)+1]}\psi\chi'\left(\frac{\varepsilon\tau+\xi\cdot v}{\alpha}\right)\\ &\quad+\int_{\mathbb{R}^3} dv\frac{\hat\zeta}{i(\varepsilon\tau+v\cdot\xi)+1}\nabla_v\psi\left[1-\chi\left(\frac{\varepsilon\tau+\xi\cdot v}{\alpha}\right)\right]\\ &:= \mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3.\end{aligned}$$

We have

$$
\begin{aligned}
|\mathscr{K}_1| &\le |\xi| \left(\int_{\mathbb{R}^3} dv \frac{\psi(1-\chi)}{[|\varepsilon\tau + v\cdot\xi| + 1]^4} \right)^{\frac{1}{2}} \|\hat{\zeta}\|_{L^2_v} \\
&\le |\xi| \left(\int_{|\varepsilon\tau+\xi\cdot v|\ge\alpha} dv \frac{1}{[|\varepsilon\tau + v\cdot\xi| + 1]^4} \right)^{\frac{1}{2}} \|\hat{\zeta}\|_{L^2_v} \\
&\le \frac{|\xi|^{\frac{1}{2}}}{\alpha^{\frac{3}{2}}} \|\hat{\zeta}\|_{L^2_v}, \\
|\mathscr{K}_2| &\le \frac{|\xi|}{\alpha^2} \left(\int_{\mathbb{R}^3} dv \left[\chi' \left(\frac{\varepsilon\tau + v\cdot\xi}{\alpha} \right) \right]^2 \right)^{\frac{1}{2}} \|\hat{\zeta}\|_{L^2_v} \\
&\le \frac{|\xi|}{\alpha^2} \left(\frac{\alpha}{|\xi|} \right)^{\frac{1}{2}} \|\hat{\zeta}\|_{L^2_v} = \frac{|\xi|^{\frac{1}{2}}}{\alpha^{\frac{3}{2}}} \|\hat{\zeta}\|_{L^2_v}, \\
|\mathscr{K}_3| &\le \left(\int_{|\varepsilon\tau+\xi\cdot v|>\alpha} dv \frac{|\nabla_v\psi|^2}{1 + |\varepsilon\tau + \xi\cdot v|^2} \right)^{\frac{1}{2}} \|\hat{\zeta}\|_{L^2_v} \le \frac{1}{\alpha^{\frac{1}{2}} |\xi|^{\frac{1}{2}}} \|\hat{\zeta}\|_{L^2_v}.
\end{aligned}
$$

Now we choose α such that

$$
\frac{\alpha^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}}} \|\hat{f}_{sm}\|_{L^2_v} = \frac{|\xi|^{\frac{1}{2}}}{\alpha^{\frac{3}{2}}} \|\hat{\zeta}\|_{L^2_v},
$$

which means

$$
\alpha = |\xi|^{\frac{1}{2}} \|\hat{\zeta}\|_{L^2_v}^{\frac{1}{2}} \|\hat{f}_{sm}\|_{L^2_v}^{-\frac{1}{2}}.
$$

With this choice, we have that,

$$
|\mathscr{T}_1| + |\mathscr{K}_1| + |\mathscr{K}_2| \le 3|\xi|^{-\frac{1}{4}} \|\hat{\zeta}\|_{L^2_v}^{\frac{1}{4}} \|\hat{f}_{sm}\|_{L^2_v}^{\frac{3}{4}}, \quad |\mathscr{K}_3| \le |\xi|^{-\frac{3}{4}} \|\hat{\zeta}\|_{L^2_v}^{\frac{3}{4}} \|\hat{f}_{sm}\|_{L^2_v}^{\frac{1}{4}}.
$$

Therefore, from (3.30)

$$
\begin{aligned}
\left\| |\xi|^{\frac{1}{4}} (|\mathscr{T}_1| + |\mathscr{K}_1| + |\mathscr{K}_2|) \right\|_{L^2_{\tau,\xi}} &\lesssim \left\| \|\hat{\zeta}\|_{L^2_v}^{\frac{1}{4}} \|\hat{f}_{sm}\|_{L^2_v}^{\frac{3}{4}} \right\|_{L^2_{\tau,\xi}} \le \varepsilon^{\frac{1}{2}} \|\Phi\|_\infty^{\frac{1}{4}} \|f\|_{L^2_{t,x,v}}^{\frac{1}{4}} \|f_{sm}\|_{L^2_{t,x,v}}^{\frac{3}{4}}, \\
\left\| |\xi|^{\frac{3}{4}} |\mathscr{K}_3| \right\|_{L^2_{\tau,\xi}} &\lesssim \left\| \|\hat{\zeta}\|_{L^2_v}^{\frac{3}{4}} \|\hat{f}_{sm}\|_{L^2_v}^{\frac{1}{4}} \right\|_{L^2_{\tau,\xi}} \le \varepsilon^{\frac{3}{2}} \|\Phi\|_\infty^{\frac{3}{4}} \|f\|_{L^2_{t,x,v}}^{\frac{3}{4}} \|f_{sm}\|_{L^2_{t,x,v}}^{\frac{1}{4}}.
\end{aligned}
$$

We also note that $\|\mathscr{K}_3\|_{L^2_{\tau,\xi}} \lesssim_\psi \|\hat{\zeta}\|_{L^2_{\tau,\xi,v}} \lesssim \varepsilon^2 \|\Phi\|_\infty \|f\|_{L^2_{t,x,v}}$.

Denote the inverse-Fourier transform by $\mathscr{F}^{-1}_{\tau,\xi}$. By the Sobolev embedding $\dot{H}^{\frac{1}{4}}(\mathbb{R}^3) \subset L^{\frac{12}{5}}(\mathbb{R}^3)$ and $\dot{H}^{\frac{3}{4}} \subset L^4$, we have

$$\begin{aligned}\|\mathscr{F}^{-1}_{\tau,\xi}(\mathscr{T}_1+\mathscr{K}_1+\mathscr{K}_2)\|_{L^2_tL_x^{\frac{12}{5}}} &\lesssim \|\mathscr{F}^{-1}_{\tau,\xi}(\mathscr{T}_1+\mathscr{K}_1+\mathscr{K}_2)\|_{L^2_t\dot{H}_x^{\frac14}}\\ &\lesssim \varepsilon^{\frac12}\|f\|^{\frac14}_{L^2_{t,x,v}}\|f_{sm}\|^{\frac34}_{L^2_{t,x,v}},\\ \|\mathscr{F}^{-1}_{\tau,\xi}\mathscr{K}_3\|_{L^2_tL^4_x} \lesssim \|\mathscr{F}^{-1}_{\tau,\xi}\mathscr{K}_3\|_{L^2_t\dot{H}_x^{\frac14}} &\lesssim \varepsilon^{\frac32}\|f\|^{\frac34}_{L^2_{t,x,v}}\|f_{sm}\|^{\frac14}_{L^2_{t,x,v}}.\end{aligned}$$

By an interpolation ($L_x^{\frac{12}{5}} \subset L^2_x \cap L^4_x$)

$$\|\mathscr{F}^{-1}_{\tau,\xi}\mathscr{K}_3\|_{L^2_tL_x^{\frac{12}{5}}} \le \|\mathscr{F}^{-1}_{\tau,\xi}\mathscr{K}_3\|^{\frac23}_{L^2_{t,x}}\|\mathscr{F}^{-1}_{\tau,\xi}\mathscr{K}_3\|^{\frac13}_{L^2_tL^4_x} \lesssim \varepsilon^{\frac{11}{6}}\|f\|^{\frac{11}{12}}_{L^2_{t,x,v}}\|f_{sm}\|^{\frac{1}{12}}_{L^2_{t,x,v}}.$$

Clearly from (3.28) and (3.27), $\|f_{sm}\|_{L^2_{t,x,v}} \le \|f\|_{L^2_{t,x,v}} + \|f_{la}\|_{L^2_{t,x,v}} \lesssim \|f\|_{L^2_{t,x,v}} + \|q\|_{L^2_{t,x,v}}$. On the other hand, from (3.21),

$$f(t,x,v) = \int_0^\infty e^{-\tau}q(t-\varepsilon\tau, X(t-\varepsilon\tau;t,x,v), V(t-\varepsilon\tau;t,x,v)).$$

Following the argument to prove (3.27), we obtain $\|f\|_{L^2_{t,x,v}} \le \|q\|_{L^2_{t,x,v}}$. Therefore we conclude

$$\|\mathscr{F}^{-1}_{\tau,\xi}(\mathscr{T}_1+\mathscr{K}_1+\mathscr{K}_2+\mathscr{K}_3)\|_{L^2_tL_x^{\frac{12}{5}}} \lesssim \varepsilon^{\frac12}(\|f\|_{L^2_{t,x,v}} + \|q\|_{L^2_{t,x,v}}),$$

and hence the estimate (3.24). □

Now we are ready to prove the main result of this section:

Proof of Proposition 3.4 Recall (1.19). We use temporary notations

$$[\zeta_0(v),\zeta_1(v),\zeta_2(v),\zeta_3(v),\zeta_4(v)] := \Big[\sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, \frac23\frac{|v|^2-3}{2}\sqrt{\mu}\Big]. \tag{3.33}$$

From (3.11),

$$\begin{aligned}&\int_{\mathbb{R}^3} f_\delta(t,x,v)\zeta_i(v)\mathrm{d}v\\ &= \int_{\mathbb{R}^3}\big[1-\chi\big(\frac{n(x)\cdot v}{\delta}\big)\chi\big(\frac{\xi(x)}{\delta}\big)\big]\chi(\delta|v|)\big\{\mathbf{1}_{t\ge0}f(t,x,v)\\ &\quad+\mathbf{1}_{t\le0}\chi(t)f_0(x,v)\big\}\zeta_i(v)\mathrm{d}v\\ &= \mathbf{1}_{t\ge0}\int_{\mathbb{R}^3}\big[1-\chi\big(\frac{n(x)\cdot v}{\delta}\big)\chi\big(\frac{\xi(x)}{\delta}\big)\big]\chi(\delta|v|)\\ &\quad\times\Big\{\sum_{j=0}^4 a_j(t,x)\zeta_j(v) + (\mathbf{I}-\mathbf{P})f(t,x,v)\Big\}\zeta_i(v)\mathrm{d}v\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{1}_{t\le 0}\int_{\mathbb{R}^3}\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi(\delta|v|)\chi(t)f_0(x,v)\zeta_i(v)\mathrm{d}v\\
&=\mathbf{1}_{t\ge 0}\Big\{a_i(t,x)+O(\delta)\sum_{j=0}^4|a_j(t,x)|+O_\delta(1)\int_{\mathbb{R}^3}|(\mathbf{I}-\mathbf{P})f(t,x,v)|\zeta_i(v)\mathrm{d}v\Big\}\\
&+\mathbf{1}_{t\le 0}\chi(t)\int_{\mathbb{R}^3}f_0(x,v)\zeta_i(v)\mathrm{d}v.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{i=0}^4\mathbf{1}_{t\ge 0}|a_i(t,x)| &\le \sum_{i=0}^4\Big|\int_{\mathbb{R}^3}f_\delta(t,x,v)\zeta_i(v)\mathrm{d}v\Big|\\
&+\mathbf{1}_{t\le 0}\chi(t)\int_{\mathbb{R}^3}|f_0(x,v)|\sum_{i=0}^4|\zeta_i(v)|\mathrm{d}v\\
&+\mathbf{1}_{t\ge 0}\Big\{O(\delta)\sum_{j=0}^4|a_j(t,x)|+O_\delta(1)\int_{\mathbb{R}^3}|(\mathbf{I}-\mathbf{P})f(t,x,v)|\sum_{i=0}^4|\zeta_i(v)|\mathrm{d}v\Big\}.
\end{aligned}
$$

Hence for all $i=0,1,2,3,4$,

$$
\begin{aligned}
|a_i(t,x)|\le &4\int_{\mathbb{R}^3}|f_\delta(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v+4\chi(t)\mathbf{1}_{t\le 0}\int_{\mathbb{R}^3}|f_0(x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v\\
&+4\int_{\mathbb{R}^3}|(\mathbf{I}-\mathbf{P})f(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v.
\end{aligned}
\tag{3.34}
$$

Now we focus on the part of f_δ in (3.34). From Lemma 3.6, there is an extension $\overline{f_\delta}$ defined by (A.1.12) such that

$$
\int_{\mathbb{R}^3}|f_\delta(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v\le\int_{\mathbb{R}^3}|\overline{f_\delta}(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v,
$$

and solves

$$
\varepsilon\partial_t\overline{f_\delta}+v\cdot\nabla_x\overline{f_\delta}+\varepsilon^2\Phi\cdot\nabla_v\overline{f_\delta}+\overline{f_\delta}=h+\overline{f_\delta},
\tag{3.35}
$$

in the sense of distributions with $h=\sum_{i=1}^4 h_i$ where h_i are defined in Lemma 3.6.

Now we apply Lemma 3.7 to (3.35) with $f=\overline{f_\delta}$ and $q=h+\overline{f_\delta}$. Then we can decompose

$$
\overline{f_\delta}=\overline{f_\delta}_{,la}+\overline{f_\delta}_{,sm},
\tag{3.36}
$$

where $\overline{f_\delta}_{,la}$ is defined, as (3.25),

$$
\overline{f_\delta}_{,la}(t,x,v)=\int_0^\infty e^{-\tau}(h+\overline{f_\delta})(t-\varepsilon\tau,x-\tau v,v)\mathrm{d}\tau,
\tag{3.37}
$$

and $\overline{f_\delta}_{,sm}:=\overline{f_\delta}-\overline{f_\delta}_{,la}$ as in (3.28).

Using (3.23) and (3.24) with $\psi = \zeta_i$ in (3.33), we deduce that

$$\begin{aligned} \Big\| \int_{\mathbb{R}^3} \zeta_i(v)\overline{f_\delta}_{,la}\mathrm{d}v \Big\|_{L^2_t L^3_x} &\lesssim \|w^{-1}h\|_{L^2_{t,x,v}} + \|w^{-1}\overline{f_\delta}\|_{L^2_{t,x,v}}, \\ \Big\| \int_{\mathbb{R}^3} \zeta_i(v)\overline{f_\delta}_{,sm}\mathrm{d}v \Big\|_{L^2_t L^{\frac{12}{5}}_x} &\lesssim \varepsilon^{\frac{1}{2}}(\|w^{-1}h\|_{L^2_{t,x,v}} + \|w^{-1}\overline{f_\delta}\|_{L^2_{t,x,v}}). \end{aligned} \tag{3.38}$$

Note that from Lemma 3.6

$$\begin{aligned} &\|w^{-1}h\|_{L^2_{t,x,v}} + \|w^{-1}\overline{f_\delta}\|_{L^2_{t,x,v}} \\ &\quad \lesssim \|g\|_{L^2_{t,x,v}} + \|f_0\|_{L^2_{x,v}} + \|[v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v]f_0\|_{L^2_{x,v}} \\ &\qquad + \|f_0\|_{L^2(\gamma)} + \|f_\gamma\|_{L^2(\mathbb{R}_+\times\gamma)}. \end{aligned} \tag{3.39}$$

Finally we set

$$\begin{aligned} \mathbf{S}_1 f(t,x) &:= \int_{\mathbb{R}^3} |\overline{f_\delta}_{,la}|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v, \\ \mathbf{S}_2 f(t,x) &:= \int_{\mathbb{R}^3} |\overline{f_\delta}_{,sm}|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v, \\ \mathbf{S}_3 f(t,x) &:= 4\int_{\mathbb{R}^3} |(\mathbf{I}-\mathbf{P})f(t,x,v)|\langle v\rangle^2\sqrt{\mu(v)}\mathrm{d}v. \end{aligned} \tag{3.40}$$

Then by (3.38) and (3.39) we conclude (3.9)–(3.10). □

3.3 Unsteady L^2−Coercivity Estimate

The main purpose of this section is to prove the following:

Proposition 3.8 *Suppose $\Phi = \Phi(x) \in C^1$, $g \in L^2(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$, and $r \in L^2(\mathbb{R}_+\times\gamma_-)$ such that, for all $t > 0$,*

$$\iint_{\Omega\times\mathbb{R}^3} g(t,x,v)\sqrt{\mu}\mathrm{d}v\mathrm{d}x = 0 = \int_{\gamma_-} r(t,x,v)\sqrt{\mu}\mathrm{d}\gamma. \tag{3.41}$$

Then, for any sufficiently small ε, there exists a unique solution to the problem

$$\varepsilon\partial_t f + v\cdot\nabla_x f + \frac{1}{\sqrt{\mu}}\varepsilon^2\Phi\cdot\nabla_v(\sqrt{\mu}f) + \varepsilon^{-1}Lf = g, \tag{3.42}$$

with $f|_{t=0} = f_0$ and $f_- = P_\gamma f + r$ on $\mathbb{R}_+\times\gamma_-$ such that

$$\iint_{\Omega\times\mathbb{R}^3} f(t,x,v)\sqrt{\mu}\mathrm{d}x\mathrm{d}v = 0, \quad \textit{for all } t \geq 0. \tag{3.43}$$

Moreover, there is $0 < \lambda \ll 1$ *such that for* $0 \le s \le t$,

$$
\begin{aligned}
&\|e^{\lambda t} f(t)\|_2^2 + \varepsilon^{-2} \int_s^t \|e^{\lambda\tau}(\mathbf{I}-\mathbf{P}) f(\tau)\|_\nu^2 \mathrm{d}\tau + \int_s^t \|e^{\lambda\tau}\mathbf{P} f(\tau)\|_2^2 \mathrm{d}\tau \\
&\quad + \varepsilon^{-1}\int_s^t |e^{\lambda\tau}(1-P_\gamma) f|_2^2 + \int_s^t |e^{\lambda\tau} f|_2^2 \\
&\quad \lesssim \|e^{\lambda s} f(s)\|_2^2 + \varepsilon^{-1}\int_s^t |e^{\lambda\tau} r|_{2,-}^2 + \int_s^t \|\nu^{-\frac{1}{2}} e^{\lambda\tau}(\mathbf{I}-\mathbf{P}) g\|_2^2 \\
&\quad + \varepsilon^{-2}\int_s^t \|e^{\lambda\tau}\,\mathbf{P} g\|_2^2.
\end{aligned} \tag{3.44}
$$

In order to prove the proposition we need the following:

Lemma 3.9 *Assume that* g *and* r *satisfy* (3.41) *and* f *satisfies* (3.42), *and* (3.43). *Then there exists a function* $G(t)$ *such that, for all* $0 \le s \le t$, $G(s) \lesssim \|f(s)\|_2^2$ *and*

$$
\begin{aligned}
\int_s^t \|\mathbf{P} f(\tau)\|_\nu^2 &\lesssim G(t) - G(s) + \int_s^t \|\frac{g(\tau)}{\sqrt{\nu}}\|_2^2 + |r(\tau)|_{2,-}^2 + \varepsilon^{-2}\int_s^t \|(\mathbf{I}-\mathbf{P}) f(\tau)\|_\nu^2 \\
&\quad + \int_s^t |(1-P_\gamma) f(\tau)|_{2,+}^2.
\end{aligned}
$$

Proof The key of the proof is to use the same choices of test functions (with extra dependence on time) of (2.70), (2.97), (2.114) and (2.118) and estimate the new contribution $\int_s^t \iint_{\Omega\times\mathbb{R}^3} \partial_t \psi f$ in the time dependent weak formulation

$$
\begin{aligned}
&\int_s^t \int_{\gamma_+} \psi f - \int_s^t \int_{\gamma_-} \psi f - \int_s^t \iint_{\Omega\times\mathbb{R}^3} \{v\cdot\nabla_x \psi + \varepsilon^2 \sqrt{\mu}\Phi\cdot\nabla_v \frac{\psi}{\sqrt{\mu}}\} f \\
&\quad - \varepsilon \int_s^t \iint_{\Omega\times\mathbb{R}^3} \partial_t \psi f = -\varepsilon \iint_{\Omega\times\mathbb{R}^3} \psi f(t) + \varepsilon \iint_{\Omega\times\mathbb{R}^3} \psi f(s) \\
&\quad + \varepsilon^{-1}\int_s^t \iint_{\Omega\times\mathbb{R}^3} [-\psi L(\mathbf{I}-\mathbf{P}) f + \psi g].
\end{aligned} \tag{3.45}
$$

We note that, with such choices $G(t) = -\iint_{\Omega\times\mathbb{R}^3} \psi f(t)$, and $|G(t)| \lesssim \|f(t)\|_2^2$. Without loss of generality we give the proof for $s = 0$.

Remark We note that (3.41), (3.42) and (3.43) are all invariant under a standard t -mollification for all $t > 0$. The estimates in *Step 1* to *Step 3* below are obtained via a t-mollification so that all the functions are smooth in t. For the notational simplicity we do not write explicitly the parameter of the regularization.

Step 1. Estimate of $\nabla_x \Delta_N^{-1}\partial_t a = \nabla_x \partial_t \varphi_a$. In the weak formulation (with time integration over $[t, t+\delta]$), if we choose the test function $\psi = \varphi\sqrt{\mu}$ with $\varphi(x)$ dependent only of x, then we get (note that Lf and g, integrated against $\varphi(x)\sqrt{\mu}$ are zero)

$$
\varepsilon \int_\Omega [a(t+\delta) - a(t)]\varphi(x) = \int_t^{t+\delta}\int_\Omega (b\cdot\nabla_x)\varphi(x) + \int_t^{t+\delta}\int_{\gamma_-} r\varphi\sqrt{\mu},
$$

where we have used the splitting (2.82) and (2.81). Taking difference quotient, we obtain for all t

$$\varepsilon \int_\Omega \varphi \partial_t a = \int_\Omega (b \cdot \nabla_x)\varphi + \int_{\gamma_-} r\varphi\sqrt{\mu}.$$

Notice that, for $\varphi = 1$, from (3.41), the right hand side of the above equation is zero. Hence, for all $t > 0$, $\int_\Omega \partial_t a(t)dx = 0$. On the other hand, for all $\varphi(x) \in H^1(\Omega)$, we have, by the trace theorem $|\varphi|_2 \lesssim \|\varphi\|_{H^1}$, $\varepsilon \left\| \int_\Omega \varphi(x)\partial_t a \mathrm{d}x \right\| \lesssim |r|_{2,-}|\varphi|_2 + \|b\|_2\|\varphi\|_{H^1} \lesssim \{\|b(t)\|_2 + |r|_{2,-}\}\|\varphi\|_{H^1}$. Therefore we conclude that, for all $t > 0$,

$$\varepsilon\|\partial_t a(t)\|_{(H^1)^*} \lesssim \|b(t)\|_2 + |r|_2,$$

where $(H^1)^* \equiv (H^1(\Omega))^*$ is the dual space of $H^1(\Omega)$ with respect to the dual pair $\langle A, B\rangle = \int_\Omega A(x)B(x)\mathrm{d}x$, for $A \in H^1$ and $B \in (H^1)^*$.

On the other hand, φ_a in (2.118) is the solution of $-\Delta\partial_t\varphi_a = \partial_t a$, $\frac{\partial}{\partial n}\partial_t\varphi_a = 0$ at $\partial\Omega$ with $\int_\Omega \partial_t a(t,x)dx = 0$ for all $t > 0$. From the standard elliptic theory,

$$\varepsilon\|\nabla_x\partial_t\varphi_a\|_2 = \varepsilon\|\Delta_N^{-1}\partial_t a(t)\|_{H^1} \lesssim \varepsilon\|\partial_t a(t)\|_{(H^1)^*} \lesssim \|b(t)\|_2 + |r|_2.$$

Therefore, we conclude, for almost all $t > 0$,

$$\|\nabla_x\partial_t\varphi_a(t)\|_2 \lesssim \varepsilon^{-1}\{\|b(t)\|_2 + |r|_2\}. \tag{3.46}$$

Step 2. Estimate of $\nabla_x\Delta^{-1}\partial_t b^j = \nabla_x\partial_t\varphi_b^i$. In (3.45), we choose a test function $\psi = \varphi(x)v_i\sqrt{\mu}$. Since $\int v_iv_j\mu(v)\mathrm{d}v = \int v_iv_j(\frac{|v|^2}{2} - \frac{3}{2})\mu(v)\mathrm{d}v = \delta_{ij}$, we get

$$\begin{aligned}
\varepsilon \int_\Omega [b_i(t+\delta) - b_i(t)]\varphi = &- \int_t^{t+\delta}\int_\gamma f\varphi v_i\sqrt{\mu} + \int_t^{t+\delta}\int_\Omega \partial_i\varphi[a+c] \\
&- \varepsilon^2\int_t^{t+\delta}\int_\Omega \Phi_i\varphi a \\
&+ \varepsilon^{-1}\int_t^{t+\delta}\iint_{\Omega\times\mathbb{R}^3}\sum_{j=1}^d v_jv_i\sqrt{\mu}\partial_j\varphi(\mathbf{I}-\mathbf{P})f \\
&+ \int_t^{t+\delta}\iint_{\Omega\times\mathbb{R}^3}\varphi v_i g\sqrt{\mu}.
\end{aligned}$$

Taking difference quotient, we obtain

$$\varepsilon \int_\Omega \partial_t b_i(t)\varphi = -\int_\gamma f(t) v_i \varphi \sqrt{\mu} + \int_\Omega \partial_i \varphi [a(t) + c(t)] - \varepsilon^2 \int_\Omega \Phi_i \varphi a \\ + \varepsilon^{-1} \iint_{\Omega\times\mathbb{R}^3} \sum_{j=1}^d v_j v_i \sqrt{\mu} \partial_j \varphi (\mathbf{I}-\mathbf{P}) f(t) + \iint_{\Omega\times\mathbb{R}^3} \varphi v_i g(t) \sqrt{\mu}.$$

For fixed $t > 0$, we choose $\varphi = \partial_t \varphi_b^i$ in (2.98) solving $-\Delta \partial_t \varphi_b^i = \partial_t b_i(t),\ \partial_t \varphi_b^i|_{\partial\Omega} = 0$. The boundary terms vanish because of the Dirichlet boundary condition on $\partial_t \varphi_b^i$. Then we have, for $t \geq 0$,

$$\begin{aligned}
&\varepsilon \int_\Omega |\nabla_x \Delta^{-1} \partial_t b_i(t)|^2 = \varepsilon \int_\Omega |\nabla_x \partial_t \varphi_b^i|^2 = -\varepsilon \int_\Omega \Delta \partial_t \varphi_b^i \partial_t \varphi_b^i \\
&\lesssim 4\eta\varepsilon \{\|\nabla_x \partial_t \varphi_b^i\|_2^2 + \|\partial_t \varphi_b^i\|_2^2\} + \frac{1}{4\eta\varepsilon}\left[\|a(t)\|_2^2 + \|c(t)\|_2^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2\right] \\
&\quad + \frac{1}{4\eta\varepsilon^3}\|(\mathbf{I}-\mathbf{P}) f(t)\|_2^2 + \frac{\varepsilon^3}{4\eta}\|\Phi\|_\infty^2 \|a\|_2^2 \\
&\lesssim 8\eta\varepsilon \|\nabla_x \partial_t \varphi_b^i\|_2^2 + \frac{1}{4\eta\varepsilon}\left[\|a(t)\|_2^2 + \|c(t)\|_2^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2\right] + \frac{1}{4\eta\varepsilon^3}\|(\mathbf{I}-\mathbf{P}) f(t)\|_2^2 \\
&\quad + \|\frac{g}{\sqrt{\nu}}\|_2^2 + \frac{\varepsilon^3}{4\eta}\|\Phi\|_\infty^2 \|a\|_2^2,
\end{aligned}$$

where we have used the Poincaré inequality. Hence, for all $t > 0$

$$\|\nabla_x \partial_t \varphi_b^i\|_2 \lesssim \varepsilon^{-1}\Big\{\|a(t)\|_2 + \|c(t)\|_2 + \|\frac{g}{\sqrt{\nu}}\|_2\Big\} + \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P}) f(t)\|_2. \tag{3.47}$$

Step 3. Estimate of $\nabla_x \Delta^{-1} \partial_t c = \nabla_x \partial_t \varphi_c$. In the weak formulation, we choose a test function $\varphi(x)(\frac{|v|^2}{2} - \frac{3}{2})\sqrt{\mu}$. Since $\int \mu(v)(\frac{|v|^2}{2} - \frac{3}{2}) = 0$, $\int \mu(v) v_i v_j (\frac{|v|^2-3}{2}) = \delta_{ij}$, $\int \mu(v)(\frac{|v|^2}{2} - \frac{3}{2})^2 \neq 0$,

$$\begin{aligned}
&\frac{3}{2}\varepsilon \int_\Omega \varphi(x) c(t+\delta, x)\mathrm{d}x - \frac{3}{2}\varepsilon \int_\Omega \varphi(x) c(t,x)\mathrm{d}x \\
&= \int_t^{t+\delta}\int_\Omega b\cdot\nabla_x\varphi - \int_t^{t+\delta}\int_\gamma \left(\frac{|v|^2}{2} - \frac{3}{2}\right)\sqrt{\mu}\varphi f \\
&\quad + \varepsilon^{-1}\int_t^{t+\delta}\iint_{\Omega\times\mathbb{R}^3} (\mathbf{I}-\mathbf{P}) f \left(\frac{|v|^2}{2} - \frac{3}{2}\right)\sqrt{\mu}(v\cdot\nabla_x)\varphi \\
&\quad + \int_t^{t+\delta}\iint_{\Omega\times\mathbb{R}^3} \varphi g\Big(\frac{|v|^2}{2} - \frac{3}{2}\Big)\sqrt{\mu} - \varepsilon^2 \int_t^{t+\delta}\int_\Omega \varphi\Phi\cdot b.
\end{aligned}$$

Taking difference quotient, we obtain

$$
\begin{aligned}
&\varepsilon \int_\Omega \varphi(x)\partial_t c(t,x)\mathrm{d}x = \frac{2}{3}\int_\Omega b(t)\cdot\nabla_x\varphi - \frac{2}{3}\int_\gamma \left(\frac{|v|^2}{2}-\frac{3}{2}\right)\sqrt{\mu}\varphi f(t) \\
&\quad + \frac{2}{3}\iint_{\Omega\times\mathbb{R}^3} \varepsilon^{-1}(\mathbf{I}-\mathbf{P})f(t)\left(\frac{|v|^2}{2}-\frac{3}{2}\right)\sqrt{\mu}(v\cdot\nabla_x)\varphi \\
&\quad + \varphi g(t)\left(\frac{|v|^2}{2}-\frac{3}{2}\right)\sqrt{\mu} - \frac{2}{3}\varepsilon^2\int_\Omega \varphi\Phi\cdot b.
\end{aligned}
$$

Note that $\partial_t\varphi_c$ in (2.70) is the solution of $-\Delta\partial_t\varphi_c = \partial_t c(t)$, $\partial_t\varphi_c|_{\partial\Omega} = 0$.

