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Hölder curves and parameterizations in the Analyst's Traveling Salesman theorem

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ABSTRACT

We investigate the geometry of sets in Euclidean and infinite-dimensional Hilbert spaces. We establish sufficient conditions that ensure a set of points is contained in the image of a $(1/s)$-Hölder continuous map $f : [0,1] \to l^2$, with $s > 1$. Our results are motivated by and generalize the "sufficient half" of the Analyst's Traveling Salesman Theorem, which characterizes subsets of rectifiable curves in $\mathbb{R}^N$ or l^2 in terms of a quadratic sum of linear approximation numbers called Jones' beta numbers. The original proof of the Analyst's Traveling Salesman Theorem depends on a well-known metric characterization of rectifiable curves from the 1920s, which is not available for higher-dimensional curves such as Hölder curves. To overcome this obstacle, we reimagine Jones' non-parametric proof and show how to construct parameterizations of the intermediate approximating curves $f_k([0,1])$. We then find conditions in terms of *tube approximations* that ensure the approximating curves converge to a Hölder curve. As an application to the geometry

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of measures, we identify conditions that guarantee fractional rectifiability of pointwise doubling measures in $\mathbb{R}^N$.

Contents

1. Introduction

The ubiquitous Traveling Salesman problem [33,27,2] is to find a tour of edges on a finite graph that returns to the initial vertex and has the shortest possible length. The Analyst's Traveling Salesman problem [32,47] is to find a rectifiable curve that contains a finite or infinite, bounded set of points in a metric space that has the shortest possible length. The former problem always has a solution yet is computationally hard, while the latter problem may or may not have any solution at all. A sophisticated example from Geometric Measure Theory of a bounded point set that is not contained in any rectifiable curve is a *Besicovitch irregular* set [11] (see §7 below); a trivial example is a solid square in the plane. Tests to decide which sets are contained in a rectifiable curve have been found in $\mathbb{R}^2$ [32], $\mathbb{R}^N$ [43], l^2 [48], the first Heisenberg group [35,36], Carnot groups [19,34], Laakso-type spaces [20], and in general metric spaces [28,29,21]. Applications of Jones' and Okikiolu's solution of the Analyst's TSP in $\mathbb{R}^N$ have been given in Complex Analysis [15,13,14], Dynamics and Probability [16,17], Geometric Measure Theory [8,9], Harmonic Analysis [50], and Metric Geometry [42,3].

Let $E \subset \mathbb{R}^N$ be a nonempty set and let $Q \subset \mathbb{R}^N$ be a bounded set of positive diameter (such as a ball or a cube). Following [32], the *Jones beta number* $\beta_E(Q)$ is defined by

$$\beta_E(Q) := \inf_{\ell} \sup_{x \in E \cap Q} \frac{\operatorname{dist}(x, \ell)}{\operatorname{diam} Q} \in [0, 1],$$

where ℓ ranges over all straight lines in $\mathbb{R}^N$, if $E \cap Q \neq \emptyset$, and by $\beta_E(Q) = 0$, if $E \cap Q = \emptyset$. Let $\Delta(\mathbb{R}^N)$ denote the family of dyadic cubes in $\mathbb{R}^N$,

$$\Delta(\mathbb{R}^N) := \{[2^k m_1, 2^k(m_1+1)] \times \cdots \times [2^k m_N, 2^k(m_N+1)] : m_1, \ldots, m_N, k \in \mathbb{Z}\}.$$

Given a cube Q and a scaling factor $\lambda > 0$, we let λQ denote the concentric dilate of Q by λ.

Analyst's Traveling Salesman Theorem *([32,43]). A bounded set $E \subset \mathbb{R}^N$ is contained in a rectifiable curve $\Gamma = f([0,1])$ if and only if*

$$S_E := \sum_{Q \in \Delta(\mathbb{R}^N)} \beta_E(3Q)^2 \operatorname{diam} Q < \infty. \tag{1.1}$$

More precisely,

(1) *If Γ is any curve containing E, then* $\operatorname{diam} E + S_E \lesssim_N \operatorname{length}(\Gamma)$.
(2) *If $S_E < \infty$, then there exists a curve $\Gamma \supset E$ such that* $\operatorname{length}(\Gamma) \lesssim_N \operatorname{diam} E + S_E$.

We may refer to statements (1) and (2) as the *necessary half* and *sufficient half* of the Analyst's Traveling Salesman theorem, respectively. The theorem is valid if the length of a curve $\Gamma = f([0,1])$ is interpreted either as the 1-dimensional Hausdorff measure of the set Γ or as the total variation of the parameterization f. A curious feature of the known proofs of the sufficient half of the Analyst's TST (see [32] or [9]) is that a rectifiable curve Γ containing the set E is constructed as the limit of piecewise linear curves Γ_k containing a 2^{-k}-net for E *without constructing a parameterization* of Γ_k or Γ. This aspect of the proof breaks the analogy with the classical TSP, in which one is asked to find a minimal tour of a graph.

In this paper, we provide a *parametric proof* of the sufficient half of the Analyst's TST, which more closely parallels the classical TSP. Beyond its intrinsic interest, the method that we provide is important, because it allows us to establish multiscale tests to ensure that a bounded set of points in $\mathbb{R}^N$ is contained in a $(1/s)$-*Hölder continuous curve* with $s \in (1, N)$. Rectifiable curves correspond precisely to the class of Lipschitz curves ($s = 1$). Remarkably, in the Hölder Traveling Salesman theorem (see §§1.1 and 5), we can replace approximation by lines in the definition of the Jones beta numbers with *approximation by thin tubes*. For a self-contained statement of the "parametric" Analyst's TST, see §6. While our focus in this paper is primarily on sets, we are motivated by open questions about the structure of Radon measures [10,6]. For applications of our Hölder Traveling Salesman theorems to fractional rectifiability of measures, see §7.

1.1. Hölder Traveling Salesman theorem(s)

A $(1/s)$-*Hölder curve* Γ in $\mathbb{R}^N$ is the image of a continuous map $f : [0,1] \to \mathbb{R}^N$ satisfying the Hölder condition,

$$|f(x) - f(y)| \leq H|x-y|^{1/s} \quad \text{for all } x, y \in [0,1],$$

where $s \in [1,\infty)$ and H is a finite constant independent of x and y. A 1-Hölder curve is also called a *Lipschitz curve* or a *rectifiable curve*. While non-trivial rectifiable curves always have topological dimension 1 and asymptotically resemble a unique tangent line $\mathcal{H}^1$ almost everywhere, $(1/s)$-Hölder curves with $s > 1$ exhibit a variety of more complicated behaviors. For example,

- an m-dimensional cube in $\mathbb{R}^N$ $(m \leq N)$ is a $(1/m)$-Hölder curve;
- the von Koch snowflake is a $\log_4(3)$-Hölder curve; and,
- the standard Sierpiński carpet is a $\log_8(3)$-Hölder curve.

In fact, Remes [45] proved that any compact, connected self-similar set $K \subset \mathbb{R}^N$ of Hausdorff dimension s that satisfies the open set condition is a $(1/s)$-Hölder curve. For related work on space-filling curves generated by graph-directed iterated function systems, see Rao and Zhang [44].

Towards a Hölder version of the Analyst's Traveling Salesman theorem, the first and third authors proved in [10] as a test case that if $s > 1$, $E \subset \mathbb{R}^N$ is bounded, and

$$\sum_{\substack{Q \in \Delta(\mathbb{R}^N) \\ Q \cap E \neq \emptyset,\, \text{side } Q \leq 1}} (\operatorname{diam} Q)^s < \infty,$$

then E is contained in a $(1/s)$-Hölder curve. By establishing a parametric version of Jones' proof of the sufficient half of the Analyst's TST, we are able to obtain the following substantial improvement.

Theorem 1.1 *(Hölder Traveling Salesman I). For all $N \geq 2$ and $s > 1$, there exists $\beta_0 \in (0,1)$ such that if $E \subset \mathbb{R}^N$ is bounded and*

$$S_E^{s,+} := \sum_{\substack{Q \in \Delta(\mathbb{R}^N) \\ \beta_E(3Q) \geq \beta_0}} (\operatorname{diam} Q)^s < \infty, \tag{1.2}$$

then E is contained in a $(1/s)$-Hölder curve. More precisely, $E \subset \Gamma = f([0,1])$ for some $(1/s)$-Hölder map $f : [0,1] \to \mathbb{R}^N$ with Hölder constant $H \lesssim_{N,s} \operatorname{diam} E + (\operatorname{diam} E)^{1-s} S_E^{s,+}$.

Condition (1.2) implies that at $\mathcal{H}^s$ almost every point, the set E asymptotically lies in sufficiently *thin tubes*. Theorem 1.1 provides a sufficient test that identifies all subsets of

some well-known Hölder curves such as snowflakes of small dimension. However, because of the richness of Hölder geometry, a condition using Jones beta numbers alone such as (1.2) cannot be expected to hold for all subsets of every Hölder curve. Indeed (1.2) fails when E is a carpet or a square. For expanded discussion and related examples, see §9.1.

Theorem 1.1 is a simplification of our main result, which is adapted to a nested sequence of separated sets in a finite or infinite-dimensional Hilbert space. See Theorem 5.1.

To estimate the size of the constant β_0 in Theorem 1.1, see Lemma 2.8 and Remark 5.6. The following variant of Theorem 1.1 is an immediate corollary, whose hypothesis does not require knowledge of β_0.

Corollary 1.2 *(Hölder Traveling Salesman II). Suppose that $N \geq 2$, $s > 1$, and $p > 0$. If $E \subset \mathbb{R}^N$ is bounded and*

$$S_E^{s,p} := \sum_{Q \in \Delta(\mathbb{R}^N)} \beta_E(3Q)^p (\operatorname{diam} Q)^s < \infty, \tag{1.3}$$

then E is contained in a $(1/s)$-Hölder curve. More precisely, $E \subset \Gamma = f([0,1])$ for some $(1/s)$-Hölder map $f : [0,1] \to \mathbb{R}^N$ with Hölder constant

$$H \lesssim_{N,s} \operatorname{diam} E + \beta_0^{-p} (\operatorname{diam} E)^{1-s} S_E^{s,p},$$

where β_0 is the constant appearing in Theorem 1.1.

A good exercise is to prove that any bounded set E in $\mathbb{R}^N$ satisfying condition (1.3) with $s > 1$ has zero s-dimensional Hausdorff measure. In §9.3, we construct a countable, compact set E (hence $\mathcal{H}^s(E) = 0$) such that E is not contained in any $(1/s)$-Hölder curve with $1 \leq s < N$. Thus, Corollary 1.2 is nonvacuous.

1.2. Overview of the proof of Theorem 1.1

In order to properly discuss the proof of Theorem 1.1, we quickly sketch the proof of the sufficient half of the Analyst's TST. The proof splits into three steps. In the first step, one uses the Jones beta numbers $\beta_E(3Q)$ (in particular, whether they are large or small) to construct a sequence of finite, connected graphs G_k in $\mathbb{R}^N$ with straight edges that converge in the Hausdorff distance to a compact, connected set G containing E. Each graph G_k is obtained by refining G_{k-1} and resembles a flat arc near points of E that look flat at scale 2^{-k}. In step two, one uses the structure of the graphs G_k and the Pythagorean theorem to prove the existence of a constant $C > 0$ such that

$$\mathcal{H}^1(G_{k+1}) \leq \mathcal{H}^1(G_k) + C \sum_{\substack{Q \in \Delta(\mathbb{R}^N) \\ \text{side } Q \simeq 2^{-k}}} \beta_E(3Q)^2 \operatorname{diam} Q. \tag{1.4}$$

Condition (1.1) and Gołąb's semicontinuity theorem (e.g. see [1]) ensure that $\mathcal{H}^1(G) \leq \liminf_{k\to\infty} \mathcal{H}^1(G_k) < \infty$. Thus, the first two parts of the proof yield a compact, connected set G containing E with $\mathcal{H}^1(G) < \infty$. The final step is to invoke Ważewski's theorem to conclude *existence* of a Lipschitz parameterization for G: if $G \subset \mathbb{R}^N$ is connected, compact, and $\mathcal{H}^1(G) < \infty$, then there exists a Lipschitz map $f : [0,1] \to \mathbb{R}^N$ such that $G = f([0,1])$ (see [1, Theorem 4.4] or [48, Lemma 3.7]). Note that the condition $\mathcal{H}^1(G) < \infty$ promotes connectedness of G to local connectedness (because G is a curve).

In the Hölder setting, there are at least two obstacles to following the approach above. First and foremost, a naive analogue of Ważewski's theorem cannot hold for Hölder maps, since the condition $\mathcal{H}^s(G) < \infty$ does not imply a continuum is locally connected when $s > 1$ (e.g. the topologist's comb). What is more, even if G is assumed to be an Ahlfors s-regular curve with finite $\mathcal{H}^s$ measure, we cannot conclude that G is a $(1/s)$-Hölder curve; we provide examples in §9.2 using a theorem of Martín and Mattila [38]. Another obstacle is the well-known failure of Gołąb's semicontinuity theorem for Hausdorff measures $\mathcal{H}^s$ with $s > 1$. Thus, in a proof of a Hölder Traveling Salesman theorem, estimating the Hausdorff measure of approximating sets has no direct use.

To overcome these obstacles, we reimagine the proof of the Analyst's TST, and in §3, give a procedure to construct a sequence of partitions $\{\mathscr{I}_k\}_{k\geq 0}$ of $[0,1]$ and a sequence of piecewise linear maps $\{f_k : [0,1] \to \mathbb{R}^N\}_{k\geq 0}$ that *parameterize* approximating graphs G_k. Each map f_k is built by carefully refining f_{k-1} to ensure that $\|f_k - f_{k-1}\|_\infty \lesssim 2^{-k}$. This guarantees that the maps f_k have a uniform limit f whose image contains the Hausdorff limit of G_k (and hence E). To prove that f is Hölder continuous, one must estimate growth of the Lipschitz constants of the maps f_k (see Appendix B for the basic method). In §4, we introduce a notion of *mass* of intervals $I \in \mathscr{I}_k$, defined using the s-power of diameters of images $f_l(J)$ of intervals $J \subset I$, $l \geq k$. This lets us record estimates in the domain of the map rather than its image, and in §4, we provide a mass-centric analogue of (1.4) that is adapted to the Hölder setting. In turn, this lets us estimate the Lipschitz constants of the maps f_k and complete the proof of the Hölder Traveling Salesman theorem in §5. For completeness, we use our method to reprove and strengthen the sufficient half of the Analyst's TST in §6.

1.3. Ważewski type theorem for flat continua

The Hahn-Mazurkiewicz Theorem (e.g. see [30, Theorem 3.30]) asserts that a set $E \subset \mathbb{R}^N$ is a continuous image of $[0,1]$ if and only if E is compact, connected, and locally connected. The Ważewski Theorem (for an attribution, see [1]) asserts that $E \subset \mathbb{R}^N$ is a Lipschitz image of $[0,1]$ if and only if E is compact, connected, and $\mathcal{H}^1(E) < \infty$. It is an easy exercise to check that every $(1/s)$-Hölder continuous image of $[0,1]$ is compact, connected, locally connected, and has $\mathcal{H}^s(E) < \infty$, but the converse fails when $s > 1$ (see §9.2 below). This motivates the following, apparently open question: Is there a metric, geometric, and/or topological characterization of Hölder curves in $\mathbb{R}^N$?

The method of proof of the Hölder Traveling Salesman theorems leads to the following Ważewski type theorem for flat continua. For the proof of Proposition 1.3, see §8. A set $E \subset \mathbb{R}^n$ is called *Ahlfors s-regular* if there exist $0 < c \leq C < \infty$ such that

$$cr^s \leq \mathcal{H}^s(E \cap B(x,r)) \leq Cr^s \quad \text{for all } x \in E \text{ and } 0 < r \leq \operatorname{diam} E. \tag{1.5}$$

We say that E is *lower (upper) Ahlfors s-regular* if the first (second) inequality in (1.5) holds for all $x \in E$ and $0 < r \leq \operatorname{diam} E$.

Proposition 1.3. *There exists a constant $\beta_1 \in (0,1)$ such that if $s > 1$ and $E \subset \mathbb{R}^N$ is compact, connected, $\mathcal{H}^s(E) < \infty$, E is lower Ahlfors s-regular with constant c, and*

$$\beta_E\left(\overline{B}(x,r)\right) \leq \beta_1 \quad \textit{for all } x \in E \textit{ and } 0 < r \leq \operatorname{diam} E, \tag{1.6}$$

then $E = f([0,1])$ for some injective $(1/s)$-Hölder continuous map $f : [0,1] \to \mathbb{R}^N$ with Hölder constant $H \lesssim_s c^{-1}\mathcal{H}^s(E)(\operatorname{diam} E)^{1-s}$.

Inclusion of lower Ahlfors regularity in the hypothesis of Proposition 1.3 is justifiable, because it holds automatically when $s = 1$, i.e. every non-trivial connected set is lower Ahlfors 1-regular. When $s > 1$, a non-trivial $(1/s)$-Hölder curve is not necessarily lower Ahlfors s-regular, and, in fact, could have zero $\mathcal{H}^s$ measure. Nevertheless, Martín and Mattila [37] proved that if Γ is a $(1/s)$-Hölder curve in $\mathbb{R}^N$ with $\mathcal{H}^s(\Gamma) > 0$, then

$$\liminf_{r \downarrow 0} \frac{\mathcal{H}^s(\Gamma \cap B(x,r))}{r^s} > 0 \quad \text{at } \mathcal{H}^s\text{-a.e. } x \in \Gamma.$$

Even if it can be weakened, the lower regularity hypothesis in Proposition 1.3 cannot be completely dropped: In §9.4, for any $s > 1$ and $\beta_1 \in (0,1)$, we find a curve $E \subset \mathbb{R}^N$ with $\mathcal{H}^s(E) < \infty$ satisfying (1.6) such that E is not contained in a $(1/s)$-Hölder curve.

Sharp estimates on the Minkowski dimension of sets satisfying (1.6) were provided by Mattila and Vuorinen [40]; for generalized *Mattila-Vuorinen type sets*, see [7].

1.4. Related work

As noted above, one motivation for this paper is to develop tools to analyze the structure of Radon measures. See §7 for background and for an application of Corollary 1.2 to the fractional rectifiability of measures.

There is considerable interest in finding higher-dimensional analogues of the Analyst's Traveling Salesman theorem, for example finding a characterization of subsets of Lipschitz images of $[0,1]^2$. This problem is still open, but some positive steps were recently taken by Azzam and Schul [4] for *Hausdorff content lower regular sets*. Also see [52].

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Part I. Proof of the Hölder Traveling Salesman theorem

In the first part of the paper, §§2–6, we establish several Hölder Traveling Salesman theorems, including Theorem 1.1 and Theorem 5.1. To start, in §2, we introduce notation and essential concepts used in the proof, including nets, flat pairs, and variation excess. In §3, we present a refined version of Jones' Traveling Salesman construction, which takes a nested sequence $(V_k)_{k=0}^{\infty}$ of $\rho_k r_0$-separated sets, approximating lines $\{\ell_{k,v} : v \in V_k\}_{k=0}^{\infty}$, and associated errors $\{\alpha_{k,v} : v \in V_k\}_{k=0}^{\infty}$ and outputs a sequence of partitions $\mathscr{I}_k$ of $[0,1]$ and piecewise linear maps f_k such that $f_k([0,1]) \supset V_k$. In §4, we define and estimate a discrete s-variation of the maps f_k, which is adapted to the partitions $\mathscr{I}_k$ of the domain. When $s > 1$, the total s-mass $\mathcal{M}_s([0,1])$ associated to the sequence of maps f_k fills the role that 1-dimensional Hausdorff measure $\mathcal{H}^1$ plays in Jones' proof of the Analyst's TST. In §5, we use the algorithm of §3 and the mass estimates of §4 to prove our main theorem (see Theorem 5.1). Finally, in §6, we use our method to obtain a stronger version of the sufficient half of the Analyst's Traveling Salesman theorem. The construction presented below can be carried out in any finite or infinite-dimensional Hilbert space.

2. Preliminaries

Given numbers $x, y \geq 0$ and parameters $a_1, \ldots, a_n$, we may write $x \lesssim_{a_1,\ldots,a_n} y$ if there exists a positive and finite constant C depending on at most $a_1, \ldots, a_n$ such that $x \leq Cy$. We write $x \simeq_{a_1,\ldots,a_n} y$ to denote $x \lesssim_{a_1,\ldots,a_n} y$ and $y \lesssim_{a_1,\ldots,a_n} x$. Similarly, we write $x \lesssim y$ or $x \simeq y$ to denote that the implicit constants are universal.

2.1. Ordering flat sets

The following lemma shows that if a discrete set is sufficiently flat at the scale of separation, then there exists a natural linear ordering of its points. Estimates (2.1) and (2.2) are consequences of the Pythagorean theorem.

