Math. Ann. (2016) 364:1025–1041
DOI 10.1007/s00208-015-1235-7

Mathematische Annalen

Homological stability for completed homology

Frank Calegari[1] · **Matthew Emerton**[2]

Received: 25 November 2014 / Revised: 19 May 2015 / Published online: 16 June 2015

Abstract We prove that the completed homology groups of $\mathrm{GL}_n(\mathbf{Z})$ in fixed degree stabilize as N goes to infinity. We also prove that the action of Hecke operators on stable cohomology is trivial, in a precisely defined sense.

Mathematics Subject Classification 11F75 · 11F80

Contents

F. Calegari was supported in part by NSF Grants DMS-0846285 and DMS-1404620.
M. Emerton was supported in part by NSF Grants DMS-1002339, DMS-1249548, and DMS-1303450.

✉ Frank Calegari
fcale@math.northwestern.edu

Matthew Emerton
emerton@math.uchicago.edu

1 Northwestern University, Evanston, IL, USA

2 University of Chicago, Chicago, IL, USA

1 Introduction

Let $\mathrm{GL}_n(\mathbf{Z}, M)$ denote the principal congruence subgroup of $\mathrm{GL}_n(\mathbf{Z})$ of level M, explicitly, the group of invertible matrices with integer coefficients which are congruent to the identity modulo M. A theorem of Charney [13] and (independently) Maazen [20] shows that, for fixed d, the homology groups

$$H_d(\mathrm{GL}_n(\mathbf{Z}, 1), \mathbf{Z}) = H_d(\mathrm{GL}_n(\mathbf{Z}), \mathbf{Z})$$

are *stable*, that is, they are independent of n for n sufficiently large. Moreover, the natural transition maps induced from the inclusions $\mathrm{GL}_n \subset \mathrm{GL}_{n+1}$ in this range are isomorphisms. In [14], Charney extends this result to prove a stability theorem for congruence subgroups. In particular, she proves that the homology groups

$$H_d\left(\mathrm{GL}_n(\mathbf{Z}, M), \mathbf{Z}\left[\frac{1}{M}\right]\right)$$

are also stable as $n \to \infty$. Note that the analogous result with coefficients in $\mathbf{R}$ is (in both cases) a consequence of a theorem of Borel [6]. The result is no longer true if one does not invert primes dividing the level. For example, if $p > 2$ is prime, it is an elementary consequence of the congruence subgroup property [5,21] that the homomorphism

$$H_1(\mathrm{GL}_n(\mathbf{Z}, p), \mathbf{Z}) \to \mathrm{GL}_n(\mathbf{Z}, p)/\mathrm{GL}_n(\mathbf{Z}, p^2) \simeq (\mathbf{Z}/p\mathbf{Z})^{n^2-1}$$

is an isomorphism for $n \geq 3$, and the latter group is certainly not stable. The starting point for the Church–Farb theory of representation stability [12] is the observation that, as a representation of $\mathrm{SL}_n(\mathbf{F}_p)$, the right hand side has a description which is independent of n (it is the adjoint representation). Alternatively, one might view this calculation as saying that all the cohomology of $\mathrm{GL}_n(\mathbf{Z}, p)$ in degree 1 (for $n \geq 3$) arises via pullback from the homology of congruence subgroups of the *local* p-adic lie group $\mathrm{GL}_n(\mathbf{Z}_p)$. This latter homology is evidently insensitive to the finer arithmetic properties of $\mathrm{GL}_n(\mathbf{Z})$, and so it is natural to try to excise the cohomology arising for "local" reasons and consider what remains. This is exactly what is achieved by replacing the homology groups by completed homology groups, as first defined and studied in [8,9,16].

Fix a tame level M, and let $\Gamma_n(p^k) = \mathrm{GL}_n(\mathbf{Z}, Mp^k)$ denote the principal congruence subgroup of level Mp^k. Recall that the completed homology and cohomology groups are defined as follows [9]:

$$\widetilde{H}_{*,n}(\mathbf{F}_p) := \varprojlim H_*(\Gamma_n(p^r), \mathbf{F}_p) \qquad \widetilde{H}^*_n(\mathbf{F}_p) := \varinjlim H^*(\Gamma_n(p^r), \mathbf{F}_p),$$
$$\widetilde{H}_{*,n}(\mathbf{Z}_p) := \varprojlim H_*(\Gamma_n(p^r), \mathbf{Z}_p) \qquad \widetilde{H}^*_n(\mathbf{Q}_p/\mathbf{Z}_p) := \varinjlim H^*(\Gamma_n(p^r), \mathbf{Q}_p/\mathbf{Z}_p).$$

Our main result is as follows:

Theorem 1.1 *The modules $\widetilde{H}_d(\mathbf{Z}_p)$ and $\widetilde{H}_d(\mathbf{F}_p)$ stabilize as $n \to \infty$ and are finite $\mathbf{Z}_p$-modules and $\mathbf{F}_p$-vector spaces respectively. Moreover, the action of $\mathrm{SL}_n(\mathbf{Q}_p)$ on either module is trivial.*

The question of computing these groups and understanding their relationship to arithmetic and K-theory will be taken up in the companion paper [7], although see Sect. 5.1 for some immediate consequences of stability of completed homology for classical homology groups. (We confine ourselves here to the non-obvious remark that all the classical homology groups at level $\Gamma_n(p^r)$ can be recovered from the completed homology groups.) It is possible to extract from our argument explicit bounds for n in terms of d which imply that $\widetilde{H}_d(\mathbf{Z}_p)$ and $\widetilde{H}_d(\mathbf{F}_p)$ are stable (see Remark 5.14); however, these bounds are presumably not optimal. It may even be the case that $n - 1 > d$ would suffice.

Our second result concerns the action of Hecke operators on cohomology classes in low degree. Let $\Gamma = \Gamma_n(M)$ for some level M. If p is prime and $\mathbf{F}$ is a field of characteristic p, then each of the cohomology groups $H^d(\Gamma, \mathbf{F})$ admits a natural action of a commutative ring $\mathbf{T}$ of Hecke operators $T_{\ell,k}$ for $1 \le k \le n$ and for primes ℓ not dividing Mp. Let $[c] \in H^d(\Gamma, \mathbf{F})$ denote an eigenclass for the action of $\mathbf{T}$ with eigenvalues $a(\ell, k) \in \mathbf{F}$. A theorem of Scholze [23] (Conjecture B of [2]) says that, associated to $[c]$, there exists a continuous semisimple Galois representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{F})$ unramified outside Mp such that

$$\sum (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k = \det(I - \rho(\mathrm{Frob}_\ell)^{-1} X). \qquad (\star)$$

We determine exactly which Galois representations arise in low degree relative to n. Let $\omega : G_{\mathbf{Q}} \to \mathbf{F}^\times$ denote the mod-p cyclotomic character.

