The Boltzmann Equation with Specular Boundary Condition in Convex Domains

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Abstract

We establish the global well-posedness and stability of the Boltzmann equation with the specular reflection boundary condition in general smooth convex domains when an initial datum is close to the Maxwellian with or without a small external potential. In particular, we have completely solved the longstanding open problem after an announcement by Shizuta and Asano in 1977. © 2017 Wiley Periodicals, Inc.

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1 Introduction

Kinetic theory describes the dynamics of any system made up of a large number of particles (e.g., gas or plasma) by a distribution function that is defined in the phase space. Among others, one of the fundamental models is the Boltzmann equation. This equation describes the dynamics of dilute collections of gas particles undergoing elastic binary collisions. In the presence of an *external potential* $-\nabla_x(\phi(t,x)+\Phi(x))$, a density of dilute charged gas particles is governed by the *Boltzmann equation*

$$\begin{aligned} \partial_t F + v\cdot\nabla_x F - \nabla_x(\phi(t,x)+\Phi(x))\cdot\nabla_v F &= Q(F,F), \\ F(0,x,v) &= F_0(x,v), \end{aligned} \tag{1.1}$$

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where $F(t,x,v)$ is a distribution function of the gas particles at a time $t \geq 0$, a position $x \in \Omega \subset \mathbb{R}^3$, and a velocity $v \in \mathbb{R}^3$. Here the collision operator Q takes the form of

$$\begin{aligned} Q(F_1, F_2) &:= Q_+(F_1, F_2) - Q_-(F_1, F_2) \\ &:= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u, \omega)\big[F_1(u')F_2(v') - F_1(u)F_2(v)\big]\mathrm{d}\omega\, \mathrm{d}u, \end{aligned}$$

where $u' = u + [(v-u)\cdot\omega]\omega$, $v' = v - [(v-u)\cdot\omega]\omega$, and $B(v-u,\omega) = |(v-u)\cdot\omega|$ (hard sphere). It is well-known (see [16]) that the following *local Maxwellian* is an equilibrium solution to (1.1):

$$\mu_E(x,v) = \mu(v)e^{-\Phi(x)}, \tag{1.2}$$

where $\mu(v) = e^{-|v|^2/2}$ is a standard *global Maxwellian*.

In many physical applications, e.g., dilute gases passing objects or charged particles inside tokamak devices, particles are interacting not only with each other but also with the boundary. Various interesting phenomena occur when gas particles interact with the boundary, such as a formation and propagation of singularities [13–15]. In the presence of the boundary, a kinetic equation has to be supplemented with boundary conditions modeling the interaction between the particles and the boundary. Among other boundary conditions (see [2, 12]), in this paper we focus on one of the most basic boundary conditions, a so-called *specular reflection boundary condition*. This boundary condition takes into account a case that if a gas particle hits a boundary, then it bounces back with the opposite normal velocity and the same tangential velocity, as a billiard ball hits a boundary and bounces back:

$$F(t,x,v) = F(t,x,R_x v) \quad \text{for } x \in \partial\Omega, \tag{1.3}$$

where $R_x v = v - 2(n(x)\cdot v)n(x)$. We note that the local Maxwellian (1.2) satisfies the boundary condition (1.3).

Despite extensive developments in the study of the Boltzmann theory, many basic boundary problems, especially regarding the specular reflection boundary condition (BC) with general domains, have remained open. In a landmark paper [21] of 1974, Ukai constructed the first global-in-time solutions near Maxwellians to the Boltzmann equation with nontrivial spatial dependence in a periodic box (no boundary). Not long after, in 1977, Shizuta and Asano announced the construction of global solutions to the Boltzmann equation (1.1) with no external potential ($\phi \equiv 0 \equiv \Phi$) near Maxwellians in *smooth convex domains* with specular reflection BC [20], but without mathematical proofs. It took more than 30 years to encounter the first mathematical resolution: Guo, in [12], developed a novel L^2-L^∞ argument to construct a unique solution to the Boltzmann equation (1.1) with no external potential for the specular reflection boundary condition. An asymptotic stability of the global Maxwellian μ is proven when an initial datum is close to μ. However, such results in [12] are established under an *extra condition*; namely, the

boundary is a level set of a *real analytic* function. Indeed, this *analyticity condition* is crucially used to verify a key part of the proof in [12]. Eventually, in this paper, we are able to establish the global well-posedness and stability of the Boltzmann equation for the specular reflection BC *without the analyticity*, and thereby we completely settle a long-standing (40 years) open question in the Boltzmann theory in the affirmative! In fact, our results even go beyond the original open question in [20]: nontrivial external potentials $\phi(t,x)$ and $\Phi(x)$ can be allowed. We discuss more on the external potential issue later.

Here we only mention some other relevant works briefly. In [1, 18], the well-posedness and asymptotic stability of the global Maxwellian are studied when the boundary condition is any convex combination of the specular reflection BC and a diffusive BC *except* the pure specular reflection boundary condition. For large-amplitude solutions, an asymptotic stability of the global Maxwellian is established in [4] with or without the boundary, provided certain a priori strong Sobolev estimates can be verified. Recently boundary regularity and singularity of solutions are extensively studied in [13–15]. We refer [19] among others for the weak solution contents.

Mathematical problems on the Boltzmann equation with an *external potential* have also drawn lots of attention. In [16], the stability of the Maxwellian μ_E in (1.2) is established with a time-dependent external potential $\Phi(x)$, which can be large, in a periodic box. The Vlasov-Poisson-Boltzmann system (VPB), which takes account of self-consistent electric fields by charged particles, is studied in [10] when solutions and fields are small perturbations in a periodic box. However, in many important physical applications (e.g., semiconductor and tokamak), the charged dilute gas interacts with the boundary. One major difficulty is that trajectories are curved and behave in a very complicated way when they hit the boundary. As the first step toward studying models of dilute charged gases interacting with a self-consistent field and boundary, in this paper we establish the global well-posedness of the Boltzmann equation coupled with small *external potentials* and the specular reflection BC.

An external potential and a boundary condition play an important role in the evolution of macroscopic quantities such as the total mass, total momentum, and total energy. Let F be a solution to (1.1) satisfying the specular reflection boundary condition (1.3). We have the total mass conservation and the evolution of the total energy as

$$\iint_{\Omega\times\mathbb{R}^3} F(t) = \iint_{\Omega\times\mathbb{R}^3} F_0, \tag{1.4}$$

$$\begin{aligned} &\iint_{\Omega\times\mathbb{R}^3}\left(\frac{|v|^2}{2}+\Phi\right)F(t) + \int_0^t\iint_{\Omega\times\mathbb{R}^3} F(s)v\cdot\nabla_x\phi(s) \\ &\quad = \iint_{\Omega\times\mathbb{R}^3}\left(\frac{|v|^2}{2}+\Phi\right)F_0. \end{aligned} \tag{1.5}$$

By normalization, without loss of generality, we assume that

$$\begin{aligned} \iint_{\Omega\times\mathbb{R}^3} F_0(x,v) &= \iint_{\Omega\times\mathbb{R}^3} \mu_E(x,v), \\ \iint_{\Omega\times\mathbb{R}^3} \left(\frac{|v|^2}{2}+\Phi(x)\right) F_0(x,v) &= \iint_{\Omega\times\mathbb{R}^3} \left(\frac{|v|^2}{2}+\Phi(x)\right)\mu_E(x,v). \end{aligned} \tag{1.6}$$

We consider a momentum for a special case: a domain Ω is *axis-symmetric* if there are vectors $x_0 \in \mathbb{R}^3$ and $\varpi \in \mathbb{R}^3$ such that

$$\{(x-x_0)\times\varpi\}\cdot n(x) = 0 \quad \text{for all } x \in \partial\Omega. \tag{1.7}$$

In the case of an axis-symmetric domain, we assume a degenerate condition for the external fields as

$$\{(x-x_0)\times\varpi\}\cdot\nabla_x(\phi(t,x)+\Phi(x)) = 0 \quad \text{for all } t \geq 0 \text{ and } x \in \Omega. \tag{1.8}$$

Then, assuming both (1.7) and (1.8), we have an evolution of an angular momentum as

$$\iint_{\Omega\times\mathbb{R}^3} \{(x-x_0)\times\varpi\}\cdot vF(t) = \iint_{\Omega\times\mathbb{R}^3} \{(x-x_0)\times\varpi\}\cdot vF_0. \tag{1.9}$$

In this case, we set

$$\iint_{\Omega\times\mathbb{R}^3} \{(x-x_0)\times\varpi\}\cdot vF_0(x,v) = 0. \tag{1.10}$$

Furthermore, the entropy

$$\mathscr{H}(F) := \iint_{\Omega\times\mathbb{R}^3} F\ln F$$

satisfies the following inequality (*H-theorem*)

$$\mathscr{H}(F(t)) - \mathscr{H}(\mu_E) \leq \mathscr{H}(F_0) - \mathscr{H}(\mu_E). \tag{1.11}$$

Now we are ready to state our main theorems.

THEOREM 1.1. *Let $w = (1+|v|)^\beta$ for $\beta > \frac{5}{2}$. Assume that the domain $\Omega \subset \mathbb{R}^3$ is C^3 and convex in* (1.15). *Assume that $\phi(t,x) \in C^{2,\gamma}_{t,x}$ and $\Phi(x) \in C^{2,\gamma}_x$ for some $0<\gamma$, $\|\phi+\Phi\|_{C^2} \ll 1$, and for $\lambda_\phi > 0$ and $\delta_\phi > 0$,*

$$\sup_{t\geq 0} e^{\lambda_\phi t}\|\phi(t)\|_{C^1} < \delta_\phi < +\infty. \tag{1.12}$$

Assume (1.6). *If $F_0 = \mu_E + \sqrt{\mu_E}\, f_0 \geq 0$ and $\|wf_0\|_\infty + |\mathscr{H}(F_0) - \mathscr{H}(\mu_E)| + \delta_\phi + \delta_\phi/\lambda_\phi \ll 1$, then there exists a unique global-in-time solution*

$$F(t) = \mu_E + \sqrt{\mu_E}\, f(t) \geq 0 \tag{1.13}$$

to (1.1) *satisfying the specular reflection boundary condition* (1.3). *Moreover,*

$$\sup_{t\geq 0}\|wf(t)\|_\infty \lesssim \|wf_0\|_\infty + |\mathscr{H}(F_0) - \mathscr{H}(\mu_E)| + \delta_\phi + \delta_\phi/\lambda_\phi. \tag{1.14}$$

Furthermore, (1.4), (1.5), *and* (1.11) *hold for all $t \geq 0$.*

Here, a C^3 domain means that for any boundary point $p \in \partial\Omega$, locally there exists a one-to-one and onto C^3 function $\eta_{\mathbf{p}}$ such that $\eta_p(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, \mathbf{x}_{p,3}) \in \partial\Omega$ if and only if $\mathbf{x}_{p,3} = 0$ (see (2.6)). The convexity is defined as follows: for $C_\Omega > 0$,

$$\sum_{i,j=1}^{2} \xi_i \xi_j \partial_i \partial_j \eta_p(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, 0) \cdot \partial_3 \eta_p(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, 0) \le -C_\Omega |\xi|^2 \quad \text{for } \xi \in \mathbb{R}^2. \tag{1.15}$$

The notation $C^{\alpha,\gamma}$ stands for the standard Hölder space in t and x.

In the presence of a time-independent external potential ($\phi \equiv 0$), the asymptotical stability of the local Maxwellian μ_E is studied.

THEOREM 1.2. *Assume the same conditions in Theorem* 1.1 *before* (1.12). *Let*

$$\phi \equiv 0. \tag{1.16}$$

Assume (1.6). *If both* (1.7) *and* (1.8) *hold, then we assume* (1.10). *If* $\|wf_0\|_\infty \ll 1$, *then there exists a unique global-in-time solution* $F = \mu_E + \sqrt{\mu_E} f \ge 0$ *to* (1.1) *with* (1.3). *Moreover, there exists* $\lambda = \lambda(\Omega, \Phi) > 0$ *such that*

$$\sup_{t \ge 0} e^{\lambda t} \|wf(t)\|_\infty \lesssim \|wf_0\|_\infty. \tag{1.17}$$

Furthermore, the total mass and energy are conserved as (1.4) *and* (1.5) *with* $\phi \equiv 0$, *and the total angular momentum is conserved as* (1.9) *if both* (1.7) *and* (1.8) *hold.*

Note that we do not have a quantitative bound of λ in (1.17). The main reason is that we use a nonconstructive method to prove L^2 coercivity in Proposition 1.4.

We remark that in both theorems we only need that the domain Ω is smooth and convex but *not real analytic*. We also note that in [16] we need a stronger C^3 assumption for the time-independent external potential to establish the well-posedness.

To illustrate the main ideas of the paper, it is convenient to play with the perturbation f. The function f in (1.13) solves

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - \nabla_x(\phi + \Phi) \cdot \nabla_v f + e^{-\Phi} L f = \\ - \left(\frac{1}{2} f + \sqrt{\mu_E} \right) v \cdot \nabla_x \phi + e^{-\frac{\Phi}{2}} \Gamma(f, f), \end{aligned} \tag{1.18}$$

and satisfies

$$f(t, x, v) = f(t, x, R_x v) \quad \text{for } x \in \partial\Omega. \tag{1.19}$$

We recall the definition of the linearized collision operator (see [2]),

$$Lf = -\frac{1}{\sqrt{\mu}} [Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)], \tag{1.20}$$

and the nonlinear collision operator,

$$\Gamma(f,g)=\frac{1}{2\sqrt{\mu}}[Q(\sqrt{\mu}f,\sqrt{\mu}g)+Q(\sqrt{\mu}g,\sqrt{\mu}f)].$$

It is well-known that (see [8])

$$Lf=\nu f-Kf,$$

where the collision frequency is defined as

$$\nu(v):=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-u)\cdot w|\mu(u)\mathrm{d}w\,\mathrm{d}u.$$

For this hard sphere case, there are positive numbers C_0 and C_1 such that, for $\langle v\rangle:=\sqrt{1+|v|^2}$,

$$C_0\langle v\rangle\le\nu(v)\le C_1\langle v\rangle. \tag{1.21}$$

Moreover, the compact operator K in $L^2(\{v\in\mathbb{R}^3\})$ is defined as

$$\begin{aligned}Kf&=\frac{1}{\sqrt{\mu}}[Q_+(\mu,\sqrt{\mu}f)+Q_+(\sqrt{\mu}f,\mu)-Q_-(\sqrt{\mu}f,\mu)]\\&=\int_{\mathbb{R}^3}\mathbf{k}(v,u)f(u)\mathrm{d}u.\end{aligned}$$

1.1 L^p-L^∞ Bootstrap Argument via the Triple Iterations

In order to handle the quadratic nonlinearity of $\Gamma(f,f)$, it is important to derive an L^∞-control of the solutions of (1.18). To illustrate the main idea, we consider a simplified linear problem

$$\partial_t f+v\cdot\nabla_x f-\nabla_x\Phi(t,x)\cdot\nabla_v f+f=\int_{|u|\le N}f(u)\mathrm{d}u. \tag{1.22}$$

Here, $\Phi(t,x)$ is a time-dependent potential and we can regard $\phi(t,x)+\Phi(x)$ in (1.1) as $\Phi(t,x)$.

We note that due to the boundary condition (1.19), the trajectory $(X(s;t,x,v),V(s;t,x,v))$ is defined as the backward billiard trajectory that is curved by the external field (or force) $-\nabla\Phi$. Let t^1 and x^1 be the first backward bouncing time and position of the trajectory sitting on a position x with a velocity v at time t. Then we define $v^1=R_{x^1}v$ where $R_{x^1}v$ is defined in (1.3). Inductively we can define the cycles (t^ℓ,x^ℓ,v^ℓ) and $X_{\mathbf{cl}}(s;t,x,v)=X(s;t^\ell,x^\ell,v^\ell)$ and $V_{\mathbf{cl}}(s;t,x,v)=V(s;t^\ell,x^\ell,v^\ell)$ for $s\in[t^{\ell+1},t^\ell]$. The Duhamel formula of (1.22) along this trajectory is given by

$$\begin{aligned}f(t,x,v)&=e^{-t}f_0(X_{\mathbf{cl}}(0;t,x,v),V_{\mathbf{cl}}(0;t,x,v))\\&\quad+\int_0^t e^{-(t-s)}\int_{|u|\le N}f(s,X_{\mathbf{cl}}(s;t,x,v),u)\mathrm{d}u\,\mathrm{d}s.\end{aligned} \tag{1.23}$$

Plugging the Duhamel formula into the integrand $f(s, X(s;t,x,v),u)$, we get

$$\begin{aligned} f(t,x,v) = &\int_0^t e^{-(t-s)} \int_0^{s-\varepsilon} e^{-(s-s')} \\ &\times \iint_{|u|\le N, |u'|\le N} f(s', X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),u),u') \mathrm{d}u' \,\mathrm{d}u \,\mathrm{d}s' \,\mathrm{d}s \\ &+ \text{initial datum's contributions} + O(\varepsilon). \end{aligned} \tag{1.24}$$

Throughout this paper, we use $O_a(A)$ for some function that depends on a and is size of A.

In the absence of a boundary and an external potential, the trajectory $X(s;t,x,v)$ is a straight line, and we can explicitly compute the Jacobian of

$$u \mapsto X(s';s,X(s;t,x,v),u),$$

which has a positive lower bound away from a small set of s. Therefore we obtain, via a change of variables,

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^p} + \text{data} + \text{small terms}. \tag{1.25}$$

Unfortunately, trajectories are very complicated when the specular reflection BC is imposed. In fact, in the case of the specular reflection BC, such a lower bound of Jacobian is only known when the domain is convex and *real analytic* in the absence of an external potential [12].

The main contribution of this paper is to establish an L^p-L^∞ bootstrap estimate as (1.25), when the domain is smooth and convex and the external potential is $C^{2,\gamma}$ and small in C^2. For the readers' convenience, we write a rough version of this result:

A ROUGH VERSION OF THEOREM 3.9. *Applying the Duhamel formula once again to* (1.24) (*triple iterations*)*, we have*

$$\begin{aligned} &f(t,x,v) \\ &= \int_0^t e^{-(t-s)} \int_0^s e^{-(s-s')} \int_0^{s'-\varepsilon} e^{-(s'-s'')} \iiint_{|u|\le N, |u_1|\le N, |u''|\le N} \\ &\quad \times f(s'', X_{\mathbf{cl}}(s'';s',X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),u),u'),u'') \mathrm{d}u'' \,\mathrm{d}u' \,\mathrm{d}u \,\mathrm{d}s'' \,\mathrm{d}s' \,\mathrm{d}s \\ &\quad + \text{initial datum's contributions} + O(\varepsilon). \end{aligned} \tag{1.26}$$

Let $(\widehat{u}_1, \widehat{u}_2)$ *and* $(\widehat{u}'_1, \widehat{u}'_2)$ *be the spherical coordinate of* $\widehat{u} = u/|u| \in \mathbb{S}^2$ *and* $\widehat{u}' = u'/|u'| \in \mathbb{S}^2$*, respectively. Then, if* s' *and* s'' *are away from some local* $C^{0,\gamma}$*-functions, then locally we can choose two distinct variables* $\{\zeta_1, \zeta_2\}$ *among* $\{|u|, \widehat{u}_1, \widehat{u}'_1, \widehat{u}'_2\}$ *such that*

$$\left| \det\left(\frac{\partial X_{\mathbf{cl}}(s'';s',X_{\mathbf{cl}}(s';s,X_{\mathbf{cl}}(s;t,x,v),u),u')}{\partial(|u'|,\zeta_1,\zeta_2)} \right) \right| \tag{1.27}$$

has a positive lower bound.

As a consequence we achieve (1.25).

We remark that the regularity of such $C^{0,\gamma}$-functions is determined and restricted crucially by the regularity of the external potential $\Phi \in C^{2,\gamma}$. Moreover, this $C^{0,\gamma}$-regularity is a (minimal) condition to guarantee that we can construct *small* ε-neighborhooods of the graph of them.

There are several key ingredients in the proof of Theorem 3.9:

Specular Basis and Geometric Decomposition. Assume that $t^{\ell+1} < s' < t^{\ell}$ and hence $X_{\mathbf{cl}}(s'; s, X_{\mathbf{cl}}(s; t, x, v), u)$ is in between ℓ-bounce and $(\ell+1)$-bounce. Then we know that

$$\partial_{|u|} X_{\mathbf{cl}}(s'; s, X_{\mathbf{cl}}(s; t, x, v), u) = v^{\ell}/|v^{\ell}| + O(\|\Phi\|_{C^2}). \tag{1.28}$$

On the other hand, for $\widehat{u} = (\widehat{u}_1, \widehat{u}_2) \in \mathbb{S}^2$, we have

$$\begin{aligned} \nabla_{\widehat{u}} X_{\mathbf{cl}}(s'; s, X_{\mathbf{cl}}(s; t, x, v), u) = \nabla_{\widehat{u}} x^{\ell} - (t^{\ell} - s')\nabla_{\widehat{u}} v^{\ell} \\ - \nabla_{\widehat{u}} t^{\ell} v^{\ell} + O(\|\Phi\|_{C^2}). \end{aligned} \tag{1.29}$$

Among other terms, $\partial_{\widehat{u}} t^{\ell}$ is the most delicate term to control since t^{ℓ} depends on all the cycles (x^l, v^l) for $l = 1, 2, \ldots, \ell - 1$. Fortunately, this harmful term appears only in the direction of $v^{\ell}/|v^{\ell}|$! Inspired by this observation we define the *specular basis* $\{\mathbf{e}_0^{\ell}, \mathbf{e}_{\perp,1}^{\ell}, \mathbf{e}_{\perp,2}^{\ell}\}$, which is an orthonormal basis with $\mathbf{e}_0^{\ell} = v^{\ell}/|v^{\ell}|$ and $\mathbf{e}_{\perp,i}^{\ell}$ perpendicular to $\mathbf{e}_0^{\ell}$. See (3.14).

Now we *decompose* $\nabla_{|u|,\widehat{u}_1,\widehat{u}_2} X_{\mathbf{cl}}(s'; s, X_{\mathbf{cl}}(s; t, x, v), u)$ into

$$\nabla X_{\mathbf{cl}} = \big(\nabla X_{\mathbf{cl}}\big)_{\parallel} + \big(\nabla X_{\mathbf{cl}}\big)_{\perp} := \big(\nabla X_{\mathbf{cl}} \cdot \mathbf{e}_0^{\ell}\big)\mathbf{e}_0^{\ell} + \nabla X_{\mathbf{cl}} - \big(\nabla X_{\mathbf{cl}}\big)_{\parallel}. \tag{1.30}$$

Then we have the following similarity relations, from (1.28) and (1.29):

$$\begin{aligned} \frac{\partial X_{\mathbf{cl}}}{\partial(|u|, \widehat{u})} &\sim \begin{pmatrix} \big(\frac{\partial X_{\mathbf{cl}}}{\partial(|u|,\widehat{u})}\big)_{\parallel} \\ \big(\frac{\partial X_{\mathbf{cl}}}{\partial(|u|,\widehat{u})}\big)_{\perp} \end{pmatrix} \\ &\sim \left(\begin{array}{c|c} -(s-s') & * - |v^{\ell}|\nabla_{\widehat{u}_1,\widehat{u}_2} t^{\ell} \\ \hline \mathbf{0}_{2\times 1} & \begin{matrix} \big(\nabla_{\widehat{u}_1,\widehat{u}_2} x^{\ell} - (t^{\ell} - s')\nabla_{\widehat{u}_1,\widehat{u}_2} v^{\ell}\big) \cdot \mathbf{e}_{\perp,1}^{\ell} \\ \big(\nabla_{\widehat{u}_1,\widehat{u}_2} x^{\ell} - (t^{\ell} - s')\nabla_{\widehat{u}_1,\widehat{u}_2} v^{\ell}\big) \cdot \mathbf{e}_{\perp,2}^{\ell} \end{matrix} \end{array}\right) \\ &\quad + O_{\Omega}(\|\Phi\|_{C^2}). \end{aligned} \tag{1.31}$$

See (3.32) for the precise form. Note that an upper right block containing $\partial_{\widehat{u}} t^{\ell}$ would have small contribution in the determinant of the full matrix since the lower left block is a zero matrix.

Due to this *geometric decomposition*, we are able to relate $\partial X_{\mathbf{cl}}/\partial(|u|, \widehat{u})$ to the mapping

$$(|u|, \widehat{u}_1, \widehat{u}_2) \;\mapsto\; (x^{\ell}, v^{\ell}). \tag{1.32}$$

Note that the map (1.32) is closely related to the billiard map [3], which turns out to be more "controllable" than $\partial X_{\mathbf{cl}}/\partial(|u|, \widehat{u})$. Moreover, the form of the first column

of (1.31) clearly guarantees that this Jacobian matrix is at least rank 1 for a small $\|\Phi\|_{C^2}$.

Diffeomorphism and Specular Matrix. By the chain rule, we can view (1.32) as the compositions of

$$(|u|, \widehat{u}_1, \widehat{u}_2) \mapsto (x^1, v^1) \mapsto (x^2, v^2) \mapsto \cdots \mapsto (x^\ell, v^\ell). \tag{1.33}$$

In the absence of external potentials, the map $(x^l, v^l) \mapsto (x^{l+1}, v^{l+1})$ is the billiard table and it is well-known that this map is diffeomorphic [3].

The quantitative study of such a map, especially in 3D domains, is performed recently in the work [14] by the first author with his other collaborators when the trajectories are very close to the boundary (grazing trajectories) in the absence of external potentials. However, these estimates cannot be sufficient for our purpose since it only can provide the information for the grazing trajectories. Moreover, the proof of [14] heavily relies on the fact that the ODE of the trajectory is autonomous. In the presence of a time-dependent external potential, however, the ODE of $(X_{\mathbf{cl}}, V_{\mathbf{cl}})$ becomes nonautonomous, which obstructs generalizing the result of [14] to the time-dependent external potential case.

We are able to overcome this difficulty by a new advance of our understanding to the derivatives of trajectories $(X_{\mathbf{cl}}, V_{\mathbf{cl}})$. In this paper, we succeed in performing the (almost) explicit computations of the Jacobian matrix of a (1.33) in the presence of a small time-dependent external potential. This also allows us to understand the role of the regularity of the external potential in (1.27). We expect that this technical improvement will allow us to generalize the work of [14].

Equipped with this quantitative estimate, we study the lower right 2×2 submatrix of (1.31). In order to use the diffeomorphism property of (1.33), we employ the *specular matrix* $\mathscr{R}$, which is a 4×4 full-rank matrix and essentially equals the Jacobian matrix of (1.33) expressed with the *specular basis*. The precise form can be computed as in (3.16), and the entries are $C^{0,\gamma}$ if the external potential is $C^{2,\gamma}$. Indeed, the lower right 2×2 submatrix of (1.31) can be written as in (3.32),

$$\begin{aligned} &\text{right upper } 2 \times 2 \text{ submatrix of } \mathscr{R} - (t^\ell - s') \\ &\times \text{right lower } 2 \times 2 \text{ submatrix of } \mathscr{R}. \end{aligned} \tag{1.34}$$

Since at least one entry of the right 4×2 submatrix of $\mathscr{R}$ should not be zero as a polynomial of s, we are able to show that $(|u|, \widehat{u}_1) \mapsto X$ is at least *rank 2* if s is away from some $C^{0,\gamma}$-function of (t, x, v) in Lemma 3.6.

Triple Iterations. Unfortunately, this *rank 2* is still not sufficient for our purpose. The key idea to overcome this difficulty is the *triple iterations* in (1.26), applying the Duhamel formula (1.23) once again to (1.24). One more iteration makes the game more feasible since now we have more free parameters to play with: $\{|u|, \widehat{u}_1, |u'|, \widehat{u}_1', \widehat{u}_2'\} \in \mathbb{R}^5$. Due to the observation (1.28), we need to choose $|u'|$ and two other free parameters $\{\zeta_1, \zeta_2\}$ so that the following map is rank 3,

$$(|u'|, \zeta_1, \zeta_2) \mapsto X(s''; s', X(s'; s, X(s; t, x, v), u), u'). \tag{1.35}$$

We use the full structure of the *specular matrix* and carefully study the quadratic polynomial (Lemma 3.5) to achieve a positive lower bound of the Jacobian of (1.35) in Lemma 3.7. The convexity of the domain (1.15) is used crucially to control the number of bounces in Lemma 3.8.

1.2 L^p-Bounds

Now we illustrate the L^p control of the Boltzmann solution. Due to the L^p-L^∞ bootstrap estimate (1.25), such L^p estimates would provide L^∞ control.

L^1-bound in the case of a time-dependent potential. In order to show the stability of μ_E in the presence of time-dependent potential ϕ, we utilize the following bound of [11, 15].

LEMMA 1.3.

$$\begin{aligned}|F-\mu_E|\mathbf{1}_{|F-\mu_E|\geq\bar{\delta}\mu_E}&\\ \leq \frac{4}{\bar{\delta}}\Big\{&(F\ln F-\mu_E\ln\mu_E)-(F-\mu_E)\\ &+\left(\frac{|v|^2}{2}+\Phi(x)\right)(F-\mu_E)\Big\}.\end{aligned}$$

Applying the L^p-L^∞ bootstrap argument via the triple iteration, the L^∞-norm of the solution is mainly bounded by the L^1-norm of $|F-\mu_E|\mathbf{1}_{|F-\mu_E|\geq\bar{\delta}\mu_E}$. By Lemma 1.3, we further bound it by the differences in the entropy, total mass, total energy of the solution, and μ_E. A new difficulty in the presence of the time-dependent potential $\phi(t,x)$ is that the total energy is not preserved anymore (1.5). Via Gronwall's inequality, we are able to prove that $\|wf(t)\|_\infty$ can grow in time at most as $e^{C(\|\phi\|_\infty+\|wf\|_\infty)t}$. Using the decay of potential and f, we can prove that the total energy is close to the initial total energy for all time. This weighted L^∞-bound is sufficient to prove the existence, uniqueness, and the stability of μ_E in Theorem 1.1.

L^2-decay in the case of a time-independent potential. It is well-known [2] that the linear operator L is only semipositive,

$$\int_{\mathbb{R}^3} Lff\,\mathrm{d}v \geq \delta_L\|\sqrt{\nu}(\mathbf{I}-\mathbf{P})f\|^2_{L^2(\mathbb{R}^3)}, \tag{1.36}$$

where $\|\cdot\|_\nu=\|\nu^{1/2}\cdot\|_{L^2}$. The null space of L is a five-dimensional subspace of $L^2(\mathbb{R}^3)$ spanned by $\{\sqrt{\mu_E}, v\sqrt{\mu_E}, |v|^2\sqrt{\mu_E}\}$ and the projection of f onto such null space is denoted by

$$\mathbf{P}f(t,x,v):=\{a(t,x)+v\cdot b(t,x)+|v|^2c(t,x)\}\sqrt{\mu_E}. \tag{1.37}$$

Due to this missing term in the lower bound of (1.36), the Boltzmann equation is degenerated dissipative. In order to prove L^2-decay, we need a coercivity estimate. Following the argument of [12, 16] we first consider

$$\partial_t f+v\cdot\nabla_x f-\nabla_x\Phi(x)\cdot\nabla_v f+e^{-\Phi(x)}Lf=0. \tag{1.38}$$

PROPOSITION 1.4. *Let* $\Phi(x) \in C^1$. *Assume that* f *solves* (1.38) *and satisfies the specular reflection BC and* (1.4)–(1.5) *with* $\phi \equiv 0$ *for* $F = \mu_E + \sqrt{\mu_E} f$. *Furthermore, for an axis-symmetric domain* (1.7) *with a degenerate potential* (1.8), *we assume* (1.9). *Then there exists* $C > 0$ *such that, for all* $N \in \mathbb{N}$,

$$\int_N^{N+1} \|\mathbf{P} f(t)\|_2^2 \, \mathrm{d}t \le C \int_N^{N+1} \|(\mathbf{I} - \mathbf{P}) f(t)\|_\nu^2 \, \mathrm{d}t, \tag{1.39}$$

where $\mathbf{P} f$ *is defined in* (1.37).

We remark that we do not need any smallness of Φ in this linear theorem. A direct consequence of (1.39) is an exponential decay in time of $\|f(t)\|_{L^2(\Omega\times\mathbb{R}^3)}$. Then following the argument of [12], we are able to show an exponential decay in time of $\|wf(t)\|_{L^\infty(\Omega\times\mathbb{R}^3)}$.

The proof of this proposition is based on the *contradiction argument* of [12, 16]. As a consequence, we do not have any quantitative estimates of C in (1.39) and the decay rate. By negating the coercivity of (1.39) and some normalization of (5.15), we obtain a weakly convergent sequence Z^m whose component orthogonal to the null space of L is vanishing as $m \to \infty$. The weak limit Z satisfies the conservation laws as (5.1)–(5.3) and the specular reflection BC (step 7 in the proof of Proposition 1.4) and

$$b(t, x) \cdot n(x) = 0 \quad \text{for almost all } x \in \partial\Omega. \tag{1.40}$$

Moreover, Z remains in the null space of L and solves the transport equation (5.24) without $e^{-\Phi} LZ$. As a consequence, the components a, b, and c of (1.37) solve the systems of [16]

$$\begin{aligned} \partial_i c &= 0, \\ \partial_t c + \partial_i b_i &= 0, \\ \partial_i b_j + \partial_j b_i &= 0, \qquad i \ne j. \\ \partial_t b_i + \partial_i a - 2c\partial_i \Phi &= 0, \\ \partial_t a - \nabla_x \Phi \cdot b &= 0, \end{aligned} \tag{1.41}$$

Unlike the case of $\Phi \equiv 0$ in [12], explicit forms of a, b, and c cannot be obtained. We use the boundary condition (1.40) and the conservation laws carefully and conclude that

$$Z(t, x, v) = 0 \quad \text{almost all } t, x, v. \tag{1.42}$$

On the other hand, due to the normalization (5.15), the L^2-norm of $\mathbf{P}Z^m$ is always 1 identically. Away from the boundary $\partial\Omega$, the weak convergence is actually strong convergence due to the velocity average lemma. For the shell-like subset of Ω, using the Duhamel form along the trajectory, we are able to bound the integration over this shell-like subset by the interior integration (Lemma 5.1). Therefore, $Z^m \to Z$ strongly and the L^2-norm of Z equals 1, which is a contradiction to (1.42).

2 Specular Trajectories with a Small Time-Dependent Potential

In (1.1), a time-dependent potential is given by $\Phi(x) + \phi(t, x)$. In this section, we write this potential as $\Phi(t, x)$ for convenience. The corresponding characteristic equation is

$$\begin{aligned} &\frac{d}{ds}X(s;t,x,v) = V(s;t,x,v),\\ &\frac{d}{ds}V(s;t,x,v) = -\nabla_x\Phi(s, X(s;t,x,v)). \end{aligned} \tag{2.1}$$

DEFINITION 2.1. We recall the standard notations from [14]. We define

$$\begin{aligned} t_{\mathbf{b}}(t,x,v) &:= \sup\{s \geq 0 : X(\tau;t,x,v) \in \Omega \text{ for all } \tau \in (t-s,t)\},\\ x_{\mathbf{b}}(t,x,v) &:= X(t - t_{\mathbf{b}}(t,x,v);t,x,v),\\ v_{\mathbf{b}}(t,x,v) &:= V(t - t_{\mathbf{b}}(t,x,v);t,x,v), \end{aligned} \tag{2.2}$$

and similarly,

$$\begin{aligned} t_{\mathbf{f}}(t,x,v) &:= \sup\left\{s \geq 0 : X(\tau;t,x,v) \in \Omega \text{ for all } \tau \in (t,t+s)\right\},\\ x_{\mathbf{f}}(t,x,v) &:= X(t + t_{\mathbf{f}}(t,x,v);t,x,v),\\ v_{\mathbf{f}}(t,x,v) &:= V(t + t_{\mathbf{f}}(t,x,v);t,x,v). \end{aligned} \tag{2.3}$$

Here, $t_{\mathbf{b}}$ and $t_{\mathbf{f}}$ are called the *backward exit time* and the *forward exit time*, respectively. We also define the specular cycle as in [14]. We set $(t^0, x^0, v^0) = (t, x, v)$. Inductively, we define

$$\begin{aligned} t^k &= t^{k-1} - t_{\mathbf{b}}(t^{k-1}, x^{k-1}, v^{k-1}),\\ x^k &= X(t^k; t^{k-1}, x^{k-1}, v^{k-1}),\\ v^k &= R_{x^k}V(t^k; t^{k-1}, x^{k-1}, v^{k-1}), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} R_{x^k}V(t^k; t^{k-1}, x^{k-1}, v^{k-1}) = V(t^k; t^{k-1}, x^{k-1}, v^{k-1})\\ - 2(n(x^k)\cdot V(t^k; t^{k-1}, x^{k-1}, v^{k-1}))n(x^k). \end{aligned}$$

We define the specular characteristics as

$$\begin{aligned} X_{\mathbf{cl}}(s;t,x,v) &= \sum_k \mathbf{1}_{s\in(t^{k+1},t^k]}X(s;t^k,x^k,v^k),\\ V_{\mathbf{cl}}(s;t,x,v) &= \sum_k \mathbf{1}_{s\in(t^{k+1},t^k]}V(s;t^k,x^k,v^k). \end{aligned} \tag{2.5}$$

For the sake of simplicity we abuse the notation of (2.5) by dropping the subscript **cl** in this section.

From the assumptions of Theorem 1.1 and Theorem 1.2, for any $p \in \partial\Omega$, there exists sufficiently small $\delta_1 > 0$ and $\delta_2 > 0$, and a 1-to-1 and onto C^3-function

$$\begin{aligned} \eta_p : \{\mathbf{x}_p \in \mathbb{R}^3 : \mathbf{x}_{p,3} < 0\} \cap B(0;\delta_1) &\to \Omega \cap B(p;\delta_2), \\ \mathbf{x}_p = (\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, \mathbf{x}_{p,3}) &\mapsto (x_1, x_2, x_3) \\ &= \eta_p(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, \mathbf{x}_{p,3}), \end{aligned} \tag{2.6}$$

and $\eta_p(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, \mathbf{x}_{p,3}) \in \partial\Omega$ if and only if $\mathbf{x}_{p,3} = 0$. We define the transformed velocity field at $\eta_p(\mathbf{x}_p)$ as

$$\mathbf{v}_i(\mathbf{x}_p) := \frac{\partial_i \eta_p(\mathbf{x}_p)}{\sqrt{g_{p,ii}(\mathbf{x}_p)}} \cdot v. \tag{2.7}$$

For any two-dimensional smooth manifold $\mathscr{S}$, we can find a local orthogonal parametrization from $\mathbb{R}^2$ to $\partial\mathscr{S}$. (See [5, cor. 2, p. 183], for example.) Therefore, we assume

$$\left\{ \frac{\partial_1 \eta_p}{\sqrt{g_{p,11}}}, \frac{\partial_2 \eta_p}{\sqrt{g_{p,22}}}, \frac{\partial_3 \eta_p}{\sqrt{g_{p,33}}} \right\} \text{ is orthonormal at } \mathbf{x}_{p,3} = 0, \tag{2.8}$$

where $g_{p,ij} := \langle \partial_i \eta_p, \partial_j \eta_p \rangle$.

For second derivative $\partial_i \partial_j \eta_p$, we define the Christoffel symbol $\Gamma^k_{p,ij}$ by

$$\partial_{ij} \eta_p = \sum_k \Gamma^k_{p,ij} \partial_k \eta_p. \tag{2.9}$$

Moreover, by reparametrization, we may assume that $g_{p,33}(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, \mathbf{x}_{p,3}) = 1$ whenever it is defined. Without loss of generality, the outward normal at the boundary is, for $x = \eta_p(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, 0) \in \partial\Omega$,

$$\begin{aligned} n(x) = n(\eta_p(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, 0)) &= \partial_3 \eta_p(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, 0) \\ &= \frac{\partial_1 \eta_p}{\sqrt{g_{p,11}}} \times \frac{\partial_2 \eta_p}{\sqrt{g_{p,22}}}\bigg|_{(\mathbf{x}_{p,1}, \mathbf{x}_{p,2}, 0)}. \end{aligned} \tag{2.10}$$

For each $k = 0, 1, 2, \ldots$, we assume that $p^k \in \partial\Omega$ is chosen to be close to x^k as in (2.6). Then we define

$$\begin{aligned} \mathbf{x}^k_{p^k} &:= (\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0) \text{ such that } x^k = \eta_{p^k}(\mathbf{x}^k_{p^k}), \\ \mathbf{v}^k_{p^k,i} &:= \mathbf{v}^k_i(\mathbf{x}^k_{p^k}) = \frac{\partial_i \eta_{p^k}(\mathbf{x}^k_{p^k})}{\sqrt{g_{p^k,ii}(\mathbf{x}^k_{p^k})}} \cdot v^k. \end{aligned} \tag{2.11}$$

Note that, due to (2.8), at the boundary,

$$v^k_i = \sum_{\ell=1}^{3} \mathbf{v}^k_{p^k,\ell} \frac{\partial_\ell \eta_{p^k,i}}{\sqrt{g_{p^k,\ell\ell}}}\bigg|_{x^k}. \tag{2.12}$$

LEMMA 2.2. *Assume that* Ω *and* Φ *are* C^2. *Consider* $(t^{k+1}, \mathbf{x}^{k+1}_{p^{k+1}}, \mathbf{v}^{k+1}_{p^{k+1}})$ *as a function of* $(t^k, \mathbf{x}^k_{p^k}, \mathbf{v}^k_{p^k})$. *Then for* $i, j = 1, 2$,

$$
\begin{aligned}
&\frac{\partial(t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,j}} \\
&\quad = \frac{-1}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \\
&\qquad \cdot \left[\partial_j \eta_{p^k}(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0) - (t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \right] \\
&\qquad + O_\Omega(\|\Phi\|_{C^2}) \frac{(t^k - t^{k+1})^2}{|\mathbf{v}^{k+1}_{p^{k+1},3}|} \big(1 + (t^k - t^{k+1})|\mathbf{v}^k_{p^k}|\big) e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2},
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},i}}{\partial \mathbf{x}^k_{p^k,j}} &= \frac{1}{\sqrt{g_{p^k,ii}(x^{k+1})}} \left[\frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^k,ii}(x^{k+1})}} + \frac{\mathbf{v}^{k+1}_{p^{k+1},i}}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \right] \\
&\qquad \cdot \left[\partial_j \eta_{p^k}(x^k) - (t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \right] \\
&\qquad + O_\Omega(\|\Phi\|_{C^2}) \left\{ 1 + \frac{|\mathbf{v}^{k+1}_{p^{k+1},i}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|} \right\} (t^k - t^{k+1})^2 \\
&\qquad\quad \times \big(1 + (t^k - t^{k+1})|\mathbf{v}^k_{p^k}|\big) e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2},
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
&\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},i}}{\partial \mathbf{x}^k_{p^k,j}} \\
&\quad = \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \cdot \frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}} \\
&\qquad + v^k \cdot \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{x}^k_{p^k,j}} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}} \left(\frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}} \right) \\
&\qquad + O_\Omega(\|\Phi\|_{C^2}) \left\{ \sum_j \left| \frac{\partial(t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,j}} \right| \right. \\
&\qquad\qquad + (t^k - t^{k+1})(1 + |\mathbf{v}^k_{p^k}|(t^k - t^{k+1})) e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2} \\
&\qquad\qquad \left. + \sum_\ell \left| \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{x}^k_{p^k,j}} \right| \right\},
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},3}}{\partial \mathbf{x}^{k}_{p^k,j}} \\
&\quad = -\frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \cdot \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \\
&\quad\quad - v^k \cdot \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{x}^k_{p^k,j}} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}} \left(\frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \right) \\
&\quad\quad + O_\Omega(\|\Phi\|_{C^2}) \Bigg\{ \sum_j \left| \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,j}} \right| \\
&\quad\quad + (t^k - t^{k+1})(1 + |\mathbf{v}^k_{p^k}|(t^k - t^{k+1})) e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2} + \sum_\ell \left| \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{x}^k_{p^k,j}} \right| \Bigg\},
\end{aligned} \tag{2.16}
$$

where

$$
\frac{\partial v^k_i}{\partial \mathbf{x}^k_{p^k,j}} = \sum_{\ell=1}^{3} \mathbf{v}^k_{p^k,\ell} \sum_{r(\neq \ell)} \frac{\sqrt{g_{p^k,rr}(x^k)}}{\sqrt{g_{p^k,\ell\ell}(x^k)}} \Gamma^r_{p^k,\ell j}(x^k) \frac{\partial_r \eta_{p^k,i}(x^k)}{\sqrt{g_{p^k,rr}(x^k)}}. \tag{2.17}
$$

For $i = 1, 2$ and $j = 1, 2, 3$,

$$
\begin{aligned}
\frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} &= \frac{(t^k - t^{k+1})}{\mathbf{v}^{k+1}_{p^{k+1},3}} \Bigg[\frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \\
&\qquad + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1})^2 e^{\|\nabla_x^2 \Phi\|_\infty |t^k - t^{k+1}|^2} \Bigg] \\
&\quad \cdot \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}},
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},i}}{\partial \mathbf{v}^k_{p^k,j}} &= -(t^k - t^{k+1}) \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \\
&\quad \cdot \frac{1}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}} \Bigg[\frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}} \\
&\qquad + \frac{\mathbf{v}^{k+1}_{p^{k+1},i}}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}}(x^{k+1})} \Bigg] \\
&\quad + O_\Omega(\|\Phi\|_{C^2}) \left(1 + \frac{|\mathbf{v}^{k+1}_{p^{k+1},i}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|} \right) (t^k - t^{k+1})^3 e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2},
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},i}}{\partial \mathbf{v}^{k}_{p^{k},j}} &= \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{v}^{k}_{p^{k},j}} \partial_\ell \left(\frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}} \right) \Bigg|_{x^{k+1}} \cdot v^k \\
&\quad + \frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \\
&\quad + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1})^2 \left(1 + \frac{|\mathbf{v}^{k+1}_{p^{k+1},i}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|} \right) \\
&\qquad \times \big(1 + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1})^2\big) e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2} \\
&\quad + O_\Omega(\|\Phi\|_{C^2}) \frac{|t^k - t^{k+1}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|} \\
&\qquad \times \big(1 + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1})^2 e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2}\big),
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},3}}{\partial \mathbf{v}^{k}_{p^{k},j}} &= -\sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{v}^{k}_{p^{k},j}} \partial_\ell \left(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}} \right) \Bigg|_{x^{k+1}} \cdot v^k \\
&\quad - \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \\
&\quad + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1})^2 \big(1 + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1})^2\big) \\
&\qquad \cdot e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2} \\
&\quad + O_\Omega(\|\Phi\|_{C^2}) \frac{|t^k - t^{k+1}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|} \big(1 + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1})^2 \\
&\qquad\qquad \cdot e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2}\big).
\end{aligned} \tag{2.21}
$$

Remark 2.3. Note that we do not need the convexity (1.15) or the smallness of the size of Φ in Lemma 2.2 .