Lemma 2.1 *([9, Lemma 8.3]). Suppose that $V \subset \mathbb{R}^N$ is a 1-separated set with $\operatorname{card}(V) \geq 2$ and there exist lines ℓ_1 and ℓ_2 and a number $\alpha \in (0, 1/16]$ such that*

$$\operatorname{dist}(v, \ell_i) \leq \alpha \qquad \text{for all } v \in V \text{ and } i = 1, 2.$$

Let π_i denote the orthogonal projection onto ℓ_i. There exist compatible identifications of ℓ_1 and ℓ_2 with $\mathbb{R}$ such that $\pi_1(v) \leq \pi_1(v')$ if and only if $\pi_2(v) \leq \pi_2(v')$ for all $v, v' \in V$. If v_1 and v_2 are consecutive points in V relative to the ordering of $\pi_1(V)$, then

$$\mathcal{H}^1([u_1, u_2]) \leq (1 + 3\alpha^2) \cdot \mathcal{H}^1([\pi_1(u_1), \pi_1(u_2)]) \quad \text{for all } [u_1, u_2] \subseteq [v_1, v_2]. \tag{2.1}$$

Moreover,

$$\mathcal{H}^1([y_1, y_2]) \leq (1 + 12\alpha^2) \cdot \mathcal{H}^1([\pi_1(y_1), \pi_1(y_2)]) \quad \text{for all } [y_1, y_2] \subseteq \ell_2. \tag{2.2}$$

Suppose that V, ℓ_1, and π_1 are given as in Lemma 2.1 and let $v, v_1, v_2 \in V$. Given an orientation of ℓ (that is, an identification of ℓ_1 with $\mathbb{R}$), we say v_1 *is to the left of* v_2 and v_2 *is to the right of* v_1 if $\pi_1(v_1) < \pi_1(v_2)$. We say v *is between* v_1 *and* v_2 if $\pi_1(v_1) \leq \pi_1(v) \leq \pi_1(v_2)$ or $\pi_1(v_2) \leq \pi_1(v) \leq \pi_1(v_1)$.

Lemma 2.2. *Suppose that $V \subset \mathbb{R}^N$ is a δ-separated set with* $\operatorname{card}(V) \geq 2$ *and there exists a line ℓ and a number $\alpha \in (0, 1/16]$ such that*

$$\operatorname{dist}(v, \ell) \leq \alpha\delta \quad \text{for all } v \in V.$$

Enumerate $V = \{v_1, \ldots, v_n\}$ so that v_{i+1} is to the right of v_i for all $1 \leq i \leq n-1$. Then

$$\sum_{i=1}^{n-1} |v_{i+1} - v_i|^s \leq (1 + 3\alpha^2)^s |v_1 - v_n|^s \quad \text{for all } s \geq 1. \tag{2.3}$$

Moreover, if $\operatorname{card}(V) \geq 3$, *then*

$$\sum_{i=1}^{n-1} |v_{i+1} - v_i|^s \leq ((1 + 3\alpha^2)|v_1 - v_n| - \delta)^s + \delta^s \quad \text{for all } s \geq 1. \tag{2.4}$$

Proof. Let π denote the orthogonal projection onto ℓ and put $x_i := \pi(v_i)$. Then

$$|x_{i+1} - x_i| \leq |v_{i+1} - v_i| \leq (1 + 3\alpha^2)|x_{i+1} - x_i| \quad \text{for all } 1 \leq i \leq n-1,$$

where the first inequality holds since projections are 1-Lipschitz and the second inequality holds by Lemma 2.1. Assume $s \geq 1$ and $\operatorname{card}(V) \geq 3$. Then

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{|v_{i+1}-v_i|^s}{(1+3\alpha^2)^s} &\le \left(\sum_{i=1}^{n-2}|x_{i+1}-x_i|^s\right)+|x_n-x_{n-1}|^s\\ &\le \left(\sum_{i=1}^{n-2}|x_{i+1}-x_i|\right)^s+|x_n-x_{n-1}|^s\\ &= (|x_n-x_1|-|x_n-x_{n-1}|)^s+|x_n-x_{n-1}|^s\\ &\le \left(|x_n-x_1|-\frac{\delta}{1+3\alpha^2}\right)^s+\left(\frac{\delta}{1+3\alpha^2}\right)^s\\ &\le \left(|v_n-v_1|-\frac{\delta}{1+3\alpha^2}\right)^s+\left(\frac{\delta}{1+3\alpha^2}\right)^s,\end{aligned}$$

where the penultimate inequality holds because for any $M>0$, $\epsilon\in(0,M)$, and $s\ge 1$, the function $f(t)=t^s+(M-t)^s$ defined on $[\epsilon,M-\epsilon]$ attains its maximum at $t=\epsilon$. This establishes (2.4). Inequality (2.3) follows from a similar (and easier) computation. □

2.2. Nets, flat pairs, and variation excess

Let $(X,|\cdot|)$ denote the Hilbert space $l^2(\mathbb{R})$ of square summable sequences or the Euclidean space $\mathbb{R}^N$ for some $N\ge 2$.

Let $\mathscr{V}=\{(V_k,\rho_k)\}_{k\ge 0}$ be a sequence of pairs of nonempty finite sets V_k in X and numbers $\rho_k>0$. Assume that there exist $x_0\in X$, $r_0>0$, $C^*\ge 1$, and $0<\xi_1\le\xi_2<1$ such that $\mathscr{V}$ satisfies the following properties.

(**V0**) When $k=0$, we have $\rho_0=1$. For all $k\ge 0$, we have $\xi_1\rho_k\le\rho_{k+1}\le\xi_2\rho_k$.
(**V1**) When $k=0$, we have $V_0\subset B(x_0,C^*r_0)$.
(**V2**) For all $k\ge 0$, we have $V_k\subset V_{k+1}$.
(**V3**) For all $k\ge 0$ and all distinct $v,v'\in V_k$, we have $|v-v'|\ge\rho_k r_0$.
(**V4**) For all $k\ge 0$ and all $v\in V_{k+1}$, there exists $v'\in V_k$ such that $|v-v'|<C^*\rho_{k+1}r_0$.

With C^* and ξ_2 given, define the associated parameter

$$A^*:=\frac{C^*}{1-\xi_2}>C^*.$$

In addition to (V0)–(V4), assume that for each $k\ge 0$ and $v\in V_k$ we are given a number $\alpha_{k,v}\ge 0$ and a straight line $\ell_{k,v}$ in X such that

$$\sup_{x\in V_{k+1}\cap B(v,30A^*\rho_k r_0)}\operatorname{dist}(x,\ell_{k,v})\le\alpha_{k,v}\rho_{k+1}r_0. \tag{**V5**}$$

We call the line $\ell_{k,v}$ an *approximating line at* (k,v).

The formulation of (V5) is motivated by [9, Proposition 3.6]. We remark that the number $\rho_{k+1}r_0$ appearing on the right hand side of (V5) is the scale of separation of

points in V_{k+1}. While we allow $X = l^2(\mathbb{R})$, each V_k can be identified with a subset of $\mathbb{R}^{N_k}$ for some increasing sequence N_k if convenient, because each V_k is finite and $V_k \subset V_{k+1}$.

Lemma 2.3. *Let $v \in V_k$ be such that $\alpha_{k,v} \leq 1/16$ and fix an orientation for $\ell_{k,v}$.*

(1) *If $v' \in V_k \cap B(v, 14A^*\rho_k r_0)$ is the first point to the right of v, then there exist fewer than $2 + 2.2C^*$ points of V_{k+1} between v and v' (inclusive).*
(2) *There exist fewer than $1.1C^*$ points of $V_{k+1} \cap B(v, C^*\rho_{k+1} r_0)$ to the right of v.*

Proof. The points in $V_{k+1} \cap B(v, 30A^*\rho_k r_0)$ are $\rho_{k+1} r_0$-separated and are linearly ordered by Lemma 2.1. Let v' be the first point in $V_k \cap B(v, 14A^*\rho_k r_0)$ to the right of v and let $w_1, \ldots, w_m$ denote the points in V_{k+1} that lie between v and v' (inclusive). By (V4), each point w_i belongs to $B(v, C^*\rho_{k+1} r_0) \cup B(v', C^*\rho_{k+1} r_0)$. Let $\pi_{k,v}$ denote the orthogonal projection onto $\ell_{k,v}$. By (V3) and (2.1), $1.1|\pi(w_i) - \pi(w_j)| > |w_i - w_j| \geq \rho_{k+1} r_0$ for all distinct i, j, since $(1 + 3(1/16)^2) < 1.1$. It follows that there are fewer than $1.1C^*$ points w_i in $V_{k+1} \cap B(v, C^*\rho_{k+1} r_0)$ to the right of v and fewer than $1.1C^*$ points w_i in $V_{k+1} \cap B(v', C^*\rho_{k+1} r_0)$ to the left of v'. The first claim follows. A similar argument gives the second claim. □

Definition 2.4 *(Flat pairs).* Fix a parameter $\alpha_0 \in (0, 1/16]$. For all $k \geq 0$, define $\mathsf{Flat}(k)$ to be the set of pairs $(v, v') \in V_k \times V_k$ such that

(1) $\rho_k r_0 \leq |v - v'| < 14A^*\rho_k r_0$,
(2) $\alpha_{k,v} < \alpha_0$ and v' is the first point in $V_k \cap B(v, 14A^*\rho_k r_0)$ to the left or to the right of v with respect to ordering induced by $\ell_{k,v}$.

Define the corresponding set of geometric line segments $\mathscr{L}_k = \{[v, v'] : (v, v') \in \mathsf{Flat}(k)\}$.

Note that the collection $\mathsf{Flat}(k)$ of flat pairs is not symmetric in the sense that $(v, v') \in \mathsf{Flat}(k)$ does imply $(v', v) \in \mathsf{Flat}(k)$, because $\alpha_{k,v}$ does not control $\alpha_{k,v'}$.

Lemma 2.5. *Let e_1, e_2, e_3 be distinct elements of $\mathscr{L}_k$ for some $k \geq 0$.*

(1) *Edges e_1 and e_2 intersect at most in a common endpoint.*
(2) *Edges e_1, e_2 and e_3 do not have a common point.*

Proof. Let $e_1 = [v_1, v_1']$, $e_2 = [v_2, v_2']$, and $e_3 = [v_3, v_3']$ represent distinct elements of $\mathscr{L}_k$, where $(v_i, v_i') \in \mathsf{Flat}(k)$ for all $i \in \{1, 2, 3\}$. If two or more of the edges intersect in a common point, say $\{e_i : i \in I_0\}$ for some $I_0 \subset \{1, 2, 3\}$ with $\mathrm{card}(I_0) \geq 2$, then those edges are contained in $B(v_j, 30A^*\rho_k r_0)$ for each $j \in I_0$, since each edge has diameter at most $14A^*\rho_k r_0$. Note that V_k is a $\rho_k r_0$ separated set, $\mathrm{dist}(v_i, \ell_{k,v_j}) \leq \alpha_{k,v_j}\rho_{k+1} r_0 <$

$\alpha_{k,v_j}\rho_k r_0$ for all $i,j \in I_0$, and $\alpha_{k,v_j} \leq 1/16$. Thus, by Lemma 2.1, the vertices $\{v_i, v_i' : i \in I_0\}$ are consistently linearly ordered according to their projections onto ℓ_{k,v_j} for each $j \in I_0$. Claims (1) and (2) follow immediately, since the segments in $\mathscr{L}_k$ emanating from a vertex $v \in V_k$ with $\alpha_{k,v} < \alpha_0$ are only drawn to the first vertex in $V_k \cap B(v, 14A^*\rho_k r_0)$ to the left or right of v with respect to the projection onto $\ell_{k,v}$. □

Given a pair $(v, v') \in \mathsf{Flat}(k)$, let $V_{k+1}(v, v')$ denote the set of all points $x \in V_{k+1} \cap B(v, 14A^*\rho_k r_0)$ such that x lies between v and v' (including v and v').

Definition 2.6 *(Variation excess).* For all $s \geq 1$, for all $k \geq 0$, and for all $(v, v') \in \mathsf{Flat}(k)$, define the *$s$-variation excess* $\tau_s(k, v, v')$ by

$$\tau_s(k,v,v')|v-v'|^s = \max\left\{\left(\sum_{i=1}^{n-1}|v_{i+1}-v_i|^s\right) - |v-v'|^s, 0\right\},$$

where $V_{k+1}(v, v') = \{v_1, \ldots, v_n\}$ with $v_1 = v$, $v_n = v'$, and v_{i+1} is the first point to the right (or left) of v_i for all $1 \leq i \leq n-1$.

Lemma 2.7. *For all $k \geq 0$ and $(v, v') \in \mathsf{Flat}(k)$, we have $\tau_1(k, v, v') \leq 3\alpha_{k,v}^2$.*

Proof. Let $V_{k+1}(v, v') = \{v_1, \ldots, v_n\}$, where $v_1 = v$, $v_n = v'$, and v_{i+1} is to the right of v_i for all $1 \leq i \leq n-1$. By Lemma 2.2, with $s = 1$,

$$\sum_{i=1}^{n-1}|v_{i+1}-v_i| \leq (1+3\alpha_{k,v}^2)|v_n - v_1| = (1+3\alpha_{k,v}^2)|v-v'|.$$

Rearranging the inequality gives $\tau_1(k, v, v') \leq 3\alpha_{k,v}^2$. □

We now demonstrate that when $s > 1$, the variation excess $\tau_s(k, v, v')$ is zero whenever the set $V_{k+1}(v, v')$ lies in a sufficiently thin tube.

Lemma 2.8 *(Tube control). For all $s > 1$, there exists $\epsilon_{s,C^*,\xi_1,\xi_2} \in (0, 1/16]$ such that if $\alpha_{k,v} \leq \epsilon_{s,C^*,\xi_1,\xi_2}$, then $\tau_s(k, v, v') = 0$ for all $(v, v') \in \mathsf{Flat}(k)$.*

Proof. Let $(v, v') \in \mathsf{Flat}(k)$ and enumerate $V_{k+1}(v, v') = \{v_1, \ldots, v_n\}$ so that $v_1 = v$, $v_n = v'$, and v_{i+1} is to the right of v_i for all $1 \leq i \leq n-1$. If $n = 2$, then $\sum_{i=1}^{n-1}|v_{i+1} - v_i|^s = |v-v'|^s$. Thus, suppose that $n \geq 3$. By Lemma 2.2, with $\delta = \rho_{k+1}r_0$ and $\alpha = \alpha_{k,v}$,

$$\begin{aligned}\sum_{i=1}^{n-1}|v_{i+1}-v_i|^s &\leq |v-v'|^s\left[\left((1+3\alpha_{k,v}^2) - \frac{\rho_{k+1}r_0}{|v-v'|}\right)^s + \left(\frac{\rho_{k+1}r_0}{|v-v'|}\right)^s\right] \\ &\leq |v-v'|^s\left[\left((1+3\alpha_{k,v}^2) - \frac{\xi_1}{14A^*}\right)^s + \left(\frac{\xi_1}{14A^*}\right)^s\right] =: A_{\alpha_{k,v}}|v-v'|^s,\end{aligned}$$

because $\xi_2^{-1}\rho_{k+1}r_0 \le |v-v'| \le 14A^*\xi_1^{-1}\rho_{k+1}r_0$ and the function

$$f(t) = ((1+3\alpha_{k,v}^2)-t)^s + t^s \quad \text{on } [\xi_1/14A^*, \xi_2]$$

takes its maximum at $t = \xi_1/14A^*$. Since $s > 1$, the coefficient

$$A_\epsilon \to \left(1 - \frac{\xi_1}{14A^*}\right)^s + \left(\frac{\xi_1}{14A^*}\right)^s < 1 \quad \text{as } \epsilon \to 0.$$

Thus, by continuity, there exists $\epsilon' > 0$ such that $A_{\epsilon'} = 1$. Let $\epsilon_{s,C^*,\xi_1,\xi_2} = \min\{\epsilon', 1/16\}$. Then $\sum_{i=1}^{n-1} |v_{i+1}-v_i|^s \le |v-v'|^s$ whenever $\alpha_{k,v} \le \epsilon_{s,C^*,\xi_1}$. □

3. Traveling Salesman algorithm

For the rest of §§3 and 4, let X denote the Hilbert space $l^2(\mathbb{R})$ or $\mathbb{R}^N$, let $(V_k)_{k\ge 0}$ be a sequence of sets in X and let $\rho_k > 0$ be a sequence of numbers satisfying (V0)–(V5) defined in §2.2. In addition, fix the parameter $\alpha_0 \in (0, 1/16]$ in Definition 2.4. For each integer $k \ge 0$, we will construct

(1) two collections of pairwise disjoint, open intervals in $[0,1]$ denoted by $\mathscr{B}_k$ (called "bridge intervals") and $\mathscr{E}_k$ ("edge intervals"),
(2) two collections of pairwise disjoint, nondegenerate closed intervals in $[0,1]$ denoted by $\mathscr{F}_k$ ("frozen point intervals") and $\mathscr{N}_k$ ("non-frozen point intervals"), and
(3) a continuous map $f_k : [0,1] \to X$

that satisfy the following properties.

(P1) The four collections $\mathscr{B}_k$, $\mathscr{E}_k$, $\mathscr{F}_k$, $\mathscr{N}_k$ are mutually disjoint and for any $x \in [0,1]$ there exists unique interval I contained in their union such that $x \in I$.
(P2) The map $f_k|I$ is affine on each $I \in \mathscr{E}_k \cup \mathscr{B}_k$ and the map $f_k|J$ is constant on each $J \in \mathscr{F}_k \cup \mathscr{N}_k$.
(P3) For all $I \in \mathscr{E}_k$, we have $\operatorname{diam} f_k(I) < 14A^*\rho_k r_0$.
(P4) The map $f_k|\bigcup \mathscr{E}_k$ is 2-to-1; that is, for every $x \in \bigcup \mathscr{E}_k$ there exists a unique $x' \in \bigcup \mathscr{E}_k \setminus \{x\}$ such that $f_k(x) = f_k(x')$.
(P5) If $(v,v') \in \mathsf{Flat}(k)$, then there exists $I \in \mathscr{E}_k$ such that $f_k(I)$ joins v with v'. Conversely, if a and b are endpoints of an interval $I \in \mathscr{E}_k$ and $\alpha_{k,f_k(a)} < \alpha_0$, then $(f_k(a), f_k(b)) \in \mathsf{Flat}(k)$.
(P6) For each $I \in \mathscr{F}_k \cup \mathscr{N}_k$, the image $f_k(I) \in V_k$ and for each $v \in V_k$ there exists a unique $I \in \mathscr{N}_k$ such that $f_k(I) = v$.
(P7) If $J \in \mathscr{N}_k$ is such that $f_k(J)$ is an endpoint of $f_k(I)$ for some $I \in \mathscr{E}_k$, then there exists $I' \in \mathscr{E}_k$ (possibly $I' = I$) such that $f_k(I') = f_k(I)$ and $J \cap \overline{I'} \ne \emptyset$.

Lemma 3.1. *Assume (P5) holds at stage $k \geq 0$. Let $a < b < a' < b'$ be such that (a, b) and (a', b') belong to $\mathscr{E}_k$, $f_k(b) = f_k(a')$, and $\alpha_{k,f_k(b)} \leq \alpha_0$. Then either*

(1) *$f_k((a, b)) = f_k((a', b'))$ or*
(2) *$f_k(b)$ lies between $f_k(a)$ and $f_k(b')$ with respect to order induced by $\ell_{k,f_k(b)}$.*

Proof. This is an immediate consequence of (P5) and Lemma 2.5. □

In §3.1, we construct $\mathscr{E}_0$, $\mathscr{B}_0$, $\mathscr{N}_0$, $\mathscr{F}_0$, and f_0. In §3.2, we formulate the inductive hypothesis. We construct the collections of intervals $\mathscr{E}_{k+1}$, $\mathscr{B}_{k+1}$, $\mathscr{N}_{k+1}$, $\mathscr{F}_{k+1}$, and f_{k+1} in §§3.3–3.8. We verify properties (P1)–(P7) in §3.9. Finally, in §3.10, we review all choices in the algorithm.

3.1. Step 0

Fix a point $v_0 \in V_0$. Let G_0 be the (not necessarily connected) graph with vertices V_0 and edges $\mathscr{L}_0$. Suppose that G_0 has components $G_0^{(1)}, \dots, G_0^{(l)}$ with $v_0 \in G_0^{(1)}$.

Case 1. Suppose that $V_0 = \{v_0\}$. Then set $\mathscr{E}_0 = \emptyset$, $\mathscr{B}_0 = \emptyset$, $\mathscr{N}_0 = \{[0, 1]\}$ and $\mathscr{F}_0 = \emptyset$. Define also $f_0 : [0, 1] \to X$ with $f_0(x) = v_0$ for all $x \in [0, 1]$. Note that properties (P1)–(P7) are trivial in this case.

Case 2. Suppose that $\operatorname{card}(V_0) \geq 2$ and that $l = 1$, that is, G_0 is connected. We apply Proposition A.1 for v_0 with $\Delta = [0, 1]$, $G = G_0$ and we obtain a collection of intervals $\mathcal{I}$ and a continuous map g. By Lemma 2.5, each point $v \in V_0$ has valence at most 2 in G_0 and there exists a component J_v of $g^{-1}(v)$ such that if e is an edge of G_0 that contains v as an endpoint, then e has a preimage $I \in \mathcal{I}$ such that $\overline{I} \cap J_v \neq \emptyset$. Let $\mathcal{N}$ be the collection of all such intervals J_v.

Set $\mathscr{E}_0 = \mathcal{I}$, $\mathscr{B}_0 = \emptyset$, $\mathscr{N}_0 = \mathcal{N}$, define $\mathscr{F}_0$ to be the components of $[0, 1] \setminus \bigcup(\mathscr{E}_0 \cup \mathscr{B}_0 \cup \mathscr{N}_0)$, and let $f_0 = g$. Properties (P1)–(P7) follow from Proposition A.1.