Theorem 1.2 *Fix an integer d, and suppose that n is sufficiently large compared to d. Let $\Gamma = \Gamma_n(M)$ for some level M prime to p. Then for any eigenclass $[c] \in H^d(\Gamma, \mathbf{F})$, there exists a character χ of conductor M and a Galois representation:*

$$\rho = \chi \otimes \left(1 \oplus \omega \oplus \omega^2 \oplus \cdots \oplus \omega^{n-1}\right),$$

such that $\rho(\mathrm{Frob}_\ell)$ satisfies $\star$.

Remark 1.3 It follows from the argument that $n \ge 2d + 6$ will suffice.

Remark 1.4 Theorem 1.2 remains true even when $p | M$; see Remark 4.7.

This theorem should be interpreted as saying that the action of Hecke on *stable* cohomology is trivial, which is indeed how we shall prove this theorem. Note that if $[c]$ comes from characteristic zero, then the result follows from a theorem of Borel [6], which shows in particular that the only rational cohomology in low degrees arises from the trivial automorphic representation.

Example 1.5 A special case of a construction due to Soulé [25] implies that $K_{22}(\mathbf{Z})$ contains an element of order 691.[1] Associated to this homotopy class is a stable class $[c]$ in $H_{22}(\mathrm{GL}_n(\mathbf{Z}), \mathbf{F}_{691})$ for all sufficiently large n. Our theorem implies that the class $[c]$ is associated to the representation $\rho := 1 \oplus \omega \oplus \cdots \oplus \omega^{n-1}$ via $\star$. On the other hand, the existence of $[c]$ corresponds—implicitly—to the existence of a *non-semisimple* Galois representation ϱ with $\varrho^{\mathrm{ss}} = 1 \oplus \omega^{11}$ such that the extension class in $\mathrm{Ext}^1(\omega^{11}, 1)$ is unramified everywhere. Is there a generalization of Ash's conjectures that predicts the existence of a *non-semisimple* Galois representations associated to Eisenstein Hecke eigenclasses?

Remark 1.6 That Theorem 1.2 (or something similar) might be true was suggested by Akshay Venkatesh in discussions with the first author. (See [3] for a discussion of this conjecture and some partial results.)

2 Arithmetic manifolds

Torsion free arithmetic groups are well known to act freely and properly discontinuously on their associated symmetric spaces. This allows one to translate questions concerning the cohomology of arithmetic groups into questions about cohomology of the associated arithmetic quotients. While one can study such spaces independently of any adelic framework, it is more natural from the perspective of Hecke operators to work in this generality, and this is the approach we adopt in this paper.

2.1 Cohomology of arithmetic quotients

Let K_∞ denote a fixed maximal compact subgroup of $\mathrm{GL}_n(\mathbf{R})$, and let K_∞^0 denote the connected component containing the identity. One has isomorphisms $K_\infty \simeq \mathrm{O}(n)$ and $K_\infty^0 \simeq \mathrm{SO}(n)$. Let $\mathbb{A}$ denote the adeles of $\mathbf{Q}$. For any finite set of places S, let $\mathbb{A}^S$ denote the adeles with the components at the places $v|S$ missing, so (for example) $\mathbb{A}^\infty$ denotes the finite adeles. Fix a compact open subgroup K^ℓ of $\mathrm{GL}_n(\mathbb{A}^{\ell,\infty})$, let K_ℓ denote a compact open subgroup of $\mathrm{GL}_n(\mathbf{Z}_\ell)$, and let $K = K^\ell K_\ell$. Let

$$Y(K) = \mathrm{GL}_n(\mathbf{Q})\backslash\mathrm{GL}_n(\mathbb{A})/K_\infty^0 K^\ell K_\ell$$

denote the corresponding arithmetic quotient.

Assume that $\mathbf{F}$ is a finite field of characteristic $p \neq \ell$. We now let $\Pi_{n,\ell}$ denote the direct limit

$$\Pi_{n,\ell} = \varinjlim_{K_\ell} H^d(Y(K), \mathbf{F}).$$

For K_ℓ sufficiently small, the quotient $Y(K)$ is a manifold consisting of a finite number of connected components, all of which are $K(\pi, 1)$ spaces. In particular, if K_ℓ is the

[1] Indeed, $K_{22}(\mathbf{Z}) \simeq \mathbf{Z}/691\mathbf{Z}$.

level ℓ^k congruence subgroup, and K^ℓ is the open subgroup of tame level M, then writing the associated space as $Y(K, \ell^k)$ there is an isomorphism

$$H^d(Y(K, \ell^k), \mathbf{F}) \simeq \bigoplus_A H^d(\Gamma(\ell^k), \mathbf{F}),$$

where $A := \mathbf{Q}^\times \backslash \mathbb{A}^{\infty,\times} / \det(K)$ is a ray class group of conductor M times a power of ℓ, and Γ is the corresponding classical congruence subgroup of level M.

In the remainder of Sects. 2, 3 and 4, we fix K^ℓ to be the open subgroup of tame level M.

The module $\Pi_{n,\ell}$ is endowed tautologically with an action of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ that is *admissible*; that is, letting $G(\ell^k)$ denote the full congruence subgroup of $\mathrm{GL}_n(\mathbf{Z}_\ell)$ of level ℓ^k, we have that

$$\dim_{\mathbf{F}} \Pi_{n,\ell}^{G(\ell^k)} < \infty$$

for any k. This property records nothing more than the finite dimensionality of the cohomology groups $H^d(Y(K, \ell^k), \mathbf{F})$.

3 Stability for $\ell \neq p$

We use the following two key inputs to prove our result. The first is as follows:

Proposition 3.1 *There is an isomorphism $\Pi_{n,\ell} \to \Pi_{n+1,\ell}$ for all $n \geq 2d + 6$.*

Proof For any fixed ℓ-power congruence subgroup, this follows immediately from the main result of Charney [14] (in particular, §5.4 Example (v), p. 2118). The theorem then follow by taking direct limits. □

Lemma 3.2 *For $n \geq 2d + 6$, the action of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ on $\Pi_{n,\ell}$ is via the determinant.*

Proof By [14], the action of $\mathrm{SL}_n(\mathbf{Z}_\ell)$ on $H^*(\Gamma(\ell^m), \mathbf{F})$ is trivial in the stable range (see also Corollary 7 of [3]), and hence the action of $\mathrm{SL}_n(\mathbf{Z}_\ell)$ on $\Pi_{n,\ell}$ is also trivial. Let V be an irreducible sub-quotient of $\Pi_{n,\ell}$. The only admissible representations of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ that are trivial on $\mathrm{SL}_n(\mathbf{Z}_\ell)$ are given by characters (the normal closure of $\mathrm{SL}_n(\mathbf{Z}_\ell)$ inside $\mathrm{SL}_n(\mathbf{Q}_\ell)$ is the entire group), and it follows that every irreducible constituent of $\Pi_{n,\ell}$ is a character. Since the action of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ on the extension of any two characters still acts through the determinant, the result follows for $\Pi_{n,\ell}$. □

Remark 3.3 One easy way to see that $\mathrm{SL}_n(\mathbf{Z}_\ell)$ must act trivially on $H^*(\Gamma(\ell^m), \mathbf{F})$ in the stable range is that the latter module has fixed dimension for all n, whereas the smallest non-trivial representation of $\mathrm{SL}_n(\mathbf{Z}_\ell)$ over $\mathbf{F}$ has unbounded dimension as n increases.