PROOF OF LEMMA 2.2. First we prove (2.13). By the definitions (2.6), (2.4), and our setting (2.11) and (2.1),

$$
\begin{aligned}
&\eta_{p^{k+1}}\big(\mathbf{x}^{k+1}_{p^{k+1},1}, \mathbf{x}^{k+1}_{p^{k+1},2}, 0\big) \\
&\quad = \eta_{p^k}\big(\mathbf{x}^{k}_{p^{k},1}, \mathbf{x}^{k}_{p^{k},2}, 0\big) + \int_{t^k}^{t^{k+1}} v^k \\
&\qquad - \int_{t^k}^{t^{k+1}} \int_{t^k}^{s} \nabla\Phi(\tau, X(\tau; t^k, x^k, v^k)) \mathrm{d}\tau \, \mathrm{d}s.
\end{aligned} \tag{2.22}
$$

We apply $\frac{\partial}{\partial \mathbf{x}^k_{p^k,j}}$ to the above equality for $j = 1, 2$ to get

$$
\begin{aligned}
&\sum_{l=1,2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}{\partial \mathbf{x}^k_{p^k,j}} \frac{\partial \eta_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}\bigg|_{x^{k+1}} \\
&\quad = -(t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \\
&\quad\quad - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,j}} \left\{ v^k - \int_{t^k}^{t^{k+1}} \nabla \Phi(s, X(s; t^k, x^k, v^k)) \mathrm{d}s \right\} \\
&\quad\quad + \left\{ \partial_j \eta_{p^k}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big) - \int_{t^k}^{t^{k+1}} \mathrm{d}s \int_{t^k}^{s} \mathrm{d}\tau \left(\frac{\partial X(\tau)}{\partial \mathbf{x}^k_{p^k,j}} \cdot \nabla_x \right) (\nabla_x \Phi(\tau)) \right\},
\end{aligned}
\tag{2.23}
$$

and then take an inner product with $\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big|_{x^{k+1}}$ to have

$$
\begin{aligned}
&\sum_{l=1,2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}{\partial \mathbf{x}^k_{p^k,j}} \frac{\partial \eta_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}\bigg|_{x^{k+1}} \cdot \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} \\
&\quad = -(t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \cdot \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} \\
&\quad\quad - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,j}} \left\{ v^k - \int_{t^k}^{t^{k+1}} \nabla \Phi(s, X(s; t^k, x^k, v^k)) \mathrm{d}s \right\} \cdot \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} \\
&\quad\quad + \left\{ \partial_j \eta_{p^k}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big) - \int_{t^k}^{t^{k+1}} \mathrm{d}s \int_{t^k}^{s} \mathrm{d}\tau \left(\frac{\partial X(\tau)}{\partial \mathbf{x}^k_{p^k,j}} \cdot \nabla_x \right) \big(\nabla_x \Phi(\tau)\big) \right\} \\
&\quad\quad\quad \cdot \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}},
\end{aligned}
\tag{2.24}
$$

where we abbreviated $X(s) = X(s; t^k, x^k, v^k)$, $V(s) = V(s; t^k, x^k, v^k)$, and $\Phi(s) = \Phi(s, X(s; t^k, x^k, v^k))$. Due to (2.8) the left-hand side equals 0.

Now we consider the right-hand side. From (2.12), we prove (2.17). We also note that

$$
\lim_{s \downarrow t^{k+1}} V(s; t^k, x^k, v^k) = v^k - \int_{t^k}^{t^{k+1}} \nabla \Phi(s, X(s; t^k, x^k, v^k)) \mathrm{d}s. \tag{2.25}
$$

Therefore, from (2.4) and (2.11),

$$
\left\{ v^k - \int_{t^k}^{t^{k+1}} \nabla \Phi(s, X(s; t^k, x^k, v^k)) \mathrm{d}s \right\} \cdot \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} = -\mathbf{v}^{k+1}_{p^{k+1},3}.
$$

From (2.24),

$$
\begin{aligned}
&\frac{\partial(t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,j}} \\
&\quad = -\frac{1}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \\
&\qquad \cdot \Bigg[\partial_j \eta_{p^k}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big) \\
&\qquad\quad - (t^k - t^{k+1}) \sum_{\ell=1}^{3} \mathbf{v}^k_{p^k,\ell} \sum_{r(\neq \ell)} \frac{\sqrt{g_{p^k,rr}(x^k)}}{\sqrt{g_{p^k,\ell\ell}(x^k)}} \Gamma^r_{p^k,\ell j}(x^k) \frac{\partial_r \eta_{p^k}(x^k)}{\sqrt{g_{p^k,rr}(x^k)}} \Bigg] \\
&\qquad + \frac{1}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \cdot \int_{t^k}^{t^{k+1}} \mathrm{d}s \int_{t^k}^{s} \mathrm{d}\tau \bigg(\frac{\partial X(\tau)}{\partial \mathbf{x}^k_{p^k,j}} \cdot \nabla_x \bigg) (\nabla_x \Phi(\tau)).
\end{aligned} \tag{2.26}
$$

Now we consider integrand $\frac{\partial X(\tau;t^k,x^k,v^k)}{\partial \mathbf{x}^k_{p^k,j}}$. From (2.22) for $t^{k+1} < \tau \le t^k$,

$$
\begin{aligned}
X(\tau; t^k, x^k, v^k) &= \eta_{p^k}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big) + v^k(\tau - t^k) \\
&\quad - \int_{t^k}^{\tau} \mathrm{d}s \int_{t^k}^{s} \mathrm{d}s' \nabla \Phi(s', X(s'; t^k, x^k, v^k)).
\end{aligned} \tag{2.27}
$$

By the direct computations, for $j = 1, 2$,

$$
\begin{aligned}
\sup_{\tau \le s' \le t^k} \left| \frac{\partial X(s'; t^k, x^k, v^k)}{\partial \mathbf{x}^k_{p^k,j}} \right| &\le \left| \partial_j \eta_{p^k}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big) \right| + |\tau - t^k| \left| \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \right| \\
&\quad + \int_{t^k}^{\tau} |s - t^k| \|\Phi\|_{C^2} \sup_{\tau \le s' \le t^k} \left| \frac{\partial X(s'; t^k, x^k, v^k)}{\partial \mathbf{x}^k_{p^k,j}} \right| \mathrm{d}s.
\end{aligned}
$$

By Gronwall's inequality and (2.17),

$$
\sup_{\tau} \left| \frac{\partial X\big(\tau; t^k, \mathbf{x}^k_{p^k}, \mathbf{v}^k_{p^k}\big)}{\partial \mathbf{x}^k_{p^k,j}} \right| \le O_\Omega(1)\big(1 + |t^k - \tau| |\mathbf{v}^k_{p^k}|\big) e^{\|\Phi\|_{C^2} |t^k - \tau|^2/2}. \tag{2.28}
$$

Using (2.26) and (2.28), we complete the proof of (2.13).

To prove (2.14), we take the inner product with $\left.\frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\right|_{x^{k+1}}$ to (2.23) to obtain

$$
\begin{aligned}
&\sum_{l=1,2} \left.\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}{\partial \mathbf{x}^{k}_{p^k,j}} \frac{\partial \eta_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}\right|_{x^{k+1}} \cdot \left.\frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\right|_{x^{k+1}} = \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},i}}{\partial \mathbf{x}^{k}_{p^k,j}} \\
&\quad = -(t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{x}^{k}_{p^k,j}} \cdot \left.\frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\right|_{x^{k+1}} \\
&\quad\quad - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{x}^{k}_{p^k,j}} \left\{ v^k - \int_{t^k}^{t^{k+1}} \nabla \Phi(s, X(s; t^k, x^k, v^k)) \mathrm{d}s \right\} \cdot \left.\frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\right|_{x^{k+1}} \\
&\quad\quad + \left\{ \partial_j \eta_{p^k}\left(\mathbf{x}^{k}_{p^k,1}, \mathbf{x}^{k}_{p^k,2}, 0\right) \right. \\
&\quad\quad\quad \left. - \int_{t^k}^{t^{k+1}} \mathrm{d}s \int_{t^k}^{s} \mathrm{d}\tau \left(\frac{\partial X(\tau)}{\partial \mathbf{x}^{k}_{p^k,j}} \cdot \nabla_x \right) (\nabla_x \Phi(\tau)) \right\} \cdot \left.\frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\right|_{x^{k+1}}.
\end{aligned} \tag{2.29}
$$

Since

$$
\left\{ v^k - \int_{t^k}^{t^{k+1}} \nabla \Phi(s, X(s; t^k, x^k, v^k)) \mathrm{d}s \right\} \cdot \left.\frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\right|_{x^{k+1}} = -\frac{\mathbf{v}^{k+1}_{p^{k+1},i}}{\sqrt{g_{p^{k+1},ii}}},
$$

from (2.8) and (2.13),

$$
\begin{aligned}
\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},i}}{\partial \mathbf{x}^{k}_{p^k,j}} &= \frac{1}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \\
&\quad \cdot \left[\partial_j \eta_{p^k}(x^k) - (t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{x}^{k}_{p^k,j}} \right] \left.\frac{\mathbf{v}^{k+1}_{p^{k+1},i}}{\sqrt{g_{p^{k+1},ii}}}\right|_{x^{k+1}} \\
&\quad + \left.\frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\right|_{x^{k+1}} \cdot \left[\partial_j \eta_{p^k}(x^k) - (t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{x}^{k}_{p^k,j}} \right] \\
&\quad + O_\Omega(\|\Phi\|_{C^2}) \frac{\left|\mathbf{v}^{k+1}_{p^{k+1},i}\right|}{\left|\mathbf{v}^{k+1}_{p^{k+1},3}\right|} (t^k - t^{k+1})^2 \\
&\quad\quad \cdot \left(1 + (t^k - t^{k+1}) \left|\mathbf{v}^{k}_{p^k}\right|\right) e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2/2} \\
&\quad + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1})^2 \left(1 + (t^k - t^{k+1}) \left|\mathbf{v}^{k}_{p^k}\right|\right) \\
&\quad\quad \cdot e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2/2}.
\end{aligned} \tag{2.30}
$$

This ends the proof of (2.14).

Now we prove (2.15) and (2.16). From (2.4) and (2.11),

$$
\begin{aligned}
\mathbf{v}^{k+1}_{p^{k+1},i} &= \frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}}\bigg|_{x^{k+1}} \cdot \lim_{s\downarrow t^{k+1}} V(s;t^k,x^k,v^k) \quad \text{for } i=1,2,\\
\mathbf{v}^{k+1}_{p^{k+1},3} &= -\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} \cdot \lim_{s\downarrow t^{k+1}} V(s;t^k,x^k,v^k).
\end{aligned} \tag{2.31}
$$

For $i, j = 1, 2$, from (2.31),

$$
\begin{aligned}
\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},i}}{\partial \mathbf{x}^k_{p^k,j}} &= \frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}}\\
&\quad \cdot \bigg[\frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} + \nabla_x \Phi(t^{k+1};t^k,x^k,v^k) \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,j}}\\
&\quad - \int_{t^k}^{t^{k+1}} \big(\partial_{\mathbf{x}^k_{p^k,j}} X(s)\cdot \nabla_x\big) \nabla_x \Phi(s, X(s;t^k,x^k,v^k))\mathrm{d}s \bigg]\\
&\quad + \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{x}^k_{p^k,j}} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}} \bigg(\frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}} \bigg)\bigg|_{x^{k+1}} \cdot \lim_{s\downarrow t^{k+1}} V(s;t^k,x^k,v^k).
\end{aligned}
$$

And for $j = 1, 2$,

$$
\begin{aligned}
\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},3}}{\partial \mathbf{x}^k_{p^k,j}} &= -\frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}}\\
&\quad \cdot \bigg[\frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} + \nabla_x \Phi(t^{k+1};t^k,x^k,v^k) \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{x}^k_{p^k,j}}\\
&\quad - \int_{t^k}^{t^{k+1}} \big(\partial_{\mathbf{x}^k_{p^k,j}} X(s)\cdot \nabla_x\big) \nabla_x \Phi(s, X(s;t^k,x^k,v^k))\mathrm{d}s \bigg]\\
&\quad - \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{x}^k_{p^k,j}} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}} \bigg(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}} \bigg)\bigg|_{x^{k+1}} \cdot \lim_{s\downarrow t^{k+1}} V(s;t^k,x^k,v^k).
\end{aligned}
$$

From (2.25) and (2.28), we prove (2.15) and (2.16).

Now we consider (2.18)–(2.21) for v-derivatives.

To prove (2.18), we take $\frac{\partial}{\partial \mathbf{v}^k_{p^k,j}}$ to (2.22) for $j = 1, 2, 3$ to get

$$
\begin{aligned}
&\sum_{l=1,2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}{\partial \mathbf{v}^k_{p^k,j}} \frac{\partial \eta_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}\bigg|_{x^{k+1}} \\
&\quad = -(t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{v}^k_{p^k,j}} \\
&\quad\quad - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} \Big\{ v^k - \int_{t^k}^{t^{k+1}} \nabla \Phi(s, X(s; t^k, x^k, v^k)) \mathrm{d}s \Big\} \\
&\quad\quad + \Big\{ - \int_{t^k}^{t^{k+1}} \mathrm{d}s \int_{t^k}^{s} \mathrm{d}\tau \Big(\frac{\partial X(\tau)}{\partial \mathbf{v}^k_{p^k,j}} \cdot \nabla_x \Big) \big(\nabla_x \Phi(\tau) \big) \Big\},
\end{aligned} \tag{2.32}
$$

and then take an inner product with $\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big|_{x^{k+1}}$ to have

$$
\begin{aligned}
&\sum_{l=1,2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}{\partial \mathbf{v}^k_{p^k,j}} \frac{\partial \eta_{p^{k+1}}}{\partial \mathbf{x}^{k+1}_{p^{k+1},l}}\bigg|_{x^{k+1}} \cdot \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} \\
&\quad = \Big\{ -(t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{v}^k_{p^k,j}} - \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} \lim_{s \downarrow t^{k+1}} V(s; t^k, x^k, v^k) \Big\} \\
&\quad\quad \cdot \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} + O_\Omega(\|\Phi\|_{C^2}) |t^k - t^{k+1}|^2 \sup_s \Big| \frac{\partial X(s)}{\partial \mathbf{v}^k_{p^k,j}} \Big|.
\end{aligned} \tag{2.33}
$$

Due to (2.8), the left-hand side equals 0. Now we consider the right-hand side. From (2.12),

$$
\frac{\partial v^k}{\partial \mathbf{v}^k_{p^k,j}} = \frac{\partial_j \eta_{p^k}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big)}{\sqrt{g_{p^k,jj}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big)}}. \tag{2.34}
$$

Now we consider $\sup_s \big| \frac{X(s;t^k,x^k,v^k)}{\partial \mathbf{v}^k_{p^k,j}} \big|$. From (2.27), for $j = 1, 2, 3$,

$$
\Big| \frac{\partial X(s)}{\partial \mathbf{v}^k_{p^k,j}} \Big| \le |t^k - s| \Big| \frac{\partial v^k}{\partial \mathbf{v}^k_{p^k,j}} \Big| + \|\nabla_x^2 \Phi\|_\infty \int_s^{t^k} |t^k - \tau| \Big| \frac{\partial X(\tau)}{\partial \mathbf{v}^k_{p^k,j}} \Big| \mathrm{d}\tau.
$$

By Gronwall's inequality and (2.34), for $t^{k+1} \le s \le t^k$,

$$
\begin{aligned}
\Big| \frac{\partial X(s; t^k, x^k, v^k)}{\partial \mathbf{v}^k_{p^k,j}} \Big| &\le |t^k - s| \Big| \frac{\partial v^k}{\partial \mathbf{v}^k_{p^k,j}} \Big| e^{\|\Phi\|_{C^2} |t^k - s|^2/2} \\
&\lesssim_\Omega |t^k - s| e^{\|\Phi\|_{C^2} |t^k - s|^2/2}.
\end{aligned} \tag{2.35}
$$

Using (2.31), (2.33), (2.34), and (2.35), we prove (2.18).

The estimate (2.19) is obtained by a similar method. For $i = 1, 2$ and $j = 1, 2, 3$, we take the inner product with $\frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\Big|_{x^{k+1}}$ to (2.32) to obtain

$$\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},i}}{\partial \mathbf{v}^k_{p^k,j}} = \left\{ -\frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} \lim_{s \downarrow t^{k+1}} V(s; t^k, x^k, v^k) - (t^k - t^{k+1}) \frac{\partial v^k}{\partial \mathbf{v}^k_{p^k,j}} \right\} \cdot \frac{\partial_i \eta_{p^{k+1}}}{g_{p^{k+1},ii}}\Big|_{x^{k+1}} + O_\Omega(\|\Phi\|_{C^2}) |t^k - t^{k+1}|^2 \sup_s \left| \frac{\partial X(s)}{\partial \mathbf{v}^k_{p^k,j}} \right|.$$

From (2.34), (2.35), and (2.18), we prove (2.19).

Now, let us prove (2.20) and (2.21). For $i = 1, 2$ and $j = 1, 2, 3$, from (2.31),

$$\begin{aligned}
\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},i}}{\partial \mathbf{v}^k_{p^k,j}} &= \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{v}^k_{p^k,j}} \partial_\ell \left(\frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}} \right)\Big|_{x^{k+1}} \cdot \lim_{s \downarrow t^{k+1}} V(s; t^k, x^k, v^k) \\
&\quad + \frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}} \cdot \Bigg(\frac{\partial v^k}{\partial \mathbf{v}^k_{p^k,j}} + \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} \nabla \Phi(t^{k+1}, x^{k+1}) \\
&\qquad - \int_{t^k}^{t^{k+1}} \left(\frac{\partial X(s)}{\partial \mathbf{v}^k_{p^k,j}} \cdot \nabla \right) \nabla \Phi(s, X(s)) \mathrm{d}s \Bigg) \\
&= \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{v}^k_{p^k,j}} \partial_\ell \left(\frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}} \right)\Big|_{x^{k+1}} \cdot v^k \\
&\quad + \frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \\
&\quad + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1}) \left| \frac{\partial \mathbf{x}^{k+1}_{p^{k+1}}}{\partial \mathbf{v}^k_{p^k,j}} \right| + O_\Omega(\|\Phi\|_{C^2}) \left| \frac{\partial (t^k - t^{k+1})}{\partial \mathbf{v}^k_{p^k,j}} \right| \\
&\quad + O_\Omega(\|\Phi\|_{C^2})(t^k - t^{k+1}) \sup_s \left| \frac{\partial X(s)}{\partial \mathbf{v}^k_{p^k,j}} \right|.
\end{aligned}$$

From (2.18), (2.19), and (2.35), we prove (2.20). The proof of (2.21) is also very similar to the above from (2.31). □

LEMMA 2.4. *Assume that $x \in \Omega$ (interior point) and $x_{\mathbf{b}}(t,x,v)$ is in the neighborhood of $p^1 \in \partial\Omega$. Then locally,*

$$\begin{aligned}\frac{\partial t_{\mathbf{b}}}{\partial x_j} &= \frac{1}{\mathbf{v}^1_{p^1,3}}\big[-e_j + O_\Omega(\|\Phi\|_{C^2})|t_{\mathbf{b}}|^2 e^{\|\nabla_x^2\Phi\|_\infty (t_{\mathbf{b}})^2/2}\big] \\ &\quad\cdot \frac{\partial_3 \eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}}, \quad j = 1,2,\end{aligned} \tag{2.36}$$

$$\begin{aligned}\frac{\partial t_{\mathbf{b}}}{\partial v_j} &= \frac{1}{\mathbf{v}^1_{p^1,3}}\left[t_{\mathbf{b}} e_j - \int_t^{t-t_{\mathbf{b}}}\int_t^s \left(\frac{\partial X(\tau)}{\partial v_j}\cdot\nabla_x\right)\big(\nabla_x\Phi(\tau)\big)\mathrm{d}\tau\mathrm{d}s\right] \\ &\quad\cdot \frac{\partial_3 \eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}} \\ &= \frac{t_{\mathbf{b}}}{\mathbf{v}^1_{p^1,3}}\big[e_j + O_\Omega(\|\Phi\|_{C^2})|t_{\mathbf{b}}|^3 e^{\|\Phi\|_{C^2}(t_{\mathbf{b}})^2/2}\big] \\ &\quad\cdot \frac{\partial_3 \eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}}, \quad j = 1,2,3,\end{aligned} \tag{2.37}$$

$$\begin{aligned}\frac{\partial \mathbf{x}^1_{p^1,i}}{\partial x_j} &= \left[e_j + O_\Omega(\|\Phi\|_{C^2})\left(1 + \frac{\big|\mathbf{v}^1_{p^1,i}\big|}{\big|\mathbf{v}^1_{p^1,3}\big|}\right)|t_{\mathbf{b}}|^2 e^{\|\nabla_x^2\Phi\|_\infty (t_{\mathbf{b}})^2/2}\right] \\ &\quad\cdot \frac{1}{\sqrt{g_{p^1,ii}(x^1)}}\left[\frac{\partial_i \eta_{p^1}(x^1)}{\sqrt{g_{p^1,ii}(x^1)}} + \frac{\mathbf{v}^1_{p^1,i}}{\mathbf{v}_{p^1,3}}\frac{\partial_3 \eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}}\right],\end{aligned} \tag{2.38}$$

$$\begin{aligned}\frac{\partial \mathbf{x}^1_{p^1,i}}{\partial v_j} &= \left[-t_{\mathbf{b}} e_j - \int_t^{t-t_{\mathbf{b}}}\int_t^s \left(\frac{\partial X(\tau)}{\partial v_j}\cdot\nabla_x\right)\nabla\Phi_x(\tau)\mathrm{d}\tau\mathrm{d}s\right] \\ &\quad\cdot \frac{1}{\sqrt{g_{p^1,ii}(x^1)}}\left[\frac{\partial_i \eta_{p^1}(x^1)}{\sqrt{g_{p^1,ii}(x^1)}} + \frac{\mathbf{v}^1_{p^1,i}}{\mathbf{v}^1_{p^1,3}}\frac{\partial_3 \eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}}\right] \\ &= -t_{\mathbf{b}}\big[e_j + O_\Omega(\|\Phi\|_{C^2})|t_{\mathbf{b}}|^2 e^{\|\nabla_x^2\Phi\|_\infty (t_{\mathbf{b}})^2/2}\big] \\ &\quad\cdot \frac{1}{\sqrt{g_{p^1,ii}(x^1)}}\left[\frac{\partial_i \eta_{p^1}(x^1)}{\sqrt{g_{p^1,ii}(x^1)}} + \frac{\mathbf{v}^1_{p^1,i}}{\mathbf{v}^1_{p^1,3}}\frac{\partial_3 \eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}}\right],\end{aligned} \tag{2.39}$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}^1_{p^1,i}}{\partial x_j} &= -\frac{\partial_i \eta_{p^1}(x^1)}{\sqrt{g_{p^1,ii}(x^1)}} \cdot \left[\int_t^{t-t_{\mathbf{b}}} \left(\frac{\partial X(s)}{\partial x_j} \cdot \nabla_x \right) (\nabla_x \Phi(s)) \mathrm{d}s \right] \\
&\quad + \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^1_{p^1,\ell}}{\partial x_j} \partial_\ell \left(\frac{\partial_i \eta_{p^1}}{\sqrt{g_{p^1,ii}}} \right) \Big|_{x^1} \cdot V(t - t_{\mathbf{b}}) \\
&= \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^1_{p^1,\ell}}{\partial x_j} \partial_\ell \left(\frac{\partial_i \eta_{p^1}}{\sqrt{g_{p^1,ii}}} \right) \Big|_{x^1} \cdot v \\
&\quad + O_\Omega(\|\Phi\|_{C^2}) t_{\mathbf{b}} (1 + |v| t_{\mathbf{b}}) e^{\|\nabla_x^2 \Phi\|_\infty (t_{\mathbf{b}})^2/2} \\
&\quad + \|\nabla_x \Phi\|_\infty |t_{\mathbf{b}}| \left(1 + \frac{|\mathbf{v}^1_{p^1,i}|}{|\mathbf{v}^1_{p^1,3}|} \right) \\
&\qquad \cdot \left(1 + O_\Omega(\|\Phi\|_{C^2})(1 + t_{\mathbf{b}}|v|)(t_{\mathbf{b}})^2 e^{\|\nabla_x^2 \Phi\|_\infty (t_{\mathbf{b}})^2/2} \right),
\end{aligned} \tag{2.40}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}^1_{p^1,i}}{\partial v_j} &= \frac{\partial_i \eta_{p^1}(x^1)}{\sqrt{g_{p^1,ii}(x^1)}} \cdot e_j + \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^1_{p^1,\ell}}{\partial v_j} \partial_\ell \left(\frac{\partial_i \eta_{p^1}}{\sqrt{g_{p^1,ii}}} \right) \Big|_{x^1} \cdot V(t - t_{\mathbf{b}}) \\
&\quad + \frac{\partial_i \eta_{p^1}(x^1)}{\sqrt{g_{p^1,ii}(x_1)}} \cdot \Big[-\nabla_x \Phi(t - t_{\mathbf{b}}; X(t - t_{\mathbf{b}}; t, x, v)) \frac{\partial t_{\mathbf{b}}}{\partial v_j} \\
&\qquad - \int_t^{t-t_{\mathbf{b}}} \left(\frac{\partial X(s)}{\partial v_j} \cdot \nabla_x \right) \nabla_x \Phi(s) \mathrm{d}s \Big] \\
&= \frac{\partial_i \eta_{p^1}(x^1)}{\sqrt{g_{p^1,ii}(x^1)}} \cdot e_j + \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^1_{p^1,\ell}}{\partial v_j} \partial_\ell \left(\frac{\partial_i \eta_{p^1}}{\sqrt{g_{p^1,ii}}} \right) \Big|_{x^1} \cdot v \\
&\quad + O_\Omega(\|\Phi\|_{C^2}) \left(1 + \frac{|\mathbf{v}^1_{p^1}|}{|\mathbf{v}^1_{p^1,3}|} \right) |t_{\mathbf{b}}| \\
&\qquad \cdot \left(1 + O_\Omega(\|\Phi\|_{C^2})(t_{\mathbf{b}})^2 e^{\|\nabla_x^2 \Phi\|_\infty (t_{\mathbf{b}})^2/2} \right).
\end{aligned} \tag{2.41}
$$

Here, e_j is the j^{th} directional unit vector in $\mathbb{R}^3$. Moreover,

$$
\begin{aligned}
\frac{\partial |\mathbf{v}^1_{p^1}|}{\partial x_j} &= O_\Omega(\|\nabla_x \Phi\|_\infty) \frac{1 + O_\Omega(\|\nabla_x^2 \Phi\|_\infty)(1 + t_{\mathbf{b}}|v|) |t_{\mathbf{b}}|^2 e^{\|\nabla_x^2 \Phi\|_\infty (t_{\mathbf{b}})^2}}{|\mathbf{v}^1_{p^1,3}|} \\
&\quad + O_\Omega(\|\Phi\|_{C^2}) t_{\mathbf{b}} e^{\|\nabla_x^2 \Phi\|_\infty (t_{\mathbf{b}})^2/2},
\end{aligned} \tag{2.42}
$$

$$\begin{aligned}\frac{\partial|\mathbf{v}^1_{p^1}|}{\partial v_j} &= \lim_{s\downarrow t^1}\frac{V_j(s;t,x,v)}{|V(s;t,x,v)|} + O_\Omega(\|\Phi\|_{C^2})(t_{\mathbf{b}})^2 e^{\|\nabla_x^2\Phi\|_\infty (t_{\mathbf{b}})^2/2}\\ &\quad + O_\Omega(\|\nabla_x\Phi\|_\infty)\frac{t_{\mathbf{b}}}{|\mathbf{v}^1_{p^1,3}|}\big\{1 + O_\Omega\big(\|\nabla_x^2\eta\|_\infty\big)|t_{\mathbf{b}}|^3 e^{\|\nabla_x^2\Phi\|_\infty (t_{\mathbf{b}})^2}\big\}.\end{aligned} \tag{2.43}$$

PROOF OF LEMMA 2.4. We have

$$\lim_{s\downarrow t^1} V(s;t,x,v) = v - \int_t^{t^1} \nabla\Phi(s, X(s;t,x,v))\mathrm{d}s, \tag{2.44}$$

$$X(\tau;t,x,v) = x + v(\tau - t) - \int_t^\tau \mathrm{d}s\int_t^s \mathrm{d}s'\nabla\Phi(s', X(s';t,x,v)). \tag{2.45}$$

Especially, when $\tau = t^1$, we get

$$X(t^1;t,x,v) = x + v(t^1 - t) - \int_t^{t^1} \mathrm{d}s\int_t^s \mathrm{d}s'\nabla\Phi(s', X(s';t,x,v)). \tag{2.46}$$

From (2.44), we have

$$\begin{aligned}\lim_{s\downarrow t^1}\frac{\partial V(s;t,x,v)}{\partial x_j} &= \frac{\partial t_{\mathbf{b}}}{\partial x_j}\nabla\Phi(t^1; X(t^1;t,x,v))\\ &\quad - \int_t^{t^1}\left(\frac{\partial X(s)}{\partial x_j}\cdot\nabla_x\right)\nabla\Phi(s)\mathrm{d}s,\end{aligned} \tag{2.47}$$

and from (2.47),

$$\sup_{\tau\le s'\le t}\left|\frac{\partial X(s';t,x,v)}{\partial x_j}\right| \le 1 + \int_t^\tau |s-t|\|\Phi\|_{C^2}\sup_{\tau\le s'\le t}\left|\frac{\partial X(s';t,x,v)}{\partial x_j}\right|\mathrm{d}s. \tag{2.48}$$

By Gronwall's inequality

$$\sup_{\tau\le s'\le t}\left|\frac{\partial X(s';t,x,v)}{\partial x_j}\right| \le O_\Omega(1)e^{\|\nabla^2\Phi\|_\infty|t-\tau|^2/2}. \tag{2.49}$$

Similarly, from (2.44), we have

$$\begin{aligned}\lim_{s\downarrow t^1}\frac{\partial V(s;t,x,v)}{\partial v_j} &= e_j + \frac{\partial t_{\mathbf{b}}}{\partial v_j}\nabla\Phi(t^1; X(t^1;t,x,v))\\ &\quad - \int_t^{t^1}\left(\frac{\partial X(s)}{\partial v_j}\cdot\nabla_x\right)\nabla\Phi(s)\mathrm{d}s,\end{aligned} \tag{2.50}$$

and from (2.45),

$$\begin{aligned}\sup_{\tau\le s'\le t}\left|\frac{\partial X(s';t,x,v)}{\partial v_j}\right| &\le |\tau - t|\\ &\quad + \int_t^\tau |s-t|\|\Phi\|_{C^2}\sup_{\tau\le s'\le t}\left|\frac{\partial X(s';t,x,v)}{\partial v_j}\right|\mathrm{d}s.\end{aligned} \tag{2.51}$$

By Gronwall's inequality

$$\sup_{\tau\le s'\le t}\left|\frac{\partial X(s';t,x,v)}{\partial v_j}\right| \le O_\Omega(1)|\tau-t|e^{\|\nabla^2\Phi\|_\infty|t-\tau|^2/2}. \tag{2.52}$$

To prove (2.36)–(2.41), these estimates are very similar to those of Lemma 2.2. It suffices for us to choose global euclidean coordinates instead of η_{p^k}. Therefore we should replace

$$\eta_{p^{k+1}}\to\eta_{p^1},\quad \eta_{p^k}\to x,\quad t^k\to t,\quad t^{k+1}\to t-t_{\mathbf{b}}=t^1,\quad \partial_{x_j}x=e_j. \tag{2.53}$$

Let us prove (2.36). For $j=1,2,3$, we apply ∂_{x_j} to (2.46) and take an inner product with $\frac{\partial_3\eta_{p^1}}{\sqrt{g_{p^1,33}}}\Big|_{x^1}$. Then we get

$$\begin{aligned}\frac{\partial t_{\mathbf{b}}}{\partial x_j} &= -\frac{1}{\mathbf{v}^1_{p^1,3}}\frac{\partial_3\eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}}\cdot e_j\\ &\quad+\frac{1}{\mathbf{v}^1_{p^1,3}}\frac{\partial_3\eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}}\cdot\int_t^{t^1}\mathrm{d}s\int_t^s\mathrm{d}\tau\left(\frac{\partial X(\tau)}{\partial\mathbf{x}_{p,j}}\cdot\nabla_x\right)\big(\nabla_x\Phi(\tau)\big).\end{aligned} \tag{2.54}$$

To prove (2.37), for $j=1,2,3$, we apply ∂_{v_j} to (2.46) and take $\cdot\frac{\partial_3\eta_{p^1}}{\sqrt{g_{p^1,33}}}\Big|_{x^1}$. Then we get

$$\begin{aligned}0&=\sum_{l=1,2}\frac{\partial\mathbf{x}^1_{p^1,l}}{\partial v_j}\frac{\partial\eta_{p^1}}{\partial\mathbf{x}^k_{p^1,l}}\bigg|_{x^1}\cdot\frac{\partial_3\eta_{p^1}}{\sqrt{g_{p^1,33}}}\bigg|_{x^1}\\ &=\left\{-(t-t^1)e_j-\frac{\partial(t-t^1)}{\partial v_j}\lim_{s\downarrow t^1}V(s;t,x,v)\right\}\cdot\frac{\partial_3\eta_{p^1}}{\sqrt{g_{p^1,33}}}\bigg|_{x^1}\\ &\quad+O_\Omega(\|\Phi\|_{C^2})|t-t^1|^2\sup_s\left|\frac{\partial X(s)}{\partial v_j}\right|.\end{aligned} \tag{2.55}$$

To prove (2.38), for $i,j=1,2$, we apply ∂_{x_j} to (2.46) and take $\cdot\frac{\partial_i\eta_{p^1}}{\sqrt{g_{p^1,ii}}}\Big|_{x^1}$. Then we use (2.48) to get

$$\begin{aligned}\frac{\partial\mathbf{x}^1_{p^1,i}}{\partial x_j}&=\frac{1}{\mathbf{v}^1_{p^1,3}}\frac{\partial_3\eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}}\cdot e_j\frac{\mathbf{v}^1_{p^1,i}}{\sqrt{g_{p^1,ii}}}\bigg|_{x^1}+\frac{\partial_i\eta_{p^1}}{g_{p^1,ii}}\bigg|_{x^1}\cdot e_j\\ &\quad+O_\Omega(\|\Phi\|_{C^2})\frac{|\mathbf{v}^1_{p^1,i}|}{|\mathbf{v}^1_{p^1,3}|}(t-t^1)^2e^{\|\Phi\|_{C^2}(t-t^1)^2/2}\\ &\quad+O_\Omega(\|\Phi\|_{C^2})(t-t^1)^2e^{\|\Phi\|_{C^2}(t-t^1)^2/2}.\end{aligned} \tag{2.56}$$

This yields (2.38).

To prove (2.39), for $i = 1, 2$ and $j = 1, 2, 3$, we apply ∂_{v_j} to (2.46) and take an inner product with $\frac{\partial_i \eta_{p^1}}{\sqrt{g_{p^1,ii}}}\Big|_{x^1}$:

$$\frac{\partial \mathbf{x}^1_{p^1,i}}{\partial v_j} = \Big\{ -\frac{\partial(t-t^1)}{\partial v_j} \lim_{s\downarrow t^1} V(s;t,x,v) - (t-t^1)\frac{\partial v}{\partial v_j} \Big\} \cdot \frac{\partial_i \eta_{p^1}}{g_{p^1,ii}}\Big|_{x^1} + O_\Omega(\|\Phi\|_{C^2})|t-t^1|^2 \sup_s \Big| \frac{\partial X(s)}{\partial \mathbf{v}^k_{p^k,j}} \Big|.$$

Then we use (2.48) to get (2.39).

Let us prove (2.40). For $i = 1, 2$ and $j = 1, 2$, we apply ∂x_j to

$$\begin{aligned} \mathbf{v}^1_{p^1,i} &= \frac{\partial_i \eta_{p^1}}{\sqrt{g_{p^1,ii}}}\Big|_{x^1} \cdot \lim_{s\downarrow t^1} V(s;t,x,v) \quad \text{for } i = 1, 2, \\ \mathbf{v}^1_{p^1,3} &= -\frac{\partial_3 \eta_{p^1}}{\sqrt{g_{p^1,33}}}\Big|_{x^1} \cdot \lim_{s\downarrow t^1} V(s;t,x,v). \end{aligned} \tag{2.57}$$

For $i, j = 1, 2$, from (2.31),

$$\begin{aligned} \frac{\partial \mathbf{v}^1_{p^1,i}}{\partial x_j} = \frac{\partial_i \eta_{p^1}(x^1)}{\sqrt{g_{p^1,ii}(x^1)}} \cdot \Big[\nabla_x \Phi(t^1;t,x,v) \frac{\partial(t-t^1)}{\partial x_j} \\ - \int_t^{t^1} \big(\partial_{x_j} X(s) \cdot \nabla_x\big) \nabla_x \Phi(s, X(s;t,x,v)) \mathrm{d}s \Big] \\ + \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^1_{p^1,\ell}}{\partial x_j} \frac{\partial}{\partial \mathbf{x}^1_{p^1,\ell}} \Big(\frac{\partial_i \eta_{p^1}}{\sqrt{g_{p^1,ii}}} \Big)\Big|_{x^1} \cdot \lim_{s\downarrow t^1} V(s;t,x,v). \end{aligned}$$

For $j = 1, 2$,

$$\begin{aligned} \frac{\partial \mathbf{v}^1_{p^1,3}}{\partial x_j} = -\frac{\partial_3 \eta_{p^1}(x^1)}{\sqrt{g_{p^1,33}(x^1)}} \cdot \Big[\nabla_x \Phi(t^1;t,x,v) \frac{\partial(t-t^1)}{\partial x_j} \\ - \int_t^{t^1} (\partial_{x_j} X(s) \cdot \nabla_x) \nabla_x \Phi(s, X(s;t,x,v)) \mathrm{d}s \Big] \\ - \sum_{\ell=1}^{2} \frac{\partial \mathbf{x}^1_{p^1,\ell}}{\partial x_j} \frac{\partial}{\partial \mathbf{x}^1_{p^1,\ell}} \Big(\frac{\partial_3 \eta_{p^1}}{\sqrt{g_{p^1,33}}} \Big)\Big|_{x^1} \cdot \lim_{s\downarrow t^1} V(s;t,x,v). \end{aligned}$$

From (2.44), (2.38), and (2.36), we prove (2.18).

The proof of (2.41) is similar to the proof above. We apply ∂v_j to (2.57) and then use (2.44), (2.39), and (2.37). We skip the details.

Let us consider (2.42). Note that $|\mathbf{v}_{p^1}^1| = \lim_{s\downarrow t^1} |V(s;t,x,v)|$ and

$$2\big|\mathbf{v}_{p^1}^1\big|\frac{\partial\big|\mathbf{v}_{p^1}^1\big|}{\partial x_j} = 2\lim_{s\downarrow t^1} V(s;t,x,v)\cdot\lim_{s\downarrow t^1}\partial_{x_j}V(s;t,x,v),$$

we have

$$\frac{\partial|\mathbf{v}_{p^1}^1|}{\partial x_j} = \lim_{s\downarrow t^1}\frac{V(s;t,x,v)}{|V(s;t,x,v)|}\cdot\lim_{s\downarrow t^1}\partial_{x_j}V(s;t,x,v). \tag{2.58}$$

We combine (2.58), (2.47), (2.36), and (2.49) to derive (2.42).

To prove (2.43), we perform a similar process to that above with ∂v_j to get

$$\frac{\partial|\mathbf{v}_{p^1}^1|}{\partial v_j} = \lim_{s\downarrow t^1}\frac{V(s;t,x,v)}{|V(s;t,x,v)|}\cdot\lim_{s\downarrow t^1}\partial_{v_j}V(s;t,x,v). \tag{2.59}$$

We combine (2.59), (2.50), (2.37), and (2.52) to derive (2.43). □

LEMMA 2.5. *We define* $(\mathbf{X}_p(s;t,x,v),\mathbf{V}_p(s;t,x,v))$ *as*

$$\begin{aligned}&\eta_p(\mathbf{X}_p(s;t,x,v)) := X(s;t,x,v),\\ &\mathbf{V}_{p,i}(s;t,x,v) := \frac{\partial_i\eta_p(\mathbf{X}_p(s;t,x,v))}{\sqrt{g_{p,ii}(\mathbf{X}_p(s;t,x,v))}}\cdot V(s;t,x,v).\end{aligned} \tag{2.60}$$

Then we have

$$\begin{aligned}&\dot{\mathbf{X}}_{p,i}(s;t,x,v)\\ &\quad=\sum_j\left(\frac{\partial_i\eta_{p,j}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}{g_{p,ii}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}+O_{\|\eta\|_{C^2}}(|\mathbf{X}_{p,3}(s)|)\right)V_j(s;t,x,v)\\ &\quad=\frac{1}{\sqrt{g_{p,ii}(\mathbf{X}_p(s;t,x,v))}}\mathbf{V}_{p,i}(s;t,x,v)\\ &\qquad+O_{\|\eta\|_{C^2}}(\max_s|\mathbf{X}_{p,3}(s)|\max_s|V(s)|),\end{aligned} \tag{2.61}$$

$$\begin{aligned}&\dot{\mathbf{V}}_{p,i}(s;t,x,v)\\ &\quad=-\sum_{m=1}^3\sum_{n(\neq i)}\sum_{\ell=1}^2\frac{1}{\sqrt{g_{p,\ell\ell}}}\partial_\ell\left(\frac{\partial_n\eta_{p,m}}{\sqrt{g_{p,nn}}}\right)\frac{\partial_i\eta_{p,m}}{\sqrt{g_{p,ii}}}\bigg|_{(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}\\ &\qquad\qquad\times\mathbf{V}_{p,\ell}(s;t,x,v)\mathbf{V}_{p,n}(s;t,x,v)\\ &\qquad+O(\|\eta\|_{C^3})\big\{\max_s|\mathbf{X}_{p,3}(s)|\max_s|V(s)|^2\\ &\qquad+\max_s|\mathbf{V}_{p,3}(s)|\max_s|V(s)|+\|\nabla_x^2\Phi\|_\infty\big\}.\end{aligned} \tag{2.62}$$

PROOF. First we prove (2.61). From (2.1),

$$\sum_\ell\partial_\ell\eta_{p,i}(\mathbf{X}_p(s;t,x,v))\dot{\mathbf{X}}_{p,\ell}(s;t,x,v) = \dot{X}_i(s;t,x,v) = V_i(s;t,x,v).$$

Note that from (2.8), for $|\mathbf{X}_{p,3}(s)| \ll 1$,

$$\begin{aligned}\big(\nabla\eta_p(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),\mathbf{X}_{p,3}(s))\big)^{-1}_{i,j} &= \frac{\partial_i\eta_{p,j}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}{g_{p,ii}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}\\ &\quad + O(\|\eta\|_{C^2})|\mathbf{X}_{p,3}(s)|,\end{aligned}$$

$$\left(\begin{array}{c}\frac{\partial_1\eta_p}{\sqrt{g_{p,11}}}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)\\ \frac{\partial_2\eta_p}{\sqrt{g_{p,22}}}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)\\ \frac{\partial_3\eta_p}{\sqrt{g_{p,33}}}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)\end{array}\right)^{-1}_{i,j} = \frac{\partial_j\eta_{p,i}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}{\sqrt{g_{p,jj}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}}. \tag{2.63}$$

We apply these to (2.60) and use (2.1) to get (2.61).

