Quadratic Hedging of Commodity and Energy Cash Flows

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Commodity and energy prices are notoriously volatile. Firms routinely trade financial contracts to hedge their cash flows that are exposed to this source of risk. When markets are incomplete, which is typical in practice, eliminating such risk is impossible and attention must thus shift to its partial mitigation. This paper reviews quadratic hedging, which is an appealing financial risk management approach for this setting, considering a single commodity or energy cash flow that occurs on a given future date and assuming that financial hedging is based on trading a risk less bond and a futures contract. This work formulates this hedging problem as a Markov decision process, derives the optimal policy using stochastic dynamic programming, and characterizes the initial optimal bond position. Further, it highlights related current and potential future research.

1. Introduction

Commodities and energy sources exhibit notoriously high price volatility (Geman 2005, Kaminski 2013, Roncoroni et al. 2015). Firms that handle them commonly trade financial instruments, especially futures, to mitigate this source of risk (see Tirole 2006, §5.4 for a theoretical analysis of the benefit of financial hedging and Pirrong 2015 for a discussion of such practices at Trafigura). In an incomplete market setting, which is the typical situation in practice (see, e.g., Swindle 2016 for a discussion in the context of energy commodities), removal of commodity price risk is unattainable and limited hedging of such risk is thus a necessity.

Quadratic hedging (Schweizer 1995, Bertsimas et al. 2001) is an attractive approach for dealing with market incompleteness when devising financial risk management policies. It is based on the idea of forming a self financing approximate replicating portfolio that is dynamically adjusted to minimize the expected quadratic hedging error. This paper reviews this methodology considering a single commodity or energy cash flow that arises on a given date in the future and assuming that financial hedging relies on trading a risk less bond and a futures contract. It formulates this hedging problem as a Markov decision process (MDP); uses stochastic dynamic programming to establish the optimal policy structure, which provides a basis for its computation in applications; and characterizes the optimal initial bond position as a discounted expectation of the future cash flow taken under a distinct martingale measure, which gives a proxy for the market value of this cash flow.

This work slightly broadens the set up of Schweizer (1995), Bertsimas et al. (2001) based on the formulation of Gugushvili (2003) and, given its focus, considering futures, rather than stock, trading, linking the findings of Bertsimas et al. (2001) to the ones of Schweizer (1995). Černý (2004) makes a similar connection when the hedging portfolio includes multiple financial contracts but no futures. Canyakmaz et al. (2017) discuss related minimum variance hedging ideas (see, e.g., Luenberger 2014, §12.10) in the context of inventory models with price risk without imposing the self financing condition on the hedging portfolio.

Section 2 introduces the hedging problem and its MDP formulation. Section 3 presents a stochastic dynamic programming approach to solve this MDP. Section 4 characterizes the optimal initial bond position. Section 5 provides a summary and mentions related ongoing and potential future research.

2. Model

This section formulates the hedging problem as an MDP based on Schweizer (1995), Bertsimas et al. (2001).

Consider I dates T_0 through T_{I-1}. The current date is T_0. Define $\mathcal{I}$ as the set $\{0, 1, \ldots, I-1\}$. On date T_i, with $i \in \mathcal{I}$, let P_i be the price of a commodity or energy futures with maturity on date T_{I-1} and $\boldsymbol{Z}_i$ be a vector of factors that have dynamics that are imperfectly correlated with the ones of this price. The sets $\mathcal{P}_i$ and $\mathcal{Z}_i$ include the values that these quantities can respectively assume. The quantity $C_{I-1}(P_{I-1}, \boldsymbol{Z}_{I-1})$ represents a given date T_{I-1} cash flow. For example, it may represent the margin from producing and selling on the wholesale market an amount of commodity or energy on this date.

The goal is to attempt to replicate the date T_{I-1} cash flow by trading a risk less bond and the given futures from dates T_0 through T_{I-2}. Denote by B_i the dollar value of the bond position on date T_i for $i \in \mathcal{I}$ (B_0 is given). Let θ_i be the number of futures held on date T_i, with $i \in \mathcal{I} \setminus \{I-1\}$. The market value of a futures position is zero when it is set up. Thus, on date T_i the market value of the portfolio (B_i, θ_i) is

$$V_i = B_i. \tag{1}$$

Let D be the per period risk free discount factor, which is assumed both deterministic and constant for simplicity. The futures position θ_i gives rise to the cash flow $(P_{i+1} - P_i)\,\theta_i$ on date T_{i+1}. Attention is restricted to self financing trading portfolios, for which the change in the market value of the bond position from date T_i to date T_{i+1}, expressed in terms of date T_{i+1} dollars, only derives from the futures position cash flow on date T_{i+1}:

$$B_{i+1} - \frac{B_i}{D} = (P_{i+1} - P_i)\,\theta_i. \tag{2}$$

It follows from (1)-(2) that the dynamics of the market value of the portfolio satisfy