We now sketch two further proofs of Lemma 3.2 (one incomplete). The first proof "explains" the triviality of the action of $\mathrm{SL}_n(\mathbf{Q}_\ell)$ by showing that $\Pi_{n,\ell}$ is very small in a precise way.

Proposition 3.4 *For sufficiently large n, there is, for all m, an isomorphism*

$$H^d(\Gamma(\ell), \mathbf{F}) \simeq H^d(\Gamma(\ell^m), \mathbf{F}).$$

Proof The main result of [14] identifies the stable cohomology groups $H^*(\Gamma(\ell), \mathbf{F})$ as well as the groups $H^*(\Gamma(\ell^m), \mathbf{F})$ with the cohomology of the homotopy fibre of the map $\mathrm{BSL}(\mathbf{Z})^+ \to \mathrm{BSL}(\mathbf{Z}/M\ell)^+$ and $\mathrm{BSL}(\mathbf{Z})^+ \to \mathrm{BSL}(\mathbf{Z}/M\ell^m)^+$ respectively (under the assumption that $\mathbf{F}$ has characteristic prime to $M\ell$). The natural map between these fibre sequences induces an isomorphism on $\mathbf{F}$-homology, trivially for $\mathrm{BSL}(\mathbf{Z})^+$ and for $\mathrm{BSL}(\mathbf{Z}/M\ell^m)^+ \to \mathrm{BSL}(\mathbf{Z}/M\ell)^+$ by Gabber's rigidity theorem [17]. By the Zeeman comparison theorem, this implies the result for the homotopy fibre. □

More details of this argument can be found in the companion paper [7] (see Remark 1.20). Alternatively, one can prove the result directly in this case, since the transfer map from $\Gamma(\ell^m)$ to $\Gamma(\ell)$ is an isomorphism over any ring such that ℓ is inverted (the group $\Gamma(\ell)/\Gamma(\ell^m)$ is an ℓ-group), and hence it suffices to show that the $\mathrm{SL}_n(\mathbf{Z}_\ell)$ action on $H^d(\Gamma(\ell^m), \mathbf{F})$ is trivial. Lemma 3.2 is an easy consequence of Proposition 3.4. For our next argument, we begin with the following:

Proposition 3.5 *Let $\widehat{V}$ be an irreducible admissible infinite dimensional representation of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ in characteristic zero. Then the Gelfand–Kirillov dimension of $\widehat{V}$ is at least $n-1$.*

Proof The Gelfand–Kirillov dimension of $\widehat{V}$ can be interpreted in two ways. On the one hand, it is that value of $d \geq 0$ for which $\dim V^{G(\ell^k)} \asymp \ell^{dk}$; on the other hand, if we pull back the Harish–Chandra character to a neighbourhood of 0 in $\mathfrak{g}$, the Lie algebra of $\mathrm{GL}_n(\mathbf{Q}_\ell)$, via the exponential map, then a result of Howe and Harish–Chandra shows that the resulting function χ admits an expansion

$$\chi = \sum_{\mathcal{O}} c_{\mathcal{O}} \widehat{\mu}_{\mathcal{O}},$$

where the sum ranges over the nilpotent $\mathrm{GL}_n(\mathbf{Q}_\ell)$-orbits in $\mathfrak{g}^\vee$, and $\widehat{\mu}_{\mathcal{O}}$ denotes the Fourier transform of a suitably normalized $\mathrm{GL}_n(\mathbf{Q}_\ell)$-invariant measure on $\mathcal{O}$ (see e.g. the introduction of [18], whose notation we are following here, for a discussion of these ideas); the Gelfand–Kirillov dimension of $\widehat{V}$ is then also equal $\dfrac{1}{2}\max_{\mathcal{O}\,|\,c_{\mathcal{O}}\neq 0} \dim \mathcal{O}$ (as one sees by pairing the characteristic function of $G(\ell^k)$—pulled back to $\mathfrak{g}$—against χ). Since $\widehat{V}$ has Gelfand–Kirillov dimension 0 if and only if it is finite dimensional, and since the minimal non-zero nilpotent orbit in $\mathfrak{g}$ has dimension $2(n-1)$ (see §1.3 p. 459 and Table 1 on p. 460 of [24]), the proposition follows. □

Let V be an irreducible sub-quotient of $\Pi_{n,\ell}$. Since the cohomology of pro-ℓ groups vanishes in characteristic $p \neq \ell$, we deduce that, for all k,

$$\dim V^{G(\ell^k)} \leq \dim \Pi_{n,\ell}^{G(\ell^k)} = \dim \Pi_{m,\ell}^{G(\ell^k)},$$

where $m = 2d + 6$ is fixed, and where the equality follows from Proposition 3.1. Yet, for a fixed m, there is the trivial inequality relating the growth of cohomology to the growth of the index (up to a constant) and thus

$$\dim \Pi_{m,\ell}^{G(\ell^k)} \ll [\mathrm{GL}_m(\mathbf{Z}_\ell) : G(\ell^k)] \ll \ell^{km^2}.$$

The bound on invariant growth then implies that the Gelfand–Kirillov dimension of $\widehat{V}$ is at most m^2. Suppose that V lifted to a (necessarily irreducible) representation $\widehat{V}$ in characteristic 0. We deduce a corresponding bound for the invariants of V. By Proposition 3.5, this is a contradiction for sufficiently large n unless $\widehat{V}$ and thus V is one-dimensional. In general, it follows from the main theorem of Vignéras ([27], p.182) that any irreducible admissible irreducible representation V of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ over $\mathbf{F}$ lifts to a *virtual* representation $[\widehat{V}]$ in characteristic 0. We *expect* that $[\widehat{V}]$ may be chosen to contain a unique irreducible representation of largest Gelfand–Kirillov dimension. If this is so, then the bound of Proposition 3.5 applies also to mod-p representations of $\mathrm{GL}_n(\mathbf{Q}_\ell)$, and it would follow that every irreducible constituent of $\Pi_{n,\ell}$ is a character. Since the action of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ on the extension of any two characters still acts through the determinant, this would yield a third proof of Proposition 3.2.

Remark 3.6 In a preliminary version of this paper, we made the error of assuming that any irreducible $\mathrm{GL}_n(\mathbf{Q}_\ell)$-representation V in characteristic p lifted to a characteristic zero representation $\widehat{V}$. We thank Jean-François Dat for pointing out to us that this is not always the case, and also for suggesting a possible approach to proving our modified expectation via induction on the ordering of partitions associated to V. Because of the availability of other arguments, however, we have not made a serious attempt to carry this out. Although this alternate argument is therefore incomplete, it is the one that is most amenable to generalization to the case of $\ell = p$ (see Sect. 5), and so—for psychological reasons—we have presented it here.