The boundary terms vanish because of the Dirichlet boundary condition on $\partial_t\varphi_c$. We follow the same procedure of estimates $\nabla_x\Delta^{-1}\partial_t a$ and $\nabla_x\Delta^{-1}\partial_t b$ to have

$$
\begin{aligned}
&\varepsilon\|\nabla_x\Delta^{-1}\partial_t c(t)\|_2^2 = \varepsilon\int_\Omega |\nabla_x\partial_t\varphi_c(x)|^2 dx = \varepsilon\int_\Omega \partial_t\varphi_c(x)\partial_t c(t,x)dx \\
&\quad \lesssim 4\eta\varepsilon\{\|\nabla_x\partial_t\varphi_c\|_2^2 + \|\partial_t\varphi_c\|_2^2\} + \frac{1}{4\eta\varepsilon}\|b(t)\|_2^2 + \frac{1}{4\eta\varepsilon^3}\|(\mathbf{I}-\mathbf{P})f(t)\|_2^2 \\
&\qquad + \|\frac{g}{\sqrt{\nu}}\|_2^2 + \frac{\varepsilon^3}{4\eta}\|\Phi\|_\infty^2\|b\|_2^2 \\
&\quad \lesssim 8\eta\varepsilon\|\nabla_x\partial_t\varphi_c\|_2^2 + \frac{1}{4\eta\varepsilon}\|b(t)\|_2^2 + \frac{1}{4\eta\varepsilon^3}\|(\mathbf{I}-\mathbf{P})f(t)\|_2^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 \\
&\qquad + \frac{\varepsilon^3}{4\eta}\|\Phi\|_\infty^2\|b\|_2^2,
\end{aligned}
$$

where we have used the Poincaré inequality. Finally we have, for all $t > 0$,

$$
\|\nabla_x\Delta^{-1}\partial_t c(t)\|_2 \lesssim \varepsilon^{-1}\Big\{\|b(t)\|_2 + \|\frac{g}{\sqrt{\nu}}\|_2\Big\} + \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f(t)\|_2. \tag{3.48}
$$

Step 4. Estimate of a, b, c contributions in (3.45). To estimate c contribution in (3.45), we plug (2.70) into (3.45) to have from (2.81)

$$
\begin{aligned}
&\int_0^t \iint_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_c)v_i\sqrt{\mu}\partial_t\partial_i\varphi_c f \\
&\quad = \sum_{j=1}^d \int_0^t \iint_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_c)v_i v_j\mu(v)\partial_t\partial_i\varphi_c b_j \\
&\qquad + \int_0^t \iint_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_c)v_i\sqrt{\mu}\partial_t\partial_i\varphi_c(\mathbf{I}-\mathbf{P})f.
\end{aligned}
$$

The second line has non-zero contribution only for $j = i$ which leads to zero by the definition of β_c in (2.83). We thus have from (3.48), for ε small,

$$\begin{aligned}
&\varepsilon \left| \int_0^t \iint_{\Omega\times\mathbb{R}^3} (|v|^2 - \beta_c) v_i \sqrt{\mu}\partial_t \partial_i \varphi_c f \right| \\
&\lesssim \varepsilon \int_0^t \left\{ \varepsilon^{-1}\Big[\|b\|_2 + \|\frac{g}{\sqrt{\nu}}\|_2 \Big] + \varepsilon^{-2} \|(\mathbf{I}-\mathbf{P})f\|_2 \right\} \|(\mathbf{I}-\mathbf{P})f\|_2 \\
&\lesssim \int_0^t \eta \|b\|_2^2 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_2^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2.
\end{aligned}$$

Combining with (2.68), we conclude, for η small,

$$\begin{aligned}
\int_0^t \|c(s)\|_2^2 \mathrm{d}s \lesssim\; & G(t) - G(0) \\
& + \int_0^t \Big\{ \varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_\nu^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + |(1-P_\gamma)f|_{2,+}^2 \\
& + |r|_2^2 + \eta\|b\|_2^2 + \varepsilon^2 \|\Phi\|_\infty \big[\|a\|_2^2 + \|b\|_2^2\big] \Big\} \mathrm{d}s.
\end{aligned}$$

To estimate b in (3.45), by (2.81), we plug (2.97) into (3.45) to get:

$$\begin{aligned}
\int_0^t \iint (v_i^2 - \beta_b)\sqrt{\mu}\partial_t \partial_j \varphi_b^j f = \int_0^t \iint & (v_i^2 - \beta_b)\mu \partial_t \partial_j \varphi_b^j \left\{ \frac{|v|^2}{2} - \frac{3}{2} \right\} c \\
& + (v_i^2 - \beta_b)\sqrt{\mu}\partial_t \partial_j \varphi_b^j (\mathbf{I}-\mathbf{P}) f,
\end{aligned}$$

where we have used (2.108) to remove the a contribution. We thus have from (3.47),

$$\begin{aligned}
&\left\| \int_0^t \iint_{\Omega\times\mathbb{R}^3} (v_i^2 - \beta_b)\sqrt{\mu}\partial_t \partial_j \varphi_b^j f \right\| \\
&\lesssim \int_0^t \left\{ \|a\|_2 + \|c\|_2 + \|(\mathbf{I}-\mathbf{P})f\|_2 + \|\frac{g}{\sqrt{\nu}}\|_2 \right\} \left\{ \|c\|_2 + \|(\mathbf{I}-\mathbf{P})f\|_2 \right\} \\
&\lesssim \int_0^t \left\{ \|(\mathbf{I}-\mathbf{P})f\|_2^2 + \|c\|_2^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + \varepsilon\|a\|_2^2 \right\}.
\end{aligned}$$

Next we plug (2.114) into (3.45) and from (3.47),

$$\begin{aligned}
&\varepsilon \int_0^t \int_{\Omega\times\mathbb{R}^3} |v|^2 v_i v_j \sqrt{\mu}\partial_t \partial_j \varphi_b^i f = \varepsilon \int_0^t \int_{\Omega\times\mathbb{R}^3} |v|^2 v_i v_j \sqrt{\mu}\partial_t \partial_j \varphi_b^i (\mathbf{I}-\mathbf{P}) f \\
&\lesssim \int_0^t \{(\|a\|_2 + \|c\|_2 + \|\frac{g}{\sqrt{\nu}}\|_2) + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_2\}\|(\mathbf{I}-\mathbf{P})f\|_2 \qquad (3.49) \\
&\lesssim \int_0^t \left\{ \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_\nu^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + \eta\big[\|a\|_2^2 + \|c\|_2^2\big] \right\}.
\end{aligned}$$

Combining this with (2.95), we conclude from (3.45) that

$$\begin{aligned}&\int_0^t \|b(s)\|_2^2 \mathrm{d}s \lesssim G(t) - G(0)\\&+ \int_0^t \Big\{\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f(s)\|_\nu^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2 + |(1-P_\gamma)f(s)|_{2,+}^2\\&+ |r(s)|_2^2 + \varepsilon^2\|\Phi\|_\infty(\|a\|_2^2 + \|c\|_2^2) + \eta\|a\|_2^2 + \|c\|_2^2(1+\eta)\Big\}\mathrm{d}s.\end{aligned} \tag{3.50}$$

Finally in order to estimate a contribution in (3.45) we plug (2.118) into (3.45). We estimate

$$\begin{aligned}&\varepsilon\int_0^t \int_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_a)v_i\mu\partial_t\partial_i\varphi_a f\\&= \varepsilon\int_0^t \int_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_a)(v_i)^2\mu\partial_t\partial_i\varphi_a b_i\\&\quad+ \varepsilon\int_0^t \int_{\Omega\times\mathbb{R}^3} (|v|^2-\beta_a)v_i\mu\partial_t\partial_i\varphi_a(\mathbf{I}-\mathbf{P})f\\&\lesssim \varepsilon\int_0^t \varepsilon^{-1}\{\|b(t)\|_{2,\Omega} + |r|_2\}\{\|b\|_2 + \|(\mathbf{I}-\mathbf{P})f\|_2\}.\end{aligned}$$

Combining this with (2.116), we conclude

$$\begin{aligned}\int_0^t \|a(s)\|_2^2 \mathrm{d}s \lesssim &G(t) - G(0) + \int_0^t \Big\{\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f(s)\|_\nu^2 + |(1-P_\gamma)f(s)|_{2,+}^2\\&+ \|\frac{g}{\sqrt{\nu}}(s)\|_2^2 + |r(s)|_2^2 + \|b\|_2^2 + \varepsilon^2\|\Phi\|_\infty\|\mathbf{P}\mathring{f}\|_2^2\Big\}\mathrm{d}s.\end{aligned} \tag{3.51}$$

From (3.49), (3.50) and (3.51), we prove the lemma for ε sufficiently small, by choosing η small. □

Now we are ready to prove the main result of this section:

Proof of Proposition 3.8 Define the approximating sequence with $f^0 \equiv f_0$, (with $\tilde{\nu} = \nu - \frac{1}{2}\varepsilon^3\Phi\cdot v$):

$$\partial_t f^{\ell+1} + \varepsilon^{-1}v\cdot\nabla_x f^{\ell+1} + \varepsilon\Phi\cdot\nabla_v f^{\ell+1} + \frac{1}{\varepsilon^2}\tilde{\nu}f^{\ell+1} - \frac{1}{\varepsilon^2}Kf^\ell = \varepsilon^{-1}g, \quad f^{\ell+1}|_{t=0} = f_0, \tag{3.52}$$

and $f_-^{\ell+1} = (1-\frac{\varepsilon}{j})P_\gamma f^\ell + r$.

Step 1. Fix j, $f^\ell \to f^j$ as $\ell\to\infty$. Notice that,

$$\begin{aligned}|(Kf^\ell, f^{\ell+1})| &\lesssim \iint_{\mathbb{R}^3\times\mathbb{R}^3} |\mathbf{k}(v,u)|^{1/2}|f^\ell(u)||\mathbf{k}(v,u)|^{1/2}|f^{\ell+1}(v)|\mathrm{d}v\mathrm{d}u\\&\lesssim \sqrt{\int_u |f^\ell(u)|^2\int_v|\mathbf{k}(v,u)|}\sqrt{\int_v |f^{\ell+1}(v)|^2\int_u|\mathbf{k}(v,u)|} \lesssim \|f^\ell\|_2^2 + \|f^{\ell+1}\|_2^2,\end{aligned}$$

where we used $\sup_u \int_v |\mathbf{k}(v,u)| + \sup_v \int_u |\mathbf{k}(v,u)| < +\infty$.

Note that, since $\int_{\gamma_-} r\sqrt{\mu} = 0$, we have

$$\left|\left(1-\frac{\varepsilon}{j}\right) P_\gamma f^\ell + r\right|_{2,-}^2 = \left|\left(1-\frac{\varepsilon}{j}\right) P_\gamma f^\ell\right|_{2,-}^2 + |r|_2^2. \tag{3.53}$$

By Green's identity (3.45) with $f^{\ell+1}$ in (3.52),

$$\begin{aligned}
&\|f^{\ell+1}(t)\|_2^2 + \varepsilon^{-2}\int_0^t \|f^{\ell+1}\|_\nu^2 + \varepsilon^{-1}\int_0^t |f^{\ell+1}|_{2,+}^2 \\
&\leq \varepsilon^{-1}\left[\left(1-\frac{\varepsilon}{j}\right)^2\right]\int_0^t |P_\gamma f^\ell|_{2,-}^2 + \varepsilon^{-1}\int_0^t |r|_2^2 + C\varepsilon^{-2}\int_0^t \max_{1\leq i\leq \ell+1}\|f^i\|_2^2 \\
&\quad + \int_0^t \|\frac{g}{\sqrt{\nu}}\|_2^2 + \|f_0\|_2^2 \\
&\leq \varepsilon^{-1}\left[\left(1-\frac{\varepsilon}{j}\right)^2\right]\int_0^t |f^\ell|_{2,+}^2 + \varepsilon^{-1}\int_0^t |r|_2^2 + C\varepsilon^{-2}\int_0^t \max_{1\leq i\leq \ell+1}\|f^i\|_2^2 \\
&\quad + \int_0^t \|\frac{g}{\sqrt{\nu}}\|_2^2 + \|f_0\|_2^2.
\end{aligned}$$

Set $\eta = (1-\frac{\varepsilon}{j})^2 < 1$. Now use this inequality to bound for $\varepsilon^{-1}\int_0^t |f^\ell|_{2,-}^2$ and iterate:

$$\begin{aligned}
&\|f^{\ell+1}(t)\|_2^2 + \varepsilon^{-2}\int_0^t \|f^{\ell+1}\|_\nu^2 + \varepsilon^{-1}\int_0^t |f^{\ell+1}|_{2,+}^2 \\
&\leq \eta\Big\{\varepsilon^{-1}\eta\int_0^t |f^{\ell-1}|_{2,+}^2 + \varepsilon^{-1}\int_0^t |r|_2^2 + C\varepsilon^{-2}\int_0^t \max_{1\leq i\leq \ell}\|f^i\|_2^2 \\
&\quad + \int_0^t \|\frac{g}{\sqrt{\nu}}\|_2^2 ds + \|f_0\|_2^2\Big\} \\
&\quad + \varepsilon^{-1}\int_0^t |r|_2^2 + C\varepsilon^{-2}\int_0^t \max_{1\leq i\leq \ell+1}\|f^i\|_2^2 + \int_0^t \|\frac{g}{\sqrt{\nu}}\|_\nu^2 + \|f_0\|_2^2 \\
&= \eta^2\varepsilon^{-1}\int_0^t |f^{\ell-1}|_{2,+}^2 + (1+\eta)\Big\{\varepsilon^{-1}\int_0^t |r|_2^2 + C\varepsilon^{-2}\int_0^t \max_{1\leq i\leq \ell+1}\|f^i\|_2^2 \\
&\quad + \int_0^t \|\frac{g}{\sqrt{\nu}}\|_2^2 + \|f_0\|_2^2\Big\} \\
&\qquad\qquad\vdots \\
&\lesssim \eta^{\ell+1}\varepsilon^{-1}\int_0^t |f^0|_{2,+}^2 + \frac{(1-\eta)^{\ell+1}}{1-\eta}\Big\{\varepsilon^{-1}\int_0^t |r|_2^2 + \varepsilon^{-2}\int_0^t \max_{1\leq i\leq \ell+1}\|f^i\|_2^2 \\
&\quad + \int_0^t \|\frac{g}{\sqrt{\nu}}\|_2^2 + \|f_0\|_2^2\Big\}.
\end{aligned}$$

We therefore have, from $f^0 \equiv f_0$,

$$\max_{1\le i\le \ell+1} \|f^i(t)\|_2^2 \lesssim_{\eta,j} \Big\{\varepsilon^{-1}\int_0^t |r|_2^2 + \varepsilon^{-2}\int_0^t \max_{1\le i\le \ell+1}\|f^i\|_2^2 + \int_0^t \|g\|_\nu^2 + \|f_0\|_2^2 + \varepsilon^{-1}t\|f_0\|_\nu^2 + t|f_0|_{2,+}^2\Big\}.$$

By Gronwall's lemma, we have, for fixed $t > 0$,

$$\max_{1\le i\le \ell+1} \|f^i(t)\|_2^2 \lesssim_{\eta,\varepsilon,j,t} \left\{\int_0^t |r|_2^2 + \int_0^t \|g(s)\|_\nu^2 \mathrm{d}s + \|f_0\|_2^2 + t\|f_0\|_\nu^2 + t|f_0|_{2,+}^2\right\}.$$

This in turns leads to

$$\begin{aligned}&\max_{1\le i\le \ell+1}\left\{\|f^i(t)\|_2^2 + \int_0^t \|f^i\|_\nu^2 + \int_0^t |f^i|_{2,+}^2\right\}\\ &\quad\lesssim_{\eta,\varepsilon,j,t} \left\{\int_0^t |r|_2^2 + \int_0^t \|g\|_\nu^2 + \|f_0\|_2^2 + t\|f_0\|_\nu^2 + t|f_0|_{2,+}^2\right\}.\end{aligned}$$

Upon taking the difference, we have

$$\begin{aligned}&\partial_t[f^{\ell+1}-f^\ell] + \varepsilon^{-1}v\cdot\nabla_x[f^{\ell+1}-f^\ell] + \varepsilon\Phi\cdot\nabla_v[f^{\ell+1}-f^\ell]\\ &\quad+\varepsilon^{-2}\tilde{\nu}[f^{\ell+1}-f^\ell] = \varepsilon^{-2}K[f^\ell - f^{\ell-1}],\end{aligned} \tag{3.54}$$

with $[f^{\ell+1}-f^\ell](0)\equiv 0$ and $f_-^{\ell+1} - f_-^\ell = (1-\frac{\varepsilon}{j})P_\gamma[f^\ell - f^{\ell-1}]$.

Applying previous iteration to $f^{\ell+1}-f^\ell$ yields

$$\begin{aligned}&\|f^{\ell+1}(t)-f^\ell(t)\|_2 + \varepsilon^{-2}\int_0^t \|f^{\ell+1}(s)-f^\ell(s)\|_\nu^2\mathrm{d}s\\ &\quad+\varepsilon^{-1}\int_0^t |f^{\ell+1}(s)-f^\ell(s)|_{2,+}^2\mathrm{d}s\\ &\le \eta\varepsilon^{-1}\int_0^t |f^\ell(s)-f^{\ell-1}(s)|_{2,+}^2\mathrm{d}s + C_K\int_0^t \|f^\ell(s)-f^{\ell-1}(s)\|_2^2\mathrm{d}s\\ &\le \eta\Big\{\varepsilon^{-1}\int_0^t |f^\ell(s)-f^{\ell-1}(s)|_{2,+}^2\mathrm{d}s + \sup_{0\le s\le T}\|f^\ell(s)-f^{\ell-1}(s)\|_2^2\Big\},\end{aligned} \tag{3.55}$$

for $TC_K < \eta < 1$. This implies that f^ℓ is Cauchy with respect to the norm

$$\varepsilon^{-1}\int_0^T |f^\ell(s)|_{2,+}^2\mathrm{d}s + \sup_{0\le s\le T}\|f^\ell(s)\|_2^2,$$

in $[0,T]$. Repeating the arguemnt for $[0,T],[T,2T]$.... we deduce that for finite t, there exists a (unique) limit function $f^\ell \to f^j$ such that

$$\begin{aligned}&\partial_t f^j+\varepsilon^{-1}v\cdot\nabla_x f^j+\frac{\varepsilon}{\sqrt{\mu}}\Phi\cdot\nabla_v(\sqrt{\mu}f^j)+\varepsilon^{-2}Lf^j=g,\quad f^j(0)=f_0,\\&f^j|_{\gamma_-}=(1-\frac{\varepsilon}{j})P_\gamma f^j+r.\end{aligned}\tag{3.56}$$

Step 2. Let $j\to\infty$. Upon using Green's identity and the boundary condition and (3.53), we deduce

$$\begin{aligned}&\|f^j(t)\|_2^2+\varepsilon^{-2}\int_0^t\|(\mathbf{I}-\mathbf{P})f^j\|_\nu^2+\varepsilon^{-1}\int_0^t|(1-P_\gamma)f^j|_{2,+}^2\\&\le\varepsilon^{-1}\int_0^t|r|_2^2+\frac{\varepsilon}{j}\varepsilon^{-1}\int_0^t|P_\gamma f^j|_{2,+}^2+\int_0^t\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2^2+\varepsilon^{-2}\int_0^t\|\,\mathbf{P}g\|_2^2\\&\quad+\frac{\varepsilon}{2}\int_0^t\int_{\Omega\times\mathbb{R}^3}|\Phi\|v\|f^j|^2+\|f_0\|_2^2.\end{aligned}$$

By the trace theorem, Lemma 2.3,

$$\begin{aligned}&\int_0^t|P_\gamma f^j|_{2,+}^2\lesssim\int_0^t|\mathbf{1}_{\gamma_+^\delta}f^j|_{2,+}^2+\int_0^t|(1-P_\gamma)f^j|_{2,+}^2\\&\lesssim\varepsilon\|f^j(0)\|_2^2+\varepsilon\int_0^t\|f^j\|_2^2+\varepsilon\int_0^t\iint_{\Omega\times\mathbb{R}^3}\big|[\partial_t+\varepsilon^{-1}v\cdot\nabla_x\\&\quad+\varepsilon\Phi\cdot\nabla_v]f^jf^j\big|+\int_0^t|(1-P_\gamma)f^j|_{2,+}^2.\\&\lesssim\ \varepsilon\|f^j(0)\|_2^2+\varepsilon\int_0^t\|f^j\|_2^2+\int_0^t\iint_{\Omega\times\mathbb{R}^3}\varepsilon\frac{|\Phi\cdot v|}{2}|f^j|^2+\varepsilon^{-2}\nu|(\mathbf{I}-\mathbf{P})f^j|^2\\&\quad+|gf^j|+\int_0^t|(1-P_\gamma)f^j|_{2,+}^2.\end{aligned}$$

From the boundary condition in (3.56), $\int_0^t|f^j|_{2,-}^2\lesssim\int_0^t|f^j|_{2,+}^2+\int_0^t|r|_{2,+}^2$. Finally from

$$\begin{aligned}&\varepsilon\int_0^t\int_{\Omega\times\mathbb{R}^3}\frac{|\Phi\cdot v|}{2}|f^j|^2\lesssim\varepsilon\|\Phi\|_\infty\|(\mathbf{I}-\mathbf{P})f^j\|_\nu^2+\varepsilon\|\Phi\|_\infty\int_0^t\|\mathbf{P}f^j(s)\|_2^2\mathrm{d}s,\\&\qquad\int_0^t\iint_{\Omega\times\mathbb{R}^3}|gf^j|\lesssim\int_0^t\|\nu^{-\frac12}(\mathbf{I}-\mathbf{P})g\|_2^2+\varepsilon^2\int_0^t\|\mathbf{P}g\|_2^2\\&\qquad\qquad+o(1)\big[\int_0^t\|\mathbf{P}f^j(s)\|_2^2\mathrm{d}s+\varepsilon^{-2}\int_0^t\|(\mathbf{I}-\mathbf{P})f^j(s)\|_\nu^2\mathrm{d}s\big],\end{aligned}$$

we get

$$\begin{aligned}&\|f^j(t)\|_2^2+\varepsilon^{-2}\int_0^t\|(\mathbf{I}-\mathbf{P})f^j\|_\nu^2+\int_0^t|f^j|_2^2\\&\lesssim\varepsilon^{-1}\int_0^t|r|_2^2+\int_0^t\|\nu^{-\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_2^2+\varepsilon^{-2}\int_0^t\|\mathbf{P}g\|_2^2+\|f_0\|_2^2.\end{aligned}\tag{3.57}$$

Since $\|\mathbf{P}f^j(s)\|_2^2\lesssim\|f^j(s)\|_2^2$, integrating (3.57) from 0 to t, we have

$$\int_0^t\|\mathbf{P}f^j\|_2^2\lesssim_t\varepsilon^{-1}\int_0^t|r|_2^2+\int_0^t\|\frac{g}{\sqrt{\nu}}\|_2^2+\|f_0\|_2^2.$$

Thus, we conclude that, for $j\gg 1$ and $0<\varepsilon\ll 1$,

$$\|f^j(t)\|_2^2+\varepsilon^{-2}\int_0^t\|f^j(s)\|_\nu^2+\int_0^t|f^j|_2^2\lesssim_{\varepsilon,t}\varepsilon^{-1}\int_0^t|r|_2^2+\int_0^t\|\frac{g}{\sqrt{\nu}}\|_2^2+\|f_0\|_2^2.\tag{3.58}$$

By taking a weak limit, we obtain a weak solution f to (3.42) with the same bound (3.58). Taking difference, we have

$$\begin{aligned}&\partial_t[f^j-f]+\varepsilon^{-1}v\cdot\nabla_x[f^j-f]+\varepsilon\frac{1}{\sqrt{\mu}}\Phi\cdot\nabla_v(\sqrt{\mu}[f^j-f])\\&\quad+\varepsilon^{-2}L[f^j-f]=0,\\&[f^j-f]_-=P_\gamma[f^j-f]+\frac{\varepsilon}{j}P_\gamma f^j,\quad[f^j-f](0)=0.\end{aligned}$$

Applying (3.58) with $r=\frac{\varepsilon}{j}P_\gamma f^j$ we obtain

$$\begin{aligned}&\|f^j(t)-f(t)\|_2^2+\varepsilon^{-1}\int_0^t\|f^j(s)-f(s)\|_\nu^2ds+\int_0^t|f^j(s)-f(s)|_2^2ds\\&\lesssim_t\frac{1}{j}\int_0^t|P_\gamma f^j|^2\to 0.\end{aligned}$$

We thus construct f as a L^2 solution to (3.42).
Step 3. Final estimate. To conclude our proposition, let $y(t)\equiv e^{\lambda t}f(t)$. We multiply (3.42) by $e^{\lambda t}$, so that y satisfies

$$\partial_t y+\varepsilon^{-1}v\cdot\nabla_x y+\frac{1}{\sqrt{\mu}}\varepsilon\Phi\cdot\nabla_v(\sqrt{\mu}y)+\varepsilon^{-2}Ly=\lambda y+e^{\lambda t}g,\quad y|_{\gamma_-}=P_\gamma y_++e^{\lambda t}r.\tag{3.59}$$

By the Green's identity,

$$
\begin{aligned}
&\frac{1}{2}\|y(t)\|_2^2+\varepsilon^{-2}\int_0^t\|(\mathbf{I}-\mathbf{P})y(s)\|_\nu^2+\varepsilon^{-1}\int_0^t|(1-P_\gamma)y(s)|_{2,+}^2\\
&\quad\le\lambda\int_0^t\|y(s)\|_2^2+\|y(0)\|_2^2+\varepsilon^{-1}\int_0^t e^{\lambda s}|r|_2^2+\int_0^t e^{\lambda s}\|\nu^{-\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_2^2\\
&\qquad+\varepsilon^{-2}\int_0^t e^{\lambda s}\|g\|_2^2+\frac{1}{2}\varepsilon\int_0^t\int_{\Omega\times\mathbb{R}^3}e^{\lambda s}|\Phi||v||y|^2.
\end{aligned}
$$

From (3.42) we know that $\iint_{\Omega\times\mathbb{R}^3}y\sqrt{\mu}=\iint_{\Omega\times\mathbb{R}^3}(\lambda y+e^{\lambda t}g)\sqrt{\mu}=0$, $\int_{\gamma_-}e^{\lambda t}r\sqrt{\mu}d\gamma$ $=0$. Applying Lemma 3.9 to (3.59), we deduce

$$
\begin{aligned}
\int_0^t\|\mathbf{P}y(s)\|_\nu^2\mathrm{d}s&\lesssim G(t)-G(0)+\varepsilon^{-2}\int_0^t\|(\mathbf{I}-\mathbf{P})y(s)\|_\nu^2\mathrm{d}s+\int_0^t e^{\lambda s}\|g\|_2^2\mathrm{d}s\\
&\quad+\lambda\int_0^t\|y\|_2^2\mathrm{d}s+\varepsilon^{-1}\int_0^t\{|(1-P_\gamma)y(s)|_{2,+}^2+e^{\lambda s}|r|_2^2\}\mathrm{d}s,
\end{aligned}
$$

where $G(t)\lesssim\varepsilon\|y(t)\|_2^2$.

Using the trace theorem and the boundary condition, as *Step 2*, we obtain

$$
\begin{aligned}
\int_0^t|P_\gamma y|_{2,+}^2&\lesssim\|y(0)\|_2^2+\int_0^t\|\frac{e^{\lambda s}g}{\sqrt{\nu}}\|_2^2+\int_0^t\|\mathbf{P}y\|_2^2\\
&\quad+\varepsilon^{-2}\int_0^t\|(\mathbf{I}-\mathbf{P})y\|_\nu^2+\int_0^t|(1-P_\gamma)y|_{2,+}^2,\\
\int_0^t|y|_{2,-}^2&\lesssim\int_0^t|y|_{2,+}^2+\int_0^t|e^{\lambda\tau}r|_{2,+}^2.
\end{aligned}
$$

All together, for $0<\lambda\ll1$ and $0<\varepsilon\ll1$, we conclude (3.44). □

3.4 L^∞ Estimate

The main goal of this section is to prove the following:

Proposition 3.10 *Let f satisfies*

$$
\begin{aligned}
\big[\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v+\varepsilon^{-1}C_0\langle v\rangle\big]|f|&\le\varepsilon^{-1}K_\beta|f|+|g|,\\
\big|f|_{\gamma_-}\big|\le P_\gamma|f|+|r|,\quad\big|f|_{t=0}\big|&\le|f_0|.
\end{aligned}
\tag{3.60}
$$

Then, for $w(v)=e^{\beta'|v|^2}$ *with* $0<\beta'\ll\beta$,

$$\|\varepsilon^{\frac12}wf(t)\|_\infty \lesssim \|\varepsilon^{\frac12}wf_0\|_\infty+\sup_{0\le s\le\infty}\|\varepsilon^{\frac12}wr(s)\|_\infty+\varepsilon^{\frac32}\sup_{0\le s\le\infty}\|\langle v\rangle^{-1}wg(s)\|_\infty$$
$$+\sup_{0\le s\le t}\|\mathbf{P}f(s)\|_{L^6(\Omega)}+\varepsilon^{-1}\sup_{0\le s\le t}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2(\Omega\times\mathbb{R}^3)}, \tag{3.61}$$

and

$$\|\varepsilon^{\frac12}wf(t)\|_\infty \lesssim \|\varepsilon^{\frac12}wf_0\|_\infty+\sup_{0\le s\le\infty}\|\varepsilon^{\frac12}wr(s)\|_\infty+\varepsilon^{\frac32}\sup_{0\le s\le\infty}\|\langle v\rangle^{-1}wg(s)\|_\infty$$
$$+\varepsilon^{-1}\sup_{0\le s\le t}\|f(s)\|_{L^2(\Omega\times\mathbb{R}^3)}. \tag{3.62}$$

We define the stochastic cycles for the unsteady case. Note that from (3.5), $\tilde{x}_{\mathbf{b}}(x,v)=x_{\mathbf{b}}(x,v)$.