Case 3. Suppose that $\operatorname{card}(V_0) \geq 2$ and that $l \geq 2$, that is, G_0 is disconnected. For each $j = 2, \dots, l$ fix a vertex u_j of $G_0^{(j)}$. Let $\{I_1, \dots, I_{2l-2}\}$ be a collection of open intervals, enumerated according to the orientation of $[0, 1]$, such that their closures are mutually disjoint and are contained in the interior of $[0, 1]$. Let also $\{J_1, \dots, J_{2l-1}\}$ be the components of $I \setminus \bigcup_{j=1}^{2l-2} I_j$ enumerated according to the orientation of $[0, 1]$. Applying Proposition A.1 for $G = G_0^{(1)}$, v_0 and $\Delta = J_1$, we obtain a family of open intervals $\mathcal{I}_1$, a map $g_1 : J_1 \to G_0^{(1)}$ and a family $\mathcal{N}_1$ of closed intervals. Similarly, for each $j = 2, \dots, l$, applying Proposition A.1 for $G = G_0^{(j)}$, u_j and $\Delta = J_{2j}$, we obtain a family of open intervals $\mathcal{I}_j$, a map $g_j : J_{2j} \to G_0^{(j)}$ and a family $\mathcal{N}_j$ of closed intervals. There exists a continuous map $g : [0, 1] \to X$ that extends the maps g_j such that

(1) $g(J_{2j+1}) = v_0$ for each $j \in \{1, \dots, l - 1\}$;
(2) $g|I_j$ is affine for each $j \in \{1, \dots, 2l - 2\}$ and $g(I_{2j-1}) = g(I_{2j}) = [u_j, v_0]$ for each $j \in \{1, \dots, l - 1\}$.

Set $\mathscr{E}_0 = \bigcup_{j=1}^{l} \mathcal{I}_j$, $\mathscr{B}_0 = \{I_1, \dots, I_{2l-2}\}$, $\mathscr{N}_0 = \bigcup_{j=1}^{l} \mathcal{N}_j$, define $\mathscr{F}_0$ to be the components of $[0,1] \setminus \bigcup(\mathscr{E}_0 \cup \mathscr{B}_0 \cup \mathscr{N}_0)$, and let $f_0|[0,1] = g$. Properties (P1)–(P7) follow from Proposition A.1.

3.2. Inductive hypothesis

Suppose that for some $k \geq 0$ we have defined collections $\mathscr{B}_k$, $\mathscr{E}_k$ of open intervals in $[0,1]$, collections $\mathscr{F}_k$, $\mathscr{N}_k$ of nondegenerate closed intervals in $[0,1]$, and a continuous map $f_k : [0,1] \to X$, which satisfy properties (P1)–(P7).

We will define a new map $f_{k+1} : [0,1] \to X$ and new collections $\mathscr{B}_{k+1}, \mathscr{E}_{k+1}, \mathscr{F}_{k+1}, \mathscr{N}_{k+1}$,

$$\mathscr{B}_{k+1} = \bigcup_{I \in \mathscr{B}_k \cup \mathscr{E}_k \cup \mathscr{F}_k \cup \mathscr{N}_k} \mathscr{B}_{k+1}(I), \qquad \mathscr{E}_{k+1} = \bigcup_{I \in \mathscr{B}_k \cup \mathscr{E}_k \cup \mathscr{F}_k \cup \mathscr{N}_k} \mathscr{E}_{k+1}(I),$$

$$\mathscr{F}_{k+1} = \bigcup_{I \in \mathscr{B}_k \cup \mathscr{E}_k \cup \mathscr{F}_k \cup \mathscr{N}_k} \mathscr{F}_{k+1}(I), \qquad \mathscr{N}_{k+1} = \bigcup_{I \in \mathscr{B}_k \cup \mathscr{E}_k \cup \mathscr{F}_k \cup \mathscr{N}_k} \mathscr{N}_{k+1}(I),$$

where $\mathscr{B}_{k+1}(I)$, $\mathscr{E}_{k+1}(I)$, $\mathscr{F}_{k+1}(I)$, $\mathscr{N}_{k+1}(I)$ are collections of intervals in I that we define below. In particular:

- In §3.3, we define the four collections and $f_{k+1}|I$ for $I \in \mathscr{B}_k$.
- In §3.4 and §3.5, we define the four collections and $f_{k+1}|I$ for $I \in \mathscr{E}_k$.
- In §3.6, we define the four collections and $f_{k+1}|I$ for $I \in \mathscr{F}_k$.
- In §3.7 and §3.8, we the four collections and $f_{k+1}|I$ for $I \in \mathscr{N}_k$.

3.3. Step $k+1$: intervals in $\mathscr{B}_k$

For any $I \in \mathscr{B}_k$ we set $\mathscr{B}_{k+1}(I) = \{I\}$, $\mathscr{E}_{k+1}(I) = \emptyset$, $\mathscr{F}_{k+1}(I) = \emptyset$, $\mathscr{N}_{k+1}(I) = \emptyset$, and we define $f_{k+1}|I = f_k|I$. In other words, bridge intervals are frozen and we make no changes on them.

3.4. Step $k+1$: intervals in $\mathscr{E}_k$ with a at least one endpoint with flat image

Here we consider those intervals $I = (a_I, b_I) \in \mathscr{E}_k$ such that one of the $\alpha_{k,f_k(a_I)}$, $\alpha_{k,f_k(b_I)}$ is less than α_0. If no such interval exists, we move to §3.5. Assume now that such intervals exist. By (P4) and the induction step, such intervals come in pairs $\{I, I'\}$ where $f_k(I) = f_k(I')$ and $f_k(I) \cap f_k(J) = \emptyset$ for all $J \in \mathscr{E}_k \setminus \{I, I'\}$. Fix now such a pair $\{I, I'\}$. We choose one of the two intervals I, I' to start with, say I.

Without loss of generality, assume that $\alpha_{k,f_k(a_I)} < \alpha_0$. Let ℓ be the approximating line for $(k, f_k(a_I))$, oriented so that $f_k(a_I)$ lies to the left of $f_k(b_I)$. Let $V_{k+1,I}$ denote the points in $V_{k+1} \cap B(f_k(a_I), 14A^*\rho_k r_0)$ that lie between $f_k(a_I)$ and $f_k(b_I)$ with respect to ℓ, including $f_k(a_I)$ and $f_k(b_I)$. Enumerate $V_{k+1,I}$ from left to right,

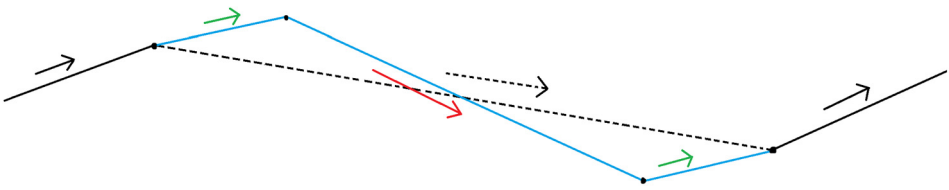

Fig. 1. The image of $f_{k+1}(I)$: black arrows denote images of intervals in $\mathscr{E}_k \cup \mathscr{B}_k$, green arrows denote images of intervals in $\mathscr{E}_{k+1}$, and red arrows denote images of intervals in $\mathscr{B}_{k+1}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$V_{k+1,I} = \{v_1, \dots, v_l\}.$$

That is, v_i lies to the left of v_{i+1} for all $i \in \{1, \dots, l-1\}$, $v_1 = f_k(a_I)$, and $v_l = f_k(b_I)$.

Remark 3.2. By Lemma 2.3, we have that $l < 2 + 2.2C^*$.

Let $\{I_1, \dots, I_{l-1}\}$ be a collection of open intervals in I with mutually disjoint closures, enumerated according to the orientation of $[0,1]$ so that the left endpoint of I_1 coincides with a_I and the right endpoint of I_{l-1} coincides with b_I. Let $\mathscr{N}_{k+1}(I)$ be the components of $[0,1] \setminus \bigcup_{i=1}^{l-1} I_i$ and $\mathscr{F}_{k+1}(I) = \emptyset$. Define

$$\mathscr{E}_{k+1}(I) = \{I_i : |v_i - v_{i+1}| < 14A^*\rho_{k+1}r_0\},$$
$$\mathscr{B}_{k+1}(I) = \{I_i : |v_i - v_{i+1}| \geq 14A^*\rho_{k+1}r_0\}.$$

Then define $f_{k+1}|\overline{I}$ continuously so that

(1) f_{k+1} is affine on each $J \in \mathscr{E}_{k+1}(I) \cup \mathscr{B}_{k+1}(I)$ and constant on each $J \in \mathscr{N}_{k+1}(I) \cup \mathscr{F}_{k+1}(I)$;
(2) for each $j = 1, \dots, l-1$, $f_{k+1}(\overline{I_j}) = [v_j, v_{j+1}]$ mapping the left endpoint of I_j onto v_j and the right endpoint of I_j onto v_{j+1}.

See Fig. 1 for the image of $f_{k+1}(I)$.

Once we have defined the four families and f_{k+1} for I, we work as follows for I'. First note that $V_{k+1,I} = V_{k+1,I'}$. Define $\psi_{I',I} : I' \to I$ to be the unique orientation-reversing linear map between I' and I. Define

$$\mathscr{E}_{k+1}(I') = \{\psi_{I',I}(J) : J \in \mathscr{E}_{k+1}(I)\} \quad \text{and} \quad \mathscr{B}_{k+1}(I') = \{\psi_{I',I}(J) : J \in \mathscr{B}_{k+1}(I)\}$$

This time, however, we set $\mathscr{F}_{k+1}(I')$ to be the components of $I' \setminus \bigcup_{i=1}^{l-1} I'_i$ and $\mathscr{N}_{k+1}(I') = \emptyset$. Define also $f_{k+1}|\overline{I'}$ continuously so that $f_{k+1}|I' = (f_{k+1}|I) \circ \psi_{I',I}$.

Lemma 3.3. *For $i = 1, 2$, let $I_i \in \mathscr{E}_k$ be an interval with at least one endpoint having flat image and let $I'_i \in \mathscr{E}_{k+1}(I_i)$. If $f_k(I_1) \neq f_k(I_2)$, then $f_{k+1}(I'_1) \cap f_{k+1}(I'_2) = \emptyset$.*

Proof. Suppose that $I_1' \in \mathscr{E}_{k+1}(I_1)$, $I_2' \in \mathscr{E}_{k+1}(I_2)$, and $f_{k+1}(I_1') \cap f_{k+1}(I_2') \neq \emptyset$. Because $f_{k+1}(I_1')$ and $f_{k+1}(I_2')$ do not include their endpoints (since intervals in $\mathscr{E}_{k+1}$ are open), we conclude that $f_{k+1}(I_1') = f_{k+1}(I_2')$ by Lemma 2.5. Now, the endpoints of the four intervals $f_k(I_1)$, $f_k(I_2)$, $f_{k+1}(I_1')$, and $f_{k+1}(I_2')$ lie in the $30A^*\rho_k r_0$ neighborhood of any flat endpoint of $f_k(I_1)$ or $f_k(I_2)$. In particular, the endpoints of the four intervals are linearly ordered by Lemma 2.1, and the endpoints of $f_{k+1}(I_i')$ lie between the endpoints of $f_k(I_i)$ by the construction of $\mathscr{E}_{k+1}(I_i)$. Because $f_{k+1}(I_1') = f_{k+1}(I_2')$, this forces $f_k(I_1) = f_k(I_2)$. □

3.5. Step $k+1$: intervals in $\mathscr{E}_k$ with no endpoints with flat image

Suppose that $I = (a_I, b_I) \in \mathscr{E}_k$ is such that $\alpha_{k,f_k(a_I)} \geq \alpha_0$ and $\alpha_{k,f_k(b_I)} \geq \alpha_0$. Then set $\mathscr{E}_{k+1}(I) = \emptyset$, $\mathscr{B}_{k+1}(I) = \{I\}$, $\mathscr{N}_{k+1}(I) = \emptyset$, $\mathscr{F}_{k+1}(I) = \emptyset$, and $f_{k+1}|I = f_k|I$. In other words, edge intervals with no endpoints with flat image become bridge intervals and remain bridge intervals for the rest of the construction.

3.6. Step $k+1$: intervals in $\mathscr{F}_k$

For any $I \in \mathscr{F}_k$ we set $\mathscr{E}_{k+1}(I) = \emptyset$, $\mathscr{B}_{k+1}(I) = \emptyset$, $\mathscr{F}_{k+1}(I) = \{I\}$, $\mathscr{N}_{k+1}(I) = \emptyset$ and we set $f_{k+1}|I = f_k|I$. In other words, frozen point intervals in $\mathscr{F}_k$ remain frozen for the rest of the construction.

3.7. Step $k+1$: intervals in $\mathscr{N}_k$ with flat image

We now consider the intervals $I \in \mathscr{N}_k$ for which $\alpha_{k,f_k(I)} < \alpha_0$. If no such interval exists we proceed to §3.8. Assume now that such intervals exist. Let I be such an interval and let ℓ be the approximating line for $(k, f_k(I))$. We consider three cases.

3.7.1. Non-terminal vertices

Suppose that there exist distinct $v, v' \in V_k \setminus \{f_k(I)\}$ such that $(f_k(I), v)$ and $(f_k(I), v')$ are in $\mathsf{Flat}(k)$. By (P5), Lemma 3.1 and the induction step, there exist $J, J' \in \mathcal{E}_k$ such that $f_k(I)$ and v are the endpoints of $f_k(J)$, while $f_k(I)$ and v' are the endpoints of $f_k(J')$. Hence, all points of V_{k+1} between v and v' are contained in the image of $f_{k+1}(J \cup J')$ defined in §3.4. Set $\mathscr{E}_{k+1}(I) = \emptyset$, $\mathscr{B}_{k+1}(I) = \emptyset$, $\mathscr{N}_{k+1}(I) = \{I\}$, $\mathscr{F}_{k+1}(I) = \emptyset$ and $f_{k+1}|I = f_k|I$.

3.7.2. 1-sided terminal vertices

Suppose that there exists unique $v \in V_k \setminus \{f_k(I)\}$ such that $(f_k(I), v) \in \mathsf{Flat}(k)$. Fix an orientation for ℓ so that v lies to the left of $f_k(I)$. As in §3.7.1, the points of V_{k+1} that lie between $f_k(I)$ and v are all contained in $f_{k+1}(J)$ for some $J \in \mathscr{E}_k$. Let $V_{k+1,I}$ denote the set that includes $f_k(I)$ and all points in $V_{k+1} \cap B(f_k(I), C^*\rho_{k+1}r_0)$ that lie to the right of $f_k(I)$. Enumerate $V_{k+1,I} = \{v_1, \ldots, v_l\}$ from left to right. That is, $v_1 = f_k(I)$ and v_l is the rightmost point of $V_{k+1,I}$

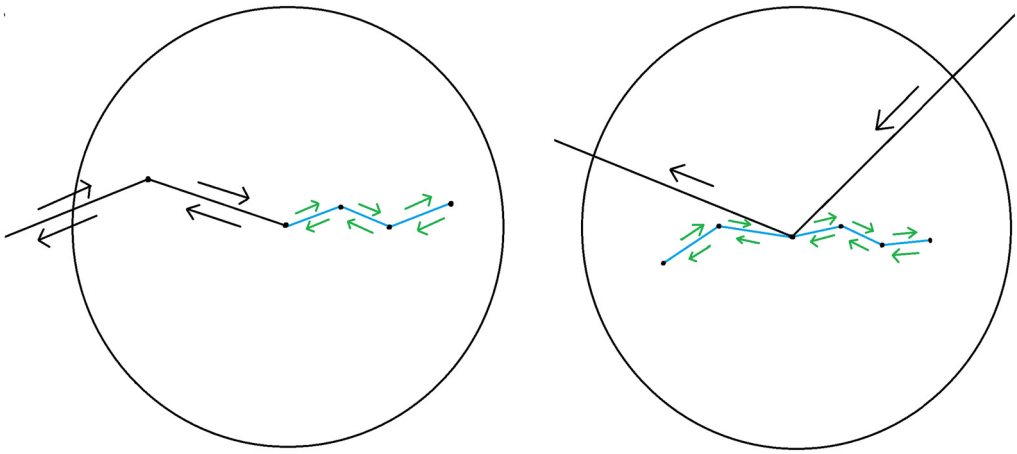

Fig. 2. The image of $f_{k+1}(I)$: on the left, we have $f_{k+1}(I)$, where I is as in §3.7.2; on the right, we have $f_{k+1}(I)$, where I is as in §3.7.3.

Remark 3.4. $\operatorname{card} V_{k+1,I} \leq 1 + 1.1C^*$ by Lemma 2.3.

If $V_{k+1,I} = \{f_k(I)\}$, then set $\mathscr{E}_{k+1}(I) = \emptyset$, $\mathscr{B}_{k+1}(I) = \emptyset$, $\mathscr{N}_{k+1}(I) = \{I\}$, $\mathscr{F}_{k+1}(I) = \emptyset$ and $f_{k+1}|I = f_k|I$.

If $V_{k+1,I} \neq \{f_k(I)\}$, then let $G_{k+1,I}$ be the graph with vertices the points in $V_{k+1,I}$ and edges the segments $\{[v_1, v_2], \ldots, [v_{l-1}, v_l]\}$. That is, $G_{k+1,I}$ forms a simple polygonal arc to the right of $f_k(I)$ joining $f_k(I)$ with v_l. Let $\mathcal{I}$ and g be the collection and map given by Proposition A.1 for $\Delta = I$, $G = G_{k+1,I}$ and $v = f_k(I)$. For each $v' \in V_{k+1,I}$ fix a component of $I \setminus \bigcup \mathscr{I}_{k+1}(I)$ that is mapped onto v' and let $\mathcal{N}$ be the collection of these components. Set $\mathscr{E}_{k+1}(I) = \mathcal{I}$, $\mathscr{B}_{k+1}(I) = \emptyset$, $\mathscr{N}_{k+1}(I) = \mathcal{N}$, and define $\mathscr{F}_{k+1}(I)$ to be the set of components of $I \setminus \bigcup(\mathscr{E}_{k+1}(I) \cup \mathscr{B}_{k+1}(I) \cup \mathscr{N}_{k+1}(I))$. Set $f_{k+1}|I = g$. See the left half of Fig. 2 for the image of $f_{k+1}(I)$.

3.7.3. 2-sided terminal vertices

Suppose that there exists no point in $V_k \setminus \{f_k(I)\}$ such that $(f_k(I), v) \in \mathsf{Flat}(k)$. That is, $V_k \cap B(f_k(I), 14A^*\rho_k r_0) = \{f_k(I)\}$. Set

$$V_{k+1,I} = V_{k+1} \cap B(f_k(I), C^*\rho_{k+1} r_0).$$

Fix an orientation for ℓ and enumerate $V_{k+1,I} = \{v_1, \ldots, v_l\}$ from left to right.

Remark 3.5. $\operatorname{card}(V_{k+1,I}) \leq 1 + 2.2C^*$ by Lemma 2.3.

If $V_{k+1,I} = \{f_k(I)\}$, then set $\mathscr{E}_{k+1}(I) = \emptyset$, $\mathscr{B}_{k+1}(I) = \emptyset$, $\mathscr{N}_{k+1}(I) = \{I\}$, $\mathscr{F}_{k+1}(I) = \emptyset$ and $f_{k+1}|I = f_k|I$. If $V_{k+1,I} \neq \{f_k(I)\}$, then let $G_{k+1,I}$ be the graph with vertices the points in $V_{k+1,I}$ and edges the segments $\{[v_1, v_2], \ldots, [v_{l-1}, v_l]\}$. The remainder of the construction proceeds in the same way as in §3.7.2. See the right half of Fig. 2 for the image of $f_{k+1}(I)$.

Lemma 3.6. *Let $I_1, I_2 \in \mathscr{N}_k$ be distinct intervals as in §3.7 and let $I_3 \in \mathscr{E}_k$ be an interval as in §3.4. If $I_i' \in \mathscr{E}_{k+1}(I_i)$ for $i = 1, 2, 3$, then the segments $f_{k+1}(I_1')$, $f_{k+1}(I_2')$, and $f_{k+1}(I_3')$ are mutually disjoint.*

Proof. This follows from similar arguments employed in the proof of Lemma 3.3. □

3.8. Step $k+1$: intervals in $\mathscr{N}_k$ with non-flat image

In this final part of the algorithm, we define $\mathscr{E}_{k+1}(I)$, $\mathscr{B}_{k+1}(I)$, $\mathscr{N}_{k+1}(I)$, $\mathscr{F}_{k+1}(I)$ and $f_{k+1}|I$ for those $I \in \mathscr{N}_k$ such that $\alpha_{k,f_k(I)} \geq \alpha_0$. Let $\{I_1, \dots, I_n\}$ be an enumeration of such intervals. The construction in this case resembles that in Step 0.