$$V_{i+1} = \frac{V_i}{D} + (P_{i+1} - P_i)\,\theta_i. \tag{3}$$

Let ψ be a self financing trading policy and Ψ be the set of all such policies. Denote by V_i^{ψ} the date T_i value of the trading portfolio for policy ψ. Given an initial bond position with market value V_0, the goal is to find a policy $\psi \in \Psi$ that minimizes the expected replication error on date T_{I-1}:

$$\min_{\psi\in\Psi} \mathbb{E}\left[\left(V_{I-1}^{\psi} - C_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right)\right)^2 \mid V_0, P_0, \boldsymbol{Z}_0\right]. \tag{4}$$

Because the market is incomplete, it is impossible to exactly replicate the date T_{I-1} cash flow.

3. Methodology

This section shows how stochastic dynamic programming can be used to characterize the solution of model (4). It is based on Bertsimas et al. (2001), Gugushvili (2003).

The set of stages is $\mathcal{I}$. The state in each stage i is the triple $(V_i, P_i, \boldsymbol{Z}_i) \in \mathbb{R} \times \mathcal{P}_i \times \mathcal{Z}_i$. The terminal conditions that define the value function in stage $I-1$ for each state $(V_{I-1}, P_{I-1}, \boldsymbol{Z}_{I-1})$ are

$$J_{I-1}\left(V_{I-1}, P_{I-1}, \boldsymbol{Z}_{I-1}\right) := \left[V_{I-1} - C_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right)\right]^2. \tag{5}$$

The Bellman equations for each other stage and state are

$$J_i\left(V_i, P_i, \boldsymbol{Z}_i\right) = \min_{\theta_i\in\mathbb{R}} \mathbb{E}\left[J_{i+1}\left(\frac{V_i}{D} + (P_{i+1} - P_i)\,\theta_i, P_{i+1}, \boldsymbol{Z}_{i+1}\right) \mid P_i, \boldsymbol{Z}_i\right], \tag{6}$$

where V_i is not included as a conditioning quantity for notational simplicity. Define $a_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right)$, $b_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right)$, and $c_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right)$ as one, $C_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right)$, and zero, respectively. The terminal conditions (5) can thus be expressed as

$$J_{I-1}\left(V_{I-1}, P_{I-1}, \boldsymbol{Z}_{I-1}\right) = a_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right)\left[V_{I-1} - b_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right)\right]^2 + c_{I-1}\left(P_{I-1}, \boldsymbol{Z}_{I-1}\right).$$

Consider stage $i \in \mathcal{I} \setminus \{I-1\}$. Make the induction hypothesis that for each stage j from $i+1$ through $I-2$ and corresponding state $(V_j, P_j, \boldsymbol{Z}_j)$ the value function $J_j\left(V_j, P_j, \boldsymbol{Z}_j\right)$ can be written as

$$J_j\left(V_j, P_j, \boldsymbol{Z}_j\right) = a_j\left(P_j, \boldsymbol{Z}_j\right)\left[V_j - b_j\left(P_j, \boldsymbol{Z}_j\right)\right]^2 + c_j\left(P_j, \boldsymbol{Z}_j\right), \tag{7}$$

for some quantities $a_j(P_j, \boldsymbol{Z}_j) \geq 0$, $b_j(P_j, \boldsymbol{Z}_j)$, and $c_j(P_j, \boldsymbol{Z}_j)$. Denote by $f_i(\theta_i)$ the objective function of the optimization on the right hand side of (6). Using (7) expressed with $j = i+1$ in (6), rearranging, dropping both the suffix of each stage $i+1$ quantity and the conditioning on stage i information when taking expectations, and replacing $\mathbb{E}$ with $\mathbb{E}_i$ for clarity yields

$$\begin{aligned} f_i(\theta_i) &= \mathbb{E}_i\left[(P_{i+1}-P_i)^2 a_{i+1}\right]\theta_i^2 + 2\mathbb{E}_i\left[(P_{i+1}-P_i)\left(\frac{V_i}{D}-b_{i+1}\right)a_{i+1}\right]\theta_i \\ &\quad +\mathbb{E}_i\left[\left(\frac{V_i}{D}-b_{i+1}\right)^2 a_{i+1}\right] + \mathbb{E}_i\left[c_{i+1}\right]. \end{aligned} \tag{8}$$