4 Hecke operators

Let $g \in \mathrm{GL}_n(\mathbb{A}^\infty)$ be invertible. Associated to g one has the Hecke operator $T(g)$, defined by considering the composition:

$$\begin{aligned} H^\bullet(Y(K), \mathbf{F}) &\to H^\bullet(Y(gKg^{-1} \cap K), \mathbf{F}) \\ &\to H^\bullet(Y(K \cap g^{-1}Kg), \mathbf{F}) \to H^\bullet(Y(K), \mathbf{F}), \end{aligned}$$

the first map coming from the obvious inclusion, the second an isomorphism coming from the conjugation by g, and the final coming from the corestriction map. The Hecke operators preserve $H^\bullet(Y(K), \mathbf{F})$, but not necessarily the cohomology of the connected components. Indeed, the action on the component group is via the determinant map on $\mathrm{GL}_n(\mathbb{A}^\infty)$ and the natural action of $\mathbb{A}^{\infty,\times}$ on A. The Hecke operator $T_{\ell,k}$ is defined by taking g to be the diagonal matrix consisting of k copies of ℓ and $n-k$ copies of 1. The algebra $\mathbf{T}$ of endomorphisms generated by $T_{\ell,k}$ on $H^*(Y(K), \mathbf{F})$ for ℓ prime to the level of K generates a commutative algebra. For any such $T = T(g)$, let $\langle T \rangle$ denote the isomorphism of $H^*(Y(K), \mathbf{F})$ that acts by permuting the components according to the image of $\det(g)$ in A.

There are many definitions of the notion of "Eisenstein" in many different contexts. For our purposes, the following very restrictive definition is appropriate:

Definition 4.1 A cohomology class $[c] \in H^d(Y(K), \mathbf{F})$ is Eisenstein if $T[c] = \langle T \rangle \deg(T)[c]$ for any Hecke operator T. A maximal ideal $\mathfrak{m}$ of $\mathbf{T}$ is Eisenstein if and only if $\mathfrak{m}$ contains $T - \deg(T)$ for all T with $\langle T \rangle = 1$ in A.

If $[c] \in H^d(\Gamma, \mathbf{F})$ is a Hecke eigenclass such that $T[c] = \deg(T)[c]$ for all $\langle T \rangle = 1$, then $[c]$ is necessarily Eisenstein, and moreover $T[c] = \chi(\langle T \rangle) \deg(T)[c]$ for some character $\chi : A \to \mathbf{F}^\times$ of A. By class field theory, χ corresponds to a finite order character of $G_{\mathbf{Q}}$ of conductor dividing M. A easy computation of $\deg(T_{\ell,k})$ then implies that $\star$ will be satisfied with ρ equals $\chi \otimes (1 \oplus \omega \oplus \cdots \oplus \omega^{n-1})$ if and only if $[c]$ is Eisenstein.

Lemma 4.2 *If $[c]$ is a Hecke eigenclass such that $T[c] = \det(T)[c]$ for almost all T with $\langle T \rangle = 1$ in A, then $[c]$ is Eisenstein.*

Proof It suffices to show that $T[c] = \det(T)[c]$ for *all* T of level prime to Mp with $\langle T \rangle = 1$ in A. This is an immediate consequence of the Cebotarev density theorem and the fact that there exists a Galois representation associated to $[c]$ by [23]. □

4.1 Proof of Theorem 1.2

Given any eigenclass $[c] \in H^d(Y(K), \mathbf{F})$, consider its image $[\iota(c)]$ in $\Pi_{n,\ell}$. If this image is non-zero, we can determine the eigenvalues of $[c]$ by determining those of $[\iota(c)]$. Since the $\mathrm{GL}_n(\mathbf{Q}_\ell)$-action on $\Pi_{n,\ell}$ factors through det, the action of any $g \in \mathrm{GL}_n(\mathbf{Q}_\ell)$ on $[\iota(c)]$, as well as the action of the $\deg(T(g))$ representatives in the double coset decomposition of g, is via $\langle T \rangle$; hence $T[\iota(c)] - \langle T \rangle \deg(T)[\iota(c)]$ is zero in $\Pi_{n,\ell}$. It follows that either $[c]$ is Eisenstein or it lies in the kernel of the map $H^d(Y(K), \mathbf{F}) \to \Pi_{n,\ell}$. Hence Theorem 1.2 follows by induction from the following Lemma.

Lemma 4.3 *Suppose that for every non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbf{T}$ the cohomology groups $H^i(Y(L), \mathbf{F})_{\mathfrak{m}}$ vanish for all $i < d$, where $L = K^\ell L_\ell$ is any compact normal open subgroup $L_\ell \subset \mathrm{GL}_n(\mathbf{Z}_\ell)$. If $[c]$ lies in the kernel of the map $H^d(Y(K), \mathbf{F}) \to \Pi_{n,\ell}$, then $[c]$ is Eisenstein.*

Proof By assumption, the class $[c]$ lies in the kernel of the map

$$H^d(Y(K), \mathbf{F}) \to H^d(Y(L), \mathbf{F})$$

for some L as above. There is a Hochschild–Serre spectral sequence:

$$E_2^{ij} = H^i(K/L, H^j(L, \mathbf{F})) \to H^{i+j}(K, \mathbf{F}).$$

This spectral sequence is compatible with Hecke operators T away from Mp and ℓ. By Lemma 4.2, we may detect whether a class is Eisenstein by considering Hecke operators away from ℓ. Localizing the spectral sequence at a non-Eisenstein maximal ideal $\mathfrak{m}$, we deduce that E_2^{ij} vanishes for $j < d$. Hence the first d rows vanish, and we are left with an isomorphism:

$$H^d(Y(K), \mathbf{F})_{\mathfrak{m}} = H^0(K/L, H^d(Y(L), \mathbf{F})_{\mathfrak{m}}) = H^d(Y(L), \mathbf{F})_{\mathfrak{m}}^{K/L}.$$

It follows that if $[c]$ is not Eisenstein, then it is trivial. The result follows. □

This completes the proof of Theorem 1.2.

Remark 4.4 By the universal coefficient theorem, one has

$$\Pi_{n,\ell}^{\vee} := \mathrm{Hom}(\Pi_{n,\ell}, \mathbf{F}) = \varprojlim H_d(Y(K), \mathbf{F}).$$

Suppose that one defines

$$\Pi_{n,\ell}^{\vee}(\mathbf{Z}_\ell) = \varprojlim H_d(Y(K), \mathbf{Z}_\ell).$$

Then it follows from Lemma 3.2 for d and $d+1$ that the action of $\mathrm{GL}_n(\mathbf{Q}_\ell)$ on $\Pi_{n,\ell}^{\vee}(\mathbf{Z}_\ell)$ is via the determinant for sufficiently large n.

Remark 4.5 One can ask whether a Hecke operator T with $\langle T \rangle$ trivial in A acts via the degree on the entire cohomology group $H^d(\Gamma, \mathbf{Z})$. Our argument shows that, for such T, the image of $T - \deg(T)$ on $H^d(\Gamma, \mathbf{Z})$ is—in the terminology of [15]—*congruence*; i.e., it lies in the kernel of the map $H^d(\Gamma, \mathbf{Z}) \to H^d(\Gamma(M), \mathbf{Z})$ for some M.