We start with I_1. Let V_{k+1,I_1} be the set of points in $V_{k+1} \cap B(f_k(I_1), C^*\rho_{k+1}r_0)$ that are not images of some $I \in \mathscr{N}_{k+1}$ defined in §3.4 and §3.7. Let $\mathscr{L}_{k+1,I_1}$ be the set of edges in $\mathscr{L}_{k+1}$ that have an endpoint in V_{k+1,I_1}. Then define $\tilde{V}_{k+1,I_1}$ to be the union of $V_{k+1,I}$ and the set of all endpoints of edges in $\mathscr{L}_{k+1,I_1}$. By the triangle inequality, the set $\tilde{V}_{k+1,I_1}$ is subset of $B(v, 15A^*\rho_{k+1}r_0)$. Finally, let G_{k+1,I_1} denote the graph with vertices $\tilde{V}_{k+1,I_1}$ and with edges $\mathscr{L}_{k+1,I_1}$. We note that the graph G_{k+1,I_1} may be connected or disconnected.

If $\tilde{V}_{k+1,I_1} = \{f_k(I_1)\}$, then we simply set $\mathscr{E}_{k+1}(I_1) = \emptyset$, $\mathscr{B}_{k+1}(I_1) = \emptyset$, $\mathscr{N}_{k+1}(I_1) = \{I_1\}$, $\mathscr{F}_{k+1}(I_1) = \emptyset$ and $f_{k+1}|I_1 = f_k|I_1$.

For the remainder of §3.8, let us assume that $\tilde{V}_{k+1,I_1}$ contains at least two points. Let $G^{(1)}_{k+1,I_1}, \dots, G^{(l_1)}_{k+1,I_1}$ denote the connected components of G_{k+1,I_1}, labeled so that $G^{(1)}_{k+1,I_1}$ is the component containing $f_k(I_1)$. There are two cases.

3.8.1. Connected graph

Suppose that G_{k+1,I_1} is connected. Apply Proposition A.1 for $\Delta = I_1$, $G = G_{k+1,I_1}$ and $v = f_k(I_1)$ to obtain a collection of intervals $\mathcal{I}$ and a continuous map g. If $v' \in V_{k+1,I_1}$, then v' has valence at most 2 by Lemma 2.5. Hence, by Proposition A.1, there exists a component $J_{v'}$ of $g^{-1}(v')$ with the following property:

> If v' is the endpoint of some $e \in \mathscr{L}_{k+1,I_1}$, then there exists $I \in \mathcal{I}$ such that $g(I) = e$ and $\overline{I} \cap J_{v'} \neq \emptyset$.

Let $\mathcal{N}$ be the collection of the fixed intervals $J_{v'}$ where $v' \in \tilde{V}_{k+1,I_1}$. Now define a set $\mathcal{E} \subseteq \mathcal{I}$ with two rules:

(1) If $e \in \mathscr{L}_{k+1,I_1}$ has both its endpoints in V_{k+1,I_1} then both components of $g^{-1}(e)$ are in $\mathcal{E}$.
(2) If $e \in \mathscr{L}_{k+1,I_1}$ has one endpoint $v' \in V_{k+1,I_1}$ and another in $\tilde{V}_{k+1,I_1} \setminus V_{k+1,I_1}$, then only one component of $g^{-1}(e)$ (one that intersects $J_{v'}$) is in $\mathcal{E}$.

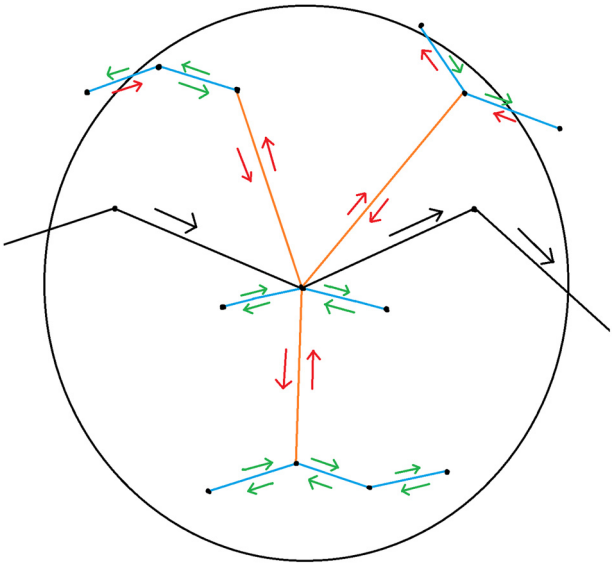

Fig. 3. The image of $f_{k+1}(I)$: Blue segments represent edges in $\mathscr{L}_{k+1}$, black arrows represent images of intervals in $\mathscr{E}_k \cup \mathscr{B}_k$, green arrows represent images of intervals in $\mathscr{E}_{k+1}(I)$ and red arrows represent images of intervals in $\mathscr{B}_{k+1}(I)$.

Set $\mathscr{E}_{k+1}(I_1) = \mathcal{E}$, $\mathscr{B}_{k+1}(I_1) = \mathcal{I} \setminus \mathcal{E}$, $\mathscr{N}_{k+1}(I_1) = \mathcal{N}$, and define $\mathscr{F}_{k+1}(I_1)$ to be the set of components of $I_1 \setminus \bigcup(\mathscr{E}_{k+1}(I_1) \cup \mathscr{B}_{k+1}(I_1) \cup \mathscr{N}_{k+1}(I_1))$ and $f_{k+1}|I_1 = g$.

3.8.2. Several components

Suppose that $l_1 \geq 2$; that is, G_{k+1,I_1} is disconnected. See Fig. 3 for the image of $f_{k+1}(I)$. We will add some edges which will make the graph connected and the preimage of these edges will be bridge intervals. To this end, for each $j \in \{2, \dots, l_1\}$ fix some point $v_j \in V_{k+1,I_1} \cap G^{(j)}_{k+1,I_1}$. Let $\{I_{1,1}, \dots, I_{1,2l_1-2}\}$ be a collection of open intervals, enumerated according to the orientation of $[0,1]$, such that their closures are mutually disjoint and are contained in the interior of I_1. Let also $\{J_{1,1}, \dots, J_{1,2l_1-1}\}$ be the components of $I_1 \setminus \bigcup_{j=1}^{2l_1-2} I_{1,j}$ enumerated according to the orientation of $[0,1]$.

Working as in §3.8.1, we obtain a family $\mathcal{I}_1$ of open intervals in $J_{1,1}$, a subset $\mathcal{E}_1 \subset \mathcal{I}_1$, a family $\mathcal{N}_1$ containing some components of $J_{1,1} \setminus \bigcup \mathcal{I}_1$ and a continuous map $g_1 : J_{1,1} \to G^{(1)}_{k+1,I_1}$. Similarly, for each $j \in \{2, \dots, l_1\}$ we obtain a family $\mathcal{I}_j$ of open intervals in $J_{1,2(j-1)}$, a subset $\mathcal{E}_j \subset \mathcal{I}_j$, a family $\mathcal{N}_j$ containing some components of $J_{1,2(j-1)} \setminus \bigcup \mathcal{I}_j$ and a continuous map $g_j : J_{1,2(j-1)} \to G^{(j)}_{k+1,I_1}$. There exists a continuous map $g : I_1 \to X$ that extends the maps g_j such that

(1) $g(J_{1,2j+1}) = f_k(I_1)$ for each $j \in \{1, \dots, l_1 - 1\}$;
(2) $g|I_{1,j}$ is affine for all $j \in \{1, \dots, 2l_1 - 1\}$ and $g(I_{1,2j-1}) = g(I_{1,2j}) = [v_j, f_k(I_1)]$ for all $j \in \{1, \dots, l_1 - 1\}$.

Define edge intervals $\mathscr{E}_{k+1}(I_1) = \bigcup_{j=1}^{l_1} \mathcal{E}_j$ and bridge intervals $\mathscr{B}_{k+1}(I_1) = \bigcup_{j=1}^{2l_1-2} I_{1,j} \cup \bigcup_{j=1}^{l_1}(\mathcal{I}_j \setminus \mathcal{E}_j)$. Set $\mathscr{N}_{k+1}(I_1) = \bigcup_{j\in\{1,2,4,\dots,2l_1\}} \mathcal{N}_j$ and define $\mathscr{F}_{k+1}(I_1)$ to be the set of components of $I_1 \setminus \bigcup(\mathscr{E}_{k+1}(I_1) \cup \mathscr{B}_{k+1}(I_1) \cup \mathscr{N}_{k+1}(I_1))$. Also, set $f_{k+1}|I_1 = g$.

3.8.3. Inductive hypothesis

Inductively, suppose that for $i \in \{1, \dots, r-1\}$ we have defined $\mathscr{E}_{k+1}(I_i)$, $\mathscr{B}_{k+1}(I_i)$, $\mathscr{F}_{k+1}(I_i)$, $\mathscr{N}_{k+1}(I_i)$ and $f_{k+1}|I_i$. We work now for I_r. Let V_{k+1,I_r} be the set of points in $V_{k+1} \cap B(f_k(I_1), C^*\rho_{k+1}r_0)$ that are not images of some $I \in \mathscr{N}_{k+1}$ defined in §3.4 or in §3.7 or for the previous intervals $I_1, \dots, I_{r-1}$. Let $\mathscr{L}_{k+1,I_r}$ be the set of edges in $\mathscr{L}_{k+1}$ that have an endpoint in V_{k+1,I_r} and let $\tilde{V}_{k+1,I_r}$ be the set of endpoints of edges in $\mathscr{L}_{k+1,I_r}$. Let now G_{k+1,I_r} be the (not necessarily connected) graph with vertices the set $\tilde{V}_{k+1,I_r}$ and with edges the set $\mathscr{L}_{k+1,I_r}$. To continue, repeat the procedure carried out for I_1 *mutatis mutandis*.

Remark 3.7. By the choice of set $\mathcal{N}$ for I_1, it follows that if $I \in \mathscr{N}_{k+1}(I_1)$ and if $f_{k+1}(I)$ is the endpoint of $f_{k+1}(J)$ for some $J \in \mathscr{E}_{k+1}(I_1)$, then there exists $J' \in \mathscr{E}_{k+1}(I_1)$ (possibly $J' = J$) such that $f_{k+1}(J) = f_{k+1}(J')$ and $\overline{I} \cap \overline{J'} \neq \emptyset$. The same is true for all I_j.

Lemma 3.8. *Let $J_1 \in \mathscr{N}_k$ be as in §3.8, let $J_2 \in \mathscr{E}_k$ be as in §3.4, and let $J_3 \in \mathscr{N}_k$ be as in §3.7. If $J_i' \in \mathscr{E}_{k+1}(J_i)$ for $i = 1, 2, 3$, then the segments $f_{k+1}(J_1')$, $f_{k+1}(J_2')$, $f_{k+1}(J_3')$ are mutually disjoint.*

Proof. By Lemma 3.6 we know that $f_{k+1}(J_2')$ and $f_{k+1}(J_3')$ are disjoint. Fix an interval $J_1 \in \mathscr{N}_k$ as in §3.8. Suppose that either $J_2 \in \mathscr{E}_k$ is as in §3.4 or $J_2 \in \mathscr{N}_k$ is as in §3.7. Let $J_1' \in \mathscr{E}_{k+1}(J_1)$ and $J_2' \in \mathscr{E}_{k+1}(J_2')$. By Lemma 2.5, either $f_{k+1}(J_1') \cap f_{k+1}(J_2') \neq \emptyset$ or $f_{k+1}(J_1') = f_{k+1}(J_2')$. However, we have defined $\mathscr{L}_{k+1,J_1}$ as those elements in $\mathscr{L}_{k+1}$ that are not contained in $f_{k+1}(J)$, where $J \in \mathscr{E}_k$ is as in §3.4 or $J \in \mathscr{N}_k$ is as in §3.7. Thus, $f_{k+1}(J_1') \cap f_{k+1}(J_2') = \emptyset$. □

3.9. Properties (P1)–(P7) for step $k+1$

We have now defined $\mathscr{B}_{k+1}$, $\mathscr{E}_{k+1}$, $\mathscr{F}_{k+1}$, $\mathscr{N}_{k+1}$ and $f_{k+1} : [0,1] \to X$. It remains to prove that f_{k+1} is continuous and that properties (P1)–(P7) are satisfied by the new collections of intervals and f_{k+1}. Properties (P1), (P2), and (P3) follow immediately from the construction.

Continuity of f_{k+1}. By design, the map f_{k+1} is continuous on every point interior to an interval in $\mathscr{E}_k \cup \mathscr{B}_k \cup \mathscr{N}_k \cup \mathscr{F}_k$. If x is an endpoint of some interval in $\mathscr{E}_k \cup \mathscr{B}_k \cup \mathscr{N}_k \cup \mathscr{F}_k$, then $f_{k+1}(x) = f_k(x)$. Thus, continuity of f_{k+1} at x follows from continuity of f_k at x.

Property (P6). The first claim of (P6), that $f_{k+1}(I) \in V_{k+1}$ for all $I \in \mathscr{N}_{k+1} \cup \mathscr{F}_{k+1}$, is immediate from the construction. To check the second claim of (P6), fix $v \in V_{k+1}$. By (V4), there exists $v' \in V_k$ such that $|v - v'| < C^*\rho_{k+1}r_0$. By the inductive step, there exists $I \in \mathscr{N}_k$ such that $f_k(I) = v'$. There are two cases.

Case 1. Suppose that $\alpha_{k,v'} < \alpha_0$. Then, following the discussion in §3.7, either $v = f_{k+1}(J)$ for some $J \in \mathscr{N}_k(J')$ and $J' \in \mathscr{E}_k$ as in §3.4, or $v = f_{k+1}(J)$ for some $J \in \mathscr{N}_k(I)$.

Case 2. Suppose that $\alpha_{k,v'} \geq \alpha_0$. Following the construction of the graph $G_{k+1,I}$ and the design of the map $f_{k+1}|I$, if v is not the image of some $J \in \mathscr{N}_{k+1}(J')$, where $J' \in \mathscr{E}_k \cup \mathscr{B}_k \cup \mathscr{N}_k \setminus \{J'\}$, then there exists $J \in \mathscr{N}_{k+1}(I)$ such that $v = f_{k+1}(J)$.

Property (P4). Fix $I \in \mathscr{E}_{k+1}$. There are three cases.

Case 1. Suppose that $I \in \mathscr{E}_{k+1}(I_0)$ for some $I_0 \in \mathscr{E}_k$. By the inductive hypothesis, there exists unique $I_0' \in \mathscr{E}_k \setminus \{I_0\}$ such that $f_k(I_0') = f_k(I_0)$ while $f_k(I_0) \cap f_k(J) = \emptyset$ for all $J \in \mathscr{E}_k \setminus \{I_0, I_0'\}$. By construction, there exists $I' \in \mathscr{E}_{k+1}(I_0')$ such that $f_{k+1}(I) = f_{k+1}(I')$. Again by construction, $f_{k+1}(I) \cap f_{k+1}(J) = \emptyset$ for all $J \in \mathscr{E}_{k+1}(I_0) \cup \mathscr{E}_{k+1}(I_0') \setminus \{I, I'\}$. By Lemma 3.3, Lemma 3.6 and Lemma 3.8, $f_{k+1}(I)$ does not intersect any $f_{k+1}(J)$ for any $J \in \mathscr{E}_{k+1}(J')$ and $J' \in \mathscr{E}_k \cup \mathscr{N}_k \setminus \{I_0, I_0'\}$.

Case 2. Suppose that $I \in \mathscr{E}_{k+1}(I_0)$ for some $I_0 \in \mathscr{N}_k$ as in §3.7. By construction, there exists $I' \in \mathscr{E}_{k+1}(I_0) \setminus \{I\}$ such that $f_{k+1}(I) = f_{k+1}(I')$ while $f_{k+1}(I) \cap f_{k+1}(J) = \emptyset$ for all $J \in \mathscr{E}_{k+1}(I_0) \setminus \{I, I'\}$. Moreover, by Lemma 3.6 and Lemma 3.8, $f_{k+1}(I)$ does not intersect any $f_{k+1}(J)$ for any $J \in \mathscr{E}_{k+1}(J')$ and $J' \in \mathscr{E}_k \cup \mathscr{N}_k \setminus \{I_0\}$.

Case 3. Suppose that $I \in \mathscr{E}_{k+1}(I_0)$ for some $I_0 \in \mathscr{N}_k$ as in §3.8. By Lemma 3.8, $f_{k+1}(I) \cap f_{k+1}(I') = \emptyset$ for all $I' \in \mathscr{E}_{k+1}(J)$ and all $J \in \mathscr{E}_k$ as in §3.4 or $J \in \mathscr{N}_k$ as in §3.7. By the construction of $f_{k+1}|I_0$, there are two possibilities.

Case 3a. Suppose that both endpoints of $f_{k+1}(I_0)$ are in V_{k_1,I_0}. Then there exists an interval $I' \in \mathscr{E}_{k+1}(I_0) \setminus \{I\}$ such that $f_{k+1}(I) = f_{k+1}(I')$. On the other hand, $f_{k+1}(I) \notin \mathscr{L}_{k+1,J}$ for any $J \in \mathscr{N}_k \setminus \{I_0\}$. Thus, by Lemma 2.5, if $J' \in \mathscr{E}_{k+1}(J)$ and $J \in \mathscr{N}_k \setminus \{I_0\}$, then $f_{k+1}(J) \cap f_{k+1}(J') = \emptyset$.

Case 3b. Suppose that only one endpoint of $f_{k+1}(I_0)$ is in V_{k_1,I_0}. In this case, by construction, $f_{k+1}(I) \cap f_{k+1}(J) = \emptyset$ for all $J \in \mathscr{E}_{k+1}(I_0) \setminus \{I\}$. Moreover, there exists unique $I_0' \in \mathscr{N}_k \setminus \{I_0\}$ as in §3.8 such that $V_{k+1,I_0'}$ contains the other endpoint of $f_{k+1}(I)$. As with I_0, there exists unique $I' \in \mathscr{E}_{k+1}(I_0')$ such that $f_{k+1}(I') = f_{k+1}(I)$ while $f_{k+1}(J) \cap f_{k+1}(I) = \emptyset$ for all $J \in \mathscr{E}_{k+1}(I_0')$. Finally, by the construction and Lemma 2.5, $f_{k+1}(I) \cap f_{k+1}(J) = \emptyset$ for all $J \in \mathscr{E}_{k+1}(J')$ and all $J' \in \mathscr{N}_k \setminus \{I_0, I_0'\}$ as in §3.8.

Property (P5). To prove the first claim in (P5), fix $(v, v') \in \mathsf{Flat}(k+1)$. Let v_0 be the point of V_k closest to v and let $I_0 \in \mathscr{N}_k$ be such that $f_k(I_0) = v_0$. There are four cases.

Case 1. Suppose that $\alpha_{k,v_0} < \alpha_0$ and v_0 is non-terminal (see §3.7.1). Then either both v and v' lie to the left of v_0 (with respect to ℓ_{k,v_0}) or both lie to the right of v_0. In any case, $[v, v']$ is the preimage of some $I \in \mathscr{E}_{k+1}(J)$ under f_{k+1} where $J \in \mathscr{E}_k$ and $f_k(J)$ is an edge with endpoint $f_k(I_0)$.

Case 2. Suppose that $\alpha_{k,v_0} < \alpha_0$ and v_0 is 2-sided terminal (see §3.7.3). Then $[v, v']$ is the preimage of some $I \in \mathscr{E}_{k+1}(I_0)$ under f_{k+1}.

Case 3. Suppose that $\alpha_{k,v_0} < \alpha_0$ and v_0 is 1-sided terminal (see §3.7.2). Then either both v and v' lie to the left of v_0 (with respect to ℓ_{k,v_0}) or both lie to the right of v_0. Depending on their position, we work as in Case 1 or Case 2.

Case 4. Suppose that $\alpha_{k,v_0} \geq \alpha_0$. By definition of graph $G_{k+1,I}$ in §3.8, the segment $[v, v']$ is the image of some $J \in \mathscr{E}_{k+1}$ under f_{k+1}.

To prove the second claim of (P5), fix $(a, b) \in \mathscr{E}_{k+1}$ such that one of its endpoints has flat image. Without loss of generality, assume $\alpha_{k+1,f_{k+1}(a)} < \alpha_0$.

Case 1. Suppose that $(a, b) \in \mathscr{E}_{k+1}(I)$ for some $I \in \mathscr{N}_k$ as in §3.8. By construction of f_{k+1} on such intervals, $(f_{k+1}(a), f_{k+1}(b)) \in \mathsf{Flat}(k+1)$.

Case 2. Suppose that $(a, b) \in \mathscr{E}_{k+1}(I)$ for some $I \in \mathscr{N}_k$ as in §3.7. By construction of f_{k+1} on such intervals, no point of $V_{k+1} \cap B(f_{k+1}(a), 14A^*\rho_{k+1}r_0)$,

$$V_{k+1} \cap B(f_{k+1}(a), 14A^*\rho_{k+1}r_0) \subset V_{k+1} \cap B(f_k(I), 30A^*\rho_k r_0),$$

lies strictly between $f_{k+1}(a)$ and $f_{k+1}(b)$ with respect to $\ell_{k,f_k(I)}$. The same is true with respect to $\ell_{k+1,f_{k+1}(a)}$ by Lemma 2.1. Thus, $(f_{k+1}(a), f_{k+1}(b)) \in \mathsf{Flat}(k+1)$.