Second, we prove (2.62). From (2.60), (2.63), and (2.61),

$$\begin{aligned}&\dot{\mathbf{V}}_{p,i}(s;t,x,v)\\ &\quad= \frac{d}{ds}\left[\frac{\partial_i\eta_p(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}{\sqrt{g_{p,ii}(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}}\cdot V(s;t,x,v)\right]\\ &\qquad+ O(\|\eta\|_{C^3})\max_s|\mathbf{X}_{p,3}(s)|\max_s|V(s)|^2\\ &\qquad+ O(\|\eta\|_{C^2})\max_s|\mathbf{V}_{p,3}(s)|\max_s|\mathbf{V}(s)|\\ &\qquad+ O(\|\eta\|_{C^2})\|\nabla_x^2\Phi\|_\infty\max_s|\mathbf{X}_{p,3}(s)|\\ &\quad= \sum_{m,n=1}^{3}\sum_{\ell=1}^{2}\frac{1}{\sqrt{g_{p,\ell\ell}}}\partial_\ell\left(\frac{\partial_i\eta_{p,m}}{\sqrt{g_{p,ii}}}\right)\frac{\partial_n\eta_{p,m}}{\sqrt{g_{p,nn}}}\bigg|_{(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}\\ &\qquad\qquad\times\mathbf{V}_{p,\ell}(s;t,x,v)\mathbf{V}_{p,n}(s;t,x,v)\\ &\qquad+ O(\|\eta\|_{C^3})\max_s|\mathbf{X}_{p,3}(s)|\max_s|V(s)|^2\\ &\qquad+ O(\|\eta\|_{C^2})\max_s|\mathbf{V}_{p,3}(s)|\max_s|\mathbf{V}(s)|\\ &\qquad+ O(\|\eta\|_{C^2})\|\nabla_x^2\Phi\|_\infty,\end{aligned}$$

where we have used

$$\begin{aligned}V_m(s;t,x,v) &= \sum_{n=1}^{3}\frac{\partial_n\eta_{p,m}}{\sqrt{g_{p,nn}}}\bigg|_{(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}\mathbf{V}_{p,n}(s;t,x,v)\\ &\quad+ C_\Omega\max_s|\mathbf{X}_{p,3}(s)|\max_s|V(s)|.\end{aligned}$$

In the case of $i = n$, we have

$$\sum_{m=1}^{3}\partial_\ell\left(\frac{\partial_i\eta_{p,m}}{\sqrt{g_{p,ii}}}\right)\frac{\partial_i\eta_{p,m}}{\sqrt{g_{p,ii}}}\bigg|_{(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)} = 0.$$

Moreover,

$$\sum_{m=1}^{3} \partial_\ell \left(\frac{\partial_i \eta_{p,m}}{\sqrt{g_{p,ii}}} \right) \frac{\partial_n \eta_{p,m}}{\sqrt{g_{p,nn}}} \Big|_{(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)} = - \sum_{m=1}^{3} \partial_\ell \left(\frac{\partial_n \eta_{p,m}}{\sqrt{g_{p,nn}}} \right) \frac{\partial_i \eta_{p,m}}{\sqrt{g_{p,ii}}} \Big|_{(\mathbf{X}_{p,1}(s),\mathbf{X}_{p,2}(s),0)}.$$

This finishes the proof for (2.62). □

LEMMA 2.6.

(i) *Let Ω be a bounded open domain in $\mathbb{R}^3$. If $|v| \geq \frac{1}{N}$ and $\|\nabla\Phi\|_\infty < \frac{\delta}{3\,\mathrm{diam}(\Omega)N^2}$ for $1 \ll N$ and $0 < \delta \ll 1$. Here $\mathrm{diam}(\Omega) := \max_{x,y\in\Omega} |x-y|$. Then*

$$t_{\mathbf{b}}(t,x,v) \leq 3N\,\mathrm{diam}(\Omega). \tag{2.64}$$

(ii) *Assume convexity in* (1.15). *Suppose*

$$\frac{1}{N} \leq |v^k| \leq N, \quad \|\nabla\Phi\|_\infty < \frac{\delta}{3\,\mathrm{diam}(\Omega)N^2} \quad \text{for } 1 \ll N,$$

and $0 < \delta \ll \frac{1}{N} \ll 1$. If either $\frac{|v^k\cdot n(x^k)|}{|v^k|} \ll 1$ or $\frac{|v^{k+1}\cdot n(x^{k+1})|}{|v^{k+1}|} \ll 1$, then we have the following estimates:

$$|v^k|(t^k - t^{k+1}) \lesssim_\Omega \min\left\{ \frac{|v^k\cdot n(x^k)|}{|v^k|}, \frac{|v^{k+1}\cdot n(x^{k+1})|}{|v^{k+1}|} \right\}, \tag{2.65}$$

$$|v^k|(t^k - t^{k+1}) \gtrsim_\Omega \min\left\{ \frac{|v^k\cdot n(x^k)|}{|v^k|}, \frac{|v^{k+1}\cdot n(x^{k+1})|}{|v^{k+1}|} \right\}. \tag{2.66}$$

PROOF. Note that if $|y-x| > \mathrm{diam}(\Omega)$ and $x \in \bar{\Omega}$, then $y \notin \bar{\Omega}$. If $s_* = t - 3N\,\mathrm{diam}(\Omega)$, then

$$\begin{aligned} |X(s_*;t,x,v) - x| &\geq \left\{ |v| - \|\nabla\Phi\|_\infty \frac{|s_*-t|}{2} \right\} |s^* - t| \\ &\geq \frac{1}{2N}\, 3N\,\mathrm{diam}(\Omega) = \frac{3}{2}\,\mathrm{diam}(\Omega). \end{aligned}$$

From (2.2), therefore,

$$t_{\mathbf{b}}(t,x,v) = \sup\{s \geq 0 : X(\tau;t,x,v) \in \Omega \text{ for all } \tau \in (t-s,t)\} \leq 3N\,\mathrm{diam}(\Omega).$$

First, we consider the case of $|v^k||t^{k+1}-t^k| > \delta$ for $0 < \delta \ll 1$. If $\|\nabla\Phi\|_\infty \le \frac{2\delta}{N^2}$, then

$$\begin{aligned} |X(t^{k+1}) - X(t^k)| &\ge |v^k||t^k - t^{k+1}| - \|\nabla\Phi\|_\infty \frac{|t^k - t^{k+1}|^2}{2} \\ &\ge |v^k||t^k - t^{k+1}| \left\{ 1 - \frac{\|\nabla\Phi\|_\infty |t^k - t^{k+1}|}{2|v^k|} \right\} \\ &\ge \frac{|v^k||t^k - t^{k+1}|}{2} \ge \frac{\delta}{2}, \end{aligned} \tag{2.67}$$

where we have used the fact that

$$\frac{\|\nabla\Phi\|_\infty |t^k - t^{k+1}|}{2|v^k|} \le \frac{2\delta}{N^2} \frac{3N \operatorname{diam}(\Omega)}{2/N} \le 3\delta \operatorname{diam}(\Omega) \ll 1. \tag{2.68}$$

On the other hand, note that

$$\begin{aligned} & n(X(t^k)) \cdot \big(X(t^{k+1}) - X(t^k) \big) \\ &= n(X(t^k)) \cdot \left(\lim_{s \uparrow t^k} V(s)(t^{k+1} - t^k) + \int_{t^k}^{t^{k+1}} \int_{t^k}^{s} -\nabla\Phi(\tau, X(\tau)) \mathrm{d}\tau \, \mathrm{d}s \right) \\ &= \mathbf{v}^k_{p^k,3}(t^{k+1} - t^k) + \|\nabla\Phi\|_\infty \frac{|t^{k+1} - t^k|^2}{2}. \end{aligned} \tag{2.69}$$

From the convexity, the left-hand side has a lower bound $C_\Omega |X(t^{k+1}) - X(t^k)|^2$. Therefore, if $\|\nabla\Phi\|_\infty \le \frac{2\delta}{N^2}$, then from (2.67) and (2.69),

$$\begin{aligned} \frac{\mathbf{v}^k_{p^k,3}}{|v^k|} &\ge \frac{1}{|v^k|} \left(\frac{C_\Omega}{|t^k - t^{k+1}|} \left(\frac{|v^k||X(t^k) - X(t^{k+1})|}{2} \right)^2 - \|\nabla\Phi\|_\infty \frac{|t^k - t^{k-1}|}{2} \right) \\ &\ge \left(\frac{C_\Omega}{4} - \frac{\|\nabla\Phi\|_\infty}{2|v^k|^2} \right) |v^k||t^k - t^{k+1}| \\ &\ge \left(\frac{C_\Omega}{4} - \frac{2\delta/N}{2/N} \right) |v^k||t^k - t^{k+1}| \\ &\ge \left(\frac{C_\Omega}{4} - \delta \right) |v^k||t^k - t^{k+1}| \\ &\ge \frac{C_\Omega}{8} |v^k||t^k - t^{k+1}|. \end{aligned}$$

Second, we consider the case of $|v^k||t^{k+1} - t^k| \le \delta$ for $0 < \delta \ll 1$. Then $|X(t^k) - X(s)| \le |v^k||t^{k+1} - t^k| + \frac{\|\nabla\Phi\|_\infty}{2} |t^{k+1} - t^k|^2 \ll 1$, and therefore we may assume that $X(s)$ can be parametrized by p^k-coordinate for all $s \in [t^{k+1}, t^k]$. From (2.61),

$$\begin{aligned} \max_s |\mathbf{X}_{p^k,3}(s)| \le{}& \left| \mathbf{v}^k_{p^k,3} \right| |t^{k+1} - t^k| \\ &+ O_{\|\eta\|_{C^2}} \big(\max_s |V(s)||t^k - t^{k+1}| \big) \times \max_s |\mathbf{X}_{p^k,3}(s)|. \end{aligned}$$

On the right-hand side, we control $\max_s |V(s)|$ by

$$\max_s |V(s)| \le |v^k| + \|\nabla\Phi\|_\infty \le 2|v^k| \quad \text{for } \|\nabla\Phi\|_\infty \le \frac{1}{N} \le |v^k|,$$

so we have

$$\max_s |V(s)||t^k - t^{k+1}| \le 2\delta \tag{2.70}$$

and

$$\max_s |\mathbf{X}_{p^k,3}(s)| \lesssim_\delta \left|\mathbf{v}^k_{p^k,3}\right||t^{k+1} - t^k|. \tag{2.71}$$

From (2.62) and (2.71),

$$\begin{aligned}\max_s |\mathbf{V}_{p^k,3}(s)| \le{}& |\mathbf{v}^k_{p^k,3}| + 4|v^k|^2|t^k - t^{k+1}| + \|\nabla\Phi\|_\infty|t^k - t^{k+1}| \\ &+ 4|\mathbf{v}^k_{p^k,3}||v^k|^2|t^k - t^{k+1}|^2 \\ &+ O_{\|\eta\|_{C^3}}(1)\max_s |\mathbf{V}_{p^k,3}(s)||v^k||t^k - t^{k+1}|.\end{aligned}$$

Now we use $|v^k||t^{k+1} - t^k| \le \delta \ll 1$ to have

$$\max_s |\mathbf{V}_{p^k,3}(s)| \le 2\left|\mathbf{v}^k_{p^k,3}\right| + 4|v^k|^2|t^k - t^{k+1}| + \|\nabla\Phi\|_\infty|t^k - t^{k+1}|. \tag{2.72}$$

Now we integrate (2.61) on $t^{k+1} \le s \le t^k$ and then use (2.62) to obtain

$$\begin{aligned}&\mathbf{v}^k_{p^k,3}(t^k - t^{k+1}) \\ &\quad= -\int_{t^{k+1}}^{t^k}\int_{t^k}^{s}\sum_{m,n=1}^{2}\mathbf{v}^k_{p^k,m}\mathbf{v}^k_{p^k,n}\frac{\partial_m\partial_n\eta_{p^k}}{\sqrt{g_{p^k,nn}}}\cdot\frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\Big|_{(\mathbf{X}_{p^k,1}(s),\mathbf{X}_{p^k,2}(s),0)}\mathrm{d}\tau\,\mathrm{d}s \\ &\qquad+ O_{\|\eta\|_{C^2}}(1)\Big[\max_s|\mathbf{X}_{p^k,3}(s)|\max_s|V(s)||t^k - t^{k+1}| + |t^{k+1} - t^k|^3\max_s|V(s)|^3 \\ &\qquad\quad+ |t^k - t^{k+1}|^2\big\{\|\nabla\Phi\|_\infty + \max_s|\mathbf{X}_{p^k,3}(s)|\max_s|V(s)|^2 \\ &\qquad\quad+ \max_s|\mathbf{V}_{p^k,3}(s)|\max_s|V(s)|\big\}\Big],\end{aligned} \tag{2.73}$$

and we use the convexity (1.15), and (2.71), (2.72), and (2.70) to derive

$$\begin{aligned}&|\mathbf{v}^k_{p^k,3}|(t^k - t^{k+1}) \\ &\quad\ge C_\Omega\frac{(t^k - t^{k+1})^2}{2}\sum_{m=1,2}|\mathbf{v}^k_{p^k,m}|^2 \\ &\qquad- O_{\|\eta\|_{C^1}}\Bigg[|\mathbf{v}^k_{p^k,3}||t^k - t^{k+1}|\underbrace{\max_s|V(s)||t^k - t^{k+1}|}_{\le 2\delta} + \underbrace{2\delta|t^k - t^{k+1}|^2\max_s|V(s)|^2}_{(*)_1} \\ &\qquad\quad+ |t^k - t^{k+1}|^2\Bigg\{\underbrace{\|\nabla\Phi\|_\infty}_{(*)_2} + \underbrace{|\mathbf{v}^k_{p^k,3}||t^k - t^{k+1}|\max_s|V(s)|^2}_{(*)_3} \\ &\qquad\quad+ 2\Bigg(\underbrace{2|\mathbf{v}^k_{p^k,3}|}_{(*)_4} + \underbrace{4|v^k|^2|t^k - t^{k+1}|}_{(*)_5} + \underbrace{\|\nabla\Phi\|_\infty|t^k - t^{k+1}|}_{(*)_6}\Bigg)\Bigg\}\Bigg].\end{aligned} \tag{2.74}$$

For $(*)_1$, we decomposed $|V(s)|$ by $\{\mathbf{v}_{p^k,\ell}\}_{\ell=1,2,3}$ and then the $\sum_{\ell=1,2}|\mathbf{v}_{p^k,\ell}|^2$ part is absorbed by $C_\Omega\frac{(t^k-t^{k+1})^2}{2}\sum_{m=1,2}|\mathbf{v}^k_{p^k,m}|^2$. $|\mathbf{v}^k_{p^k,3}|^2$ is absorbed by the left-hand side by the fact $|v^k||t^{k+1}-t^k|\le\delta\ll 1$. For $(*)_2$, since $|v^k|\ge 1$,

$$\begin{aligned}\|\nabla\Phi\|_\infty|t^k-t^{k+1}|^2 &\le \big(|\mathbf{v}^k_{p^k,3}||v^k|+|\mathbf{v}^k_{p^k,\|}|^2\big)N^2\|\nabla\Phi\|_\infty|t^k-t^{k+1}|^2\\ &\quad+|\mathbf{v}^k_{p^k,\|}|^2N^2\|\nabla\Phi\|_\infty|t^k-t^{k+1}|^2\\ &\le \underbrace{N^2\|\nabla\Phi\|_\infty}_{\lesssim O(1)}\underbrace{|v^k||t^k-t^{k+1}|}_{\le\delta\ll 1}\big|\mathbf{v}^k_{p^k,3}\big||t^k-t^{k+1}|,\end{aligned}\tag{2.75}$$

so $(*)_2$ is absorbed by the left-hand side. For $(*)_3$, it is also absorbed by the left-hand side from (2.70). For $(*)_4$, it is also absorbed by the left-hand side from the facts $|v^k||t^{k+1}-t^k|\le\delta\ll 1$ and $\delta\ll\frac{1}{N}$. For $(*)_5$, we perform decomposition as we did in $(*)_1$ and apply $|v^k||t^{k+1}-t^k|\le\delta\ll 1$ and $\delta\ll\frac{1}{N}$ to be absorbed by the left-hand side and $C_\Omega\frac{(t^k-t^{k+1})^2}{2}\sum_{m=1,2}|\mathbf{v}^k_{p^k,m}|^2$. For $(*)_6$, it is also absorbed by the left-hand side similarly to the $(*)_2$ case. Finally, we conclude (2.65).

Assume that x^{k+1} and x^k are close enough, i.e., $|x^{k+1}-x^k|\le\|\Phi\|_{C^1}^{1/2}\ll 1$. From

$$\begin{aligned}&\eta_{p^k}\big(\mathbf{x}^{k+1}_{p^k}\big)-\eta_{p^k}\big(\mathbf{x}^k_{p^k}\big)\\ &=\int_0^{-(t^k-t^{k+1})}V\big(t^k+s;t^k,\eta_{p^k}\big(\mathbf{x}^k_{p^k}\big),v^k\big)\mathrm{d}s\\ &=v^k(t^{k+1}-t^k)\\ &\quad-\int_0^{-(t^k-t^{k+1})}\int_0^s\nabla\Phi\big(t^k+\tau;X\big(t^k+\tau;t^k,\eta_{p^k}\big(\mathbf{x}^k_{p^k}\big),v^k\big)\big)\mathrm{d}\tau\,\mathrm{d}s,\end{aligned}\tag{2.76}$$

we have

$$\eta_{p^k}\big(\mathbf{x}^{k+1}_{p^k}\big)-\eta_{p^k}\big(\mathbf{x}^k_{p^k}\big)=v^k(t^{k+1}-t^k)+O(\|\Phi\|_{C^1})|t^{k+1}-t^k|^2.$$

By the expansion, $\eta_{p^k}(\mathbf{x}^{k+1}_{p^k})-\eta_{p^k}(\mathbf{x}^k_{p^k})=(\mathbf{x}^{k+1}_{p^{k+1}}-\mathbf{x}^k_{p^k})\cdot\nabla\eta_{p^k}(\mathbf{x}^{k+1}_{p^k})$. For $|t^{k+1}-t^k|\le 1$, $|v^k|\ge\frac{1}{N}$, and $\|\Phi\|_{C^2}\le\frac{1}{4N}$ for $N\gg 1$,

$$\begin{aligned}|\mathbf{x}^{k+1}_{p^{k+1}}-\mathbf{x}^k_{p^k}|&\le\big|(\nabla\eta_{p^k}(\mathbf{x}^{k+1}_{p^k}))^{-1}\big||t^{k+1}-t^k|\big\{|v^k|+O(\|\Phi\|_{C^1})\big\}\\ &\lesssim_{\Omega,N}|v^k||t^{k+1}-t^k|.\end{aligned}\tag{2.77}$$

On the other hand, from (2.76) $\cdot\, n_{p^k}(\mathbf{x}^{k+1}_{p^k})$, we have

$$\big[\eta_{p^k}\big(\mathbf{x}^{k+1}_{p^k}\big)-\eta_{p^k}\big(\mathbf{x}^k_{p^k}\big)\big]\cdot n_{p^k}\big(\mathbf{x}^{k+1}_{p^k}\big)=\mathbf{v}^k_{p^k,3}(t^{k+1}-t^k)+O(\|\Phi\|_{C^2})|t^{k+1}-t^k|^2.$$

By the expansion, the left-hand side equals

$$\begin{aligned}
&[\eta_{p^k}(\mathbf{x}_{p^k}^{k+1}) - \eta_{p^k}(\mathbf{x}_{p^k}^{k})] \cdot n_{p^k}(\mathbf{x}_{p^k}^{k+1}) \\
&\quad = \left[(\mathbf{x}_{p^{k+1}}^{k+1} - \mathbf{x}_{p^k}^{k}) \cdot \nabla \eta_{p^k}(\mathbf{x}_{p^k}^{k+1})\right] \cdot n_{p^k}(\mathbf{x}_{p^k}^{k+1}) + O(\|\eta\|_{C^2})|\mathbf{x}_{p^{k+1}}^{k+1} - \mathbf{x}_{p^k}^{k}|^2 \\
&\quad \lesssim_\Omega |\mathbf{x}_{p^{k+1}}^{k+1} - \mathbf{x}_{p^k}^{k}|^2,
\end{aligned}$$

where we have used the fact that $\nabla \eta_{p^{k+1}}(\mathbf{x}_{p^k}^{k+1}) \perp n_{p^k}(\mathbf{x}_{p^k}^{k+1})$. Therefore, if $|\mathbf{v}_{p^k,3}^k| > \varepsilon$ and $\|\Phi\|_{C^2} \ll_\varepsilon 1$,

$$\begin{aligned}
|\mathbf{x}_{p^{k+1}}^{k+1} - \mathbf{x}_{p^k}^{k}|^2 &\gtrsim_\Omega |\mathbf{v}_{p^k,3}^k (t^{k+1} - t^k) + O(\|\Phi\|_{C^2})|t^{k+1} - t^k|^2| \\
&\gtrsim_\Omega \{|\mathbf{v}_{p^k,3}^k| - O(\|\Phi\|_{C^2})\}|t^{k+1} - t^k| \\
&\gtrsim_\Omega |\mathbf{v}_{p^k,3}^k||t^{k+1} - t^k|.
\end{aligned} \tag{2.78}$$

From (2.77) and (2.78), we prove (2.66) when x^{k+1} and x^k are close enough.

Assume x^{k+1} and x^k are not close, i.e., $|x^{k+1} - x^k| \geq \|\Phi\|_{C^1}^{1/2}$. From (2.1) and $|t^k - t^{k+1}| \leq 1$, $|v^k| \geq \frac{1}{N}$, and $\|\Phi\|_{C^2} \leq \frac{1}{4N}$ for $N \gg 1$,

$$|t^k - t^{k+1}||v^k| \geq |x^{k+1} - x^k| - O(\|\Phi\|_{C^1})|t^k - t^{k+1}|^2 \gtrsim \|\Phi\|_{C^1}^{1/2}.$$

This proves (2.66). □

LEMMA 2.7. *Assume* (2.6) *and* (1.15) *hold. Suppose* $x \in \overline{\Omega}$, $\frac{1}{N} \leq |v| \leq N$, $\|\nabla\Phi\|_\infty < \frac{\delta}{3\,\mathrm{diam}(\Omega)N^2}$ *for* $1 \ll N$, *and* $0 < \delta \ll \frac{1}{N} \ll 1$. *Assume* $t \in [M, M+1]$ *for* $M \in \mathbb{N}$. *For all* $i \in \mathbb{N}$ *with* $t^i \in [M-1, t]$,

$$\begin{aligned}
&\max\{1 - C_\Omega|v^k||t^k - t^{k+1}|, c_{\Omega,N}\}\mathbf{v}_{p^k,3}^k \\
&\quad \leq \mathbf{v}_{p^{k+1},3}^{k+1} \leq \min\{1 + C_\Omega|v^k||t^k - t^{k+1}|, C_{\Omega,N}\}\mathbf{v}_{p^k,3}^k,
\end{aligned} \tag{2.79}$$

and

$$\begin{aligned}
&\prod_{j=1}^{k} \max\{1 - C_\Omega|v^j||t^j - t^{j+1}|, c_{\Omega,N}\}\mathbf{v}_{p^1,3}^1 \\
&\quad \leq \mathbf{v}_{p^{k+1},3}^{k+1} \leq \prod_{j=1}^{k} \min\{1 + C_\Omega|v^j||t^j - t^{j+1}|, C_{\Omega,N}\}\mathbf{v}_{p^1,3}^1.
\end{aligned} \tag{2.80}$$

Moreover,

$$\sup\{k \in \mathbb{N} : |t - t^k| \leq 1\} \lesssim_{\Omega,N,\delta} 1. \tag{2.81}$$

PROOF.

Step 1. We claim that if $\mathbf{v}^k_{p^k,3} \ll |v^k|$, then

$$
\begin{aligned}
&t^k - t^{k+1} \\
&= \frac{-2\mathbf{v}^{k+1}_{p^{k+1},3}}{\sum_{m,n=1}^{2} \frac{\partial_m\partial_n\eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},mm}}\sqrt{g_{p^{k+1},nn}}}\Big|_{x^{k+1}} \cdot \frac{\partial_3\eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big|_{x^{k+1}} \mathbf{v}^{k+1}_{p^{k+1},m}\mathbf{v}^{k+1}_{p^{k+1},n}} \\
&\quad + C_\Omega \frac{|t^k - t^{k+1}|}{|v^k|^2}\{\|\nabla\Phi\|_\infty + |\mathbf{v}^k_{p^k,3}||v^k|\}.
\end{aligned} \tag{2.82}
$$

Due to (2.65) and its proof, if $\mathbf{v}^k_{p^k,3} \ll |v^k|$, then

$$
X(s; t^{k+1}, x^{k+1}, v^{k+1}) \sim x^{k+1} \sim p^{k+1} \quad \text{for all } t^{k+1} \le s \le t^k.
$$

By the expansion of

$$
\big(\mathbf{X}_{p^{k+1}}(s; t^{k+1}, x^{k+1}, v^{k+1}), \mathbf{V}_{p^{k+1}}(s; t^{k+1}, x^{k+1}, v^{k+1})\big)
$$

in (2.62) around $s = t^{k+1}$,

$$
\begin{aligned}
&\dot{\mathbf{V}}_{p^{k+1},3}(s; t^{k+1}, x^{k+1}, v^{k+1}) \\
&= -\sum_{n=1}^{2}\sum_{\ell=1}^{2} \frac{\partial_\ell\partial_n\eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},\ell\ell}}\sqrt{g_{p^{k+1},nn}}}\Big|_{x^{k+1}} \cdot \frac{\partial_3\eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big|_{x^{k+1}} \mathbf{v}^{k+1}_{p^{k+1},\ell}\mathbf{v}^{k+1}_{p^{k+1},n} \\
&\quad + O(\|\eta\|_{C^3})\big\{\max_s |V(s)|^3|t^k - t^{k+1}| + \|\nabla_x\Phi\|_\infty \max_s |V(s)||t^k - t^{k+1}| \\
&\qquad + \max_s |\mathbf{X}_{p^{k+1},3}(s)| \max_s |V(s)|^2\big\} \\
&\qquad + O(\|\eta\|_{C^2})\big\{\max_s |\mathbf{V}_{p^{k+1},3}(s)| \max_s |V(s)| + \|\nabla_x^2\Phi\|_\infty\big\}.
\end{aligned}
$$

Note that from Lemma 2.6 and (2.71), the last three lines above are bounded from above by $|v^{k+1}||\mathbf{v}^{k+1}_{p^{k+1}}| + \|\nabla_x^2\Phi\|_\infty$. Then from (2.61), (2.62), (2.71), and (2.72),

$$
\begin{aligned}
&-\frac{(t^k - t^{k+1})^2}{2} \sum_{m,n=1}^{2} \frac{\partial_m\partial_n\eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},mm}}\sqrt{g_{p^{k+1},nn}}}\Big|_{x^{k+1}} \\
&\qquad \cdot \frac{\partial_3\eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big|_{x^{k+1}} \mathbf{v}^{k+1}_{p^{k+1},m}\mathbf{v}^{k+1}_{p^{k+1},n} \\
&= \mathbf{v}^{k+1}_{p^{k+1},3}(t^k - t^{k+1}) + O_{\|\eta\|_{C^3}}(|t^k - t^{k+1}|^2)\{\|\nabla\Phi\|_\infty + |\mathbf{v}^{k+1}_{p^{k+1},3}||v^{k+1}|\}.
\end{aligned}
$$

This proves (2.82).

Step 2. We claim that for $\mathbf{v}^{k+1}_{p^{k+1},3} \ll |v^{k+1}|$,

$$
\frac{\partial\mathbf{v}^{k+1}_{p^{k+1},3}}{\partial\mathbf{v}^k_{p^k,3}} = 1 + O_\Omega\big(\|\Phi\|_{C^1}|t^k - t^{k+1}|\big) + O_\Omega(|v^{k+1}|(t^k - t^{k+1})). \tag{2.83}
$$

From Lemma 2.2,

$$
\begin{aligned}
&\frac{\partial \mathbf{v}^{k+1}_{p^{k+1},3}}{\partial \mathbf{v}^{k}_{p^{k},3}} \\
&\quad = -\sum_{\ell=1}^{2}\Bigg\{-(t^k - t^{k+1})\frac{\partial_3 \eta_{p^k}(x^k)}{\sqrt{g_{p^k,33}(x^k)}}\cdot\frac{1}{\sqrt{g_{p^{k+1},\ell\ell}(x^{k+1})}}\Bigg[\frac{\partial_\ell \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},\ell\ell}(x^{k+1})}} \\
&\qquad\qquad + \frac{\mathbf{v}^{k+1}_{p^{k+1},\ell}}{\mathbf{v}^{k+1}_{p^{k+1},3}}\frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}}\Bigg]\Bigg\}\partial_\ell\left(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\right)\Bigg|_{x^{k+1}}\cdot v^k \\
&\qquad - \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{x^{k+1}}\cdot\frac{\partial_3 \eta_{p^k}}{\sqrt{g_{p^k,33}}}\Bigg|_{x^k} \\
&\qquad + O_\Omega(\|\Phi\|_{C^2})|v^k|\left(1+\frac{|\mathbf{v}^{k+1}_{p^{k+1},i}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|}\right)(t^k - t^{k+1})^3 e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2} \\
&\quad = \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{x^{k+1}}\cdot\frac{\partial_3 \eta_{p^k}}{\sqrt{g_{p^k,33}}}\Bigg|_{x^k} \\
&\qquad \times \underbrace{\left\{-1+\frac{t^k - t^{k+1}}{\mathbf{v}^{k+1}_{p^{k+1},3}}\sum_{\ell=1}^{2}\frac{\mathbf{v}^{k+1}_{p^{k+1},\ell}}{\sqrt{g_{p^{k+1},\ell\ell}}}\partial_\ell\left(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\right)\Bigg|_{x^{k+1}}\cdot v^k\right\}}_{(*)} \\
&\qquad + O_\Omega(1)|v^k|(t^k - t^{k+1})\left|\frac{\partial_3 \eta_{p^k}}{\sqrt{g_{p^k,33}}}\Bigg|_{x^k}\cdot\frac{\partial_\ell \eta_{p^{k+1}}}{g_{p^{k+1},\ell\ell}}\Bigg|_{x^{k+1}}\right| \\
&\qquad + O_\Omega(\|\Phi\|_{C^2})\left(1+\frac{|\mathbf{v}^{k+1}_{p^{k+1},i}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|}\right)|v^k|(t^k - t^{k+1})^3 e^{\|\Phi\|_{C^2}(t^k - t^{k+1})^2}.
\end{aligned}
$$

Consider $(*)$. For $\ell, j = 1, 2$, from (2.8),

$$
\begin{aligned}
&\partial_\ell\left(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\right)\cdot\frac{\partial_j \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},jj}}}\Bigg|_{x^{k+1}} = \\
&\quad \underbrace{\partial_\ell\left(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\cdot\frac{\partial_j \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},jj}}}\right)\Bigg|_{x^{k+1}}}_{=0} - \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Bigg|_{x^{k+1}}\cdot\frac{\partial_\ell \partial_j \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},jj}}}\Bigg|_{x^{k+1}},
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\partial_\ell\left(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\right)\Bigg|_{x^{k+1}}\cdot v^k \\
&\quad = \partial_\ell\left(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\right)\cdot v^{k+1} + O_{\|\eta\|_{C^2}}(\|\Phi\|_{C^1})|t^k - t^{k+1}| =
\end{aligned}
$$

$$= -\sum_{j=1}^{2} \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}} \cdot \frac{\partial_\ell \partial_j \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},jj}}}\bigg|_{x^{k+1}} \mathbf{v}^{k+1}_{p^{k+1},j}$$
$$+ O_{\|\eta\|_{C^2}}(\|\Phi\|_{C^1})|t^k - t^{k+1}|.$$

Combining the above with (2.82), we conclude that

$$(*) = -1 + 2 + O_{\|\eta\|_{C^2}}(\|\Phi\|_{C^1}) \frac{|v^{k+1}|^2 |t^k - t^{k+1}|^2}{\left|\mathbf{v}^{k+1}_{p^{k+1},3}\right|}$$
$$= 1 + O_{\|\eta\|_{C^2}}(\|\Phi\|_{C^1}|t^k - t^{k+1}|).$$

Note that

$$\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} \cdot \frac{\partial_3 \eta_{p^k}}{\sqrt{g_{p^k,33}}}\bigg|_{x^k} = 1 + O_\Omega(1) \max_s |V(s)|(t^k - t^{k+1}),$$
$$\frac{\partial_\ell \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},\ell\ell}}}\bigg|_{x^{k+1}} \cdot \frac{\partial_3 \eta_{p^k}}{\sqrt{g_{p^k,33}}}\bigg|_{x^k} = O_\Omega(1) \max_s |V(s)|(t^k - t^{k+1}), \quad \text{for } \ell = 1, 2.$$

Taking the above all together, we prove (2.83).

Step 3. We now prove (2.79). For $\mathbf{v}^{k+1}_{p^{k+1},3} \ll |v^{k+1}|$, by the expansion and (2.83),

$$\mathbf{v}^{k+1}_{p^{k+1},3}\big(t^k, \mathbf{x}^k_{p^k}; \mathbf{v}^k_{p^k,1}, \mathbf{v}^k_{p^k,2}, \mathbf{v}^k_{p^k,3}\big)$$
$$= \mathbf{v}^{k+1}_{p^{k+1},3}\big(t^k, \mathbf{x}^k_{p^k}; \mathbf{v}^k_{p^k,1}, \mathbf{v}^k_{p^k,2}, 0\big)$$
$$+ \int_0^{\mathbf{v}^k_{p^k,3}} \frac{\partial \mathbf{v}^{k+1}_{p^{k+1},3}}{\partial \mathbf{v}^k_{p^k,3}}\big(t^k, \mathbf{x}^k_{p^k}; \mathbf{v}^k_{p^k,1}, \mathbf{v}^k_{p^k,2}, \tau\big)\mathrm{d}\tau$$
$$= 0 + \mathbf{v}^k_{p^k,3} \times (2.83).$$

This proves $\mathbf{v}^{k+1}_{p^{k+1},3} = (1 + O_{\|\eta\|_{C^2}}|v^k||t^k - t^{k+1}|)\mathbf{v}^k_{p^k,3}$.

Now we consider the case of $\mathbf{v}^{k+1}_{p^{k+1},3} \gtrsim |v^{k+1}|$. Clearly

$$\mathbf{v}^k_{p^k,3} \le \left|\mathbf{v}^k_{p^k}\right| \le |v^{k+1}| + \|\nabla_x \Phi\|_\infty |t^k - t^{k+1}| \le \frac{1}{2}|v^{k+1}| \lesssim \mathbf{v}^{k+1}_{p^{k+1},3}$$

for sufficiently small $\|\nabla_x \Phi\|_\infty$. This proves (2.79). Then we prove (2.80) by induction in k. Also, the proof of (2.81) is a direct consequence of (2.80):

$$\mathbf{v}^{k+1}_{p^{k+1},3} \ge \big(1 + C_\Omega |v^k||t^k - t^{k+1}|\big)^{-1} \mathbf{v}^k_{p^k,3} \ge e^{-C_\Omega |v^k||t^k - t^{k+1}|}\mathbf{v}^k_{p^k,3}$$
$$\ge e^{-C_\Omega \sum_{i=1}^k |v^i||t^i - t^{i+1}|}\mathbf{v}^1_{p^1,3} \ge e^{-C_\Omega N}\delta. \qquad \square$$

LEMMA 2.8. *Assume* $\frac{1}{N} \le |v| \le N$, $\|\nabla\Phi\|_\infty < \frac{\delta}{3\,\mathrm{diam}(\Omega)N^2}$ *for* $1 \ll N$, *and* $0 < \delta \ll \frac{1}{N} \ll 1$. *Also, we assume* $|t^k - t^{k+1}| \le 1$. *Then*

$$\left|\det\begin{bmatrix} \nabla_{\mathbf{x}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1}} & \nabla_{\mathbf{v}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1}} \\ \nabla_{\mathbf{x}^k_{p^k}} \mathbf{v}^{k+1}_{p^{k+1}} & \nabla_{\mathbf{v}^k_{p^k}} \mathbf{v}^{k+1}_{p^{k+1}} \end{bmatrix}_{5\times5}\right| =$$

$$(1 + O_{\Omega,N}(\|\Phi\|_{C^2}))\frac{\sqrt{g_{p^k,11}(x^k)}\sqrt{g_{p^k,22}(x^k)}}{\sqrt{g_{p^{k+1},11}(x^{k+1})}\sqrt{g_{p^{k+1},22}(x^{k+1})}}\frac{|\mathbf{v}^k_{p^k,3}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|}$$

for the mapping $(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, \mathbf{v}^k_{p^k}) \mapsto (\mathbf{x}^{k+1}_{p^{k+1},1}, \mathbf{x}^k_{p^{k+1},2}, \mathbf{v}^k_{p^{k+1}})$.

PROOF. From Lemma 2.2 and Lemma 2.6,

$$\left[\begin{array}{c|c} \nabla_{\mathbf{x}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1},1} & \nabla_{\mathbf{v}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1},1} \\ \nabla_{\mathbf{x}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1},2} & \nabla_{\mathbf{v}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1},2} \\ \hline \nabla_{\mathbf{x}^k_{p^k}} \mathbf{v}^{k+1}_{p^{k+1}} & \nabla_{\mathbf{v}^k_{p^k}} \mathbf{v}^{k+1}_{p^{k+1}} \end{array}\right]_{5\times5}$$

$$= \underbrace{\left[\begin{array}{c|c} \left[\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},i}}{\partial \mathbf{x}^k_{p^k,j}}\right]_{i=1,2,j=1,2} & \left[\frac{\partial \mathbf{x}^{k+1}_{p^{k+1},i}}{\partial \mathbf{v}^k_{p^k,j}}\right]_{i=1,2,j=1,2,3} \\ \hline \begin{array}{c} \frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}}\Big|_{x^{k+1}} \cdot \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \\ + \sum_{\ell=1}^2 \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{x}^k_{p^k,j}} \partial_\ell\Big(\frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}}\Big)\Big|_{x^{k+1}} \cdot v^k \end{array} & \begin{array}{c} \sum_{\ell=1}^2 \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{v}^k_{p^k,j}} \partial_\ell\Big(\frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}}\Big)\Big|_{x^{k+1}} \cdot v^k \\ + \frac{\partial_i \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},ii}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \end{array} \\ \hline \begin{array}{c} -\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big|_{x^{k+1}} \cdot \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} \\ - \sum_{\ell=1}^2 \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{x}^k_{p^k,j}} \partial_\ell\Big(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big)\Big|_{x^{k+1}} \cdot v^k \end{array} & \begin{array}{c} - \sum_{\ell=1}^2 \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},\ell}}{\partial \mathbf{v}^k_{p^k,j}} \partial_\ell\Big(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\Big)\Big|_{x^{k+1}} \cdot v^k \\ - \frac{\partial_3 \eta_{p^{k+1}}(x^{k+1})}{\sqrt{g_{p^{k+1},33}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \end{array} \end{array}\right]}_{:=\mathscr{A}}$$

$$+ \left[\begin{array}{c|c} 0 & 0 \\ \hline C_N\|\Phi\|_{C^2} & C_N\|\Phi\|_{C^2} \end{array}\right]_{5\times5}.$$

Now for $i = 1$ and $i = 2$, we multiply

$$\partial_\ell\left(\frac{\partial_i \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}}\right)\Big|_{x^{k+1}} \cdot v^k$$

to the ℓ^{th} row for $\ell = 1, 2$, and then subtract this from the $(i+2)^{\text{th}}$ row. Similarly, we multiply

$$\partial_\ell\left(\frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},ii}}}\right)\Big|_{x^{k+1}} \cdot v^k$$

to the ℓ^{th} row for $\ell = 1, 2$ and then subtract this from the 5^{th} row. Hence, rewriting the first two rows using Lemma 2.2, the resulting row echelon form of matrix $\mathscr{A}$ is

$$\left[\begin{array}{c|c} \left[\partial_j \eta_{p^k}(x^k) - (t^k - t^{k+1})\frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}}\right] & -(t^k - t^{k+1})\frac{\partial_j \eta_{p^k}}{\sqrt{g_{p^k,jj}}}\Big|_{x^k} \\ \frac{1}{\sqrt{g_{p^k+1,ii}}}\left[\frac{\partial_i \eta_{p^k+1}}{\sqrt{g_{p^k+1,ii}}} + \frac{\mathbf{v}^{k+1}_{p^k+1,i}}{\mathbf{v}^{k+1}_{p^k+1,3}}\frac{\partial_3 \eta_{p^k+1}}{\sqrt{g_{p^k+1,33}}}\right]\Big|_{x^{k+1}} & \cdot\frac{1}{\sqrt{g_{p^k+1,ii}}}\left[\frac{\partial_i \eta_{p^k+1}}{\sqrt{g_{p^k+1,ii}}} + \frac{\mathbf{v}^{k+1}_{p^k+1,i}}{\mathbf{v}^{k+1}_{p^k+1,3}}\frac{\partial_3 \eta_{p^k+1}}{\sqrt{g_{p^k+1,33}}}\right]\Big|_{x^{k+1}} \\ \hline \frac{\partial_i \eta_{p^k+1}}{\sqrt{g_{p^k+1,ii}}}\Big|_{x^{k+1}} \cdot \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} & \frac{\partial_i \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,ii}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \\ -\frac{\partial_3 \eta_{p^k+1}}{\sqrt{g_{p^k+1,33}}}\Big|_{x^{k+1}} \cdot \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} & -\frac{\partial_3 \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,33}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \end{array}\right] + \left[\begin{array}{c|c} 0 & 0 \\ \hline C_N\|\Phi\|_{C^2} & C_N\|\Phi\|_{C^2} \end{array}\right]_{5\times 5}.$$

Then we use rows 3 through 5 to remove the following parts in rows 1 and 2:

$$\left[\begin{array}{c|c} -(t^k - t^{k+1})\frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} & -(t^k - t^{k+1})\frac{\partial_j \eta_{p^k}}{\sqrt{g_{p^k,jj}}} \\ \cdot\frac{1}{\sqrt{g_{p^k+1,ii}}}\left[\frac{\partial_i \eta_{p^k+1}}{\sqrt{g_{p^k+1,ii}}} + \frac{\mathbf{v}^{k+1}_{p^k+1,i}}{\mathbf{v}^{k+1}_{p^k+1,3}}\frac{\partial_3 \eta_{p^k+1}}{\sqrt{g_{p^k+1,33}}}\right] & \cdot\frac{1}{\sqrt{g_{p^k+1,ii}}}\left[\frac{\partial_i \eta_{p^k+1}}{\sqrt{g_{p^k+1,ii}}} + \frac{\mathbf{v}^{k+1}_{p^k+1,i}}{\mathbf{v}^{k+1}_{p^k+1,3}}\frac{\partial_3 \eta_{p^k+1}}{\sqrt{g_{p^k+1,33}}}\right] \end{array}\right].$$

Via this process, we obtain the following row echelon form of matrix $\mathscr{A}$:

$$\left[\begin{array}{c|c} \partial_j \eta_{p^k}(x^k)\frac{1}{\sqrt{g_{p^k+1,ii}}} \cdot\left[\frac{\partial_i \eta_{p^k+1}}{\sqrt{g_{p^k+1,ii}}} + \frac{\mathbf{v}^{k+1}_{p^k+1,i}}{\mathbf{v}^{k+1}_{p^k+1,3}}\frac{\partial_3 \eta_{p^k+1}}{\sqrt{g_{p^k+1,33}}}\right]\Big|_{x^{k+1}} & 0 \\ \hline \frac{\partial_i \eta_{p^k+1}}{\sqrt{g_{p^k+1,ii}}}\Big|_{x^{k+1}} \cdot \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} & \frac{\partial_i \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,ii}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \\ -\frac{\partial_3 \eta_{p^k+1}}{\sqrt{g_{p^k+1,33}}}\Big|_{x^{k+1}} \cdot \frac{\partial v^k}{\partial \mathbf{x}^k_{p^k,j}} & -\frac{\partial_3 \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,33}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \end{array}\right] + \left[\begin{array}{c|c} O_\Omega(\|\Phi\|_{C^2})|v^k|(t^k - t^{k+1})\left(1 + \frac{|\mathbf{v}^{k+1}_{p^k+1}|}{|\mathbf{v}^{k+1}_{p^k+1,3}|}\right) & O_\Omega(\|\Phi\|_{C^2})|v^k|(t^k - t^{k+1})\left(1 + \frac{|\mathbf{v}^{k+1}_{p^k+1}|}{|\mathbf{v}^{k+1}_{p^k+1,3}|}\right) \\ \hline O_{\Omega,N}(\|\Phi\|_{C^2}) & O_{\Omega,N}(\|\Phi\|_{C^2}) \end{array}\right]_{5\times 5}. \tag{2.84}$$