Definition 3.11 Define, for free variables $v_k\in\mathbb{R}^3$, from (3.5)

$$\begin{aligned}
\tilde{t}_1&=t-\tilde{t}_{\mathbf{b}}(x,v)=t-\varepsilon t_{\mathbf{b}}(x,v),\\
\tilde{x}_1&=Y(\tilde{t}_1;t,x,v)=\tilde{x}_{\mathbf{b}}(x,v)=x_{\mathbf{b}}(x,v)=x_1,\\
\tilde{t}_2&=t-\tilde{t}_{\mathbf{b}}(x,v)-\tilde{t}_{\mathbf{b}}(x_1,v_1)=t-\varepsilon t_{\mathbf{b}}(x,v)-\varepsilon t_{\mathbf{b}}(x_1,v_1),\\
\tilde{x}_2&=Y(\tilde{t}_2;\tilde{t}_1,x_1,v_1)=\tilde{x}_{\mathbf{b}}(x_1,v_1)=x_{\mathbf{b}}(x_1,v_1)=x_2,\\
&\vdots\\
\tilde{t}_{k+1}&=\tilde{t}_k-\tilde{t}_{\mathbf{b}}(x_k,v_k)=\tilde{t}_k-\varepsilon t_{\mathbf{b}}(x_k,v_k),\\
\tilde{x}_{k+1}&=Y(\tilde{t}_{k+1};\tilde{t}_k,x_k,v_k)=\tilde{x}_{\mathbf{b}}(x_k,v_k)=x_{\mathbf{b}}(x_k,v_k)=x_{k+1}.
\end{aligned}$$

and

$$\begin{aligned}
t-\tilde{t}_1&=\varepsilon t_{\mathbf{b}}(x,v)=\varepsilon(t-t_1),\\
t-\tilde{t}_2&=\varepsilon t_{\mathbf{b}}(x,v)+\varepsilon t_{\mathbf{b}}(x_1,v_1)=\varepsilon(t-t_2),\\
&\vdots\\
t-\tilde{t}_k&=\varepsilon(t-t_k).
\end{aligned}$$

Set

$$\begin{aligned}
Y_{\mathbf{cl}}(s;t,x,v)&:=\sum_k \mathbf{1}_{[\tilde{t}_{k+1},\tilde{t}_k)}(s)Y(s;\tilde{t}_k,x_k,v_k),\\
W_{\mathbf{cl}}(s;t,x,v)&:=\sum_k \mathbf{1}_{[\tilde{t}_{k+1},\tilde{t}_k)}(s)W(s;\tilde{t}_k,x_k,v_k).
\end{aligned}$$

Clearly

$$[Y_{\mathbf{cl}}(s;t,x,v),W_{\mathbf{cl}}(s;t,x,v)]=\left[X_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon};t,x,v\right),V_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon};t,x,v\right)\right]. \tag{3.63}$$

The following lemma is a generalized version of Lemma 23 of [31].

Lemma 3.12 ([31]) *Assume* $\Phi = \Phi(x) \in C^1$. *For sufficiently large* $T_0 > 0$, *there exist constant* $C_1, C_2 > 0$, *independent of* T_0, *such that for* $k = C_1 T_0^{5/4}$,

$$\sup_{(t,x,v)\in[0,\varepsilon T_0]\times\bar{\Omega}\times\mathbb{R}^3} \int_{\prod_{\ell=1}^{k-1}\mathcal{V}_\ell} \mathbf{1}_{\tilde{t}_k(t,x,v_1,v_2,\dots,v_{k-1})>0} \Pi_{\ell=1}^{k-1} \mathrm{d}\sigma_\ell < \left\{\frac{1}{2}\right\}^{C_2 T_0^{5/4}}. \tag{3.64}$$

Proof Since $\tilde{t}_k(\varepsilon T_0, x, v, v_1, v_2, \dots, v_{k-1}) = \tilde{t}_k(t, x, v, v_1, v_2, \dots, v_{k-1}) + \{\varepsilon T_0 - t\}$, for $0 \le t \le \varepsilon T_0$,

$$\mathbf{1}_{\tilde{t}_k(t,x,v,v_1,v_2,\dots,v_{k-1})>0} \le \mathbf{1}_{\tilde{t}_k(\varepsilon T_0,x,v,v_1,v_2,\dots,v_{k-1})>0} = \mathbf{1}_{\varepsilon T_0 - \tilde{t}_k(\varepsilon T_0,x,v,v_1,v_2,\dots,v_{k-1})<\varepsilon T_0}.$$

Note that, from Definition 3.11, for any $T_1, T_2 > 0$

$$\begin{aligned}\left\{T_1 - \tilde{t}_k(T_1, x, v, v_1, v_2, \dots, v_{k-1})\right\} &= \tilde{t}_{\mathbf{b}}(x, v) + \tilde{t}_{\mathbf{b}}(x_1, v_1) + \cdots + \tilde{t}_{\mathbf{b}}(x_{k-1}, v_{k-1}) \\ &= \varepsilon t_{\mathbf{b}}(x, v) + \varepsilon t_{\mathbf{b}}(x_1, v_1) + \cdots \\ &\quad + \varepsilon t_{\mathbf{b}}(x_{k-1}, v_{k-1}) \\ &= \varepsilon\left\{T_2 - t_k(T_2, x, v, v_1, v_2, \dots, v_{k-1})\right\}.\end{aligned}$$

Therefore, with $T_1 = \varepsilon T_0$ and $T_2 = T_0$,

$$\varepsilon T_0 > \varepsilon T_0 - \tilde{t}_k(\varepsilon T_0, x, v, v_1, v_2, \dots, v_{k-1}) = \varepsilon\{T_0 - t_k(T_0, x, v, v_1, v_2, \dots, v_{k-1})\},$$

and

$$\begin{aligned}\mathbf{1}_{\varepsilon T_0 - \tilde{t}_k(\varepsilon T_0,x,v,v_1,v_2,\dots,v_{k-1})<\varepsilon T_0} &= \mathbf{1}_{T_0 - t_k(T_0,x,v,v_1,v_2,\dots,v_{k-1})<T_0} \\ &= \mathbf{1}_{t_k(T_0,x,v,v_1,v_2,\dots,v_{k-1})>0}.\end{aligned}$$

Hence,

$$\begin{aligned}&\sup_{(t,x,v)\in[0,\varepsilon T_0]\times\bar{\Omega}\times\mathbb{R}^3} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{\tilde{t}_k(t,x,v,v_1,v_2,\dots,v_{k-1})>0} \Pi_{j=1}^{k-1} \mathrm{d}\sigma_j \\ &\le \sup_{(x,v)\in\bar{\Omega}\times\mathbb{R}^3} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{t_k(T_0,x,v,v_1,v_2,\dots,v_{k-1})>0} \Pi_{j=1}^{k-1} \mathrm{d}\sigma_j.\end{aligned}$$

This proves (3.64). □

Now we are ready to prove the main result of this section:

Proof of Proposition 3.10 Define, for $w(v)$ and h as (2.32).

Then, from (3.60),

$$
\begin{aligned}
&\Big[\partial_t+\varepsilon^{-1}v\cdot\nabla_x+\varepsilon\Phi\cdot\nabla_v+\varepsilon^{-2}C_0\langle v\rangle+\frac{\varepsilon\Phi\cdot\nabla_v w}{w}\Big]|\varepsilon^{\frac12}h|\\
&\quad\le\varepsilon^{-2}\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\beta}(v,u)|\varepsilon^{\frac12}h(u)|\mathrm{d}u+\varepsilon^{-\frac12}|wg|.
\end{aligned}
\tag{3.65}
$$

We have the boundary condition as (2.34).
Step 1. We claim, for $t\in[n\varepsilon T_0,(n+1)\varepsilon T_0]$ with all $n\in\mathbb{N}$ and T_0 in Lemma 3.12

$$
\begin{aligned}
&|\varepsilon^{\frac12}h(t,x,v)|\\
&\le\ CT_0^{5/2}e^{-\frac{C_0(t-n\varepsilon T_0)}{\varepsilon^2}}\|\varepsilon^{\frac12}h(n\varepsilon T_0)\|_\infty+C_{T_0}\varepsilon^{\frac12}\sup_{0\le s\le t}\|wr(s)\|_\infty\\
&\quad+C_{T_0}T_0^{5/2}\varepsilon^{\frac32}\|\langle v\rangle^{-1}wg(s)\|_\infty\\
&\quad+CT_0^{5/2}\sup_{n\varepsilon T_0\le s\le(n+1)\varepsilon T_0}\|\mathbf{P}f(s)\|_{L^6(\Omega)}\\
&\quad+CT_0^{5/2}\varepsilon^{-1}\sup_{n\varepsilon T_0\le s\le(n+1)\varepsilon T_0}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2(\Omega\times\mathbb{R}^3)}\\
&\quad+\Big[CT_0^{5/4}\Big\{\frac12\Big\}^{C_2T_0^{5/4}}+o(1)CT_0^{5/2}\Big]\sup_{n\varepsilon T_0\le s\le(n+1)\varepsilon T_0}\|\varepsilon^{\frac12}h(s)\|_\infty,
\end{aligned}
\tag{3.66}
$$

and

$$
\begin{aligned}
&|\varepsilon^{\frac12}h(t,x,v)|\\
&\le\ CT_0^{5/2}e^{-\frac{C_0(t-n\varepsilon T_0)}{\varepsilon^2}}\|\varepsilon^{\frac12}h(n\varepsilon T_0)\|_\infty+\varepsilon^{\frac12}\sup_{0\le s\le t}\|wr(s)\|_\infty\\
&\quad+CT_0^{5/2}\varepsilon^{\frac32}\|\langle v\rangle^{-1}wg(s)\|_\infty\\
&\quad+CT_0^{5/2}\frac{1}{\varepsilon}\sup_{n\varepsilon T_0\le s\le(n+1)\varepsilon T_0}\|f(s)\|_{L^2(\Omega\times\mathbb{R}^3)}\\
&\quad+\Big[CT_0^{5/4}\Big\{\frac12\Big\}^{C_2T_0^{5/4}}+o(1)CT_0^{5/2}\Big]\sup_{n\varepsilon T_0\le s\le(n+1)\varepsilon T_0}\|\varepsilon^{\frac12}h(s)\|_\infty.
\end{aligned}
\tag{3.67}
$$

We first prove (3.66). From (3.65), for $\tilde t_1(t,x,v)<s\le t$,

$$
\begin{aligned}
&\frac{d}{ds}\Big[e^{-\int_s^t\frac{C_0}{\varepsilon^2}\langle V(t-\frac{t-\tau}{\varepsilon};t,x,v)\rangle\mathrm{d}\tau}\varepsilon^{\frac12}h^{\ell+1}(s,X_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon};t,x,v),V_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon};t,x,v))\Big]\\
&\le e^{-\int_s^t\frac{C_0}{\varepsilon^2}\langle V(t-\frac{t-\tau}{\varepsilon};t,x,v)\rangle\mathrm{d}\tau}\\
&\quad\times\frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}\mathbf{k}_{\tilde\beta}\left(V_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon};t,x,v\right),v'\right)\left|\varepsilon^{\frac12}h\left(s,X_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon};t,x,v\right),v'\right)\right|\mathrm{d}v'\\
&\quad+e^{-\int_s^t\frac{C_0}{\varepsilon^2}\langle V(t-\frac{t-\tau}{\varepsilon};t,x,v)\rangle\mathrm{d}\tau}\varepsilon^{-\frac12}\left|wg\left(s,X_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon};t,x,v\right),V_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon};t,x,v\right)\right)\right|.
\end{aligned}
$$

Along the stochastic cycles, for $k = C_1 T_0^{5/4}$, we deduce the following estimate:

$$
\begin{aligned}
&|\varepsilon^{\frac{1}{2}} h^{\ell+1}(t,x,v)| \\
&\le \mathbf{1}_{\{\tilde{t}_1<0\}} e^{-\int_0^t \frac{C_0\langle V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon})\rangle}{\varepsilon^2}\mathrm{d}\tau} \\
&\quad \times \left|\varepsilon^{\frac{1}{2}} h^{\ell+1}\left(0, X_{\mathbf{cl}}\left(t-\frac{t}{\varepsilon}\right), V_{\mathbf{cl}}\left(t-\frac{t}{\varepsilon}\right)\right)\right| \qquad (3.68)\\
&\quad + \int_{\max\{0,\tilde{t}_1\}}^t \mathrm{d}s\, \frac{e^{-\int_s^t \frac{C_0\langle V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon})\rangle}{\varepsilon^2}\mathrm{d}\tau}}{\varepsilon^2} \\
&\quad \times \int_{\mathbb{R}^3} \mathrm{d}v'\, \mathbf{k}_{\tilde{\beta}}\left(V_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon}\right), v'\right) \\
&\quad \times \left|\varepsilon^{\frac{1}{2}} h^{\ell}\left(s, X_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon}\right), v'\right)\right| \qquad (3.69)\\
&\quad + \int_{\max\{0,\tilde{t}_1\}}^t \mathrm{d}s\, \frac{e^{-\int_s^t \frac{C_0\langle V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon})\rangle}{\varepsilon^2}\mathrm{d}\tau}}{\varepsilon^2} \\
&\quad \times \varepsilon^2 \varepsilon^{-\frac{1}{2}} \left| wg\left(s, X_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon}\right), V_{\mathbf{cl}}\left(t-\frac{t-s}{\varepsilon}\right)\right)\right| \qquad (3.70)\\
&\quad + \mathbf{1}_{\{\tilde{t}_1\ge 0\}} e^{-\int_{\tilde{t}_1}^t \frac{C_0\langle V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon})\rangle}{\varepsilon^2}\mathrm{d}\tau} \varepsilon^{\frac{1}{2}} |wr(\tilde{t}_1(t,x,v))| \\
&\quad + \mathbf{1}_{\{\tilde{t}_1\ge 0\}} \frac{e^{-\int_{\tilde{t}_1}^t \frac{C_0\langle V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon})\rangle}{\varepsilon^2}\mathrm{d}\tau}}{\tilde{w}(v_1)} \int_{\Pi_{j=1}^{k-1}\mathscr{V}_j} H, \qquad (3.71)
\end{aligned}
$$

$(X_{\mathbf{cl}}(\cdot), V_{\mathbf{cl}}(\cdot)) = (X_{\mathbf{cl}}(\cdot;t,x,v), V_{\mathbf{cl}}(\cdot;t,x,v))$, and where $\mathscr{V}_j$ is defined in (2.29). And H is given by

$$
\begin{aligned}
&\sum_{l=1}^{k-1} \mathbf{1}_{\tilde{t}_{l+1}\le 0<\tilde{t}_l} \left|\varepsilon^{\frac{1}{2}} h\left(0, X_{\mathbf{cl}}\left(\tilde{t}_l-\frac{\tilde{t}_l}{\varepsilon}\right), V_{\mathbf{cl}}\left(\tilde{t}_l-\frac{\tilde{t}_l}{\varepsilon}\right)\right)\right| \\
&\times \Pi_{m=1}^{l-1} \frac{\tilde{w}(v_m)}{\tilde{w}\left(V_{\mathbf{cl}}\left(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon}; v_m\right)\right)} \mathrm{d}\Sigma_l(0) \qquad (3.72)\\
&+ \sum_{l=1}^{k-1} \int_{\max\{0,\tilde{t}_{l+1}\}}^{\tilde{t}_l} \mathrm{d}\tau\, \mathbf{1}_{\tilde{t}_l>0} \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \mathbf{k}_{\tilde{\beta}}\left(V_{\mathbf{cl}}\left(\tilde{t}_l-\frac{\tilde{t}_l-\tau}{\varepsilon}\right), u\right) \\
&\times \left|\varepsilon^{\frac{1}{2}} h\left(\tau, X_{\mathbf{cl}}\left(\tilde{t}_l-\frac{\tilde{t}_l-\tau}{\varepsilon}\right), u\right)\right| \times \Pi_{m=1}^{l-1} \frac{\tilde{w}(v_m)}{\tilde{w}\left(V_{\mathbf{cl}}\left(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon}; v_m\right)\right)} \mathrm{d}u\,\mathrm{d}\Sigma_l(\tau)
\end{aligned}
\qquad (3.73)
$$

$$
\begin{aligned}
&+\sum_{l=1}^{k-1}\int_{\max\{0,\tilde{t}_{l+1}\}}^{\tilde{t}_l}\mathbf{1}_{\tilde{t}_l>0}\Pi_{m=1}^{l-1}\frac{\tilde{w}(v_m)}{\tilde{w}\left(V_{\mathbf{cl}}\left(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};v_m\right)\right)}\\
&\times\varepsilon^{-\frac{1}{2}}\left|wg\left(\tau,X_{\mathbf{cl}}\left(\tilde{t}_l-\frac{\tilde{t}_l-\tau}{\varepsilon}\right),V_{\mathbf{cl}}\left(\tilde{t}_l-\frac{\tilde{t}_l-\tau}{\varepsilon}\right)\right)\right|\mathrm{d}\Sigma_l(\tau)\mathrm{d}\tau \qquad (3.74)\\
&+\sum_{l=1}^{k-1}\mathbf{1}_{\tilde{t}_l>0}\varepsilon^{\frac{1}{2}}w(v_l)|r(\tilde{t}_l,x_{l+1},v_l)|\Pi_{m=1}^{l-1}\frac{\tilde{w}(v_m)}{\tilde{w}\left(V_{\mathbf{cl}}\left(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};v_m\right)\right)}\mathrm{d}\Sigma_l(\tilde{t}_{l+1}) \qquad (3.75)\\
&+\mathbf{1}_{\tilde{t}_k>0}|\varepsilon^{\frac{1}{2}}h(\tilde{t}_k,x_k,v_{k-1})|\Pi_{m=1}^{k-2}\frac{\tilde{w}(v_m)}{\tilde{w}\left(V_{\mathbf{cl}}\left(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};v_m\right)\right)}\mathrm{d}\Sigma_{k-1}(\tilde{t}_k), \qquad (3.76)
\end{aligned}
$$

where $(X_{\mathbf{cl}}(\cdot),V_{\mathbf{cl}}(\cdot))=(X_{\mathbf{cl}}(.;\tilde{t}_l,x_l,v_l),V_{\mathbf{cl}}(.;\tilde{t}_l,x_l,v_l))$ and, $\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};v_m))=\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};\tilde{t}^m,x_m,v_m))$ and $\mathrm{d}\Sigma_{k-1}(\tilde{t}_k)$ is evaluated at $s=\tilde{t}_k$ of

$$
\begin{aligned}
\mathrm{d}\Sigma_l(s) := &\left\{\prod_{j=l+1}^{k-1}\mathrm{d}\sigma_j\right\}\left\{e^{-\int_s^{\tilde{t}_l}\frac{C_0\langle V_{\mathbf{cl}}(\tilde{t}_l-\frac{\tilde{t}_l-\tau}{\varepsilon};\tilde{t}_l,x_l,v_l)\rangle}{\varepsilon^2}\mathrm{d}\tau}\tilde{w}(v_l)\mathrm{d}\sigma_l\right\}\\
&\times\prod_{j=1}^{l-1}\left\{e^{-\int_{\tilde{t}_{j+1}}^{\tilde{t}_j}\frac{C_0\langle V_{\mathbf{cl}}(\tilde{t}_j-\frac{\tilde{t}_j-\tau}{\varepsilon};\tilde{t}_j,x_j,v_j)\rangle}{\varepsilon^2}\mathrm{d}\tau}\mathrm{d}\sigma_j\right\}.
\end{aligned} \qquad (3.77)
$$

We note that

$$
\left|V_{\mathbf{cl}}\left(\tilde{t}_m-\frac{\tilde{t}_{m+1}-t_m}{\varepsilon};\tilde{t}_m,x_m,v_m\right)-v_m\right|\le\frac{\tilde{t}_{m+1}-\tilde{t}_m}{\varepsilon}\varepsilon^2\|\Phi\|_\infty\le\varepsilon T_0,
$$

Hence for $\varepsilon T_0<1$, we have

$$
\frac{\tilde{w}(v_m)}{\tilde{w}\left(V_{\mathbf{cl}}\left(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};\tilde{t}_m,x_m,v_m\right)\right)}\lesssim 1. \qquad (3.78)
$$

From our choice $k=C_1T_0^{5/4}$,

$$
\begin{aligned}
&(3.68)+(3.72)\lesssim C_1T_0^{5/4}e^{-\frac{C_0}{\varepsilon^2}t}\|\varepsilon^{\frac{1}{2}}h_0\|_\infty,\\
&(3.71)+(3.75)\lesssim C_1T_0^{5/4}\varepsilon\sup_{0\le s\le t}\|\varepsilon^{\frac{1}{2}}wr(s)\|_\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&(3.70) + (3.74)\\
&\lesssim \varepsilon^2\varepsilon^{-\frac{1}{2}} \sup_{0\le s\le t} \big\|\langle v\rangle w g(s)\big\|_\infty \times \Bigg\{ \int_0^t \frac{\langle V_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon};t,x,v)\rangle}{\varepsilon^2} e^{-\int_s^t \frac{C_0\langle V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon};t,x,v)\rangle}{\varepsilon^2}\mathrm{d}\tau}\mathrm{d}s\\
&\quad + C_1T_0^{5/4}\sup_l \int_0^{\tilde t_l} \frac{\langle V_{\mathbf{cl}}(\tilde t_l-\frac{\tilde t_l-\tau}{\varepsilon};\tilde t_l,x_l,v_l)\rangle}{\varepsilon^2} e^{-\int_s^{\tilde t_l} \frac{C_0\langle V_{\mathbf{cl}}(\tilde t_l-\frac{\tilde t_l-\tau}{\varepsilon};\tilde t_l,x_l,v_l)\rangle}{\varepsilon^2}\mathrm{d}\tau}\mathrm{d}\tau\Bigg\}\\
&\lesssim C_1T_0^{5/4}\varepsilon^{\frac{3}{2}} \sup_{0\le s\le t} \big\|\langle v\rangle w g(s)\big\|_\infty \times \int_0^t \frac{d}{ds} e^{-\int_s^t \frac{C_0\langle V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon};t,x,v)\rangle}{\varepsilon^2}\mathrm{d}\tau}\mathrm{d}s\\
&\lesssim C_1T_0^{5/4}\varepsilon^{\frac{3}{2}} \sup_{0\le s\le t} \big\|\langle v\rangle w g(s)\big\|_\infty,
\end{aligned}
$$

where we have used the fact that $\mathrm{d}\sigma_j$ is a probability measure of $\mathscr{V}_j$.

Now we focus on (3.69) and (3.73). For $N > 1$, we can choose $m = m(N) \gg 1$ and define $\mathbf{k}_m(v,u)$ as (2.48). We split $\mathbf{k}_{\tilde\beta}(v,u) = [\mathbf{k}_{\tilde\beta}(v,u) - \mathbf{k}_m(v,u)] + \mathbf{k}_m(v,u)$, and the first difference would lead to a small contribution in (3.69) and (3.73) as, for $N \gg_{T_0} 1$,

$$
\frac{k}{N}\sup_{0\le s\le t}\|\varepsilon^{\frac{1}{2}}h(s)\|_\infty = \frac{C_1T_0^{5/4}}{N}\sup_{0\le s\le t}\|\varepsilon^{\frac{1}{2}}h(s)\|_\infty.
$$

We further split the time integrations in (3.69) and (3.73) as $[\tilde t_l - \kappa\varepsilon^2, \tilde t_l]$ and $[\max\{0,\tilde t_{l+1}\}, \tilde t_l - \kappa\varepsilon^2]$:

$$
(3.69) = \underbrace{\int_{t-\kappa\varepsilon^2}^t} + \int_{\max\{0,\tilde t_1\}}^{t-\kappa\varepsilon^2}, \quad (3.73) = \mathbf{1}_{\{\tilde t_1\ge 0\}}\int_{\Pi_{j=1}^{k-1}\mathscr{V}_j}\sum_{l=1}^{k-1}\Bigg\{\underbrace{\int_{\tilde t_l-\kappa\varepsilon^2}^{\tilde t_l}} + \int_{\max\{0,\tilde t_{l+1}\}}^{\tilde t_l-\kappa\varepsilon^2}\Bigg\}.
$$

The first small-in-time contributions of both (3.69) and (3.73), underbraced terms, are bounded by

$$
\begin{aligned}
&\kappa\varepsilon^2\frac{1}{\varepsilon^2}\sup_v\int_{|v'|\le N}\mathbf{k}_m(v,v')\mathrm{d}v'\sup_{0\le s\le t}\|\varepsilon^{\frac{1}{2}}h(s)\|_\infty\\
&\quad\lesssim \kappa\sup_{0\le s\le t}\|\varepsilon^{\frac{1}{2}}h(s)\|_\infty, C_1T_0^{5/4}\kappa\varepsilon^2\frac{1}{\varepsilon^2}\sup_v\int_{|v'|\le N}\mathbf{k}_m(v,v')\mathrm{d}v'\sup_{0\le s\le t}\|\varepsilon^{\frac{1}{2}}h(s)\|_\infty\\
&\quad\lesssim \kappa C_1T_0^{5/4}\sup_{0\le s\le t}\|\varepsilon^{\frac{1}{2}}h(s)\|_\infty,
\end{aligned}
$$

which would be small contribution if $\kappa \ll_{T_0} 1$.

For (3.76), by Lemma 3.12,

$$
\begin{aligned}
(3.76) \lesssim &\sup_{0\le s\le t}\|\varepsilon^{\frac{1}{2}}h(s)\|_\infty\\
&\times\sup_{(t,x,v)\in[0,\varepsilon T_0]\times\bar\Omega\times\mathbb{R}^3}\int_{\prod_{j=1}^{k-1}\mathscr{V}_j}\mathbf{1}_{\tilde t_k(t,x,v,v_1,v_2,\dots,v_{k-1})>0}\Pi_{j=1}^{k-1}\mathrm{d}\sigma_j.
\end{aligned}
$$

Then, by Lemma 3.12,

$$
\begin{aligned}
(3.76) &\lesssim \{1+O(\varepsilon)\}^{C_1 T_0^{5/4}} \sup_{0\le s\le t} \|\varepsilon^{\frac12} h(s)\|_\infty \\
&\quad\times \sup_{(x,v)\in\bar{\Omega}\times\mathbb{R}^3} \int_{\prod_{j=1}^{k-1}\mathcal{V}_j} \mathbf{1}_{t_k(T_0,x,v,v_1,v_2,\dots,v_{k-1})>0} \Pi_{j=1}^{k-1} \mathrm{d}\sigma_j \\
&\lesssim \left\{\frac45\right\}^{C_2 T_0^{5/4}} \sup_{0\le s\le t} \|\varepsilon^{\frac12} h(s)\|_\infty .
\end{aligned}
$$

Overall, for $(t,x,v)\in[0,\varepsilon T_0]\times\bar{\Omega}\times\mathbb{R}^3$,

$$
\begin{aligned}
|\varepsilon^{\frac12} h(t,x,v)| &\lesssim \int_{\max\{0,\tilde{t}_1(x,v)\}}^{t-\kappa\varepsilon^2} \mathrm{d}s\, \frac{e^{-\frac{C_0}{\varepsilon^2}(t-s)}}{\varepsilon^2} \\
&\quad\times \int_{|v'|\le m} \mathrm{d}v' \underbrace{\left|\varepsilon^{\frac12} h\Big(s, X_{\mathbf{cl}}\Big(t-\frac{t-s}{\varepsilon}; t,x,v\Big), v'\Big)\right|} \\
&\quad+ \mathbf{1}_{\{\tilde{t}_1\ge 0\}} \frac{e^{-\frac{C_0}{\varepsilon^2}(t-\tilde{t}_1)}}{\tilde{w}(v)} \int_{\Pi_{j=1}^{k-1}\mathcal{V}_j} \sum_{\ell=1}^{k-1} \int_{\max\{0,\tilde{t}_{\ell+1}\}}^{\tilde{t}_\ell-\kappa\varepsilon^2} \mathbf{1}_{\tilde{t}_\ell>0} \frac{1}{\varepsilon^2} \\
&\quad\times \int_{|v''|\le m} \underbrace{\left|\varepsilon^{\frac12} h\Big(\tau, X_{\mathbf{cl}}\Big(\tilde{t}_\ell-\frac{\tilde{t}_\ell-\tau}{\varepsilon}; \tilde{t}_\ell, x_\ell, v_\ell\Big), v''\Big)\right|} \mathrm{d}v''\mathrm{d}\Sigma_\ell(\tau)\mathrm{d}\tau \\
&\quad+ C T_0^{5/4}\Big\{ e^{-\frac{C_0}{\varepsilon^2}t} \varepsilon^{\frac12}\|h_0\|_\infty + \varepsilon^{\frac12} \sup_{0\le s\le t}\|wr(s)\|_\infty \\
&\quad+ \varepsilon^{-\frac12} \sup_{0\le s\le t} \|\langle v\rangle w g(s)\|_\infty \Big\} \\
&\quad+ o(1) C T_0^{5/4} \sup_{0\le s\le t} \|\varepsilon^{\frac12} h(s)\|_\infty + \left\{\frac45\right\}^{C_2 T_0^{5/4}} \sup_{0\le s\le t} \|\varepsilon^{\frac12} h(s)\|_\infty .
\end{aligned}
\tag{3.79}
$$

Note that the similar estimate holds for the underbraced terms in (3.79). We plug these estimates into the underbraced terms of (3.79) to conclude

$$
|\varepsilon^{\frac12} h^{\ell+1}(t,x,v)| \le \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 .
$$