Case 3. Suppose that $(a, b) \in \mathscr{E}_{k+1}(I)$ for some $I \in \mathscr{E}_k$ as in §3.4. The argument is similar to Case 2

Property (P7). To check the final property, fix $J \in \mathscr{N}_{k+1}$ and choose $I \in \mathcal{E}_{k+1}$ such that $f_{k+1}(J)$ is an endpoint of $f_{k+1}(I)$. There are several cases.

Case 1. Suppose $J \in \mathscr{N}_{k+1}(J_0)$ for some $J_0 \in \mathscr{N}_k$ as in §3.8. Then there exists $I' \in \mathscr{E}_{k+1}(J_0)$ such that $f_{k+1}(I') = f_{k+1}(I)$. By the construction of $\mathscr{N}_{k+1}(J_0)$ in §3.8, $\overline{I'} \cap J \neq \emptyset$.

Case 2. Suppose $J \in \mathscr{N}_{k+1}(J_0)$ for some $J_0 \in \mathscr{E}_k$ as in §3.4. By (P4), there exists $I' \in \mathscr{E}_{k+1}(J_0)$ such that $f_{k+1}(I') = f_{k+1}(I)$. The interval I' satisfies $\overline{I'} \cap J \neq \emptyset$.

Case 3. Suppose $J \in \mathscr{N}_{k+1}(J_0)$ for some $J' \in \mathscr{N}_k$ as in §3.7. There are three subcases.

Case 3a. Suppose that $f_{k+1}(J) \neq f_k(J_0)$. Then by the choice of $\mathscr{N}_{k+1}(J_0)$, there exists $I' \in \mathscr{E}_{k+1}(J_0)$ such that $f_{k+1}(I') = f_{k+1}(I)$ and $\overline{I'} \cap \overline{J} \neq \emptyset$.

Case 3b. Suppose that $f_{k+1}(J) = f_k(J_0)$ and there exists $\tilde{I} \in \mathscr{E}_{k+1}(J_0)$ such that $f_{k+1}(\tilde{I}) = f_{k+1}(I)$. As in Case 3a, the claim follows from the choice of $\mathscr{N}_{k+1}(J_0)$.

Case 3c. Suppose that $f_{k+1}(J) = f_k(J_0)$ and there exists no $\tilde{I} \in \mathscr{E}_{k+1}(J_0)$ such that $f_{k+1}(\tilde{I}) = f_{k+1}(I)$. In this case, $f_k(J_0)$ is the endpoint of $f_k(I_0)$ for some $I_0 \in \mathcal{E}_k$. By the inductive hypothesis and (P4), there exists at least one and at most two intervals $I_0' \in \mathscr{E}_k$ such that $f_k(I_0) = f_k(I_0')$ and $\overline{I_0'} \cap \overline{J_0} \neq \emptyset$. On one hand, if there is only one interval I_0', then J_0 is as in §3.7.1 and $J = J_0$. Hence there exists $I' \in I_0'$ such that $f_{k+1}(I') = f_{k+1}(I)$ and $\overline{I'} \cap J \neq \emptyset$. On the other hand, if there are two intervals I_0', I_0'', then one of them has a closure which intersects J, say I_0'. Then there exists $I' \in I_0'$ such that $f_{k+1}(I') = f_{k+1}(I)$ and $\overline{I'} \cap J \neq \emptyset$.

3.10. Choices in the Traveling Salesman algorithm

In §§2.2 and 3, we made a series of implicit and explicit choices.

(C0) The choice of $\alpha_0 \in (0, 1/16]$ determines the set $\mathsf{Flat}(k)$ of flat pairs. The constant 14 in the definition of $\mathsf{Flat}(k)$ is chosen to facilitate the estimates in §4 (see (E2)),

but has not been optimized. The constant 30 in the definition of $\alpha_{k,v}$ is chosen to be larger than $(1 + 3(1/16)^2) \cdot 2 \cdot 14$. For example, see the proof of Lemma 2.5.

(C1) If $I, I' \in \mathscr{E}_k$ satisfy $f_k(I) = f_k(I')$ and are as in §3.4 (that is, I has at least one endpoint x with $\alpha_{k,f_k(x)} < \alpha_0$), then either $\mathscr{N}_{k+1}(I) = \mathscr{F}_{k+1}(I') = \emptyset$ or vice-versa.

(C2) If $I \in \mathscr{N}_k$ is as in §3.7.3 (i.e. $\alpha_{k,f_k(I)} < \alpha_0$ and $V_k \cap B(f_k(I), 14A^*\rho_k r_0) = \{f_k(I)\}$), then there may exist up to two different ways to parameterize the graph $G_{k+1,I}$ therein.

(C3) If $I \in \mathscr{N}_k$ is as in §3.8 (i.e. $\alpha_{k,f_k(I)} \geq \alpha_0$) and $G^{(1)}_{k+1,I}, \dots, G^{(l)}_{k+1,I}$ are the graph components of the graph $G_{k+1,I}$ therein, then

(C3a) we choose the order in which we parameterize the graph components and

(C3b) in each graph component, there exists up to two choices of parameterization.

Similar choices are made in the step 0.

(C4) We get to choose the enumeration of intervals I in $\mathscr{N}_k$ such that $\alpha_{k,f_k(I)} \geq \alpha_0$.

The algorithm can be made more flexible by permitting four additional choices. Let $\tilde{\alpha}_0 \in (0, \alpha_0)$ and $\tilde{A} > 14A^*$.

(C5) Suppose that $I \in \mathscr{N}_k$.
- If $\alpha_{k,f_k(I)} < \tilde{\alpha}_0$, then we treat I as in §3.7; i.e., we treat $f_k(I)$ as a flat vertex.
- If $\alpha_{k,f_k(I)} \geq \alpha_0$, then we treat I as in §3.8; i.e., we treat $f_k(I)$ as a non-flat vertex.
- If $\alpha_{k,f_k(I)} \in [\tilde{\alpha}_0, \alpha_0)$, then we can either treat I as in §3.7 or as in §3.8.

(C6) Suppose that $v \in V_k$ is chosen to be considered "flat" by (C5). Let ℓ be the approximating line for (k, v) and let $v' \in V_k$ be such that there exists no $v'' \in V_k \cap B(v, \tilde{A}\rho_k r_0)$ such that $\pi_\ell(v'')$ is between v and v'.
- If $|v - v'| < 14A^*\rho_k r_0$, then $(v, v') \in \mathsf{Flat}(k)$.
- If $|v - v'| \geq \tilde{A}\rho_k r_0$, then $(v, v') \notin \mathsf{Flat}(k)$.
- If $|v - v'| \in [14A^*\rho_k r_0, \tilde{A}\rho_k r_0)$, then we are free to choose whether (v, v') is contained in $\mathsf{Flat}(k)$ or not.

(C7) Similarly to (C6), suppose that $I \in \mathscr{E}_k$ is as in §3.4; i.e., $f_k(I)$ has at least one endpoint x whose image is "flat" by (C5). Let $\{v_1, \dots, v_l\}$ and $\{I_1, \dots, I_{l-1}\}$ be as in §3.4.
- If $|v_i - v_{i+1}| < 14A^*\rho_{k+1} r_0$, then we set $I_i \in \mathscr{E}_{k+1}$.
- If $|v_i - v_{i+1}| \geq \tilde{A}\rho_{k+1} r_0$, then we set $I_i \in \mathscr{B}_{k+1}$.
- If $|v_i - v_{i+1}| \in [14A^*\rho_{k+1} r_0, \tilde{A}\rho_{k+1} r_0)$, then we can choose in each instance whether $I_i \in \mathscr{E}_{k+1}(I)$ or $I_i \in \mathscr{B}_{k+1}(I)$.

(C8) Suppose that $\{I_1, \dots, I_n\}$ are the intervals in $\mathscr{N}_k$ that have a non-flat image. Suppose also that we have defined f_{k+1} on $I_1, \dots, I_{r-1}$ and on intervals in $\mathscr{N}_k$ that have an image chosen to be flat. Let $v \in V_{k+1}$ be a point which is not the image of some $I \in \mathscr{N}_{k+1}(J)$, where $J \in \{I_1, \dots, I_{r-1}\}$ or $J \in \mathscr{N}_k$ is as in §3.7.
- If $v \in B(f_k(I_r), 14A^*\rho_k r_0)$, then $v \in V_{k+1,I_r}$.

- If $v \in B(f_k(I_r), \tilde{A}\rho_k r_0) \setminus B(f_k(I_r), 14A^*\rho_k r_0)$, then we may choose whether $v \in V_{k+1,I_r}$ or not.

Note that (C5) has subsequent implications on the treatment of intervals $I \in \mathscr{E}_k$. For instance, if both endpoints if I have images chosen to be non-flat, then $I \in \mathscr{B}_{k+1}$; otherwise, we treat I as in §3.4. Similarly, (C6) gives us the set $\mathscr{L}_k$ and together with (C8) affects the parametrization near non-flat vertices.

Remark 3.9 *(Coherence).* In the original, non-parametric Analyst's Traveling Salesman construction, Jones [32] required the coherence property (V2), i.e. $V_k \subset V_{k+1}$. The first author and Schul [9] established a non-parametric Traveling Salesman construction, which replaced (V2) with the weaker property that for all $v \in V_k$, the set

$$v' \in V_{k+1} \cap B(v, C^*\rho_k r_0)$$

is nonempty. This relaxation was crucial for the proof of the main result in [9], which characterized Radon measures in $\mathbb{R}^N$ that are carried by rectifiable curves. We would like to emphasize that in the parametric Traveling Salesman construction described above, we heavily rely on (V2). At this time, we do not know how to build a parameterization under the relaxed condition of [9].

4. Mass of intervals

In this section, we use the construction of §3, to assign mass to intervals defined in §3. The total mass $\mathcal{M}_s$ on the domain of the maps fills the role that the Hausdorff measure $\mathcal{H}^1$ of the image plays in the proof of the sufficient half of the Analyst's TST given in [32] or [9]. The main result of this section is Proposition 4.11, which bounds the total mass of $[0,1]$ by a sum involving the flatness approximation errors $\alpha_{k,v}$ and variation excess $\tau_s(k,v,v')$ defined in §2.2. For each $k \geq 0$, set

$$\mathscr{I}_k := \mathscr{E}_k \cup \mathscr{B}_k \cup \mathscr{N}_k \cup \mathscr{F}_k \qquad \text{and} \qquad \mathscr{I} := \bigcup_{k\geq 0} \mathscr{I}_k.$$

For each $I \in \mathscr{I}_k$, set $\mathscr{I}_{k+1}(I) := \mathscr{E}_{k+1}(I) \cup \mathscr{B}_{k+1}(I) \cup \mathscr{N}_{k+1}(I) \cup \mathscr{F}_{k+1}(I)$.

Remark 4.1. If $I \in \mathscr{I}_k \cap \mathscr{I}_m$ for some $m \neq k$, then $f_k|I = f_m|I$.

4.1. Trees over intervals

Given $k \geq 0$ and $I \in \mathscr{I}_k$, we define a *finite tree T over* (k,I) to be a finite subset of $\bigcup_{m\geq 0}(\{m\} \times \mathscr{I}_m)$ satisfying the following three conditions.

(1) The pair $(k,I) \in T$. If $(m,J) \in T$, then $m \geq k$ and $J \subset I$.

(2) If $(m, J) \in T$ and there exists $J' \in \mathscr{I}_{m+1}(J)$ such that $(m+1, J') \cap T$, then $\{m+1\} \times \mathscr{I}_{m+1}(J) \subseteq T$.
(3) If $(m, J) \in T$ for some $m > k$ and $J \in \mathscr{I}_m(J')$, then $(m-1, J') \in T$.

The first condition says that the root of the tree is (k, I) and its elements are descendants of (k, I). The second condition says that if one child $(m+1, J')$ of (m, J) is in T, then every child of (m, J) is in T. The third condition says that if (m, J) is in T, then all its ancestors up to (k, I) are in T.

We extend this notion to the entire domain by defining a *finite tree* T over $[0, 1]$ to be a set of the form

$$T = \{[0,1]\} \cup \bigcup_{I \in \mathscr{I}_0} T_I,$$

where T_I is a finite tree over $(0, I)$. A finite tree over $[0, 1]$ may be thought to belong to step $k = -1$ of the construction.

Let T be a finite tree over (k, I). The *boundary* ∂T *of* T is defined by

$$\partial T := \{(m, J) \in T : (\{m+1\} \times \mathscr{I}_{m+1}(I)) \cap T = \emptyset\}.$$

The *depth* $m(T)$ *of* T is the integer defined by

$$m(T) := \max\{m \geq 0 : (m, J) \in T\} = \max\{k \geq 0 : (m, J) \in \partial T\}.$$

If $m(T) \geq 1$, the *parent tree* $p(T)$ is defined by

$$p(T) := T \setminus (\{m(T)\} \times \mathscr{I}_{m(T)})$$

Note that $m(p(T)) = m(T) - 1$.

Remark 4.2. If T is a finite tree over (k, I) and $\partial T = \{(k_1, J_1), \ldots, (k_n, J_n)\}$, then the intervals $J_1, \ldots, J_n$ partition I. That is, for all $x \in I$, there exists a unique $i \in \{1, \ldots, n\}$ such that $x \in J_i$.

4.2. Mass of intervals

For all $s \geq 1$, $k \geq 0$, and intervals $I \in \mathscr{I}_k$, define the *s-mass* $\mathcal{M}_s(k, I)$ *of* (k, I) by

$$\mathcal{M}_s(k, I) := \sup_T \sum_{(k', I') \in \partial T} (\operatorname{diam} f_{k'}(I'))^s \in [0, \infty],$$

where the supremum is taken over all finite trees over (k, I). This notion extends to $[0, 1]$ by assigning

$$\mathcal{M}_s([0,1]) := \sum_{I \in \mathscr{I}_0} \mathcal{M}_s(0, I) \in [0, \infty].$$

Lemma 4.3. *Let $k \geq 0$ and $I \in \mathscr{I}_k$.*

(1) *If $I \in \mathscr{B}_k$, then $\mathcal{M}_s(k, I) = (\operatorname{diam} f_k(I))^s$.*
(2) *If $I \in \mathscr{F}_k$, then $\mathcal{M}_s(k, I) = 0$.*
(3) *$\mathcal{M}_s(k, I) \geq \sum_{I' \in \mathscr{I}_{k+1}(I)} \mathcal{M}_s(k+1, I')$.*
(4) *If $I \in \mathscr{I}_k \cap \mathscr{I}_m$ for some $m \geq 0$, then $\mathcal{M}_s(k, I) = \mathcal{M}_s(m, I)$.*

Before proving Lemma 4.3, we make two clarifying remarks. First, it is possible for an interval $I \in \mathscr{N}_k$ to have $\mathcal{M}_s(k, I) > 0$ even though $\operatorname{diam} f_k(I)^s = 0$. This happens whenever $I \in \mathscr{N}_k$ and $\mathscr{E}_{k+1}(I) \cup \mathscr{B}_{k+1}(I)$ is non-empty. Second, Lemma 4.3(4) implies that given $I \in \mathscr{I}_k$, the mass $\mathcal{M}_s(k, I)$ is defined independently of the step of the construction in which I appears. Nevertheless, we include the step k in definition of the mass to improve exposition of the estimates in §4.5 and §4.6.

Proof of Lemma 4.3. For the first claim, note that if $I \in \mathscr{B}_k$ and $m \geq k+1$, then $\mathscr{I}_m(I) = \{I\}$. Therefore, if T is a finite tree over I of depth m, then $\partial T = \{(m, I)\}$. Thus, $\mathcal{M}_s(k, I) = \sup_{m \geq k} (\operatorname{diam} f_m(I))^s = (\operatorname{diam} f_k(I))^s$.

For the second claim, note that if $I \in \mathscr{F}_k$ and $m \geq k+1$, then $\mathscr{I}_m(I) = \{I\}$ and $f_m(I)$ is a point. Therefore, if T is a finite tree over I of depth m, then $\partial T = \{(m, I)\}$. Thus, $\mathcal{M}_s(k, I) = \sup_{m \geq k} (\operatorname{diam} f_m(I))^s = 0$.

For the third claim, let us first assume that $\mathcal{M}_s(k+1, J) = \infty$ for some $J \in \mathscr{I}_{k+1}(I)$. Fix $M > 0$ and find a finite tree T_J over $(k+1, J)$ such that

$$\sum_{(k', J') \in T_J} (\operatorname{diam} f_{k'}(J'))^s > M.$$

The collection $T = T_J \cup \{(k, I)\} \cup (\{k+1\} \times \mathscr{I}_{k+1}(I))$ is a finite tree over (k, I) and $\partial T_J \subset \partial T$. Hence

$$\mathcal{M}_s(k, I) \geq \sum_{(k', I') \in \partial T} (\operatorname{diam} f_{k'}(I'))^s \geq \sum_{(k', J') \in \partial T_J} (\operatorname{diam} f_{k'}(J'))^s > M.$$

We conclude that $\mathcal{M}_s(k, I) = \infty$.

Alternatively, assume that $\mathcal{M}_s(k+1, J)$ is finite for all $J \in \mathscr{I}_{k+1}(I)$. Fix $\epsilon > 0$. For each interval $J \in \mathscr{I}_{k+1}(I)$, let T_J be a finite tree over $(k+1, J)$ such that

$$\sum_{(k', J') \in \partial T_J} (\operatorname{diam} f_{k'}(J'))^s > \mathcal{M}_s(k+1, J) - \frac{\epsilon}{\operatorname{card}(\mathscr{I}_{k+1}(I))}.$$

Then the collection $T = \{(k, I)\} \cup \bigcup_{J \in \mathscr{I}_{k+1}(I)} T_J$ is a finite tree over (k, I) with $\partial T = \bigcup_{J \in \mathscr{I}_{k+1}(I)} \partial T_J$. Therefore,

$$\mathcal{M}_s(k,I) \geq \sum_{J\in\mathscr{I}_{k+1}(I)} \sum_{(k',J')\in\partial T_J} (\operatorname{diam} f_{k'}(J'))^s > \sum_{J\in\mathscr{I}_{k+1}(I)} \mathcal{M}_s(k+1,J) - \epsilon.$$

The third claim follows by taking $\epsilon \downarrow 0$.

For the fourth claim, suppose that $I \in \mathscr{I}_k \cap \mathscr{I}_m$ for some m and k, say without loss of generality that $m > k$. Because $I \in \mathscr{I}_k \cap \mathscr{I}_m$, we have $\mathscr{I}_n(I) = \{I\}$ for all $k \leq n \leq m$. Thus, iterating the third claim, $\mathcal{M}_s(k,I) \geq \mathcal{M}_s(m,I)$. For the opposite inequality, let T be a finite tree over (k,I). If $(m,I) \notin T$, then $T = \{(k,I),\dots,(l,I)\}$ for some $k \leq l \leq m-1$ and we define $T' = \{(m,I)\}$. If $(m,I) \in T$, then $T \supset \{(k,I),\dots,(m-1,I)\}$ and we set $T' = T \setminus \{(k,I),\dots,(m-1,I)\}$ so that T' is a finite tree over (m,I) and $\partial T = \partial T'$. In either case,

$$\sum_{(k',I')\in\partial T} (\operatorname{diam} f_{k'}(I'))^s = \sum_{(k',I')\in\partial T'} (\operatorname{diam} f_{k'}(I'))^s$$

and it follows that $\mathcal{M}_s(k,I) \leq \mathcal{M}_s(m,I)$. □

When $s = 1$, the 1-mass is comparable to the Hausdorff measure $\mathcal{H}^1$ of the image.

Lemma 4.4. *For each $k \geq 0$ and each $I \in \mathscr{I}_k$,*

$$\mathcal{M}_1(k,I) = \lim_{m\to\infty} \sum_{\substack{J\in\mathscr{I}_m \\ J\subset I}} \operatorname{diam} f_m(J) \geq \limsup_{m\to\infty} \mathcal{H}^1(f_m(I)).$$

If there exists $n \in \mathbb{N}$ such that $f_m|\bigcup \mathscr{B}_k$ is at most n-to-1 for all $m \geq k$, then

$$\mathcal{M}_1(k,I) \simeq_n \liminf_{m\to\infty} \mathcal{H}^1(f_m(I)).$$

Proof. Fix $k \geq 0$ and $I \in \mathscr{I}_k$. By definition of the mass and (P2),

$$\mathcal{M}_1(k,I) \geq \limsup_{m\to\infty} \sum_{\substack{J\in\mathscr{I}_m \\ J\subset I}} \operatorname{diam} f_m(J) \geq \limsup_{m\to\infty} \mathcal{H}^1(f_m(I)).$$

To establish the other direction, let T be a finite tree over I of depth $m \geq k$ and enumerate with $\partial T = \{(k_1,I_1),\dots,(k_n,I_n)\}$, where each $k_i \leq m$. For each $i = 1,\dots,n$, let $\mathcal{I}_i$ be the set of all intervals $J \in \mathscr{I}_m$ such that $J \subset I_i$. Then

$$\sum_{\substack{J\in\mathscr{I}_m \\ J\subset I}} \operatorname{diam} f_m(J) = \sum_{i=1}^{n} \sum_{J\in\mathcal{I}_i} \operatorname{diam} f_m(J) \geq \sum_{i=1}^{n} \operatorname{diam} f_{k_i}(I_i) = \sum_{(l,J)\in\partial T} \operatorname{diam} f_l(J).$$

Therefore,

$$\sup_{m\geq k} \sum_{\substack{J\in\mathscr{I}_m \\ J\subset I}} \operatorname{diam} f_m(J) \leq \mathcal{M}_1(k,I) = \sup_T \sum_{(l,J)\in\partial T} \operatorname{diam} f_l(J) \leq \liminf_{m\to\infty} \sum_{\substack{J\in\mathscr{I}_m \\ J\subset I}} \operatorname{diam} f_m(J).$$

This shows that

$$\mathcal{M}_1(k,I) = \lim_{m\to\infty} \sum_{\substack{J\in\mathscr{I}_m \\ J\subset I}} \operatorname{diam} f_m(J).$$

By (P4), the maps $f_m|\bigcup \mathscr{E}_m$ are 2-to-1. In §3, we did not examine overlaps of images of bridge intervals. By modifying the algorithm, the overlap of images of bridge intervals can be made 2-to-1 (see the proof of Proposition 5.7). Nevertheless, suppose that we know the overlaps of images of bridge intervals is at most n-to-1 for some $n \geq 2$. Then

$$\mathcal{M}_1(k,I) = \lim_{m\to\infty} \sum_{\substack{J\in\mathscr{I}_m \\ J\subset I}} \operatorname{diam} f_m(J) \leq \frac{1}{n} \liminf_{m\to\infty} \mathcal{H}^1(f_m(I)). \quad \square$$

While Lemma 4.4 does not hold when $s > 1$, we always have the following comparison between the s-mass and the Hausdorff measure $\mathcal{H}^s$ of the closure of the points in $\bigcup_{k=0}^{\infty} V_k$.