Note that the type of elementary row echelon operation we used preserves the determinant. Therefore, we compute determinants of two matrices. The determinant of the lower right 3×3 block of the first matrix in (2.84) is given by

$$\begin{aligned} -1 &= \det\left[\begin{array}{c} \frac{\partial_i \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,ii}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x_{p^k})}{\sqrt{g_{p^k,jj}(x^k)}} \\ -\frac{\partial_3 \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,33}(x^{k+1})}} \cdot \frac{\partial_j \eta_{p^k}(x^k)}{\sqrt{g_{p^k,jj}(x^k)}} \end{array}\right]_{3\times 3} \\ &= \det\left[\begin{array}{c} \frac{\partial_1 \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,11}(x^{k+1})}} \\ \frac{\partial_2 \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,22}(x^{k+1})}} \\ -\frac{\partial_3 \eta_{p^k+1}(x^{k+1})}{\sqrt{g_{p^k+1,33}(x^{k+1})}} \end{array}\right]_{3\times 3} \det\left[\begin{array}{ccc} \frac{\partial_1 \eta_{p^k}(x^k)}{\sqrt{g_{p^k,11}(x^k)}} & \frac{\partial_2 \eta_{p^k}(x^k)}{\sqrt{g_{p^k,22}(x^k)}} & \frac{\partial_3 \eta_{p^k}(x^k)}{\sqrt{g_{p^k,33}(x^k)}} \end{array}\right]_{3\times 3}. \end{aligned} \tag{2.85}$$

In order to evaluate the determinant of the upper left 2×2 matrix, we use a basic linear algebra result: Let $A_1, A_2, B_1, B_2 \in \mathbb{R}^3$. Then

$$\left|\det\begin{pmatrix} A_1 \cdot B_1 & A_1 \cdot B_2 \\ A_2 \cdot B_1 & A_2 \cdot B_2 \end{pmatrix}\right| = \big|(A_1 \times A_2) \cdot (B_1 \times B_2)\big|. \tag{2.86}$$

From (2.86), the determinant of upper left 2×2 submatrix of the first matrix in (2.84) equals

$$
\begin{aligned}
&\left| \left(\partial_1 \eta_{p^k} \times \partial_2 \eta_{p^k}\right)\big|_{x^k} \right. \\
&\quad \cdot \left(\frac{1}{\sqrt{g_{p^{k+1},11}}} \left[\frac{\partial_1 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},11}}}\bigg|_{x^{k+1}} + \frac{\mathbf{v}^{k+1}_{p^{k+1},1}}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} \right] \right. \\
&\quad \left. \left. \times \frac{1}{\sqrt{g_{p^{k+1},22}}} \left[\frac{\partial_2 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},22}}}\bigg|_{x^{k+1}} + \frac{\mathbf{v}^{k+1}_{p^{k+1},2}}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_3 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},33}}}\bigg|_{x^{k+1}} \right] \right) \right| \\
&= \frac{\sqrt{g_{p^k,11}(x^k)}\sqrt{g_{p^k,22}(x^k)}}{\sqrt{g_{p^{k+1},11}(x^{k+1})}\sqrt{g_{p^{k+1},22}(x^{k+1})}} \qquad (2.87) \\
&\quad \times \left| n_{p^k}(x^k) \cdot \left(n_{p^{k+1}} - \frac{\mathbf{v}^{k+1}_{p^{k+1},2}}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_2 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},22}}} - \frac{\mathbf{v}^{k+1}_{p^{k+1},1}}{\mathbf{v}^{k+1}_{p^{k+1},3}} \frac{\partial_1 \eta_{p^{k+1}}}{\sqrt{g_{p^{k+1},11}}} \right) \right| \\
&= \frac{\sqrt{g_{p^k,11}(x^k) g_{p^k,22}(x^k)}}{\sqrt{g_{p^{k+1},11}(x^{k+1}) g_{p^{k+1},22}(x^{k+1})}} \frac{|\mathbf{v}^k_{p^k,3}| + \|\nabla \Phi\|_\infty |t^k - t^{k+1}|}{|\mathbf{v}^{k+1}_{p^{k+1},3}|}. \qquad (2.88)
\end{aligned}
$$

Since the determinant of the second matrix in (2.84) is the size of $\|\Phi\|_{C^2}$, we finish the proof from (2.84), (2.85), and (2.88). □

LEMMA 2.9. *We define, for all k,*

$$
|\mathbf{v}^k_{p^k}| = \sqrt{\left(\mathbf{v}^k_{p^k,1}\right)^2 + \left(\mathbf{v}^k_{p^k,2}\right)^2 + \left(\mathbf{v}^k_{p^k,3}\right)^2}, \quad \hat{\mathbf{v}}^k_{p^k,1} = \frac{\mathbf{v}^k_{p^k,1}}{|\mathbf{v}^k_{p^k}|}, \quad \hat{\mathbf{v}}^k_{p^k,2} = \frac{\mathbf{v}^k_{p^k,2}}{|\mathbf{v}^k_{p^k}|}, \qquad (2.89)
$$

where $\mathbf{v}^k_{p^k} = \mathbf{v}^k_{p^k}(t,x,v)$ are defined in (2.11). *Assume* (1.15), $\frac{1}{N} \le |v| \le N$, $\|\Phi\|_{C^2_x} < \frac{\delta_1}{3\,\mathrm{diam}(\Omega) N^2}$ *for* $1 \ll N$, $0 < \delta_1 \ll \frac{1}{N} \ll 1$, *and* $|\mathbf{v}^1_{p^1,3}(t,x,v)| > \delta_2 > 0$. *If* $|t - t^k| \le 1$, *then*

$$
\left| \det \left[\begin{array}{cc|cc}
\frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,2}} & \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \hat{\mathbf{v}}^1_{p^1,1}} & \frac{\partial \mathbf{x}^k_{p^k,1}}{\partial \hat{\mathbf{v}}^1_{p^1,2}} \\
\frac{\partial \mathbf{x}^k_{p^k,2}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \mathbf{x}^k_{p^k,2}}{\partial \mathbf{x}^1_{p^1,2}} & \frac{\partial \mathbf{x}^k_{p^k,2}}{\partial \hat{\mathbf{v}}^1_{p^1,1}} & \frac{\partial \mathbf{x}^k_{p^k,2}}{\partial \hat{\mathbf{v}}^1_{p^1,2}} \\
\hline
\frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \mathbf{x}^1_{p^1,2}} & \frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \hat{\mathbf{v}}^1_{p^1,1}} & \frac{\partial \hat{\mathbf{v}}^k_{p^k,1}}{\partial \hat{\mathbf{v}}^1_{p^1,2}} \\
\frac{\partial \hat{\mathbf{v}}^k_{p^k,2}}{\partial \mathbf{x}^1_{p^1,1}} & \frac{\partial \hat{\mathbf{v}}^k_{p^k,2}}{\partial \mathbf{x}^1_{p^1,2}} & \frac{\partial \hat{\mathbf{v}}^k_{p^k,2}}{\partial \hat{\mathbf{v}}^1_{p^1,1}} & \frac{\partial \hat{\mathbf{v}}^k_{p^k,2}}{\partial \hat{\mathbf{v}}^1_{p^1,2}}
\end{array} \right] \right| > \epsilon_{\Omega,N,\delta_1,\delta_2} > 0, \qquad (2.90)
$$

where $t^1 = t^1(t,x,v)$, $\mathbf{x}^1_{p^1,i} = \mathbf{x}^1_{p^1,i}(t,x,v)$, $\widehat{\mathbf{v}}^1_{p^1,i} = \widehat{\mathbf{v}}^1_{p^1,i}(t,x,v)$, *and*

$$\mathbf{x}^k_{p^k,i} = \mathbf{x}^k_{p^k,i}\big(t^1, \mathbf{x}^1_{p^1,1}, \mathbf{x}^1_{p^1,2}, \widehat{\mathbf{v}}^1_{p^1,1}, \widehat{\mathbf{v}}^1_{p^1,2}, |\mathbf{v}^1_{p^1}|\big),$$
$$\widehat{\mathbf{v}}^k_{p^k,i} = \widehat{\mathbf{v}}^k_{p^k,i}\big(t^1, \mathbf{x}^1_{p^1,1}, \mathbf{x}^1_{p^1,2}, \widehat{\mathbf{v}}^1_{p^1,1}, \widehat{\mathbf{v}}^1_{p^1,2}, |\mathbf{v}^1_{p^1}|\big).$$

Here the constant $\epsilon_{\Omega,N,\delta_1,\delta_2} > 0$ *does not depend on* t *or* x.

PROOF. *Step* 1. We compute

$$\begin{aligned} J_i^{i+1} &:= \frac{\partial\big(\mathbf{x}^{i+1}_{p^{i+1},1}, \mathbf{x}^{i+1}_{p^{i+1},2}, \widehat{\mathbf{v}}^{i+1}_{p^{i+1},1}, \widehat{\mathbf{v}}^{i+1}_{p^{i+1},2}, |\mathbf{v}^{i+1}_{p^{i+1}}|\big)}{\partial\big(\mathbf{x}^{i}_{p^{i},1}, \mathbf{x}^{i}_{p^{i},2}, \widehat{\mathbf{v}}^{i}_{p^{i},1}, \widehat{\mathbf{v}}^{i}_{p^{i},2}, |\mathbf{v}^{i}_{p^{i}}|\big)} \\ &= \underbrace{\frac{\partial\big(\mathbf{x}^{i}_{p^{i},1}, \mathbf{x}^{i}_{p^{i},2}, \mathbf{v}^{i}_{p^{i}}\big)}{\partial\big(\mathbf{x}^{i}_{p^{i},1}, \mathbf{x}^{i}_{p^{i},2}, \widehat{\mathbf{v}}^{i}_{p^{i},1}, \widehat{\mathbf{v}}^{i}_{p^{i},2}, |\mathbf{v}^{i}_{p^{i}}|\big)}}_{=Q_i} \underbrace{\frac{\partial\big(\mathbf{x}^{i+1}_{p^{i+1},1}, \mathbf{x}^{i+1}_{p^{i+1},2}, \mathbf{v}^{i+1}_{p^{i+1}}\big)}{\partial\big(\mathbf{x}^{i}_{p^{i},1}, \mathbf{x}^{i}_{p^{i},2}, \mathbf{v}^{i}_{p^{i}}\big)}}_{=P_i} \\ &\quad \times \underbrace{\frac{\partial\big(\mathbf{x}^{i+1}_{p^{i+1},1}, \mathbf{x}^{i+1}_{p^{i+1},2}, \widehat{\mathbf{v}}^{i+1}_{p^{i+1},1}, \widehat{\mathbf{v}}^{i+1}_{p^{i+1},2}, |\mathbf{v}^{i+1}_{p^{i+1}}|\big)}{\partial\big(\mathbf{x}^{i+1}_{p^{i+1},1}, \mathbf{x}^{i+1}_{p^{i+1},2}, \mathbf{v}^{i+1}_{p^{i+1}}\big)}}_{=Q_{i+1}}. \end{aligned} \tag{2.91}$$

For Q_i,

$$Q_i = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{\partial \mathbf{v}^i_{p^i,1}}{\partial \widehat{\mathbf{v}}^i_{p^i,1}} & \frac{\partial \mathbf{v}^i_{p^i,1}}{\partial \widehat{\mathbf{v}}^i_{p^i,2}} & \frac{\partial \mathbf{v}^i_{p^i,1}}{\partial |\mathbf{v}^i_{p^i}|} \\ 0 & 0 & \frac{\partial \mathbf{v}^i_{p^i,2}}{\partial \widehat{\mathbf{v}}^i_{p^i,1}} & \frac{\partial \mathbf{v}^i_{p^i,2}}{\partial \widehat{\mathbf{v}}^i_{p^i,2}} & \frac{\partial \mathbf{v}^i_{p^i,2}}{\partial |\mathbf{v}^i_{p^i}|} \\ 0 & 0 & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial \widehat{\mathbf{v}}^i_{p^i,1}} & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial \widehat{\mathbf{v}}^i_{p^i,2}} & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial |\mathbf{v}^i_{p^i}|} \end{array}\right] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & |\mathbf{v}^i_{p^i}| & 0 & \frac{\partial \mathbf{v}^i_{p^i,1}}{\partial |\mathbf{v}^i_{p^i}|} \\ 0 & 0 & 0 & |\mathbf{v}^i_{p^i}| & \frac{\partial \mathbf{v}^i_{p^i,2}}{\partial |\mathbf{v}^i_{p^i}|} \\ 0 & 0 & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial \widehat{\mathbf{v}}^i_{p^i,1}} & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial \widehat{\mathbf{v}}^i_{p^i,2}} & \frac{\partial \mathbf{v}^i_{p^i,3}}{\partial |\mathbf{v}^i_{p^i}|} \end{array}\right]. \tag{2.92}$$

For Q_{i+1},

$$Q_{i+1} = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{\partial \widehat{\mathbf{v}}^{i+1}_{p^{i+1},1}}{\partial \mathbf{v}^{i+1}_{p^{i+1},1}} & \frac{\partial \widehat{\mathbf{v}}^{i+1}_{p^{i+1},1}}{\partial \mathbf{v}^{i+1}_{p^{i+1},2}} & \frac{\partial \widehat{\mathbf{v}}^{i+1}_{p^{i+1},1}}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \\ 0 & 0 & \frac{\partial \widehat{\mathbf{v}}^{i+1}_{p^{i+1},2}}{\partial \mathbf{v}^{i+1}_{p^{i+1},1}} & \frac{\partial \widehat{\mathbf{v}}^{i+1}_{p^{i+1},2}}{\partial \mathbf{v}^{i+1}_{p^{i+1},2}} & \frac{\partial \widehat{\mathbf{v}}^{i+1}_{p^{i+1},2}}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \\ 0 & 0 & \frac{\partial |\mathbf{v}^{i+1}_{p^{i+1}}|}{\partial \mathbf{v}^{i+1}_{p^{i+1},1}} & \frac{\partial |\mathbf{v}^{i+1}_{p^{i+1}}|}{\partial \mathbf{v}^{i+1}_{p^{i+1},2}} & \frac{\partial |\mathbf{v}^{i+1}_{p^{i+1}}|}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \end{array}\right] = \tag{2.93}$$

$$= \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \left|\mathbf{v}^{i+1}_{p^{i+1}}\right|^{-1} & 0 & \frac{\partial \hat{\mathbf{v}}^{i+1}_{p^{i+1},1}}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \\ 0 & 0 & 0 & \left|\mathbf{v}^{i+1}_{p^{i+1}}\right|^{-1} & \frac{\partial \hat{\mathbf{v}}^{i+1}_{p^{i+1},2}}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \\ 0 & 0 & \frac{\partial \left|\mathbf{v}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},1}} & \frac{\partial \left|\mathbf{v}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},2}} & \frac{\partial \left|\mathbf{v}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \end{array}\right].$$

Note that for $\ell = 1, 2$,

$$\frac{\partial \hat{\mathbf{v}}^{i+1}_{p^{i+1},\ell}}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} = \mathbf{v}^{i+1}_{p^{i+1},\ell} \frac{\partial}{\partial \mathbf{v}^{i+1}_{p^{i+1},3}} \left(\frac{1}{|\mathbf{v}^{i+1}_{p^{i+1}}|} \right) = \frac{\mathbf{v}^{i+1}_{p^{i+1},\ell} \mathbf{v}^{i+1}_{p^{i+1},3}}{|\mathbf{v}^{i+1}_{p^{i+1}}|^3}, \tag{2.94}$$

and for $k = 1, 2, 3$,

$$\frac{\partial \left|\mathbf{v}^{i+1}_{p^{i+1}}\right|}{\partial \mathbf{v}^{i+1}_{p^{i+1},k}} = -\frac{\mathbf{v}^{i+1}_{p^{i+1},k}}{|\mathbf{v}^{i+1}_{p^{i+1}}|}. \tag{2.95}$$

From (2.93), (2.94), and (2.95),

$$\begin{aligned} \det Q_{i+1} &= \frac{1}{|\mathbf{v}^{i+1}_{p^{i+1}}|} \left(-\frac{\mathbf{v}^{i+1}_{p^{i+1},3}}{|\mathbf{v}^{i+1}_{p^{i+1}}|^2} + \frac{\left(\mathbf{v}^{i+1}_{p^{i+1},2}\right)^2 \mathbf{v}^{i+1}_{p^{i+1},3}}{\left|\mathbf{v}^{i+1}_{p^{i+1}}\right|^4} \right) \\ &\quad + \frac{\mathbf{v}^{i+1}_{p^{i+1},1} \mathbf{v}^{i+1}_{p^{i+1},3}}{\left|\mathbf{v}^{i+1}_{p^{i+1}}\right|^3} \frac{\mathbf{v}^{i+1}_{p^{i+1},1}}{\left|\mathbf{v}^{i+1}_{p^{i+1}}\right|^2} \\ &= -\frac{\left(\mathbf{v}^{i+1}_{p^{i+1},3}\right)^3}{\left|\mathbf{v}^{i+1}_{p^{i+1}}\right|^5}. \end{aligned} \tag{2.96}$$

By taking the inverse and changing index $i + 1$ to i, we get

$$\det Q_i = -\frac{\left|\mathbf{v}^{i}_{p^{i}}\right|^5}{\left(\mathbf{v}^{i}_{p^{i},3}\right)^3}. \tag{2.97}$$

From (2.91), (2.97), (2.96), and Lemma 2.8, we get

$$\begin{aligned} &\left| \det \left[\begin{array}{cc} \nabla_{\mathbf{x}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1}} & \nabla_{\mathbf{v}^k_{p^k}} \mathbf{x}^{k+1}_{p^{k+1}} \\ \nabla_{\mathbf{x}^k_{p^k}} \mathbf{v}^{k+1}_{p^{k+1}} & \nabla_{\mathbf{v}^k_{p^k}} \mathbf{v}^{k+1}_{p^{k+1}} \end{array}\right]_{5\times 5} \right| \\ &\quad = (1 + O_{\Omega,N}(\|\Phi\|_{C^2})) \frac{\sqrt{g_{p^k,11}(x^k)} \sqrt{g_{p^k,22}(x^k)}}{\sqrt{g_{p^{k+1},11}(x^{k+1})} \sqrt{g_{p^{k+1},22}(x^{k+1})}} \frac{\left|\mathbf{v}^{k}_{p^k,3}\right|}{\left|\mathbf{v}^{k+1}_{p^{k+1},3}\right|}, \end{aligned}$$

$$
\begin{aligned}
\left|\det J_i^{i+1}\right| &= |\det Q_i \det P_i \det Q_{i+1}| \\
&= \frac{\left|\mathbf{v}^i_{p^i}\right|^5}{\left(\mathbf{v}^i_{p^i,3}\right)^3}(1+O_{\Omega,N}(\|\Phi\|_{C^2}))\frac{\sqrt{g_{p^i,11}}\sqrt{g_{p^i,22}}\big|_{x^i}}{\sqrt{g_{p^{i+1},11}}\sqrt{g_{p^{i+1},22}}\big|_{x^{i+1}}} \\
&\quad \cdot \frac{\left|\mathbf{v}^i_{p^i,3}\right|}{\left|\mathbf{v}^{i+1}_{p^{i+1},3}\right|}\frac{\left(\mathbf{v}^{i+1}_{p^{i+1},3}\right)^3}{\left|\mathbf{v}^{i+1}_{p^{i+1}}\right|^5} \\
&= (1+O_{\Omega,N}(\|\Phi\|_{C^2}))\frac{\sqrt{g_{p^i,11}}\sqrt{g_{p^i,22}}\big|_{x^i}}{\sqrt{g_{p^{i+1},11}}\sqrt{g_{p^{i+1},22}}\big|_{x^{i+1}}}\frac{\left|\mathbf{v}^{i+1}_{p^{i+1},3}\right|^2}{\left|\mathbf{v}^i_{p^i,3}\right|^2} \\
&\quad + O_{\Omega,N}(\|\Phi\|_{C^2}).
\end{aligned} \tag{2.98}
$$

Therefore,

$$
\begin{aligned}
\left|\det J_1^k\right| &= (1+O_{\Omega,N}(\|\Phi\|_{C^2}))\frac{\sqrt{g_{p^1,11}}\sqrt{g_{p^1,22}}\big|_{x^1}}{\sqrt{g_{p^k,11}}\sqrt{g_{p^k,22}}\big|_{x^k}}\frac{\left|\mathbf{v}^k_{p^k,3}\right|^2}{\left|\mathbf{v}^1_{p^1,3}\right|^2} \\
&\quad + O_{\Omega,N}(\|\Phi\|_{C^2}).
\end{aligned} \tag{2.99}
$$

Step 2. From (2.35),

$$
\begin{aligned}
&2|\mathbf{v}^{i+1}_{p^{i+1}}|\frac{\partial|\mathbf{v}^{i+1}_{p^{i+1}}|}{\partial\mathbf{v}^i_{p^i,n}} \\
&= \frac{\partial|V(t^{i+1};t^i,x^i,v^i)|^2}{\partial\mathbf{v}^i_{p^i,n}} = 2\frac{\partial V(t^{i+1};t^i,x^i,v^i)}{\partial\mathbf{v}^i_{p^i,n}}\cdot V(t^{i+1};t^i,x^i,v^i) \\
&= 2\Bigg(\frac{\partial_n\eta_{p^i}}{\sqrt{g_{p^i,nn}}}\bigg|_{x^i}+\frac{\partial(t^i-t^{i+1})}{\partial\mathbf{v}^i_{p^i,n}}\nabla_x\Phi(t^{i+1},x^{i+1}) \\
&\qquad + O_\Omega(\nabla_x^2\Phi\|_\infty)|t^i-t^{i+1}|^2e^{\|\nabla_x^2\Phi\|_\infty(t^i-t^{i+1})^2}\Bigg)\quad\cdot V(t^{i+1};t^i,x^i,v^i) \\
&= 2\mathbf{v}^i_{p^i,n}+2\left|\frac{\partial(t^i-t^{i+1})}{\partial\mathbf{v}^i_{p^i,n}}\right|\|\nabla_x\Phi\|_\infty|\mathbf{v}^{i+1}_{p^{i+1}}|+O_\Omega(\|\nabla_x\Phi\|_\infty)(t^i-t^{i+1}) \\
&\quad + O_\Omega(\|\nabla_x^2\Phi\|_\infty)|v^i|(t^i-t^{i+1})^2e^{\|\nabla_x^2\Phi\|_\infty(t^i-t^{i+1})^2}.
\end{aligned} \tag{2.100}
$$

Then by Lemma 2.2 and $\left|\frac{\partial(t^i-t^{i+1})}{\partial\mathbf{v}^i_{p^i,n}}\right| \lesssim_{\Omega,N} 1$, we get

$$
\frac{\partial|\mathbf{v}^{i+1}_{p^{i+1}}|}{\partial\mathbf{v}^i_{p^i,n}} = \frac{\mathbf{v}^i_{p^i,n}}{|\mathbf{v}^{i+1}_{p^{i+1}}|}+O_{\Omega,N}(\|\Phi\|_{C^2})\quad\text{for } n=1,2. \tag{2.101}
$$

From (2.35), for $n = 1, 2$,

$$
\begin{aligned}
2\big|\mathbf{v}^{i+1}_{p^{i+1}}\big|\frac{\partial\big|\mathbf{v}^{i+1}_{p^{i+1}}\big|}{\partial \mathbf{x}^{i}_{p^{i},n}} &= \frac{\partial |V(t^{i+1}; t^{i}, x^{i}, v^{i})|^{2}}{\partial \mathbf{x}^{i}_{p^{i},n}} \\
&= 2\frac{\partial V(t^{i+1}; t^{i}, x^{i}, v^{i})}{\partial \mathbf{x}^{i}_{p^{i},n}} \cdot V(t^{i+1}; t^{i}, x^{i}, v^{i}) \\
&= 2\bigg(\frac{\partial (t^{i} - t^{i+1})}{\partial \mathbf{x}^{i}_{p^{i},n}} \nabla_{x}\Phi(t^{i+1}, x^{i+1}) \\
&\qquad + O_{\Omega}(\|\nabla_{x}^{2}\Phi\|_{\infty})|t^{i} - t^{i+1}| e^{\|\nabla_{x}^{2}\Phi\|_{\infty}(t^{i}-t^{i+1})^{2}} \bigg) \\
&\quad \cdot V(t^{i+1}; t^{i}, x^{i}, v^{i}) \\
&\le O_{N,\Omega}(\|\Phi\|_{C^{2}}),
\end{aligned}
$$

where we have used $|\partial(t^{i} - t^{i+1})/\partial \mathbf{x}^{i}_{p^{i},n}| \lesssim_{N,\Omega} 1$ for $n = 1, 2$ from Lemma 2.2. This proves

$$
\frac{\partial\big|\mathbf{v}^{i+1}_{p^{i+1}}\big|}{\partial \mathbf{x}^{i}_{p^{i},n}} = O_{N,\Omega}(\|\Phi\|_{C^{2}}) \quad \text{for } n = 1, 2. \tag{2.102}
$$

Meanwhile,

$$
\begin{aligned}
\frac{\partial\big|\mathbf{v}^{i+1}_{p^{i+1}}\big|}{\partial\big|\mathbf{v}^{i}_{p^{i}}\big|} &= \sum_{\ell=1}^{2} \hat{\mathbf{v}}^{i}_{p^{i},\ell} \frac{\partial\big|\mathbf{v}^{i+1}_{p^{i+1}}\big|}{\partial \mathbf{v}^{i}_{p^{i},\ell}} + \sqrt{1 - \big(\hat{\mathbf{v}}^{i}_{p^{i},1}\big)^{2} - \big(\hat{\mathbf{v}}^{i}_{p^{i},2}\big)^{2}} \frac{\mathbf{v}^{i}_{p^{i},3}}{\big|\mathbf{v}^{i+1}_{p^{i+1}}\big|} \\
&\quad + O_{\Omega,N}(\|\Phi\|_{C^{2}}) \\
&= \sum_{\ell=1}^{2} \hat{\mathbf{v}}^{i}_{p^{i},\ell} \frac{\mathbf{v}^{i}_{p^{i},\ell}}{\big|\mathbf{v}^{i}_{p^{i}}\big|} + \sqrt{1 - \big(\hat{\mathbf{v}}^{i}_{p^{i},1}\big)^{2} - \big(\hat{\mathbf{v}}^{i}_{p^{i},2}\big)^{2}} \frac{\mathbf{v}^{i}_{p^{i},3}}{\big|\mathbf{v}^{i}_{p^{i}}\big|} + O_{\Omega,N}(\|\Phi\|_{C^{2}}) \\
&= 1 + O_{\Omega,N}(\|\Phi\|_{C^{2}}).
\end{aligned} \tag{2.103}
$$

Step 3. From (2.99), (2.101), (2.102), and (2.103),

$$\begin{aligned}\left|\det J_1^k\right| &= (1+O_{\Omega,N}(\|\Phi\|_{C^2}))\frac{\sqrt{g_{p^1,11}}\sqrt{g_{p^1,22}}\big|_{x^1}}{\sqrt{g_{p^k,11}}\sqrt{g_{p^k,22}}\big|_{x^k}}\frac{\left|\mathbf{v}^k_{p^k,3}\right|^2}{\left|\mathbf{v}^1_{p^1,3}\right|^2}\\ &\quad + O_{\Omega,N}(\|\Phi\|_{C^2})\\ &= \left|\det\left[\begin{array}{c|c|c} \frac{\partial(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2})}{\partial(\mathbf{x}^1_{p^1,1},\mathbf{x}^1_{p^1,2})} & \frac{\partial(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2})}{\partial(\widehat{\mathbf{v}}^1_{p^1,1},\widehat{\mathbf{v}}^1_{p^1,2})} & (*)\\ \hline \frac{\partial(\widehat{\mathbf{v}}^k_{p^k,1},\widehat{\mathbf{v}}^k_{p^k,2})}{\partial(\mathbf{x}^1_{p^1,1},\mathbf{x}^1_{p^1,2})} & \frac{\partial(\widehat{\mathbf{v}}^k_{p^k,1},\widehat{\mathbf{v}}^k_{p^k,2})}{\partial(\widehat{\mathbf{v}}^1_{p^1,1},\widehat{\mathbf{v}}^1_{p^1,2})} & (*)\\ \hline (*) & (*) & 1+(*)\end{array}\right]_{5\times5}\right|,\\ &\quad \text{where } (*) := O_{\Omega,N}(\|\Phi\|_{C^2}),\\ &= \left|\det\left[\frac{\partial(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},\widehat{\mathbf{v}}^k_{p^k,1},\widehat{\mathbf{v}}^k_{p^k,2})}{\partial(\mathbf{x}^1_{p^1,1},\mathbf{x}^1_{p^1,2},\widehat{\mathbf{v}}^1_{p^1,1},\widehat{\mathbf{v}}^1_{p^1,2})}\right]_{4\times4}\right| + O_{\Omega,N}(\|\Phi\|_{C^2}).\end{aligned}$$

Note that from (2.81), $k \lesssim_{\Omega,N,\delta_{1,2}} 1$ and $|\mathbf{v}^k_{p^k,3}| \lesssim_{\Omega,N,\delta_{1,2}} |\mathbf{v}^1_{p^1,3}|$. Therefore, we conclude (2.90). □

3 Transversality via the Geometric Decomposition and Triple Iterations

LEMMA 3.1. *Assume* $Y : (y_1, y_2) \mapsto Y(y_1, y_2) \in \mathbb{R}^3$ *is a* C^1*-map locally. For any* $t, s \geq 0$ *with* $s \in [t-1, t]$, $|n(x^1(t, Y(y_1, y_2), v)) \cdot v^1(t, Y(y_1, y_2), v)| > \delta$, $\frac{1}{N} \leq |v| \leq N$, $\frac{1}{N} \leq |v_3|$, $t^{k+1}(t, Y(y_1, y_2), v) < s < t^k(t, Y(y_1, y_2), v)$, *and* $\|\nabla\Phi\|_\infty < \frac{\delta}{3\,\mathrm{diam}(\Omega)N^2}$, *we have*

$$\begin{aligned}(3.1)\quad &\partial_{|v|}[X_i(s;t,Y(y_1,y_2),v)] =\\ &\quad -(t-s)\sum_{\ell=1}^{3}\frac{\partial_\ell\eta_{p^k,i}}{\sqrt{g_{p^k,\ell\ell}}}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\frac{\mathbf{v}^k_{p^k,\ell}}{\left|\mathbf{v}^k_{p^k}\right|} + O_{\delta,N}(\|\Phi\|_{C^2}),\end{aligned}$$

and for $\partial \in \{\partial_{\widehat{v}_1}, \partial_{\widehat{v}_2}, \partial_{y_1}, \partial_{y_2}\}$,

$$
\begin{aligned}
&\partial[X_i(s;t,Y(y_1,y_2),v)]\\
&\quad= -\partial t^k |\mathbf{v}^k_{p^k}| \sum_{\ell=1}^{3} \frac{\partial_\ell \eta_{p^k,i}}{\sqrt{g_{p^k,\ell\ell}}}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big)\frac{\mathbf{v}^k_{p^k\ell}}{|\mathbf{v}^k_{p^k}|}\\
&\qquad+ \sum_{\ell=1}^{2} \partial \mathbf{x}^k_{p^k,\ell}\partial_\ell \eta_{p^k,i}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big)\\
&\qquad- (t^k - s)|\mathbf{v}^k_{p^k}| \sum_{j=1}^{2}\Bigg(\sum_{\ell=1}^{3} \frac{\partial}{\partial \mathbf{x}^k_{p^k,j}}\Bigg[\frac{\partial_\ell \eta_{p^k,i}}{\sqrt{g_{p^k,\ell\ell}}}\Bigg]\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big)\widehat{\mathbf{v}}^k_{p^k,\ell}\Bigg)\partial \mathbf{x}^k_{p^k,j}\\
&\qquad- (t^k - s)|\mathbf{v}^k_{p^k}| \sum_{j=1}^{2}\Bigg[\frac{\partial_j \eta_{p^k,i}}{\sqrt{g_{p^k,11}}}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big)\\
&\qquad\qquad- \frac{\partial_3 \eta_{p^k,i}}{\sqrt{g_{p^k,33}}}\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0\big)\frac{\widehat{\mathbf{v}}^k_{p^k,j}}{\widehat{\mathbf{v}}^k_{p^k,3}}\Bigg]\partial \widehat{\mathbf{v}}^k_{p^k,j}\\
&\qquad+ O_{\delta,N}(\|\Phi\|_{C^2}).
\end{aligned}
\tag{3.2}
$$

Here

$$
t^k = t^k(t, Y(y_1,y_2), v), \quad \mathbf{x}^k_{p^k} = \mathbf{x}^k_{p^k}(t, Y(y_1,y_2), v),
$$
$$
\textit{and} \quad \mathbf{v}^k_{p^k} = \mathbf{v}^k_{p^k}(t, Y(y_1,y_2), v).
$$

PROOF.

Step 1. We claim that

$$
\begin{aligned}
&\frac{\partial((t^j - t^{j+1})|v^k|)}{\partial |v|} = O_{\Omega,N}(\|\Phi\|_{C^2}), \quad \frac{\partial |v^k|}{\partial |v|} = 1 + O_{\Omega,N}(\|\Phi\|_{C^2}),\\
&\frac{\partial \mathbf{x}^j_{p^j,i}}{\partial |v|} = O_{\Omega,N}(\|\Phi\|_{C^2}), \quad \frac{\partial \widehat{\mathbf{v}}^j_{p^j,i}}{\partial |v|} = O_{\Omega,N}(\|\Phi\|_{C^2}).
\end{aligned}
\tag{3.3}
$$

By the chain rule,

$$
\begin{aligned}
\begin{bmatrix} \nabla_{y_1,y_2,\widehat{v}_1,\widehat{v}_2,|v|} t^k \\ \nabla_{y_1,y_2,\widehat{v}_1,\widehat{v}_2,|v|} \mathbf{x}^k_{p^k} \\ \nabla_{y_1,y_2,\widehat{v}_1,\widehat{v}_2,|v|} \widehat{\mathbf{v}}^k_{p^k} \\ \nabla_{y_1,y_2,\widehat{v}_1,\widehat{v}_2,|v|} |\mathbf{v}^k_{p^k}| \end{bmatrix} &= \begin{bmatrix} \nabla_{t^{k-1},\mathbf{x}^{k-1}_{p^{k-1}},\widehat{\mathbf{v}}^{k-1}_{p^{k-1}},|\mathbf{v}^{k-1}_{p^{k-1}}|} t^k \\ \nabla_{t^{k-1},\mathbf{x}^{k-1}_{p^{k-1}},\widehat{\mathbf{v}}^{k-1}_{p^{k-1}},|\mathbf{v}^{k-1}_{p^{k-1}}|} \mathbf{x}^k_{p^k} \\ \nabla_{t^{k-1},\mathbf{x}^{k-1}_{p^{k-1}},\widehat{\mathbf{v}}^{k-1}_{p^{k-1}},|\mathbf{v}^{k-1}_{p^{k-1}}|} \widehat{\mathbf{v}}^k_{p^k} \\ \nabla_{t^{k-1},\mathbf{x}^{k-1}_{p^{k-1}},\widehat{\mathbf{v}}^{k-1}_{p^{k-1}},|\mathbf{v}^{k-1}_{p^{k-1}}|} |\mathbf{v}^k_{p^k}| \end{bmatrix} \times \cdots\\
&\quad \cdots \times \begin{bmatrix} \nabla_{t^1,\mathbf{x}^1_{p^1},\widehat{\mathbf{v}}^1_{p^1},|\mathbf{v}^1_{p^1}|} t^2 \\ \nabla_{t^1,\mathbf{x}^1_{p^1},\widehat{\mathbf{v}}^1_{p^1},|\mathbf{v}^1_{p^1}|} \mathbf{x}^2_{p^2} \\ \nabla_{t^1,\mathbf{x}^1_{p^1},\widehat{\mathbf{v}}^1_{p^1},|\mathbf{v}^1_{p^1}|} \widehat{\mathbf{v}}^2_{p^2} \\ \nabla_{t^1,\mathbf{x}^1_{p^1},\widehat{\mathbf{v}}^1_{p^1},|\mathbf{v}^1_{p^1}|} |\mathbf{v}^2_{p^2}| \end{bmatrix} \begin{bmatrix} \nabla_{y_1,y_2,\widehat{v}_1,\widehat{v}_2,|v|} t^1 \\ \nabla_{y_1,y_2,\widehat{v}_1,\widehat{v}_2,|v|} \mathbf{x}^1_{p^1} \\ \nabla_{y_1,y_2,\widehat{v}_1,\widehat{v}_2,|v|} \widehat{\mathbf{v}}^1_{p^1} \\ \nabla_{y_1,y_2,\widehat{v}_1,\widehat{v}_2,|v|} |\mathbf{v}^1_{p^1}| \end{bmatrix}.
\end{aligned}
\tag{3.4}
$$

We claim that

$$\begin{bmatrix} \nabla_{t^j,\mathbf{x}^j_{p^j},\widehat{\mathbf{v}}^j_{p^j},|\mathbf{v}^j_{p^j}|} t^{j+1} \\ \nabla_{t^j,\mathbf{x}^j_{p^j},\widehat{\mathbf{v}}^j_{p^j},|\mathbf{v}^j_{p^j}|} \mathbf{x}^{j+1}_{p^{j+1}} \\ \nabla_{t^j,\mathbf{x}^j_{p^j},\widehat{\mathbf{v}}^j_{p^j},|\mathbf{v}^j_{p^j}|} \widehat{\mathbf{v}}^{j+1}_{p^{j+1}} \\ \nabla_{t^j,\mathbf{x}^j_{p^j},\widehat{\mathbf{v}}^j_{p^j},|\mathbf{v}^j_{p^j}|} |\mathbf{v}^{j+1}_{p^{j+1}}| \end{bmatrix} = \tag{3.5}$$

$$\left[\begin{array}{c|cccc|c} 1 & 0 & 0 & 0 & 0 & -\frac{t^{j+1}}{|\mathbf{v}^j_{p^j}|} \\ \hline 0 & & & & & 0 \\ 0 & & & & & 0 \\ 0 & & O_{\Omega,N,\delta}(1) & & & 0 \\ 0 & & & & & 0 \\ 0 & & & & & 1 \end{array}\right] + O_{\Omega,N}(\|\Phi\|_{C^2}).$$

Once (3.5) is proven, from the chain rule (3.4) and Lemma 2.4, we conclude (3.3).

From (2.1),

$$\begin{aligned} v^j(t^j - t^{j+1}) &= \eta_{p^{j+1}}\big(\mathbf{x}^{j+1}_{p^{j+1},1}, \mathbf{x}^{j+1}_{p^{j+1},2}, 0\big) - \eta_{p^j}\big(\mathbf{x}^j_{p^j,1}, \mathbf{x}^j_{p^j,2}, 0\big) \\ &\quad - \int_{t^j}^{t^{j+1}} \int_{t^j}^{s} \nabla_x \Phi(\tau, X(\tau; t^j, x^j, v^j)) \mathrm{d}\tau \, \mathrm{d}s. \end{aligned}$$

Taking $\frac{\partial}{\partial t^j}$ directly to the above equality, we derive

$$\begin{aligned} v^j\left(1 - \frac{\partial t^{j+1}}{\partial t^j}\right) &= -\frac{\partial t^{j+1}}{\partial t^j} \int_{t^j}^{t^{j+1}} \nabla_x \Phi(\tau, X(\tau; t^j, x^j, v^j)) \mathrm{d}\tau \\ &\quad + (t^{j+1} - t^j) \nabla_x \Phi(t^j, x^j), \end{aligned}$$

and $\frac{\partial t^{j+1}}{\partial t^j} = 1 + \|\nabla_x \Phi\|_\infty |t^j - t^{j+1}| / |\mathbf{v}^{j+1}_{p^{j+1}}|$. Now from Lemma 2.6,

$$\big|\mathbf{v}^{j+1}_{p^{j+1}}\big| = |v| + O(\|\nabla \Phi\|_\infty)|t - t^{j+1}| \geq \frac{1}{N} + \frac{\delta \times 3N \operatorname{diam}(\Omega)}{3 \operatorname{diam}(\Omega) N^2} \gtrsim \frac{1}{N}.$$

Therefore we conclude that

$$\frac{\partial t^{j+1}}{\partial t^j} = 1 + O_{\Omega,N}(\|\Phi\|_{C^2}). \tag{3.6}$$

From (2.22), we derive

$$\begin{aligned} \frac{\partial \mathbf{x}^{j+1}_{p^{j+1},i}}{\partial t^j} &= \left(\frac{\partial t^{j+1}}{\partial t^j} - 1\right) \mathbf{v}^{j+1}_{p^{j+1},i} \\ &\quad + \|\Phi\|_{C^2} |t^j - t^{j+1}|^2 \sup_{t^{j+1} \leq \tau \leq t^j} \left| \frac{\partial X(\tau; t^j, x^j, v^j)}{\partial t^j} \right| \\ &= O_{\Omega,N}(\|\Phi\|_{C^2}), \end{aligned} \tag{3.7}$$

where we have used the fact that

$$\sup_{t^{j+1}\le\tau\le t^j}\left|\frac{\partial X(\tau;t^j,x^j,v^j)}{\partial t^j}\right|\lesssim |v^j|+\|\nabla_x\Phi\|_\infty|t^j-t^{j+1}|\le C_{N,\Omega},$$

which is proved similarly to the proof of (2.28).

From (2.1), we have

$$|v^{j+1}|^2=|v^j|^2-2\int_{t^j}^{t^{j+1}}v^j\cdot\nabla_x\Phi(\tau,X(\tau;t^j,x^j,v^j))\mathrm{d}\tau+\left|\int_{t^j}^{t^{j+1}}\nabla_x\Phi(\tau)\right|^2.$$

Then

$$\begin{aligned}&2|v^{j+1}|\frac{\partial|v^{j+1}|}{\partial t^j}\\&\quad=-2\frac{\partial t^{j+1}}{\partial t^j}v^j\cdot\nabla_x\Phi(t^{j+1})+2v^j\cdot\nabla_x\Phi(t^j)\\&\qquad+2\left\{\|\nabla_x^2\Phi\|_\infty|v^j|(t^j-t^{j+1})+\sup_{t^{j+1}\le\tau\le t^j}\left|\frac{\partial X(\tau)}{\partial t^j}\right|\right\}\sup_{t^{j+1}\le\tau\le t^j}\left|\frac{\partial X(\tau)}{\partial t^j}\right|,\end{aligned}$$

and hence

$$\frac{\partial|v^{j+1}|}{\partial t^j}=O_{\Omega,N}(\|\Phi\|_{C^2}).\tag{3.8}$$

From (2.31), we prove

$$\begin{aligned}&\frac{\partial\hat{\mathbf{v}}^{j+1}_{p^{j+1},i}}{\partial t^j}\\&\quad=O_{\|\eta\|_{C^2}}\left(\left|\frac{\partial\mathbf{x}^j_{p^j}}{\partial t^j}\right|\right)\{|v^j|+\|\nabla\Phi\|_\infty(t^j-t^{j+1})\}+O_{\|\eta\|_{C^1}}(\|\nabla_x\Phi\|_\infty)\\&\quad=O_{N,\Omega}(\|\Phi\|_{C^2}).\end{aligned}\tag{3.9}$$

We already have estimates for $\partial t^{j+1}/\partial\mathbf{x}^j_{p^j,i}$ in Lemma 2.2.

From Lemma 2.2,

$$\frac{\partial t^{j+1}}{\partial|\mathbf{v}^j_{p^j}|}=-\frac{t^{j+1}}{|\mathbf{v}^j_{p^j}|}+O_{\Omega,N}(\|\Phi\|_{C^2})\quad\text{and}\quad\frac{\partial t^{j+1}}{\partial\hat{\mathbf{v}}^j_{p^j,i}}=O_{\Omega,N}(\|\Phi\|_{C^2}).$$

Moreover, from the conditions $|n(x^1(t,Y(y_1,y_2),v))\cdot v^1(t,Y(y_1,y_2),v)|>\delta$, $\frac{1}{N}\le|v|\le N$, Lemma 2.7, (2.80), and (2.81), we have

$$\left|\mathbf{v}^j_{p^j,3}(t,Y(y_1,y_2),v)\right|\gtrsim\delta.\tag{3.10}$$

Then, from Lemma 2.2,

$$\left|\frac{\partial\left(\mathbf{x}^{j+1}_{p^{j+1}},\hat{\mathbf{v}}^{j+1}_{p^{j+1}}\right)}{\partial\left(\mathbf{x}^j_{p^j},\hat{\mathbf{v}}^j_{p^j},|\mathbf{v}^j_{p^j}|\right)}\right|=O_{\Omega,N,\delta}(1).\tag{3.11}$$

From (3.6) to (3.11), we prove (3.5).