The optimization on the right hand side of (6) can thus be written as

$$\min_{\theta_i\in\mathbb{R}} f_i(\theta_i). \tag{9}$$

If the coefficient of the quadratic term in (8) is zero then so is the one of the linear term in this expression. That is, the objective function in (9) does not depend on θ_i. In this degenerate case define the optimal value of θ_i in (9) to be $\theta_i^* := 0$. Otherwise the induction hypothesis implies that the term $\mathbb{E}_i\left[(P_{i+1}-P_i)^2 a_{i+1}\right]$ in (8) is strictly positive, so that the minimizer in (9) is

$$\theta_i^* = \frac{\mathbb{E}_i\left[(P_{i+1}-P_i)(b_{i+1}-V_i/D)a_{i+1}\right]}{\mathbb{E}_i\left[(P_{i+1}-P_i)^2 a_{i+1}\right]}. \tag{10}$$

Replacing θ_i with θ_i^* in (8) and using (10) leads to

$$f_i(\theta_i^*) = -\frac{\left(\mathbb{E}_i\left[(P_{i+1}-P_i)(b_{i+1}-V_i/D)a_{i+1}\right]\right)^2}{\mathbb{E}_i\left[(P_{i+1}-P_i)^2 a_{i+1}\right]} + \mathbb{E}_i\left[\left(\frac{V_i}{D}-b_{i+1}\right)^2 a_{i+1}\right] + \mathbb{E}_i\left[c_{i+1}\right]. \tag{11}$$

This expression applies also to the degenerate case by defining $0/0$ as zero, a convention used throughout. Define

$$p_i := \frac{\mathbb{E}_i\left[(P_{i+1}-P_i)a_{i+1}b_{i+1}\right]}{\mathbb{E}_i\left[(P_{i+1}-P_i)^2 a_{i+1}\right]}, \tag{12}$$

$$q_i := \frac{\mathbb{E}_i\left[(P_{i+1}-P_i)a_{i+1}\right]}{\mathbb{E}_i\left[(P_{i+1}-P_i)^2 a_{i+1}\right]}, \tag{13}$$

where the suffix $(P_i, \boldsymbol{Z}_i)$ is omitted from the two quantities being defined. Using (12)-(13) and adopting a similar simplification, further define

$$a_i := \frac{1}{D^2}\mathbb{E}_i\left[\left(1-(P_{i+1}-P_i)q_i\right)^2 a_{i+1}\right], \tag{14}$$

$$b_i := \frac{1}{a_i D}\mathbb{E}_i\left[\left(b_{i+1}-(P_{i+1}-P_i)p_i\right)\left(1-(P_{i+1}-P_i)q_i\right)a_{i+1}\right], \tag{15}$$

$$c_i := \mathbb{E}_i\left[c_{i+1}\right] + \mathbb{E}_i\left[\left(b_{i+1}-(P_{i+1}-P_i)p_i\right)^2 a_{i+1}\right] - a_i b_i^2.$$

Some algebra shows that $J_i(V_i, P_i, \boldsymbol{Z}_i)$, which corresponds to the right hand side of (11), can be written as

$$a_i(P_i, \boldsymbol{Z}_i)[V_i - b_i(P_i, \boldsymbol{Z}_i)]^2 + c_i(P_i, \boldsymbol{Z}_i),$$

where the suffix $(P_i, \boldsymbol{Z}_i)$ for $a_i(P_i, \boldsymbol{Z}_i)$, $b_i(P_i, \boldsymbol{Z}_i)$, and $c_i(P_i, \boldsymbol{Z}_i)$ is reinstated. The induction hypothesis and (14) imply that $a_i(P_i, \boldsymbol{Z}_i)$ is (weakly) positive. The properties inductively assumed for each state in stages $i+1$ through $I-2$ thus also hold for each state in stage i. The principle of mathematical induction implies that in every stage and state the value function can be expressed as

$$J_i(V_i, P_i, \boldsymbol{Z}_i) = a_i(P_i, \boldsymbol{Z}_i)[V_i - b_i(P_i, \boldsymbol{Z}_i)]^2 + c_i(P_i, \boldsymbol{Z}_i),$$

with $a_i(P_i, \boldsymbol{Z}_i) \geq 0$. Using (12)-(13) and appending the suffix $(P_i, \boldsymbol{Z}_i)$ to both p_i and q_i, the right hand side of (10) can be written as

$$p_i(P_i, \boldsymbol{Z}_i) - \frac{q_i(P_i, \boldsymbol{Z}_i)}{D} V_i, \tag{16}$$

which is the optimal decision rule for stage i. It is linear in the value of the portfolio, V_i. The definitions (12)-(13) and the recursive expressions (14)-(15) form the basis for the computation of (16) in applications.

4. Discussion

This section characterizes the optimal initial bond position, linking the material in Bertsimas et al. (2001) to the one in Schweizer (1995).