Here, using $w(u)\lesssim_m 1$ for $|u|\le m$,

$$
\begin{aligned}
\mathbf{I}_1 &\lesssim_m \int_{\max\{0,\tilde{t}_1\}}^{t-\kappa\varepsilon^2} \mathrm{d}s\, \frac{e^{-\frac{C_0}{\varepsilon^2}(t-s)}}{\varepsilon^2} \int_{|v'|\le m} \mathrm{d}v' \int_{\max\{0,\tilde{t}_1'\}}^{s-\kappa\varepsilon^2} \mathrm{d}s'\, \frac{e^{-\frac{C_0(s-s')}{\varepsilon^2}}}{\varepsilon^2} \int_{|u|\le m} \mathrm{d}u \\
&\quad\times \left|\varepsilon^{\frac12} h\Big(s', X_{\mathbf{cl}}\Big(s-\frac{s-s'}{\varepsilon}; s, X_{\mathbf{cl}}\Big(t-\frac{t-s}{\varepsilon}; t,x,v\Big), v'\Big), u\Big)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\max\{0,\tilde t_1\}}^{t-\kappa\varepsilon^2} \mathrm{d}s\, \frac{e^{-\frac{C_0}{\varepsilon^2}(t-s)}}{\varepsilon^2} \int_{|v'|\le m} \mathrm{d}v'\, \mathbf{1}_{\{\tilde t_1'\ge 0\}} \frac{e^{-\frac{C_0}{\varepsilon^2}(s-\tilde t_1')}}{\tilde w(v)} \\
&\times \int_{\Pi_{j=1}^{k-1}\mathscr{V}_j'} \sum_{\ell'=1}^{k-1} \int_{\max\{0,\tilde t'_{\ell'+1}\}}^{\tilde t'_{\ell'}-\kappa\varepsilon^2} \mathbf{1}_{\tilde t'_{\ell'}>0} \frac{1}{\varepsilon^2} \\
&\times \Big|\varepsilon^{\frac12} h\big(\tau, X_{\mathbf{cl}}\big(\tilde t'_{\ell'} - \frac{\tilde t'_{\ell'}-\tau}{\varepsilon}; \tilde t'_{\ell'}, x'_{\ell'}, v'_{\ell'}\big), u\big)\Big| \mathrm{d}u \mathrm{d}\Sigma_{\ell'}(\tau)\mathrm{d}\tau,
\end{aligned}
$$

where

$$
\tilde t'_{\ell'} := \tilde t_{\ell'}\left(s, X_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon}; t, x, v), v'\right),
$$

$$
x'_{\ell'} := x_{\ell'}\left(X_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon}; t, x, v), v'\right), \quad v'_{\ell'} := v_{\ell'}\left(X_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon}; t, x, v), v'\right).
$$

Moreover

$$
\begin{aligned}
\mathbf{I}_2 \lesssim_m & \mathbf{1}_{\{\tilde t_1\ge 0\}} \frac{e^{-\frac{C_0}{\varepsilon^2}(t-\tilde t_1)}}{\tilde w(v)} \int_{\Pi_{j=1}^{k-1}\mathscr{V}_j} \sum_{\ell=1}^{k-1} \int_{\max\{0,\tilde t_{\ell+1}\}}^{\tilde t_\ell-\kappa\varepsilon^2} \mathrm{d}\Sigma_\ell(\tau)\mathrm{d}\tau\, \mathbf{1}_{\tilde t_\ell>0} \frac{1}{\varepsilon^2} \int_{|v''|\le m} \mathrm{d}v'' \\
&\times \int_{\max\{0,\tilde t_1''\}}^{\tau-\kappa\varepsilon^2} \mathrm{d}s''\, \frac{e^{-\frac{C_0}{\varepsilon^2}(\tau-s'')}}{\varepsilon^2} \\
&\times \int_{|u|\le m} \mathrm{d}u\, \Big|\varepsilon^{\frac12} h\big(s'', X_{\mathbf{cl}}\big(\tau - \frac{\tau-s''}{\varepsilon}; \tau, X_{\mathbf{cl}}\big(\tilde t_\ell - \frac{\tilde t_\ell-\tau}{\varepsilon}; \tilde t_\ell, x_\ell, v_\ell\big), v''\big), u\big)\Big| \\
&+ \mathbf{1}_{\{\tilde t_1\ge 0\}} \frac{e^{-\frac{C_0}{\varepsilon^2}(t-\tilde t_1)}}{\tilde w(v)} \int_{\Pi_{j=1}^{k-1}\mathscr{V}_j} \sum_{\ell=1}^{k-1} \int_{\max\{0,\tilde t_{\ell+1}\}}^{\tilde t_\ell-\kappa\varepsilon^2} \mathrm{d}\Sigma_\ell(\tau)\mathrm{d}\tau\, \mathbf{1}_{\tilde t_\ell>0} \frac{1}{\varepsilon^2} \int_{|v''|\le m} \mathrm{d}v'' \\
&\times \mathbf{1}_{\tilde t_1''\ge 0} \frac{e^{-\frac{C_0}{\varepsilon^2}(\tau-\tilde t_1'')}}{\tilde w(v'')} \int_{\prod_{j=1}^{k-1}\mathscr{V}_j''} \sum_{\ell''=1}^{k-1} \int_{\max\{0,\tilde t''_{\ell''+1}\}}^{\tilde t''_{\ell''}-\kappa\varepsilon^2} \mathbf{1}_{\tilde t''_{\ell''}>0} \frac{1}{\varepsilon^2} \\
&\times \int_{|u|\le m} \Big|\varepsilon^{\frac12} h\big(\tau'', X_{\mathbf{cl}}\big(\tilde t''_{\ell''} - \frac{\tilde t''_{\ell''}-\tau''}{\varepsilon}; \tilde t''_{\ell''}, x''_{\ell''}, v''_{\ell''}\big), u\big)\Big| \mathrm{d}u \mathrm{d}\Sigma''_{\ell''}(\tau'')\mathrm{d}\tau'',
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde t''_{\ell''} &:= \tilde t_{\ell''}\left(\tau, X_{\mathbf{cl}}\left(\tilde t_\ell - \frac{\tilde t_\ell-\tau}{\varepsilon}; \tilde t_\ell, x_\ell, v_\ell\right), v''\right), \\
x''_{\ell''} &:= x_{\ell''}\left(X_{\mathbf{cl}}\left(\tilde t_\ell - \frac{\tilde t_\ell-\tau}{\varepsilon}; \tilde t_\ell, x_\ell, v_\ell\right), v''\right), \\
v''_{\ell''} &:= v_{\ell''}\left(X_{\mathbf{cl}}(\tilde t_\ell - \frac{\tilde t_\ell-\tau}{\varepsilon}; \tilde t_\ell, x_\ell, v_\ell), v''\right).
\end{aligned}
$$

Furthermore

$$\mathbf{I}_3 \lesssim CT_0^{5/2}\left\{e^{-\frac{C_0}{\varepsilon^2}t}\|\varepsilon^{\frac12}h_0\|_\infty+\varepsilon^{\frac12}\sup_{0\le s\le t}\|wr(s)\|_\infty+\varepsilon^{\frac32}\sup_{0\le s\le t}\|\langle v\rangle wg(s)\|_\infty\right\}$$
$$+\,o(1)CT_0^{5/2}\sup_{0\le s\le t}\|\varepsilon^{\frac12}h(s)\|_\infty+T_0^{5/4}\left\{\frac12\right\}^{C_2T_0^{5/4}}\sup_{0\le s\le t}\|\varepsilon^{\frac12}h(s)\|_\infty.$$

This bound of $\mathbf{I}_3$ is already included in the RHS of (3.66) and (3.67).

Now we focus on $\mathbf{I}_1$ and $\mathbf{I}_2$. Consider the change of variables

$$v'_{\ell'}\mapsto X_{\mathbf{cl}}\left(\tilde t'_{\ell'}-\frac{\tilde t'_{\ell'}-\tau}{\varepsilon};\tilde t'_{\ell'},x'_{\ell'},v'_{\ell'}\right).$$

For $0\le\tilde t'_{\ell'}\le\tau-\kappa\varepsilon^2\le\tau\le\varepsilon T_0$,

$$\begin{aligned}\frac{\partial X_i(\tilde t'_{\ell'}-\frac{\tilde t'_{\ell'}-\tau}{\varepsilon};\tilde t'_{\ell'})}{\partial v'_j}&=-\frac{\tilde t'_{\ell'}-\tau}{\varepsilon}\delta_{ij}\\&\quad+\int_{\tilde t'_{\ell'}}^{\tilde t'_{\ell'}-\frac{\tilde t'_{\ell'}-\tau}{\varepsilon}}\mathrm{d}\tau'\int_{\tilde t'_{\ell'}}^{\tau'}\mathrm{d}\tau''\varepsilon^2\sum_m\partial_m\Phi_i\left(X\left(\tau'';\tilde t'_{\ell'}\right)\right)\frac{\partial X_m}{\partial v'_j}\left(\tau'';\tilde t'_{\ell'}\right)\\&=-\frac{\tilde t'_{\ell'}-\tau}{\varepsilon}\delta_{ij}+O(1)\|\Phi\|_{C^1}\varepsilon^2\left(\frac{\tilde t'_{\ell'}-\tau}{\varepsilon}\right)^3e^{C_\Phi\frac{|\tilde t'_{\ell'}-\tau|}{\varepsilon}}\\&=-\frac{\tilde t'_{\ell'}-\tau}{\varepsilon}\Big[\delta_{ij}+O(1)\|\Phi\|_{C^1}\varepsilon^2T_0^2e^{C_\Phi T_0}\Big],\end{aligned}$$

and therefore

$$\begin{aligned}&\det\nabla_{v'_{\ell'}}X_{\mathbf{cl}}(\tilde t'_{\ell'}-\frac{\tilde t'_{\ell'}-\tau}{\varepsilon};\tilde t'_{\ell'},x'_{\ell'},v'_{\ell'})\\&\quad=\left(\frac{-\tilde t'_{\ell'}+\tau}{\varepsilon}\right)^3\det\left(\delta_{ij}+O(1)\|\Phi\|_{C^1}\varepsilon^2T_0^2e^{C_\Phi T_0}\right)\gtrsim\kappa^3\varepsilon^3.\end{aligned}$$

We have similar change of variables for $v''_{\ell''}\mapsto X_{\mathbf{cl}}(\tilde t''_{\ell''}-\frac{\tilde t''_{\ell''}-\tau''}{\varepsilon};\tilde t''_{\ell''},x''_{\ell''},v''_{\ell''})$, $v'\mapsto X_{\mathbf{cl}}(s-\frac{s-s'}{\varepsilon};s,X_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon};t,x,v),v')$, and $v''\mapsto X_{\mathbf{cl}}(\tau-\frac{\tau-s''}{\varepsilon};\tau,X_{\mathbf{cl}}(\tilde t_\ell-\frac{\tilde t_\ell-\tau}{\varepsilon};\tilde t_\ell,x_\ell,v_\ell),v'')$.

Hence,

$$\mathbf{I}_1+\mathbf{I}_2\lesssim T_0^{5/2}\frac{1}{\varepsilon}\sup_{0\le s\le t}\|f(s)\|_{L^2(\Omega\times\mathbb{R}^3)},$$

where we applied the above change of variables as

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{|u|\le m} \mathbf{k}_m\big(V_{\mathbf{cl}}\big(\tilde{t}''_{\ell''} - \frac{\tilde{t}''_{\ell''}-\tau''}{\varepsilon}; \tilde{t}''_{\ell''}, x''_{\ell''}, v''_{\ell''}\big), u\big) \\
&\quad\times \big|f\big(\tau'', X_{\mathbf{cl}}\big(\tilde{t}''_{\ell''} - \frac{\tilde{t}''_{\ell''}-\tau''}{\varepsilon}; \tilde{t}''_{\ell''}, x''_{\ell''}, v''_{\ell''}\big), u\big)\big| \mathrm{d}u \mathrm{d}v''_{\ell''} \\
&\quad\le \Big[\iint_{\{|u|\le m\}\times\mathbb{R}^3} \mathbf{1}_{|V_{\mathbf{cl}}(\tilde{t}''_{\ell''} - \frac{\tilde{t}''_{\ell''}-\tau''}{\varepsilon}; \tilde{t}''_{\ell''}, x''_{\ell''}, v''_{\ell''})|\le m} \mathrm{d}u\mathrm{d}v''_{\ell''}\Big]^{1/2} \\
&\qquad\times\Big[\int_{v''_{\ell''}}\int_u \big|f\big(\tau'', X_{\mathbf{cl}}(\tilde{t}''_{\ell''} - \frac{\tilde{t}''_{\ell''}-\tau''}{\varepsilon}; \tilde{t}''_{\ell''}, x''_{\ell''}, v''_{\ell''}), u\big)\big|^2 \mathrm{d}u\mathrm{d}v''_{\ell''}\Big]^{1/2} \\
&\quad\lesssim_m \Big[\int_\Omega\int_u \big|f\big(\tau'', y, u\big)\big|^2 \frac{1}{\kappa^3\varepsilon^3}\mathrm{d}u\mathrm{d}y\Big]^{1/2} \lesssim_m \frac{1}{\varepsilon^{3/2}}\|f(\tau'')\|_{L^2(\Omega\times\mathbb{R}^3)},
\end{aligned}$$

where we have used $\mathbf{1}_{|V_{\mathbf{cl}}(\tilde{t}''_{\ell''} - \frac{\tilde{t}''_{\ell''}-\tau''}{\varepsilon}; \tilde{t}''_{\ell''}, x''_{\ell''}, v''_{\ell''})|\le m} \le \mathbf{1}_{|v''_{\ell''}|\le 2m}$ and

$$\begin{aligned}
|v''_{\ell''}| &\le |V_{\mathbf{cl}}(\tilde{t}''_{\ell''} - \frac{\tilde{t}''_{\ell''}-\tau''}{\varepsilon}; \tilde{t}''_{\ell''}, x''_{\ell''}, v''_{\ell''})| + \varepsilon^2\|\Phi\|_\infty \frac{|\tilde{t}''_{\ell''}-\tau''|}{\varepsilon} \\
&\le m + \varepsilon^2\|\Phi\|_\infty T_0 \le 2m.
\end{aligned}$$

Moreover

$$\mathbf{I}_1 + \mathbf{I}_2 \lesssim T_0^{5/2}\sup_{0\le s\le t}\|\mathbf{P}f(s)\|_{L^6(\Omega\times\mathbb{R}^3)} + T_0^{5/2}\frac{1}{\varepsilon}\sup_{0\le s\le t}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2(\Omega\times\mathbb{R}^3)},$$

where we have used the above change of variables for $\mathbf{P}f$ as

$$\begin{aligned}
&\int_{v''_{\ell''}}\int_u \big|\mathbf{P}f\big(\tau'', X_{\mathbf{cl}}(\tilde{t}''_{\ell''} - \frac{\tilde{t}''_{\ell''}-\tau''}{\varepsilon}; \tilde{t}''_{\ell''}, x''_{\ell''}, v''_{\ell''})\big)\psi(u)\big|\mathrm{d}u\mathrm{d}v''_{\ell''} \\
&\quad\lesssim_m \Big[\int_{v''_{\ell''}}\int_u \big|\mathbf{P}f\big(\tau'', X_{\mathbf{cl}}(\tilde{t}''_{\ell''} - \frac{\tilde{t}''_{\ell''}-\tau''}{\varepsilon}; \tilde{t}''_{\ell''}, x''_{\ell''}, v''_{\ell''})\big)\big|^2\mathrm{d}v''_{\ell''}\Big]^{1/2} \\
&\quad\lesssim_m \Big[\int_\Omega \big|\mathbf{P}f\big(\tau'', y\big)\big|^6\frac{1}{\kappa^3\varepsilon^3}\mathrm{d}y\Big]^{1/6} \lesssim_m \frac{1}{\varepsilon^{\frac{1}{2}}}\|\mathbf{S}_1 f(\tau'')\|_{L^6(\Omega)}.
\end{aligned}$$

All together we prove our claims (3.66) and (3.67).

Step 2. Applying (3.66) successively,

$$\begin{aligned}
&\|\varepsilon^{\frac{1}{2}}h(n\varepsilon T_0)\|_\infty \\
&\quad\le C T_0^{5/2}e^{-\frac{C_0T_0}{\varepsilon}}\|\varepsilon^{\frac{1}{2}}h((n-1)\varepsilon T_0)\|_\infty + \sup_{(n-1)\varepsilon T_0\le s\le n\varepsilon T_0} D(s) \\
&\quad\le \Big[C T_0^{5/2}e^{-\frac{C_0T_0}{\varepsilon}}\Big]^2\|\varepsilon^{\frac{1}{2}}h((n-2)\varepsilon T_0)\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{1}\Big[CT_0^{5/2}e^{-\frac{C_0T_0}{\varepsilon}}\Big]^j \sup_{(n-2)\varepsilon T_0\le s\le n\varepsilon T_0} D(s)\\
&\vdots\\
&\le \Big[CT_0^{5/2}e^{-\frac{C_0T_0}{\varepsilon}}\Big]^n\|\varepsilon^{\frac12}h_0\|_\infty+\sum_{j=0}^{n-1}\Big[CT_0^{5/2}e^{-\frac{C_0T_0}{\varepsilon}}\Big]^j\sup_{0\le s\le n\varepsilon T_0} D(s),
\end{aligned}
$$

where

$$
\begin{aligned}
D(s) := {}&\|\varepsilon^{\frac12}wr(s)\|_\infty + CT_0^{5/2}\|\langle v\rangle^{-1}\varepsilon^{\frac32}wg(s)\|_\infty + CT_0^{5/2}\|\mathbf{P}f(s)\|_{L^6(\Omega)}\\
&+CT_0^{5/2}\frac{1}{\varepsilon}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2(\Omega\times\mathbb{R}^3)}\\
&+\big[CT_0^{5/4}\Big\{\frac12\Big\}^{C_2T_0^{5/4}}+o(1)CT_0^{5/4}\big]\|\varepsilon^{\frac12}h(s)\|_\infty.
\end{aligned}
$$

Clearly $\sum_j\Big[CT_0^{5/2}e^{-\frac{C_0T_0}{\varepsilon}}\Big]^j<\infty$.

Combining the above estimate with (3.66), for $t\in[n\varepsilon T_0,(n+1)\varepsilon T_0]$, and absorbing the last term,

$$
\begin{aligned}
&CT_0^{5/2}e^{-\frac{C_0(t-n\varepsilon T_0)}{\varepsilon^2}}\sum_{j=0}^{n-1}\Big[CT_0^{5/2}e^{-\frac{C_0T_0}{\varepsilon}}\Big]^j\\
&\quad\big[CT_0^{5/4}\Big\{\frac12\Big\}^{C_2T_0^{5/4}}+o(1)CT_0^{5/4}\big]\sup_{0\le s\le t}\|\varepsilon^{\frac12}h(s)\|_\infty\\
&\lesssim \frac{T_0^{5/2}}{1-CT_0^{5/2}e^{-\frac{C_0T_0}{\varepsilon}}}\big[CT_0^{5/4}\Big\{\frac12\Big\}^{C_2T_0^{5/4}}+o(1)CT_0^{5/2}\big]\times\sup_{0\le s\le t}\|\varepsilon^{\frac12}h(s)\|_\infty\\
&\lesssim o(1)\times\sup_{0\le s\le t}\|\varepsilon^{\frac12}h(s)\|_\infty,
\end{aligned}
$$

where we used

$$
T_0^{5/2}\big[CT_0^{5/4}\Big\{\frac12\Big\}^{C_2T_0^{5/4}}+o(1)CT_0^{5/2}\big]\ll 1,
$$

to conclude (3.61). Similarly we can prove (3.62). □

3.5 L^6 Estimate

Proposition 3.13 *Let f satisfy the assumptions of Proposition 3.8. Then*

$$
\begin{aligned}
\sup_{0\le s\le t}\|e^{\lambda s}\mathbf{P}f\|_{L^6_{x,v}} \lesssim{}& \lambda\|f_0\|_2+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_0\|_\nu+\varepsilon^{-\frac12}\|(1-P_\gamma)f_0\|_{2,\gamma}\\
&+\mathscr{D}_\lambda[f](t)^{\frac12}+(\lambda+\varepsilon)\mathscr{E}_\lambda[f](t)^{\frac12}+\|e^{\lambda t}r\|_{L^\infty_tL^2(\gamma)}+\|\varepsilon^{\frac32}e^{\lambda t}\langle v\rangle^{-1}wr\|_{L^\infty_tL^\infty(\gamma)}
\end{aligned}
$$

$$+\|e^{\lambda t}\nu^{-\frac{1}{2}}g\|_{L^\infty_t L^2_{x,v}}$$
$$+\varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1}wg\|_{L^\infty_{t,x,v}}+\lambda\varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1}wf\|_{L^\infty_{t,x,v}}+\varepsilon^{\frac{5}{2}}\|e^{\lambda t}\langle v\rangle^{-1}w\partial_t f\|_{L^\infty_{t,x,v}},$$

where $\mathscr{E}_\lambda[\cdot]$ *and* $\mathscr{D}_\lambda[\cdot]$ *are defined in* (1.53) *and* (1.54).

Proof By moving $\varepsilon\partial_t f$ on the right hand side of (3.42), we can use (2.136) to obtain, for any $t>0$

$$\begin{aligned}\|\mathbf{P}e^{\lambda t}f\|_{L^6_{x,v}}\le \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})e^{\lambda t}f\|_\nu+\varepsilon^{-\frac{1}{2}}|e^{\lambda t}(1-P_\gamma)f|_{L^2(\gamma)}\\ +|e^{\lambda t}r|_{L^2(\gamma)}+|\varepsilon^{\frac{1}{2}}e^{\lambda t}\langle v\rangle^{-1}wr|_{L^\infty(\gamma)}\\ +\|e^{\lambda t}\nu^{-\frac{1}{2}}\big[\lambda f+g-\varepsilon\partial_t f\big]\|_2\\ +\varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1}w[\lambda f+g-\varepsilon\partial_t f]\|_\infty.\end{aligned}\tag{3.80}$$

Since

$$\begin{aligned}&\sup_{s\in[0,t]}\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})e^{\lambda s}f\|_\nu^2\\ &\le\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f_0\|_\nu^2+2\varepsilon^{-2}\int_0^t ds\|(\mathbf{I}-\mathbf{P})e^{\lambda s}f\|_\nu\|(\mathbf{I}-\mathbf{P})e^{\lambda s}\partial_t f\|_\nu\\ &\le\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f_0\|_\nu^2+\mathscr{D}_\lambda(t),\end{aligned}\tag{3.81}$$

and, similarly

$$\begin{aligned}&\sup_{s\in[0,t]}\varepsilon^{-1}|e^{\lambda t}(1-P_\gamma)f|_{2,\gamma}^2\\ &\quad\le\varepsilon^{-1}|(1-P_\gamma)f_0|_{2,\gamma}^2\\ &\qquad+2\varepsilon^{-1}\int_0^t ds|e^{\lambda s}(1-P_\gamma)f|_{2,\gamma}|(\mathbf{I}-\mathbf{P})e^{\lambda s}(1-P_\gamma)\partial_t f|_{2,\gamma}\\ &\quad\le\varepsilon^{-1}|(1-P_\gamma)f_0|_{2,\gamma}+\mathscr{D}_\lambda(t),\end{aligned}$$

the first two terms in the right hand side of (3.80) are bounded by

$$\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_0\|_\nu+\varepsilon^{-1/2}|(1-P_\gamma)f_0|_{2,\gamma}+\sqrt{\mathscr{D}_\lambda(t)}.$$

The third and fourth terms in the right hand side of (3.80) are bounded by

$$|e^{\lambda t}r|_{L^\infty_t L^2(\gamma)}+\varepsilon^{\frac{1}{2}}e^{\lambda t}\langle v\rangle^{-1}wr|_{L^\infty_t L^\infty(\gamma)}.$$

We have

$$\|e^{\lambda t}\nu^{-\frac{1}{2}}[\lambda f+g-\varepsilon\partial_t f]\|^2_{L^2_{t,x,v}}\le\lambda\|f_0\|_2^2+(\lambda+\varepsilon)\mathscr{D}(t)+\|e^{\lambda t}\nu^{-\frac{1}{2}}g\|^2_{L^\infty_t L^2_{x,v}},$$

and

$$
\begin{aligned}
&\varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1}w[\lambda f+g-\varepsilon\partial_t f]\|_{L^\infty_{t,x,v}}\\
&\quad\le \varepsilon^{\frac{3}{2}}\lambda\|e^{\lambda t}\langle v\rangle^{-1}wf\|_{L^\infty_{t,x,v}}+\varepsilon^{\frac{5}{2}}\|e^{\lambda t}\langle v\rangle^{-1}w\partial_t f\|_{L^\infty_{t,x,v}}+\varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1}wg\|_{L^\infty_{t,x,v}}.
\end{aligned}
$$

Collecting the bounds we conclude the proof. □

Corollary 3.14 *For λ and ε sufficiently small we have*

$$
\begin{aligned}
&\sup_{0\le s\le t}\|e^{\lambda s}\mathbf{P}f\|_{L^6_x}\lesssim \lambda\|f_0\|_2+\lambda\varepsilon\|\varepsilon^{\frac{1}{2}}wf_0\|_\infty\\
&\quad+\varepsilon^2\|\varepsilon^{\frac{1}{2}}w\partial_t f(0)\|_\infty+\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_0\|_\nu+\varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f_0|_{2,\gamma}\\
&\quad+\sqrt{\mathscr{D}_\lambda[f](t)}+(\lambda+2\varepsilon)\sqrt{\mathscr{E}_\lambda[f](t)}+|e^{\lambda t}r|_{L^\infty_t L^2(\gamma)}\\
&\quad|\varepsilon^{\frac{3}{2}}e^{\lambda t}\langle v\rangle wr|_{L^\infty_t L^\infty(\gamma)}\\
&\quad+\|e^{\lambda t}\nu^{-\frac{1}{2}}g\|_{L^\infty_t L^2_{x,v}}+\varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1}wg\|^2_{L^\infty_{t,x,v}}\\
&\quad+\sup_{0\le s\le\infty}\|\varepsilon^{\frac{5}{2}}wr_t(s)\|_\infty+\varepsilon^{\frac{7}{2}}\|\langle v\rangle^{-1}wg_t(s)\|_\infty.
\end{aligned}
$$

Proof We use (3.61) to bound

$$
\begin{aligned}
\lambda\varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1}wf\|^2_{L^\infty_{t,x,v}}\le{}&\lambda\varepsilon\|\varepsilon^{\frac{1}{2}}wf_0\|_\infty+\lambda\varepsilon\sup_{0\le s\le\infty}\|\varepsilon^{\frac{1}{2}}wr(s)\|_\infty\\
&+\lambda\varepsilon\varepsilon^{\frac{3}{2}}\sup_{0\le s\le\infty}\|\langle v\rangle^{-1}wg(s)\|_\infty\\
&+\lambda\varepsilon\sup_{0\le s\le t}\|\mathbf{P}f(s)\|_{L^6(\Omega\times\mathbb{R}^3)}\\
&+\lambda\varepsilon\varepsilon^{-1}\sup_{0\le s\le t}\|(\mathbf{I}-\mathbf{P})f(s)\|_{L^2(\Omega\times\mathbb{R}^3)},
\end{aligned}
\tag{3.82}
$$

and (3.81) to bound the last term with $\sqrt{\mathscr{D}_\lambda(t)}$. Moreover, we use (3.62) written for $\partial_t f$ to bound

$$
\begin{aligned}
&\varepsilon^{\frac{5}{2}}\|e^{\lambda t}\langle v\rangle^{-1}w\partial_t f\|_{L^\infty_{t,x,v}}\le\varepsilon^2\|\varepsilon^{\frac{1}{2}}w\partial_t f(0)\|_\infty+\varepsilon^2\sup_{0\le s\le\infty}\|\varepsilon^{\frac{1}{2}}wr_t(s)\|_{L^\infty_{t,x,v}}\\
&\quad+\varepsilon^2\varepsilon^{\frac{3}{2}}\|\langle v\rangle^{-1}wg_t(s)\|_\infty+\varepsilon\|e^{\lambda t}\partial_t f\|_{L^\infty_t L^2_{xv}},
\end{aligned}
\tag{3.83}
$$

and we bound the last term with $\varepsilon\sqrt{\mathscr{E}_\lambda[f](t)}$. Combining previous estimates we conclude the proof. □

3.6 Estimates of the Collision Operators

Lemma 3.15 *Given f and g in L^2, assume that, for $t > 0$*

$$|a_i(f)| \le \mathbf{S}_1 f(t,x) + \mathbf{S}_2 f(t,x) + \mathbf{S}_3 f(t,x),$$
$$|a_i(g)| \le \mathbf{S}_1 g(t,x) + \mathbf{S}_2 g(t,x) + \mathbf{S}_3 g(t,x),$$

where $\mathbf{S}_i f, \mathbf{S}_i g \ge 0$ and $a_i(f)$ are defined as $[a_0, a_1, a_2, a_3, a_4] = [a, b_1, b_2, b_3, c]$ in (1.19).

Then

$$\begin{aligned}
&\|\nu^{-\frac{1}{2}}\Gamma_\pm(f,g)\|_{L^2_{t,x,v}} \\
&\lesssim \varepsilon^{\frac{1}{2}}\big[\varepsilon^{\frac{1}{2}}\|wg\|_{L^\infty_{t,x,v}}\big]\big\{\big[\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}}\big] + \varepsilon^{-1}\|e^{\lambda t}\mathbf{S}_3 f\|_{L^2_{t,x,v}}\big\} \\
&\quad + \|\mathbf{P}g\|_{L^\infty_t L^6_x}\|\mathbf{S}_1 f\|_{L^2_t L^3_x} + \varepsilon^{1/4}[\varepsilon^{1/2}\|g\|_\infty]^{1/2}\|\mathbf{P}g\|^{1/2}_{L^\infty_t L^6_{x,v}}\big[\varepsilon^{-1/2}\|\mathbf{S}_2 f\|_{L^2_t L_x^{\frac{12}{5}}}\big] \\
&\quad + \varepsilon^{\frac{1}{2}}\big[\varepsilon^{\frac{1}{2}}\|wf\|_{L^\infty_{t,x,v}}\big]\big[\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}}\big],
\end{aligned} \tag{3.84}$$

and

$$\begin{aligned}
&\|\nu^{-\frac{1}{2}}\Gamma_\pm(f,g)\|_{L^2_{t,x,v}} \\
&\lesssim \varepsilon^{\frac{1}{2}}\big[\varepsilon^{\frac{1}{2}}\|wg\|_\infty\big]\big\{\big[\varepsilon^{-1}\|e^{\lambda t}\mathbf{S}_3 f\|_{L^2_{t,x,v}}\big] + \big[\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}}\big]\big\} \\
&\quad + \|\mathbf{P}g\|_{L^\infty_t L^6_{x,v}}\|\mathbf{S}_1 f\|_{L^2_t L^3_x} \\
&\quad + \varepsilon^{1/4}\big[\varepsilon^{1/2}\|e^{\lambda t}g\|_\infty\big]^{1/2}\|e^{\lambda t}\mathbf{P}g\|^{1/2}_{L^\infty_t L^6_{x,v}}\big[\varepsilon^{-1/2}\|e^{\lambda t}\mathbf{S}_2 f\|_{L^2_t L_x^{\frac{12}{5}}}\big] \\
&\quad + \big[\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_{L^\infty_t L^2_{x,v}}\big]^{1/3}\big[\varepsilon^{\frac{1}{2}}\|wg\|^{2/3}_\infty\big]\|\mathbf{P}f\|_{L^2_t L^3_{x,v}},
\end{aligned} \tag{3.85}$$

and

$$\begin{aligned}
&\|\nu^{-\frac{1}{2}}\Gamma_\pm(f,g)\|_{L^2_{t,x,v}} \\
&\lesssim \varepsilon^{\frac{1}{2}}\big[\varepsilon^{\frac{1}{2}}\|wg\|_{L^\infty_{t,x,v}}\big]\big\{\big[\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}}\big] + \varepsilon^{-1}\|e^{\lambda t}\mathbf{S}_3 f\|_{L^2_{t,x,v}}\big\} \\
&\quad + \varepsilon^{1/4}\big[\varepsilon^{1/2}\|wg\|_\infty\big]^{1/2}\|\mathbf{P}g\|^{1/2}_{L^\infty_t L^6_{x,v}}\big[\varepsilon^{-1/2}\|\mathbf{S}_2 f\|_{L^2_t L_x^{\frac{12}{5}}}\big] \\
&\quad + \Big\{\|\mathbf{P}g\|_{L^\infty_t L^6_x} + \big[\varepsilon^{1/2}\|wg\|_\infty\big]^{2/3}\Big[\sqrt{\mathscr{D}_\lambda[g](\infty)} + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})g|_{t=0}\|_\nu\Big]^{1/3}\Big\} \\
&\quad \times \|\mathbf{S}_1 f\|_{L^2_t L^3_x}.
\end{aligned} \tag{3.86}$$

Proof First we prove (3.84). We decompose

$$|f(t,x,v)| \le |\mathbf{P}f(t,x,v)| + |(\mathbf{I}-\mathbf{P})f(t,x,v)|, \tag{3.87}$$

and $|g(t,x,v)|$ in the same way. We use the same decomposition of (2.138) replacing the $L^2_{x,v}$ norm with $L^2_{t,x,v}$ norm.