Remark 4.6 The main theorem and its proof remain valid, and essentially unchanged, if one replaces $\mathrm{GL}_n(\mathbf{Z})$ by $\mathrm{GL}_n(\mathcal{O}_F)$ for any number field F.

Remark 4.7 Theorem 1.2 remains valid if M is divisible by p. To see this, one may (using Theorem 1.1) repeat the entire argument replacing the usual cohomology groups with completed cohomology. The key point is to notice that Proposition 3.4 (and hence Lemma 3.2 and Proposition 3.1) also holds for completed cohomology. To prove this, one may argue exactly as in Remark 3.3, namely, by Theorem 1.1, the completed cohomology groups at tame level ℓ^m (with any additional fixed tame level M) will be finite dimensional and independent of n for sufficiently large n, and hence the action of $\mathrm{SL}_n(\mathbf{Z}_\ell)$ on these groups will necessarily be trivial, and so (by the transfer map) these groups will not depend on m for $m \geq 1$.

5 Stability for $\ell = p$

It is natural to wonder whether the methods of the previous sections can be extended to $\ell = p$. One obvious obstruction is that the naïve notion of stability fails, even for $d = 1$ (as noted in the introduction). We proceed instead by using completed cohomology, whose definition for a finite field $\mathbf{F}$ we recall below. It is completed cohomology that is the correct analogue of the modules $\Pi_{n,\ell}$ defined above.

We fix, once and for all, a tame level M, and let $\Gamma(p^k) = \Gamma_n(p^k)$ denote the principal congruence subgroup of $\mathrm{SL}_n(\mathbf{Z})$ of level Mp^k.

Definition 5.1 The completed cohomology groups $\widetilde{H}^d_n$ are defined as follows [9]:

$$\widetilde{H}^d_n := \varinjlim H^d(\Gamma(p^k), \mathbf{F})$$

Although this definition is formally the same as $\Pi_{n,\ell}$, the theory when $\ell = p$ is quite different to the $\ell \neq p$ case, due to the non-semisimple nature of representations of pro-p groups on mod-p vector spaces. However, we still prove the following result:

Theorem 5.2 (Stability of completed cohomology) *For n sufficiently large, the modules $\widetilde{H}^d_n$ are finite dimensional over* $\mathbf{F}$ *and are independent of n.*

Remark 5.3 We have an isomorphism $\widetilde{H}_{d,n} := \mathrm{Hom}(\widetilde{H}^d_n, \mathbf{F}) \simeq \varprojlim H_d(\Gamma(p^k), \mathbf{F})$. If we define

$$\widetilde{H}_{d,n}(\mathbf{Z}_p) := \varprojlim H_d(\Gamma(p^k), \mathbf{Z}_p),$$

then Theorem 5.2 also implies, by Nakayama's lemma, that $\widetilde{H}_{d,n}(\mathbf{Z}_p)$ is a finite $\mathbf{Z}_p$-module for sufficiently large n. Hence Theorem 5.2 implies Theorem 1.1.

Remark 5.4 Theorem 5.2 is immediate when $d = 0$ and $d = 1$. Indeed, for $d = 0$ one has $\widetilde{H}^0_n(\mathbf{F}) = \mathbf{F}$ for all n, while for $d = 1$ one has $\widetilde{H}^1_n(\mathbf{F}) = 0$ for all $n \geq 3$ by the congruence subgroup property.

The proof of Theorem 5.2 will occupy most of the remainder of the paper. We begin by recalling some facts concerning non-commutative Iwasawa theory. Let $K = \mathrm{SL}_n(\mathbf{Z}_p)$ and let $K(p^k)$ denote the principal congruence subgroups of K. By construction, the module $\widetilde{H}^d_n$ is naturally a module over the completed group ring $\Lambda = \Lambda_{\mathbf{F}_p} := \mathbf{F}_p[[K(p)]]$. If M is a Λ-module, then let $M^\vee := \mathrm{Hom}(M, \mathbf{F}_p)$ be the dual Λ-module. By Nakyama's Lemma, $\widetilde{H}_{d,n}$ is finitely generated, which implies (by definition) that $\widetilde{H}^d_n$ is co-finitely generated. The ring Λ is Auslander regular [26], which implies that there is a nice notion of dimension and co-dimension of finitely generated Λ-modules. One characterization of dimension for modules is given by the following result of Ardakov and Brown [1]:

Proposition 5.5 *If M is finitely generated and $M^\vee$ is the co-finitely generated dual, then M has dimension at most m if and only if, as k increases without bound,*

$$\dim(M^\vee)^{K(p^k)} \ll p^{mk}.$$

The following is the natural analogue of Proposition 3.5 in this context.

Conjecture 5.6 *Let M be a finitely generated* $\mathbf{Z}_p[[K(p)]]$*-module. If M is infinite dimensional over* $\mathbf{F}$*, then the dimension of M is at least $n - 1$.*

Remark 5.7 The analogue of this conjecture for $\Lambda_{\mathbf{Q}_p}$ is true by Theorem A of [4]. In particular, if any finitely generated pure $\Lambda_{\mathbf{F}_p}$-module M is the reduction of a p-torsion free $\Lambda_{\mathbf{Z}_p} = \mathbf{Z}_p[[K(p)]]$-module, then the conjecture is true. Such a lifting always exists for *commutative* regular local rings, but it is unclear whether one should expect it to hold for Λ.

Our arguments proceed in a manner quite similar to the $\ell \neq p$ case. One missing ingredient is that Proposition 3.1 is no longer valid. We use the central stability results of Putman [22] as a replacement.

Let (V_n) be a collection of representations of S_n. Recall from [22] that a sequence

$$\ldots \longrightarrow V_{n-1} \xrightarrow[\phi_{n-1}]{} V_n \xrightarrow[\phi_n]{} V_{n+1} \longrightarrow \ldots$$

is *centrally stable* if:

(1) The ϕ_* are equivariant with respect to the natural inclusions on symmetric groups,
(2) For each n, the morphism $\mathrm{Ind}_{S_n}^{S_{n+1}} V_n \to V_{n+1}$ induced by ϕ_n identifies V_{n+1} with the (unique) maximal quotient of $\mathrm{Ind}_{S_n}^{S_{n+1}} V_n$ such that the 2-cycle $(n, n+1)$ acts trivially on $\phi_{n-1}(V_{n-1})$.

Lemma 5.8 *If Theorem 5.2 holds for $j < d$, then the modules $\widetilde{H}_{d,n}$ are centrally stable for sufficiently large n.*

Proof We first give an argument which is essentially taken directly from Putman [22], and we follow his argument closely. The lemma follows immediately from Theorem 2.1 of [22] with the extra hypothesis that the characteristic p is sufficiently large compared to d. We shall explain how the extra assumption (that the completed cohomology groups $\widetilde{H}_{j,n}$ in smaller degree $j < d$ are actually stable rather than centrally stable) renders the use of this supplementary characteristic hypothesis unnecessary.