Lemma 4.5. *For all $s \geq 1$, $\mathcal{H}^s\left(\overline{\bigcup_{k=0}^{\infty} V_k}\right) \lesssim_{s,C^*,\xi_2} \mathcal{M}_s([0,1])$.*

Proof. Fix $\delta > 0$ and choose $m \in \mathbb{N}$ sufficiently large such that $2C^*\xi_2^{m+1} r_0/(1-\xi_2) \leq \delta$. By (V0), (V2), and (V4), the collection $\{B(v, C^*\rho_{m+1}r_0/(1-\xi_2)) : v \in V_m\}$ is a cover of $\overline{\bigcup_{k=0}^{\infty} V_k}$ with elements of diameter at most $2C^*\rho_{m+1}r_0/(1-\xi_2) \leq 2C^*\xi_2^{m+1}r_0 \leq \delta$. Let T be the maximal finite tree over $[0,1]$ of depth m, i.e., $T = \bigcup_{k=0}^{m} \mathscr{I}_k$. Then

$$\begin{aligned}\mathcal{H}^s_\delta\left(\overline{\bigcup_{k=0}^{\infty} V_k}\right) &\leq \sum_{v\in V_m} \left(\frac{2C^*\rho_{m+1}r_0}{1-\xi_2}\right)^s \leq \left(\frac{2C^*\xi_2}{1-\xi_2}\right)^s \sum_{(m,I)\in\partial T} (\operatorname{diam} f_m(I))^s \\ &\lesssim_{s,C^*,\xi_2} \mathcal{M}_s([0,1]).\end{aligned}$$

Taking $\delta \downarrow 0$ completes the proof. $\square$

We include Lemma 4.4 and Lemma 4.5 for completeness. We will not use either lemma in any of the estimates below.

4.3. Terminal vertices and phantom mass

Let $I \in \mathscr{N}_k$ be an interval such that $\alpha_{k,f_k(I)} < \alpha_0$. We classify $f_k(I)$ according to the arrangement of nearby points in V_{k+1}.

- If I is as in §3.7.1, then $f_k(I)$ is called a *non-terminal vertex in V_k*.
- If I is as in §3.7.2, then $f_k(I)$ is called a *1-sided terminal vertex in V_k*.
- If I is as in §3.7.3, then $f_k(I)$ is called a *2-sided terminal vertex in V_k*.

Motivated by [32] and [9], for each $k \geq 0$ we will define a set $\mathscr{P}_k \subset \{k\} \times \mathscr{N}_k$ and for each $(k, I) \in \mathscr{P}_k$ define a number $p_{k,I} > 0$, which we call the *phantom mass* at (k, I). The phantom mass $p_{k,I}$ will let us pay for the length of edges between vertices in V_{k+1} nearby $f_k(I)$ that do not lie between vertices in V_k nearby $f_k(I)$ (i.e., the blue edges in Fig. 2). To start, define an auxiliary parameter P depending only on s, C^*, and ξ_2 by requiring that $[P + 2(1.1C^*)^s]\,\xi_2^s = P$. That is,

$$P = \frac{2(1.1C^*)^s}{1 - \xi_2^s}. \tag{4.1}$$

For each $k \geq 0$, define

$$\mathscr{P}_k = \{(k, I) : I \in \mathscr{N}_k,\ \alpha_{k,f_k(I)} < \alpha_0 \text{ and } f_k(I) \text{ is 1- or 2-sided terminal in } V_k\}.$$

For each $k \geq 0$ and $(k, I) \in \mathscr{P}_k$, assign

$$p_{k,I} := \begin{cases} 2P\rho_k^s r_0^s, & \text{if } f_k(I) \text{ is 2-sided terminal} \\ P\rho_k^s r_0^s, & \text{if } f_k(I) \text{ is 1-sided terminal.} \end{cases}$$

Lemma 4.6. *Let $I \in \mathscr{N}_k$ be an interval such that $\alpha_{k,f_k(I)} < \alpha_0$. If $f_k(I)$ is 1-sided terminal, then*

$$\sum_{J \in \mathscr{I}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s < 2(1.1C^*)^s \rho_{k+1}^s r_0^s.$$

If $f_k(I)$ is 2-sided terminal, then

$$\sum_{J \in \mathscr{I}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s < 4(1.1C^*)^s \rho_{k+1}^s r_0^s.$$

Proof. Suppose $v = f_k(I)$ is 1-sided terminal and let $\{v_1, \ldots, v_n\}$ be an enumeration of the points in $V_{k+1} \cap B(v, C^*\rho_{k+1}r_0)$ starting from $v_1 = v$ and moving consecutively towards the terminal direction. Then

$$\begin{aligned} \sum_{J \in \mathscr{I}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s = 2\sum_{i=1}^{n} |v_{i+1} - v_i|^s &\leq 2(1 + 3\alpha_{k,v}^2)^s |v_1 - v_n|^s \\ &< 2(1.1)^s (C^*\rho_{k+1}r_0)^s \end{aligned}$$

by Lemma 2.2, since $1 + 3\alpha_{k,v}^2 \leq 1 + 3(1/16)^2 < 1.1$. The case that v is 2-sided terminal follows from a similar computation. □

4.4. Special bridge intervals

Given $k \geq 1$, we define

$$\mathscr{B}_k^* := \{(k, I) : I \in \mathscr{B}_k(J),\ J \in \mathscr{E}_{k-1} \text{ is as in §3.4}\}.$$

Recall from §3.4 that if $(k, I) \in \mathscr{B}_k^*$, then $14A^*\rho_k r_0 \leq \operatorname{diam} f_k(I) < 14A^*\rho_{k-1} r_0$.

Lemma 4.7. *If $I \in \mathscr{B}_k^*$, then*

$$\frac{1}{6}(\operatorname{diam} f_k(I))^s \geq P\rho_k^s r_0^s.$$

Proof. Because $\operatorname{diam} f_k(I) \geq 14A^*\rho_k r_0$, it suffices to check that $(14A^*)^s \geq 6P$. Recalling the definition of A^*, we find that

$$(14A^*)^s = \left(\frac{14C^*}{1-\xi_2}\right)^s \geq \frac{12(1.1C^*)^s}{1-\xi_2^s} = 6P.$$

Here $1/(1-\xi_2)^s \geq 1/(1-\xi_2^s)$ because $0 < \xi_2 < 1$ and $s \geq 1$. □

Let T be a finite tree over $[0,1]$, let m be the depth of T and let $0 \leq k \leq m$ be an integer. Define

$$\mathscr{B}_k^*(T) := \{(k, I) \in \mathscr{B}_k^* :\ \text{there exists } (k, J) \in T \cap (\{k\} \times \mathscr{N}_k) \text{ such that } J \cap \overline{I} \neq \emptyset\}.$$

Although the sets $\mathscr{B}_k^*(T)$ are not necessarily subsets of ∂T, we show in the next lemma that each element in ∂T generates at most two elements in $\bigcup_{k=1}^m \mathscr{B}_k^*(T)$.

Lemma 4.8. *Let T be a finite tree over $[0,1]$ with depth $m \geq 1$ and let $k \leq m$.*

(1) *For each $(k, I) \in \mathscr{B}_k^*(T)$, there exists a unique $(l, J) \in \partial T$ such that $I \subset J$. In fact, $J \in \mathscr{E}_l \cup \mathscr{B}_l$.*
(2) *For each $(l, J) \in \partial T$ such that $J \in \mathscr{E}_l \cup \mathscr{B}_l$, there exist at most two distinct $(k, I) \in \bigcup_{i=0}^m \mathscr{B}_i^*(T)$ such that $I \subset J$.*

Proof. The first claim of (1) follows immediately from Remark 4.2. For the second claim, let $(k, I) \in \mathscr{B}_k^*(T)$. There are two cases.

Case 1. If $(k, I) \in T$, then $\mathscr{I}_n(I) = \{I\}$ and $I \in \mathscr{B}_n$ for all $n > k$, since I is a bridge interval. Therefore, $(l, I) \in \partial T$ for some $l \leq m$ and $I \in \mathscr{B}_l$.

Case 2. Suppose now that $(k, I) \notin T$. If J was in $\mathscr{N}_l \cup \mathscr{F}_l$, then the closure of I would be contained in the interior of J and I would intersect only intervals $I' \in \mathscr{I}_{l'}$ for which $(l', I') \notin T$, which is a contradiction.

For (2), fix $(l, J) \in \partial T$ with $J \in \mathscr{E}_l \cup \mathscr{B}_l$ and assume that $\{(k_i, J_i) : i = 1, 2, 3\}$ are distinct elements in $\bigcup_{j=0}^{m} \mathscr{B}_j^*(T)$ such that $J_i \subset J$ for $i = 1, 2, 3$. Without loss of generality, assume that J_2 is between J_1 and J_3, in the orientation of J. Then there exists $l' \geq l + 1$ and $J' \subseteq J$ such that $J' \in \mathscr{N}_{l'}$ and $(l', J') \in T$. But then $(l, J) \notin \partial T$ which is a contradiction. □

We now impose an additional restriction on α_0.

Lemma 4.9. *For all C^*, ξ_1, and ξ_2, there exists $\alpha_1 \in (0, 1/16]$ such that if $\alpha_0 \leq \alpha_1$, then for all $k \geq 0$, $(v, v') \in \mathsf{Flat}(k)$, and $y, y' \in V_{k+1}(v, v')$, we have $|y - y'| \leq |v - v'|$.*

Proof. Enumerate $V_{k+1}(v, v') = \{v_1, \ldots, v_n\}$ from left to right, so that $v_1 = v$ and $v_n = v'$. Let $y = v_l$ and $y' = v_m$ for some $1 \leq l < m \leq n$. If $y = v$ and $y' = v'$, the conclusion is trivial. Thus, let us suppose that there exists at least one point to the left of y or the right of y', say without loss of generality that $l \geq 2$. Let $x_i = \pi_{\ell_{k,v}}(v_i)$ for all $1 \leq i \leq n$. Then, arguing as in the proof of Lemma 2.2,

$$\frac{|v_1 - v_2| + |v_l - v_m|}{1 + 3\alpha_0^2} \leq |x_1 - x_2| + |x_l - x_m| \leq |x_1 - x_n| \leq |v - v'|.$$

Because V_{k+1} is ρ_{k+1}-separated,

$$\begin{aligned} |v_l - v_m| \leq (1 + 3\alpha_0^2)|v - v'| - |v_1 - v_2| &\leq (1 + 3\alpha_0^2)|v - v'| - \rho_{k+1} r_0 \\ &= |v - v'| \left(1 + 3\alpha_0^2 - \frac{\rho_{k+1} r_0}{|v - v'|}\right). \end{aligned}$$

Now $|v - v'| \leq 14 A^* \rho_k r_0 \leq 14 A^* \xi_1^{-1} \rho_{k+1} r_0$. Hence $|v_l - v_m| \leq |v - v'|$ provided that

$$1 + 3\alpha_0^2 - \frac{\xi_1}{14 A^*} \leq 1.$$

Thus, we can take

$$\alpha_1 := \min\left\{\frac{1}{16}, \left(\frac{\xi_1}{42 A^*}\right)^{1/2}\right\}. \quad \square \tag{4.2}$$

Together Lemma 4.8 and Lemma 4.9 yield the following result.

Corollary 4.10. *Assume $\alpha_0 \leq \alpha_1$. If T is a finite tree over $[0, 1]$ of depth m, then,*

$$\sum_{k=1}^{m} \sum_{(k,J) \in \mathscr{B}_k^*(T)} (\operatorname{diam} f_k(J))^s \leq 2 \sum_{(l,J) \in \partial T} (\operatorname{diam} f_l(J))^s.$$

4.5. An upper bound for the total mass

Here we prove the following proposition which gives an upper bound for the mass of $[0,1]$ in terms of the variation excess $\tau_s(k,v,v')$ of flat pairs $(v,v') \in \mathsf{Flat}(k)$ defined in §2.2.

Proposition 4.11. *Assume that $\alpha_0 \le \alpha_1$ (see Lemma 4.9). For all $s \ge 1$,*

$$\mathcal{M}_s([0,1]) \lesssim_{s,C^*,\xi_2} r_0^s + \sum_{k=0}^{\infty} \sum_{(v,v')\in\mathsf{Flat}(k)} \tau_s(k,v,v')\rho_k^s r_0^s + \sum_{k=0}^{\infty} \sum_{\substack{v\in V_k \\ \alpha_{k,v}\ge\alpha_0}} \rho_k^s r_0^s.$$

The proof of Proposition 4.11 reduces to proving the following lemma (cf. [9, (9.4)]). Recall the definition of the parent tree from §4.1.

Lemma 4.12. *Let $k \ge 1$ be an integer, let T be a finite tree over $[0,1]$ of depth $k+1$ and let $p(T)$ be the parent tree. There exists a constant $C > 0$ depending only on s, C^*, and ξ_2 such that*

$$\begin{aligned}
&\sum_{(l,I)\in\partial T} (\operatorname{diam} f_l(I))^s + \sum_{(k+1,I)\in\mathscr{P}_{k+1}\cap\partial T} p_{k+1,I} \\
&\le \sum_{(l,I)\in\partial p(T)} (\operatorname{diam} f_l(I))^s + \sum_{(k,I)\in\mathscr{P}_k\cap\partial p(T)} p_{k,I} + \frac{1}{3} \sum_{(k+1,I)\in\mathscr{B}_{k+1}^*(T)} (\operatorname{diam} f_{k+1}(I))^s \\
&\quad + C \sum_{\substack{v\in V_k \\ \alpha_{k,v}\ge\alpha_0}} \rho_k^s r_0^s + C \sum_{\substack{w\in V_{k+1} \\ \alpha_{k+1,w}\ge\alpha_0}} \rho_{k+1}^s r_0^s + C \sum_{(v,v')\in\mathsf{Flat}(k)} \tau_s(k,v,v')\rho_k^s r_0^s.
\end{aligned} \tag{E}$$

We prove Lemma 4.12 in §4.6. Assuming that Lemma 4.12 holds, here is the proof of Proposition 4.11.

Proof of Proposition 4.11. Assume that $\alpha_1 \le \alpha_0$. Let T be a finite tree over $[0,1]$ of depth m. Set $T_m = T$ and for each $1 \le k \le m$ (if any) set $T_{k-1} = p(T_k)$. Note that $T_0 = \{0\} \times \mathscr{I}_0$. By Lemma 4.12, for all $1 \le k \le m$ (if any),

$$\begin{aligned}
&\sum_{(l,I)\in\partial T_k} (\operatorname{diam} f_l(I))^s + \sum_{(k,I)\in\mathscr{P}_k\cap\partial T_k} p_{k,I} \\
&\le \sum_{(l,I)\in\partial T_{k-1}} (\operatorname{diam} f_l(I))^s + \sum_{(k-1,I)\in\mathscr{P}_{k-1}\cap\partial T_{k-1}} p_{k-1,I} + \frac{1}{3} \sum_{(k,I)\in\mathscr{B}_k^*(T_k)} (\operatorname{diam} f_k(I))^s \\
&\quad + C \sum_{\substack{v\in V_{k-1} \\ \alpha_{k-1,v}\ge\alpha_0}} \rho_{k-1}^s r_0^s + C \sum_{\substack{w\in V_k \\ \alpha_{k,w}\ge\alpha_0}} \rho_k^s r_0^s + C \sum_{(v,v')\in\mathsf{Flat}(k-1)} \tau_s(k-1,v,v')\rho_{k-1}^s r_0^s.
\end{aligned}$$

Iterating the latter inequality,

$$\sum_{(l,I)\in\partial T}(\operatorname{diam} f_l(I))^s \le \sum_{I\in\mathscr{E}_0\cup\mathscr{B}_0}(\operatorname{diam} f_0(I))^s + \sum_{I\in\mathscr{P}_0} p_{0,I} + \frac{1}{3}\sum_{k=1}^{m}\sum_{I\in\mathscr{B}_k^*(T)}(\operatorname{diam} f_k(I))^s + 2C\sum_{k=0}^{m}\sum_{\substack{v\in V_k\\ \alpha_{k,v}\ge\alpha_0}}\rho_k^s r_0^s + C\sum_{k=1}^{m}\sum_{(v,v')\in\mathsf{Flat}(k)}\tau_s(k,v,v')\rho_k^s r_0^s.$$

Since $\alpha_0 \le \alpha_1$, we obtain $\frac{1}{3}\sum_{k=1}^{m}\sum_{I\in\mathscr{B}_k^*(T)}(\operatorname{diam} f_k(I))^s \le \frac{2}{3}\sum_{(l,I)\in\partial T}(\operatorname{diam} f_l(I))^s$ by Corollary 4.10. This is the only place in the proof of the Proposition that we use the restriction $\alpha_0 \le \alpha_1$. Therefore,

$$\sum_{(l,I)\in\partial T}(\operatorname{diam} f_l(I))^s \le 3\sum_{I\in\mathscr{E}_0\cup\mathscr{B}_0}(\operatorname{diam} f_0(I))^s + 3\sum_{I\in\mathscr{P}_0} p_{0,I} + 6C\sum_{k=0}^{m}\sum_{\substack{v\in V_k\\ \alpha_{k,v}\ge\alpha_0}}\rho_k^s r_0^s + 3C\sum_{k=0}^{m-1}\sum_{(v,v')\in\mathsf{Flat}(k)}\tau_s(k,v,v')\rho_k^s r_0^s.$$

There are now two alternatives. On one hand, suppose that $\alpha_{0,v} < \alpha_0$ for some $v \in V_0$. Then V_0 projects onto an $(1+3(1/16)^2)^{-1}r_0$ separated set in $\ell_{0,v}$ of diameter at most $2C^*r_0$ by Lemma 2.1. Hence $\operatorname{card} V_0 \lesssim_{C^*} 1$ and

$$\sum_{I\in\mathscr{E}_0\cup\mathscr{B}_0}(\operatorname{diam} f_0(I))^s + \sum_{I\in\mathscr{P}_0} p_{0,I} \lesssim_{s,C^*,\xi_2} r_0^s.$$

On the other hand, suppose that $\alpha_{0,v} \ge \alpha_0$ for all $v \in V_0$. Then

$$\sum_{I\in\mathscr{E}_0\cup\mathscr{B}_0}(\operatorname{diam} f_0(I))^s + \sum_{I\in\mathscr{P}_0} p_{0,I} \lesssim_{s,C^*,\xi_2} \sum_{\substack{v\in V_0\\ \alpha_{0,v}\ge\alpha_0}} r_0^s.$$

In either case, we arrive at

$$\sum_{(l,I)\in\partial T}(\operatorname{diam} f_l(I))^s \lesssim_{s,C^*,\xi_2} r_0^s + \sum_{k=0}^{m-1}\sum_{(v,v')\in\mathsf{Flat}(k)}\tau_s(k,v,v')\rho_k^s r_0^s + \sum_{k=0}^{m}\sum_{\substack{v\in V_k\\ \alpha_{k,v}\ge\alpha_0}}\rho_k^s r_0^s$$
$$\lesssim_{s,C^*,\xi_2} r_0^s + \sum_{k=0}^{\infty}\sum_{(v,v')\in\mathsf{Flat}(k)}\tau_s(k,v,v')\rho_k^s r_0^s + \sum_{k=0}^{\infty}\sum_{\substack{v\in V_k\\ \alpha_{k,v}\ge\alpha_0}}\rho_k^s r_0^s.$$

Since T was an arbitrary tree over $[0,1]$, we obtain the desired bound on $\mathcal{M}_s([0,1])$. □

4.6. Proof of Lemma 4.12

The proof is divided into five estimates (E1), (E2), (E3), (E4) and (E5), whose sum gives (E). Towards this end, we split the left hand side of (E) into smaller sums by making the following four decompositions.