The first two terms of the RHS of (2.138) is bounded by

$$\begin{aligned}
&\varepsilon\|wg\|_{L^\infty_{t,x,v}}\|\nu^{-1/2}\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})f|,w^{-1})\|_{L^2_{t,x,v}}\\
&\quad+\varepsilon\|wf\|_{L^\infty_{t,x,v}}\|\nu^{-1/2}\Gamma_\pm(\varepsilon^{-1}|(\mathbf{I}-\mathbf{P})g|,w^{-1})\|_{L^2_{t,x,v}}\\
&\lesssim \varepsilon^{\frac12}\big[\varepsilon^{\frac12}\|wg\|_{L^\infty_{t,x,v}}\big]\big[\varepsilon^{-1}\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}}\big]\\
&\quad+\varepsilon^{\frac12}\big[\varepsilon^{\frac12}\|wf\|_{L^\infty_{t,x,v}}\big]\big[\varepsilon^{-1}\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})g\|_{L^2_{t,x,v}}\big].
\end{aligned}$$

To bound the last term of (2.138) we note that

$$|\mathbf{P}f|\lesssim \nu^2\sqrt{\mu}\Big[\sum_{i=1}^3\mathbf{S}_i f\Big]. \tag{3.88}$$

From $\|\nu^{-1/2}\Gamma(\mu^{0+},\mu^{0+})\|_{L^p_v}<\infty$, we get, for any $1\le p\le\infty$

$$\|\nu^{-1/2}\Gamma_\pm(|\mathbf{S}_i f|\nu^2\sqrt{\mu},|\mathbf{P}g|)\|_{L^2_{t,x,v}}\lesssim \Big\|\mathbf{S}_i f\|\mathbf{P}g\|_{L^p_v}\Big\|_{L^2_{t,x}}.$$

We estimate $\Big\|\mathbf{S}_i f\|\mathbf{P}g\|_{L^p_v}\Big\|_{L^2_{t,x}}$ for $i=1,2,3$:

$$\begin{aligned}
\|\mathbf{S}_1 f\|\mathbf{P}g\|_{L^6_v}\|_{L^2_{t,x}}&\lesssim \Big\|\|\mathbf{S}_1 f\|_{L^3_x}\|\mathbf{P}g\|_{L^6_{x,v}}\Big\|_{L^2_t}\lesssim\|\mathbf{S}_1 f\|_{L^2_tL^3_x}\|\mathbf{P}g\|_{L^\infty_tL^6_{x,v}},\\
\Big\|\mathbf{S}_2 f\|\mathbf{P}g\|_{L^6_v}\Big\|_{L^2_{t,x}}&\lesssim\Big\|\|\mathbf{S}_2 f\|_{L_x^{\frac{12}{5}}}\|\mathbf{P}g\|_{L^{12}_xL^6_v}\Big\|_{L^2_t}\le\|\mathbf{S}_2 f\|_{L^2_tL_x^{\frac{12}{5}}}\|\mathbf{P}g\|_{L^\infty_tL^{12}_xL^6_v}\\
&\lesssim\|\mathbf{S}_2 f\|_{L^2_tL_x^{\frac{12}{5}}}\varepsilon^{-\frac14}\sqrt{\|\mathbf{P}g\|_{L^\infty_tL^6_{x,v}}\varepsilon^{\frac12}\|g\|_{L^\infty_{t,x,v}}},\\
\Big\|\mathbf{S}_3 f\|\mathbf{P}g\|_{L^\infty_v}\Big\|_{L^2_{t,x}}&\lesssim\|\mathbf{S}_3 f\|_{L^2_{t,x}}\|g\|_\infty.
\end{aligned}$$

By collecting the estimates we obtain the estimate (3.84).

Now we prove (3.85). All the estimates are same as (3.84) except

$$\|\nu^{-1/2}\Gamma_\pm(|\mathbf{P}f|,|(\mathbf{I}-\mathbf{P})g|)\|_{L^2_{t,x,v}}. \tag{3.89}$$

By Holder inequality,

$$\begin{aligned}
(3.89)&\lesssim\|\mathbf{P}f\|_{L^2_tL^3_{x,v}}\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})g\|_{L^\infty_tL^6_{x,v}}\\
&\lesssim\|\mathbf{P}f\|_{L^2_tL^3_{x,v}}\big[\varepsilon^{\frac12}\|g\|_\infty\big]^{2/3}\big[\varepsilon^{-1}\|\nu^{\frac12}(\mathbf{I}-\mathbf{P})g\|_{L^\infty_tL^2_{x,v}}\big]^{1/3}.
\end{aligned}$$

The proof of (3.86) is same as the proof of (3.84) except (3.89). By (3.88),

$$\begin{aligned}(3.89) &\lesssim \|\mathbf{S}_1 f\|_{L^2_t L^3_x} \|\nu^{-\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_{L^\infty_t L^6_{x,v}} + \big[\varepsilon^{-\frac{1}{2}} \|\mathbf{S}_2 f\|_{L^2_t L_x^{\frac{12}{5}}}\big]\big[\varepsilon^{\frac{1}{2}}\|wg\|_\infty\big] \\ &\quad + \varepsilon^{\frac{1}{2}}\big[\varepsilon^{-1}\|\mathbf{S}_3 f\|_{L^2_{t,x,v}}\big]\big[\varepsilon^{\frac{1}{2}}\|g\|_\infty\big] \\ &\lesssim \|\mathbf{S}_1 f\|_{L^2_t L^3_x}\big[\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_{L^\infty_t L^2_{x,v}}\big]^{1/3}\big[\varepsilon^{\frac{1}{2}}\|g\|_\infty\big]^{2/3} \\ &\quad + \big[\varepsilon^{-\frac{1}{2}}\|\mathbf{S}_2 f\|_{L^2_t L_x^{\frac{12}{5}}}\big]\big[\varepsilon^{\frac{1}{2}}\|wg\|_\infty\big] + \varepsilon^{\frac{1}{2}}\big[\varepsilon^{-1}\|\mathbf{S}_3 f\|_{L^2_{t,x,v}}\big]\big[\varepsilon^{\frac{1}{2}}\|wg\|_\infty\big].\end{aligned}$$

Using (3.81), we deduce

$$\varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g\|_{L^\infty_t L^2_{x,v}} \lesssim \varepsilon^{-1}\|\nu^{\frac{1}{2}}(\mathbf{I}-\mathbf{P})g_0\|_{L^2_{x,v}} + \sqrt{\mathscr{D}_0[g](\infty)}.$$

By collecting the terms we prove (3.86). □

In order to estimate $\Gamma(f, \partial_t g)$ we will need the following *commutation* property:

Lemma 3.16 *For* $t \geq 0$,

$$\partial_t(\mathbf{S}_1 f) \leq \mathbf{S}_1(\partial_t f), \quad \partial_t(\mathbf{S}_2 f) \leq \mathbf{S}_2(\partial_t f). \tag{3.90}$$

Proof From the definition of $\mathbf{S}_1 f(t,x)$ in (3.8),

$$\partial_t[\mathbf{S}_1 f(t,x)] = 2\int_{\mathbb{R}^3} \mathrm{sgn}(f_\delta(t,x,v))\partial_t f_\delta(t,x,v)\nu^2\sqrt{\mu(v)}\mathrm{d}v.$$

From the definition of f_δ in (3.11), for $t \geq 0$

$$\begin{aligned}&\partial_t f_\delta(t,x,v)|_{t\geq 0} \\ &\quad = \left[1-\chi\left(\frac{n(x)\cdot v}{\delta}\right)\chi\big(\frac{\xi(x)}{\delta}\big)\right]\chi(\delta|v|)\big\{\mathbf{1}_{t\in[0,\infty)}\partial_t f(t,x,v) \\ &\qquad + \mathbf{1}_{t\in(-\infty,0]}\chi'(t)f_0(x,v)\big\}\big|_{t\geq 0} \\ &\quad = \left[1-\chi\left(\frac{n(x)\cdot v}{\delta}\right)\chi\big(\frac{\xi(x)}{\delta}\big)\right]\chi(\delta|v|)\mathbf{1}_{t\in[0,\infty)}\partial_t f(t,x,v) \\ &\quad = \left[1-\chi\left(\frac{n(x)\cdot v}{\delta}\right)\chi\big(\frac{\xi(x)}{\delta}\big)\right]\chi(\delta|v|)\big\{\mathbf{1}_{t\in[0,\infty)}\partial_t f(t,x,v) \\ &\qquad + \mathbf{1}_{t\in(-\infty,0]}\chi(t)\partial_t f_0(x,v)\big\}\big|_{t\geq 0} \\ &\quad = [\partial_t f]_\delta(t,x,v)|_{t\geq 0}.\end{aligned}$$

Therefore, for $t \geq 0$,

$$\partial_t[\mathbf{S}_1 f(t,x)] \leq 2\int_{\mathbb{R}^3} |\partial_t f_\delta(t,x,v)| v^2\sqrt{\mu(v)}\mathrm{d}v \leq 2\int_{\mathbb{R}^3} |[\partial_t f]_\delta(t,x,v)| v^2\sqrt{\mu(v)}\mathrm{d}v$$
$$= \mathbf{S}_1\partial_t f(t,x).$$

Similarly we show the second inequality. □

3.7 Global-in-Time Validity

Proof of Theorem 1.3 For the construction of the solution and the energy estimate, we consider $\tilde{f}^\ell(t,x,v)$ solving, for $\ell \in \mathbb{N}$,

$$\begin{aligned}
&\partial_t[e^{\lambda t}\tilde{f}^{\ell+1}] + \varepsilon^{-1}v\cdot\nabla_x[e^{\lambda t}\tilde{f}^{\ell+1}] + \varepsilon\Phi\cdot\nabla_v[e^{\lambda t}\tilde{f}^{\ell+1}] + \varepsilon^{-2}L[e^{\lambda t}\tilde{f}^{\ell+1}]\\
&\quad = \lambda[e^{\lambda t}\tilde{f}^{\ell+1}] + \varepsilon^{-1}L_{f_w+f_s}[e^{\lambda t}\tilde{f}^\ell] + e^{-\lambda t}\varepsilon^{-1}\Gamma(e^{\lambda t}\tilde{f}^\ell, e^{\lambda t}\tilde{f}^\ell)\\
&\qquad + \varepsilon\frac{\Phi\cdot v}{2}[e^{\lambda t}\tilde{f}^{\ell+1}],\\
&e^{\lambda t}\tilde{f}^{\ell+1}|_{\gamma_-} = P_\gamma e^{\lambda t}\tilde{f}^{\ell+1} + \varepsilon e^{\lambda t}\mathscr{Q}\tilde{f}^\ell, \quad e^{\lambda t}\tilde{f}^{\ell+1}|_{t=0} = \tilde{f}_0.
\end{aligned} \tag{3.91}$$

Here we set $\tilde{f}^0(t,x,v) := 0$.

Clearly $e^{\lambda t}\partial_t\tilde{f}^\ell$ solves

$$\begin{aligned}
&\partial_t[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}] + \varepsilon^{-1}v\cdot\nabla_x[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}] + \varepsilon\Phi\cdot\nabla_v[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}] + \varepsilon^{-2}L[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}]\\
&= \lambda[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}] + \varepsilon^{-1}L_{f_w+f_s}[e^{\lambda t}\partial_t\tilde{f}^\ell]\\
&\quad + e^{-\lambda t}\varepsilon^{-1}[\Gamma(e^{\lambda t}\partial_t\tilde{f}^\ell, e^{\lambda t}\tilde{f}^\ell) + \Gamma(e^{\lambda t}\tilde{f}^\ell, e^{\lambda t}\partial_t\tilde{f}^\ell)] + \varepsilon\frac{\Phi\cdot v}{2}[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}],\\
&e^{\lambda t}\partial_t\tilde{f}^{\ell+1}(t,x,v)|_{\gamma_-} = P_\gamma e^{\lambda t}\partial_t\tilde{f}^{\ell+1} + \varepsilon e^{\lambda t}\mathscr{Q}\partial_t\tilde{f}^\ell, \qquad e^{\lambda t}\partial_t\tilde{f}^{\ell+1}|_{t=0} = \partial_t\tilde{f}_0.
\end{aligned} \tag{3.92}$$

As in the steady case, from (1.51) and $\int_{n\cdot v\gtrless 0} M_w|n\cdot v|\mathrm{d}v = 1 = \int_{n\cdot v\gtrless 0}\sqrt{2\pi}\mu|n\cdot v|\mathrm{d}v$,

$$\mathbf{P}\big(\varepsilon^{-1}L_{f_s}\tilde{f} + \varepsilon^{-1}\Gamma(\tilde{f},\tilde{f})\big) = 0, \quad \int_{\mathbb{R}^3}\int_{n\cdot v<0}\mathscr{Q}\tilde{f}\{n\cdot v\}\mathrm{d}v \;=\; 0.$$

Note that Proposition 3.8 guarantees the solvability of such linear problems (3.91) and (3.92).

Now define the quantity that we want to bound in the iteration scheme of (3.91):

$$
\begin{aligned}
[\![\tilde{f}^\ell]\!] :=& \sqrt{\mathscr{E}_\lambda[\tilde{f}^\ell](\infty)} + \sqrt{\mathscr{D}_\lambda[\tilde{f}^\ell](\infty)} + \varepsilon^{\frac{1}{2}}\|e^{\lambda t}\tilde{f}^\ell\|_\infty + \varepsilon^{\frac{3}{2}}\|\varepsilon^{\lambda t}\partial_t\tilde{f}^\ell\|_\infty \\
&+ \|e^{\lambda t}\mathbf{P}\tilde{f}^\ell\|_{L^\infty_t L^6_{x,v}} \\
&+ \|\mathbf{S}_1\tilde{f}^\ell\|_{L^2_t L^3_x} + \|\mathbf{S}_1\partial_t\tilde{f}^\ell\|_{L^2_t L^3_x} + \varepsilon^{-\frac{1}{2}}\|\mathbf{S}_2\tilde{f}^\ell\|_{L^2_t L_x^{\frac{12}{5}}} \\
&+ \varepsilon^{-\frac{1}{2}}\|\mathbf{S}_2\partial_t\tilde{f}^\ell\|_{L^2_t L_x^{\frac{12}{5}}},
\end{aligned}
\tag{3.93}
$$

where $\mathbf{S}_1\tilde{f}^\ell$ and $\mathbf{S}_2\tilde{f}^\ell$ are defined in (3.40). For the sake of simplicity temporally we denote $\mathscr{E}^\ell_\lambda = \mathscr{E}_\lambda[\tilde{f}^\ell](\infty)$ and $\mathscr{D}^\ell_\lambda = \mathscr{D}_\lambda[\tilde{f}^\ell](\infty)$.

For $0 < \eta_0 \ll 1$ and $0 < c_0 \ll 1$, we assume (induction hypothesis) that

$$
\begin{aligned}
\sup_{0\le j\le\ell} [\![\tilde{f}^j]\!] < \eta_0, \|\tilde{f}_0\|_{L^2_{x,v}} + \|\tilde{f}_0\|_{L^\infty_{x,v}} + \|\tilde{\partial}_t f_0\|_{L^2_{x,v}} + \|\tilde{\partial}_t f_0\|_{L^\infty_{x,v}} < c_0\eta_0, \\
\|\mathbf{P} f_s\|_{L^6_x} + [\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_s\|_{L^2_{x,v}}] < \eta_0.
\end{aligned}
\tag{3.94}
$$

The condition for the steady solution f_s can be achieved by choosing further smaller $\|\vartheta_w\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Phi\|_{H^1(\Omega)}$ in Theorem 1.2.

In order to show that (3.94) holds for all ℓ, it suffices to show that (3.94) holds for $j = \ell + 1$. Throughout *Step 1* to *Step 4* we claim

$$
[\![\tilde{f}^{\ell+1}]\!] \le [\![\tilde{f}^\ell]\!]^2 + o(1)[\![\tilde{f}^{\ell+1}]\!] + o(1).
\tag{3.95}
$$

This clearly proves (3.94) for all ℓ.

Step 1. We prove the crucial estimates involving operator Γ. We apply (3.94) repeatedly. Applying (3.84) with $f = e^{\lambda t}\tilde{f}^\ell = g$,

$$
\|\nu^{-\frac{1}{2}}e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^\ell, e^{\lambda t}\tilde{f}^\ell)\|_{L^2_{t,x,v}} \lesssim (1+\varepsilon^{1/4}+\varepsilon^{1/2})[\![\tilde{f}^\ell]\!]^2.
\tag{3.96}
$$

Again applying (3.86) with $f = e^{\lambda t}\tilde{f}^\ell$, $g = e^{\lambda t}\partial_t\tilde{f}^\ell$,

$$
\begin{aligned}
&\|\nu^{-\frac{1}{2}}e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^\ell, e^{\lambda t}\partial_t\tilde{f}^\ell)\|_{L^2_{t,x,v}} \\
&\lesssim \varepsilon^{\frac{1}{2}}[\varepsilon^{\frac{1}{2}}\|e^{\lambda t}w\tilde{f}^\ell\|_{L^\infty_{t,x,v}}] \\
&\quad\times\{[\varepsilon^{-1}\|\nu^{-\frac{1}{2}}e^{\lambda t}(\mathbf{I}-\mathbf{P})\partial_t\tilde{f}^\ell\|_{L^2_{t,x,v}}] + \varepsilon^{-1}\|e^{\lambda t}\mathbf{S}_3\partial_t\tilde{f}^\ell\|_{L^2_{t,x,v}}\} \\
&+\varepsilon^{1/4}[\varepsilon^{1/2}\|e^{\lambda t}\tilde{f}^\ell\|_\infty]^{1/2}\|e^{\lambda t}\mathbf{P}\tilde{f}^\ell\|^{1/2}_{L^\infty_t L^6_{x,v}} \\
&\quad\times[\varepsilon^{1/2}\|we^{\lambda t}\tilde{f}^\ell\|_\infty][\varepsilon^{-1/2}\|e^{\lambda t}\mathbf{S}_2\tilde{\partial}_t f^\ell\|_{L^2_t L_x^{\frac{12}{5}}}] \\
&+\|e^{\lambda t}\mathbf{P}\tilde{f}^\ell\|_{L^\infty_t L^6_x}\|e^{\lambda t}\mathbf{S}_1\tilde{\partial}_t f^\ell\|_{L^2_t L^3_x} \\
&+[\varepsilon^{1/2}\|we^{\lambda t}\tilde{f}^\ell\|_\infty]^{2/3}\Big[\sqrt{\mathscr{D}_\lambda[\tilde{f}^\ell](\infty)} + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_0\|_\nu\Big]^{1/3}
\end{aligned}
$$

$$
\begin{aligned}
&\times \|e^{\lambda t}\mathbf{S}_1\tilde{\partial}_t f^\ell\|_{L^2_t L^3_x}\\
&\lesssim (1+\varepsilon^{\frac12})[\![\![\tilde f^\ell]\!]\!]^2 + c_0\eta_0 [\![\![\tilde f^\ell]\!]\!].
\end{aligned}\tag{3.97}
$$

Recall (1.51). Applying (3.85) with $g=(f_w+f_s)$ and $f=e^{\lambda t}\tilde f^\ell$,

$$
\begin{aligned}
&\|\nu^{-\frac12}L_{f_w+f_s}e^{\lambda t}\tilde f^\ell\|_{L^2_{t,x,v}} \lesssim \left(1+\varepsilon^{\frac14}+\varepsilon^{\frac12}\right)\eta_0[\![\![\tilde f^\ell]\!]\!].\\
&\quad\lesssim \varepsilon^{\frac12}\left[\varepsilon^{\frac12}\|wg\|_\infty\right]\left\{[\varepsilon^{-1}\|e^{\lambda t}\mathbf{S}_3 f\|_{L^2_{t,x,v}}]+[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2_{t,x,v}}]\right\}\\
&\qquad+\|\mathbf{P}g\|_{L^\infty_t L^6_{x,v}}\|\mathbf{S}_1 f\|_{L^2_t L^3_x}\\
&\qquad+\varepsilon^{1/4}[\varepsilon^{1/2}\|g\|_\infty]^{1/2}\|\mathbf{P}g\|^{1/2}_{L^\infty_t L^6_{x,v}}[\varepsilon^{-1/2}\|e^{\lambda t}\mathbf{S}_2 f\|_{L^2_t L^{\frac{12}{5}}_x}]\\
&\qquad+[\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})g\|_{L^\infty_t L^2_{x,v}}]^{1/3}[\varepsilon^{\frac12}\|wg\|^{2/3}_\infty]\|\mathbf{P}f\|_{L^2_t L^3_{x,v}}.
\end{aligned}\tag{3.98}
$$

Again applying (3.85) with $f=(f_w+f_s)$ and $g=e^{\lambda t}\partial_t\tilde f^\ell$,

$$
\|\nu^{-\frac12}L_{f_w+f_s}e^{\lambda t}\partial_t\tilde f^\ell\|_{L^2_{t,x,v}} \lesssim (1+\varepsilon^{\frac14}+\varepsilon^{\frac12})\eta_0[\![\![\tilde f^\ell]\!]\!].\tag{3.99}
$$

Step 2. From (3.44), (1.28), (3.96), (3.98), (3.97), and (3.99)

$$
\begin{aligned}
&\|e^{\lambda t}\tilde f^{\ell+1}(t)\|_2^2+\frac1\varepsilon\int_0^t |e^{\lambda s}(1-P_\gamma)\tilde f^{\ell+1}|_2^2+\frac{1}{\varepsilon^2}\int_0^t\|e^{\lambda s}(\mathbf{I}-\mathbf{P})\tilde f^{\ell+1}\|_\nu^2\\
&\quad+\int_0^t\|e^{\lambda s}\mathbf{P}\tilde f^{\ell+1}\|_2^2\\
&\lesssim \|\tilde f^{\ell+1}(0)\|_2^2+\varepsilon^{-1}\int_0^t|e^{\lambda s}\varepsilon\mathcal{Q}\tilde f^\ell|^2_{2,-}+\int_0^t e^{-\lambda s}\|\nu^{-\frac12}\Gamma(e^{\lambda s}\tilde f^\ell,e^{\lambda s}\tilde f^\ell)\|_2^2\\
&\quad+\int_0^t\|\nu^{-\frac12}L_{f_w+f_s}e^{\lambda s}\tilde f^\ell\|_2^2\\
&\lesssim (c_0\eta_0)^2+\varepsilon\|\vartheta_w\|_\infty[\![\![\tilde f^\ell]\!]\!]^2+[\![\![\tilde f^\ell]\!]\!]^4+\eta_0^2[\![\![\tilde f^\ell]\!]\!]^4,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|e^{\lambda t}\partial_t\tilde f^{\ell+1}(t)\|_2^2+\frac1\varepsilon\int_0^t |e^{\lambda s}(1-P_\gamma)\partial_t\tilde f^{\ell+1}|_2^2+\frac{1}{\varepsilon^2}\int_0^t\|e^{\lambda s}(\mathbf{I}-\mathbf{P})\partial_t\tilde f^{\ell+1}\|_\nu^2\\
&\quad+\int_0^t\|e^{\lambda s}\mathbf{P}\partial_t\tilde f^{\ell+1}\|_2^2\\
&\lesssim \|\partial_t\tilde f^{\ell+1}(0)\|_2^2+\varepsilon^{-1}\int_0^t|e^{\lambda s}\varepsilon\mathcal{Q}\partial_t\tilde f^\ell|^2_{2,-}\\
&\quad+\int_0^t e^{-\lambda s}\|\nu^{-\frac12}\Gamma(e^{\lambda s}\partial_t\tilde f^\ell,e^{\lambda s}\tilde f^\ell)\|_2^2+\int_0^t\|\nu^{-\frac12}L_{f_w+f_s}e^{\lambda s}\partial_t\tilde f^\ell\|_2^2\\
&\lesssim (c_0\eta_0)^2+\varepsilon\|\vartheta_w\|_\infty[\![\![\tilde f^\ell]\!]\!]^2+[\![\![\tilde f^\ell]\!]\!]^4+\eta_0^2[\![\![\tilde f^\ell]\!]\!]^4.
\end{aligned}
$$

Therefore we conclude

$$\mathscr{E}_{\lambda}^{\ell+1}+\mathscr{D}_{\lambda}^{\ell+1}\lesssim\left\{[\![\tilde{f}^{\ell}]\!]^{2}+o(1)[\![\tilde{f}^{\ell}]\!]+o(1)(\eta_{0})\right\}^{2}. \tag{3.100}$$

Step 3. We apply Proposition 3.4 to (3.91): Set

$$\begin{aligned}
&f=e^{\lambda t}\tilde{f}^{\ell+1},\\
&g=-\varepsilon^{-1}L[e^{\lambda t}\tilde{f}^{\ell+1}]+\varepsilon\lambda[e^{\lambda t}\tilde{f}^{\ell+1}]+L_{f_w+f_s}[e^{\lambda t}\tilde{f}^{\ell}]+e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell},e^{\lambda t}\tilde{f}^{\ell})\\
&\qquad+\varepsilon^{2}\frac{\Phi\cdot v}{2}[e^{\lambda t}\tilde{f}^{\ell+1}].
\end{aligned}$$

Then, from (3.96) and (3.98),

$$\begin{aligned}
&\|\mathbf{S}_1e^{\lambda t}\tilde{f}^{\ell+1}\|_{L_t^2L_x^3}+\varepsilon^{-\frac{1}{2}}\|\mathbf{S}_2e^{\lambda t}\tilde{f}^{\ell+1}\|_{L_t^2L_x^{\frac{12}{5}}}\\
&\lesssim\Big\|w^{-1}\Big[-\varepsilon^{-1}L[e^{\lambda t}\tilde{f}^{\ell+1}]+\varepsilon\lambda[e^{\lambda t}\tilde{f}^{\ell+1}]+L_{f_w+f_s}[e^{\lambda t}\tilde{f}^{\ell}]\\
&\quad+e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell},e^{\lambda t}\tilde{f}^{\ell})+\varepsilon^{2}\frac{\Phi\cdot v}{2}[e^{\lambda t}\tilde{f}^{\ell+1}]\Big]\Big\|_{L^2_{t,x,v}}\\
&\quad+\|e^{\lambda t}\tilde{f}^{\ell+1}\|_{L^2_{t,x,v}}+\|e^{\lambda t}\tilde{f}^{\ell+1}\|_{L^2_tL^2_\gamma}+\|f_0\|_{L^2_{x,v}}\\
&\quad+\|[v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v]f_0\|_{L^2_{x,v}}+\|f_0\|_{L^2(\gamma)}\\
&\lesssim(1+\varepsilon\lambda+\varepsilon^2\|\Phi\|_\infty)\mathscr{E}_{\lambda}^{\ell+1}+(c_0\eta_0+[\![\tilde{f}^{\ell}]\!])[\![\tilde{f}^{\ell}]\!]+c_0\eta_0.
\end{aligned}$$

From (3.100)

$$\|\mathbf{S}_1e^{\lambda t}\tilde{f}^{\ell+1}\|_{L_t^2L_x^3}+\varepsilon^{-\frac{1}{2}}\|\mathbf{S}_2e^{\lambda t}\tilde{f}^{\ell+1}\|_{L_t^2L_x^{\frac{12}{5}}}\lesssim[\![\tilde{f}^{\ell}]\!]^2+o(1)\eta_0. \tag{3.101}$$

Similarly, we apply Proposition 3.4 to (3.92): Set

$$\begin{aligned}
&f=e^{\lambda t}\partial_t\tilde{f}^{\ell+1},\\
&g=-\varepsilon^{-1}L[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}]+\varepsilon\lambda[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}]+L_{f_w+f_s}[e^{\lambda t}\partial_t\tilde{f}^{\ell}]\\
&\qquad+e^{-\lambda t}[\Gamma(e^{\lambda t}\partial_t\tilde{f}^{\ell},e^{\lambda t}\tilde{f}^{\ell})+\Gamma(e^{\lambda t}\tilde{f}^{\ell},e^{\lambda t}\partial_t\tilde{f}^{\ell})]+\varepsilon^{2}\frac{\Phi\cdot v}{2}[e^{\lambda t}\partial_t\tilde{f}^{\ell+1}].
\end{aligned}$$

Then

$$\|\mathbf{S}_1e^{\lambda t}\partial_t\tilde{f}^{\ell+1}\|_{L_t^2L_x^3}+\varepsilon^{-\frac{1}{2}}\|\mathbf{S}_2e^{\lambda t}\partial_t\tilde{f}^{\ell+1}\|_{L_t^2L_x^3}\lesssim[\![\tilde{f}^{\ell}]\!]^2+o(1)\eta_0. \tag{3.102}$$