Step 2. Recall that, from (2.1), for $t^{k+1} \le s < t^k$

$$
\begin{aligned}
&X_i(s;t,Y(x_1,x_2),v)\\
&\quad= X_i\big(s;t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k}\big)\\
&\quad= \eta_{p^k,i}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)-(t^k-s)|v^k|\widehat{v}^k_{,i}\\
&\qquad-\int_{t^k}^{s}\int_{t^k}^{\tau}\partial_i\Phi\big(\tau';X\big(\tau';t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k}\big)\big)\mathrm{d}\tau'\,\mathrm{d}\tau,
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&V_i\big(s;t,Y(x_1,x_2),v\big)\\
&\quad= V_i\big(s;t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k}\big)\\
&\quad= |v^k|\widehat{v}^k_{,i}-\int_{t^k}^{s}\partial_i\Phi\big(\tau;X\big(\tau;t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k}\big)\big)\mathrm{d}\tau,
\end{aligned}
$$

where the specular cycles are defined in (2.4) as

$$
\big(t^k,\mathbf{x}^k_{p^k},\mathbf{v}^k_{p^k}\big)=\big(t^k(t,Y(y_1,y_2),v),\mathbf{x}^k_{p^k}(t,Y(y_1,y_2),v),\mathbf{v}^k_{p^k}(t,Y(y_1,y_2),v)\big).
$$

By direct computations, for $\partial=\partial_{|v|}$,

$$
\begin{aligned}
&\partial_{|v|}[X_i(s;t,Y(x_1,x_2),v)]\\
&\quad=\sum_{\ell=1}^{2}\partial_{|v|}\mathbf{x}^k_{p^k,\ell}\cdot\partial_\ell\eta_{p^k,i}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\\
&\qquad+\partial_{|v|}\big[(t-t^k)|v^k|\big]\widehat{v}^k-(t-s)\partial_{|v|}|v^k|\widehat{v}^k-(t^k-s)|v^k|\partial_{|v|}[\widehat{v}^k]\\
&\qquad-\int_{t^k}^{s}\int_{t^k}^{\tau}\Bigg(\partial_{|v|}t^k\partial_{t^k}X(\tau';t^k)\\
&\qquad\qquad+\sum_{\ell=1}^{2}\partial_{|v|}\mathbf{x}^k_{p^k,\ell}\partial_{\mathbf{x}^k_{p^k,\ell}}X(\tau';t^k)+\partial_{|v|}\mathbf{v}^k_{p^k,\ell}\partial_{\mathbf{v}^k_\ell}X(\tau';t^k)\Bigg)\\
&\qquad\qquad\cdot\nabla\partial_i\Phi\big(\tau';X\big(\tau';t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k}\big)\big)\mathrm{d}\tau'\,\mathrm{d}\tau\\
&\qquad+\partial_{|v|}t^k(s-t^k)\lim_{\tau'\uparrow t^k}\partial_i\Phi\big(\tau';X\big(\tau';t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k}\big)\big),
\end{aligned}
$$

where we have used the abbreviated notation $X(\tau';t^k)$ for $X(\tau';t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k})$. From (3.3), we bound the first, second, fourth, fifth, and last line of the right-hand side by $O_{\Omega,N}(\|\Phi\|_{C^2})$. Finally, we apply (3.3) to the third line and conclude (3.1).

Step 3. First we compute $\partial\hat{v}^k$ with any arbitrary derivative ∂. Note that from (2.7) and (2.10), $\hat{\mathbf{v}}^k_{p^k,3} > 0$ and $\hat{\mathbf{v}}^k_{p^k,3} = \sqrt{1-|\hat{\mathbf{v}}^k_{p^k,1}|^2 - |\hat{\mathbf{v}}^k_{p^k,2}|^2}$. Therefore

$$\begin{aligned}
\partial\hat{v}^k &= \partial\Bigg[\sum_{\ell=1}^{2}\frac{\partial_\ell\eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\hat{\mathbf{v}}^k_{p^k,\ell}\\
&\qquad+\frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\sqrt{1-|\hat{\mathbf{v}}^k_{p^k,1}|^2-|\hat{\mathbf{v}}^k_{p^k,2}|^2}\Bigg]\\
&=\sum_{\ell=1}^{3}\sum_{m=1}^{2}\partial\mathbf{x}^k_{p^k,m}\partial_m\left[\frac{\partial_\ell\eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\hat{\mathbf{v}}^k_{p^k,\ell}\\
&\quad+\sum_{\ell=1}^{2}\frac{\partial_\ell\eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\big(\mathbf{x}^k_{p,1},\mathbf{x}^k_{p,2},0\big)\partial\big[\hat{\mathbf{v}}^k_{p^k,\ell}\big]\\
&\quad-\frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\frac{1}{\sqrt{1-|\hat{\mathbf{v}}^k_{p^k,1}|^2-|\hat{\mathbf{v}}^k_{p^k,2}|^2}}\\
&\qquad\cdot\big[\hat{\mathbf{v}}^k_{p^k,1}\partial\big[\hat{\mathbf{v}}^k_{p^k,1}\big]+\hat{\mathbf{v}}^k_{p^k,2}\partial\big[\hat{\mathbf{v}}^k_{p^k,2}\big]\big]\\
&=\sum_{j=1}^{2}\Bigg(\sum_{\ell=1}^{3}\frac{\partial}{\partial\mathbf{x}^k_{p^k,j}}\left[\frac{\partial_\ell\eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\hat{\mathbf{v}}^k_{p^k,\ell}\Bigg)\partial\mathbf{x}^k_{p^k,j}\\
&\quad+\sum_{j=1}^{2}\Bigg[\frac{\partial_j\eta_{p^k}}{\sqrt{g_{p^k,11}}}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\\
&\qquad\qquad-\frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)\frac{\hat{\mathbf{v}}^k_{p^k,j}}{\hat{\mathbf{v}}^k_{p^k,3}}\Bigg]\partial\big[\hat{\mathbf{v}}^k_{p^k,j}\big].
\end{aligned}$$

From (3.12), for $\partial\in\{\partial_{\hat{v}_1},\partial_{\hat{v}_2},\partial_{y_1},\partial_{y_2}\}$,

$$\begin{aligned}
&\partial[X_i(s;t,Y(y_1,y_2),v)]\\
&\quad=\sum_{\ell=1}^{2}\partial\mathbf{x}^k_{p^k,\ell}\cdot\partial_\ell\eta_{p^k,i}\big(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0\big)-\partial t^k\big|\mathbf{v}^k_{p^k}\big|\hat{v}^k\\
&\qquad-(t^k-s)\partial|\mathbf{v}^k_{p^k}|\hat{v}^k-(t^k-s)|\mathbf{v}^k_{p^k}|\partial\hat{v}^k\\
&\qquad-\int_{t^k}^{s}\int_{t^k}^{\tau}\Bigg(\partial t^k\partial_{t^k}X(\tau';t^k)+\sum_{\ell=1}^{2}\partial\mathbf{x}^k_{p^k,\ell}\partial_{\mathbf{x}^k_{p^k,\ell}}X(\tau';t^k)\\
&\qquad\qquad\qquad+\partial\mathbf{v}^k_{p^k,\ell}\big(\mathbf{x}^k_{p^k}\big)\partial_{\mathbf{v}^k_{p^k,\ell}}X(\tau';t^k)\Bigg)\\
&\qquad\qquad\cdot\nabla\partial_i\Phi\big(\tau';X\big(\tau';t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k}\big)\big)\mathrm{d}\tau'\,\mathrm{d}\tau\\
&\qquad+\partial t^k(s-t^k)\lim_{\tau'\uparrow t^k}\partial_i\Phi\big(\tau';X\big(\tau';t^k,\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},0,\mathbf{v}^k_{p^k}\big)\big).
\end{aligned}$$

We can easily conclude (3.2) by (3.3) and Step 2. □

Recall the specular cycles (t^k, x^k, v^k) in (2.4) and η_{p^k} in (2.6). Assume

$$n(x^k)\cdot v^k \neq 0. \tag{3.13}$$

DEFINITION 3.2 (Specular Basis). We define the *specular basis*, which is an orthonormal basis of $\mathbb{R}^3$, as

$$\begin{aligned}
\mathbf{e}_0^k &:= \frac{v^k}{|v^k|},\\
\mathbf{e}_{\perp,1}^k &:= \mathbf{e}_0^k \times \frac{\partial_2\eta_{p^k}(x^k)}{\sqrt{g_{p^k,22}(x^k)}} \Bigg/ \left|\mathbf{e}_0^k \times \frac{\partial_2\eta_{p^k}(x^k)}{\sqrt{g_{p^k,22}(x^k)}}\right|,\\
\mathbf{e}_{\perp,2}^k &:= \mathbf{e}_0^k \times \mathbf{e}_{\perp,1}^k.
\end{aligned} \tag{3.14}$$

DEFINITION 3.3 (Specular Matrix). For fixed $k \in \mathbb{N}$ and a C^1-map

$$Y : (y_1, y_2) \mapsto Y(y_1, y_2),$$

assume (3.13) with $x^k = x^k(t, Y(y_1, y_2), |v|, \hat{v}_1, \hat{v}_2)$ and $v^k = v^k(t, Y(y_1, y_2), |v|, \hat{v}_1, \hat{v}_2)$. We define the 4×4 specular transition matrix $\mathcal{S}^{k,p^k,Y} = \mathcal{S}^{k,p^k,Y}(t, y_1, y_2, |v|, \hat{v}_1, \hat{v}_2)$ as

$$\mathcal{S}^{k,p^k,Y} := \left[\begin{array}{c|c} \mathcal{S}_1^{k,p^k,Y} & 0_{2\times 2} \\ \hline \mathcal{S}_2^{k,p^k,Y} & \mathcal{S}_3^{k,p^k,Y} \end{array}\right]_{4\times 4}, \tag{3.15}$$

where

$$\mathcal{S}_1^{k,p^k,Y} := \begin{bmatrix} \partial_1\eta_{p^k}\cdot\mathbf{e}_{\perp,1}^k & \partial_2\eta_{p^k}\cdot\mathbf{e}_{\perp,1}^k \\ \partial_1\eta_{p^k}\cdot\mathbf{e}_{\perp,2}^k & \partial_2\eta_{p^k}\cdot\mathbf{e}_{\perp,2}^k \end{bmatrix}_{2\times 2},$$

$$\mathcal{S}_2^{k,p^k,Y} := \begin{bmatrix} \left(\sum_{\ell=1}^3 \partial_1\left[\frac{\partial_\ell\eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\hat{\mathbf{v}}_{p^k,\ell}^k\right)\cdot\mathbf{e}_{\perp,1}^k & \left(\sum_{\ell=1}^3 \partial_2\left[\frac{\partial_\ell\eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\hat{\mathbf{v}}_{p,\ell}^k\right)\cdot\mathbf{e}_{\perp,1}^k \\ \left(\sum_{\ell=1}^3 \partial_1\left[\frac{\partial_\ell\eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\hat{\mathbf{v}}_{p^k,\ell}^k\right)\cdot\mathbf{e}_{\perp,2}^k & \left(\sum_{\ell=1}^3 \partial_2\left[\frac{\partial_\ell\eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}}\right]\hat{\mathbf{v}}_{p^k,\ell}^k\right)\cdot\mathbf{e}_{\perp,2}^k \end{bmatrix}_{2\times 2},$$

$$\mathcal{S}_3^{k,p^k,Y} := \begin{bmatrix} \left[\frac{\partial_1\eta_{p^k}}{\sqrt{g_{p^k,11}}} - \frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\frac{\hat{\mathbf{v}}_{p^k,1}^k}{\hat{\mathbf{v}}_{p^k,3}^k}\right]\cdot\mathbf{e}_{\perp,1}^k & \left[\frac{\partial_2\eta_{p^k}}{\sqrt{g_{p^k,22}}} - \frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\frac{\hat{\mathbf{v}}_{p^k,2}^k}{\hat{\mathbf{v}}_{p^k,3}^k}\right]\cdot\mathbf{e}_{\perp,1}^k \\ \left[\frac{\partial_1\eta_{p^k}}{\sqrt{g_{p^k,11}}} - \frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\frac{\hat{\mathbf{v}}_{p^k,1}^k}{\hat{\mathbf{v}}_{p^k,3}^k}\right]\cdot\mathbf{e}_{\perp,2}^k & \left[\frac{\partial_2\eta_{p^k}}{\sqrt{g_{p^k,22}}} - \frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\frac{\hat{\mathbf{v}}_{p^k,2}^k}{\hat{\mathbf{v}}_{p^k,3}^k}\right]\cdot\mathbf{e}_{\perp,2}^k \end{bmatrix}_{2\times 2},$$

where η_{p^k} and g_{p^k} are evaluated at $x^k(t, Y(y_1, y_2), |v|, \hat{v}_1, \hat{v}_2)$. We also define the 4×4 *specular matrix* $\mathcal{R}^{k,p^k,Y} = \mathcal{R}^{k,p^k,Y}(t, y_1, y_2, |v|, \hat{v}_1, \hat{v}_2)$ as

$$\mathcal{R}^{k,p^k,Y} := \mathcal{S}^{k,p^k,Y}\frac{\partial\left(\mathbf{x}_{p^k,1}^k, \mathbf{x}_{p^k,2}^k, \hat{\mathbf{v}}_{p^k,1}^k, \hat{\mathbf{v}}_{p^k,1}^k\right)}{\partial(y_1, y_2, \hat{v}_1, \hat{v}_2)} \tag{3.16}$$

where $\mathbf{x}_{p^k}^k = \mathbf{x}_{p^k}^k(t, Y(y_1, y_2), |v|, \hat{v}_1, \hat{v}_2)$, $\mathbf{v}_{p^k}^k = \mathbf{v}_{p^k}^k(t, Y(y_1, y_2), |v|, \hat{v}_1, \hat{v}_2)$.

Finally, we state the result that is a crucial ingredient in the proofs of Lemma 3.6 and Lemma 3.7. For $n \times m$ matrix A, we use the notation $A_{i,j}$ for its (i,j)-entry. Recall that $e_3 = (0,0,1) \in \mathbb{R}^3$ and $v_3 = v \cdot e_3$.

LEMMA 3.4. *Let a C^1-map $Y : (y_1, y_2) \mapsto Y(y_1, y_2) \in \bar{\Omega}$ with $\|Y\|_{C^1} \lesssim 1$. Assume $\frac{1}{N} \le |v| \le N$, $\frac{1}{N} \le |v_3|$, $\frac{1}{N} < |n(x^1(t, Y(y_1, y_2), v)) \cdot e_3|$, and $\|\Phi\|_{C^2_x} < \frac{\delta_1}{3\,\mathrm{diam}(\Omega) N^2}$ for $1 \ll N$ and $0 < \delta_1 \ll 1$. We also assume a nongrazing condition*

$$|v^1(t, Y(y_1, y_2), v) \cdot n(x^1(t, Y(y_1, y_2), v))| > \delta_2 > 0 \tag{3.17}$$

and nondegenerate condition

$$\left| \left(\frac{\partial Y(y_1, y_2)}{\partial y_1} \times \frac{\partial Y(y_1, y_2)}{\partial y_2} \right) \cdot R_{x^1(t, Y(y_1, y_2), v)} v^1(t, Y(y_1, y_2), v) \right| > \delta_3 > 0. \tag{3.18}$$

Fix $k \in \mathbb{N}$ with $|t - t^k| \le 1$. Then the following results hold:

(i) *For some constant $\varrho_{\Omega, N, \delta_1, \delta_2, \delta_3} > 0$, there exists at least one $i \in \{1, 2, 3, 4\}$ such that*

$$\left| \mathscr{R}^{k, p^k, Y}_{i,3}(t, Y(y_1, y_2), v) \right| > \varrho_{\Omega, N, \delta_1, \delta_2, \delta_3}. \tag{3.19}$$

(ii) *There exist $i, j \in \{1, 2, 3, 4\}$ with $i < j$ such that*

$$\left| \det \begin{bmatrix} \mathscr{R}^{k, p^k, Y}_{3,i} & \mathscr{R}^{k, p^k, Y}_{3,j} \\ R^{k, p^k, Y}_{4,i} & \mathscr{R}^{k, p^k, Y}_{4,j} \end{bmatrix} (t, Y(y_1, y_2), v) \right| > \varrho_{\Omega, N, \delta_1, \delta_2, \delta_3}. \tag{3.20}$$

PROOF.

Step 1. We claim that

$$\left| \det \mathscr{R}^{k, p^k, Y}(t, Y(y_1, y_2), v) \right| \gtrsim_{\Omega, N, \delta_1, \delta_2, \delta_3} 1. \tag{3.21}$$

Note that from (3.16) and (3.15),

$$\begin{aligned} \det(\mathscr{R}^{k, p^k, Y}) = {} & \det\big(\mathscr{S}^{k, p^k, Y}_1\big) \det\big(\mathscr{S}^{k, p^k, Y}_3\big) \\ & \times \det\left(\frac{\partial\big(\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, \hat{\mathbf{v}}^k_{p^k,1}, \hat{\mathbf{v}}^k_{p^k,1}\big)}{\partial(y_1, y_2, \hat{v}_1, \hat{v}_2)} \right). \end{aligned}$$

By (2.86) and (3.14),

$$\begin{aligned} \det\big(\mathscr{S}^{k, p^k, Y}_1\big) &= \big|(\partial_1 \eta_{p^k} \times \partial_1 \eta_{p^k}) \cdot \big(\mathbf{e}^k_{\perp,1} \times \mathbf{e}^k_{\perp,2}\big)\big| \\ &= \sqrt{g_{p^k,11}(x^k) g_{p^k,22}(x^k)} \left| \frac{\mathbf{v}^k_{p^k}}{|\mathbf{v}^k_{p^k}|} \cdot n(x^k) \right| \\ &= \sqrt{g_{p^k,11}(x^k) g_{p^k,22}(x^k)} \frac{|\mathbf{v}^k_{p^k,3}|}{|\mathbf{v}^k_{p^k}|}, \end{aligned}$$

$$
\begin{aligned}
\det(\mathcal{S}_3^{k,p^k,Y}) &= \left|\left(\left[\frac{\partial_1\eta_{p^k}}{\sqrt{g_{p^k,11}}} - \frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\frac{\hat{\mathbf{v}}^k_{p^k,1}}{\hat{\mathbf{v}}^k_{p^k,3}}\right]\right.\right. \\
&\qquad \left.\left.\times\left[\frac{\partial_2\eta_{p^k}}{\sqrt{g_{p^k,22}}} - \frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\frac{\hat{\mathbf{v}}^k_{p^k,2}}{\hat{\mathbf{v}}^k_{p^k,3}}\right]\right)\cdot\left(\mathbf{e}^k_{\perp,1}\times\mathbf{e}^k_{\perp,2}\right)\right| = \\
&= \frac{1}{|\hat{\mathbf{v}}^k_{p^k,3}|}\left|\left(\hat{\mathbf{v}}^k_{p^k,1}\frac{\partial_1\eta_{p^k}}{\sqrt{g_{p^k,11}}} + \hat{\mathbf{v}}^k_{p^k,2}\frac{\partial_2\eta_{p^k}}{\sqrt{g_{p^k,22}}} + \hat{\mathbf{v}}^k_{p^k,3}\frac{\partial_3\eta_{p^k}}{\sqrt{g_{p^k,33}}}\right)\right. \\
&\qquad \left.\cdot\left(\mathbf{e}^k_{\perp,1}\times\mathbf{e}^k_{\perp,2}\right)\right| \\
&= \frac{1}{|\mathbf{v}^k_{p^k,3}|}\mathbf{v}^k_{p^k}\cdot\frac{\mathbf{v}^k_{p^k}}{|\mathbf{v}^k_{p^k}|} = \frac{|\mathbf{v}^k_{p^k}|}{|\mathbf{v}^k_{p^k,3}|}.
\end{aligned}
$$

From the chain rule and (2.90),

$$
\begin{aligned}
&\left|\det\left(\frac{\partial(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},\hat{\mathbf{v}}^k_{p^k,1},\hat{\mathbf{v}}^k_{p^k,1})}{\partial(y_1,y_2,\hat{v}_1,\hat{v}_2)}\right)\right| \\
&= \left|\det\left(\frac{\partial(\mathbf{x}^k_{p^k,1},\mathbf{x}^k_{p^k,2},\hat{\mathbf{v}}^k_{p^k,1},\hat{\mathbf{v}}^k_{p^k,2})}{\partial(\mathbf{x}^1_{p^1,1},\mathbf{x}^1_{p^1,2},\hat{\mathbf{v}}^1_{p^k,1},\hat{\mathbf{v}}^1_{p^k,2})}\right)\det\left(\frac{\partial(\mathbf{x}^1_{p^1,1},\mathbf{x}^1_{p^1,2},\hat{\mathbf{v}}^1_{p^1,1},\hat{\mathbf{v}}^1_{p^1,2})}{\partial(y_1,y_2,\hat{v}_1,\hat{v}_2)}\right)\right| \\
&\geq \epsilon_{\Omega,N,\delta_1,\delta_2,\delta_3}\left|\det\left(\frac{\partial(\mathbf{x}^1_{p^1,1},\mathbf{x}^1_{p^1,2},\hat{\mathbf{v}}^1_{p^1,1},\hat{\mathbf{v}}^1_{p^1,2})}{\partial(y_1,y_2,\hat{v}_1,\hat{v}_2)}\right)\right|.
\end{aligned}
\tag{3.22}
$$

Note that

$$
\frac{\partial\left(\mathbf{x}^1_{p^1,1},\mathbf{x}^1_{p^1,2},\hat{\mathbf{v}}^1_{p^1,1},\hat{\mathbf{v}}^1_{p^1,2}\right)}{\partial(y_1,y_2,\hat{v}_1,\hat{v}_2)} = \left[\begin{array}{cc|cc}
\frac{\partial Y}{\partial y_1}\cdot\nabla_x\mathbf{x}^1_{p^1,1} & \frac{\partial Y}{\partial y_2}\cdot\nabla_x\mathbf{x}^1_{p^1,1} & \frac{\partial\mathbf{x}^1_{p^1,1}}{\partial\hat{v}_1} & \frac{\partial\mathbf{x}^1_{p^1,1}}{\partial\hat{v}_2} \\
\frac{\partial Y}{\partial y_1}\cdot\nabla_x\mathbf{x}^1_{p^1,2} & \frac{\partial Y}{\partial y_2}\cdot\nabla_x\mathbf{x}^1_{p^1,2} & \frac{\partial\mathbf{x}^1_{p^1,2}}{\partial\hat{v}_1} & \frac{\partial\mathbf{x}^1_{p^1,2}}{\partial\hat{v}_2} \\
\hline
\frac{\partial Y}{\partial y_1}\cdot\nabla_x\hat{\mathbf{v}}^1_{p^1,1} & \frac{\partial Y}{\partial y_2}\cdot\nabla_x\hat{\mathbf{v}}^1_{p^1,1} & \frac{\partial\hat{\mathbf{v}}^1_{p^1,1}}{\partial\hat{v}_1} & \frac{\partial\hat{\mathbf{v}}^1_{p^1,1}}{\partial\hat{v}_2} \\
\frac{\partial Y}{\partial y_1}\cdot\nabla_x\hat{\mathbf{v}}^1_{p^1,2} & \frac{\partial Y}{\partial y_2}\cdot\nabla_x\hat{\mathbf{v}}^1_{p^1,2} & \frac{\partial\hat{\mathbf{v}}^1_{p^1,2}}{\partial\hat{v}_1} & \frac{\partial\hat{\mathbf{v}}^1_{p^1,2}}{\partial\hat{v}_2}
\end{array}\right].
$$

From Lemma 2.4 and (3.17),

$$
\begin{aligned}
\nabla_x\hat{\mathbf{v}}^1_{p^1,i} &= \frac{1}{|\mathbf{v}^1_{p^1}|}\left\{\nabla_x\mathbf{x}^1_{p^1}\cdot\nabla\left(\frac{\partial_i\eta_{p^1}}{\sqrt{g_{p^1,ii}}}\right)\Big|_{x^1}\cdot v + O_{\Omega,N,\delta_2}(\|\Phi\|_{C^2})\right\} \\
&\quad + \frac{O_{\Omega,N}(\|\Phi\|_{C^2})}{|\mathbf{v}^1_{p^1}|^2|\mathbf{v}^1_{p^1,3}|} \\
&= \nabla_x\mathbf{x}^1_{p^1}\cdot\nabla\left(\frac{\partial_i\eta_{p^1}}{\sqrt{g_{p^1,ii}}}\right)\Big|_{x^1}\cdot\frac{v}{|\mathbf{v}^1_{p^1}|} + O_{\Omega,N,\delta_1,\delta_2}(\|\Phi\|_{C^2}),
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\hat{v}}\hat{\mathbf{v}}^1_{p^1,i} &= \frac{1}{|\mathbf{v}^1_{p^1}|}\nabla_{\hat{v}}\mathbf{v}^1_{p^1,i} - \frac{\mathbf{v}^1_{p^1,i}}{|\mathbf{v}^1_{p^1}|^2}\nabla_{\hat{v}}|\mathbf{v}^1_{p^1}| \\
&= \frac{1}{|\mathbf{v}^1_{p^1}|}\left\{\nabla_{\hat{v}}\mathbf{x}^1_{p^1}\cdot\nabla\left(\frac{\partial_i\eta_{p^1}}{\sqrt{g_{p^1,ii}}}\right)\Big|_{x^1}\cdot v + O_{\Omega,N,\delta_1,\delta_2}(\|\Phi\|_{C^2})\right\} \\
&\quad + \frac{1}{|\mathbf{v}^1_{p^1}|}\left\{|v|e_j\cdot\frac{\partial_i\eta_{p^1}}{\sqrt{g_{p^1,ii}}}\Big|_{x^1} + O(\|\Phi\|_{C^1})\left|\frac{\partial t_{\mathbf{b}}}{\partial\hat{\mathbf{v}}_j}\right| + O_N(\|\Phi\|_{C^2})t_{\mathbf{b}}\right\} \\
&\quad + O_{\Omega,N,\delta_1,\delta_2}(\|\Phi\|_{C^2}) \\
&= \frac{|v|}{|\mathbf{v}^1_{p^1}|}\frac{\partial_i\eta_{p^1}}{\sqrt{g_{p^1,ii}}}\Big|_{x^1}\cdot e_j + \nabla_{\hat{v}}\mathbf{x}^1_{p^1}\cdot\nabla\left(\frac{\partial_i\eta_{p^1}}{\sqrt{g_{p^1,ii}}}\right)\Big|_{x^1}\cdot\frac{v}{|\mathbf{v}^1_{p^1}|} \\
&\quad + O_{\Omega,N,\delta_1,\delta_2}(\|\Phi\|_{C^2}).
\end{aligned}
$$

Then by Gaussian elimination,

$$
\begin{aligned}
&\det\left(\frac{\partial(\mathbf{x}^1_{p^1,1},\mathbf{x}^1_{p^1,2},\hat{\mathbf{v}}^1_{p^1,1},\hat{\mathbf{v}}^1_{p^1,2})}{\partial(y_1,y_2,\hat{v}_1,\hat{v}_2)}\right) = \\
&\det\left(\left[\begin{array}{cc|cc}
\frac{\partial Y}{\partial y_1}\cdot\nabla_x\mathbf{x}^1_{p^1,1} & \frac{\partial Y}{\partial y_2}\cdot\nabla_x\mathbf{x}^1_{p^1,1} & \frac{\partial\mathbf{x}^1_{p^1,1}}{\partial\hat{v}_1} & \frac{\partial\mathbf{x}^1_{p^1,1}}{\partial\hat{v}_2} \\
\frac{\partial Y}{\partial y_1}\cdot\nabla_x\mathbf{x}^1_{p^1,2} & \frac{\partial Y}{\partial y_2}\cdot\nabla_x\mathbf{x}^1_{p^1,2} & \frac{\partial\mathbf{x}^1_{p^1,2}}{\partial\hat{v}_1} & \frac{\partial\mathbf{x}^1_{p^1,2}}{\partial\hat{v}_2} \\
\hline
0 & 0 & \frac{\partial_1\eta_{p^1}}{\sqrt{g_{p^1,11}}}\cdot e_1 & \frac{\partial_1\eta_{p^1}}{\sqrt{g_{p^1,11}}}\cdot e_2 \\
0 & 0 & \frac{\partial_2\eta_{p^2}}{\sqrt{g_{p^2,22}}}\cdot e_1 & \frac{\partial_2\eta_{p^2}}{\sqrt{g_{p^2,22}}}\cdot e_2
\end{array}\right]\right) \\
&\qquad + [O_{\Omega,N,\delta_1,\delta_2}(\|\Phi\|_{C^2})]_{4\times4}).
\end{aligned}
$$

From (3.10) and (2.2), all the entries of the above matrix are bound and hence the determinant of the Jacobian matrix equals

$$
\begin{aligned}
&\det\left(\left[\begin{array}{cc|cc}
\frac{\partial Y}{\partial y_1}\cdot\nabla_x\mathbf{x}^1_{p^1,1} & \frac{\partial Y}{\partial y_2}\cdot\nabla_x\mathbf{x}^1_{p^1,1} & \frac{\partial\mathbf{x}^1_{p^1,1}}{\partial\hat{v}_1} & \frac{\partial\mathbf{x}^1_{p^1,1}}{\partial\hat{v}_2} \\
\frac{\partial Y}{\partial y_1}\cdot\nabla_x\mathbf{x}^1_{p^1,2} & \frac{\partial Y}{\partial y_2}\cdot\nabla_x\mathbf{x}^1_{p^1,2} & \frac{\partial\mathbf{x}^1_{p^1,2}}{\partial\hat{v}_1} & \frac{\partial\mathbf{x}^1_{p^1,2}}{\partial\hat{v}_2} \\
\hline
0 & 0 & \frac{\partial_1\eta_{p^1}}{\sqrt{g_{p^1,11}}}\cdot e_1 & \frac{\partial_1\eta_{p^1}}{\sqrt{g_{p^1,11}}}\cdot e_2 \\
0 & 0 & \frac{\partial_2\eta_{p^2}}{\sqrt{g_{p^2,22}}}\cdot e_1 & \frac{\partial_2\eta_{p^2}}{\sqrt{g_{p^2,22}}}\cdot e_2
\end{array}\right]\right) \\
&\quad + O_{\Omega,N,\delta_1,\delta_2}(\|\Phi\|_{C^2}).
\end{aligned} \tag{3.23}
$$

From (2.86), the determinant equals

$$\left|\left(\frac{\partial Y}{\partial y_1}\times\frac{\partial Y}{\partial y_2}\right)\cdot\left(\nabla_x \mathbf{x}^1_{p^1,1}\times\nabla_x \mathbf{x}^1_{p^1,2}\right)\right|\times\left|\left(\frac{\partial_1\eta_{p^1}}{\sqrt{g_{p^1,11}}}\times\frac{\partial_2\eta_{p^1}}{\sqrt{g_{p^1,22}}}\right)\cdot e_3\right| + O_{\Omega,N,\delta_1,\delta_2}(\|\Phi\|_{C^2}). \tag{3.24}$$

From (2.4)

$$\begin{aligned}&\nabla_x \mathbf{x}^1_{p^1,1}\times\nabla_x \mathbf{x}^1_{p^1,2}\\ &\quad=\frac{1}{\sqrt{g_{p^1,11}(x^1)g_{p^1,22}(x^1)}}\\ &\qquad\times\left[n_{p^1}(x^1)-\frac{\mathbf{v}^1_{p^1,1}}{\mathbf{v}^1_{p^1,3}}\frac{\partial_1\eta_{p^1}(x^1)}{\sqrt{g_{p^1,11}(x^1)}}-\frac{\mathbf{v}^1_{p^1,2}}{\mathbf{v}^1_{p^1,3}}\frac{\partial_2\eta_{p^1}(x^1)}{\sqrt{g_{p^1,22}(x^1)}}\right]\\ &\qquad+O_\Omega(\|\nabla_x^2\Phi\|_\infty)\left(1+\frac{|v|}{|\mathbf{v}^1_{p^1,3}|}\right)\frac{|v|}{|\mathbf{v}^1_{p^1,3}|}.\end{aligned}$$

From (2.10) and (2.11), the first line of the right-hand side above equals

$$\frac{1}{\sqrt{g_{p^1,11}(x^1)g_{p^1,22}(x^1)}}\frac{-R_{x^1}v^1}{\mathbf{v}^1_{p^1,3}},$$

while the second line is bounded by $O_{\Omega,N,\delta_2}(\|\nabla_x^2\Phi\|_\infty)(|v|/|\mathbf{v}^1_{p^1,3}|)$ from (3.17). From the assumptions of the lemma, including (3.18), we derive a lower bound as

$$(3.24)\gtrsim_\Omega \frac{\delta_3}{N}\times\frac{1}{N}+O(\|\Phi\|_{C^2}).$$

By choosing sufficiently small $\|\Phi\|_{C^2}$ we prove (3.21).

Step 2. Assume $|\mathscr{R}^{k,p^k,Y}_{i,3}|\ll 1$ for all $i\in\{1,2,3,4\}$. Then

$$|\det\mathscr{R}^{k,p^k,Y}|\le\left|\sum_{i=1}^4(-1)^{i+3}\mathscr{R}^{k,p^k,Y}_{i,3}M_{i,3}\right|\le 4\max_i|M_{i,3}|\times\max_i|\mathscr{R}^{k,p^k,Y}_{i,3}|,$$

where the minor $M_{i,j}$ is defined to be the determinant of the 3×3 matrix that results from $\mathscr{R}^{k,p^k,Y}$ by removing the i^{th} row and the j^{th} column. Note that $|M_{i,3}|\lesssim_{\Omega,N,\delta_1,\delta_2}1$. From (3.21) we prove (3.19).

Step 3. Note that

$$\begin{aligned}|\det \mathscr{R}^{k,p^k,Y}| &\le 12 \max_i \left|\mathscr{R}_{1,i}^{k,p^k,Y}\right| \times \max_i \left|\mathscr{R}_{2,i}^{k,p^k,Y}\right| \\ &\quad \times \max_{i,j} \left|\det \begin{bmatrix} \mathscr{R}_{3,i}^{k,p^k,Y} & \mathscr{R}_{3,j}^{k,p^k,Y} \\ \mathscr{R}_{4,i}^{k,p^k,Y} & \mathscr{R}_{4,j}^{k,p^k,Y} \end{bmatrix}\right| \\ &\lesssim_{\Omega,N,\delta_1,\delta_2} \max_{i,j} \left|\det \begin{bmatrix} \mathscr{R}_{3,i}^{k,p^k,Y} & \mathscr{R}_{3,j}^{k,p^k,Y} \\ \mathscr{R}_{4,i}^{k,p^k,Y} & \mathscr{R}_{4,j}^{k,p^k,Y} \end{bmatrix}\right|.\end{aligned}$$

From (3.21), we prove (3.20). □

LEMMA 3.5. *Assume that $a(z)$, $b(z)$, and $c(z)$ are $C^{0,\gamma}$-functions of $z \in \mathbb{R}^n$ locally. We consider $G(z,s) := a(z)s^2 + b(z)s + c(z)$.*

(i) *Assume $|a| \ge \min|a| > 0$. Define*

$$\begin{aligned}\varphi_1(z) &:= \frac{-b(z)}{2a(z)}, \quad \varphi_2(z) := \mathbf{1}_{b^2-4ac>0} \frac{-b(z) + \sqrt{b^2(z) - 4a(z)c(z)}}{2a(z)}, \\ \varphi_3(z) &:= \mathbf{1}_{b^2(z)-4a(z)c(z)>0} \frac{-b(z) - \sqrt{b^2(z) - 4a(z)c(z)}}{2a(z)}.\end{aligned} \tag{3.25}$$

Then $\varphi_i(z) \in C^{0,\gamma}$ with $\|\varphi_i\|_{C^{0,\gamma}} \le C(\min|a|, \|a\|_{C^{0,\gamma}}, \|b\|_{C^{0,\gamma}}, \|c\|_{C^{0,\gamma}})$ for $i = 1,2,3$ such that if $|s| \le 1$ and $\min_{i=1,2,3}|s - \varphi_i(z)| > \delta$, then $|G(z,s)| \gtrsim \min|a| \times \delta^2$.

(ii) *Assume $a \equiv 0$ and $\min|b| > 0$. Define*

$$\varphi_4(z) := \frac{-c(z)}{b(z)}. \tag{3.26}$$

Then $\varphi_4(z) \in C^{0,\gamma}$ with $\|\varphi_4\|_{C^{0,\gamma}} \le C(\min|b|, \|b\|_{C^{0,\gamma}}, \|c\|_{C^{0,\gamma}})$. Moreover, if $|s| \le 1$ and $|s - \varphi_4(z)| > \delta$, then $|G(z,s)| \gtrsim \min|b| \times \delta$.

(iii) *Assume $a \equiv 0$ and $\min|c| > 0$. Define*

$$\varphi_5(z) := \mathbf{1}_{|b(z)|>\frac{\min|c|}{4}} \frac{-c(z)}{b(z)}. \tag{3.27}$$

Then $\varphi_5(z) \in C^{0,\gamma}$ with $\|\varphi_5\|_{C^{0,\gamma}} \le C(\min|b|, \|b\|_{C^{0,\gamma}}, \|c\|_{C^{0,\gamma}})$. Moreover, if $|s| \le 1$ and $|s - \varphi_5(z)| > \delta$, then $|G(z,s)| \ge \min\left\{\frac{\min|c|}{2}, \frac{\min|c|}{4} \times \delta\right\}$.

PROOF. We consider (i). Without loss of generality we may assume that $a \ge \min a > 0$. Clearly if $a(z) \ge \min a > 0$, then φ_i is $C^{0,\gamma}$ and $\|\varphi_i\|_{C^{0,\gamma}} \le C(\min a, \|a\|_{C^{0,\gamma}}, \|b\|_{C^{0,\gamma}}, \|c\|_{C^{0,\gamma}})$ for $i = 1,2,3$.

We claim that

$$\min_{|s-\varphi_1|>\delta} \{|G(z,s+\delta) - G(z,s)|, |G(z,s) - G(z,s-\delta)|\} \gtrsim \min a \times \delta^2.$$

Since $G(z,s)$ is symmetric around $s=\varphi_1$, it suffices to prove the above estimate for $s \geq \varphi_1$. First, we consider the difference $G(z,s+\delta)-G(z,s)$ for $s \geq -\frac{b}{2a}$ and $\delta > 0$. Note $\partial_s[G(z,s+\delta)-G(z,s)] = 2a\delta > 0$. Therefore, for any z,

$$\min_{s\geq -\frac{b}{2a}} [G(z,s+\delta)-G(z,s)] = \left[G\left(z,-\frac{b}{2a}+\delta\right)-G\left(z,-\frac{b}{2a}\right)\right] \geq \min a \times \delta^2.$$

Second, we consider $G(z,s)-G(z,s-\delta)$ for $s \geq -\frac{b}{2a}+\delta$ and $\delta > 0$. Since $\partial_s[G(z,s+\delta)-G(z,s)] = 2a\delta > 0$,

$$\min_{s\geq -\frac{b}{2a}+\delta} [G(z,s)-G(z,s-\delta)] = \left[G\left(z,-\frac{b}{2a}+\delta\right)-G\left(z,-\frac{b}{2a}\right)\right] \geq \min a \times \delta^2,$$

and thus we prove the claim.

Finally, we consider φ_2 and φ_3. We split the argument into two cases with small number δ: $\delta < \frac{\sqrt{b^2-4ac}}{2a}$ and $\delta \geq \frac{\sqrt{b^2-4ac}}{2a}$.

Case 1. If $\delta < \frac{\sqrt{b^2-4ac}}{2a} = \frac{\varphi_2-\varphi_3}{2}$,

$$\left\{s : \min_{i=2,3} |s-\varphi_i| > \delta\right\} = \{s < \varphi_3-\delta\} \cup \{\varphi_3+\delta < s < \varphi_2-\delta\} \cup \{\varphi_2+\delta < s\},$$

where $\{\varphi_3+\delta < s < \varphi_2-\delta\}$ is not the empty set. For $\{s > \varphi_2+\delta\}$,

$$\begin{aligned}
|G(z,s)| &= \int_{\varphi_2}^{s} \partial_s G(z,t)\mathrm{d}t = \int_{\varphi_2}^{s} (2at+b)\mathrm{d}t \\
&= \int_{\varphi_2}^{s} 2a\left(t-\frac{b}{2a}\right)\mathrm{d}t = \int_{\varphi_2}^{s} 2a(t-\varphi_1)\mathrm{d}t, \quad t-\varphi_1 := r, \\
&= \int_{\varphi_2-\varphi_1}^{s-\varphi_1} 2ar\,\mathrm{d}t \geq (\min a)(s-\varphi_2)(s-\varphi_1+\varphi_2-\varphi_1) \\
&\geq (\min a)\delta(s-\varphi_2+(\varphi_2-\varphi_1)) \geq (\min a)\delta(s-\varphi_2) \\
&\geq (\min a)\delta^2.
\end{aligned}$$

By symmetry, we get the same estimate for the $\{s < \varphi_3-\delta\}$ case.

On the other hand, for $\{\varphi_3+\delta < s < \varphi_2-\delta\}$, it suffices to consider $\{\varphi_1 \leq s < \varphi_2-\delta\}$ because the $\{\varphi_3+\delta < s \leq \varphi_1\}$ case is the same by symmetry. We have a lower bound as follows:

$$\begin{aligned}
|G(z,s)| &= \int_{s}^{\varphi_2} \partial_s G(z,t)\mathrm{d}t = \int_{s}^{\varphi_2} 2a(t-\varphi_1)\mathrm{d}t, \quad t-\varphi_1 := r, \\
&= \int_{s-\varphi_1}^{\varphi_2-\varphi_1} 2ar\,\mathrm{d}t \geq (\min a)(\varphi_2-s)(\varphi_2-\varphi_1+s-\varphi_1) \\
&\geq (\min a)\delta(\delta+(s-\varphi_1)) \\
&\geq (\min a)\delta^2.
\end{aligned}$$

Case 2. If $\delta \geq \frac{\sqrt{b^2-4ac}}{2a} = \frac{\varphi_2-\varphi_3}{2}$,

$$\{s : \min_{i=2,3} |s-\varphi_i| > \delta\} = \{s < \varphi_3 - \delta\} \cup \{s > \varphi_2 + \delta\}.$$

Note that $\{\varphi_3 + \delta < s < \varphi_2 - \delta\}$ is empty set because, if $s - \varphi_3 = |s-\varphi_3| > \delta$,

$$\varphi_2 - s = |s-\varphi_2| = (\varphi_2 - \varphi_3) + (\varphi_3 - s) < 2\delta + (-\delta) = \delta.$$

For $\{s < \varphi_3 - \delta\}$ and $\{s > \varphi_2 + \delta\}$, we already checked that $|G(z,s)| \gtrsim \min|a| \times \delta^2$ holds in Case 1.

Finally, we conclude that

$$|G(z,s)| = |G(z,s) - G(z,\varphi_i)| \gtrsim \min a \times \delta^2 \ \text{ for } \min_{i=2,3} |s-\varphi_i| > \delta,$$

when $-1 < s < 2$.

Now we consider (ii). Clearly φ_4 is $C^{0,\gamma}$ for this case, and

$$\begin{aligned} |G(z,s)| &\geq \min\left\{\left|b(z)\left(\frac{-c(z)}{b(z)} + \delta\right) + c(z)\right|, \left|b(z)\left(\frac{-c(z)}{b(z)}\delta\right) + c(z)\right|\right\} \\ &\geq \min|b| \times \delta. \end{aligned}$$

Now we consider (iii). First, if $|b| < \frac{\min|c|}{2}$, then $|\varphi_5(z)| \geq \frac{|c(z)|}{\min|c|/2} \geq 2$. Therefore,

$$|G(z,s)| \geq \min\{|G(z,1)|, |G(z,-1)|\} \geq |c(z)| - |b(z)| \geq \frac{\min|c|}{2}.$$

Consider the case of $|b| > \frac{\min|c|}{4}$. If $|s - \varphi_5(s)| > \delta$, then

$$\begin{aligned} |G(z,s)| &\geq \min\left\{\left|b(z)\left(\frac{-c(z)}{b(z)} + \delta\right) + c(z)\right|, \left|b(z)\left(\frac{-c(z)}{b(z)} - \delta\right) + c(z)\right|\right\} \\ &= \min|b| \times \delta \geq \frac{\min|c|}{2} \times \delta. \end{aligned}$$

□

LEMMA 3.6. *Assume* Ω *is* C^3 (2.6) *and convex* (1.15)*, and* Φ *is* $C^{2,\gamma}_{t,x}$ *for some* $0 < \gamma < 1$. *We also assume that* $\|\Phi\|_{C^2_x} \ll \delta_1$. *Let* $t^0 \geq 0$, $x^0 \in \bar{\Omega}$, $v^0 \in \mathbb{R}^3$, *and*

$$\frac{1}{N} \leq |v^0| \leq N, \ \ \frac{1}{N} \leq |v^0_3|, \ \ \frac{1}{N} \leq |n(x^1)\cdot e_3|, \ \ |n(x^1)\cdot v^1| > \delta_2 > 0, \tag{3.28}$$

where $(x^1, v^1) = (x^1(t^0, x^0, v^0), v^1(t^0, x^0, v^0))$.