Firstly, ∂T can be partitioned as $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{F} \cup (\partial T \cap \partial p(T))$, where

$$\begin{aligned}
\mathcal{E}_1 &= \{(k+1, I) \in \partial T : I \in \mathscr{E}_{k+1}(J) \text{ and } J \in \mathscr{N}_k \text{ is as in §3.8}\},\\
\mathcal{E}_2 &= \{(k+1, I) \in \partial T : I \in \mathscr{E}_{k+1}(J) \text{ and } J \in \mathscr{N}_k \text{ is as in §3.7}\},\\
\mathcal{E}_3 &= \{(k+1, I) \in \partial T : I \in \mathscr{E}_{k+1}(J) \text{ and } J \in \mathscr{E}_k \text{ is as in §3.4}\},\\
\mathcal{B}_1 &= \{(k+1, I) \in \partial T : I \in \mathscr{B}_{k+1}(J) \text{ and } J \in \mathscr{N}_k \text{ is as in §3.8}\},\\
\mathcal{B}_2 &= \{(k+1, I) \in \partial T : I \in \mathscr{B}_{k+1}(J) \text{ and } J \in \mathscr{E}_k \text{ is as in §3.5}\},\\
\mathcal{B}_3 &= \{(k+1, I) \in \partial T : I \in \mathscr{B}_{k+1}(J) \text{ and } J \in \mathscr{E}_k \text{ is as in §3.4}\},\\
\mathcal{B}_4 &= \{(k+1, I) \in \partial T : I \in \mathscr{B}_{k+1}(J) \text{ and } J \in \mathscr{B}_k \text{ is as in §3.3}\},\\
\mathcal{F} &\subseteq \{k+1\} \times (\mathscr{N}_{k+1} \cup \mathscr{F}_{k+1}).
\end{aligned}$$

Secondly, $\mathscr{P}_{k+1} \cap \partial T$ can be partitioned as $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$, where

$$\begin{aligned}
\mathcal{P}_1 &= \{(k+1, I) \in \partial T \cap \mathscr{P}_{k+1} : I \in \mathscr{N}_{k+1}(J) \text{ and } J \in \mathscr{N}_k \text{ is as in §3.8}\},\\
\mathcal{P}_2 &= \{(k+1, I) \in \partial T \cap \mathscr{P}_{k+1} : I \in \mathscr{N}_{k+1}(J) \text{ and } J \in \mathscr{N}_k \text{ is as in §3.7}\},\\
\mathcal{P}_3 &= \{(k+1, I) \in \partial T \cap \mathscr{P}_{k+1} : I \in \mathscr{N}_{k+1}(J) \text{ and } J \in \mathscr{E}_k \text{ is as in §3.4}\}.
\end{aligned}$$

Thirdly, $\partial p(T)$ can be partitioned as $\mathcal{E}'_1 \cup \mathcal{E}'_2 \cup \mathcal{F}' \cup (\partial T \cap \partial p(T))$, where

$$\begin{aligned}
\mathcal{E}'_1 &= \{(k, I) \in \partial p(T) \setminus \partial T : I \in \mathscr{E}_k \text{ is as in §3.5}\},\\
\mathcal{E}'_2 &= \{(k, I) \in \partial p(T) \setminus \partial T : I \in \mathscr{E}_k \text{ is as in §3.4}\},\\
\mathcal{B}'_1 &= \{(k, I) \in \partial p(T) \setminus \partial T : I \in \mathscr{B}_k \text{ is as in §3.3}\},\\
\mathcal{F}' &\subseteq \{k\} \times (\mathscr{N}_k \cup \mathscr{F}_k).
\end{aligned}$$

Fourthly, set $\mathscr{B}^{**}_{k+1}(T) = \mathscr{B}^{*}_{k+1}(T) \sqcup \mathscr{B}^{*}_{k+1}(T)$. Then define collections $\mathcal{B}^*_1$ and $\mathcal{B}^*_2$ as follows. If $I \in \mathscr{N}_k$, $\alpha_{k,f_k(I)} < \alpha_0$, $\mathscr{I}_{k+1}(I) \subset \partial T$, and $f_k(I)$ is an endpoint of the image $f_{k+1}(J)$ of $(k+1, J) \in \mathscr{B}^*_{k+1}$, then we include a copy of $(k+1, J)$ from $\mathscr{B}^{**}_{k+1}(T)$ in $\mathcal{B}^*_1$. If $K \in \mathscr{N}_k$ and $f_k(K)$ is the endpoint of $f_{k+1}(I)$ for some $(k+1, I) \in \mathcal{B}_3$ that lies strictly between the endpoints of the image $f_k(J)$ of the associated edge interval $J \in \mathscr{E}_k$, then we include a copy of $(k+1, I)$ from $\mathscr{B}^{**}_{k+1}(T)$ in $\mathcal{B}^*_2$. Because each bridge has only two endpoints, we can choose the included copies so that $\mathcal{B}^*_1 \cup \mathcal{B}^*_2 \subset \mathscr{B}^{**}_{k+1}$.

Before proceeding to the estimates, we remark that

$$\sum_{(k,I)\in\mathcal{F}'} (\operatorname{diam} f_k(I))^s = \sum_{(k+1,I)\in\mathcal{F}} (\operatorname{diam} f_{k+1}(I))^s = 0,$$

because f_k is constant on each interval in $\mathscr{N}_k \cup \mathscr{F}_k$ by (P2).

Estimate 1. Here we deal with phantom masses and new intervals coming from some $I \in \mathscr{N}_k$ whose image is not flat. In particular, we will show that there exists C_1 depending only on s, C^*, and ξ_2 such that

$$\begin{aligned}\sum_{(k+1,I)\in\mathcal{E}_1\cup\mathcal{B}_1} (\operatorname{diam} f_{k+1}(I))^s + \sum_{(k+1,J)\in\mathcal{P}_1} p_{k+1,J} \\ \le C_1 \sum_{\substack{v\in V_k\\ \alpha_{k,v}\ge\alpha_0}} \rho_k^s r_0^s + C_1 \sum_{\substack{w\in V_{k+1}\\ \alpha_{k+1,w}\ge\alpha_0}} \rho_{k+1}^s r_0^s.\end{aligned} \tag{E1}$$

Since

$$\{I \in \mathscr{N}_k : \alpha_{k,f_k(I)} \ge \alpha_0 \text{ and } \mathscr{I}_{k+1}(I) \subset \partial T\} \subseteq \{I \in \mathscr{N}_k \cap \partial p(T) : \alpha_{k,f_k(I)} \ge \alpha_0\},$$

inequality (E1) follows from the inequality

$$\begin{aligned}\sum_{\substack{I\in\mathscr{N}_k\\ \{k+1\}\times\mathscr{I}_{k+1}(I)\subset\partial T\\ \alpha_{k,f_k(I)}\ge\alpha_0}} \left(\sum_{J\in\mathscr{E}_{k+1}(I)\cup\mathscr{B}_{k+1}(I)} (\operatorname{diam} f_{k+1}(I))^s + \sum_{\substack{J\in\mathscr{N}_{k+1}(I)\\ (k+1,J)\in\mathscr{P}_{k+1}}} p_{k+1,J} \right) \\ \le C_1 \sum_{\substack{I\in\mathscr{N}_k\\ \{k+1\}\times\mathscr{I}_{k+1}(I)\subset\partial T\\ \alpha_{k,f_k(I)}\ge\alpha_0}} \rho_k^s r_0^s + C_1 \sum_{\substack{I\in\mathscr{N}_k\\ \{k+1\}\times\mathscr{I}_{k+1}(I)\subset\partial T\\ \alpha_{k,f_k(I)}\ge\alpha_0}} \sum_{\substack{J\in\mathscr{N}_{k+1}(I)\\ \alpha_{k+1,f_{k+1}(J)}\ge\alpha_0}} \rho_{k+1}^s r_0^s.\end{aligned} \tag{4.3}$$

To prove (4.3), fix any $I \in \mathscr{N}_k$ such that $\{k+1\} \times \mathscr{I}_{k+1}(I) \subset \partial T$ and $\alpha_{k,f_k(I)} \ge \alpha_0$. Recall that $\mathcal{N}_{k+1}(I)$ is in one-to-one correspondence with $V_{k+1,I}$ defined in §3.8. There are now two possibilities. On one hand, suppose that $\alpha_{k+1,w} < \alpha_0$ for some $w \in V_{k+1,I}$. Then $V_{k+1,I}$ projects onto an $(1+3(1/16)^2)^{-1}\rho_{k+1}r_0$ separated set in $\ell_{k+1,w}$ of diameter at most $2C^*\rho_{k+1}r_0$ by Lemma 2.1. Hence $\operatorname{card} \mathscr{I}_{k+1}(I) \lesssim_{C^*} 1$ and

$$\sum_{J\in\mathscr{E}_{k+1}(I)\cup\mathscr{B}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s + \sum_{\substack{J\in\mathscr{N}_{k+1}(I)\\ (k+1,J)\in\mathscr{P}_{k+1}}} p_{k+1,J} \lesssim_{s,C^*,\xi_2} \rho_{k+1}^s r_0^s \lesssim_{s,C^*,\xi_2} \rho_k^s r_0^s.$$

On the other hand, suppose that $\alpha_{k+1,w} \ge \alpha_0$ for all $w \in V_{k+1,I}$. Then

$$\sum_{J\in\mathscr{E}_{k+1}(I)\cup\mathscr{B}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s + \sum_{\substack{J\in\mathscr{N}_{k+1}(I)\\ (k+1,J)\in\mathscr{P}_{k+1}}} p_{k+1,J} \lesssim_{s,C^*,\xi_2} \sum_{J\in\mathscr{N}_{k+1}(I)} \rho_{k+1}^s r_0^s.$$

Because the sets $V_{k+1,I}$ for different $I \in \mathscr{N}_k$ are pairwise disjoint (see §3.8), summing over all such I yields (4.3).

Estimate 2. We estimate the new phantom masses and new intervals coming from some $I \in \mathscr{N}_k$ such that $\alpha_{k,f_k(I)} < \alpha_0$ and $\mathscr{I}_{k+1}(I) \subset \partial T$. In particular, we show that

$$\begin{aligned} &\sum_{(k+1,I)\in\mathcal{E}_2} (\operatorname{diam} f_{k+1}(I))^s + \sum_{(k+1,I)\in\mathcal{P}_2} p_{k+1,I} \\ &\qquad \le \sum_{(k,I)\in\mathscr{P}_k\cap\partial p(T)} p_{k,I} + \frac{1}{6} \sum_{(k+1,I)\in\mathcal{B}_1^*} (\operatorname{diam} f_{k+1}(I))^s. \end{aligned} \tag{E2}$$

This estimate is responsible for the choice of the constant $14A^*$ appearing in the definition of $\mathsf{Flat}(k)$, and thus, for the constant $30A^*$ appearing in the definition of $\alpha_{k,v}$. Inequality (E2) is equivalent to

$$\begin{aligned} &\sum_{\substack{I\in\mathscr{N}_k \\ \{k+1\}\times\mathscr{N}_{k+1}(I)\subset\partial T \\ \alpha_{k,f_k(I)}<\alpha_0}} \left(\sum_{J\in\mathscr{E}_{k+1}(I)} (\operatorname{diam} f_{k+1}(I))^s + \sum_{\substack{J\in\mathscr{N}_{k+1}(I) \\ (k+1,J)\in\mathscr{P}_{k+1}}} p_{k+1,J} \right) \\ &\qquad \le \sum_{\substack{I\in\mathscr{N}_k \\ \{k+1\}\times\mathscr{N}_{k+1}(I)\subset\partial T \\ \alpha_{k,f_k(I)}<\alpha_0}} p_{k,I} + \frac{1}{6} \sum_{(k+1,I')\in\mathcal{B}_1^*} (\operatorname{diam} f_{k+1}(I))^s. \end{aligned} \tag{4.4}$$

To prove (4.4) fix $I \in \mathscr{N}_k$ such that $\{k+1\} \times \mathscr{N}_{k+1}(I) \subset \partial T$ and $\alpha_{k,f_k(I)} < \alpha_0$. There are nine cases (1a, 1b, 2a, 2b, 2c, 2d, 3a, 3b, 3c). We sincerely apologize to the reader.

For the first four cases (1a, 1b, 2a, 2b), we show that

$$\sum_{J\in\mathscr{E}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s + \sum_{\substack{J\in\mathscr{N}_{k+1}(I) \\ (k+1,J)\in\mathscr{P}_{k+1}}} p_{k+1,J} \le p_{k,I}. \tag{4.5}$$

Case 1. Suppose $f_k(I)$ is 2-sided terminal in V_k.

Case 1a. If $\mathscr{N}_{k+1}(I) = \{I\}$, then $f_{k+1}(I)$ is 2-sided terminal in V_{k+1}, $\mathscr{E}_{k+1}(I) = \emptyset$ and the new phantom mass $p_{k+1,I} = 2P\rho_{k+1}^s r_0^s$ is dominated by the old phantom mass $p_{k,I} = 2P\rho_k^s r_0^s$. Hence (4.5) holds.

Case 1b. Assume that $\mathscr{N}_{k+1}(I)$ contains at least two elements (see Fig. 2 above). In this case, at most two elements of $\mathcal{N}_{k+1}(I)$ map to 1-sided terminal vertices in V_{k+1}. By Lemma 4.6,

$$\begin{aligned} p_{k+1,J_1} + p_{k+1,J_2} + \sum_{J\in\mathscr{I}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s &\le 2P\rho_{k+1}^s r_0^s + 4(1.1C^*)^s \rho_{k+1}^s r_0^s \\ &\le 2P\rho_k^s r_0^s = p_{k,I}, \end{aligned}$$

because P was chosen to be sufficiently large such that

$$[P + 2(1.1C^*)^s]\xi_2^s \leq P.$$

Thus, (4.5) holds in this case, as well.

Case 2. Suppose that $f_k(I)$ is 1-sided terminal in V_k. Let (v_1, v') be the unique element in $\mathsf{Flat}(k)$ with $v_1 = f_k(I)$ and let v_2 be the first vertex in V_{k+1} between v_1 and v' in the direction going from v_1 to v'. By property (P7), we can find an interval L in $\mathscr{E}_k$ such that $f_k(L) = (v_1, v')$ and $I \cap \overline{L} \neq \emptyset$. Let K be an interval in $\mathscr{I}_{k+1}(L)$ such that $f_k(K) = (v_1, v_2)$. There will be four cases, depending on whether $\mathscr{N}_{k+1}(I)$ contains one or more elements and whether K belongs to $\mathscr{E}_{k+1}(L)$ or $\mathscr{B}_{k+1}(L)$.

Case 2a. Suppose that $\mathscr{N}_{k+1}(I) = \{I\}$ and $K \in \mathscr{E}_{k+1}(L)$. Then $f_{k+1}(I)$ is 1-sided terminal in V_{k+1}, $\mathscr{E}_{k+1}(I) = \emptyset$, and the new phantom mass $p_{k+1,I} = P\rho_{k+1}^s r_0^s$ is dominated by the old phantom mass $p_{k,I} = P\rho_k^s r_0^s$. Hence (4.5) holds.

Case 2b. Suppose that $\mathscr{N}_{k+1}(I)$ contains at least two elements (see Fig. 2 above) and $K \in \mathscr{E}_{k+1}(L)$. Then at most one element of $\mathcal{N}_{k+1}(K)$ maps to a 1-sided terminal vertex in V_{k+1}. By Lemma 4.6,

$$\begin{aligned} p_{k+1,J_1} + \sum_{J \in \mathscr{I}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s &\leq P\rho_{k+1}^s r_0^s + 2(1.1C^*)^s \rho_{k+1}^s r_0^s \\ &\leq P\rho_k^s r_0^s = p_{k,I}, \end{aligned}$$

because P was chosen to be sufficiently large such that

$$[P + 2(1.1C^*)^s]\xi_2^s \leq P.$$

Thus, (4.5) holds, once again.

Case 2c. Suppose that $\mathscr{N}_{k+1}(I) = \{I\}$ and $K \in \mathscr{B}_{k+1}(L)$. Then $f_{k+1}(I)$ is 2-sided terminal in V_{k+1} and $\mathscr{E}_{k+1}(I) = \emptyset$. The new phantom mass that must be paid for is $p_{k+1,I} = 2P\rho_{k+1}^s r_0^s$. In this case, we pay for one half of $p_{k+1,I}$ with $p_{k,I} = P\rho_k^s r_0^s$ and use Lemma 4.7 to pay for the other half of $p_{k+1,I}$ with $\frac{1}{6}(\operatorname{diam} f_{k+1}(K))^s$, where $K \in \mathscr{B}_{k+1}^*(T)$. That is,

$$p_{k+1,I} \leq p_{k,I} + \frac{1}{6}(\operatorname{diam} f_{k+1}(K))^s.$$

Case 2d. Suppose that $\mathscr{N}_{k+1}(I)$ contains at least two points and $K \in \mathscr{B}_{k+1}(L)$. Then $f_{k+1}(I)$ is 1-sided terminal in V_{k+1}, $\mathscr{E}_{k+1}(I)$ is nonempty, and up to one of the new vertices drawn could be 1-sided terminal in V_{k+1}, as well. In this case,

$$\underbrace{p_{k+1,I} + p_{k+1,J_1} + \sum_{J \in \mathscr{E}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s}_{} \leq \frac{1}{6}(\operatorname{diam} f_{k+1}(K))^s + p_{k,I},$$

by Lemma 4.7, Lemma 4.6, and the choice of P.

Case 3. Suppose $f_k(I)$ is not terminal in V_k. Then $\mathscr{E}_{k+1}(I) = \emptyset$. It remains to pay for $p_{k+1,I}$ as needed. Let L_1, L_2, K_1, and K_2 be defined by analogy with L and K from Case 2, but corresponding to the two distinct flat pairs $(f_k(I), v')$ and $(f_k(I), v'')$. There are three cases, depending on whether K_1 and K_2 are both edge intervals, one of K_1 or K_2 is an edge interval and the other is a bridge interval, or both K_1 and K_2 are bridge intervals.

Case 3a. Suppose that K_1 belongs to $\mathscr{E}_{k+1}(L_1)$ and K_2 belongs to $\mathscr{E}_{k+1}(L_2)$. Then $f_{k+1}(I)$ is non-terminal in V_{k+1}. Hence $(k+1, I) \notin \mathscr{P}_{k+1}$ and both sides of (4.5) are zero. In other words, there is nothing to pay for in Case 3a.

Case 3b. Suppose that one of K_1 or K_2 is an edge interval and the other is a bridge interval, say without loss of generality that $K_1 \in \mathscr{E}_{k+1}(L_1)$ and $K_2 \in \mathscr{B}_{k+1}(L_2)$. Then $f_{k+1}(I)$ is 1-sided terminal in V_{k+1} and $p_{k+1,I} = P\rho_{k+1}^s r_0^s \leq \frac{1}{6}(\operatorname{diam} f_{k+1}(K_2))^s$ by Lemma 4.7.

Case 3c. Suppose that K_1 belongs to $\mathscr{B}_{k+1}(L_1)$ and K_2 belongs to $\mathscr{B}_{k+1}(L_2)$. Then $f_{k+1}(I)$ is 2-sided terminal in V_{k+1} and

$$p_{k+1,I} = 2P\rho_{k+1}^s r_0^s \leq \frac{1}{6}(\operatorname{diam} f_{k+1}(K_1))^s + \frac{1}{6}(\operatorname{diam} f_{k+1}(K_2))^s.$$

Adding up the estimates in the nine cases, we obtain (E2).

Estimate 3. On one hand, $(k, I) \in \mathcal{B}_1'$ if and only if $(k+1, I) \in \mathcal{B}_4$ (see §3.3). When $(k, I) \in \mathcal{B}_1'$, we have $\operatorname{diam} f_k(I) = \operatorname{diam} f_{k+1}(I)$. Thus,

$$\sum_{(k+1,I)\in\mathcal{B}_4} (\operatorname{diam} f_{k+1}(I))^s = \sum_{(k,I)\in\mathcal{B}_1'} (\operatorname{diam} f_k(I))^s.$$

On the other hand, when both endpoints of the image of an edge interval are non-flat, the edge interval becomes a bridge interval and pays for itself (see §3.5):

$$\sum_{(k+1,I)\in\mathcal{B}_2} (\operatorname{diam} f_{k+1}(I))^s = \sum_{(k,I)\in\mathcal{E}_1'} (\operatorname{diam} f_k(I))^s.$$

All together, we have

$$\sum_{(k+1,I)\in\mathcal{B}_2\cup\mathcal{B}_4} (\operatorname{diam} f_{k+1}(I))^s = \sum_{(k,I)\in\mathcal{E}_1'\cup\mathcal{B}_1'} (\operatorname{diam} f_k(I))^s. \tag{E3}$$

Estimate 4. Next, we control the new phantom masses at endpoints of images $f_{k+1}(J)$ of bridge intervals $J \in \mathscr{B}_{k+1}(I)$ coming from some edge interval $I \in \mathscr{E}_k$ as in §3.4 such that the endpoint lies between the endpoints of $f_k(I)$. Specifically, we show that

$$\sum_{(k+1,I)\in\mathcal{P}_3} p_{k+1,I} \leq \frac{1}{6} \sum_{(k+1,I)\in\mathcal{B}_2^*} (\operatorname{diam} f_{k+1}(I))^s. \tag{E4}$$

Inequality (E4) is equivalent to

$$\sum_{(k,I)\in\mathcal{E}_2'} \sum_{\substack{J\in\mathscr{N}_{k+1}(I)\\(k+1,J)\in\mathscr{P}_{k+1}}} p_{k+1,J} \le \frac{1}{6} \sum_{(k+1,I)\in\mathcal{B}_2^*} (\operatorname{diam} f_{k+1}(I))^s. \tag{4.6}$$

To prove (4.6), fix $(k,I)\in\mathcal{E}_2'$ and $J\in\mathscr{N}_{k+1}(I)$ be such that $(k+1,J)\in\mathscr{P}_{k+1}$. Then $f_{k+1}(J)$ lies strictly between the endpoints of $f_k(I)$ and $f_{k+1}(J)$ is either 1- or 2-sided terminal in V_{k+1}. On one hand, if $f_{k+1}(J)$ is 1-sided terminal, then there exists precisely one element $(k+1,K)\in\mathcal{B}_2^*$ such that $f_{k+1}(J)$ is an endpoint of $f_{k+1}(K)$. In this case,

$$p_{k+1,J} = P\rho_{k+1}^s r_0^s \le \frac{1}{6}(\operatorname{diam} f_{k+1}(K))^s$$

by Lemma 4.7. On the other hand, if $f_{k+1}(J)$ is 2-sided terminal, then there exist two elements $(k+1,K_1)$ and $(k+1,K_2)$ in $\mathcal{B}_2^*$ such that $f_{k+1}(J)$ is the common endpoint of $f_{k+1}(K_1)$ and $f_{k+1}(K_2)$. In this case,

$$p_{k+1,J} = 2P\rho_{k+1}^s r_0^s \le \frac{1}{6}(\operatorname{diam} f_{k+1}(K_1))^s + \frac{1}{6}(\operatorname{diam} f_{k+1}(K_2))^s$$

by Lemma 4.7.