Putman considers a spectral sequence (Theorem 5.2 of [22], as applied in §5.3) at a fixed level, which we may take to be $\Gamma(p^k)$. These spectral sequences admit compatible maps when one varies the level. Taking the inverse limit of these sequences, one obtains a corresponding spectral sequence for completed homology. In order for this sequence to degenerate at the relevant terms on page 2, it suffices to show that the appropriate $E^2_{i,j}$ are actually zero, as in Claim 4 (§5.3) of [22]. At finite level, Putman establishes this vanishing in Claim 3, which invokes Proposition 4.5 of [22], and it is this proposition which requires an assumption on the characteristic. However, we are assuming in addition that the modules $\widetilde{H}_{j,n}$ for $j < d$ are finite dimensional vector spaces that are *stable*, namely, they are (eventually) independent of n and have a trivial S_n-action. Thus the required claim reduces to showing that the $(n+1)$st central stability complex for the *trivial* sequence:

$$\mathbf{F} \to \mathbf{F} \to \mathbf{F} \to \cdots$$

of modules for the symmetric group is exact, without any assumption on the characteristic. This is a very special case of Proposition 6.1 of [22], but can be verified directly in this case. Ultimately, the assumption on the characteristic is used (see Theorem E, §7 of [22]) for the implication: central stability ⇒ Specht stability, however, one can make this deduction unconditionally in the case of the trivial (stable) sequence. Hence one deduces—as in Putman—that the $\widetilde{H}_{d,n}$ are centrally stable.

We now give an alternate proof. Theorem D of [11] gives a direct proof that the groups $H_d(\Gamma_n(p^k), \mathbf{F})$ are centrally stable. If one can establish that the range (in n) where these groups become centrally stable is bounded independently of k, then one gets an immediate and direct proof that the $\widetilde{H}_{d,n}$ are centrally stable without any induction hypothesis. However, this is a direct consequence of Theorem C′ of [10], proving the Lemma. □

Corollary 5.9 *If Theorem 5.2 holds for $j < d$, then the dimension of $\widetilde{H}_{d,n}$ as a $\Lambda_{\mathbf{F}_p}$-module is bounded independently of n.*

Proof This is obvious for any fixed collection of n. Yet, for n sufficiently large, central stability implies that the natural map

$$\mathrm{Ind}_{S_n}^{S_{n+1}} \widetilde{H}_{d,n} \to \widetilde{H}_{d,n+1}$$

induced by the $n+1$ embeddings of $\mathrm{SL}(n)$ into $\mathrm{SL}(n+1)$ is surjective. Passing to cohomology, the natural maps $\widetilde{H}^d_{n+1} \to \widetilde{H}^d_n$ induce homomorphisms from the $\Gamma_{n+1}(p^k)$-invariants of $\widetilde{H}^d_{n+1}$ to the $\Gamma_n(p^k)$-invariants of $\widetilde{H}^d_n$. In particular, the dimension of the $\Gamma_{n+1}(p^k)$-invariants of $\widetilde{H}^d_{n+1}$ is at most $n+1$ times the dimension of the $\Gamma_n(p^k)$-invariants of $\widetilde{H}^d_n$, and so it follows by Proposition 5.5 that $\widetilde{H}_{d,n+1}$ has Iwasawa dimension bounded by the Iwasawa dimension of $\widetilde{H}_{d,n}$. □

Remark 5.10 Using Putman's spectral sequence, one may prove unconditionally that

$$\dim_{\mathbf{F}} H^d(\Gamma(p^k), \mathbf{F}) \ll p^{mk}$$

for some constant m that does not depend on n (although the implied constant does depend on n). This leads to an alternate proof of Corollary 5.9 via the Hochschild–Serre spectral sequence and Proposition 5.5.

We prove Theorem 5.2 by induction. By the proceeding corollary, the dimension of $\widetilde{H}_{d,n}$ must be bounded independently of n. If we assume Conjecture 5.6, then we immediately deduce that this dimension is actually zero for sufficiently large n, and hence $\widetilde{H}_{d,n}$ is finite and the action of $\mathrm{SL}_n(\mathbf{Q}_p)$ on $\widetilde{H}_{d,n}$ is trivial. In particular, the action of S_n must also be either trivial, or via the sign character, for sufficiently large n. By Lemma 5.8, we also deduce that the sequence $\widetilde{H}_{d,n}$ is centrally stable. This rules out the possibility that S_n acts via the sign character (for large enough n), since these do not fit into a centrally stable sequence. It follows that we must have a centrally stable sequence of trivial S_n-representations. Yet a centrally stable sequence of trivial modules is stable, and hence $\widetilde{H}_{d,n}$ stabilizes.

How may we upgrade this to an unconditional proof? One thing to observe is that the modules $\widetilde{H}_{d,n}$ have an action of $\mathrm{SL}_n(\mathbf{Q}_p)$ that has not been exploited. For technical reasons, it is actually more convenient to have an action of the group $\mathrm{GL}_n(\mathbf{Q}_p)$. In order to upgrade the action of $\mathrm{SL}_n(\mathbf{Q}_p)$ to $\mathrm{GL}_n(\mathbf{Q}_p)$, one should take the direct limit (in cohomology) as follows:

$$\lim_{\substack{\rightarrow \\ K_p}} H^d(Y(K), \mathbf{F}).$$

From now on, we use $\widetilde{H}_n^d$ to denote this limit (rather than the limit over the cohomology of congruence subgroups that it denoted up till now).

With this new definition, $\widetilde{H}_n^d$ is cofinitely generated over $\mathbf{F}_p[[\mathrm{GL}_n(\mathbf{Z}_p)]]$ rather than $\Lambda_{\mathbf{F}_p} = \mathbf{F}_p[[\mathrm{SL}_n(\mathbf{Z}_p)]]$. It then suffices to show that these completions are eventually stable and finite over the ring $\mathbf{F}_p[[\det(\mathrm{GL}_n(\mathbf{Z}_p))]] = \mathbf{F}_p[[\mathbf{Z}_p^\times]]$. Instead of appealing to a general conjecture concerning finitely generated $\Lambda_{\mathbf{F}_p}$-modules, we only need consider such modules whose dual admits a smooth admissible action of $\mathrm{GL}_n(\mathbf{Q}_p)$. In particular, it suffices to show that the only irreducible $\mathrm{GL}_n(\mathbf{Q}_p)$-representations that occur as sub-quotients of these completions are either trivial or have large dimension. So, instead of proving Conjecture 5.6, it suffices to prove the following:

Conjecture 5.11 *Let π be an irreducible infinite-dimensional smooth admissible representation of* $\mathrm{GL}_n(\mathbf{Q}_p)$*, and let* $\pi^\vee = \mathrm{Hom}(\pi, \mathbf{F}_p)$*. Then the dimension of $\pi^\vee$ as a $\Lambda_{\mathbf{F}_p}$-module is $\geq n - 1$.*

Although this conjecture is presumably easier than Conjecture 5.6, we still do not know how to prove it. Instead, we prove the following two lemmas:

Lemma 5.12 *Let π be an irreducible infinite-dimensional smooth admissible representation of* $\mathrm{GL}_n(\mathbf{Q}_p)$*, and let* $\pi^\vee = \mathrm{Hom}(\pi, \mathbf{F}_p)$*. If π is not a supercuspidal representation, then the dimension of $\pi^\vee$ as a $\Lambda_{\mathbf{F}_p}$-module is $\geq n - 1$.*