Fix $k \in \mathbb{N}$ *with* $t^k \geq t - 1$. *Then there exists* $\varepsilon > 0$ *and finitely many* $C^{0,\gamma}$*-functions* $\psi^k_i : B_\varepsilon(t,x,v) \to \mathbb{R}$ *with* $\|\psi^k_i\|_{C^{0,\gamma}_{t,x}} \lesssim_{\delta_1,\delta_2,\Omega,N} 1$ *and there exists a constant* $\epsilon_{\delta_1,\delta_2,\Omega,N} > 0$,

$$\begin{aligned} &\text{if } \min_i \left|s - \psi^k_i(t,x,v)\right| > \delta_* \text{ and} \\ &(s;t,x,v) \in \left[\max\left\{t-1, t^{k+1}\right\}, \min\left\{t - \tfrac{1}{N}, t^k\right\}\right] \times B_\varepsilon(t^0,x^0,v^0), \\ &\text{then } \left|\partial_{|v|} X(s;t,x,v) \times \partial_{\widehat{v}_1} X(s;t,x,v)\right| > \epsilon_{\delta_1,\delta_2,\Omega,N,\delta_*}. \end{aligned} \tag{3.29}$$

Here $\widehat{v}_1 = v_1/|v|,\ \widehat{v}_2 = v_2/|v|$.

It is important that this lower bound $\epsilon_{\delta_1,\delta_2,\Omega,N}$ not depend on time t.

PROOF.

Step 1. For (t^0, x^0, v^0) in the assumption we choose (t, x, v) with $|(t, x, v) - (t^0, x^0, v^0)| \ll 1$.

For each x with $|x - x^0| \ll 1$, we set a C^1-map $Y_x : (y_1, y_2) \mapsto Y(y_1, y_2) \in \mathbb{R}^3$ such that

$$Y_x(y_1, y_2) := x + y_1 \mathbf{e}^0_{\perp,1}(t^0, x^0, v^0) + y_2 \mathbf{e}^0_{\perp,2}(t^0, x^0, v^0). \tag{3.30}$$

We claim that

$$\left| \left(\frac{\partial Y_x(y_1, y_2)}{\partial y_1} \times \frac{\partial Y_x(y_1, y_2)}{\partial y_2} \right) \cdot R_{x^1(t, Y_x(y_1,y_2), v)} v^1(t, Y_x(y_1, y_2), v) \right| \gtrsim_{\Omega,N,\delta_1,\delta_2} 1. \tag{3.31}$$

Using the definition of the *specular basis* (3.14), we equate the left-hand side of (3.31) to

$$\left| \left(\mathbf{e}^0_{\perp,1}(t^0, x^0, v^0) \times \mathbf{e}^0_{\perp,2}(t^0, x^0, v^0) \right) \cdot R_{x^1(t^0,x^0,v^0)} v^1(t^0, x^0, v^0) \right| = \left| \frac{v^0(t^0, x^0, v^0)}{|v^0(t^0, x^0, v^0)|} \cdot \lim_{s \downarrow t^1} V(s; t^0, x^0, v^0) \right|.$$

For a small potential, we conclude (3.31).

Step 2. Fix k with $|t^k(t, x, v) - t| \le 1$. Then we fix the orthonormal basis $\{\mathbf{e}^k_0, \mathbf{e}^k_{\perp,1}, \mathbf{e}^k_{\perp,2}\}$ of (3.14) with $x^k = x^k(t, x, v)$, $v^k = v^k(t, x, v)$. Note that this orthonormal basis $\{\mathbf{e}^k_0, \mathbf{e}^k_{\perp,1}, \mathbf{e}^k_{\perp,2}\}$ depends on (t, x, v).

For $t^{k+1} < s < t^k$, recall the forms of $\frac{\partial X(s)}{\partial |v|}$ and $\frac{\partial X(s)}{\partial \widehat{v}_j}$ in (3.1) and (3.2) of Lemma 3.1, where

$$X(s) = X(s; t^k, x^k, v^k).$$

Recall the *specular matrix* (3.16) with $Y = Y_x$ in (3.30). Using the *specular basis* (3.14) and the *specular matrix* (3.16), we rewrite (3.1) and (3.2) as

$$\begin{bmatrix} \frac{\partial X(s)}{\partial |v|} \cdot \mathbf{e}^k_0 & \frac{\partial X(s)}{\partial \widehat{v}_1} \cdot \mathbf{e}^k_0 & \frac{\partial X(s)}{\partial \widehat{v}_2} \cdot \mathbf{e}^k_0 \\ \frac{\partial X(s)}{\partial |v|} \cdot \mathbf{e}^k_{\perp,1} & \frac{\partial X(s)}{\partial \widehat{v}_1} \cdot \mathbf{e}^k_{\perp,1} & \frac{\partial X(s)}{\partial \widehat{v}_2} \cdot \mathbf{e}^k_{\perp,1} \\ \frac{\partial X(s)}{\partial |v|} \cdot \mathbf{e}^k_{\perp,2} & \frac{\partial X(s)}{\partial \widehat{v}_1} \cdot \mathbf{e}^k_{\perp,2} & \frac{\partial X(s)}{\partial \widehat{v}_2} \cdot \mathbf{e}^k_{\perp,2} \end{bmatrix} = \tag{3.32}$$

$$\left[\begin{array}{c|c} -(t-s) & \begin{array}{c} -\left|\mathbf{v}^k_{p^k}\right| \nabla_{\widehat{v}_1,\widehat{v}_2} t^k + \nabla_{\widehat{v}_1,\widehat{v}_2} \mathbf{x}^k_{p^k,\ell} \partial_\ell \eta_{p^k} \big|_{x^k} \cdot \mathbf{e}^k_0 \\ -(t^k - s)|\mathbf{v}^k_{p^k}| \sum_{j=1}^2 \left(\sum_{\ell=1}^3 \partial_j \left(\frac{\partial_\ell \eta_{p^k}}{\sqrt{g_{p^k,\ell\ell}}} \right) \Big|_{x^k} \widehat{\mathbf{v}}^k_{p^k,\ell} \right) \nabla_{\widehat{v}_1,\widehat{v}_2} \mathbf{x}^k_{p^k,j} \end{array} \\ \hline \mathbf{0}_{2,1} & \begin{bmatrix} \mathscr{R}^{k,p^k,Y}_{1,3} & \mathscr{R}^{k,p^k,Y}_{1,4} \\ \mathscr{R}^{k,p^k,Y}_{2,3} & \mathscr{R}^{k,p^k,Y}_{2,4} \end{bmatrix} - (t^k - s)|\mathbf{v}^k_{p^k}| \begin{bmatrix} \mathscr{R}^{k,p^k,Y}_{3,3} & \mathscr{R}^{k,p^k,Y}_{3,4} \\ \mathscr{R}^{k,p^k,Y}_{4,3} & \mathscr{R}^{k,p^k,Y}_{4,4} \end{bmatrix} \end{array}\right] + O_{\Omega,N,\delta_2}(\|\Phi\|_{C^2}).$$

From (3.10) and (2.2), all the entries of the above matrix are bounded. By direct computation we obtain

$$
\begin{aligned}
&\partial_{|v|}X(s)\times\partial_{\widehat{v}_1}X(s)\\
&\quad=-(t-s)\{\mathscr{R}^{k,p^k,Y}_{1,3}-(t^k-s)\big|\mathbf{v}^k_{p^k}\big|\mathscr{R}^{k,p^k,Y}_{3,3}\}\mathbf{e}^k_{\perp,2}\\
&\qquad+(t-s)\{\mathscr{R}^{k,p^k,Y}_{2,3}-(t^k-s)\big|\mathbf{v}^k_{p^k}\big|\mathscr{R}^{k,p^k,Y}_{4,3}\}\mathbf{e}^k_{\perp,1}\\
&\qquad+O_{\Omega,N,\delta_2}(\|\Phi\|_{C^2}).
\end{aligned}
\tag{3.33}
$$

Here $\mathscr{R}^{k,p^k,Y}_{i,j}$, t^k, $\mathbf{v}^k_{p^k}$, and $\mathbf{e}^k_{\perp,i}$ depend on (t,x,v) but not on s.

Step 3. Recall Lemma 3.4. From (3.28) and (3.31), we can choose nonzero constants δ_1, δ_2, and δ_3 for a large N. Applying Lemma 3.4 and (3.19), we conclude that, for some $i\in\{1,2,3,4\}$,

$$
\big|\mathscr{R}^{k,p^k,Y}_{i,3}(t^0,x^0,v^0)\big|>\varrho_{\Omega,N,\delta_1,\delta_2}>0.
\tag{3.34}
$$

Now we claim that $\mathscr{R}^{k,p^k,Y}_{i,j}(t,x,v)\in C^{0,\gamma}_{t,x,v}$ if $|(t,x,v)-(t^0,x^0,v^0)|\ll 1$. Since the domain is convex (1.15) and $|n(x^1(t^0,x^0,v^0))\cdot v^1(t^0,x^0,v^0)|>\delta_2$ in (3.28), utilizing Lemma 2.7 we deduce that if $|(t,x,v)-(t^0,x^0,v^0)|\ll 1$ then $|n(x^l)\cdot v^l|\gtrsim\delta_2$ for all $1\le l\le k$. By Lemma 2.2, (t^l,x^l,v^l) is $C^{0,\gamma}$ for all $1\le l\le k$. Hence, from (3.15) and (3.16), we conclude our claim.

Finally, we choose a small constant $\varepsilon>0$ such that, for some $i\in\{1,2,3,4\}$ satisfying (3.34),

$$
\big|\mathscr{R}^{k,p^k,Y}_{i,3}(t,x,v)\big|>\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{2}\quad\text{for }|(t,x,v)-(t^0,x^0,v^0)|<\varepsilon.
\tag{3.35}
$$

Step 4. With $N\gg 1$, from (3.35), we divide the cases as follows:

$$
\begin{aligned}
&\big|\mathscr{R}^{k,p^k,Y}_{i,3}\big|>\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{2}\quad\text{for some } i\in\{1,2\},\\
&\big|\mathscr{R}^{k,p^k,Y}_{j,3}\big|\ge\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{2}\quad\text{for some } j\in\{3,4\}.
\end{aligned}
\tag{3.36}
$$

We split the first case (3.36) further into two more cases as

$$
\begin{aligned}
&\min_{i=1,2}\big|\mathscr{R}^{k,p^k,Y}_{i,3}\big|>\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{2}\quad\text{and}\quad\max_{i=1,2}\big|\mathscr{R}^{k,p^k,Y}_{i+2,3}\big|<\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{4N},\\
&\min_{i=1,2}\big|\mathscr{R}^{k,p^k,Y}_{i,3}\big|>\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{2}\quad\text{and}\quad\max_{i=1,2}\big|\mathscr{R}^{k,p^k,Y}_{i+2,3}\big|\ge\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{4N}.
\end{aligned}
\tag{3.37}
$$

Set the other case

$$
\big|\mathscr{R}^{k,p^k,Y}_{j,3}\big|\ge\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{2}\quad\text{for some } j\in\{3,4\}.
\tag{3.38}
$$

Then clearly (3.37) and (3.38) cover all the cases.

Step 5. We consider the case of (3.37). Then, from (3.33),

$$
\begin{aligned}
&|\partial_{|v|}X(s)\times\partial_{\widehat{v}_1}X(s)|\\
&\quad\ge \big||v^k|\mathscr{R}^{k,p^k,Y}_{i+2,3}(t^k-s)-\mathscr{R}^{k,p^k,Y}_{i,3}\big|(t-s)+O_{\Omega,N,\delta}(\|\Phi\|_{C^2})\\
&\quad= \big|\underbrace{|v^k|\mathscr{R}^{k,p^k,Y}_{i+2,3}(t-s)+\big[-\mathscr{R}^{k,p^k,Y}_{i,3}+(t^k-t)|v^k|\mathscr{R}^{k,p^k,Y}_{i+2,3}\big]}\big|(t-s)\\
&\qquad+O_{\Omega,\delta_2,N}(\|\Phi\|_{C^2}).
\end{aligned}
\tag{3.39}
$$

Let us consider the underbraced term above. We define

$$\tilde{s}=t-s, \tag{3.40}$$

and set

$$a\equiv 0,\quad b:=|v^k|\mathscr{R}^{k,p^k,Y}_{i+2,3},\quad c:=-\mathscr{R}^{k,p^k,Y}_{i,3}+(t^k-t)|v^k|\mathscr{R}^{k,p^k,Y}_{i+2,3}. \tag{3.41}$$

Note that $\mathscr{R}^{k,p^k,Y}_{i,3}$, $\mathscr{R}^{k,p^k,Y}_{i+2,3}$, $|v^k|$, and t^k depend only on (t,x,v).

Hence we regard the underbraced term of (3.39) as an affine function of $\tilde{s}$,

$$b(t,x,v)\tilde{s}+c(t,x,v). \tag{3.42}$$

Note that from (3.37)

$$|c(t,x,v)|\ge\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{2}-N\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{4N}\ge\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{4}.$$

Now we apply (iii) of Lemma 3.5. With $\varphi_5(t,x,v)$ in (3.27), if $|\tilde{s}-\varphi_5(t,x,v)|>\delta_*$, then $|b(t,x,v)\tilde{s}+c(t,x,v)|\ge\frac{\varrho_{\Omega,N,\delta}}{4}\times\delta_*$. We set

$$\psi_5(t,x,v)=t-\varphi_5(t,x,v). \tag{3.43}$$

From (3.40),

$$
\begin{aligned}
&\text{if } |s-\psi_5(t,x,v)|>\delta_*, \text{ then}\\
&|b(t,x,v)(t-s)+c(t,x,v)|\ge\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{4}\times\delta_*.
\end{aligned}
\tag{3.44}
$$

Now we consider the case of (3.38). From (3.33),

$$
\begin{aligned}
&|\partial_{|v|}X(s)\times\partial_{\widehat{v}_1}X(s)|\\
&\quad\ge \big||v^k|\mathscr{R}^{k,p^k,Y}_{j,3}(t-s)\\
&\qquad+\big[-\mathscr{R}^{k,p^k,Y}_{j-2,3}+(t^k-t)|v^k|\mathscr{R}^{k,p^k,Y}_{j,3}\big]\big|(t-s)\\
&\qquad+O_{\Omega,N,\delta}(\|\Phi\|_{C^2}).
\end{aligned}
\tag{3.45}
$$

We set $\tilde{s}$ as in (3.40) and set

$$a\equiv 0,\quad b:=|v^k|\mathscr{R}^{k,p^k,Y}_{j,3},\quad c:=-\mathscr{R}^{k,p^k,Y}_{j-2,3}+(t^k-t)|v^k|\mathscr{R}^{k,p^k,Y}_{j,3}. \tag{3.46}$$

From (3.38) and (3.46)

$$|b(t,x,v)|\ge\frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{8N^2}.$$

We apply (ii) of Lemma 3.5 to this case: With $\varphi_4(t,x,v)$ in (3.26), we set

$$\psi_4(t,x,v) = t - \varphi_4(t,x,v), \tag{3.47}$$

and

$$\begin{aligned} &\text{if } |s-\psi_4(t,x,v)| > \delta_*, \text{ then} \\ &|b(t,x,v)(t-s)+c(t,x,v)| \gtrsim \frac{\varrho_{\Omega,N,\delta_1,\delta_2}}{8N^2} \times \delta_*. \end{aligned} \tag{3.48}$$

From (3.44), (3.39), (3.48), and (3.45), we conclude the proof of Lemma 3.6. □

LEMMA 3.7. *Assume Ω is C^3* (2.6) *and convex* (1.15)*, and Φ is $C^{2,\gamma}_{t,x}$ for some $0<\gamma<1$. Assume the conditions of Lemma* 3.4.

Let a C^1-map $Y_x : (y_1,y_2) \mapsto Y_x(y_1,y_2) \in \bar{\Omega}$ with $Y_x(0,0) = x$ and $\|Y\|_{C^1_{x,y_1,y_2}} \lesssim 1$. We assume that

$$\left| \left(\frac{\partial Y_{x^0}(0,0)}{\partial y_1} \times \frac{\partial Y_{x^0}(0,0)}{\partial y_2} \right) \cdot R_{x^1(t,x^0,v^0)} v^1(t,x^0,v^0) \right| > \delta_3 > 0. \tag{3.49}$$

For $k \in \mathbb{N}$ with $t^k \geq t-1$, there exist $\varepsilon > 0$, finitely many $C^{0,\gamma}$-functions $\psi_i^k : B_\varepsilon(t,x,v) \to \mathbb{R}$ with $\|\psi_i^k\|_{C^{0,\gamma}_{t,x}} \lesssim 1$, and a constant $\epsilon_{\delta_1,\delta_2,\delta_3,N,\Omega} > 0$, and $\{\zeta_1,\zeta_2\} \subset \{\widehat{v}_1,\widehat{v}_2,y_1,y_2\}$ such that if $\min_i |s-\psi_i^k(t,Y_x(y_1,y_2),v)| > \delta_*$ and

$$(s;t,Y_x(y_1,y_2),v) \in \left[\max\{t-1,t^{k+1}\}, \min\left\{ t-\frac{1}{N}, t^k \right\} \right] \times B_\varepsilon(t^0,x^0,v^0), \tag{3.50}$$

then

$$\det\left(\frac{\partial X(s;t,Y_x(y_1,y_2),|v|,\widehat{v}_1,\widehat{v}_2)}{\partial(|v|,\zeta_1,\zeta_2)} \right) > \epsilon_{\delta_1,\delta_2,\delta_3,N,\Omega,\delta_*} > 0.$$

PROOF.

Step 1. Recall the *specular basis* $\{\mathbf{e}_0^k, \mathbf{e}_{\perp,1}^k, \mathbf{e}_{\perp,2}^k\}$ in (3.14) with

$$x^k = x^k(t,Y_x(y_1,y_2),|v|,\widehat{v}_1,\widehat{v}_2) \quad \text{and} \quad v^k = v^k(t,Y_x(y_1,y_2),|v|,\widehat{v}_1,\widehat{v}_2) :$$

$$\frac{\partial X(s;t,Y_x(y_1,y_2),|v|,\widehat{v}_1,\widehat{v}_2)}{\partial(|v|,y_1,y_2,\widehat{v}_1,\widehat{v}_2)} =$$

$$\begin{bmatrix} \mathbf{e}_0^k & \mathbf{e}_{\perp,1}^k & \mathbf{e}_{\perp,2}^k \end{bmatrix} \underbrace{\begin{bmatrix} \frac{\partial X}{\partial|v|}\cdot\mathbf{e}_0^k & \frac{\partial X}{\partial y_1}\cdot\mathbf{e}_0^k & \frac{\partial X}{\partial y_2}\cdot\mathbf{e}_0^k & \frac{\partial X}{\partial \widehat{v}_1}\cdot\mathbf{e}_0^k & \frac{\partial X}{\partial \widehat{v}_2}\cdot\mathbf{e}_0^k \\ \frac{\partial X}{\partial|v|}\cdot\mathbf{e}_{\perp,1}^k & \frac{\partial X}{\partial y_1}\cdot\mathbf{e}_{\perp,1}^k & \frac{\partial X}{\partial y_2}\cdot\mathbf{e}_{\perp,1}^k & \frac{\partial X}{\partial \widehat{v}_1}\cdot\mathbf{e}_{\perp,1}^k & \frac{\partial X}{\partial \widehat{v}_2}\cdot\mathbf{e}_{\perp,1}^k \\ \frac{\partial X}{\partial|v|}\cdot\mathbf{e}_{\perp,2}^k & \frac{\partial X}{\partial y_1}\cdot\mathbf{e}_{\perp,2}^k & \frac{\partial X}{\partial y_2}\cdot\mathbf{e}_{\perp,2}^k & \frac{\partial X}{\partial \widehat{v}_1}\cdot\mathbf{e}_{\perp,2}^k & \frac{\partial X}{\partial \widehat{v}_2}\cdot\mathbf{e}_{\perp,2}^k \end{bmatrix}}.$$

From (3.1) and (3.2), using the *specular basis* (3.14) and the *specular matrix* (3.16), we rewrite the underbraced term as

$$\left[\begin{array}{c|c} -(t-s) & \begin{array}{c} -|v^k|\nabla_{\widehat{v}_1,\widehat{v}_2,y_1,y_2}t^k+\nabla_{\widehat{v}_1,\widehat{v}_2,y_1,y_2}\mathbf{x}^k_{p^k,\ell}\partial_\ell\eta_{p^k}\cdot\mathbf{e}^k_0 \\ -(t^k-s)|v^k|\sum_{j=1}^2\left(\sum_{\ell=1}^3\frac{\partial}{\partial\mathbf{x}^k_{p^k,j}}\left[\frac{\partial_\ell\eta_{pk}}{\sqrt{g}_{p^k,\ell\ell}}\right]\widehat{\mathbf{v}}^k_{p^k,\ell}\right)\nabla_{\widehat{v}_1,\widehat{v}_2,y_1,y_2}\mathbf{x}^k_{p^k,j} \end{array} \\ \hline \begin{array}{c}0\\0\end{array} & *_{2\times4} \end{array}\right] + O_{\delta,N}(\|\Phi\|_{C^2}),$$

where the lower right 2×4-submatrix equals

$$\begin{bmatrix} \mathscr{R}^{k,p^k,Y_x}_{1,1} & \mathscr{R}^{k,p^k,Y_x}_{1,2} & \mathscr{R}^{k,p^k,Y_x}_{1,3} & \mathscr{R}^{k,p^k,Y_x}_{1,4} \\ \mathscr{R}^{k,p^k,Y_x}_{2,1} & \mathscr{R}^{k,p^k,Y_x}_{2,2} & \mathscr{R}^{k,p^k,Y_x}_{2,3} & \mathscr{R}^{k,p^k,Y_x}_{2,4} \end{bmatrix} - (t^k-s)|v^k|\begin{bmatrix} \mathscr{R}^{k,p^k,Y_x}_{3,1} & \mathscr{R}^{k,p^k,Y_x}_{3,2} & \mathscr{R}^{k,p^k,Y_x}_{3,3} & \mathscr{R}^{k,p^k,Y_x}_{3,4} \\ \mathscr{R}^{k,p^k,Y_x}_{4,1} & \mathscr{R}^{k,p^k,Y_x}_{4,2} & \mathscr{R}^{k,p^k,Y_x}_{4,3} & \mathscr{R}^{k,p^k,Y_x}_{4,4} \end{bmatrix}. \tag{3.51}$$

Here $\mathscr{R}^{k,p^k,Y_x}_{i,j}$ is defined in (3.16) with $x^k = x^k(t, Y_x(y_1,y_2), |v|, \widehat{v}_1, \widehat{v}_2)$ and $v^k = v^k(t, Y_x(y_1,y_2), |v|, \widehat{v}_1, \widehat{v}_2)$.

Step 2. From Lemma 3.4, there exist $i<j$ such that (3.20) holds. We choose ζ_1,ζ_2 to be the i^{th} component and j^{th} component of $\{y_1,y_2,\widehat{v}_1,\widehat{v}_2\}$. For the sake of simplicity, we abuse the notation as

$$\begin{bmatrix} \mathscr{R}^{k,p^k,Y_x}_{3,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{3,\zeta_2} \\ \mathscr{R}^{k,p^k,Y_x}_{4,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{4,\zeta_2} \end{bmatrix} = \begin{bmatrix} \mathscr{R}^{k,p^k,Y_x}_{3,i} & \mathscr{R}^{k,p^k,Y_x}_{3,j} \\ \mathscr{R}^{k,p^k,Y_x}_{4,i} & \mathscr{R}^{k,p^k,Y_x}_{4,j} \end{bmatrix}.$$

Note that

$$\det\left(\frac{\partial X(s;t,Y_x(y_1,y_2),|v|,\widehat{v}_1,\widehat{v}_2)}{\partial(|v|,\zeta_1,\zeta_2)}\right) = \det\left(\left[\begin{array}{c|c} (s-t) & \begin{array}{c} -|v^k|\nabla_{\zeta_1,\zeta_2}t^k+\nabla_{\zeta_1,\zeta_2}\mathbf{x}^k_{p^k,\ell}\partial_\ell\eta_{p^k}\cdot\mathbf{e}^k_0 \\ -(t^k-s)|v^k|\sum_{j=1}^2\left(\sum_{\ell=1}^3\frac{\partial}{\partial\mathbf{x}^k_{p^k,j}}\left[\frac{\partial_\ell\eta_{p^k}}{\sqrt{g}_{p^k,\ell\ell}}\right]\widehat{\mathbf{v}}^k_{p^k,\ell}\right)\nabla_{\zeta_1,\zeta_2}\mathbf{x}^k_{p^k,j} \end{array} \\ \hline \begin{array}{c}0\\0\end{array} & \begin{bmatrix} \mathscr{R}^{k,p^k,Y_x}_{1,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{1,\zeta_2} \\ \mathscr{R}^{k,p^k,Y_x}_{2,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{2,\zeta_2} \end{bmatrix} - (t^k-s)|v^k|\begin{bmatrix} \mathscr{R}^{k,p^k,Y_x}_{3,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{3,\zeta_2} \\ \mathscr{R}^{k,p^k,Y_x}_{4,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{4,\zeta_2} \end{bmatrix} \end{array}\right]\right) + \big[O_{\delta,N}(\|\Phi\|_{C^2})\big]_{3\times3}.$$

From (3.10) and (2.2), all the entries of the above matrix are bound and hence the determinant of above matrix equals

$$-(t-s)\underbrace{\det\left(\begin{bmatrix} \mathscr{R}^{k,p^k,Y_x}_{1,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{1,\zeta_2} \\ \mathscr{R}^{k,p^k,Y_x}_{2,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{2,\zeta_2} \end{bmatrix} - (t^k-s)|v^k|\begin{bmatrix} \mathscr{R}^{k,p^k,Y_x}_{3,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{3,\zeta_2} \\ \mathscr{R}^{k,p^k,Y_x}_{4,\zeta_1} & \mathscr{R}^{k,p^k,Y_x}_{4,\zeta_2} \end{bmatrix}\right)} + O(\|\Phi\|_{C^2}). \tag{3.52}$$

The underbraced term equals

$$
\begin{aligned}
&(t^k-s)^2|v^k|^2\det\left(\begin{bmatrix}\mathscr{R}^{k,p^k,Y_x}_{3,\xi_1} & \mathscr{R}^{k,p^k,Y_x}_{3,\xi_2}\\ \mathscr{R}^{k,p^k,Y_x}_{4,\xi_1} & \mathscr{R}^{k,p^k,Y_x}_{4,\xi_2}\end{bmatrix}\right)\\
&\quad+\det\left(\begin{bmatrix}\mathscr{R}^{k,p^k,Y_x}_{1,\xi_1} & \mathscr{R}^{k,p^k,Y_x}_{1,\xi_2}\\ \mathscr{R}^{k,p^k,Y_x}_{2,\xi_1} & \mathscr{R}^{k,p^k,Y_x}_{2,\xi_2}\end{bmatrix}\right)\\
&\quad-(t^k-s)|v^k|\Big(\mathscr{R}^{k,p^k,Y_x}_{3,\xi_1}\mathscr{R}^{k,p^k,Y_x}_{2,\xi_2}+\mathscr{R}^{k,p^k,Y_x}_{1,\xi_1}\mathscr{R}^{k,p^k,Y_x}_{4,\xi_2}\\
&\qquad\qquad-\mathscr{R}^{k,p^k,Y_x}_{1,\xi_2}\mathscr{R}^{k,p^k,Y_x}_{4,\xi_1}-\mathscr{R}^{k,p^k,Y_x}_{3,\xi_2}\mathscr{R}^{k,p^k,Y_x}_{2,\xi_1}\Big).
\end{aligned}
\tag{3.53}
$$

We define

$$\tilde{s}=t^k(t,Y_x(y_1,y_2),v)-s,$$

and we regard (3.53) as a quadratic polynomial of $\tilde{s}$. Then the coefficient of $\tilde{s}^2$ is

$$|v^k|^2\left|\det\left(\begin{bmatrix}\mathscr{R}^{k,p^k,Y_x}_{3,\xi_1} & \mathscr{R}^{k,p^k,Y_x}_{3,\xi_2}\\ \mathscr{R}^{k,p^k,Y_x}_{4,\xi_1} & \mathscr{R}^{k,p^k,Y_x}_{4,\xi_2}\end{bmatrix}\right)\right|,$$

which depends only on $(t,Y_x(y_1,y_2),v)$. From (3.20) and $|v|\geq\frac{1}{N}$, we have a lower bound of $\frac{\delta_2}{N^2}$.

We apply (iii) of Lemma 3.5: There exist C^1-functions $\psi_1(t,Y_x(y_1,y_2),v)$, $\psi_2(t,Y_x(y_1,y_2),v)$, and $\psi_3(t,Y_x(y_1,y_2),v)$ so that if $|\tilde{s}-\psi_i(t,Y_x(y_1,y_2),v)|>\delta_2$ for all $i=1,2,3$, then the absolute value of (3.53) has a positive lower bound. Set

$$\psi_i=t^k-\phi_i.$$

Using $|t-s|>\frac{1}{N}$ and (3.52) we prove (3.50). □

LEMMA 3.8. *Assume Ω is convex in* (1.15) *and $\|\Phi\|_{C^1}\ll 1$. Choose $N\gg 1$, $0<\delta\ll 1$, and then $\delta_1=\delta_1(\Omega,N,\delta,\|\nabla_x\Phi\|_\infty)>0$ as in* (3.57). *There exist collections of open subsets $\{\mathscr{O}_i\}_{i=1}^{I_{\Omega,N,\delta,\delta_1}}$ of Ω and $\{\mathscr{V}_i(\mathbf{q}_1,\mathbf{q}_2)\}_{i=1}^{I_{\Omega,N,\delta,\delta_1}}$ of $\mathbb{R}^3$, where $\mathbf{q}_1$ and $\mathbf{q}_2$ are two independent vectors in $\mathbb{R}^3$, with $I_{\Omega,N,\delta,\delta_1}<\infty$ such that $\bar{\Omega}\subset\bigcup_i\mathscr{O}_i$ and $\int_{\mathbb{R}^3\setminus\mathscr{V}_i(\mathbf{q}_1,\mathbf{q}_2)}e^{-|v|^2/100}\mathrm{d}v\leq O_\Omega(\frac{1}{N})+O_\Omega(\delta_1)$. Moreover,*

$$
\begin{aligned}
K_i:=\sup\big\{&k\in\mathbb{N}: t^k(t,x,v)\geq T,\\
&(t,x,v)\in[T,T+1]\times\mathscr{O}_i\times\mathbb{R}^3\setminus\mathscr{V}_i(\mathbf{q}_1,\mathbf{q}_2)\big\}<\infty.
\end{aligned}
\tag{3.54}
$$

If $(x,v)\in\mathscr{O}_i\times\mathbb{R}^3\setminus\mathscr{V}_i(\mathbf{q}_1,\mathbf{q}_2)$ for some i, then

$$|n(x^1(t,x,v))\cdot v^1(t,x,v)|>\min\left\{\frac{\delta_1}{4},C_{\Omega,N,\|\nabla_x\Phi\|_\infty}\delta\right\}\tag{3.55}$$

and

$$|(\mathbf{q}_1 \times \mathbf{q}_2) \cdot v| \geq \frac{1}{N}. \tag{3.56}$$

PROOF. We construct $\mathscr{O}_i$ and $\mathscr{V}_i(\mathbf{q}_1, \mathbf{q}_2)$. Choose $\mathfrak{x} \in \bar{\Omega}$ and $v \in \mathbb{R}^3$ with $\frac{1}{N} \leq |v| \leq N$, $\frac{1}{N} \leq |v_3|$ for $N \gg 1$. We split the cases $|v||t - t^1(t, \mathfrak{x}, v)| \geq \delta$ and $|v||t - t^1(t, \mathfrak{x}, v)| \leq 2\delta$ for some $0 < \delta \ll 1$. For the first case, from (2.65),

$$\delta \leq |v||t - t^1(t, \mathfrak{x}, v)| \lesssim_\Omega \frac{|v^1(t, \mathfrak{x}, v) \cdot n(x^1(t, \mathfrak{x}, v))|}{|v^1(t, \mathfrak{x}, v)|},$$

and hence $|v^1(t, \mathfrak{x}, v) \cdot n(x^1(t, \mathfrak{x}, v))| > C_{\Omega, N, \|\nabla_x \Phi\|_\infty} \delta$. For the second case,

$$\begin{aligned}
&|n(x^1(t, \mathfrak{x}, v)) \cdot v^1(t, \mathfrak{x}, v)| \\
&\quad = |n(\mathfrak{x}) \cdot v| + O_{\|\eta\|_{C^3}}(|x^1(t, \mathfrak{x}, v) - \mathfrak{x}|) \times \{|v| + \|\nabla_x \Phi\|_\infty\} + \|\nabla_x \Phi\|_\infty \\
&\quad = |n(\mathfrak{x}) \cdot v| + O_{N, \|\eta\|_{C^3}}(\delta) + O_{N, \|\eta\|_{C^3}}(\|\nabla_x \Phi\|_\infty),
\end{aligned}$$

where we have used the fact $|x^1(t, \mathfrak{x}, v) - \mathfrak{x}| = |v||t - t^1| + \|\nabla_x \Phi\|_\infty |t - t^1|^2$.

Let us choose

$$\delta_1 = 2\big| O_{N, \|\eta\|_{C^3}}(\delta) + O_{N, \|\eta\|_{C^3}}(\|\nabla_x \Phi\|_\infty)\big| \quad \text{for } \delta \ll_{N,\Omega} 1,\ \|\nabla_x \Phi\|_\infty \ll_{N,\Omega} 1. \tag{3.57}$$

Then $|n(x^1(t, \mathfrak{x}, v)) \cdot v^1(t, \mathfrak{x}, v)| \geq \frac{\delta_1}{2}$ for $|n(\mathfrak{x}) \cdot v| \geq \delta_1$. Condition (3.56) is independent of position x. Note that, from Lemma 2.4, (t^1, x^1, v^1) is continuous locally. Therefore, we can choose $r_\mathfrak{x} > 0$ such that if $x \in B(\mathfrak{x}, r_\mathfrak{x}) \cap \bar{\Omega}$, $\frac{1}{N} \leq |v| \leq N$, $\frac{1}{N} \leq |v_3|$, $|n(\mathfrak{x}) \cdot v| \geq 2\delta_1$, and $|(\mathbf{q}_1 \times \mathbf{q}_2) \cdot v| \geq \frac{1}{N}$, then we have (3.55) and (3.56). Since $\bar{\Omega}$ is a compact subset of $\mathbb{R}^3$, we extract finite points $\{\mathfrak{x}_i\}_{i=1}^{I_{\Omega,N,\delta,\delta_1}}$ with $I_{\Omega,N,\delta,\delta_1} < \infty$ such that $\{B(\mathfrak{x}_i, r_{\mathfrak{x}_i})\}_{i=1}^{I_{\Omega,N,\delta,\delta_1}}$ is an open covering of $\bar{\Omega}$. We define

$$\begin{aligned}
\mathscr{O}_i &:= B(\mathfrak{x}_i, r_{\mathfrak{x}_i}), \\
\mathscr{V}_i(\mathbf{q}_1, \mathbf{q}_2) &:= \Big\{ v \in \mathbb{R}^3 : |v| \leq \frac{1}{N}, |v| \geq N, |v_3| \leq \frac{1}{N}, |n(\mathfrak{x}) \cdot v| \leq 2\delta_1, \\
&\qquad \text{or } |(\mathbf{q}_1 \times \mathbf{q}_2) \cdot v| \leq \frac{1}{N} \Big\},
\end{aligned} \tag{3.58}$$

for two independent vectors $\mathbf{q}_1, \mathbf{q}_2$ in $\mathbb{R}^3$. Clearly we already proved that if $(x, v) \in \mathscr{O}_i \times \mathbb{R}^3 \setminus \mathscr{V}_i(\mathbf{q}_1, \mathbf{q}_2)$ for some $i = 1, 2, \ldots, I_{\Omega,N,\delta,\delta_1}$, then we have (3.55). Moreover, $\int_{\mathscr{V}_i(\mathbf{q}_1,\mathbf{q}_2)} e^{-|v|^2/100} \mathrm{d}v < O(\frac{1}{N}) + O(\delta_1)$ from our construction. From (2.81), we prove (3.54). □

Now we are ready to prove the main theorem.

THEOREM 3.9. *Fix arbitrary* $(t,x,v)\in[T,T+1]\times\Omega\times\mathbb{R}^3$. *Recall* M, δ, δ_1, *and* $\mathscr{O}_i$, $\mathscr{V}_i(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2)$, *which are chosen in Lemma* 3.8. *For each* $i=1,2,\dots,I_{\Omega,N,\delta,\delta_1}$, *there exist* $\delta_2>0$ *and a* $C^{0,\gamma}$*-function* $\psi^{\ell_0,\vec{\ell},i,k}$ *for* $k\le K_i$ *in* (3.54) *where* $\psi^{\ell_0,\vec{\ell},i,k}$ *is defined locally around* $(T+\delta_2\ell_0, X(T+\delta_2\ell_0;t,x,v),\delta_2\vec{\ell})$ *with*

$$(\ell_0,\vec{\ell})=(\ell_0,\ell_1,\ell_2,\ell_3)\in\left\{0,1,\dots,\left\lfloor\frac{1}{\delta_2}\right\rfloor+1\right\}\times\left\{-\left\lfloor\frac{N}{\delta_2}\right\rfloor-1,\dots,0,\dots,\left\lfloor\frac{N}{\delta_2}\right\rfloor+1\right\}^3$$

and $\|\psi^{\ell_0,\vec{\ell},i,k}\|_{C^{0,\gamma}}\le C_{N,\Omega,\delta,\delta_1,\delta_2,\|\Phi\|_{C^{2,\gamma}}}<\infty$.

Moreover, if

$$(X(s;t,x,v),u)\in\mathscr{O}_i\times\mathbb{R}^3\setminus\mathscr{V}_i(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2)\quad\text{for } i=1,2,\dots,I_{\Omega,N,\delta,\delta_1},\tag{3.59}$$

$$(s,u)\in[T+(\ell_0-1)\delta_2,T+(\ell_0+1)\delta_2]\times B(\delta_2\vec{\ell};2\delta_2),\tag{3.60}$$

$$s'\in\left[t^{k+1}(T+\delta_2\ell_0;X(T+\delta_2\ell_0;t,x,v),\delta_2\vec{\ell})+\frac{1}{N},\ t^{k}(T+\delta_2\ell_0;X(T+\delta_2\ell_0;t,x,v),\delta_2\vec{\ell})-\frac{1}{N}\right],\tag{3.61}$$

and

$$\left|s'-\psi^{\ell_0,\vec{\ell},i,k}(T+\delta_2\ell_0,X(T+\delta_2\ell_0;t,x,v),\delta_2\vec{\ell})\right|>N^2(1+\|\psi^{\ell_0,\vec{\ell},i,k}\|_{C^{0,\gamma}})(\delta_2)^\gamma,\tag{3.62}$$

then

$$\left|\partial_{|u|}X(s';s,X(s;t,x,v),u)\times\partial_{\hat{u}_1}X(s';s,X(s;t,x,v),u)\right|>\epsilon_{\Omega,N,\|\Phi\|_{C^2},\delta_1,\delta_2,\delta_2}.\tag{3.63}$$

Here $\epsilon_{\Omega,N,\|\Phi\|_{C^2},\delta_1,\delta_2,\delta_2}>0$ *does not depend on* T, t, x, *or* v.

For each $j=1,2,\dots,I_{\Omega,N,\delta,\delta_1}$ *in Lemma* 3.8, *there exist* $\delta_3>0$ *and* $C^{0,\gamma}$*-functions*

$$\psi_1^{\ell_0,\vec{\ell},i,k,j,m_0,\vec{m},k'},\quad\psi_2^{\ell_0,\vec{\ell},i,k,j,m_0,\vec{m},k'},\quad\psi_3^{\ell_0,\vec{\ell},i,k,j,m_0,\vec{m},k'},\tag{3.64}$$

for $k'\le K^j$ *in* (3.54) *where* $\psi_n^{\ell_0,\vec{\ell},i,k,j,m_0,\vec{m},k'}$ *is defined locally around*

$$\left(T+\delta_3m_0;X(T+\delta_3m_0;T+\delta_2\ell_0,X(T+\delta_2\ell_0;t,x,v),\delta_2\vec{\ell}),\delta_3\vec{m}\right)$$

for some

$$(m_0,\vec{m}) = (m_0,m_1,m_2,m_3) \in \left\{0,1,\dots,\left\lfloor\frac{1}{\delta_3}\right\rfloor+1\right\} \times \left\{-\left\lfloor\frac{N}{\delta_3}\right\rfloor-1,\dots,0,\dots,\left\lfloor\frac{N}{\delta_3}\right\rfloor+1\right\}^3$$

with $0<\delta_3\ll 1$.

Moreover, if we assume (3.59), (3.60), (3.61), *and* (3.62),

$$(X(s';s,X(s;t,x,v),u),u') \in \mathscr{O}_j\times\mathbb{R}^3\setminus\mathscr{V}_j(\partial_{|u|}X,\partial_{\widehat{u}_1}X) \quad \text{for some } j=1,2,\dots,I_{\Omega,N,\delta,\delta_1} \text{ in Lemma 3.8,} \tag{3.65}$$

$$s''\in\Big[t^{k'+1}\big(T+\delta_3m_0;X(T+\delta_3m_0;T+\delta_2\ell_0,X(T+\delta_2\ell_0;t,x,v),\delta_2\vec{\ell}),\delta_3\vec{m}\big)+\frac{1}{N}, \underbrace{t^{k'}\big(T+\delta_3m_0;X(T+\delta_3m_0;T+\delta_2\ell_0,X(T+\delta_2\ell_0;t,x,v),\delta_2\vec{\ell}),\delta_3\vec{m}\big)}_{(**)}-\frac{1}{N}\Big], \tag{3.66}$$

and

$$\min_{n=1,2,3}\big|s''-\psi_n^{\ell_0,\vec{\ell},i,k,j,m_0,\vec{m},k'}(**)\big| > N^2\big(1+\max_{n=1,2,3}\big\|\psi_n^{\ell_0,\vec{\ell},i,k,j,m_0,\vec{m},k'}\big\|_{C^{0,\gamma}}\big)(\delta_3)^\gamma, \tag{3.67}$$

where $(**)$ *is defined in* (3.66). *Then for each* ℓ_0, $\vec{\ell}$, i, k, j, m_0, $\vec{m}$, *and* k' *we can choose two distinct variables* $\{\zeta_1,\zeta_2\}\subset\{|u|,\widehat{u}_1,\widehat{u}_1',\widehat{u}_2'\}$ *such that* $(|u'|,\zeta_1,\zeta_2)\mapsto X(s'';s',X(s';s,X(s;t,x,v),u),u')$ *is one-to-one locally and*

$$\left|\det\left(\frac{\partial X(s'';s',X(s';s,X(s;t,x,v),u),u')}{\partial(|u'|,\zeta_1,\zeta_2)}\right)\right| > \epsilon'_{\Omega,N,\|\Phi\|_{C^2},\delta_1,\delta_2,\delta_3}. \tag{3.68}$$

Here $\epsilon'_{\Omega,N,\|\Phi\|_{C^2},\delta_1,\delta_2,\delta_3}>0$ *does not depend on* T, t, x, *or* v.

PROOF.

Step 1. Fix any arbitrary $(t,x,v)\in[T,T+1]\times\Omega\times\mathbb{R}^3$. Assume that $s\in[T,t]$ and $(X(s;t,x,v),u)\in\mathscr{O}_i\times\mathbb{R}^3\setminus\mathscr{V}_i(\widehat{\mathbf{e}}_1,\widehat{\mathbf{e}}_2)$ for some i, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard unit vectors $(1,0,0)$ and $(0,1,0)$ in global coordinates. Due to Lemma 3.8, $(X(s';s,X(s;t,x,v),u),V(s';s,X(s;t,x,v),u))$ is well-defined for all $s'\in[T,s]$ and $|n(x^k(s,X(s;t,x,v),u))\cdot v^k(s,X(s;t,x,v),u)|\gtrsim_{\Omega,N}1$ for all k with $|t-t^k(s,X(s;t,x,v),u)|\le 1$.