Step 4. We apply Proposition 3.10 to (3.91): Set $f=e^{\lambda t}\tilde{f}^{\ell+1}$. Note $\varepsilon^{-1}Le^{\lambda t}\tilde{f}^{\ell+1}=\varepsilon^{-1}\nu(v)e^{\lambda t}\tilde{f}^{\ell+1}-\varepsilon^{-1}\int_{\mathbb{R}^3}\mathbf{k}(v,u)e^{\lambda t}\tilde{f}^{\ell+1}(u)\mathrm{d}u$ with $\nu(v)\sim\langle v\rangle$ and $|\mathbf{k}(v,u)|\lesssim\mathbf{k}_\beta(v,u)$. Moreover,

$$\varepsilon^{-1}\nu(v)-\varepsilon\lambda-\varepsilon\frac{\Phi\cdot v}{2}\gtrsim\varepsilon^{-1}C\langle v\rangle-\varepsilon\lambda-\varepsilon\|\Phi\|_\infty|v|\gtrsim\varepsilon^{-1}C_0\langle v\rangle.$$

Therefore (3.60), the condition of Proposition 3.10, is satisfied with the following setting

$$g = L_{f_s}[e^{\lambda t}\tilde{f}^{\ell}] + e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell}), \quad r = \varepsilon e^{\lambda t}\mathscr{Q}\tilde{f}^{\ell} \quad f_0 = \tilde{f}_0.$$

By Proposition 3.10, from (3.61),

$$\begin{aligned}
&\varepsilon^{\frac{1}{2}}\|e^{\lambda t}w\tilde{f}^{\ell+1}\|_{L^\infty_{t,x,v}} \\
&\lesssim \varepsilon^{\frac{1}{2}}\|w\tilde{f}^{\ell+1}(0)\|_\infty + \varepsilon^{\frac{1}{2}}\max_{0\le j\le \ell}\sup_{0\le t\le\infty}\varepsilon\|e^{\lambda t}w\tilde{f}^{j}\|_\infty + \varepsilon\sup_{0\le t\le\infty}\varepsilon^{1/2}\|e^{\lambda t}w\mathscr{Q}\tilde{f}^{\ell}\|_\infty \\
&\quad + \varepsilon^{\frac{3}{2}}\sup_{0\le t\le\infty}\Big\|w\nu^{-1}\Big[L_{f_w+f_s}[e^{\lambda t}\tilde{f}^{\ell}] + \Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell})\Big]\Big\|_\infty \\
&\quad + \|e^{\lambda s}\mathbf{P}\tilde{f}^{\ell+1}(s)\|_{L^\infty_t L^6_{x,v}} + \frac{1}{\varepsilon}\|e^{\lambda s}(\mathbf{I}-\mathbf{P})\tilde{f}^{\ell+1}(s)\|_{L^\infty_t L^2_{x,v}}.
\end{aligned}$$

Using $|w\Gamma_\pm(w^{-1}, w^{-1})| \lesssim \langle v\rangle \lesssim \nu$, we obtain

$$\begin{aligned}
&\varepsilon^{\frac{3}{2}}\sup_{0\le t\le\infty}\Big\|w\nu^{-1}e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell})\Big\|_\infty \\
&\lesssim \varepsilon^{1/2}\left[\sup_{0\le t\le\infty}\|\varepsilon^{\frac{1}{2}}we^{\lambda t}\tilde{f}^{\ell}\|_\infty\right]^2|\nu^{-1}w\Gamma(w^{-1}, w^{-1})| \\
&\lesssim \varepsilon^{1/2}[\sup_{0\le t\le\infty}\|\varepsilon^{\frac{1}{2}}we^{\lambda t}\tilde{f}^{\ell}\|_\infty]^2 \lesssim \varepsilon^{1/2}[\![\tilde{f}^{\ell}]\!]^2, \\
&\quad |\varepsilon^{\frac{3}{2}}w\nu^{-1}L_{f_w+f_s}(e^{\lambda t}\tilde{f}^{\ell})| \lesssim \varepsilon^{\frac{1}{2}}|w\nu^{-1}\Gamma_\pm(\varepsilon^{\frac{1}{2}}(f_w+f_s), e^{\lambda t}\varepsilon^{\frac{1}{2}}\tilde{f}^{\ell})| \\
&\lesssim \varepsilon^{\frac{1}{2}}\|w\varepsilon^{\frac{1}{2}}(f_w+f_s)\|_\infty\|we^{\lambda t}[\varepsilon^{\frac{1}{2}}\tilde{f}^{\ell}]\|_\infty|\nu^{-1}w\Gamma_\pm(w^{-1}, w^{-1})| \\
&\lesssim \varepsilon^{\frac{1}{2}}\|w\varepsilon^{\frac{1}{2}}(f_w+f_s)\|_\infty\|w\varepsilon^{\frac{1}{2}}\tilde{f}^{\ell}\|_\infty \\
&\lesssim \varepsilon^{1/2}\eta_0[\![\tilde{f}^{\ell}]\!].
\end{aligned}$$

Using Corollary 3.14 with $f = e^{\lambda t}\tilde{f}^{\ell+1}$, $g = \big[e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell}) + e^{-\lambda t}L_{f_s}e^{\lambda t}\partial_t\tilde{f}^{\ell}\big]$, $r = e^{\lambda t}\varepsilon\mathscr{Q}\tilde{f}^{\ell}$ and $\partial_t g = \varepsilon^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\partial_t\tilde{f}^{\ell}) + e^{-\lambda t}\Gamma(e^{\lambda t}\partial_t\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell}) + e^{-\lambda t}L_{f_s}e^{\lambda t}\partial_t\tilde{f}^{\ell}$, $\partial_t r = e^{\lambda t}\varepsilon\mathscr{Q}\partial_t\tilde{f}^{\ell}$, we have

$$\begin{aligned}
&\|\mathbf{P}\tilde{f}^{\ell+1}e^{\lambda t}\|_{L^\infty_t L^6_{x,v}} \le \lambda\|f_0\|_2 + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f_0\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f_0|_{2,\gamma} \\
&\quad + \mathscr{D}^{\ell+1}_\lambda(t)^{\frac{1}{2}} + (\lambda + 2\varepsilon)\mathscr{E}^{\ell+1}_\lambda(t)^{\frac{1}{2}} + |e^{\lambda t}\varepsilon e^{\lambda t}\mathscr{Q}\tilde{f}^{\ell}|_{L^\infty_t L^2(\gamma)} \\
&\quad + |\varepsilon^{\frac{3}{2}}e^{\lambda t}\langle v\rangle^{-1}w\varepsilon e^{\lambda t}\mathscr{Q}\tilde{f}^{\ell}|_{L^\infty_t L^\infty(\gamma)} \\
&\quad + \|e^{\lambda t}\nu^{-\frac{1}{2}}\big[e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell}) + e^{-\lambda t}L_{f_w+f_s}e^{\lambda t}\tilde{f}^{\ell}\big]\|_{L^\infty_t L^2_{x,v}} \\
&\quad + \varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1}w\big[e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell}) + e^{-\lambda t}L_{f_w+f_s}e^{\lambda t}\tilde{f}^{\ell}\big]\|_{L^\infty_{t,x,v}} \\
&\quad + \varepsilon^{\frac{5}{2}}|e^{\lambda t}\langle v\rangle^{-1}w\varepsilon\mathscr{Q}\tilde{f}^{\ell}|_{L^\infty_{t,\gamma}}
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^{\frac{7}{2}}\|\langle v\rangle w\big[e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\partial_t\tilde{f}^{\ell})+e^{-\lambda t}\Gamma(e^{\lambda t}\partial_t\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell})\\
&+e^{-\lambda t}L_{f_w+f_s}e^{\lambda t}\partial_t\tilde{f}^{\ell}\big]\|_{L^\infty_{t,x,v}}.
\end{aligned}
$$

By (3.96)–(3.99), and (3.100),

$$
\varepsilon^{\frac{1}{2}}\|e^{\lambda t}w\tilde{f}^{\ell+1}\|_{L^\infty_{t,x,v}}+\|\mathbf{P}\tilde{f}^{\ell+1}e^{\lambda t}\|^2_{L^\infty_t L^6_x}\lesssim [\![\tilde{f}^{\ell}]\!]^2+o(1)\eta_0. \tag{3.103}
$$

Now we consider $\partial_t\tilde{f}^{\ell+1}$. Apply Proposition 3.10 to (3.92): Set $f=e^{\lambda t}\partial_t\tilde{f}^{\ell+1}$. Note $\varepsilon^{-1}Le^{\lambda t}\partial_t\tilde{f}^{\ell+1}=\varepsilon^{-1}\nu(v)e^{\lambda t}\partial_t\tilde{f}^{\ell+1}-\varepsilon^{-1}\int_{\mathbb{R}^3}\mathbf{k}(v,u)e^{\lambda t}\partial_t\tilde{f}^{\ell+1}(u)\mathrm{d}u$ with $\nu(v)\sim\langle v\rangle$ and $|\mathbf{k}(v,u)|\lesssim\mathbf{k}_\beta(v,u)$. For Proposition 3.10 we set

$$
\begin{aligned}
&g=L_{f_w+f_s}[e^{\lambda t}\partial_t\tilde{f}^{\ell}]+e^{-\lambda t}\Gamma(e^{\lambda t}\partial_t\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell})+e^{-\lambda t}\varepsilon^{1/2}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\partial_t\tilde{f}^{\ell}),\\
&r=\varepsilon e^{\lambda t}\partial_t\mathscr{Q}\partial_t\tilde{f}^{\ell},\quad f_0=\partial_t\tilde{f}_0.
\end{aligned}
$$

From (3.62),

$$
\begin{aligned}
&\varepsilon^{\frac{1}{2}}\|e^{\lambda t}w\partial_t\tilde{f}^{\ell+1}\|_{L^\infty_{t,x,v}}\\
&\lesssim\varepsilon^{\frac{1}{2}}\|w\partial_t\tilde{f}^{\ell+1}(0)\|_\infty+\varepsilon^{\frac{1}{2}}\varepsilon\|we^{\lambda t}\mathscr{Q}\tilde{\partial}_t f\|_{L^\infty_{t,\gamma}}+\frac{1}{\varepsilon}|e^{\lambda t}\partial_t\tilde{f}^{\ell+1}(t)|_{L^\infty_t L^2_{x,v}}\\
&\quad+\varepsilon^{\frac{3}{2}}\sup_{0\le t\le\infty}\Big\|w\nu^{-1}\Big[L_{f_w+f_s}[e^{\lambda t}\partial_t\tilde{f}^{\ell}]+e^{-\lambda t}\Gamma(e^{\lambda t}\partial_t\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell})\\
&\quad+e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell}_t)\Big]\Big\|_\infty.
\end{aligned}
$$

From $|w\Gamma_\pm(w^{-1},w^{-1})|\lesssim\langle v\rangle\lesssim\nu$,

$$
\begin{aligned}
&\varepsilon^{\frac{3}{2}}\sup_{0\le t\le\infty}\big\|w\nu^{-1}e^{-\lambda t}\Gamma(e^{\lambda t}\partial_t\tilde{f}^{\ell}, e^{\lambda t}\tilde{f}^{\ell})\big\|_\infty\\
&\quad+\varepsilon^{\frac{3}{2}}\sup_{0\le t\le\infty}\big\|w\nu^{-1}e^{-\lambda t}\Gamma(e^{\lambda t}\tilde{f}^{\ell}, e^{\lambda t}\partial_t\tilde{f}^{\ell})\big\|_\infty\\
&\lesssim\varepsilon^{1/2}\Big[\|\varepsilon^{\frac{1}{2}}we^{\lambda t}\tilde{f}^{\ell}\|_\infty\Big]\Big[\|\varepsilon^{\frac{3}{2}}we^{\lambda t}\tilde{f}^{\ell}_t\|_\infty\Big]|\nu^{-1}w\Gamma(w^{-1},w^{-1})|\\
&\lesssim\varepsilon^{1/2}\Big[\|\varepsilon^{\frac{1}{2}}we^{\lambda t}\tilde{f}^{\ell}\|_\infty\Big]\Big[\|\varepsilon^{\frac{1}{2}}we^{\lambda t}\tilde{f}^{\ell}_t\|_\infty\Big],\\
&\quad|\varepsilon^{\frac{3}{2}}w\nu^{-1}L_{f_w+f_s}e^{\lambda t}\tilde{f}^{\ell}_t|\lesssim\varepsilon^{\frac{1}{2}}\|w(f_w+f_s)\|_\infty\|we^{\lambda t}\varepsilon^{\frac{1}{2}}\partial_t\tilde{f}^{\ell}\|_\infty.
\end{aligned}
$$

Altogether

$$\begin{aligned}
&\varepsilon^{\frac{3}{2}}\|e^{\lambda t} w\partial_t \tilde{f}^{\ell+1}\|_{L^\infty_{t,x,v}} \lesssim \varepsilon^{\frac{3}{2}}\|w\partial_t \tilde{f}^{\ell+1}(0)\|_\infty + \varepsilon^{\frac{3}{2}}\|\vartheta_w\|_\infty \|e^{\lambda t} w \tilde{f}^\ell_t\|_\infty \\
&\quad + \|\varepsilon^{\lambda t}\tilde{\partial}_t f^{\ell+1}\|_{L^\infty_t L^2_{x,v}} \\
&\quad + \varepsilon^{\frac{3}{2}}\left[\|\varepsilon^{\frac{1}{2}} we^{\lambda t}\tilde{f}^\ell\|_\infty\right]\left[\|\varepsilon^{\frac{1}{2}} we^{\lambda t}\partial_t \tilde{f}^\ell\|_\infty + \|w(f_w+f_s)\|_\infty \|we^{\lambda t}\varepsilon^{\frac{1}{2}}\partial_t \tilde{f}^\ell\|_\infty\right],
\end{aligned}$$

and therefore

$$\varepsilon^{\frac{3}{2}}\|e^{\lambda t} w\partial_t \tilde{f}^{\ell+1}\|_{L^\infty_{t,x,v}} < [\![\tilde{f}^\ell]\!]^2 + o(1)\eta_0. \tag{3.104}$$

Collecting (3.100), (3.101), (3.102), (3.103), and (3.104), we prove the claim (3.95). *Step 5.* We repeat *Step1* $\sim$ *Step4* for $\tilde{R}^{\ell+1} - \tilde{R}^\ell$ to show that $\tilde{R}^\ell$ is Cauchy sequence in $L^\infty \cap L^2$ for fixed ε. Now we pass a limit $\ell \to \infty$ in $L^\infty \cap L^2$ to conclude the existence. The proof of uniqueness is standard. (See [17] for details)
Step 6. The proof that the components of the limit $\tilde{f}_1$ satisfy the unsteady INSF system 1.49 is achieved similarly to the steady case and will not be repeated here. □

3.8 Positivity of Solutions

In this section, we prove the non-negativity of F_s in the main theorem. The proof is based on the asymptotical stability of F_s (Proposition 3.4) and the non-negativity of unsteady solution.

Proof of the non-negativity of $F(t, x, v)$ in Theorem 1.3 and $F_s(x, v)$ in Theorem 1.1

We use the positivity-preserving sequence as in [17,31]. Set $F^0(t, x, v) = F_0(x, v) \geq 0$ and for $\ell \geq 0$

$$\begin{aligned}
&\partial_t F^{\ell+1} + \frac{1}{\varepsilon} v\cdot\nabla_x F^{\ell+1} + \varepsilon\Phi\cdot\nabla_v F^{\ell+1} + \frac{1}{\varepsilon^2}\nu(F^\ell)F^{\ell+1} = \frac{1}{\varepsilon^2}Q_+(F^\ell, F^\ell),\\
&F^{\ell+1}(x, v)|_{\gamma_-} = M_w \int_{n(x)\cdot u>0} F^{\ell+1}(x, u)\{n(x)\cdot u\}\mathrm{d}u,\\
&\quad \times F^{\ell+1}(t, x, v)|_{t=0} = F_0(x, v),
\end{aligned}$$

where $\nu(F)(v) = \int_{\mathbb{R}^3} du \int_{\mathbb{S}^2} d\omega B(v-u, \omega)F(v_*)$.

Note that $F^{\ell+1} \geq 0$ for all ℓ by proof of Theorem 4 in page 807 of [31].
Step 1. We set $F^\ell = \mu + \varepsilon f^\ell\sqrt{\mu}$ and let $F^0(t, x, v) := F_0(x, v)$. We claim that, there exists $0 < T = T(\|\varepsilon w f^\ell(t_0)\|_\infty) \ll 1$ and $C_1 = C_1(T) \gg 1$ for any $t_0 \geq 0$ such that

$$\begin{aligned}
\sup_{t_0\leq t\leq t_0+\varepsilon^2 T} \|\varepsilon w f^{\ell+1}(t)\|_\infty &\leq C_1\big\{\|\varepsilon w f^{\ell+1}(t_0)\|_\infty \\
&\quad + \big(\sup_{t_0\leq t\leq t_0+\varepsilon^2 T}\|\varepsilon w f^\ell(t)\|_\infty\big)^2 + O(\varepsilon^2)\|A\|_\infty\big\}.
\end{aligned} \tag{3.105}$$

It suffices to show (3.105) for $t_0 = 0$. Clearly (3.105) holds for $\ell = 0$. Now we assume (3.105) for $0 \leq l \leq \ell$.

Clearly, $f^{\ell+1}$ solves

$$\begin{aligned}
&\partial_t f^{\ell+1} + \varepsilon^{-1} v \cdot \nabla_x f^{\ell+1} + \varepsilon \Phi \cdot \nabla_v f^{\ell+1} + \varepsilon \frac{\Phi \cdot v}{2} f^{\ell+1} + \varepsilon^{-2} \nu f^{\ell+1} - \varepsilon^{-2} K f^{\ell} \\
&\quad = \quad \varepsilon^{-1} \Gamma_+(f^{\ell}, f^{\ell}) - \varepsilon^{-1} \nu(f^{\ell}\sqrt{\mu}) f^{\ell+1} + A, \\
&f^{\ell+1}|_{t=0} = f_0,
\end{aligned} \tag{3.106}$$

where $A = 2\Phi \cdot v\sqrt{\mu}$. The boundary condition is given by

$$f^{\ell+1}|_{\gamma_-} = P_\gamma f^{\ell+1} + \varepsilon \mathscr{Q} f^{\ell+1}. \tag{3.107}$$

Define $h^{\ell}(t, x, v) := w(v)^{-1} f^{\ell}(t, x, v)$. Note that

$$\check{\nu}(v) := \nu(v) - \varepsilon \frac{\|\Phi\|_\infty |v|}{2} - \varepsilon \nu(\sqrt{\mu}) \|w f^{\ell}\|_{L^\infty_{t,x,v}} \geq \frac{\nu(v)}{2} \gtrsim \langle v \rangle.$$

We define $\check{\mathbf{k}}$ such that $\int_{\mathbb{R}^3} \check{\mathbf{k}}(v, u) \frac{w(v)}{w(u)} f^{\ell}(u) \mathrm{d}u = K f^{\ell} + \varepsilon \nu(f^{\ell}\sqrt{\mu})[f_w]\sqrt{\mu} - \varepsilon[\Gamma_+(f_w, f^{\ell}) + \Gamma_+(f^{\ell}, f_w)]$. Then $\check{\mathbf{k}}(v, u) \lesssim \mathbf{k}_\beta(v, u)$, where $\mathbf{k}_\beta(v, u)$ is defined in (2.26).

Then, for $\tilde{t}_1 \leq s \leq t$

$$\begin{aligned}
&\frac{d}{ds}\Big[|\varepsilon h^{\ell+1}(s, Y_{\mathbf{cl}}(s; t, x, v), W_{\mathbf{cl}}(s; t, x, v))| e^{-\int_s^t \varepsilon^{-2} \check{\nu}(W_{\mathbf{cl}}(\tau; t, x, v)) \mathrm{d}\tau}\Big] \\
&\quad \lesssim \Big\{ \varepsilon^{-2} \int_{\mathbb{R}^3} \mathbf{k}_\beta(W_{\mathbf{cl}}(s; t, x, v), u) |\varepsilon h^{\ell}(s, Y_{\mathbf{cl}}(s; t, x, v), u)| \mathrm{d}u \\
&\qquad + \varepsilon^{-2} \langle W_{\mathbf{cl}}(s; t, x, v) \rangle \|\varepsilon h^{\ell}(s)\|_\infty^2 + |A| \Big\} e^{-\int_s^t \varepsilon^{-2} \check{\nu}(W_{\mathbf{cl}}(\tau; t, x, v)) \mathrm{d}\tau} \\
&\quad \lesssim \big\{1 + \langle W_{\mathbf{cl}}(s; t, x, v) \rangle \|\varepsilon h^{\ell}(s)\|_\infty \big\} \|\varepsilon h^{\ell}(s)\|_\infty \varepsilon^{-2} e^{-\int_s^t \varepsilon^{-2} \check{\nu}(W_{\mathbf{cl}}(\tau; t, x, v)) \mathrm{d}\tau} \\
&\qquad + \|A\|_\infty \varepsilon^2 \varepsilon^{-2} e^{-\int_s^t \varepsilon^{-2} \check{\nu}(W_{\mathbf{cl}}(\tau; t, x, v)) \mathrm{d}\tau},
\end{aligned}$$

where we used the fact $\int_{\mathbb{R}^3} \mathbf{k}_\beta(W_{\mathbf{cl}}(s; t, x, v), u) \mathrm{d}u \lesssim 1$ and $w\Gamma(\frac{\varepsilon h^{\ell}}{w}, \frac{\varepsilon h^{\ell}}{w})(v) \lesssim \langle v \rangle \|\varepsilon h^{\ell}\|_\infty^2$.

Then, for $t \in [0, \varepsilon^2 T]$,

$$\begin{aligned}
&|\varepsilon h^{\ell+1}(t, x, v)| \\
&\quad \lesssim \mathbf{1}_{\{\tilde{t}_1 < 0\}} e^{-CT} ||\varepsilon h^{\ell+1}(0)\|_\infty \\
&\qquad + \int_{\max\{0, \tilde{t}_1(x,v)\}}^t \mathrm{d}s \, \frac{e^{-\int_s^t \frac{\check{\nu}(V_{\mathbf{cl}}(t - \frac{t-\tau}{\varepsilon}; t, x, v))}{\varepsilon^2} \mathrm{d}\tau}}{\varepsilon^2} \\
&\qquad \times \Big\{ \Big(1 + \Big\langle V_{\mathbf{cl}}\Big(t - \frac{t-s}{\varepsilon}; t, x, v\Big) \Big\rangle \Big) \|\varepsilon h^{\ell}(s)\|_\infty) \|\varepsilon h^{\ell}(s)\|_\infty + \varepsilon^2 \|A\|_\infty \Big\}
\end{aligned}$$

$$+\mathbf{1}_{\{\tilde{t}_1\geq 0\}}e^{-\int_{\tilde{t}_1}^{t}\frac{\tilde{\nu}(V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon};t,x,v))}{\varepsilon^2}\mathrm{d}\tau}O(\varepsilon)\mu(\tilde{v}_1)^{\frac{1}{2}-}$$
$$+\mathbf{1}_{\{\tilde{t}_1\geq 0\}}\frac{e^{-\int_{\tilde{t}_1}^{t}\frac{\tilde{\nu}(t-\frac{t-\tau}{\varepsilon};t,x,v)}{\varepsilon^2}\mathrm{d}\tau}}{\tilde{w}(v_1)}\int_{\Pi_{j=1}^{k-1}\mathscr{V}_j}H,$$

where H is given by

$$\sum_{l=1}^{k-1}\mathbf{1}_{\tilde{t}_{l+1}\leq 0<\tilde{t}_l}\|\varepsilon h^{\ell+1}(0)\|_\infty\Pi_{m=1}^{l-1}\frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};v_m))}\mathrm{d}\Sigma_l(0)$$
$$+\sum_{l=1}^{k-1}\int_{\max\{0,\tilde{t}_{l+1}\}}^{\tilde{t}_l}\mathbf{1}_{\tilde{t}_l>0}\Big\{(1+\langle V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon})\rangle)\|\varepsilon h^{\ell}(\tau)\|_\infty^2+\varepsilon^2\|A\|_\infty\Big\}$$
$$\times\Pi_{m=1}^{l-1}\frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};v_m))}\mathrm{d}\Sigma_l(\tau)\mathrm{d}\tau$$
$$+\sum_{l=1}^{k-1}\mathbf{1}_{\tilde{t}_l>0}O(\varepsilon)\mu(v_l)^{\frac{1}{2}-}\Pi_{m=1}^{l-1}\frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};v_m))}\mathrm{d}\Sigma_l(\tilde{t}_{l+1})$$
$$+\mathbf{1}_{\tilde{t}_k>0}\|\varepsilon h^{\ell+1}(\tilde{t}_k)\|_\infty\Pi_{m=1}^{k-2}\frac{\tilde{w}(v_m)}{\tilde{w}(V_{\mathbf{cl}}(\tilde{t}_{m+1}-\frac{\tilde{t}_{m+1}}{\varepsilon};v_m))}\mathrm{d}\Sigma_{k-1}(\tilde{t}_k),$$

and $\mathrm{d}\Sigma_{k-1}(\tilde{t}_k)$ is evaluated at $s=\tilde{t}_k$ of

$$\mathrm{d}\Sigma_l(s):=\{\Pi_{j=l+1}^{k-1}\mathrm{d}\sigma_j\}\left\{e^{-\int_s^{\tilde{t}_l}\frac{\check{\nu}(V_{\mathbf{cl}}(\tilde{t}_l-\frac{\tilde{t}_l-\tau}{\varepsilon};\tilde{t}_l,x_l,v_l))}{\varepsilon^2}\mathrm{d}\tau}\tilde{w}(v_l)\mathrm{d}\sigma_l\right\}$$
$$\times\Pi_{j=1}^{l-1}\left\{e^{-\int_{\tilde{t}_{j+1}}^{\tilde{t}_j}\frac{\check{\nu}(V_{\mathbf{cl}}(\tilde{t}_j-\frac{\tilde{t}_j-\tau}{\varepsilon};\tilde{t}_j,x_j,v_j))}{\varepsilon^2}\mathrm{d}\tau}\mathrm{d}\sigma_j\right\}.$$

Recall (3.78). With the choice of $k=C_1T_0^{5/4}$ (clearly $0\leq t\leq\varepsilon^2T\ll\varepsilon T_0$), for $t\in[0,\varepsilon^2T]$,

$$\begin{aligned}&|\varepsilon h^{\ell+1}(t,x,v)|\\&\lesssim C_1T_0^{5/4}\Big\{e^{-CT}\|\varepsilon h^{\ell+1}(0)\|_\infty+T\sup_{0\leq s\leq t}\|\varepsilon h^{\ell}(s)\|_\infty+\varepsilon^2\|A\|_\infty+O(\varepsilon^2)\\&\quad+\underbrace{\int_0^t\frac{\check{\nu}(V_{\mathbf{cl}}(t-\frac{t-s}{\varepsilon};t,x,v))}{\varepsilon^2}e^{-\int_s^t\frac{\check{\nu}(V_{\mathbf{cl}}(t-\frac{t-\tau}{\varepsilon};t,x,v))}{\varepsilon^2}\mathrm{d}\tau}\mathrm{d}s}_{\lesssim 1}\times\sup_{0\leq s\leq t}\|\varepsilon h^{\ell}(s)\|_\infty^2\Big\}\\&\quad+T_0^{5/4}\left\{\frac{1}{2}\right\}^{C_2T_0^{5/4}}\sup_{0\leq s\leq t}\|\varepsilon h^{\ell+1}(s)\|_\infty.\end{aligned}$$

For $T_0\gg 1,\ 0<T\ll 1$, and $0<\varepsilon\ll 1$, we then established (3.105).

Using (3.105) with a small initial datum and an induction in ℓ, we deduce

$$\sup_{0\le t\le \varepsilon^2 T} \|\varepsilon h^{\ell+1}(t)\|_\infty \lesssim \|\varepsilon h_0\|_\infty + \varepsilon^2\|A\|_\infty + O(\varepsilon^2),$$

for $C_1 \ge 10C_{T_0}$.
Step 2. From *Step 2*, $wf^\ell \to wf$ weak-$*$ in $L^\infty([0,\varepsilon^2 T]\times\Omega\times\mathbb{R}^3)$ up to subsequence. Clearly f satisfies the bound (3.105). On the other hand, applyig the argument of previous step to $f^{\ell+1} - f^\ell$, from (3.105) we can prove that wf^ℓ is a Cauchy sequence in $L^\infty([0,\varepsilon^2 T]\times\Omega\times\mathbb{R}^3)$. It is standard to show that f solves (3.106) and (3.107) with $f^{\ell+1} = f = f^\ell$. Therefore $F = \mu + \varepsilon\varphi\sqrt{\mu}$ solves the Boltzmann equation with diffuse BC. Since the unique solution f has a uniform-in-time bound from Proposition 1.3, we can continue the *Step 2* for $[\varepsilon^2 T, 2\varepsilon^2 T]$, $[2\varepsilon^2 T, 3\varepsilon^2 T]$, $\cdots$, to conclude $wr^\ell \to wr$ in $L^\infty(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$. Therefore $F^\ell \to F \ge 0$ a.e.
Step 3. Let, for sufficiently large m,

$$F_0(x,v) = \mu + \sqrt{\mu}\varepsilon(f_w + f_s + \tilde f(0)) + \mu^{\frac12+\beta^-}\mathbf{1}_{|v|>m|\log\varepsilon|}.$$

Clearly, by the L^∞ estimate of f_s we have $F_0 \ge 0$ [30]. Moreover, F_0 satisfies the assumptions of Theorem 1.3. By Theorem 1.3, we have $\|F(t) - F_s\|_{L^2} \lesssim e^{-\lambda t}$. Then, as $t\to\infty$, for any non negative test function $\psi(x,v)$,

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} F_s(x,v)\psi(x,v)\mathrm{d}x\mathrm{d}v\\
&\quad= \iint_{\Omega\times\mathbb{R}^3} F(t,x,v)\psi(x,v)\mathrm{d}x\mathrm{d}v\\
&\qquad+ O(1)\iint_{\Omega\times\mathbb{R}^3} |F_s(x,v) - F(t,x,v)|\psi(x,v)\mathrm{d}x\mathrm{d}v\\
&\quad\ge 0 - O(1)\|F(t) - F_s\|_{L^2(\Omega\times\mathbb{R}^3)}\\
&\quad\ge 0.
\end{aligned}$$

This proves $F_s(x,v)\ge 0$ a.e. □

3.9 Local-in-Time Validity

Proof of Theorem 1.6 We fix a time interval $[0,T]$ and a $L^\infty_t C^2_{x,v}$ solution to the INSF system in this interval with Dirichlet boundary conditions on $\partial\Omega$. Under the assumptions of Theorem 1.6, are well defined the functions

$$\begin{aligned}
f_1 &= \sqrt{\mu}\left[\rho + u\cdot v + \theta\frac{|v|^2-3}{2}\right],\\
f_2 &= \frac12\sum_{i,j=1}^3 \mathscr{A}_{ij}[\partial_{x_i}u_{j,s} + \partial_{x_j}u_{i,s}] + \sum_{i=1}^3 \mathscr{B}_i\partial_{x_i}\theta - L^{-1}[\Gamma(f_1,f_1)]\\
&\quad+ \frac{|v|^2-3}{2}\theta_2\sqrt{\mu},
\end{aligned}\tag{3.108}$$

with $\theta_2 = p - \fint p - \theta\rho$. We write the solution to the Boltzmann equation as

$$F = \mu + \varepsilon\sqrt{\mu}[f_1 + \varepsilon^2 f_2 + \varepsilon^{\frac{1}{2}} R],$$

so that R satisfies the equation

$$\begin{aligned} &\partial_t R + \varepsilon^{-1} v \cdot \nabla_x R + \varepsilon \Phi \cdot \nabla_v R + \varepsilon^{-2} L R \\ &\quad = \varepsilon^{-1} L_{f_1+\varepsilon f_2} R + \varepsilon^{-1/2}\Gamma(R, R) + \varepsilon \frac{\Phi \cdot v}{2} R + \varepsilon^{-1/2} A, \end{aligned} \tag{3.109}$$

where

$$\begin{aligned} A = &-(\mathbf{I} - \mathbf{P})[v \cdot \nabla_x (f_2 + \tilde{f}_2)] - 2\Gamma(f_1, f_2) \\ &- \varepsilon\big\{\partial_t f_2 + \Phi \cdot \frac{1}{\sqrt{\mu}} \nabla_v\big[\sqrt{\mu}(f_1 + \varepsilon f_2))\big] - \Gamma(f_2, f_2)\big\}, \end{aligned} \tag{3.110}$$

and the boundary condition

$$R|_{\gamma_-} = P_\gamma R + \varepsilon \mathscr{Q} R + \varepsilon^{1/2} r, \tag{3.111}$$

where $r := \varepsilon^{-2}\mu^{-\frac{1}{2}}\{\mathscr{P}^w_\gamma[\mu + (\varepsilon f_1 + \varepsilon^2 f_2)\sqrt{\mu}] - [\mu + (\varepsilon f_1 + \varepsilon^2 f_2)\sqrt{\mu}]\big|_\gamma\} = O(1)$.