Lemma 5.13 *Suppose that Theorem 5.2 holds for $j < d$, and suppose that n is sufficiently large. Then any infinite-dimensional irreducible smooth admissible representation π that occurs as a sub-quotient of $\widetilde{H}_{n+1}^d$ is not supercuspidal.*

Taken together, these lemmas serve as a replacement for either Conjecture 5.6 or 5.11, and imply the main theorem. To recapitulate the argument, from Lemma 5.13, we deduce that every irreducible constituent of $\widetilde{H}_{n+1}^d$ is not supercuspidal. From Lemma 5.12, we deduce that either $\widetilde{H}_{n+1}^d$ contains a representation of dimension at least n, which contradicts the uniformly bounded dimension of $\widetilde{H}_{n+1}^d$ for all n, or the only irreducible constituents of $\widetilde{H}_{n+1}^d$ are finite, in which case the action of $\mathrm{SL}_n(\mathbf{Q}_p)$ is also trivial.

Proof of Lemma 5.12 Let G denote $\mathrm{GL}_n(\mathbf{Q}_p)$ and let B denote a Borel of G. The Lemma is easy to verify directly for the Steinberg representation, for which the Iwasawa dimension of the dual is equal to $\dim(G/B)$. By the main theorem of [19], we may therefore assume that there exists a proper parabolic P with Levi M such that:

$$\pi = \mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r) = \mathrm{Ind}_P^G(\sigma).$$

The dual of such an induced representation has dimension equal to the sum of the dimension of the dual of σ and the dimension of G/P, and hence has dimension at least equal to that of G/P. It is easy enough to prove this directly, but we instead give a proof here using the characterization of dimension coming from Proposition 5.5 (which applies just as well to $\mathrm{GL}_n(\mathbf{Z}_p)$ as it does to $\mathrm{SL}_n(\mathbf{Z}_p)$).

Using the Iwasawa decomposition $G = KP$ (where now K denotes $\mathrm{GL}_n(\mathbf{Z}_p)$), we see that we may rewrite π (thought of as a K-representation) as $\pi = \mathrm{Ind}_{P\cap K}^{K}(\sigma)$. A short calculation then shows, since the level p^k congruence subgroup $K(p^k)$ is normal in K, that

$$\pi^{K(p^k)} = \mathrm{Ind}_{P(\mathbf{Z}/p^k)}^{G(\mathbf{Z}/p^k)}\big(\sigma^{M\cap K(p^k)}\big).$$

Now for $k > 0$, the representation inside the induction is nonzero. Hence

$$\dim \pi^{K(p^k)} \geq \big[G\big(\mathbf{Z}/p^k\big) : P\big(\mathbf{Z}/p^k\big)\big] \sim p^{mk},$$

where $m = \dim(G/P)$. As claimed, this gives a lower bound of $\dim(G/P)$ on the Iwasawa dimension of π.

The minimum value of $\dim(G/P)$ amongst all proper parabolics is $n-1$, and so the lemma follows. □

Proof of Lemma 5.13 Let $M \subset \mathrm{GL}_{n+1}(\mathbf{Q}_p)$ denote the Levi subgroup $\mathbf{Q}_p^\times \times \mathrm{GL}_n(\mathbf{Q}_p)$ of $\mathrm{GL}_{n+1}(\mathbf{Q}_p)$, and let P be the corresponding parabolic. Associated to P is a natural arithmetic group $\Gamma_P \subset \mathrm{GL}_{n+1}(\mathbf{Z})$. Let $\Gamma_P(p^k) = \Gamma_P \cap \Gamma_{n+1}(p^k)$. The group $\Gamma_P(p^k)$ is a semi-direct product:

$$\Gamma_P\big(p^k\big) \simeq \big(p^k\mathbf{Z}\big)^n \rtimes \Gamma_n\big(p^k\big).$$

Let us compute the completed cohomology groups of Γ_P. There is a Hochschild–Serre spectral sequence:

$$H^i\big(\Gamma_n\big(p^k\big), H^j\big(\big(p^k\mathbf{Z}\big)^n, \mathbf{F}_p\big)\big) \Rightarrow H^{i+j}\big(\Gamma_P\big(p^k\big), \mathbf{F}_p\big).$$

There is an isomorphism $H^j(p^n\mathbf{Z}, \mathbf{F}_p) = \wedge^j(\mathbf{F}_p)^n$, but the composition maps from $\Gamma_P(p^k)$ to $\Gamma_P(p^{k+1})$ induce the zero map on these cohomology groups unless $j = 0$. Hence, taking direct limits, we get an isomorphism:

$$\widetilde{H}_P^d := \varinjlim H^d(\Gamma_P(p^n), \mathbf{F}_p) \simeq \widetilde{H}_n^d.$$

Omit the degree d from the notation. We therefore have maps as follows:

$$\widetilde{H}_{n+1} \to \widetilde{H}_P \to \widetilde{H}_n.$$

The first map is P-equivariant, the second is an isomorphism, and the composition is the natural map. It follows that there exists a G-equivariant map

$$\widetilde{H}_{n+1} \to \mathrm{Ind}_P^G \widetilde{H}_n.$$

Note that $S_{n+1} \cap P = S_n$, and hence the right hand side includes the induction from $\widetilde{H}_n$ from S_n to S_{n+1}. If n is sufficiently large with respect to d, then we deduce, by central stability (which we know under our hypotheses by Lemma 5.8), that we actually have an injection:

$$\widetilde{H}_{n+1} \to \mathrm{Ind}_P^G \widetilde{H}_n.$$

To complete the proof of the lemma, it suffices to show that no constituent of the right hand side is supercuspidal. But this also follows immediately from Herzig's classification of irreducible admissible representations of G [19], since $\widetilde{H}_n$ is a smooth admissible M-representation, where M is the Levi of P. □

Remark 5.14 It is possible to give explicit bounds on n in terms of d which guarantee that $\widetilde{H}_{d,n}$ is stable. Suppose that the sequence $\widetilde{H}_{d,n}$ is centrally stable for n at least m. Then, using (the proof of) Corollary 5.9, we deduce that the dimensions of $\widetilde{H}_{d,n}$ for $n \geq m$ are at most the dimension of $\widetilde{H}_{d,m}$, which is at most the dimension of $\Lambda_{m,\mathbf{F}_p}$, which is $m^2 - 1$. Then, using Lemmas 5.13 and 5.12, we deduce that $\widetilde{H}_{d,n}$ is finite as long as $n - 1 > m^2 - 1$, or when $n > m^2$. Using the bounds on m coming from [10], it follows that one can take m to be $11 \cdot 2^{d-2}$, and so $\widetilde{H}_{d,n}$ is independent of n for $n \geq 121 \cdot 4^{d-2}$.