We note that, from $X(s;t,x,v)=X(\overline{s};t,x,v)+\int_{\overline{s}}^{s}V(\tau;t,x,v)\mathrm{d}\tau$,

$$\begin{aligned}&|\psi^k(s,X(s;t,x,v),u)-\psi^k(\overline{s},X(\overline{s};t,x,v),\overline{u})|\\ &\quad\le\|\psi^k\|_{C^{0,\gamma}_{t,x,v}}\{|s-\overline{s}|^\gamma+|X(s;t,x,v)-X(\overline{s};t,x,v)|^\gamma+|u-\overline{u}|^\gamma\}\\ &\quad\le\|\psi^k\|_{C^{0,\gamma}_{t,x,v}}\{|s-\overline{s}|^\gamma+(1+N^\gamma+\|\nabla_x\Phi\|_\infty^\gamma)|u-\overline{u}|^\gamma\}.\end{aligned} \tag{3.69}$$

For $0 < \delta_2 \ll 1$ we split

$$[T, T+1] = \bigcup_{\ell_0=0}^{[\delta_2^{-1}]+1} \big[T + (\ell_0 - 1)\delta_2, T + (\ell_0 + 1)\delta_2\big],$$

$$\mathbb{R}^3 \setminus \mathscr{V}_i(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2) = \bigcup_{|\vec{\ell}|=0}^{[N/\delta_2^{-2}]+1} B(\vec{\ell}\delta_2; 2\delta_2) \cap \mathbb{R}^3 \setminus \mathscr{V}_i(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2).$$

From (3.69), if

$$\begin{aligned}(s,u) \in [T + (\ell_0 - 1)\delta_2, T + (\ell_0 + 1)\delta_2] \\ \times \{B(\vec{\ell}\delta_2; 2\delta_2) \cap \mathbb{R}^3 \setminus \mathscr{V}_i\},\end{aligned}$$

then

$$\begin{aligned}&|\psi^k(T + \ell_0\delta, X(T + \ell_0\delta; t, x, v), (\ell_1\delta, \ell_2\delta, \ell_3\delta)) - \psi^k(s, X(s; t, x, v), u)| \\ &\quad \le \|\psi^k\|_{C^{0,\gamma}}(2 + N^\gamma + \|\nabla_x \Phi\|_\infty^\gamma)(\delta_2)^\gamma.\end{aligned}$$

Therefore, if (3.62) holds, then

$$\begin{aligned}&|s' - \psi^k(s, X(s; t, x, v), u)| \\ &\quad \ge \big|s' - \psi^k\big(T + \ell_0\delta, X(T + \ell_0\delta; t, x, v), \vec{\ell}\delta\big)\big| \\ &\qquad - \big|\psi^k\big(T + \ell_0\delta, X(T + \ell_0\delta; t, x, v), \vec{\ell}\delta\big) \\ &\qquad\quad - \psi^k(s, X(s; t, x, v), u)\big| \\ &\quad \gtrsim (N^2 - N^\gamma)\|\psi^k\|_{C^{0,\gamma}}(\delta_2)^\gamma \gtrsim_N \|\psi^k\|_{C^{0,\gamma}}(\delta_2)^\gamma.\end{aligned} \tag{3.70}$$

Consider the mapping $u \mapsto X(s'; s, X(s; t, x, v), u)$. Note that from Lemma 3.8 we verify the condition of Lemma 3.6. Applying Lemma 3.6, we construct the $C^{0,\gamma}$-function $\psi^k : B_\varepsilon(s, X(s; t, x, v), u) \to \mathbb{R}$ for $k \le K^i$ such that if $|s' - \psi^k(s, X(s; t, x, v), u)| > (\delta_2)^\gamma$, then

$$\begin{aligned}\big|\partial_{|u|} X(s'; s, X(s; t, x, v), u) \times \partial_{\hat{u}_1} X(s'; s, X(s; t, x, v), u)\big| &> \epsilon_{\Omega, N, \|\Phi\|_{C^2}, \delta_1, (\delta_2)^\gamma} \\ &> 0.\end{aligned}$$

Clearly if (3.62) holds, then from (3.70) we have $|s' - \psi^k(s, X(s; t, x, v), u)| > (\delta_2)^\gamma$.

Step 2. Assume all the conditions of (3.59)–(3.62) and (3.65). Applying Lemma 3.7, we construct (3.64). From (3.69)

$$\begin{aligned}&|\psi(s', X(s'; s, X(s; t, x, v), u), u') - \psi(\bar{s'}, X(\bar{s'}; s, X(s; t, x, v), u), \bar{u'})| \\ &\quad \le \|\psi\|_{C^{0,\gamma}}\{|s' - \bar{s'}|^\gamma + (1 + N^\gamma + \|\nabla_x \Phi\|_\infty^\gamma)|u' - \bar{u'}|^\gamma\}.\end{aligned} \tag{3.71}$$

For $0 < \delta_3 \ll 1$, we split

$$[T, T+1] = \bigcup_{m_0=0}^{[\delta_3^{-1}]+1} \big[T + (m_0-1)\delta_3, T + (m_0+1)\delta_3\big],$$

$$\mathbb{R}^3 \setminus \mathscr{V}_j(\partial_{|u|}X, \partial_{\widehat{u}_1}X) = \bigcup_{|\vec{m}|=0}^{[N/\delta_3^{-2}]+1} B(\vec{m}\delta_3; 2\delta_3) \cap \mathbb{R}^3 \setminus \mathscr{V}_j(\partial_{|u|}X, \partial_{\widehat{u}_1}X).$$

From (3.71) if

$$(s', u') \in [T + (m-1)\delta, T + (m+1)\delta] \times \{B(\vec{m}\delta; 2\delta) \cap \mathbb{R}^3 \setminus \mathscr{V}_j(\partial_{|u|}X, \partial_{\widehat{u}_1}X)\},$$

$$(X(s; t, x, v), u) \in \mathscr{O}_i \times \mathbb{R}^3 \setminus \mathscr{V}_i(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2),$$

$$(X(s'; s, X(s; t, x, v), u), u') \in \mathscr{O}_j \times \mathbb{R}^3 \setminus \mathscr{V}_j(\partial_{|u|}X, \partial_{\widehat{u}_1}X),$$

then

$$|s' - \phi(T + \ell\delta, X(T + \ell\delta; t, x, v), \vec{\ell}\delta)| \gtrsim_N \|\phi\|_{C^{0,\gamma}}\delta^\gamma,$$

$$|s'' - \psi(T + m\delta, X(T + m\delta; T + \ell\delta, X(T + \ell\delta; t, x, v), \vec{m}\delta), \vec{\ell}\delta)| \gtrsim_N \delta^\gamma.$$

Consider the mapping

$$(u, u') \mapsto X(s''; s', X(s'; s, X(s; t, x, v), u), u').$$

Note that from Lemma 3.8 we verify the condition of Lemma 3.7. For each i, j and $\ell_0, \ell_1, \ell_2, \ell_3$ and m_0, m_1, m_2, m_3, in applying Lemma 3.7, we can choose two variables $\{\zeta_1, \zeta_2\} \subset \{|u|, \widehat{u}_1, \widehat{u}_1', \widehat{u}_2'\}$ so that (3.68) holds. □

4 A Time-Dependent Potential

THEOREM 4.1 (Local existence). *For a sufficiently small $\delta_0 > 0$ and $\delta_\phi > 0$ there exists $T^* > 0$ such that if $\|wf_0\|_\infty \le \delta_0$ and $\|\phi\|_{C^1} \le \delta_\phi$, then there exists a unique solution $f(t, x, v)$ to* (1.18) *in $[0, T^*) \times \Omega \times \mathbb{R}^3$ such that*

$$\sup_{0 \le t \le T^*} \|wf(t)\|_\infty \le 2(\delta_0 + C\delta_\phi), \tag{4.1}$$

and $\|wf(t)\|_\infty$ is continuous over $[0, T^*)$. If $F_0 = \mu_E + \sqrt{\mu}_E f_0 \ge 0$, then $F = \mu_E + \sqrt{\mu_E} f \ge 0$.

PROOF. For the proof we use a sequence of $F^0 \equiv 0$ and for $\ell \ge 0$

$$\partial_t F^{\ell+1} + v \cdot \nabla_x F^{\ell+1} - \nabla_x(\phi + \Phi) \cdot \nabla_v F^{\ell+1} = Q_+(F^\ell, F^\ell) - \nu(F^\ell)F^{\ell+1}, \quad F|_{t=0} = F_0,$$

$$F^{\ell+1}(t, x, v) = F^{\ell+1}(t, x, R_x v) \quad \text{on } \partial\Omega.$$

Note that

$$\frac{d}{ds}e^{-\int_s^t \nu(F^\ell)(\tau,X(\tau;t,x,v),X(\tau;t,x,v))\mathrm{d}\tau}F^{\ell+1}(s,X(s),V(s)) = Q_+(F^\ell,F^\ell)(s,X(s),V(s)),$$

where $X(s) = X(s;t,x,v)$ and $V(s) := V(s;t,x,v)$ satisfy (2.1) and (2.4). Note that if $F^\ell \geq 0$, then $\nu(F^\ell) \geq 0$ and $Q_+(F^\ell,F^\ell) \geq 0$. Therefore, if $F^\ell \geq 0$ and $F_0 \geq 0$, then

$$F^{\ell+1} \geq 0 \quad \text{for all } \ell. \tag{4.2}$$

From $F^{\ell+1} = \mu_E + \sqrt{\mu_E} f^{\ell+1}$,

$$\begin{aligned}
&\partial_t f^{\ell+1} + v\cdot\nabla_x f^{\ell+1} - \nabla_x(\phi+\Phi)\cdot\nabla_v f^{\ell+1} \\
&\quad + e^{-\Phi}\nu f^{\ell+1} + \frac{f^{\ell+1}}{2} v\cdot\nabla_x\phi \\
&\qquad = e^{-\Phi}Kf^\ell - \sqrt{\mu_E} v\cdot\nabla_x\phi + e^{-\frac{\Phi}{2}}\Gamma_+(f^\ell,f^\ell) - e^{-\frac{\Phi}{2}}\Gamma_-(f^\ell,f^{\ell+1}).
\end{aligned} \tag{4.3}$$

For $h^\ell := wf^\ell$

$$\begin{aligned}
&\partial_t h^{\ell+1} + v\cdot\nabla_x h^{\ell+1} - \nabla_x(\phi+\Phi)\cdot\nabla_v h^{\ell+1} \\
&\quad + \frac{h^{\ell+1}}{w}\nabla(\phi+\Phi)\cdot\nabla_v w + e^{-\Phi}\nu h^{\ell+1} + \frac{h^{\ell+1}}{2} v\cdot\nabla_x\phi \\
&\qquad = e^{-\Phi}K_w h^\ell - w\sqrt{\mu_E} v\cdot\nabla_x\phi + we^{-\frac{\Phi}{2}}\Gamma_+\left(\frac{h^\ell}{w},\frac{h^\ell}{w}\right) \\
&\qquad\quad - we^{-\frac{\Phi}{2}}\Gamma_-\left(\frac{h^\ell}{w},\frac{h^{\ell+1}}{w}\right).
\end{aligned} \tag{4.4}$$

We claim that we can choose $0 < T^* \ll 1$ such that for all ℓ

$$\sup_{0\leq t\leq T^*} \|h^\ell(t)\|_\infty \leq 2(\delta_0 + C\delta_\phi). \tag{4.5}$$

We define

$$\begin{aligned}
E(v,t,x) &:= \exp\left\{-\int_s^t \nu_E(\tau,X(\tau;t,x,v),V(\tau;t,x,v))\mathrm{d}\tau\right\} \\
&:= \exp\left\{-\int_s^t \left[e^{-\Phi(X(\tau))}\nu(V(\tau)) + \frac{1}{2}V(\tau)\cdot\nabla\phi(\tau,X(\tau))\right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{w}\nabla_x(\phi(\tau,X(\tau)) + \Phi(X(\tau))\cdot\nabla_v w(V(\tau))\right]\mathrm{d}\tau\right\},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
G^{\ell+1} &:= -w\sqrt{\mu_E} V\cdot\nabla_x\phi + we^{-\frac{\Phi}{2}}\Gamma_+\left(\frac{h^\ell}{w},\frac{h^\ell}{w}\right) \\
&\quad - we^{-\frac{\Phi}{2}}\Gamma_-\left(\frac{h^\ell}{w},\frac{h^{\ell+1}}{w}\right).
\end{aligned}$$

Along the trajectory,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(E(v,t,s)h^{\ell+1}(s,X(s;t,x,v),V(s;t,x,v))\Big) = E(v,t,s)\big[e^{-\Phi(X(s))}K_w h^{\ell+1} + G^{\ell+1}\big](s,X(s;t,x,v),V(s;t,x,v)).$$

By integrating from 0 to t, we obtain

$$\begin{aligned} &h^{\ell+1}(t,x,v) \\ &= E(v,t,0)h^{\ell+1}(0,X(0),V(0)) + \int_0^t E(v,t,s)G^{\ell+1}(s)\mathrm{d}s \\ &\quad + \int_0^t E(v,t,s)e^{-\Phi(X(s))}\int_{\mathbb{R}^3} k_w(u,V(s))h^{\ell+1}(s,X(s;t,x,v),u)\mathrm{d}u\,\mathrm{d}s. \end{aligned} \tag{4.7}$$

From (1.21),

$$\langle V(\tau;t,x,v)\rangle \lesssim_{\phi,\Phi} \nu_E(\tau,X(\tau;t,x,v),V(\tau;t,x,v)) \lesssim_{\phi,\Phi} \langle V(\tau;t,x,v)\rangle. \tag{4.8}$$

Recall the standard estimates (see lemma 4 and lemma 5 in [14]):

$$\int_{\mathbb{R}^3} |k_w(v,u)|\mathrm{d}u \le C_K\langle v\rangle^{-1}, \qquad \left| w\Gamma_{\pm}\Big(\frac{h_1}{w},\frac{h_2}{w}\Big)(v)\right| \lesssim \langle v\rangle |h_1||h_2|. \tag{4.9}$$

Therefore,

$$\begin{aligned} &|G^{\ell+1}(s;t,x,v)| \\ &\lesssim_{\Phi} \|\nabla_x\phi(s)\|_\infty e^{-\frac{|V(s)|^2}{8}} \\ &\quad + \langle V(s;t,x,v)\rangle\{\|h^\ell(s)\|_\infty + \|h^{\ell+1}(s)\|_\infty\}\|h^\ell(s)\|_\infty. \end{aligned} \tag{4.10}$$

From (4.8) and (4.10), we deduce that

$$\begin{aligned} &\sup_{0\le t\le T}\|h^{\ell+1}(t)\|_\infty \\ &\le \delta_0 + C\delta_\phi + C_K T\sup_{0\le s\le t}\|h^{\ell+1}(s)\|_\infty \\ &\quad + CT\Big\{\sup_{0\le s\le t}\|h^{\ell+1}(s)\|_\infty + \sup_{0\le s\le t}\|h^\ell(s)\|_\infty\Big\}\sup_{0\le s\le t}\|h^\ell(s)\|_\infty. \end{aligned}$$

Choose $T^* > 0$ such that $C_K T^* \ll 1$. Then from (4.5) for h^ℓ

$$\left(1 - \frac{1}{10} - 2(\delta_0 + C\delta_\phi)\right)\times \sup_{0\le t\le T}\|h^{\ell+1}(t)\|_\infty \le \delta_0 + C\delta_\phi + C(\delta_0 + C\delta_\phi)^2,$$

and we prove the same upper bound of (4.5) for $h^{\ell+1}$ for sufficiently small δ_0 and δ_ϕ.

We can show that h^ℓ is a Cauchy sequence in $L^\infty([0,T^*);L^\infty(\Omega\times\mathbb{R}^3))$ by repeating the argument with $h^{\ell+1}-h^\ell$. Then we pass a limit $\ell\to\infty$ to prove the existence and (4.1). Using (4.2) and this limit we prove $F\ge 0$. Assume h_1 and h_2 solve the same equation (4.4). Following the same proof as for (4.5), we prove that

$\sup_{0\le t\le T^*}\|h_1-h_2\|_\infty \le o(1)\sup_{0\le t\le T^*}\|h_1-h_2\|_\infty$. Hence $h_1\equiv h_2$, and we conclude the uniqueness.

For $0<\varepsilon\ll 1$, from (4.4) with $h=h^{\ell+1}$

$$\begin{aligned}\|h(t+\varepsilon)\|_\infty-\|h(t)\|_\infty &\le \|h(t+\varepsilon)-h(t)\|_\infty\\ &\lesssim \varepsilon\{\|h_0\|_\infty+\|h\|_\infty+\|h\|_\infty^2+\|\phi\|_{C^1}\}.\end{aligned}$$

Hence $\|wf(t)\|_\infty$ is continuous on $[0,T^*)$. □

LEMMA 4.2. *For $w(v)=(1+|v|)^\beta$ with $\beta>2$,*

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\Gamma_+(\psi,\psi)\varphi\,\mathrm{d}v\right| \lesssim \|w\psi\|_\infty\|\psi\|_{L^2(\mathbb{R}^3)}\|\varphi\|_{L^2(\mathbb{R}^3)},\\ &\left|\int_{\mathbb{R}^3}\Gamma_-(\psi,\psi)\varphi\,\mathrm{d}v\right| \lesssim \|w\psi\|_\infty\big\{\|\psi\|_{L^2(\mathbb{R}^3)}\|\varphi\|_{L^2(\mathbb{R}^3)}\\ &\qquad\qquad +\|(\mathbf{I}-\mathbf{P})\psi\|_\nu\|\varphi\|_\nu\big\}.\end{aligned}\tag{4.11}$$

PROOF. Via the well-known Carleman representation (for example, see (32) in [15]), we have

$$\begin{aligned}&\Gamma_+(\psi,\psi)(v)\\ &=\frac{1}{\sqrt{\mu(v)}}Q_+(\sqrt{\mu}\psi,\sqrt{\mu}\psi)(v)\\ &=2\int_{\mathbb{R}^3}\psi(v')\frac{1}{|v-v'|^2}\\ &\quad\cdot\int_{E_{vv'}}\psi(v_1')e^{-\frac{|-v+v'+v_1'|^2}{4}}B\left(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|}\right)\mathrm{d}v_1'\,\mathrm{d}v',\end{aligned}$$

where $E_{vv'}$ is a hyperplane containing $v\in\mathbb{R}^3$ and is perpendicular to $\frac{v'-v}{|v'-v|}\in\mathbb{S}^2$, i.e.,

$$E_{vv'}:=\big\{v_1'\in\mathbb{R}^3:(v_1'-v)\cdot(v'-v)=0\big\}.$$

For the internal integration over $E_{vv'}$, using lemma 6 and (34) in [15], we bound it above as

$$\int_{E_{vv'}}\cdots\,\mathrm{d}v_1'\lesssim\|w\psi\|_\infty\frac{1+|v-v'|}{w(v-v')}\lesssim\|w\psi\|_\infty\langle v-v'\rangle^{-(\beta-1)},$$

where we have used

$$w(v_1')^{-1}e^{-\frac{|-v+v'+v_1'|^2}{4}}\lesssim w(v-v')^{-1}.$$

Note that $\int_{\mathbb{R}^3} \frac{\langle v-v'\rangle^{-(\beta-1)}}{|v-v'|^2}\mathrm{d}v' \lesssim 1$ for $\beta > 2$. Hence we conclude that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \Gamma_+(\psi,\psi)\varphi\,\mathrm{d}v\right| \\
&\quad\lesssim \|w\psi\|_\infty \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\langle v-v'\rangle^{-(\beta-1)}}{|v-v'|^2}|\psi(v')||\varphi(v)|\mathrm{d}v'\,\mathrm{d}v \\
&\quad\lesssim \|w\psi\|_\infty \left[\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} \frac{\langle v-v'\rangle^{-(\beta-1)}}{|v-v'|^2}\,\mathrm{d}v\right)|\psi(v')|^2\,\mathrm{d}v'\right]^{\frac{1}{2}} \\
&\qquad\times\left[\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} \frac{\langle v-v'\rangle^{-(\beta-1)}}{|v-v'|^2}\,\mathrm{d}v'\right)|\varphi(v)|^2\,\mathrm{d}v\right]^{\frac{1}{2}} \\
&\quad\lesssim \|w\psi\|_\infty \|\psi\|_{L^2(\mathbb{R}^3)}\|\varphi\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

For the Γ_- estimate, we have

$$\begin{aligned}
\int_{\mathbb{R}^3}|\Gamma_-(\psi,\psi)(v)\varphi(v)|\mathrm{d}v &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v-u||\psi(u)|\sqrt{\mu(u)}\mathrm{d}u\times|\psi(v)||\varphi(v)|\mathrm{d}v \\
&\lesssim \|w\psi\|_\infty\int_{\mathbb{R}^3}\langle v\rangle\{|\mathbf{P}\psi(v)|+|(\mathbf{I}-\mathbf{P})\psi(v)|\}|\varphi(v)|\mathrm{d}v \\
&\lesssim \|w\psi\|_\infty\{\|\psi\|_{L^2(\mathbb{R}^3)}\|\varphi\|_{L^2(\mathbb{R}^3)}+\|(\mathbf{I}-\mathbf{P})\psi\|_\nu\|\varphi\|_\nu\},
\end{aligned}$$

where we have used the fact, for all $p\in[1,\infty]$,

$$\|\langle v\rangle\mathbf{P}\psi\|_{L^p(\mathbb{R}^3)} \lesssim \left\|\langle v\rangle^3\sqrt{\mu(v)}\int_{\mathbb{R}^3}\psi(u)\langle u\rangle^2\sqrt{\mu(u)}\mathrm{d}u\right\|_{L^p(\mathbb{R}^3)} \lesssim \|\psi\|_{L^p(\mathbb{R}^3)}.$$

□

LEMMA 4.3. *Let f solve* (1.18)*. Then there exists a constant $C>0$ not depending on f_0, f, or ϕ such that, for all $t\ge 0$,*

$$\begin{aligned}
\|f(t)\|_2^2 \le C\left(\|f(0)\|_2^2+\int_0^t\|\phi(s)\|_\infty\right) \\
\times\left(1+C(\|\phi\|_\infty+\|wf\|_\infty)te^{C(\|\phi\|_\infty+\|wf\|_\infty)t}\right).
\end{aligned} \tag{4.12}$$

PROOF.

$$\begin{aligned}
&\|f(t)\|_2^2+\int_0^t\iint_{\Omega\times\mathbb{R}^3} e^{-\Phi}fLf \\
&\quad=\|f(0)\|_2^2-\underbrace{\int_0^t\iint_{\Omega\times\mathbb{R}^3}\frac{v\cdot\nabla_x\phi}{2}|f|^2}_{\text{(I)}}-\underbrace{\int_0^t\iint_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x\phi\mathbf{P}f\sqrt{\mu_E}}_{\text{(II)}} \\
&\qquad+\underbrace{\int_0^t\iint_{\Omega\times\mathbb{R}^3} e^{-\frac{\Phi}{2}}\Gamma(f,f)(\mathbf{I}-\mathbf{P})f}_{\text{(III)}}.
\end{aligned}$$

By the decomposition $f = \mathbf{P}f + (\mathbf{I}-\mathbf{P})f$ and the strong decay-in-v of $\mathbf{P}f$ in (1.37),

$$\begin{aligned}(\mathrm{I}) &\le \|\phi\|_\infty \Big\{ \int_0^t \iint_{\Omega\times\mathbb{R}^3} |v||\mathbf{P}f|^2 + \int_0^t \|(\mathbf{I}-\mathbf{P})f\|_\nu^2 \Big\} \\ &\lesssim \|\phi\|_\infty \Big\{ \int_0^t \|f\|_2^2 + \int_0^t \|(\mathbf{I}-\mathbf{P})f\|_\nu^2 \Big\},\end{aligned}$$

$$(\mathrm{II}) \lesssim \|\phi\|_\infty \int_0^t \|f\|_2^2 + \int_0^t \|\phi(s)\|_\infty \,\mathrm{d}s.$$

From (4.11)

$$\begin{aligned}(\mathrm{III}) &\lesssim \int_0^t \int_\Omega \|wf(s,x,\cdot)\|_\infty \|(\mathbf{I}-\mathbf{P})f(s,x,\cdot)\|_\nu^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t \int_\Omega \|wf(s,x,\cdot)\|_\infty \|f(s,x,\cdot)\|_2^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\lesssim \|wf\|_\infty \int_0^t \|(\mathbf{I}-\mathbf{P})f(s)\|_\nu^2 \,\mathrm{d}s + \|wf\|_\infty \int_0^t \|f(s)\|_2^2 \,\mathrm{d}s.\end{aligned}$$

Using (1.36) and collecting the terms, we deduce that, for some constant $C > 0$,

$$\begin{aligned}\|f(t)\|_2^2 &\le \|f(t)\|_2^2 + (\delta_L - \|\phi\|_\infty - \|wf\|_\infty) \int_0^t \|(\mathbf{I}-\mathbf{P})f\|_\nu^2 \\ &\le \|f(0)\|_2^2 + C(\|\phi\|_\infty + \|wf\|_\infty) \int_0^t \|f\|_2^2 + C\int_0^t \|\phi\|_\infty.\end{aligned} \tag{4.13}$$

By Gronwall's inequality we conclude (4.12). □

LEMMA 4.4. *Assume* $F = \mu_E + \sqrt{\mu_E} f$ *solves* (1.1) *and satisfies* (1.5). *Assume* (1.12) *and*

$$\lambda_\phi \gg \delta_\phi + \|wf\|_\infty. \tag{4.14}$$

Then

$$\begin{aligned}&\Big| \iint_{\Omega\times\mathbb{R}^3} \Big(\frac{|v|^2}{2} + \Phi(x) \Big) F(t,x,v) \mathrm{d}x\,\mathrm{d}v \\ &\quad - \iint_{\Omega\times\mathbb{R}^3} \Big(\frac{|v|^2}{2} + \Phi(x) \Big) F_0(x,v) \mathrm{d}x\,\mathrm{d}v \Big| \\ &\quad \lesssim \frac{\delta_\phi}{\lambda_\phi} \big\{ 1 + \|f(0)\|_2^2 + \|wf\|_\infty \big\}.\end{aligned} \tag{4.15}$$

PROOF. The proof is a direct consequence of two previous lemmas ((1.5) and (4.12)) and the exponential decay-in-time of $\phi(t)$ in (1.12): The LHS of (4.15) is bounded by

$$
\begin{aligned}
&\le \int_0^t \iint_{\Omega\times\mathbb{R}^3} \{\mu_E(x,v) + \sqrt{\mu_E(x,v)}|f(s,x,v)|\}|v||\nabla_x\phi(s,x)|\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\le \delta_\phi \int_0^t e^{-\lambda_\phi s}\Big\{1 + C\Big(\|f(0)\|_2^2 + \int_0^t \|\phi(s)\|_\infty\Big) \\
&\qquad\qquad \times \big(1 + C(\|\phi\|_\infty + \|wf\|_\infty)se^{C(\|\phi\|_\infty+\|wf\|_\infty)s}\big)\Big\}\mathrm{d}s \\
&\le \delta_\phi \int_0^t e^{-\lambda_\phi s}\big\{1 + C\big(\|f(0)\|_2^2 + \delta_\phi/\lambda_\phi\big) \\
&\qquad\qquad \times \big(1 + C(\delta_\phi + \|wf\|_\infty)se^{C(\delta_\phi+\|wf\|_\infty)s}\big)\big\}\mathrm{d}s \\
&\le \delta_\phi/\lambda_\phi\big\{1 + C\big(\|f(0)\|_2^2 + \delta_\phi/\lambda_\phi\big)\big\} \\
&\quad + C\delta_\phi(\delta_\phi + \|wf\|_\infty)\int_0^t e^{-[\lambda_\phi - C(\delta_\phi+\|wf\|_\infty)]s} \\
&\lesssim \frac{\delta_\phi}{\lambda_\phi}\big\{1 + \|f(0)\|_2^2 + \|wf\|_\infty\big\}.
\end{aligned}
$$

□

LEMMA 4.5 ([11, 15]). *Recall μ_E in* (1.2)*. Then*

$$
\begin{aligned}
&|F-\mu_E|\mathbf{1}_{|F-\mu_E|\ge\overline{\delta}\mu_E} \\
&\le \frac{4}{\overline{\delta}}\Big\{(F\ln F - \mu_E\ln\mu_E) \\
&\qquad - (F-\mu_E) + \Big(\frac{|v|^2}{2} + \Phi(x)\Big)(F-\mu_E)\Big\}.
\end{aligned}
\tag{4.16}
$$

PROOF. The proof is based on the proof of lemma 4 of [15] and the argument on page 147 of [11].

By the Taylor expansion, for $t, s > 0$

$$
\frac{1}{\max\{t,s\}}\frac{|t-s|^2}{2} \le \int_s^t\int_s^{s_1}\frac{1}{s_2}\,\mathrm{d}s_2\,\mathrm{d}s_1 = t\ln t - s\ln s - (1+\ln s)(t-s). \tag{4.17}
$$

Note that if $F(t,x,v) - \mu_E(x,v) \ge \overline{\delta}\mu_E(x,v)$ with $0 \le \overline{\delta} \ll 1$, then $F \ge (1+\overline{\delta})\mu_E$ and hence

$$\max\{F, \mu_E\} = (1+\overline{\delta})\mu_E.$$

If $\mu_E(x,v) - F(t,x,v) \ge -\overline{\delta}\mu_E(x,v)$, then $(1+\overline{\delta})\mu_E \ge F$ and

$$\max\{F, \mu_E\} \le (1+\overline{\delta})\mu_E.$$

Therefore, if $|F-\mu_E| \ge \overline{\delta}\mu_E$ then

$$\begin{aligned}\frac{|F-\mu_E|}{\max\{F,\mu_E\}} \times \frac{|F-\mu_E|}{2} &\ge \frac{\overline{\delta}\mu_E}{(1+\overline{\delta})\mu_E} \times \frac{|F-\mu_E|}{2} \\ &\ge \frac{\overline{\delta}}{2} \times \frac{|F-\mu_E|}{2} \ge \frac{\overline{\delta}}{4}|F-\mu_E|.\end{aligned}$$

Hence, from (4.17), we deduce that

$$\begin{aligned}&|F-\mu_E|\mathbf{1}_{|F-\mu_E|\ge\overline{\delta}\mu_E} \\ &\quad\le \frac{4}{\overline{\delta}}\frac{1}{\max\{F,\mu_E\}}\frac{|F-\mu_E|^2}{2} \\ &\quad\le \frac{4}{\overline{\delta}}\big\{F(t)\ln F(t) - \mu_E \ln \mu_E - (1+\ln\mu_E)(F-\mu_E(x,v))\big\} \\ &\quad\le \frac{4}{\overline{\delta}}\big\{(F\ln F - \mu_E\ln\mu_E) - (F-\mu_E) + (|v|^2/2+\Phi(x))(F-\mu_E)\big\},\end{aligned}$$

where we have used $\ln\mu_E = -(\frac{|v|^2}{2}+\Phi(x))$. □

PROOF OF THEOREM 1.1. Denote

$$T_1 := \sup\{t\ge 0 : \|wf(t)\|_\infty \le 2(\delta_0 + C\delta_\phi)\}.$$

Note that (1.4), (1.5), and (1.11) hold for $0\le t\le T_1$. Note that f and h satisfy (1.18) and (4.4) with $h^{\ell+1}=h=h^\ell$. Then we have (4.7) with $h^{\ell+1}=h=h^\ell$.

We apply the Duhamel formula (4.7) *three times*, for $0\le t\le T_1$, and decompose the integrand h as

$$h = h\mathbf{1}_{|F-\mu_E|\ge\overline{\delta}\mu_E} + w\frac{F-\mu_E}{\sqrt{\mu_E}}\mathbf{1}_{|F-\mu_E|\le\overline{\delta}\mu_E}, \tag{4.18}$$

for sufficiently small $0<\overline{\delta}\ll 1$, to get

$$\begin{aligned}&h(t,x,v) \\ &\quad= E(v,t,0)h(0) + \int_0^t E(v,t,s)G(s)\mathrm{d}s \\ &\qquad+ \int_0^t E(v,t,s)e^{-\Phi(X(s))}\int_u k_w(u,v)h(s,X(s),u)\mathrm{d}u\,\mathrm{d}s \\ &\quad= E(v,t,0)h(0) + \int_0^t E(v,t,s)G(s)\mathrm{d}s \\ &\qquad+ \int_0^t E(v,t,s)e^{-\Phi(X(s))} \\ &\qquad\quad\times \int_u k_w(u,v)E(u,s,0)\Big\{h(0) + \int_0^s E(u,s,s')G(s')\mathrm{d}s'\Big\} \\ &\qquad+ \int_0^t E(v,t,s)e^{-\Phi(X(s))}\int_u k_w(u,v)\int_0^s E(u,s,s')e^{-\Phi(X(s'))} \\ &\qquad\quad\times \int_{u'} k_w(u',u)h(s',X(s'),u')\mathrm{d}u'\,\mathrm{d}s'\,\mathrm{d}u\,\mathrm{d}s =\end{aligned}$$

$$
\begin{aligned}
(4.19)\quad &= E(v,t,0)h(0) + \int_0^t E(v,t,s)G(s)\mathrm{d}s \\
&+ \int_0^t E(v,t,s)e^{-\Phi}\int_u k_w(u,v)E(u,s,0)h(0)\mathrm{d}u\,\mathrm{d}s \\
&+ \int_0^t\int_0^s\int_u E(v,t,s)e^{-\Phi}k_w(u,v)E(u,s,0)E(u,s,s')G(s')\mathrm{d}u\,\mathrm{d}s'\,\mathrm{d}s \\
&+ \int_0^t E(v,t,s)e^{-\Phi(X(s))}\int_u k_w(u,v) \\
&\quad\times\int_0^s E(u,s,s')e^{-\Phi(X(s'))}\int_{u'} k_w(u',u)E(u',s',0)h(0) \\
&+ \int_0^t E(v,t,s)e^{-\Phi(X(s))}\int_u k_w(u,v) \\
&\quad\times\int_0^s E(u,s,s')e^{-\Phi(X(s'))}\int_{u'} k_w(u',u)\int_0^{s'} E(u',s',s'')G(s'') \\
&+ \int_0^t E(v,t,s)e^{-\Phi(X(s))}\int_u k_w(u,v)\int_0^s E(u,s,s')e^{-\Phi(X(s'))}\int_{u'} k_w(u',u) \\
&\quad\times\int_0^{s'} E(u',s',s'')e^{-\Phi(X(s''))}\int_{u''} k_w(u'',u')h(s'',X(s''),u'')\mathbf{1}_{|F-\mu_E|\le\bar{\delta}\mu_E} \\
&+ \underline{\int_0^t E(v,t,s)e^{-\Phi(X(s))}\int_u k_w(u,v)\int_0^s E(u,s,s')e^{-\Phi(X(s'))}\int_{u'} k_w(u',u)} \\
&\quad\underline{\times\int_0^{s'} E(u',s',s'')e^{-\Phi(X(s''))}\int_{u''} k_w(u'',u')h(s'',X(s''),u'')\,\mathbf{1}_{|F-\mu_E|\ge\bar{\delta}\mu_E}}_{(*)},
\end{aligned}
$$

where we abbreviated notation as follows:,

$$
\begin{gathered}
X(s) := X(s;t,x,v), \quad X(s') := X'(s';s,X(s;t,x,v),u), \\
X(s'') := X(s'';s',X'(s';s,X(s;t,x,v),u),u'),
\end{gathered}
$$

and where we use definitions similar to those in (4.6):

$$
\begin{aligned}
E(v,t,s) &:= \exp\left\{-\int_s^t \nu_E(\tau,X(\tau;t,x,v),V(\tau;t,x,v))\mathrm{d}\tau\right\} \\
&:= \exp\left\{-\int_s^t\left[e^{-\Phi(X(\tau))}\nu(V(\tau)) + \frac{1}{2}V(\tau)\cdot\nabla\phi(\tau,X(\tau))\right.\right. \\
&\qquad\left.\left. + \frac{1}{w}\nabla_x(\phi(\tau,X(\tau)) + \Phi(X(\tau))\cdot\nabla_v w(V(\tau))\right]\mathrm{d}\tau\right\}, \qquad (4.20)
\end{aligned}
$$

$$
G := -w\sqrt{\mu_E}V\cdot\nabla_x\phi + we^{-\frac{\Phi}{2}}\Gamma\left(\frac{h}{w},\frac{h}{w}\right).
$$

Under the assumption that $\delta_\phi + \delta_\phi/\lambda_\phi \ll 1$,

$$
E(v,t,s) \le e^{-\frac{1}{2}e^{-\|\Phi\|_C}\nu(v)(t-s)} := e^{-\nu_\Phi(v)(t-s)}, \qquad (4.21)
$$

where we define $\nu_\Phi(v) := \frac{1}{2}e^{-\|\Phi\|_C}\nu(v)$. For (4.19), every term except $(*)$ is controlled by

$$C_{\Phi,\lambda_\phi}\big(\delta_\phi + \overline{\delta} + \|h(0)\|_\infty + \sup_{0\le s\le t}\|h(s)\|_\infty^2\big). \tag{4.22}$$

For $(*)$, we choose $m(N)$ so that

$$k_{w,m}(u,v) := \mathbf{1}_{\{|u-v|\ge\frac{1}{m},|u|\le m\}}k_w(u,v) \tag{4.23}$$

satisfies $\int_{\mathbb{R}^3}|k_{w,m}(u,v)-k_w(u,v)|\mathrm{d}u \le \frac{1}{N}$ for sufficiently large $N \ge 1$. Then, by splitting k_w,

$$\begin{aligned}(*) \le{}& \underbrace{\int_0^t\int_0^s\int_0^{s'} e^{-\nu_\Phi(0)(t-s'')}\int_u k_{w,m}(u,v)\int_{u'}k_{w,m}(u',u)}\\ &\underbrace{\times\int_{u''}k_{w,m}(u'',u')h(s'',X''(s''),u'')\mathbf{1}_{|F-\mu_E|\ge\overline{\delta}\mu_E}\,\mathrm{d}u''\,\mathrm{d}u'\,\mathrm{d}u\,\mathrm{d}s''\,\mathrm{d}s'\,\mathrm{d}s}_{(**)}\\ &+O_\Omega\Big(\frac{1}{N}\Big)\sup_{0\le s\le t}\|h(s)\|_\infty.\end{aligned} \tag{4.24}$$

We analyze $(**)$. We use Theorem 3.9; then

$$\begin{aligned}&\exists i_s \in \{1,2,\dots,I_{\Omega,N}\} \text{ such that } X(s)\in\mathscr{O}_{i_s},\\ &\exists j_{s,s'} \in \{1,2,\dots,I_{\Omega,N}\} \text{ such that } X(s';s,X(s;t,x,v),u)\in\mathscr{O}_{j_{s,s'}},\end{aligned} \tag{4.25}$$

and then we can define the following sets for fixed n, $\vec{n}$, i, k, m, $\vec{m}$, j, and k', where Theorem 3.9 does not work.

$$\begin{aligned}R_1 &:= \big\{u \mid u\notin B(\vec{n}\delta;2\delta)\cap\{\mathbb{R}^3\setminus\mathscr{V}_{i_s}(\widehat{\mathbf{e}_1},\widehat{\mathbf{e}_2})\}\big\},\\ R_2 &:= \big\{s' \mid |s-s'|\le\delta^\gamma\big\},\\ R_3 &:= \big\{s' \mid |s'-\psi_1^{n,\vec{n},i,k,m,\vec{m},j,k'}(n\delta,X(n\delta;t,x,v),\vec{n}\delta)|\lesssim_N \delta^\gamma\|\psi_1\|_{C^{0,\gamma}}\big\},\\ R_4 &:= \big\{u' \mid u'\notin B(\vec{m}\delta;2\delta)\cap\{\mathbb{R}^3\setminus\mathscr{V}_{j_{s,s'}}(\partial_{|u|}X,\partial_{\widehat{u}_1}X)\}\big\},\\ R_5 &:= \big\{s'' \mid |s'-s''|\le\delta^\gamma\big\},\\ R_6 &:= \big\{s'' \mid \min_{r=1,2}|s''-\psi_r^{n,\vec{n},i,k,m,\vec{m},j,k'}(m\delta,X(m\delta;n\delta,X(n\delta;t,x,v),\vec{m}\delta),\vec{n}\delta)|\\ &\qquad\lesssim_N \delta^\gamma\min_{r=1,2}\|\psi_r\|_{C^{0,\gamma}}\big\}.\end{aligned} \tag{4.26}$$

Therefore,

$$\begin{aligned}(**) = {}& \underbrace{\sum_{n=0}^{[t/\delta]+1}\sum_{|\vec{n}|\le N}\sum_{m=0}^{[t/\delta]+1}\sum_{|\vec{m}|\le N}\sum_{k}^{K_{i_s}}\sum_{k'}^{K'_{j_{s,s'}}}\int_{(n-1)\delta}^{(n+1)\delta}\int_{t^{k+1}+\delta^\gamma}^{t^k-\delta^\gamma}\int_{t^{k'+1}+\delta^\gamma}^{t^{k'}-\delta^\gamma}e^{-\nu_\Phi(0)(t-s'')}}\\ &\underbrace{\times\int_{|u|\le N,|u'|\le N,|u''|\le N}|h(s'',X(s''),u'')|\,\mathbf{1}_{|F-\mu_E|\ge\overline{\delta}\mu_E}\,\mathbf{1}_{R_1^c\cap R_2^c\cap R_3^c\cap R_4^c\cap R_5^c\cap R_6^c}}_{(\mathrm{MAIN})}\\ &+B+R,\end{aligned} \tag{4.27}$$

where the B term corresponds to where the trajectory is near bouncing points and R corresponds to where (u,s',u',s'') is in one of R_1 through R_6. So we have the

following small estimates for B and R:

$$
\begin{aligned}
B \le & \int_0^t \int_0^s \int_0^{s'} e^{-\nu_\Phi(0)(t-s'')} \int_{|u|\le N} k_{w,m}(u,v) \int_{|u'|\le N} k_{w,m}(u',u) \\
& \times \int_{|u''|\le N} k_{w,m}(u'',u') h(s'', X''(s''), u'') \mathbf{1}_{|F-\mu_E|\ge \overline{\delta}\mu_E} \\
& \times \mathbf{1}_{|s'-t^k|\le \delta^\gamma \text{ or } |s''-t^{k'}|\le \delta^\gamma} \\
\le & C_N \delta^\gamma \sup_{0\le s\le t} \|h(s)\|_\infty,
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
R \le & \int_0^t \int_0^s \int_0^{s'} e^{-\nu_\Phi(0)(t-s'')} \int_{|u|\le N} k_{w,m}(u,v) \int_{|u'|\le N} k_{w,m}(u',u) \\
& \times \int_{|u''|\le N} k_{w,m}(u'',u') h(s'', X''(s''), u'') \mathbf{1}_{|F-\mu_E|\ge \overline{\delta}\mu_E} \\
& \times \mathbf{1}_{R_1\cup R_2\cup R_3\cup R_4\cup R_5\cup R_6} \\
\le & C_N \delta^\gamma \sup_{0\le s\le t} \|h(s)\|_\infty,
\end{aligned}
$$

For (MAIN) in (4.27), we are away from two sets B and R. Under the condition of

$$
(u, s', u', s'') \in R_1^c \ \cap \ R_2^c \ \cap R_3^c \ \cap R_4^c \ \cap R_5^c \ \cap R_6^c,
$$

where R_i^c is the complement of R_i. Indices n, $\vec{n}$, i_s, k, m, $\vec{m}$, $j_{s,s'}$, and k' are determined so that

$$
\begin{aligned}
t &\in [(n-1)\delta, (n+1)\delta], \\
X(s;t,x,v) &\in \mathscr{O}_{i_s}, \\
X(s';s,X(s;t,x,v),u) &\in \mathscr{O}_{j_{s,s'}}, \\
u &\in B(\vec{n}\delta; 2\delta) \cap \mathbb{R}^3 \setminus \mathscr{V}_{i_s}(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2), \\
u' &\in B(\vec{m}\delta; 2\delta) \cap \mathbb{R}^3 \setminus \mathscr{V}_{j_{s,s'}}(\partial_{|u|}X, \partial_{\hat{u}_1}X).
\end{aligned}
$$

We can apply Theorem 3.9, which gives a local time-independent lower bound of

$$
\left| \det\left(\frac{\partial(X(s''))}{\partial(|u'|, \zeta_1, \zeta_2)} \right) \right| \ge \epsilon'_\delta.
$$

Note that $\{\zeta_1, \zeta_2\} \subset \{|u|, \hat{u}_1, \hat{u}'_1, \hat{u}'_2\}$ are chosen variables in Theorem 3.9 and $\{\zeta_3, \zeta_4\} \subset \{|u|, \hat{u}_1, \hat{u}'_1, \hat{u}'_2\}$ are unchosen variables. Let us use $\mathscr{P}$ to denote the projection of $B(\vec{n}\delta; 2\delta) \cap \mathbb{R}^3 \setminus \mathscr{V}_{i_s}(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2) \times B(\vec{m}\delta; 2\delta) \cap \mathbb{R}^3 \setminus \mathscr{V}_{j_{s,s'}}(\partial_{|u|}X, \partial_{\hat{u}_1}X)$ into $\mathbb{R}^3$, which corresponds to the $(|u'|, \zeta_1, \zeta_2)$ components. If we choose sufficiently small δ, there exist small $r_{\delta,n,\vec{n},i,k,m,\vec{m},j,k'}$ such that there exist one-to-one

map $\mathscr{M}$,

$$\mathscr{M} : \mathscr{P}\big(B(\vec{n}\delta;2\delta)\cap\mathbb{R}^3\setminus\mathscr{V}_{i_s}(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2)\times B(\vec{m}\delta;2\delta)\cap\mathbb{R}^3\setminus\mathscr{V}_{j_{s,s'}}(\partial_{|u|}X,\partial_{\widehat{u}_1}X)\big)$$
$$\mapsto B(X(s'';s',X(s';s,X(s;t,x,v),u),u'),r_{\delta,n,\vec{n},i,k,m,\vec{m},j,k'}).$$