We use the iteration scheme

$$\begin{aligned} &\partial_t R^{\ell+1} + \varepsilon^{-1} v \cdot \nabla_x R^{\ell+1} + \varepsilon \Phi \cdot \nabla_v R^{\ell+1} + \varepsilon^{-2} L R^{\ell+1} \\ &\quad = \varepsilon^{-1} L_{f_1+\varepsilon f_2} R^{\ell+1} + \varepsilon^{-1/2}\Gamma(R^\ell, R^\ell) + \varepsilon \frac{\Phi \cdot v}{2} R^{\ell+1} + \varepsilon^{-1/2} A, \\ &R^{\ell+1}|_{\gamma_-} = P_\gamma R^{\ell+1} + \varepsilon \mathscr{Q} R^\ell + \varepsilon^{1/2} r, \quad R^{\ell+1}|_{t=0} = R_0. \end{aligned} \tag{3.112}$$

Here we set $R^0(t, x, v) := R_0(x, v)$.

Clearly $R^\ell_t := \partial_t R^\ell$ solves

$$\begin{aligned} &\partial_t R^{\ell+1}_t + \varepsilon^{-1} v \cdot \nabla_x R^{\ell+1}_t + \varepsilon \Phi \cdot \nabla_v R^{\ell+1}_t + \varepsilon^{-2} L R^{\ell+1}_t \\ &\quad = \varepsilon^{-1} L_{f_1+\varepsilon f_2} R^{\ell+1}_t + \varepsilon^{-1} L_{\partial_t f_1+\varepsilon\partial_t f_2} R^{\ell+1} + \varepsilon^{-1/2}[\Gamma(R^\ell_t, R^\ell) + \Gamma(R^\ell, R^\ell_t)] \\ &\quad\quad + \varepsilon \frac{\Phi \cdot v}{2} R^{\ell+1}_t + \varepsilon^{-1/2}\partial_t A, \\ &R^{\ell+1}_t(t, x, v)|_{\gamma_-} = P_\gamma R^{\ell+1}_t + \varepsilon \mathscr{Q} R^\ell_t + \varepsilon^{1/2}\partial_t r, \quad R^{\ell+1}_t|_{t=0} = \partial_t R_0. \end{aligned} \tag{3.113}$$

We use

$$\begin{aligned} \|\langle v\rangle^{-1/2} L_{f_1+\varepsilon f_2} R^\ell\|_{L^2_t([0,t])L^2_{x,v}} &\lesssim \|w[f_1 + \varepsilon f_2]\|_{L^\infty_{t,x,v}} \|R^\ell\|_{L^2_t([0,t])L^2_{x,v}} \\ \|\langle v\rangle^{-1/2} L_{f_1+\varepsilon f_2} R^\ell_t\|_{L^2_t([0,t])L^2_{x,v}} &\lesssim \|w[f_1 + \varepsilon f_2]\|_{L^\infty_{t,x,v}} \|R^\ell_t\|_{L^2_t([0,t])L^2_{x,v}}, \\ \|\langle v\rangle^{-1/2} L_{\partial_t f_1+\varepsilon\partial_t f_2} R^\ell\|_{L^2_t([0,t])L^2_{x,v}} &\lesssim \|w[\partial_t f_1 + \varepsilon\partial_t f_2]\|_{L^\infty_{t,x,v}} \|R^\ell\|_{L^2_t([0,t])L^2_{x,v}}. \end{aligned}$$

In all the estimates the time interval is restricted to $[0, T]$. Let

$$\mathscr{K} = \max\{\|w[f_1 + \varepsilon f_2]\|_{L^\infty_{t,x,v}}, \|w[\partial_t f_1 + \varepsilon \partial_t f_2]\|_{L^\infty_{t,x,v}}\},$$

which is finite by the properties of $\mathscr{A}$ and $\mathscr{B}$ provide in [13]. We use Corollary 3.14 and Lemma 3.15 and repeat the Step 1 in Sect. 3.7. Step 2 is replaced by the inequality

$$\begin{aligned} &\|R^{\ell+1}(t)\|^2_{L^2_{x,v}} + \varepsilon^{-1}\|(1-P_\gamma)R^{\ell+1}\|^2_{L^2_t L^2(\gamma)} + \varepsilon^{-2}\int_0^t \|(\mathbf{I}-\mathbf{P})R^{\ell+1}\|^2_\nu \\ &\lesssim \|R_0\|^2_{L^2_{x,v}} + \mathscr{K}\|R^{\ell+1}\|^2_{L^2_{t,x,v}} + \varepsilon^{\frac{1}{2}}\|\Gamma(R^\ell, R^\ell)\|^2_{L^2_{t,x,v}} \\ &\quad + \varepsilon^{\frac{1}{2}}\|A\|^2_{L^2_{t,x,v}} + \|r\|^2_{L^2_t L^2_\gamma}. \end{aligned} \tag{3.114}$$

Using Gronwall's inequality, we thus obtain for $t \in [0, T]$,

$$\begin{aligned} \|R^{\ell+1}(t)\|^2_{L^2_{x,v}} \lesssim\ & T e^{\mathscr{K} t}\varepsilon^{\frac{1}{2}}\|\Gamma(R^\ell, R^\ell)\|^2_{L^\infty_t L^2_{x,v}} \\ & + e^{\mathscr{K} t}\Big\{T\big[\varepsilon^{\frac{1}{2}}\|A\|^2_{L^\infty_t L^2_{x,v}} + \|r\|^2_{L^\infty_t L^2_\gamma}\big] + \|R_0\|^2_{L^2_{x,v}}\Big\}. \end{aligned} \tag{3.115}$$

A similar estimate holds for R_t. Steps 3 and 4 are the same as in Sect. 3.7, but for the fact that the non linear term is multiplied by an extra factor $\varepsilon^{\frac{1}{2}}$. We conclude that, for any fixed T there is an $\varepsilon(T)$ such that $T e^{\mathscr{K} T}\varepsilon^{\frac{1}{2}} \ll 1$ and the inductive hypothesis holds for $R^{\ell+1}$ for any $\varepsilon \leq \varepsilon(T)$.

We repeat this process with $\tilde{R}^{\ell+1} - \tilde{R}^\ell$ to show that $\tilde{R}^\ell$ is a Cauchy sequence in $L^2 \cap L^\infty$. Then it is standard to conclude the existence and uniqueness. □

Acknowledgements We thank Prof. Fujun Zhou for pointing out some inaccuracies in previous versions of this paper. Y. Guo's research is supported in part by NSFC Grant 10828103, NSF Grant DMS-0905255, and BICMR. C. Kim's research is supported in part by NSF DMS-1501031, KAIST-CMC, the Herchel Smith Foundation, and the University of Wisconsin-Madison Graduate School with funding from the Wisconsin Alumni Research Foundation. R. Marra is partially supported by MIUR-Prin.

Appendix A: Extensions and Compactness

A.1 Extension

Proof of Lemma 3.6 Step 1. In the sense of distributions on $[0, \infty) \times \Omega \times \mathbb{R}^3$,

$$\begin{aligned} &\varepsilon\partial_t f_\delta + v\cdot\nabla_x f_\delta + \varepsilon^2\Phi\cdot\nabla_v f_\delta \\ &= \left[1 - \chi\left(\frac{n(x)\cdot v}{\delta}\right)\chi\left(\frac{\xi(x)}{\delta}\right)\right]\chi(\delta|v|) \\ &\quad \times \left[\mathbf{1}_{t\in[0,\infty)} g + \mathbf{1}_{t\in(-\infty,0]}\chi(t)\left\{\varepsilon\frac{\chi'(t)}{\chi(t)} + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\right\} f_0(x,v)\right] \end{aligned}$$

$$
\begin{aligned}
&+\big[\mathbf{1}_{t\in[0,\infty)} f + \mathbf{1}_{t\in(-\infty,0]}\chi(t) f_0(x,v)\big]\{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\} \\
&\times\left(\left[1-\chi(\frac{n(x)\cdot v}{\delta})\chi\left(\frac{\xi(x)}{\delta}\right)\right]\chi(\delta|v|)\right).
\end{aligned} \tag{A.1.1}
$$

Note that,

$$
\begin{aligned}
&\Big|\{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\}\big[1-\chi\big(\frac{n(x)\cdot v}{\delta}\big)\chi\big(\frac{\xi(x)}{\delta}\big)\big]\chi(\delta|v|)\Big| \\
&= \Big| -\frac{1}{\delta}\{v\cdot\nabla_x n(x)\cdot v + \varepsilon^2\Phi\cdot n(x)\}\chi'\big(\frac{n(x)\cdot v}{\delta}\big)\chi\big(\frac{\xi(x)}{\delta}\big)\chi(\delta|v|) \\
&\quad -\frac{1}{\delta} v\cdot\nabla_x\xi(x)\chi'\big(\frac{\xi(x)}{\delta}\big)\chi\big(\frac{n(x)\cdot v}{\delta}\big)\chi(\delta|v|) \\
&\quad +\varepsilon^2\delta\Phi\cdot\frac{v}{|v|}\chi'(\delta|v|)\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\big(\frac{\xi(x)}{\delta}\big)\Big]\Big| \\
&\le \frac{4}{\delta}(|v|^2\|\xi\|_{C^2} + \varepsilon^2\|\Phi\|_\infty)\chi(\delta|v|) + \frac{C_\Omega}{\delta}|v|\chi(\delta|v|) + \varepsilon^2\delta\|\Phi\|_\infty\mathbf{1}_{|v|\le 2\delta^{-1}} \\
&\lesssim \delta^{-3}\mathbf{1}_{|v|\le 2\delta^{-1}}.
\end{aligned} \tag{A.1.2}
$$

This proves the second line of (3.20). Since $\big[1-\chi(\frac{n(x)\cdot v}{\delta})\chi(\frac{\xi(x)}{\delta})\big]\chi(\delta|v|)\le 1$, we prove the first line of (3.20) directly.
Step 2. We claim that if $0\le\xi(x)\le\tilde{C}\delta^4$, $|n(x)\cdot v|>\delta$ and $|v|\le\frac{1}{\delta}$ then either $\xi(\tilde{x}_{\mathbf{f}}(x,v))=\tilde{C}\delta^4$ or $\xi(\tilde{x}_{\mathbf{b}}(x,v))=\tilde{C}\delta^4$.

To show this, if $v\cdot n(x)\ge\delta$, we take $s>0$, while if $v\cdot n(x)\le -\delta$ then we take $s<0$. From (2.8),

$$
\begin{aligned}
&\xi(X(s;0,x,v)) = \xi(x) + \int_0^s V(\tau;0,x,v)\cdot\nabla_x\xi(X(\tau;0,x,v))\mathrm{d}\tau \\
&= \xi(x) + \int_0^s \{v + O(1)\varepsilon^2\|\Phi\|_\infty\tau\}\cdot\{\nabla_x\xi(x) + O(1)\|\xi\|_{C^2}(|v|+\varepsilon^2\|\Phi\|_\infty\tau)\tau\} \\
&= \xi(x) + v\cdot\nabla_x\xi(x)s + O(1)\|\xi\|_{C^2}\big\{|v|^2s^2 + \varepsilon^2\|\Phi\|_\infty s^2 \\
&\quad + \varepsilon^2\|\Phi\|_\infty|v|s^3 + \varepsilon^4\|\Phi\|_\infty^2 s^4\big\}.
\end{aligned}
$$

From $\xi(x)\ge 0$,

$$
\begin{aligned}
\xi(X(s;0,x,v)) &\ge \delta|s|\Big\{1-\frac{\|\xi\|_{C^2}}{\delta}\Big[|v|^2|s| + \varepsilon^2\|\Phi\|_\infty|s| + \varepsilon^2\|\Phi\|_\infty|v||s|^2 \\
&\qquad + \varepsilon^4\|\Phi\|_\infty^2|s|^3\Big]\Big\} \\
&\ge \delta|s|\Big\{1-\frac{\|\xi\|_{C^2}}{\delta}\Big[\frac{1}{\delta^2}|s| + \varepsilon^2\|\Phi\|_\infty|s| + \varepsilon^2\|\Phi\|_\infty\frac{1}{\delta}|s|^2 \\
&\qquad + \varepsilon^4\|\Phi\|_\infty^2|s|^3\Big]\Big\} \\
&\ge \delta|s|\Big\{1-\Big[\frac{1}{4} + \frac{\varepsilon^2\delta^2\|\Phi\|_\infty}{4} + \frac{\varepsilon^2\delta^4\|\Phi\|_\infty}{16} \\
&\qquad + \frac{\varepsilon^4\delta^8\|\Phi\|_\infty^2}{64}\Big]\Big\} \ \ge\ \frac{\delta|s|}{2},
\end{aligned} \tag{A.1.3}
$$

for $0 \leq |s| \leq \frac{\delta^3}{4(1+\|\xi\|_{C^2})}$ and $0 < \varepsilon \ll 1$. Therefore

$$\xi(Y(s;0,x,v)) = \xi\left(X\left(\frac{s}{\varepsilon};0,x,v\right)\right) \geq \delta\frac{|s|}{2\varepsilon},$$

for all $0 \leq |s| \leq \frac{\varepsilon\delta^3}{4(1+\|\xi\|_{C^2})}$ with $0 < \varepsilon \ll 1$. Especially with $\varepsilon s_* = +\frac{\varepsilon\delta^3}{4(1+\|\xi\|_{C^2})}$ for $n(x)\cdot v > \delta$ and $\varepsilon s_* = -\frac{\varepsilon\delta^3}{4(1+\|\xi\|_{C^2})}$ for $n(x)\cdot v < \delta$,

$$\xi(Y(s_*;0,x,v)) > \tilde{C}\delta^4.$$

Therefore, by the intermediate value theorem, we prove our claim.
Step 3. We define $f_E(t,x,v)$ for $(x,v) \in [\mathbb{R}^3\backslash\bar{\Omega}] \times \mathbb{R}^3$:

$$\begin{aligned} f_E(t,x,v) \;&:=\; f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big)\chi\big(t^*_{\mathbf{b}}(x,v)\big) \\ &\qquad \text{if } x^*_{\mathbf{b}}(x,v) \in \partial\Omega, \\ &:=\; f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big)\chi\big(t^*_{\mathbf{f}}(x,v)\big) \\ &\qquad \text{if } x^*_{\mathbf{f}}(x,v) \in \partial\Omega, \\ &:=\; 0 \qquad \text{if } x^*_{\mathbf{b}}(x,v) \notin \partial\Omega \text{ and } x^*_{\mathbf{f}}(x,v) \notin \partial\Omega. \end{aligned} \tag{A.1.4}$$

We check that f_E is well-defined. It suffices to prove the following:

$$\begin{aligned} &\text{If } x^*_{\mathbf{b}}(x,v) \in \partial\Omega \text{ and } x^*_{\mathbf{f}}(x,v) \in \partial\Omega \\ &\text{then } f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big) = 0 \\ &\quad = f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big). \end{aligned}$$

If $|n(x^*_{\mathbf{b}}(x,v))\cdot v^*_{\mathbf{b}}(x,v)| \leq \delta$ or $|v^*_{\mathbf{b}}(x,v)| \geq \frac{1}{\delta}$ then $f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) = 0$ due to (3.12). If $n(x^*_{\mathbf{b}}(x,v))\cdot v^*_{\mathbf{b}}(x,v) > \delta$ and $|v^*_{\mathbf{b}}(x,v)| \leq \frac{1}{\delta}$ then, due to *Step 2*, $\xi(x^*_{\mathbf{f}}(x,v)) = \xi(x^*_{\mathbf{f}}(x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))) = \tilde{C}\delta^4$ so that $x^*_{\mathbf{f}}(x,v) \notin \partial\Omega$.

On the other hand, if $|n(x^*_{\mathbf{f}}(x,v))\cdot v^*_{\mathbf{f}}(x,v)| \leq \delta$ or $|v^*_{\mathbf{f}}(x,v)| \geq \frac{1}{\delta}$ then $f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v)) = 0$ due to (3.12). If $n(x^*_{\mathbf{f}}(x,v))\cdot v^*_{\mathbf{f}}(x,v) < -\delta$ and $|v^*_{\mathbf{f}}(x,v)| \leq \frac{1}{\delta}$ then, due to *Step 2*, $\xi(x^*_{\mathbf{b}}(x,v)) = \xi(x^*_{\mathbf{b}}(x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))) = \tilde{C}\delta^4$ so that $x^*_{\mathbf{b}}(x,v) \notin \partial\Omega$.

Note that

$$f_E(t,x,v) = f_\delta(t,x,v) \quad \text{for all } x \in \partial\Omega. \tag{A.1.5}$$

If $x \in \partial\Omega$ and $n(x)\cdot v > \delta$ then $(x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) = (x,v)$. From the definition (A.1.4), for those (x,v), we have $f_E(t,x,v) = f_\delta(t,x,v)$. If $x \in \partial\Omega$ and $n(x)\cdot v < -\delta$ then $(x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x.v)) = (x,v)$. From the definition (A.1.4), we conclude (A.1.5) again. Otherwise, if $-\delta < n(x)\cdot v < \delta$ then $f_E|_{\partial\Omega} \equiv 0 \equiv f_\delta|_{\partial\Omega}$.

Step 4. We claim that $f_E(x,v) \in L^2([\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3)$.

From the definition of (A.1.4), we have $f_E(x,v) \equiv 0$ if $x^*_{\mathbf{b}}(x,v) \notin \partial\Omega$ and $x^*_{\mathbf{f}}(x,v) \notin \partial\Omega$. Therefore, from (A.1.4),

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3} |f_E(t,x,v)|^2 \mathrm{d}x\mathrm{d}v\mathrm{d}t \\
&\quad= \int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3} \mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}|f_E|^2 + \int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3} \mathbf{1}_{x^*_{\mathbf{f}}\in\partial\Omega}|f_E|^2 \\
&\quad= \int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3} \mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}|f_\delta(t-\varepsilon t^*_{\mathbf{b}}, x^*_{\mathbf{b}}, v^*_{\mathbf{b}})|^2 \\
&\qquad\times \Big|\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\Big|^2|\chi(t^*_{\mathbf{b}})|^2\mathrm{d}x\mathrm{d}v\mathrm{d}t \qquad\qquad (\text{A.1.6})\\
&\qquad+ \int_{-\infty}^{\infty}\iint_{[\mathbb{R}^3\backslash\Omega]\times\mathbb{R}^3} \mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}|f_\delta(t+\varepsilon t^*_{\mathbf{f}}, x^*_{\mathbf{f}}, v^*_{\mathbf{f}})|^2 \\
&\qquad\times \Big|\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\Big|^2|\chi(t^*_{\mathbf{f}})|^2\mathrm{d}x\mathrm{d}v\mathrm{d}t, \qquad\qquad (\text{A.1.7})
\end{aligned}
$$

where $(t^*_{\mathbf{b}}, x^*_{\mathbf{b}}, v^*_{\mathbf{b}})$ and $(t^*_{\mathbf{f}}, x^*_{\mathbf{f}}, v^*_{\mathbf{f}})$ are evaluated at (x,v).

By (2.11),

$$
\begin{aligned}
(A.1.6) &\leq \int_{-\infty}^{\infty}\mathrm{d}t\int_{\partial\Omega}\int_{n(x)\cdot v>0}\int_0^{\min\{t^*_{\mathbf{f}}(x,v),1\}} \mathrm{d}S_x\mathrm{d}v\mathrm{d}s\{|n(x)\cdot v| + O(\varepsilon)(1+|v|)s\} \\
&\quad\times\Big|f_\delta\big(t-\varepsilon s, x^*_{\mathbf{b}}(X(s;0,x,v), V(s;0,x,v)), \\
&\quad\times v^*_{\mathbf{b}}(X(s;0,x,v), V(s;0,x,v))\big)\Big|^2 \\
&\leq \int_{-\infty}^{\infty}\mathrm{d}t\int_{\partial\Omega}\int_{n(x)\cdot v>0}\int_0^1 \Big|f_\delta(t,x,v)\Big|^2\{|n(x)\cdot v| + O(\varepsilon)(1+|v|)s\}\mathrm{d}s\mathrm{d}v\mathrm{d}S_x \\
&\lesssim \int_{-\infty}^{\infty}\mathrm{d}t\int_{\partial\Omega}\int_{n(x)\cdot v>0} \Big|f_\delta(t,x,v)\Big|^2|n(x)\cdot v|\mathrm{d}v\mathrm{d}S_x \lesssim \|f_\delta\|^2_{L^2(\mathbb{R}\times\partial\Omega\times\mathbb{R}^3)},
\end{aligned}
$$

where we have used the fact, from (3.11), $O(\varepsilon)(1+|v|)|s| \leq O(\varepsilon)(1+\frac{1}{\delta}) \lesssim \delta \lesssim |n(x)\cdot v|$ for $(x,v)\in \mathrm{supp}(f_\delta)$, and, for $n(x)\cdot v>0$, $x\in\partial\Omega$, and $0\leq s\leq t^*_{\mathbf{f}}(x,v)$,

$$
\begin{aligned}
&(x^*_{\mathbf{b}}(X(s;0,x,v), V(s;0,x,v)), v^*_{\mathbf{b}}(X(s;0,x,v), V(s;0,x,v))) \\
&\quad= (x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) = (x,v),
\end{aligned}
$$

and $t^*_{\mathbf{b}}(X(s;0,x,v), V(s;0,x,v)) = s$ and the change of variables $t-\varepsilon s \mapsto t$. Similarly we can show (A.1.7) $\lesssim \|f_\delta\|^2_{L^2(\mathbb{R}\times\partial\Omega\times\mathbb{R}^3)}$.

Step 5. We claim that, in the sense of distributions on $\mathbb{R}\times[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3$,

$$
\begin{aligned}
&\varepsilon\partial_t f_E + v\cdot\nabla_x f_E + \varepsilon^2\Phi\cdot\nabla_v f_E \\
&\quad= \frac{1}{\tilde{C}\delta^4}v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\Big[f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi(t^*_{\mathbf{b}}(x,v))
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\
&+ f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\chi(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\Big] \\
&+ f_\delta(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\
&- f_\delta(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}.
\end{aligned} \tag{A.1.8}
$$

Note that

$$
\begin{aligned}
&[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) \\
&\quad = \underbrace{[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v](t-\varepsilon t^*_{\mathbf{b}}(x,v))} \\
&\qquad \times \partial_t f(t-\varepsilon t^*_{\mathbf{b}}(x,v), x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v)) \\
&\qquad + [v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(s, x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))|_{s=t-\varepsilon t^*_{\mathbf{b}}(x,v)}, \\
&[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v)) \\
&\quad = \underbrace{[\varepsilon\partial_t + v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v](t+\varepsilon t^*_{\mathbf{f}}(x,v))} \\
&\qquad \times \partial_t f(t+\varepsilon t^*_{\mathbf{f}}(x,v), x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v)) \\
&\qquad + [v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v] f(s, x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))|_{s=t+\varepsilon t^*_{\mathbf{f}}(x,v)}.
\end{aligned}
$$

The underbraced terms vanish because $[v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v](t-\varepsilon t^*_{\mathbf{b}}(x.v)) = \frac{d}{ds}\Big|_{s=0}(t-\varepsilon t^*_{\mathbf{b}}(X(s;0,x,v),V(s;0,x,v))) = \frac{d}{ds}\Big|_{s=0}(t-\varepsilon s) = -\varepsilon$, and $[v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v](t+\varepsilon t^*_{\mathbf{f}}(x.v)) = \frac{d}{ds}\Big|_{s=0}(t+\varepsilon t^*_{\mathbf{f}}(X(s;0,x,v),V(s;0,x,v))) = \frac{d}{ds}\Big|_{s=0}(t-\varepsilon s+\varepsilon t^*_{\mathbf{f}}(x,v)) = -\varepsilon$. Moreover, in the sense of distributions on $[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3$,

$$
\begin{aligned}
&v\cdot\nabla_x f_E + \varepsilon^2\Phi\cdot\nabla_v f_E \\
&= \frac{1}{\tilde{C}\delta^4} v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\Big[f_\delta(x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\
&\quad + f_\delta(x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\chi(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\Big] \\
&\quad + f_\delta(x^*_{\mathbf{b}}(x,v), v^*_{\mathbf{b}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega} \\
&\quad - f_\delta(x^*_{\mathbf{f}}(x,v), v^*_{\mathbf{f}}(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}.
\end{aligned} \tag{A.1.9}
$$

For $\phi\in C^\infty_c([\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3)$, we choose small $t>0$ such that $X(s;0,x,v)\in\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}$ for all $|s|\le t$ and all $(x,v)\in\operatorname{supp}(\phi)$. Then, from (A.1.4), for $(X(s),V(s)) = (X(s;0,x,v),V(s;0,x,v))$,

$$
\begin{aligned}
&\frac{d}{ds} f_E(X(s), V(s)) \\
&\quad = \frac{d}{ds}\Big[f_\delta(x_{\mathbf{b}}^*(X(s), V(s)), v_{\mathbf{b}}^*(X(s), V(s)))\chi(t_{\mathbf{b}}^*(X(s), V(s)))\mathbf{1}_{x_{\mathbf{b}}^*(X(s),V(s))\in\partial\Omega} \\
&\qquad + f_\delta(x_{\mathbf{f}}^*(X(s), V(s)), v_{\mathbf{f}}^*(X(s), V(s)))\chi(t_{\mathbf{f}}^*(X(s), V(s)))\mathbf{1}_{x_{\mathbf{f}}^*(X(s),V(s))\in\partial\Omega}\Big] \\
&\qquad \times \chi\big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\big) \\
&\qquad + \Big[f_\delta(x_{\mathbf{b}}^*(X(s), V(s)), v_{\mathbf{b}}^*(X(s), V(s)))\chi(t_{\mathbf{b}}^*(X(s), V(s)))\mathbf{1}_{x_{\mathbf{b}}^*(X(s),V(s))\in\partial\Omega} \\
&\qquad + f_\delta(x_{\mathbf{f}}^*(X(s), V(s)), v_{\mathbf{f}}^*(X(s), V(s)))\chi(t_{\mathbf{f}}^*(X(s), V(s)))\mathbf{1}_{x_{\mathbf{f}}^*(X(s),V(s))\in\partial\Omega}\Big] \\
&\qquad \times \frac{d}{ds}\chi\big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\big).
\end{aligned}
$$

From $(x_{\mathbf{b}}^*(X(s;0,x,v), V(s;0,x,v)), v_{\mathbf{b}}^*(X(s;0,x,v), V(s;0,x,v))) = (x_{\mathbf{b}}^*(x,v), v_{\mathbf{b}}^*(x,v))$ and $(x_{\mathbf{f}}^*(X(s;0,x,v), V(s;0,x,v)), v_{\mathbf{f}}^*(X(s;0,x,v), V(s;0,x,v))) = (x_{\mathbf{f}}^*(x,v), v_{\mathbf{f}}^*(x,v))$ and $t_{\mathbf{f}}^*(X(s;0,x,v), V(s;0,x,v)) = t_{\mathbf{f}}^*(x,v) - s$ and $t_{\mathbf{b}}^*(X(s;0,x,v), V(s;0,x,v)) = t_{\mathbf{b}}^*(x,v) + s$,

$$
\begin{aligned}
&\frac{d}{ds} f_E(X(s), V(s)) \\
&\quad = \Big[f_\delta(x_{\mathbf{b}}^*(x,v), v_{\mathbf{b}}^*(x,v))\chi'(t_{\mathbf{b}}^*(X(s), V(s)))\mathbf{1}_{x_{\mathbf{b}}^*(x,v)\in\partial\Omega} \\
&\qquad - f_\delta(x_{\mathbf{f}}^*(x,v), v_{\mathbf{f}}^*(x,v))\chi'(t_{\mathbf{f}}^*(X(s), V(s)))\mathbf{1}_{x_{\mathbf{f}}^*(x,v)\in\partial\Omega}\Big]\chi\big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\big) \\
&\qquad + \Big[f_\delta(x_{\mathbf{b}}^*(x,v), v_{\mathbf{b}}^*(x,v))\chi(t_{\mathbf{b}}^*(X(s), V(s)))\mathbf{1}_{x_{\mathbf{b}}^*(x,v)\in\partial\Omega} \\
&\qquad + f_\delta(x_{\mathbf{f}}^*(x,v), v_{\mathbf{f}}^*(x,v))\chi(t_{\mathbf{f}}^*(X(s), V(s)))\mathbf{1}_{x_{\mathbf{f}}^*(x,v)\in\partial\Omega}\Big] \\
&\qquad \times \frac{1}{\tilde{C}\delta^4} V(s)\cdot\nabla_x\xi(X(s))\chi'\big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\big).
\end{aligned}
\tag{A.1.10}
$$

By the change of variables $(x, v) \mapsto (X(s;0,x,v), V(s;0,x,v))$, for sufficiently small s,

$$
\begin{aligned}
&-\iint_{[\Omega_{\tilde{C}\delta^4}\setminus\bar{\Omega}]\times\mathbb{R}^3} f_E(x,v)\{v\cdot\nabla_x + \varepsilon^2\Phi\cdot\nabla_v\}\phi(x,v)\mathrm{d}x\mathrm{d}v \\
&\quad = -\iint_{[\Omega_{\tilde{C}\delta^4}\setminus\bar{\Omega}]\times\mathbb{R}^3} f_E(X(s), V(s))\{V(s)\cdot\nabla_X + \varepsilon^2\Phi\cdot\nabla_V\}\phi(X(s), V(s))\mathrm{d}x\mathrm{d}v \\
&\quad = -\iint_{[\Omega_{\tilde{C}\delta^4}\setminus\bar{\Omega}]\times\mathbb{R}^3} f_E(X(s), V(s))\frac{d}{ds}\phi(X(s), V(s))\mathrm{d}x\mathrm{d}v.
\end{aligned}
\tag{A.1.11}
$$

Since the change of variables $(x, v) \mapsto (X(s;0,x,v), V(s;0,x,v))$ has unit Jacobian, it follows that, for s sufficiently small,

$$\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E((X(s),V(s))\phi(X(s),V(s))) = \iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E(x,v)\phi(x,v),$$

and hence
$$\frac{d}{ds}\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_E((X(s),V(s))\phi(X(s),V(s)) = 0.$$

Therefore we can move the s-derivative on f_E: By (A.1.10),

$$\begin{aligned}
&(A.1.11)\\
&= \iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} \frac{d}{ds} f_E(X(s),V(s))\phi(X(s),V(s))\mathrm{d}x\mathrm{d}v\\
&= \iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} \Big[f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi'(t^*_{\mathbf{b}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&\quad - f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi'(t^*_{\mathbf{f}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\Big]\\
&\quad \times \chi\big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\big)\phi(X(s),V(s))\\
&\quad + \iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} \Big[f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi(t^*_{\mathbf{b}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&\quad + f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi(t^*_{\mathbf{f}}(X(s),V(s)))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\Big]\\
&\quad \times \frac{1}{\tilde{C}\delta^4} V(s)\cdot\nabla_x\xi(X(s))\chi'\big(\frac{\xi(X(s))}{\tilde{C}\delta^4}\big)\phi(X(s),V(s)).
\end{aligned}$$

From the change of variable $(X(s;0,x,v),V(s;0,x,v))\mapsto(x,v)$,

$$\begin{aligned}
(A.1.11) = &\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} \big[f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi'(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&- f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi'(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\big]\chi\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big)\phi(x,v)\\
&+ \iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} \Big[f_\delta(x^*_{\mathbf{b}}(x,v),v^*_{\mathbf{b}}(x,v))\chi(t^*_{\mathbf{b}}(x,v))\mathbf{1}_{x^*_{\mathbf{b}}(x,v)\in\partial\Omega}\\
&+ f_\delta(x^*_{\mathbf{f}}(x,v),v^*_{\mathbf{f}}(x,v))\chi(t^*_{\mathbf{f}}(x,v))\mathbf{1}_{x^*_{\mathbf{f}}(x,v)\in\partial\Omega}\Big]\\
&\times \frac{1}{\tilde{C}\delta^4} v\cdot\nabla_x\xi(x)\chi'\big(\frac{\xi(x)}{\tilde{C}\delta^4}\big)\phi(x,v).
\end{aligned}$$

Hence (A.1.8) is proved.