5.1 Consequences for classical cohomology groups

The homology (or cohomology) of arithmetic groups with $\mathbf{F}$ coefficients can be recovered from completed homology via the Hochschild–Serre spectral sequence [9]. For example, if we take our arithmetic group to be the principal congruence subgroup $\Gamma(p)$ of $\mathrm{SL}_n(\mathbf{Z})$, then, recalling that $G(p)$ is the principal subgroup of $\mathrm{SL}_n(\mathbf{Z}_p)$, this spectral sequence has the form

$$E_2^{i,j} := H^i\big(G(p), \widetilde{H}^j\big) \implies H^{i+j}\big(\Gamma(p), \mathbf{F}\big).$$

As noted above, $\widetilde{H}^0 = \mathbf{F}$, and Theorem 5.2 implies that each $\widetilde{H}^j$ is finite with trivial $G(p)$-action. Hence Theorem 5.2 implies that the cohomology group $H^d(\Gamma(p), \mathbf{F})$ has a filtration with graded pieces consisting of three types of classes: those arising via pullback from $H^d(G(p), \mathbf{F})$, those arising via (higher) transgressions from H^j of $G(p)$ for $j < d$, and those arising from $\widetilde{H}^d$ (which do not depend on n). In this optic, the notion of *representation stability* developed by Church and Farb [12] is then seen (in this context) to have its origin in the mod-p cohomology of p-adic Lie groups, rather than in the properties of arithmetic groups.

One interpretation of Theorem 5.2 is that the $\ell = p$ and $\ell \neq p$ theories after completion are quite similar. The difference in phenomenology between these two cases is then a consequence of the vanishing of $H^i(G(\ell^k), \mathbf{F})$ when $k \geq 1$ and the characteristic p of $\mathbf{F}$ is not ℓ. Another way to state these results is to say that the only irreducible $\mathrm{GL}_n(\mathbb{A}^\infty)$-representations occurring inside cohomology in sufficiently small degree are one-dimensional.

5.2 Relation to K-theory

Even knowing that $\widetilde{H}_d$ is finite, it is not at all apparent how to compute it. One might wonder whether there is some interpretation of $\widetilde{H}_d$ in terms of K-theory (some related speculation along these lines occurs in §8.3 of [15]). This question is taken up in the companion paper [7].

Acknowledgments The first author would like to thank Akshay Venkatesh, and both authors are grateful to Andy Putman for answering questions about his paper [22]. We would also like to thank Konstantin Ardakov, Kevin Buzzard, Thomas Church, Jean-François Dat, Jordan Ellenberg, Benson Farb, Toby Gee, Florian Herzig, Akshay Venkatesh, and Simon Wadsley for interesting conversations and helpful remarks. An earlier version of this preprint from 2012 established Theorem 1.1 *conditionally* on Conjecture 5.6; the unconditional proof of Theorem 1.1 was found in October of 2013. The first author would especially like to thank Florian Herzig for some remarks concerning admissible representations that led to the proof of Theorem 1.1. Finally, we would like to thank the reviewer for useful remarks, in particular the suggestion to include Remark 4.7.

References

1. Ardakov, K., Brown, K.A.: Ring-theoretic properties of Iwasawa algebras: a survey. In: Fesenko, I. (ed.) A Collection of Manuscripts Written in Honour of John H. Coates on the Occasion of his Sixtieth Birthday. Series: Documenta Mathematica, vol. 4, pp. 7–33. Deutsche Mathematiker-Vereinigung, Bielefeld, Germany. ISBN 9783936609288
2. Ash, A.: Galois representations attached to mod p cohomology of $\mathrm{GL}(n, Z)$. Duke Math. J. **65**(2), 235–255 (1992)
3. Ash, A.: Hecke operators and the stable homology of $GL(n)$. Commun. Algebra **40**(4), 1369–1387 (2012)
4. Ardakov, K., Wadsley, S.J.: On irreducible representations of compact p-adic analytic groups (2012) (preprint)
5. Bass, H., Lazard, M., Serre, J.P.: Sous-groupes d'indice fini dans $SL(n, Z)$. Bull. Am. Math. Soc. **70**, 385–392 (1964)
6. Borel, A.: Stable real cohomology of arithmetic groups. Ann. Sci. École Norm. Sup. (4) **7**(2), 235–272 (1975)
7. Calegari, F.: The stable homology of congruence subgroups. Geom. Topol. (2015) (to appear)
8. Calegari, F., Emerton, M.: Bounds for multiplicities of unitary representations of cohomological type in spaces of cusp forms. Ann. Math. (2) **170**(3), 1437–1446 (2009)
9. Calegari, F., Emerton, M.: Completed cohomology—a survey. Non-abelian fundamental groups and Iwasawa theory. London Math. Soc. Lecture Note Ser., vol. 393, pp. 239–257. Cambridge Univ. Press, Cambridge (2012)
10. Church, T., Ellenberg, J.S.: Homology of FI-modules (2015) (preprint)
11. Church, T., Ellenberg, J.S., Farb, B., Nagpal, R.: FI-modules over Noetherian rings. Geom. Topol. **18**(5), 2951–2984 (2014)
12. Church, T., Farb, B.: Representation theory and homological stability. Adv. Math. **245**, 250–314 (2013)
13. Charney, R.M.: Homology stability for GL_n of a Dedekind domain. Invent. Math. **56**(1), 1–17 (1980)
14. Charney, R.: On the problem of homology stability for congruence subgroups. Commun. Algebra **12**(17–18), 2081–2123 (1984)
15. Calegari, F., Venkatesh, A.: Towards a Torsion Langlands correspondence (2012) (preprint)
16. Emerton, M.: On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. Invent. Math. **164**(1), 1–84 (2006)
17. Gabber, O.: K-theory of Henselian local rings and Henselian pairs, Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989). Contemp. Math., vol. 126, pp. 59–701. Amer. Math. Soc., Providence (1992)
18. Gan, W.T., Savin, G.: On minimal representations definitions and properties. Represent. Theory **9**(3), 46–93 (2005) (electronic)

19. Herzig, F.: The classification of irreducible admissible mod p representations of a p-adic GL_n. Invent. Math. **186**(2), 373–434 (2011)
20. Maazen, H.: Stabilité de l'homologie de GL_n. C. R. Acad. Sci. Paris Sér. A-B **288**(15), 707–708 (1979)
21. Mennicke, J.L.: Finite factor groups of the unimodular group. Ann. Math. (2) **81**, 31–37 (1965)
22. Putman, A.: Stability in the homology of congruence subgroups. Invent. Math. (2015) (to appear)
23. Scholze, P.: On torsion in the cohomology of locally symmetric varieties. Ann. Math. (2015) (to appear)
24. Smith, S.P.: Krull dimension of factor rings of the enveloping algebra of a semisimple Lie algebra. Math. Proc. Cambridge Philos. Soc. **93**(3), 459–466 (1983)
25. Soulé, C.: K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale. Invent. Math. **55**(3), 251–295 (1979)
26. Venjakob, O.: On the structure theory of the Iwasawa algebra of a p-adic Lie group. J. Eur. Math. Soc. **4**(3), 271–311 (2002)
27. Vignéras, M.F.: Correspondance de Langlands semi-simple pour $GL(n, F)$ modulo $\ell \neq p$. Invent. Math. **144**(1), 177–223 (2001)