So we perform a change of variable for (MAIN) in (4.27) to obtain

$$\begin{aligned}
&\text{(MAIN)}\\
&\le \sum_{n=0}^{[t/\delta]+1}\sum_{|\vec{n}|\le N}\sum_{m=0}^{[t/\delta]+1}\sum_{|\vec{m}|\le N}\sum_{k}^{K_{i_s}}\sum_{k'}^{K'_{j_{s,s'}}}\int_{(n-1)\delta}^{(n+1)\delta}\int_{t^{k+1}+\delta^\nu}^{t^k-\delta^\nu}\int_{t^{k'+1}+\delta^\nu}^{t^{k'}-\delta^\nu} e^{-\nu_\Phi(0)(t-s'')}\\
&\quad\times\int_{u''}\mathrm{d}u''\int_{\widehat{u}_2,\zeta_3,\zeta_4}\mathbf{1}_{|u|\le N,|u'|\le N,|u''|\le N}\,\mathrm{d}\widehat{u}_2\,\mathrm{d}\zeta_3\,\mathrm{d}\zeta_4\\
&\quad\times\int_{|u'|,\zeta_1,\zeta_2}\mathrm{d}|u'|\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2|h(s'',X(s''),u'')|\mathbf{1}_{|F-\mu_E|\ge\bar{\delta}\mu_E}\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}s''\\
&\le \sum_{n=0}^{[t/\delta]+1}\sum_{|\vec{n}|\le N}\sum_{m=0}^{[t/\delta]+1}\sum_{|\vec{m}|\le N}\sum_{k}^{K_{i_s}}\sum_{k'}^{K'_{j_{s,s'}}}\int_{(n-1)\delta}^{(n+1)\delta}\int_{t^{k+1}+\delta^\nu}^{t^k-\delta^\nu}\int_{t^{k'+1}+\delta^\nu}^{t^{k'}-\delta^\nu} e^{-\nu_\Phi(0)(t-s'')}\\
&\quad\times\int_{\widehat{u}_2,\zeta_3,\zeta_4}\mathbf{1}_{|u|\le N,|u'|\le N,|u''|\le N}\,\mathrm{d}\widehat{u}_2\,\mathrm{d}\zeta_3\,\mathrm{d}\zeta_4\,\mathbf{1}_{|F-\mu_E|\ge\bar{\delta}\mu_E}\\
&\quad\times\int_{u''}\int_{B(X(s''),r_{\delta,n,\vec{n},i,k,m,\vec{m},j,k'})}|h^{\ell+1}(s'',x,u'')|\frac{1}{\epsilon'_{\Omega,N,\|\Phi\|_{C^2},\delta}}\,\mathrm{d}x\,\mathrm{d}u''\,\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}s''\\
&\le C_{N,\delta,\Phi,\phi,\Omega}\int_0^t e^{-\nu_\Phi(0)(t-s'')}\int_\Omega\int_{|u''|\le N}|h(s'',x,u'')|\,\mathbf{1}_{|F-\mu_E|\ge\bar{\delta}\mu_E}\,\mathrm{d}u''\,\mathrm{d}x\,\mathrm{d}s''\\
&\le C_{N,\delta,\Phi,\phi,\Omega}\sup_{0\le s''\le t}\big\|h(s'')\mathbf{1}_{|F-\mu_E|\ge\bar{\delta}\mu_E}\big\|_{L^1(\Omega\times B_N)}\\
&\le C_{N,\delta,\Phi,\phi,\Omega}\Big\|\frac{w}{\sqrt{\mu}}\Big\|_{L^\infty(B_N)}\sup_{0\le s''\le t}\big\|(F(s'')-\mu_E)\mathbf{1}_{|F-\mu_E|\ge\bar{\delta}\mu_E}\big\|_{L^1(\Omega\times B_N)}.
\end{aligned}\tag{4.29}$$

From (4.16) and (1.11), we can further bound (4.29) by

$$\begin{aligned}
&\le C_{N,\delta,\Phi,\phi,\Omega}\frac{1}{\bar{\delta}}\Big\|\frac{w}{\sqrt{\mu}}\Big\|_{L^\infty(B_N)}\\
&\quad\times\sup_{0\le s''\le t}\Big\{\mathscr{H}(F(0))-\mathscr{H}(\mu_E)-\iint(F(s'')-\mu_E)\\
&\qquad+\iint\Big(\frac{|v|^2}{2}+\Phi(x)\Big)(F(s'')-\mu_E)\Big\}.
\end{aligned}\tag{4.30}$$

Finally, utilizing (1.4), (1.6), and (4.15), we deduce that

$$\begin{aligned}
\text{(MAIN)}&\le C_{N,\delta,\Phi,\phi,\Omega}\frac{1}{\bar{\delta}}\Big\|\frac{w}{\sqrt{\mu}}\Big\|_{L^\infty(B_N)}\\
&\quad\times\Big(\mathscr{H}(F(0))-\mathscr{H}(\mu_E)+\frac{\delta_\phi}{\lambda_\phi}\big\{1+\sup_{0\le s''\le t}\|wf(s'')\|_\infty\big\}\Big).
\end{aligned}\tag{4.31}$$

For $\delta_0, \delta_\phi, \frac{\delta_\phi}{\lambda_\phi} \ll 1$, we collect (4.22), (4.24), (4.27), (4.28), and (4.31) to get

$$\begin{aligned}\sup_{0\le t\le T_1}\|wf(t)\|_\infty \lesssim \|wf_0\|_\infty + \delta_\phi + \overline{\delta} + \mathscr{H}(0) - \mathscr{H}(\mu_E)\\ + \frac{\delta_\phi}{\lambda_\phi} + \sup_{0\le t\le T_1}\|wf(t)\|_\infty^2.\end{aligned} \tag{4.32}$$

By choosing small data we deduce $\sup_{0\le t\le T_1}\|wf(t)\|_\infty < 2(\delta_0 + C\delta_\phi) \ll 1$ from (4.1). By continuity of $\|wf(t)\|_\infty$ in Theorem 4.1 and a uniform bound, we conclude

$$T_1 = \infty,$$

and this proves the global-in-time existence. □

5 A Time-Independent Potential

First we derive L^2-coercivity for the homogeneous linear Boltzmann of (1.18)

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\Phi(x)\cdot\nabla_v f + e^{-\Phi}Lf = 0,$$

with specular reflection boundary condition on the boundary $\partial\Omega$. From (1.6),

$$\iint_{\Omega\times\mathbb{R}^3} f(t)\sqrt{\mu_E} = \iint_{\Omega\times\mathbb{R}^3} f_0\sqrt{\mu_E} = 0, \tag{5.1}$$

$$\iint_{\Omega\times\mathbb{R}^3} f(t)\left(\frac{|v|^2}{2}+\Phi\right)\sqrt{\mu_E} = \iint_{\Omega\times\mathbb{R}^3} f_0\left(\frac{|v|^2}{2}+\Phi\right)\sqrt{\mu_E} = 0. \tag{5.2}$$

If the domain is axis-symmetric (1.7) and Φ is degenerated (1.8), then

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}^3} f(t)\{(x-x^0)\times\varpi\}\cdot v\sqrt{\mu_E} =\\ \iint_{\Omega\times\mathbb{R}^3} f_0\{(x-x^0)\times\varpi\}\cdot v\sqrt{\mu_E} = 0.\end{aligned} \tag{5.3}$$

We prove Proposition 1.4 by the contradiction argument of the proof of proposition 11 in [12]. We first study the geometric lemma, which allows estimating near the boundary via the interior bound, and postpone the proof of the proposition. Define the distance function toward the boundary as

$$\mathrm{dist}(x,\partial\Omega) = \inf\{|x-y| : y\in\partial\Omega\}, \tag{5.4}$$

which is well-defined if $\mathrm{dist}(x,\partial\Omega) \ll 1$. In this case there exists a unique $x_* \in \partial\Omega$ satisfying $|x^* - x| = \mathrm{dist}(x,\partial\Omega)$. We also define

$$n(x) = n(x_*) \tag{5.5}$$

for $x\in\Omega$ with $\mathrm{dist}(x,\partial\Omega) \ll 1$.

LEMMA 5.1. *Let g be a (distributional) solution to*

$$\partial_t g + v\cdot\nabla_x g + E\cdot\nabla_v g = G, \tag{5.6}$$

where $E = E(t,x) \in C^{1,\gamma}$. Then, for a sufficiently small $\varepsilon > 0$,

$$\int_\varepsilon^{1-\varepsilon} \|\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)<\varepsilon^4}\mathbf{1}_{|n(x)\cdot v|>\varepsilon} g(t)\|_2^2\,\mathrm{d}t \lesssim \int_0^1 \|\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)>\varepsilon^3/2} g(t)\|_2^2\,\mathrm{d}t + \int_0^1 \iint_{\Omega\times\mathbb{R}^3} |gG|. \tag{5.7}$$

Note that this lemma is true even for a time-dependent external field case.

PROOF. For $x \in \bar\Omega$ with $\mathrm{dist}(x,\partial\Omega) < \varepsilon^4$, $n(x)\cdot v < -\varepsilon$, and $y \in \partial\Omega$ with $|y - x_*| \ll 1$,

$$\begin{aligned} |X(t+\varepsilon;t,x,v) - y| &\geq |(X(t+\varepsilon;t,x,v) - y)\cdot n(x_*)| \\ &= \left|(x-y)\cdot n(x_*) + v\cdot n(x_*)\varepsilon^2 + O(1)\|E\|_\infty \frac{\varepsilon^4}{2}\right| \\ &\geq \varepsilon^3 - \varepsilon^4 - O(1)\|E\|_\infty\frac{\varepsilon^4}{2} \geq \frac{\varepsilon^3}{2}. \end{aligned}$$

Hence

$$\mathrm{dist}(X(t+\varepsilon;t,x,v),\Omega) = \inf_{y\in\partial\Omega,|y-x_*|\ll 1} |X(t+\varepsilon;t,x,v) - y| \geq \frac{\varepsilon^3}{2}. \tag{5.8}$$

We can prove the exact same lower bound of $|X(t-\varepsilon;t,x,v)-y|$ when $n(x)\cdot v > \varepsilon$. Hence we conclude, for $x \in \bar\Omega$ with $\mathrm{dist}(x,\partial\Omega) < \varepsilon^4$ and $n(x)\cdot v > \varepsilon$,

$$\mathrm{dist}(X(t-\varepsilon;t,x,v),\Omega) = \inf_{y\in\partial\Omega,|y-x_*|\ll 1} |X(t-\varepsilon;t,x,v) - y| \geq \frac{\varepsilon^3}{2}. \tag{5.9}$$

Moreover, it is well-known that $(x,v) \mapsto (X(t+\varepsilon;t,x,v), V(t+\varepsilon;t,x,v))$ is a local diffeomorphism if $\mathrm{dist}(x,\partial\Omega) < \varepsilon^4$ and $n(x)\cdot v < -\varepsilon$. And, $(x,v) \mapsto (X(t-\varepsilon;t,x,v), V(t+\varepsilon;t,x,v))$ is also a local diffeomorphism if $\mathrm{dist}(x,\partial\Omega) < \varepsilon^4$ and $n(x)\cdot v > \varepsilon$ hold, since they never hit the boundary. These diffeomorphisms satisfy

$$\mathrm{Jac}\left(\frac{\partial(X(t\pm\varepsilon;t,x,v), V(t\pm\varepsilon;t,x,v))}{\partial(x,v)}\right) = 1. \tag{5.10}$$

Note that

$$\|X(t+\varepsilon;t,\cdot,\cdot)\|_{C^{1,\gamma}} \leq \frac{\varepsilon^2}{2}\|E\|_{C^{1,\gamma}},\quad \|V(t+\varepsilon;t,\cdot,\cdot)\|_{C^{1,\gamma}} \leq \varepsilon\|E\|_{C^{1,\gamma}}.$$

By expansions, we conclude that there exist sufficiently small $\delta > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $(X(t+\varepsilon;t,\cdot,\cdot), V(t+\varepsilon;t,\cdot,\cdot))$ is one-to-one in $\{(x,v) \in \bar\Omega\times\mathbb{R}^3 : \mathrm{dist}(x,\partial\Omega) < \varepsilon^4,\ n(x)\cdot v < -\varepsilon,\ |x-x^0| + |v-v^0| < \delta\}$ and $(X(t-\varepsilon;t,\cdot,\cdot), V(t-\varepsilon;t,\cdot,\cdot))$ is so in $\{(x,v) \in \bar\Omega\times\mathbb{R}^3 : \mathrm{dist}(x,\partial\Omega) < \varepsilon^4,\ n(x)\cdot v > \varepsilon,\ |x-x^0| + |v-v^0| < \delta\}$.

On the other hand, if $X(t+\varepsilon;t,\tilde{x},\tilde{v}) = X(t+\varepsilon;t,x,v)$ and $V(t+\varepsilon;t,\tilde{x},\tilde{v}) = V(t+\varepsilon;t,x,v)$, then $|v-\tilde{v}| \le \|E\|_\infty \varepsilon$ and $|x-\tilde{x}| \le 2\|E\|_\infty \varepsilon^2$. We deduce the same conclusion if $X(t-\varepsilon;t,\tilde{x},\tilde{v}) = X(t-\varepsilon;t,x,v)$ and $V(t-\varepsilon;t,\tilde{x},\tilde{v}) = V(t-\varepsilon;t,x,v)$. Hence for a sufficiently small ε, such (x,v) and $(\tilde{x},\tilde{v})$ are close as $|(x,v)-(\tilde{x},\tilde{v})| < \delta$. From the local one-to-one property in the previous sentence we conclude $(x,v) = (\tilde{x},\tilde{v})$.

Now we are ready to prove (5.7). Note that

$$\frac{d}{ds}|g(s,X(s;t,x,v),V(s;t,x,v))|^2 = 2(gG)(s,X(s;t,x,v),V(s;t,x,v)).$$

For (x,v) with $\mathrm{dist}(x,\partial\Omega) < \varepsilon^4$ and $n(x)\cdot v < -\varepsilon$, taking integration $s \in [t,t+\varepsilon]$ along the trajectory,

$$|g(t,x,v)|^2 = |g(t+\varepsilon,X(t+\varepsilon),V(t+\varepsilon))|^2 - 2\int_t^{t+\varepsilon} 2(gG)(s,X(s),V(s))\mathrm{d}s.$$

From (5.8), $\mathrm{dist}(X(t+\varepsilon),\partial\Omega) \ge \varepsilon^3/2$. Using (5.10) and the one-to-one property of $(x,v) \mapsto (X(s),V(s))$ for any fixed $|s| \le \varepsilon$, we take an integration over $\mathrm{dist}(x,\partial\Omega) < \varepsilon^4$ and $n(x)\cdot v < -\varepsilon$ and conclude that

$$\|\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)<\varepsilon^4}\mathbf{1}_{n(x)\cdot v<-\varepsilon}g(t)\|_2^2 = \|\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)>\varepsilon^3/2}g(t+\varepsilon)\|_2^2 + \int_t^{t+\varepsilon}\iint_{\Omega\times\mathbb{R}^3}|g(s)G(s)|. \tag{5.11}$$

For the other case, $\mathrm{dist}(x,\partial\Omega) < \varepsilon^4$ and $n(x)\cdot v > \varepsilon$, we repeat the same argument but change ε to $-\varepsilon$ and conclude that

$$\|\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)<\varepsilon^4}\mathbf{1}_{n(x)\cdot v>\varepsilon}g(t)\|_2^2 = \|\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)>\varepsilon^3/2}g(t-\varepsilon)\|_2^2 + \int_{t-\varepsilon}^{t}\iint_{\Omega\times\mathbb{R}^3}|g(s)G(s)|. \tag{5.12}$$

Finally by $\int_\varepsilon^{1-\varepsilon}(5.11)\mathrm{d}t$ and $\int_\varepsilon^{1-\varepsilon}(5.12)\mathrm{d}t$, we conclude (5.7). □

PROOF OF PROPOSITION 1.4. First, it is easy to check that equation (1.38) is translation invariant in time; i.e., $\tilde{f}(t,x,v) := f(t+c,x,v)$ also solves the same equation for any c. Note that this is not true for the time-dependent potential case anymore, unless the potential is periodic in time. Therefore it suffices to prove coercivity for finite time interval $t \in [0,1]$ and so we claim (1.39) for $N = 0$.

Step 1. Assume that Proposition 1.4 is wrong. This means for any $m \gg 1$ there exists a solution f^m to (1.38) satisfying the specular reflection BC that solves

$$\partial_t f^m + v\cdot\nabla_x f^m - \nabla_x\Phi\cdot\nabla_v f^m + e^{-\Phi}Lf^m = 0 \quad \text{for } t \in [0,1] \tag{5.13}$$

and satisfies

$$\int_0^1 \|\mathbf{P} f^m(t)\|_2^2 \,\mathrm{d}t \geq m \int_0^1 \|(\mathbf{I}-\mathbf{P}) f^m(t)\|_\nu^2 \,\mathrm{d}t. \tag{5.14}$$

We define the normalized form of f^m by

$$Z^m(t,x,v) := \frac{f^m(t,x,v)}{\sqrt{\int_0^1 \|\mathbf{P} f^m(t)\|_2^2 \,\mathrm{d}t}}. \tag{5.15}$$

Then Z^m solves

$$\partial_t Z^m + v\cdot\nabla_x Z^m - \nabla_x\Phi\cdot\nabla_v Z^m + e^{-\Phi} L Z^m = 0, \tag{5.16}$$

$$Z^m(t,x,v) = Z^m(t,x,R_x v) \quad x\in\partial\Omega, \tag{5.17}$$

and

$$\frac{1}{m} \geq \int_0^1 \|(\mathbf{I}-\mathbf{P}) Z^m(t)\|_\nu^2 \,\mathrm{d}t. \tag{5.18}$$

Step 2. We claim that

$$\sup_m \sup_{0\leq t\leq 1} \|Z^m(t)\|_2^2 < \infty. \tag{5.19}$$

From (5.16), for $0 \leq t \leq 1$,

$$\|Z^m(t)\|_2^2 + \int_0^t e^{-\Phi}(LZ^m, Z^m) = \|Z^m(0)\|_2^2. \tag{5.20}$$

From the nonnegativity of L,

$$\sup_{0\leq t\leq 1} \|Z^m(t)\|_2^2 \leq \|Z^m(0)\|_2^2. \tag{5.21}$$

On the other hand, by integration $\int_0^1$ (5.20)dt and utilizing (5.18) and (5.15),

$$\|Z^m(0)\|_2^2 \lesssim_\Phi \int_0^1 \|Z^m\|_2^2 + \int_0^1 \|(\mathbf{I}-\mathbf{P})Z^m\|_\nu^2 \lesssim_\Phi 1 + \frac{1}{m}. \tag{5.22}$$

Therefore, we prove the claim (5.19) from (5.21) and (5.22).

Step 3. Therefore, the sequence $\{Z^m\}_{m\gg 1}$ is uniformly bounded from above by $\sup_{0\leq t\leq 1}\|g(t)\|_\nu^2 \,\mathrm{d}t$. By the weak compactness of L^2-space, there exists a weak limit Z such that

$$Z^m \rightharpoonup Z \quad \text{in } L^\infty\big([0,1]; L^2_\nu(\Omega\times\mathbb{R}^3)\big) \cap L^2\big([0,1]; L^2_\nu(\Omega\times\mathbb{R}^3)\big). \tag{5.23}$$

Therefore, in the sense of distributions, Z solves

$$\partial_t Z + v\cdot\nabla_x Z - \nabla_x\Phi\cdot\nabla_v Z = 0. \tag{5.24}$$

Now we consider the limit of the linear conservation laws. Note that, taking a weak limit $Z^m \rightharpoonup Z$ in $L^\infty_t L^2_{x,v}$ of (5.23) and using (5.1), (5.2), and (5.15), we deduce linear conservation laws, for almost every $t \in [0,1]$,

$$\iint_{\Omega\times\mathbb{R}^3} Z(t)\sqrt{\mu_E} = 0, \quad \iint_{\Omega\times\mathbb{R}^3} Z(t)\left(\frac{|v|^2}{2}+\Phi\right)\sqrt{\mu_E} = 0. \tag{5.25}$$

In the case that both (1.7) and (1.8) hold, from (5.3),

$$\iint_{\Omega\times\mathbb{R}^3} \{(x-x^0)\times\varpi\}\cdot v Z(t)\sqrt{\mu_E} = 0. \tag{5.26}$$

On the other hand, since

$$\mathbf{P}Z^m \rightharpoonup \mathbf{P}Z \quad \text{and} \quad (\mathbf{I}-\mathbf{P})Z^m \to 0 \quad \text{in} \int_0^1 \|\cdot\|_\nu^2 \,\mathrm{d}t,$$

we know that weak limit Z has only a hydrodynamic part, i.e.,

$$Z(t,x,v) = \{a(t,x) + v\cdot b(x,v) + |v|^2 c(t,x)\}\sqrt{\mu_E}, \tag{5.27}$$

and

$$\int_0^1 \|Z\|_\nu^2 \,\mathrm{d}t \le \liminf_{m\to\infty}\int_0^1 \|Z^m\|_\nu^2 \,\mathrm{d}t \le 1 + \frac{1}{m} \to 1. \tag{5.28}$$

Step 4. Let $\chi_\varepsilon : \bar{\Omega} \to [0,1]$ be a smooth function such that $\chi_\varepsilon(x) = 1$ if $\mathrm{dist}(x,\partial\Omega) > 2\varepsilon^4$ and $\chi_\varepsilon(x) = 0$ if $\mathrm{dist}(x,\partial\Omega) < \varepsilon^4$. From (5.16),

$$[\partial_t + v\cdot\nabla_x](\chi_\varepsilon Z^m) = \nabla_x\Phi\cdot\nabla_v(\chi_\varepsilon Z^m) + v\cdot\nabla_x\chi_\varepsilon Z^m - e^{-\Phi}L(\chi_\varepsilon Z^m).$$

From the standard average lemma in [8], $\chi_\varepsilon Z^m$ is compact, i.e.,

$$\chi_\varepsilon Z^m \to \chi_\varepsilon Z \quad \text{strongly in } L^2([0,1]; L^2_\nu(\Omega\times\mathbb{R}^3)). \tag{5.29}$$

Step 5. First we claim that

$$\int_\varepsilon^{1-\varepsilon} \left\|(Z^m(t,x,v) - Z(t,x,v))\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)<\varepsilon^4}\mathbf{1}_{|n(x)\cdot v|>\varepsilon}\right\|_2^2 \lesssim \int_0^1 \left\|\big(Z^m(t,x,v) - Z(t,x,v)\big)\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)>\varepsilon^3/2}\right\|_2^2 + O\left(\frac{1}{\sqrt{m}}\right). \tag{5.30}$$

We consider the equation of $Z^m - Z$. By subtracting (5.16) from (5.24),

$$(Z^m - Z) + e^{-\Phi}LZ^m = 0. \tag{5.31}$$

Now we apply Lemma 5.1 to (5.31) by equating g and G with $Z^m - Z$ and the right-hand side of (5.31), respectively. Then

$$\begin{aligned}\int_\varepsilon^{1-\varepsilon} &\|\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)<\varepsilon^4}\mathbf{1}_{|n(x)\cdot v|>\varepsilon}(Z^m - Z)(t)\|_2^2\,\mathrm{d}t \\ &\lesssim \int_0^1 \|\mathbf{1}_{\mathrm{dist}(x,\partial\Omega)>\varepsilon^3/2}(Z^m - Z)(t)\|_2^2\,\mathrm{d}t \\ &\quad + \int_0^1 \iint_{\Omega\times\mathbb{R}^3} |Z^m - Z|\langle v\rangle|(\mathbf{I}-\mathbf{P})Z^m|.\end{aligned}$$

Using Hölder's inequality, we bound the last line of the above estimate by

$$\sqrt{m}\int_0^1 \|(\mathbf{I}-\mathbf{P})Z^m\|_\nu^2 + \frac{1}{\sqrt{m}}\int_0^1 \|Z^m\|_\nu^2 + \|Z\|_\nu^2.$$

By (5.19) and (5.18), we conclude (5.30).

On the other hand, from (5.18), (5.27), and (5.19),

$$\begin{aligned}&\int_\varepsilon^{1-\varepsilon} \|(Z^m - Z)\mathbf{1}_{|n(x)\cdot v|\le\varepsilon}\|_2^2 \\ &\le \int_\varepsilon^{1-\varepsilon} \|(\mathbf{I}-\mathbf{P})Z^m\|_\nu^2 + O(\varepsilon)\int_\varepsilon^{1-\varepsilon} \|\mathbf{P}Z^m\|_2^2 + \|\mathbf{P}Z\|_2^2 \\ &\le \frac{1}{m} + O(\varepsilon).\end{aligned} \tag{5.32}$$

Step 6. For given $\varepsilon > 0$, we can choose $m \gg_\varepsilon 1$ such that

$$\begin{aligned}&\int_0^1 \iint_{\Omega\times\mathbb{R}^3} |Z^m - Z|^2 \\ &\le \int_{1-\varepsilon}^1 \iint_{\Omega\times\mathbb{R}^3} + \int_0^\varepsilon \iint_{\Omega\times\mathbb{R}^3} + \int_\varepsilon^{1-\varepsilon} \iint_{\Omega_\varepsilon\times\mathbb{R}^3} \\ &\quad + \int_\varepsilon^{1-\varepsilon} \iint_{\substack{\Omega\setminus\Omega_\varepsilon\times\mathbb{R}^3 \\ \cap\{|n(x)\cdot v|<\varepsilon \text{ or } |v|\ge\varepsilon^{-1}\}}} + \int_\varepsilon^{1-\varepsilon} \iint_{\substack{\Omega\setminus\Omega_\varepsilon\times\mathbb{R}^3 \\ \cap\,\{|n(x)\cdot v|\ge\varepsilon \text{ and } |v|\le\varepsilon^{-1}\}}} \\ &< C\varepsilon,\end{aligned}$$

where we have used (5.19), (5.29), (5.30), and (5.32). Therefore, we conclude that $Z^m \to Z$ strongly in $L^2([0,1]\times\Omega\times\mathbb{R}^3)$ and hence

$$\int_0^1 \|Z\|_2^2 = 1. \tag{5.33}$$

Step 7. We consider the boundary condition of Z. Fix a small constant $\delta > 0$. In order to control Z in $\{(x,v)\in\gamma_\pm : |n(x)\cdot v| < \delta\}$, we use smooth functions $\phi_\pm^\delta : \bar\Omega\times\mathbb{R}^3 \to [0,1]$ where $\phi_\pm^\delta \equiv 1$ on $\{(x,v)\in\gamma_\pm : |n(x)\cdot v| < \delta\}$ and $\phi_\pm^\delta \equiv 0$ on $\{(x,v)\in\gamma_\pm : |n(x)\cdot v| > 2\delta\}$, respectively.

From the weak formulation, we have $(\partial_t + v\cdot\nabla_x - \nabla_x\Phi\cdot\nabla_v)|Z|^2 = 0$. Testing it with $\phi^\delta_\pm$, we obtain

$$\int_0^1\int_\gamma |Z|^2\phi^\delta_\pm(n\cdot v) = -\iint_{\Omega\times\mathbb{R}^3}\phi^\delta_\pm|Z(1)|^2 + \iint_{\Omega\times\mathbb{R}^3}\phi^\delta_\pm|Z(0)|^2 + \int_0^1\iint_{\Omega\times\mathbb{R}^3} -v\cdot\nabla_x\phi^\delta_\pm|Z|^2 + \nabla_x\Phi\cdot\nabla_v\phi^\delta_\pm|Z|^2.$$

From (5.27) and (5.28), we deduce that $Z\in L^2(\{(x,v)\in\gamma_\pm : |n(x)\cdot v|<\delta\})$, and $a,b,c\in L^2([0,1]\times\partial\Omega)$ such that

$$\int_0^1\int_{\partial\Omega}|a|^2+|b|^2+|c|^2 \lesssim \int_0^1\|Z\|_\nu^2 + \sup_{0\le t\le 1}\|Z(t)\|_2^2. \tag{5.34}$$

Now we claim that

$$Z(t,x,v) = Z(t,x,R_xv) \quad \text{almost every } [\delta,1-\delta]\times\gamma_-. \tag{5.35}$$

Let $\phi : \bar\Omega\times\mathbb{R}^3\to\mathbb{R}$ be a smooth bounded function with strong decay in v. Moreover, we assume that this test function is an even function in $\phi(n(x)\cdot v)$ at the boundary. Testing (5.24) with such a function ϕ, we have

$$\int_0^1\int_\gamma Z\phi(n(x)\cdot v) = -\iint_{\Omega\times\mathbb{R}^3}(Z(1)-Z(0))\phi + \int_0^1\iint_{\Omega\times\mathbb{R}^3} Z(-v\cdot\nabla_x\phi \quad + \nabla_x\Phi\cdot\nabla_v\phi). \tag{5.36}$$

On the other hand, employing the same test function, from (5.16) and (5.17), we conclude that

$$0 = -\iint_{\Omega\times\mathbb{R}^3}(Z^m(1)-Z^m(0))\phi + \int_0^1\iint_{\Omega\times\mathbb{R}^3} Z^m(-v\cdot\nabla_x\phi+\nabla_x\Phi\cdot\nabla_v\phi) + \int_0^1\iint_{\Omega\times\mathbb{R}^3} e^\Phi LZ^m\phi.$$

By passing to the limit $m\to\infty$, from (5.27) and (5.18), we realize that the right-hand side of (5.36) equals 0. Therefore, we conclude that

$$\int_0^1\int_\gamma Z\phi(n(x)\cdot v) = 0. \tag{5.37}$$

for any smooth function ϕ that is even in $n(x)\cdot v$ at the boundary. This proves (5.35).

Finally, combining (5.35), (5.27), and (5.34), we prove (1.40).

Step 8. We claim (1.42). We consider the system of a, b, and c that is obtained by plugging (5.27) in (5.24). From [16], in the sense of distributions, they solve (1.41).

The first equation of (1.41) implies that c is only a function of t, i.e., $c = c(t)$. From the first three equations of (1.41) we can get

$$b(t,x) = -\partial_t c(t)x + \varpi(t) \times x + m(t). \tag{5.38}$$

The proof of (5.38) is based on direct computations. (See lemma 12 in [12] for the details.)

From the second equation of (1.41), we obtain $\nabla_x \cdot b = -3c'(t)$. By the divergence theorem and (1.40),

$$-3c'(t)|\Omega| = \int_{\partial\Omega} b \cdot n = 0.$$

Therefore, $c'(t) = 0$, $c(t) = c_0$, and $b = \varpi(t) \times x + m(t)$. We conclude

$$c(t,x) = c_0. \tag{5.39}$$

We split into two cases: $\varpi = 0$ and $\varpi \neq 0$.

Case of $\varpi = 0$. If $\varpi = 0$, then $b(t) = m(t)$. From (1.40) we deduce that

$$b(t) \equiv m(t) \equiv 0. \tag{5.40}$$

Then from the last equation of (1.41), $a = a(x)$. From the fourth equation of (1.41), for some constant C, we obtain that

$$a(t,x) = 2c_0\Phi(x) + C. \tag{5.41}$$

Plugging (5.39) and (5.41) into the conservation laws (5.25),

$$\begin{aligned} 0 &= \iint \big(2c_0\Phi(x) + C + c_0|v|^2\big)\mu_E \\ &= \iint \big(2c_0\Phi(x) + C + c_0|v|^2\big)\left(\frac{|v|^2}{2} + \Phi(x)\right)\mu_E. \end{aligned}$$

From the direct computations, we deduce $c_0 = 0 = C$ and hence (1.42).

Case of $\varpi \neq 0$. From (1.40), at the boundary,

$$b(t,x) \cdot n(x) = (\varpi(t) \times x + m(t)) \cdot n(x) = 0.$$

Since $m(t)$ is a fixed vector for given t, we decompose $m(t)$ into the parallel and orthogonal components to $\varpi(t)$ as

$$m(t) = \alpha(t)\varpi(t) - \varpi(t) \times x_0(t).$$

Then

$$\begin{aligned} &b(t,x) \cdot n(x) \\ &\quad = (\varpi(t) \times x + m(t)) \cdot n(x) \\ &\quad = \big(\varpi(t) \times (x - x_0(t))\big) \cdot n(x) + \alpha(t)\varpi(t) \cdot n(x) = 0 \quad \forall x \in \partial\Omega. \end{aligned} \tag{5.42}$$

Choose t with $\varpi(t) \neq 0$. We can pick $x' \in \partial\Omega$ such that $\varpi(t) \parallel n(x')$. Then the first term of the right-hand side in (5.42) is 0. Hence we deduce, from (5.38) and (5.39), that

$$\alpha(t) = 0 \quad \text{and} \quad b(t,x) = \varpi(t) \times (x - x^0(t)). \tag{5.43}$$

This yields

$$\big(\varpi(t) \times (x - x_0(t))\big) \cdot n(x) = 0 \quad \forall x \in \partial\Omega. \tag{5.44}$$

The equality (5.44) implies that Ω is axis symmetric with the origin $x_0(t)$ and the axis $\varpi(t)$. From (5.26) and (5.43),

$$0 = \iint_\Omega |\varpi \times (x - x_0(t)) \cdot v|^2 \mu e^{-\Phi} \, \mathrm{d}x \, \mathrm{d}v.$$

Therefore, we conclude that $\varpi(t) \equiv 0$ for all t. This proves $b(t,x) \equiv 0$. Then we follow the argument for the case of $\varpi = 0$ and deduce (1.42).

Step 9. Finally, we deduce a contradiction from (5.33) and (1.42). Hence we prove the theorem. □

Once such a coercivity is proven, we can directly deduce an exponential decay.

COROLLARY 5.2. *Assume the same conditions in Proposition* 1.4. *Then there exists* $\lambda > 0$ *such that a solution of* (1.38) *satisfies*

$$\sup_{0 \le t} e^{\lambda t} \| f(t) \|_2^2 \lesssim \| f_0 \|_2^2. \tag{5.45}$$

PROOF OF COROLLARY 5.2. Assume that $0 \le t \le 1$. From the energy estimate of (1.38) in a time interval $[0, N]$,

$$\| f(N) \|_2^2 + \int_0^N \iint_{\Omega \times \mathbb{R}^3} e^{-\Phi} f L f \le \| f(0) \|_2^2. \tag{5.46}$$

From (1.38), for any $\lambda > 0$

$$[\partial_t + v \cdot \nabla_x - \nabla_x \Phi \cdot \nabla_v](e^{\lambda t} f) + e^{-\Phi} L(e^{\lambda t} f) = \lambda e^{\lambda t} f. \tag{5.47}$$

By the energy estimate,

$$\begin{aligned} &\| e^{\lambda t} f(N) \|_2^2 + \underbrace{\int_0^N \iint_{\Omega \times \mathbb{R}^3} e^{-\Phi} e^{2\lambda s} f L f}_{\text{(I)}} \\ &\quad - \lambda \int_0^N \iint_{\Omega \times \mathbb{R}^3} |e^{\lambda s} f(s)|^2 \le \| f(0) \|_2^2. \end{aligned} \tag{5.48}$$

First, we consider (I) in (5.48). From (1.36), the term (I) in (5.48) is bounded below by

$$\begin{aligned}(\mathrm{I}) &\geq \delta_L \int_0^N \int_\Omega e^{-\Phi} \int_{\mathbb{R}^3} \langle v\rangle |e^{\lambda s}(\mathbf{I}-\mathbf{P})f|^2 \\ &\geq \delta_L e^{-\|\Phi\|_\infty} \int_0^N \|e^{\lambda s}(\mathbf{I}-\mathbf{P})f\|_\nu^2.\end{aligned}$$

By time translation, we apply (1.39) to obtain

$$\begin{aligned}&\geq \frac{\delta_L e^{-\|\Phi\|_\infty}}{2} \int_0^N \|e^{\lambda s}(\mathbf{I}-\mathbf{P})f\|_\nu^2 + \frac{\delta_L e^{-\|\Phi\|_\infty}}{2C} \int_0^N \|e^{\lambda s}\mathbf{P}f\|_2^2 \\ &\geq \frac{\delta_L e^{-\|\Phi\|_\infty}}{2C} \int_0^N \|e^{\lambda s} f\|_2^2.\end{aligned}$$

Therefore, we derive

$$e^{\lambda N}\|f(N)\|_2^2 + \left(\frac{\delta_L e^{-\|\Phi\|_\infty}}{2C} - \lambda\right) \int_0^N \|e^{\lambda s} f\|_2^2 \leq \|f(0)\|_2^2. \tag{5.49}$$

On the other hand, from the energy estimate of (1.38) in a time interval $[N, t]$, using (1.36), we have

$$\|f(t)\|_2^2 \leq \|f(N)\|_2^2. \tag{5.50}$$

Finally, choosing $\lambda \ll 1$, from (5.49) and (5.50), we conclude that

$$e^{\lambda t}\|f(t)\|_2^2 = e^{\lambda(t-N)} e^{\lambda N}\|f(N)\|_2^2 \leq 2\|f(0)\|_2^2 \tag{5.51}$$

and prove (5.45). □

PROOF OF THEOREM 1.2. We sketch the proof of the nonlinear L^∞-decay. Note that we have shown a local existence result in (4.1) and the global stability theorem, Theorem 1.1, so we perform an exponential decaying a priori estimate for a nonlinear problem to finish proof.

Note that for small $\|\Phi\|_{C^1} = \delta_\Phi \ll 1$, we have

$$e^{-\Phi}\nu(v) + \frac{1}{w}\nabla_y \Phi \cdot \nabla_v w \geq \frac{1}{2} e^{-\delta_\Phi}\nu(v).$$

This inequality implies

$$\begin{aligned} e^{-\int_s^t e^{-\Phi(X)}\nu(V)\mathrm{d}\tau - \int_s^t \frac{1}{w}\nabla_y\Phi\cdot\nabla_v w} &\leq e^{-\frac{1}{2}e^{-\delta_\Phi}\nu(v)(t-s)} \\ &:= e^{-\frac{1}{2}\nu_\Phi(v)(t-s)}, \end{aligned} \tag{5.52}$$

where we defined $\nu_\Phi(v) := e^{-\delta_\Phi}\nu(v)$. Then, similar to the proof of Theorem 1.1,

$$
\begin{aligned}
&h(t,x,v)\\
&\quad= E(v,t,T)h(T) + \int_T^t E(v,t,s)e^{-\Phi}\int_u k_w(u,v)E(u,s,T)h(T)\mathrm{d}u\,\mathrm{d}s\\
&\qquad+\int_T^t E(v,t,s)e^{-\Phi(X(s))}\int_u k_w(u,v)\int_T^s E(u,s,s')e^{-\Phi(X(s'))}\\
&\qquad\times\int_{u'} k_w(u',u)E(u',s',T)h(T)\mathrm{d}u'\mathrm{d}s'\mathrm{d}u\,\mathrm{d}s\\
&\qquad+\underline{\int_T^t E(v,t,s)e^{-\Phi(X(s))}\int_u k_w(u,v)\int_T^s E(u,s,s')e^{-\Phi(X(s'))}}\\
&\qquad\times\underline{\int_{u'} k_w(u',u)\int_T^{s'} E(u',s',s'')e^{-\Phi(X(s''))}}\\
&\qquad\times\underline{\int_{u''} k_w(u'',u')h(s'',X(s'';s',X(s';s,X(s;t,x,v),u),u'),u'')\mathrm{d}u''\mathrm{d}s''\mathrm{d}u'\mathrm{d}s'\mathrm{d}u\,\mathrm{d}s}_{\text{(IV)}},
\end{aligned}
$$

where we defined,

$$
E(v,t,s) := e^{-\int_s^t e^{-\Phi(X(s;t,x,v))}\nu(V(s;t,x,v))\mathrm{d}\tau-\int_s^t \frac{1}{w}\nabla_x\Phi(X(s;t,x,v))\cdot\nabla_v w(V(s;t,x,v))}.
$$

Except for (IV), the rest of the terms are clearly bounded by

$$
e^{-\frac{1}{2}\nu_\Phi(0)(t-T)}\|h(T)\|_\infty. \tag{5.53}
$$

The estimate for (IV) is obtained by a change of variable similar to (4.29) in the proof of Theorem 1.1. Using definition (4.23) and performing a change of variable, we obtain

$$
\begin{aligned}
\text{(IV)} &\lesssim C_{N,\Omega,\Phi,\beta}\int_T^t\int_{X''}\int_{u''} h(s'',X''(s''),u'')\mathrm{d}u''\,\mathrm{d}X''\,\mathrm{d}s\\
&\quad+ C_{N,\Omega,\Phi}\delta^\gamma\sup_{s\in[T,t]}\|h(s)\|_\infty\\
&\lesssim C_{N,\Omega,\Phi,\beta}\int_T^t\|f(s)\|_{L^2_{x,v}}\,\mathrm{d}s + C_{N,\Omega,\Phi}\delta^\gamma\sup_{s\in[T,t]}\|h(s)\|_\infty.
\end{aligned} \tag{5.54}
$$

Hence

$$
\sup_{s\in[T,t]}\|h(s)\|_\infty \lesssim_{N,\Omega,\Phi,\beta} e^{-\frac{1}{2}\nu_\Phi(0)(t-T)}\|h(T)\|_\infty + \int_T^t\|f(s)\|_2\,\mathrm{d}s. \tag{5.55}
$$

We assume that $m \le t < m+1$ and define $\lambda^* := \min\{\frac{\nu_\Phi(0)}{2},\lambda\}$, where λ is some constant from Corollary 5.2. We use (5.55) repeatedly for each time step,

$[k, k+1)$, $k \in \mathbb{N}$, and Corollary 5.2 to perform an L^2-L^∞ bootstrap, i.e.,

$$
\begin{aligned}
\|h(t)\|_\infty \lesssim_{N,\Omega,\Phi,\beta}\ & e^{-m\frac{\nu_\Phi(0)}{2}}\|h(0)\|_\infty \\
&+ \sum_{k=0}^{m-1} e^{-k\frac{\nu_\Phi(0)}{2}} \int_{m-1-k}^{m-k} \|f(s)\|\mathrm{d}s \\
\lesssim_{N,\Omega,\Phi,\beta}\ & e^{-m\frac{\nu_\Phi(0)}{2}}\|h(0)\|_\infty \\
&+ \sum_{k=0}^{m-1} e^{-k\frac{\nu_\Phi(0)}{2}} \int_{m-1-k}^{m-k} e^{-\lambda(m-1-k)}\|f(0)\|\mathrm{d}s \\
\le\ & C_{N,\Omega,\Phi,\beta} e^{-\frac{\lambda^*}{2}t}\|h(0)\|_\infty.
\end{aligned} \tag{5.56}
$$

For a nonlinear problem, from Duhamel principle,

$$
\begin{aligned}
h :=\ & U(t)h_0 + \int_0^t U(t-s) w e^{-\frac{\Phi}{2}} \Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s, \\
\|h(t)\|_\infty \lesssim_{N,\Omega,\Phi,\beta}\ & e^{-\frac{\lambda^*}{2}t}\|h(0)\|_\infty \\
&+ \left\| \int_0^t U(t-s) w e^{-\frac{\Phi}{2}} \Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s \right\|_\infty,
\end{aligned} \tag{5.57}
$$

where $U(t)$ is a linear solver for a linearized Boltzmann equation. Inspired by [12], we use Duhamel's principle again, i.e.,

$$
U(t-s) = G(t-s) + \int_s^t G(t-s_1) K_w U(s_1 - s)\mathrm{d}s_1, \tag{5.58}
$$

where $G(t)$ is linear solver for the system

$$
\begin{gathered}
\partial_t h + v\cdot\nabla_x h - \nabla_x\Phi\cdot\nabla_v h + \frac{h}{w}\nabla\Phi\cdot\nabla_v w + e^{-\Phi} v h = 0 \\
\text{and} \quad |G(t)h_0| \le e^{-\frac{1}{2}\nu_\Phi(v)t}|h_0|.
\end{gathered} \tag{5.59}
$$

For the last term in (5.57),

$$
\begin{aligned}
&\left\| \int_0^t U(t-s) w e^{-\frac{\Phi}{2}} \Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s \right\|_\infty \\
&\le \left\| \int_0^t G(t-s) w \Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s \right\|_\infty \\
&\quad + \left\| \int_0^t \int_s^t G(t-s_1) K_w U(s_1-s) w \Gamma\left(\frac{h}{w}, \frac{h}{w}\right)(s)\mathrm{d}s_1\,\mathrm{d}s \right\|_\infty \\
&\le C_\Phi e^{-\frac{\lambda^*}{2}t}\Big(\sup_{0\le s\le\infty} e^{\frac{\lambda^*}{2}s}\|h(s)\|_\infty\Big)^2.
\end{aligned} \tag{5.60}
$$

Therefore, for sufficiently small $\|h_0\|_\infty \ll 1$, we have a uniform bound

$$\sup_{0\le t\le\infty} e^{\frac{\lambda^*}{2}t}\|h(t)\|_\infty \ll 1. \tag{5.61}$$

From this small uniform bound, we get global decay and uniqueness. Positivity was already proved in Theorem 4.1. □

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