On the other hand, following the bounds of (A.1.6) and (A.1.7) in *Step 4* we prove the third line of (3.20).

Step 6. We define $\bar{f}(t,x,v)$ for $(t,x,v)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$:

$$\bar{f}(t,x,v) := f_\delta(t,x,v)\mathbf{1}_{(x,v)\in\bar{\Omega}\times\mathbb{R}^3} + f_E(t,x,v)\mathbf{1}_{(x,v)\in[\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3}. \tag{A.1.12}$$

For $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$, by Lemma 3.3,

$$
\begin{aligned}
&-\int_{-\infty}^{\infty}\mathrm{d}t\iint_{\mathbb{R}^3\times\mathbb{R}^3}\bar{f}(t,x,v)\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi(t,x,v)\mathrm{d}x\mathrm{d}v\\
&=-\int_{-\infty}^{\infty}\mathrm{d}t\iint_{\Omega\times\mathbb{R}^3}f_\delta(t,x,v)\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi(t,x,v)\mathrm{d}x\mathrm{d}v\\
&\quad-\int_{-\infty}^{\infty}\mathrm{d}t\iint_{[\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3}f_E(t,x,v)\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}\phi(t,x,v)\mathrm{d}x\mathrm{d}v\\
&=\int_{-\infty}^{\infty}\mathrm{d}t\int_\gamma f_\delta(t,x,v)\phi(t,x,v)\{n(x)\cdot v\}\mathrm{d}S_x\mathrm{d}v\\
&\quad+\int_{-\infty}^{\infty}\mathrm{d}t\int_\gamma f_E(t,x,v)\phi(t,x,v)\{-n(x)\cdot v\}\mathrm{d}S_x\mathrm{d}v\\
&\quad+\int_{-\infty}^{\infty}\mathrm{d}t\iint_{\Omega\times\mathbb{R}^3}\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}f_\delta(t,x,v)\phi(t,x,v)\mathrm{d}x\mathrm{d}v\\
&\quad+\int_{-\infty}^{\infty}\mathrm{d}t\iint_{[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3}\{\varepsilon\partial_t+v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\}f_E(x,v)\phi(x,v)\mathrm{d}x\mathrm{d}v,
\end{aligned}
$$

where the contributions of $\{t=\infty\}$ and $\{t=-\infty\}$ vanish since $\phi(t)\in C_c^\infty(\mathbb{R})$.
From (A.1.5), the boundary contributions are cancelled:

$$
\int_{-\infty}^{\infty}\int_\gamma f_\delta(t,x,v)\phi(t,x,v)\mathrm{d}\gamma\mathrm{d}t-\int_{-\infty}^{\infty}\int_\gamma f_E(t,x,v)\phi(t,x,v)\mathrm{d}\gamma\mathrm{d}t=0.
$$

Further from (A.1.1) and (A.1.8), we prove that $\bar{f}$ solves (3.13) in the sense of distributions on $\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3$. □

A.2 Compactness of $K\mathscr{L}^{-1}$

Proof of Lemma 2.11 From Proposition 2.10, $\mathscr{L}^{-1}$ maps L^2 to L^2 so that $\sup_n\|f^n\|_{L^2}<+\infty$.

Step 1. We approximate K by a compactly supported smooth K_N.
For any $N\gg1$, by the Hölder inequality,

$$
\begin{aligned}
&\|K(\mathbf{1}_{\{|u|>N\text{ or }|v|>N\text{ or }|v-u|<\frac{1}{N}\}}f)\|_2\\
&=\Big\|\int_{\mathbb{R}^3}\mathbf{k}(v,u)\mathbf{1}_{\{|u|>N\text{ or }|v|>N\text{ or }|v-u|<\frac{1}{N}\}}f(u)\mathrm{d}u\Big\|_2\\
&\le\sup_v\sqrt{\int_{|u|>N}|\mathbf{k}(v,u)|\mathrm{d}u}\times\sup_u\sqrt{\int_{\mathbb{R}^3}|\mathbf{k}(v,u)|\mathrm{d}v}\times\|f\|_2\\
&\quad+\sup_v\sqrt{\int_{\mathbb{R}^3}|\mathbf{k}(v,u)|\mathrm{d}u}\times\sup_u\sqrt{\int_{|v|>N}|\mathbf{k}(v,u)|\mathrm{d}v}\times\|f\|_2
\end{aligned}
$$

$$
\begin{aligned}
&+\sup_v \sqrt{\int_{\mathbb{R}^3} |\mathbf{k}(v,u)|\mathrm{d}u} \times \sup_u \sqrt{\int_{\mathbb{R}^3} \frac{e^{-\theta|v-u|^2}}{|v-u|}\mathbf{1}_{|v-u|<\frac{1}{N}}\mathrm{d}v} \\
&\times \|f\|_2 \lesssim o(1)\|f\|_2,
\end{aligned}
$$

where we have used the fact $\int_{\mathbb{R}^3}\mathbf{k}(v,u)\mathrm{d}u \lesssim 1$, $\int_{\mathbb{R}^3}\mathbf{k}(v,u)\mathrm{d}v \lesssim 1$, and $\int_{\mathbb{R}^3}\frac{e^{-\beta|v|^2}}{|v|}1_{|v|<\frac{1}{N}} = o(1)$.

Note that

$$
\mathbf{k}(v,u) = \mathbf{k}_1(v,u) - \mathbf{k}_2(v,u), \quad \text{with} \quad \mathbf{k}_1, \mathbf{k}_2 > 0,
$$

and therefore $\mathbf{k}_i(v,u)\mathbf{1}_{\{|u|\le N \ \& \ |v|\le N \ \& \ |v-u|\ge\frac{1}{N}\}} > \delta$ for $0<\delta\ll 1$.

We now define $K_N f := \int \mathbf{k}_N f$ with

$$
\begin{aligned}
\mathbf{k}_N(v,u) = \ &\mathbf{k}_{1,N}(v,u) - \mathbf{k}_{2,N}(v,u) \\
:= \ &\mathbf{k}_1(v,u)\mathbf{1}_{\{|u|\le N \ \& \ |v|\le N \ \& \ |v-u|\ge\frac{1}{N}\}} * \phi_{\frac{1}{N}}(v)\phi_{\frac{1}{N}}(u) \\
&-\mathbf{k}_2(v,u)\mathbf{1}_{\{|u|\le N \ \& \ |v|\le N \ \& \ |v-u|\ge\frac{1}{N}\}} * \phi_{\frac{1}{N}}(v)\phi_{\frac{1}{N}}(u) \\
= &\int_{\mathbb{R}^3}\mathrm{d}u'\phi_{\frac{1}{N}}(u-u')\int_{\mathbb{R}^3}\mathrm{d}v'\phi_{\frac{1}{N}}(v-v')\mathbf{k}_1(v,u)\mathbf{1}_{\left\{|u|\le N \ \& \ |v|\le N \ \& \ |v-u|\le\frac{1}{N}\right\}} \\
&-\int_{\mathbb{R}^3}\mathrm{d}u'\phi_{\frac{1}{N}}(u-u')\int_{\mathbb{R}^3}\mathrm{d}v'\phi_{\frac{1}{N}}(v-v')\mathbf{k}_2(v,u)\mathbf{1}_{\left\{|u|\le N \ \& \ |v|\le N \ \& \ |v-u|\le\frac{1}{N}\right\}},
\end{aligned}
\tag{A.2.1}
$$

where $\phi_{\frac{1}{N}}$ is the standard mollifiers. In particular, for $i=1,2$,

$$
\begin{aligned}
&\sup_v \int_{\mathbb{R}^3}\left|\mathbf{k}_{i,N}(v,u) - \mathbf{k}_i(v,u)\mathbf{1}_{\{|u|\le N \ \& \ |v|\le N \ \& \ |v-u|\le\frac{1}{N}\}}\right|\mathrm{d}u = o(1), \\
&\sup_u \int_{\mathbb{R}^3}\left|\mathbf{k}_{i,N}(v,u) - \mathbf{k}_i(v,u)\mathbf{1}_{\{|u|\le N \ \& \ |v|\le N \ \& \ |v-u|\le\frac{1}{N}\}}\right|\mathrm{d}v = o(1).
\end{aligned}
$$

Consequently, $\|K_N f - Kf\|_2 \lesssim o(1)\|f\|_2$.

We denote

$$
\bar{\mathbf{k}}_N(v,u) := \mathbf{k}_{1,N}(v,u) + \mathbf{k}_{2,N}(v,u), \quad \bar{K}_N f(v) := \int_{\mathbb{R}^3}\bar{\mathbf{k}}_N(v,u)f(u)\mathrm{d}u.
$$

Note that $\mathbf{k}_{1,N}, \mathbf{k}_{2,N} \ge 0$ and hence $\bar{\mathbf{k}}_N \ge 0$ and $\bar{\mathbf{k}}_N \ge |\mathbf{k}_N|$.

Step 2.

We fix $\delta \ll 1$. Given f^n, we define f^n_δ as

$$
f^n_\delta(x,v) := \Big[1-\chi\big(\frac{n(x)\cdot v}{\delta}\big)\chi\big(\frac{\xi(x)}{\delta}\big)\Big]\chi(\delta|v|)f^(x,v),
\tag{A.2.2}
$$

and extend it to the whole space $\mathbb{R}^3\times\mathbb{R}^3$. We follow the process in the proof of Lemma 3.6 and we only pinpoint the difference.

Similarly to *Step 3* of the proof of Lemma 3.6, we define $f_E^n(x,v)$ for $(x,v)\in[\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3$

$$\begin{aligned}
&f_E^n(x,v)\\
&:= e^{-\lambda t_{\mathbf{b}}^*(x,v)+\int_0^{-t_{\mathbf{b}}^*(x,v)}\left[\frac{\nu(X(\tau),V(\tau))}{\varepsilon}-\frac12\varepsilon^2\Phi(X(\tau))\cdot V(\tau)\right]\mathrm{d}\tau}\\
&\quad\times f_\delta^n(x_{\mathbf{b}}^*(x,v),v_{\mathbf{b}}^*(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi(t_{\mathbf{b}}^*(x,v))\quad\text{ for } x_{\mathbf{b}}^*(x,v)\in\partial\Omega,\\
&:= e^{\lambda t_{\mathbf{f}}^*(x,v)+\int_0^{t_{\mathbf{f}}^*(x,v)}\left[\frac{\nu(X(\tau),V(\tau))}{\varepsilon}-\frac12\varepsilon^2\Phi(X(\tau))\cdot V(\tau)\right]\mathrm{d}\tau}\\
&\quad\times f_\delta^n(x_{\mathbf{f}}^*(x,v),v_{\mathbf{f}}^*(x,v))\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi(t_{\mathbf{f}}^*(x,v))\quad\text{ for } x_{\mathbf{f}}^*(x,v)\in\partial\Omega,\\
&:= 0\quad\text{for}\quad x_{\mathbf{b}}^*(x,v)\notin\partial\Omega\text{ and } x_{\mathbf{f}}^*(x,v)\notin\partial\Omega,
\end{aligned}\tag{A.2.3}$$

where $(X(\tau),V(\tau))=(X(\tau;0,x,v),V(\tau;0,x,v))$.

Then

$$f_E^n(x,v)=f_\delta^n(x,v)\quad\text{for all } x\in\partial\Omega,\tag{A.2.4}$$

because, if $x\in\partial\Omega$ and $n(x)\cdot v>\delta$, then $t_{\mathbf{b}}^*(x,v)=0$. If $x\in\partial\Omega$ and $n(x)\cdot v<\delta$ then $t_{\mathbf{f}}^*(x,v)=0$.

We define $\bar{f}^n(x,v)$ as

$$\bar{f}(x,v)\ :=\ f_\delta(x,v)\mathbf{1}_{(x,v)\in\bar{\Omega}\times\mathbb{R}^3}+f_E(x,v)\mathbf{1}_{(x,v)\in[\mathbb{R}^3\backslash\bar{\Omega}]\times\mathbb{R}^3}.\tag{A.2.5}$$

Note that $\bar{f}^n$ solves

$$\lambda\bar{f}^n+v\cdot\nabla_x\bar{f}^n+\frac{1}{\varepsilon}\nu\bar{f}^n+\varepsilon^2\Phi\cdot\nabla_v\bar{f}^n-\frac12\varepsilon^2\Phi\cdot v\bar{f}^n\ =\ h^n\ =\ h_1^n+h_2^n+h_3^n+h_4^n,$$

where

$$\begin{aligned}
&h_1^n(x,v)=\mathbf{1}_{(x,v)\in\Omega\times\mathbb{R}^3}\left[1-\chi\left(\frac{n(x)\cdot v}{\delta}\right)\chi\left(\frac{\xi(x)}{\delta}\right)\chi\left(\frac{\xi(x)}{\delta}\right)\right]\chi(\delta|v|)g^n,\\
&h_2^n(x,v)=\mathbf{1}_{(x,v)\in\Omega\times\mathbb{R}^3}f^n\left\{v\cdot\nabla_x+\varepsilon^2\Phi\cdot\nabla_v\right\}\\
&\quad\times\Big(\Big[1-\chi\Big(\frac{n(x)\cdot v}{\delta}\Big)\chi\Big(\frac{\xi(x)}{\delta}\Big)\Big]\chi\Big(\frac{\xi(x)}{\delta}\Big)\chi(\delta|v|)\Big),\\
&h_3^n(x,v)=\mathbf{1}_{\ (x,v)\in[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3}\ \frac{1}{\tilde{C}\delta^4}v\cdot\nabla_x\xi(x)\chi'\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\\
&\quad\times\Big[f_\delta^n(x_{\mathbf{b}}^*(x,v),v_{\mathbf{b}}^*(x,v))e^{-\lambda t_{\mathbf{b}}^*(x,v)+\int_0^{-t_{\mathbf{b}}^*(x,v)}\left[\frac{\nu(X(\tau),V(\tau))}{\varepsilon}-\frac12\varepsilon^2\Phi(X(\tau))\cdot V(\tau)\right]\mathrm{d}\tau}\mathbf{1}_{x_{\mathbf{b}}^*(x,v)\in\partial\Omega}\\
&\quad+f_\delta^n(x_{\mathbf{f}}^*(x,v),v_{\mathbf{f}}^*(x,v))\mathbf{1}_{x_{\mathbf{f}}^*(x,v)\in\partial\Omega}e^{\lambda t_{\mathbf{f}}^*(x,v)+\int_0^{t_{\mathbf{f}}^*(x,v)}\left[\frac{\nu(X(\tau),V(\tau))}{\varepsilon}-\frac12\varepsilon^2\Phi(X(\tau))\cdot V(\tau)\right]\mathrm{d}\tau}\Big],\\
&h_4^n(x,v)=\mathbf{1}_{\ (x,v)\in[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3}f_\delta^n(x_{\mathbf{b}}^*(x,v),v_{\mathbf{b}}^*(x,v))\\
&\quad\times e^{-\lambda t_{\mathbf{b}}^*(x,v)+\int_0^{-t_{\mathbf{b}}^*(x,v)}\left[\frac{\nu(X(\tau),V(\tau))}{\varepsilon}-\frac12\varepsilon^2\Phi(X(\tau))\cdot V(\tau)\right]\mathrm{d}\tau}\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t_{\mathbf{b}}^*(x,v))\mathbf{1}_{x_{\mathbf{b}}^*(x,v)\in\partial\Omega}
\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{1}_{(x,v)\in[\Omega_{\tilde{C}\delta^4}\backslash\bar{\Omega}]\times\mathbb{R}^3} f_\delta^n(x_{\mathbf{f}}^*(x,v), v_{\mathbf{f}}^*(x,v)) \\
&\times e^{\lambda t_{\mathbf{f}}^*(x,v)+\int_0^{t_{\mathbf{f}}^*(x,v)}\left[\frac{\nu(X(\tau),V(\tau))}{\varepsilon}-\frac{1}{2}\varepsilon^2\Phi(X(\tau))\cdot V(\tau)\right]\mathrm{d}\tau}\chi\Big(\frac{\xi(x)}{\tilde{C}\delta^4}\Big)\chi'(t_{\mathbf{f}}^*(x,v))\mathbf{1}_{x_{\mathbf{f}}^*(x,v)\in\partial\Omega},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|h_1^n\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \lesssim \|g^n\|_{L^2(\Omega\times\mathbb{R}^3)}, \quad \|h_2^n\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \lesssim_\delta \|f^n\|_{L^2(\Omega\times\mathbb{R}^3)}, \\
&\|h_3^n\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} + \|h_4^n\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \lesssim_\delta \|f_\delta^n\|_{L^2(\gamma)}.
\end{aligned}
$$

Step 3. For $(x,v)\in\operatorname{supp}(\bar{f})$, we can choose a fixed $T>0$ such that

$$
X(T;0,x,v)\notin\operatorname{supp}(\bar{f}) \text{ and } X(T;0,x,v)\notin\operatorname{supp}(h), \tag{A.2.6}
$$

so that

$$
\bar{f}(X(T;0,x,v),V(T;0,x,v))=0.
$$

Directly,

$$
|X(T;0,x,v)-x| = |vT+O(\varepsilon^2)\|\Phi\|_\infty T^2| \geq \delta T - O(\varepsilon^2)\|\Phi\|_\infty T^2.
$$

We choose $T=\frac{C}{\delta}$ for large but fixed $C\gg 1$ such that $|X(T;0,x,v)-x|\geq C-O(\frac{\varepsilon^2}{\delta^2})\geq\frac{C}{2}\gg 1$. This proves our claim (A.2.6).

With this choice T,

$$
\begin{aligned}
\bar{f}^n(x,v) = -\int_0^T & h^n(X(s;0,x,v),V(s;0,x,v)) \\
& e^{\lambda s+\int_0^s\left[\frac{\nu(V(\tau;0,x,v))}{\varepsilon}-\frac{1}{2}\varepsilon^2\Phi(X(\tau;0,x,v))\right]\mathrm{d}\tau}\mathrm{d}s.
\end{aligned}
$$

By the averaging lemma [23], for any given v,

$$
\int \mathbf{k}_N(v,u)\bar{f}^n(x,u)\mathrm{d}u \in H^{1/4}(\mathbb{R}^3).
$$

Since $\bar{f}^n$'s support is bounded uniformly, by a diaganolization argument, it follows that there exists a weak limit $\bar{f}\in L^2$ of $\bar{f}^n$ such that for any rational point v

$$
\int \mathbf{k}_N(v,u)\bar{f}^n(x,u)\mathrm{d}u \to \int \mathbf{k}_N(v,u)\bar{f}(x,u)\mathrm{d}u \quad \text{strongly in } L_x^2. \tag{A.2.7}
$$

Since $\mathbf{k}_N(v,u)$ is smooth in v with compact support, we deduce (A.2.7) for all $v\in\mathbb{R}^3$.
Step 4. Finally,

$$
Kf^n = K_N f^n + (Kf^n - K_N f^n) = K_N f_\delta^n + K_N(f^n - f_\delta^n) + (Kf^n - K_N f^n).
$$

From *Step 3*,

$$K_N f_\delta^n = K_N \bar{f}^n|_\Omega \to K_N \bar{f}|_\Omega \quad \text{strongly in } L^2.$$

Note that

$$1-\Big[1-\chi\big(\frac{n(x)\cdot v}{\delta}\big)\chi\big(\frac{\xi(x)}{\delta}\big)\Big]\chi(\delta|v|) \le \mathbf{1}_{|v|\ge\frac{1}{\delta}} + \mathbf{1}_{|v|\le\frac{1}{\delta},\mathrm{dist}(x,\partial\Omega)<\frac{\delta}{2},|n(x)\cdot v|<\delta}. \tag{A.2.8}$$

Hence, from (A.2.1),

$$\|K_N(f^n - f_\delta^n)\|_{L^2_{x,v}} \lesssim \|f^n - f_\delta^n\|_{L^2_{x,v}} \lesssim \big\|\chi_\delta^c\big\|_\infty \|f^n\|_{L^2_{x,v}} \lesssim O(\delta),$$
$$\|Kf^n - K_N f^n\|_{L^2_{x,v}} \lesssim o(1)\|f^n\|_{L^2_{x,v}} \lesssim o(1).$$

We conclude the proof by choosing $\delta \ll 1,\ 1 \ll N$ then letting $n \to \infty$. □

Appendix B: List of symbols

$\|\cdot\|_p$	the $L^p(\bar\Omega\times\mathbb{R}^3)$ norm	
$\|\cdot\|_{L^p}$	the $L^p(\bar\Omega)$ norm	
$\|\cdot\|_{L^p_y}$	the $L^p(\{y\in Y\})$ norm	
$(\cdot,\cdot)$	$L^2(\bar\Omega\times\mathbb{R}^3)$ inner product or $L^2(\mathbb{R}^3)$ inner product	
$\bar\Omega$	$\Omega\cup\partial\Omega$	
$\|\cdot\|_\nu$	$\|\nu^{1/2}\cdot\|_2$	
$W^{k,p}$	Sobolev space	
H^k	a Sobolev space $W^{k,p}$ with $p=2$	
$\|\cdot\|_{L^pL^q}$	a norm of $\|\cdot\|_{L^p(L^q)}$ which equals $\big\|\|\cdot\|_{L^q}\big\|_{L^p}$	
$\mathrm{d}\gamma$	$\|n(x)\cdot v\|\mathrm{d}S(x)\mathrm{d}v$ with $\mathrm{d}S(x)$ the surface measure of $\partial\Omega$	
$L^p(\partial\Omega\times\mathbb{R}^3)$	L^p-norm on $\partial\Omega\times\mathbb{R}^3$ with a measure $\mathrm{d}\gamma$	
$\|f\|_p^p$	$\int_\gamma \|f(x,v)\|^p\mathrm{d}\gamma$ or $\int_{\partial\Omega}\|f(x)\|^p\mathrm{d}S(x)$	
$f_\gamma, f_\pm$	a trace of f on γ, a trace of f on $\gamma_\pm$	
$\|f\|_{p,\pm}$	$\|f\mathbf{1}_{\gamma_\pm}\|_p$	
$X\lesssim_\alpha Y$	$X\le C_\alpha Y$ where C_α only depends a parameter α	
$X\ll_a Y$	$X\le C_a Y$ where $C_a>0$ is sufficiently small	
$0+$	Some positive number $\delta>0$ without specifying the size	
$\vec{G}$	A force field	(1.1)
Φ	A force field	(1.9)
M_w	A wall Maxwellian	(1.3)
$\mathscr{P}_\gamma^w$	The diffuse reflection boundary operator	(1.5)
μ	The global Maxwellian	(1.8)
T_w	A wall temperature	(1.9)
ϑ_w	An ε-order variation of the wall temperature around 1	(1.9)
Θ_w	An extension of ϑ_w in Ω	(1.10)
ρ_w	A function defined in (1.11)	(1.11)
f_w	A correction term at the boundary	(1.11)
$\mathbf{P}$	A L^2_v-projection on the null space of L	
$a,b,c,$	Coefficients of $\mathbf{P}$	(1.19)
$\mathbf{I}-\mathbf{P}$	$(\mathbf{I}-\mathbf{P})f := f-\mathbf{P}f$	

$\mathfrak{M}$	Mach number	
φ_ε	A correction on the boundary	(1.22)
r	$\mu^{-1/2}\mathscr{P}_\gamma^w(\sqrt{\mu}\varphi_\varepsilon) - \varphi_\varepsilon$	(1.25)
$P_\gamma f$	$\sqrt{2\pi}\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u$	(1.26)
$\mathscr{Q}$	$\varepsilon^{-1}\big[\mu^{-1/2}\mathscr{P}_\gamma^w(\sqrt{\mu}f) - P_\gamma f\big]$	(1.27)
f_s	An ε-order correction of steady solutions substracted f_w	(1.32)
L_1, A_s	Terms defined in (1.34), (1.35)	
L^{-1}	L^{-1} is an inverse with $\mathbf{P}L^{-1} \equiv 0$	
$w(v)$	$e^{\beta\|v\|^2}$ with $0 < \beta \ll 1$	Theorem 1.1
g_1, g_2	Defined in (1.41)	
$\tilde{f}$	An unsteady perturbation around steady solutions	(1.48)
L_ϕ	$L_\phi\psi := -[\Gamma(\phi,\psi) + \Gamma(\psi,\phi)]$	
$\mathscr{E}_\lambda[\cdot](t)$	An energy term in L^2 energy estimate	(1.53)
$\mathscr{D}_\lambda[\cdot](t)$	A dissipation term in L^2 energy estimate	(1.54)
ξ	A function defined in (2.3)	(2.3)
$n(x)$	The outward normal at $x \in \partial\Omega$	(2.5)
$t_{\mathbf{b}}, x_{\mathbf{b}}$	A backward exit time, a backward exit point	(2.9)
$t_{\mathbf{f}}, x_{\mathbf{f}}$	A forward exit time, a forward exit point	(2.10)
$t_i(t,\dots,v_{i-1})$	i-th backward exit time	Definition 2.7
$X_{\mathbf{cl}}, V_{\mathbf{cl}}$	A stochastic trajectory	Definition 2.7
$\gamma_\pm^\delta$	Non-grazing phase boundary	(2.15)
$[\![\cdot]\!]$	A closing norm for the steady case	(2.20)
$\mathbf{k}_\beta$	A function defined in (2.26)	(2.26)
K_β	$K_\beta g := \int_{\mathbb{R}^3}\mathbf{k}_\beta(v,u)g(u)\mathrm{d}u$	
β'	A number $0 < \beta' \ll \beta$	Proposition 2.6
$\mathscr{V}(x), \mathscr{V}_j$	velocity sections of γ_+	(2.28), (2.29)
$\mathrm{d}\sigma, \mathrm{d}\sigma_j$	Measures defined on the velocity sections of $\mathscr{V}(x), \mathscr{V}_j$	(2.28)
$\tilde{w}$	$w^{-1}\mu^{-1/2}$	(2.35)
$\mathrm{d}\Sigma_l$	a stochastic measure	(2.46)
Y, W	Scaled trajectory for the dynamic problem	(3.2)
$\tilde{t}_{\mathbf{b}}, \tilde{x}_{\mathbf{b}}$	Scaled backward exit time and position	(3.4)
$\tilde{t}_{\mathbf{f}}, \tilde{x}_{\mathbf{f}}$	Scaled forward exit time and position	(3.4)
f_δ	An interior and non-grazing parts of f	(3.11)
f_{la}, f_{sm}	Functions defined in (3.22)	
$\mathbf{S}_i f$	Functions defined in (3.40) and (3.40)	
$[\![\![\cdot]\!]\!]$	A closing norm for the unsteady case	(3.93)
$\overline{f_\delta}$	Defined in (A.1.12)	

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