Model (4) takes the initial bond position V_0 as given. Denote as $\mathbb{V}_0(P_0, \boldsymbol{Z}_0)$ the value of this position that leads to the smallest replication error. This value is known as the minimal production cost for the cash flow $C_{I-1}(P_{I-1}, \boldsymbol{Z}_{I-1})$. That is, it belongs to

$$\arg\min_{V_0 \in \mathbb{R}} \min_{\psi \in \Psi} \mathbb{E}\left[\left(V_{I-1}^{\psi} - C_{I-1}(P_{I-1}, \boldsymbol{Z}_{I-1})\right)^2 \mid V_0, P_0, \boldsymbol{Z}_0\right].$$

The quantity $\mathbb{V}_0(P_0, \boldsymbol{Z}_0)$ is $b_0(P_0, \boldsymbol{Z}_0)$ and the minimal approximate replication error is $c_0(P_0, \boldsymbol{Z}_0)$.

To characterize $\mathbb{V}_0(P_0, \boldsymbol{Z}_0)$, consider expressions (14)-(15), which, using the same notational simplifications adopted in §3, after some algebra can be equivalently written as

$$a_i = \frac{1}{D^2}\mathbb{E}_i[(1 - (P_{i+1} - P_i)q_i)a_{i+1}], \tag{17}$$

$$b_i = \frac{\mathbb{E}_i[(1 - (P_{i+1} - P_i)q_i)a_{i+1}b_{i+1}]}{a_i D}. \tag{18}$$

An induction argument starting from stage $I-2$ establishes that these terms satisfy

$$a_i = \frac{1}{D^{2[I-(i+1)]}} \mathbb{E}_i \left[\prod_{k=i}^{I-2} \left(1 - \left(P_{k+1} - P_k\right) q_k\right) \right], \tag{19}$$

$$b_i = \frac{\mathbb{E}_i \left[C_{I-1} \prod_{k=i}^{I-2} \left(1 - (P_{k+1} - P_k) q_k\right) \right]}{a_i D^{I-(i+1)}}, \tag{20}$$

with $C_{I-1}(P_{I-1}, \boldsymbol{Z}_{I-1})$ written without suffix in (20) to reduce the notational burden. Substituting (19) in (20) and simplifying yields

$$b_i = D^{I-(i+1)} \frac{\mathbb{E}_i \left[C_{I-1} \prod_{k=i}^{I-2} \left(1 - (P_{k+1} - P_k) q_k\right) \right]}{\mathbb{E}_i \left[\prod_{k=i}^{I-2} \left(1 - \left(P_{k+1} - P_k\right) q_k\right) \right]}. \tag{21}$$

The quantity $\prod_{k=i}^{I-2} \left(1 - (P_{k+1} - P_k) q_k\right) / \mathbb{E}_i \left[\prod_{k=i}^{I-2} \left(1 - \left(P_{k+1} - P_k\right) q_k\right) \right]$ is a signed measure known as the variance optimal measure. In particular, it is a martingale measure. That is, denoting by $\tilde{\mathbb{E}}_i$ conditional expectation under this measure, it holds that $\tilde{\mathbb{E}}_i[P_j] = P_i$ for j equal to $i+1$ through $I-1$. Expression (21) evaluated for $i=0$ and the equality $\mathbb{V}_0(P_0, \boldsymbol{Z}_0) = b_0(P_0, \boldsymbol{Z}_0)$ imply

$$\mathbb{V}_0(P_0, \boldsymbol{Z}_0) = D^{I-1} \tilde{\mathbb{E}} \left[C_{I-1}(P_{I-1}, \boldsymbol{Z}_{I-1}) \mid P_0, \boldsymbol{Z}_0 \right]. \tag{22}$$

That is, the minimal production cost of the cash flow $C_{I-1}(P_{I-1}, \boldsymbol{Z}_{I-1})$ is its expected value under the variance optimal measure deflated by applying the risk free discount factor. The condition (22) is consistent with risk neutral valuation in complete markets, in which case the market value of a risky cash flow is its expected value under the so called risk neutral measure, which is also a martingale measure, discounted using the risk free rate (see, e.g., Smith and McCardle 1999; Luenberger 2014, §14.5; Secomandi and Seppi 2014; and references therein). The quantity $\mathbb{V}_0(P_0, \boldsymbol{Z}_0)$ can thus be interpreted as a proxy for the market value on date T_0 of the cash flow $C_{I-1}(P_{I-1}, \boldsymbol{Z}_{I-1})$.

5. Conclusions

This work reviews quadratic hedging of commodity and energy cash flows in incomplete markets. Focusing on a single future cash flow to be hedged by trading a risk less bond and a futures contract, it formulates the hedging problem as an MDP and discusses both the structure of the optimal hedging policy and the characterization of the optimal initial bond position. Secomandi (2018) considers the case of linked cash flows that occur on multiple dates, in which case optimization of the operating policy that generates them is relevant. Future research could deal with applications in realistic settings, e.g., in the context of commodity and energy merchant operations (Secomandi and Seppi 2014, 2016, Secomandi 2017).

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