Remark 4.13. In Estimates 2 and Estimate 4, each endpoint of the image $f_{k+1}(I)$ of $(k+1,I)\in\mathcal{B}_1^*\cup\mathcal{B}_2^*$ is used once and each $f_{k+1}(I)$ has only two endpoints. Hence

$$\frac{1}{6}\sum_{(k+1,I)\in\mathcal{B}_1^*\cup\mathcal{B}_2^*} (\operatorname{diam} f_{k+1}(I))^s \le \frac{1}{6}\sum_{(k+1,I)\in\mathscr{B}_{k+1}^{**}(T)} = \frac{1}{3}\sum_{(k+1,I)\in\mathscr{B}_{k+1}^{*}(T)} .$$

Estimate 5. In this final estimate, we deal with new intervals in ∂T coming from an edge interval in $\partial p(T)$ which has an endpoint with flat image. We will show that

$$\begin{aligned}&\sum_{(k+1,I)\in\mathcal{E}_3\cup\mathcal{B}_3} (\operatorname{diam} f_{k+1}(I))^s\\ &\quad\le \sum_{(k,I)\in\mathcal{E}_2'} (\operatorname{diam} f_{k+1}(I))^s + (14A^*)^s \sum_{(v,v')\in\mathsf{Flat}(k)} \tau_s(k,v,v')\rho_k^s r_0^s.\end{aligned} \tag{E5}$$

For each $I\in\mathcal{E}_2'$, pick an endpoint x_I of I such that $\alpha_{k,f_k(x_I)} < \alpha_0$ and let y_I be the other endpoint of I. Estimate (E5) follows immediately from

$$\begin{aligned}&\sum_{(k,I)\in\mathcal{E}_2'} \sum_{J\in\mathscr{E}_{k+1}(I)\cup\mathscr{B}_{k+1}(I)} (\operatorname{diam} f_{k+1}(J))^s\\ &\quad\le \sum_{(k,I)\in\mathcal{E}_2'} ((\operatorname{diam} f_k(I))^s + (14A^*)^s\tau_s(k,f_k(x_I),f_k(y_I)))\rho_k^s r_0^s.\end{aligned} \tag{4.7}$$

To show (4.7), fix $(k, I) \in \mathcal{E}_2'$ and enumerate $\mathscr{E}_{k+1}(I) \cup \mathscr{B}_{k+1}(I) = \{I_1, \dots, I_n\}$. Let $v = f_k(x_I)$ and let $v' = f_k(y_I)$. By definition of $\tau_s(k, v, v')$ and (P3), we have

$$\begin{aligned}\sum_{i=1}^{n-1} (\operatorname{diam} f_{k+1}(I_i))^s &\le |v - v'|^s + \tau_s(k, v, v')|v - v'|^s \\ &\le (\operatorname{diam} f_k(I))^s + (14A^*)^2 \tau_s(k, v, v')\rho_k^s r_0^s.\end{aligned}$$

Summing over all pairs $(k, I) \in \mathcal{E}_2'$, we obtain (4.7).

Adding (E1), (E2), (E3), (E4), and (E5), we arrive at (E). This completes the proof of Lemma 4.12.

5. Hölder parametrization

In §5.1, we prove the following theorem, which is the paper's main result. Afterwards, in §5.2, we derive several corollaries, including Theorem 1.1. In §5.3, we state and prove a refinement of Theorem 5.1 that gives an essentially 2-to-1 curve. In §5.4, we show that replacing (5.1) with a Carleson type condition produces an upper Ahlfors regular curve.

Given parameters C^*, ξ_1, and ξ_2, let α_1 be defined by (4.2). That is,

$$\alpha_1 = \min\left\{\frac{1}{16}, \left(\frac{\xi_1(1-\xi_2)}{42C^*}\right)^{1/2}\right\}.$$

Theorem 5.1 *(Hölder Traveling Salesman with Nets). Assume that $X = l^2(\mathbb{R})$ or $X = \mathbb{R}^N$ for some $N \ge 2$. Let $s \ge 1$, let $\mathscr{V} = (V_k, \rho_k)_{k\ge 0}$ be a sequence of finite sets V_k in X and numbers $\rho_k > 0$ that satisfy properties (V0)–(V5) defined in §2.2. If $\alpha_0 \in (0, \alpha_1]$ and*

$$S_{\mathscr{V}}^s := \sum_{k=0}^{\infty} \sum_{(v,v')\in \mathsf{Flat}(k)} \tau_s(k, v, v')\rho_k^s + \sum_{k=0}^{\infty} \sum_{\substack{v\in V_k \\ \alpha_{k,v}\ge\alpha_0}} \rho_k^s < \infty, \tag{5.1}$$

then there exists a $(1/s)$-Hölder map $f : [0, 1] \to X$ such that $f([0, 1]) \supset \bigcup_{k\ge 0} V_k$ and the Hölder constant of f satisfies $H \lesssim_{s,C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}}^s)$.

5.1. Proof of Theorem 5.1

In this subsection, w will always denote a finite word in the alphabet $\mathbb{N} = \{1, 2, \dots\}$, including the empty word $\emptyset$. We denote the length of a word w by $|w|$.

The conclusion holds trivially if $\bigcup_{k\ge 0} V_k$ is a singleton. Thus, in addition to $S_{\mathscr{V}}^s < \infty$, we may assume that $\bigcup_{k\ge 0} V_k$ contains at least two points. Because $\alpha_0 \le \alpha_1$, Proposition 4.11 gives

$$0 < \mathcal{M}_s([0, 1]) \lesssim_{s,C^*,\xi_2} r_0^s(1 + S_{\mathscr{V}}^s) < \infty. \tag{5.2}$$

To proceed, we start by renaming the intervals in $\{[0,1]\} \cup \mathscr{I}$. Denote $\Delta_\emptyset = [0,1]$ and write $\mathscr{I}_0 = \{\Delta_1, \dots, \Delta_{n_\emptyset}\}$, enumerated according to the orientation of $[0,1]$. Inductively, suppose that for some word w with $|w| = k$, we have defined $\Delta_w \in \mathscr{I}_k$. Suppose also that

$$\mathscr{I}_{k+1}(\Delta_w) = \{J_1, \dots, J_{n_w}\},$$

enumerated according to the orientation of $[0,1]$. Then for each $i \in \{1, \dots, n_w\}$, denote $\Delta_{wi} = J_i$. Denote by $\mathcal{W}$ the set of all finite words with letters from $\mathbb{N}$ for which an interval Δ_w has been defined.

Next, we use the masses of intervals defined in §4 to modify the length of intervals Δ_w. Define $\Delta'_\emptyset = [0,1]$. Let $\{\Delta'_1, \dots, \Delta'_{n_\emptyset}\}$ be a partition of $\Delta'_\emptyset$, enumerated according to the orientation of $[0,1]$, satisfying

(1) Δ'_i is open (resp. closed) if and only if Δ_i is open (resp. closed), and
(2) $\operatorname{diam} \Delta'_i = \mathcal{M}_s(0, \Delta_i)/\mathcal{M}_s([0,1])$.

These intervals exist, because $\mathcal{M}_s([0,1]) = \sum_{i=1}^{n_\emptyset} \mathcal{M}_s(0, \Delta_i)$. Inductively, suppose that an interval $\Delta'_w \subset [0,1]$ has been defined for some $w \in \mathcal{W}$ such that

$$\operatorname{diam} \Delta'_w \geq \frac{\mathcal{M}_s(|w|-1, \Delta_w)}{\mathcal{M}_s([0,1])}$$

and Δ'_w is open (resp. closed) if and only if Δ_w is open (resp. closed). Let $\{\Delta'_{w1}, \dots, \Delta'_{wn_w}\}$ be a partition of Δ'_w, enumerated according to the orientation of $[0,1]$, satisfying

(1) Δ'_{wi} is open (resp. closed) if and only if Δ_{wi} is open (resp. closed), and
(2) $\operatorname{diam} \Delta'_{wi} \geq \mathcal{M}_s(|w|, \Delta_{wi})/\mathcal{M}_s([0,1])$.

This partition exists by Lemma 4.3(3).

Define the family

$$\mathscr{E}'_k = \{\Delta'_w : \Delta_w \in \mathscr{E}_k\}$$

and similarly define the families $\mathscr{B}'_k$, $\mathscr{N}'_k$, $\mathscr{F}'_k$, and $\mathscr{I}'_k$. For each $k \geq 0$, define a continuous map $F_k : [0,1] \to X$ by

$$F_k|\Delta'_w = (f_k|\Delta_w) \circ \phi_w \qquad \text{for all } w \in \mathcal{W},$$

where ϕ_w is the unique increasing affine homeomorphism mapping Δ'_w onto Δ_w when Δ'_w is nondegenerate and ϕ_w maps to any point in Δ_w when Δ'_w is a singleton. (The latter possibility occurs only when Δ'_w belongs to $\mathscr{F}'_k$ or $\mathscr{N}'_k$.)

We now prove two auxiliary results for the sequence $(F_k)_{k \geq 0}$.

Lemma 5.2. *For all $k \geq 0$ and $x \in [0,1]$, we have $|F_{k+1}(x) - F_k(x)| \leq 30A^*\xi_2 r_0 \rho_k$.*

Proof. Fix $x \in [0,1]$ and $w \in \mathcal{W}$ such that $x \in \Delta_w$ and $|w| = k$. Let also $i \in \{1, \dots, n_w\}$ be such that $x \in \Delta'_{wi}$.

If $\Delta'_w \in \mathscr{B}'_k$, then $\Delta'_w = \Delta'_{wi}$, $\Delta_w = \Delta_{wi}$, and $F_k|\Delta'_w = F_{k+1}|\Delta'_{wi}$. We conclude that $|F_{k+1}(x) - F_k(x)| = 0$.

If $\Delta'_w \in \mathscr{F}'_k$, then $\Delta'_w = \Delta'_{wi} = \{x\}$, and $F_k(x) = F_{k+1}(x)$. Hence $|F_{k+1}(x) - F_k(x)| = 0$.

If $\Delta'_w \in \mathscr{E}'_k$, then $|F_{k+1}(x) - F_k(x)| \leq 2 \operatorname{diam} F_k(\Delta'_w) = 2 \operatorname{diam} f_k(\Delta_w) \leq 28A^* \rho_{k+1} r_0$.

If $\Delta'_w \in \mathscr{N}'_k$, then $|F_{k+1}(x) - F_k(x)| \leq 2 \operatorname{diam} f_k(\Delta_w) \leq \operatorname{diam} \tilde{V}_{k+1,I} \leq 30A^* \rho_{k+1} r_0$, where $\tilde{V}_{k+1,I}$ is a set defined in §3.8. □

Lemma 5.3. *For all $k \geq 0$ and $x, y \in [0,1]$,*

$$|F_k(x) - F_k(y)| \leq \mathcal{M}_s([0,1]) r_0^{1-s} \rho_k^{1-s} |x - y| \lesssim_{s,C^*,\xi_2} r_0(1 + S_\mathscr{V}^s) \rho_k^{1-s} |x - y|.$$

Proof. Fix $k \geq 0$ and $x, y \in [0,1]$. Without loss of generality, we may assume that $x < y$. We consider three cases. In the first two cases, the points x and y belong to the same interval Δ'_w, $|w| = k$, while in the third case they belong to different intervals.

Case 1. If $x, y \in \Delta'_w \in \mathscr{N}'_k \cup \mathscr{F}'_k$, then $|F_k(x) - F_k(y)| = 0|x - y|$, because the map $F_k|\Delta'_w$ is constant.

Case 2. Suppose that $x, y \in \Delta'_w \in \mathscr{E}'_k \cup \mathscr{B}'_k$. Since $F_k|\Delta'_w$ is affine,

$$\begin{aligned} |F_k(x) - F_k(y)| &\leq \mathcal{M}_s([0,1]) \frac{\operatorname{diam} f_k(\Delta_w)}{\mathcal{M}_s(|w| - 1, \Delta_w)} |x - y| \\ &\leq \mathcal{M}_s([0,1]) \operatorname{diam} f_k(\Delta_w)^{1-s} |x - y| \\ &\leq \mathcal{M}_s([0,1]) r_0^{1-s} \rho_k^{1-s} |x - y|, \end{aligned}$$

by (V3) and the assumption $s \geq 1$. Thus, by (5.2),

$$|F_k(x) - F_k(y)| \lesssim_{s,C^*,\xi_2} r_0(1 + S_\mathscr{V}^s) \rho_k^{1-s} |x - y|.$$

Case 3. Suppose that $x \in \Delta'_w$ and $y \in \Delta'_u$ for some $\Delta'_w, \Delta'_u \in \mathscr{I}'_k$ with $\Delta'_w \cap \Delta'_u = \emptyset$. By the preceding cases and the Fundamental Theorem of Calculus,

$$\begin{aligned} |F_k(x) - F_k(y)| &\leq \int_x^y |\nabla F_k(t)| \, dt \leq \mathcal{M}_s([0,1]) r_0^{1-s} \rho_k^{1-s} |x - y| \\ &\lesssim_{s,C^*,\xi_2} r_0(1 + S_\mathscr{V}^s) \rho_k^{1-s} |x - y|. \quad \square \end{aligned}$$

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Define $F : [0,1] \to X$ pointwise by

$$F(x) := F_0(x) + \sum_{k=0}^{\infty} (F_{k+1}(x) - F_k(x)).$$

By Lemma 5.2, F is well defined and continuous in all $[0,1]$. By Lemma 5.2, Lemma 5.3, and Lemma B.1 from the appendix, F is $(1/s)$-Hölder continuous with Hölder constant

$$H \leq \frac{1}{\xi_1} \left(\mathcal{M}_s([0,1]) r_0^{1-s} + 60 A^* r_0 \frac{\xi_2}{1-\xi_2} \right) \lesssim_{s,C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}}^s).$$

Finally, for any integer $k \geq 0$ and any integer $m \geq k$, we have $V_k \subset F_m([0,1])$. Therefore, $V_k \subset F([0,1])$ for all integers $k \geq 0$. □

5.2. Corollaries to Theorem 5.1 and proof of Theorem 1.1

Corollary 5.4 *(Tube approximation). For all $s > 1$, $C^* \geq 1$, and $0 < \xi_1 < \xi_2 < 1$, there exists $\alpha^* > 0$ with the following property. Assume that $X = l^2(\mathbb{R})$ or $X = \mathbb{R}^N$ for some $N \geq 2$. Let $\mathscr{V} = (V_k, \rho_k)_{k \geq 0}$ be a sequence of finite sets in X and numbers $\rho_k > 0$ satisfying properties (V0)–(V5) of §2.2 with constants C^*, ξ_1, and ξ_2. If*

$$S_{\mathscr{V}}^{s,+} := \sum_{k=0}^{\infty} \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha^*}} \rho_k^s < \infty,$$

then there exists a $(1/s)$-Hölder map $f : [0,1] \to X$ such that $\bigcup_{k \geq 0} V_k \subset f([0,1])$ and the Hölder constant of f satisfies $H \lesssim_{s,C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}}^{s,+})$.

Proof. By Lemma 2.8, there exists $\epsilon_{s,C^*,\xi_1,\xi_2} \in (0, 1/16]$ such that if $k \geq 0$, $v \in V_k$, and $\alpha_{k,v} \leq \epsilon_{s,C^*,\xi,\xi_2}$, then $\tau_s(k, v, v') = 0$ for all $(v, v') \in \mathsf{Flat}(k)$. Set $\alpha^* = \min\{\epsilon_{s,C^*,\xi_1,\xi_2}, \alpha_1\}$ (a careful inspection shows $\epsilon_{s,C^*,\xi_1,\xi_2}$ is strictly smaller than α_1). Thus, with $\alpha_0 = \alpha^*$,

$$\sum_{k=0}^{\infty} \sum_{(v,v') \in \mathsf{Flat}(k)} \tau_s(k, v, v') \rho_k^s + \sum_{k=0}^{\infty} \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha^*}} \rho_k^s = \sum_{k=0}^{\infty} \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha^*}} \rho_k^s < \infty.$$

The conclusion follows immediately by Theorem 5.1. □

Corollary 5.5. *Alternatively, if*

$$S_{\mathscr{V}}^{s,p} := \sum_{k=0}^{\infty} \sum_{v \in V_k} \alpha_{k,v}^p \, \rho_k^s < \infty \quad \text{for some } p > 0,$$

then there exists a $(1/s)$-Hölder map $f : [0,1] \to X$ such that $\bigcup_{k\geq 0} V_k \subset f([0,1])$ and the Hölder constant of f satisfies $H \lesssim_{s,C^*,\xi_1,\xi_2} r_0(1 + (\alpha^*)^{-p} S^{s,p}_{\mathscr{V}})$.

Proof. Inspecting the definitions of the two sums, $S^{s,+}_{\mathscr{V}} \leq (\alpha^*)^{-p} S^{s,p}_{\mathscr{V}}$. □

We now turn to the proof of Theorem 1.1.

Remark 5.6. For all $x \in \mathbb{R}^N$ and $r > 0$, a minimal dyadic cube Q in $\mathbb{R}^N$ such that $x \in Q$ and $3Q$ contains $B(x,r)$ satisfies $\operatorname{diam} 3Q \leq Cr$ for some $C = C(N) > 0$.

Proof of Theorem 1.1. Let $N \geq 2$ and $s > 1$ be given. Fix $\beta_0 > 0$ to be specified below. Assume that $E \subset \mathbb{R}^N$ is a bounded set such that

$$S^{s,+}_E = \sum_{\substack{Q\in\Delta(\mathbb{R}^N)\\ \beta_E(3Q)\geq\beta_0}} (\operatorname{diam} Q)^s < \infty.$$

Pick any $x_0 \in E$ and set $r_0 = \operatorname{diam} E$. Define $V_0 = \{x_0\}$. Assume that V_k has been defined for some k. Choose a maximal $2^{-(k+1)}$-separated set in E such that $V_{k+1} \supset V_k$. Then the sequence $\mathscr{V} = (V_k, 2^{-k})_{k\geq 0}$ satisfies conditions (V0)–(V4) in §2.2 with $C^* = 2$ and $\xi_1 = \xi_2 = 1/2$. Note that

$$A^* = \frac{C^*}{1-\xi_2} = 4, \quad 30A^* = 120.$$

For all $k \geq 0$ and $v \in V_k$, let $Q_{k,v}$ be a minimal dyadic cube such that $v \in Q_{k,v}$ and $3Q_{k,v}$ contains $B(v, 120 \cdot 2^{-k} r_0)$ and choose $\ell_{k,v}$ be a line such that

$$\sup_{x\in E\cap 3Q} \operatorname{dist}(x, \ell_{k,v}) = \beta_E(3Q) \operatorname{diam} 3Q.$$

Then, by Remark 5.6, there exists $C = C(N) > 0$ such that

$$\alpha_{k,v} := \frac{1}{2^{-k+1} r_0} \sup_{x\in V_{k+1}\cap B(v,120\cdot 2^{-k} r_0)} \operatorname{dist}(x, \ell_{k,v}) \leq \frac{\operatorname{diam} 3Q}{2^{-k+1} r_0} \beta_E(3Q) \leq 240 C \beta_E(3Q).$$

We now specify that $\beta_0 = \alpha^*/240C$, where α^* is the constant from Corollary 5.4 and C the constant from Remark 5.6. Because each dyadic cube Q in $\mathbb{R}^N$ is associated to some (k,v) at most $C(n)$ times, it follows that

$$S^{s,+}_{\mathscr{V}} \lesssim_N r_0^{-s} S^{s,+}_E < \infty.$$

Therefore, by Corollary 5.4, there exists a $(1/s)$-Hölder map $f : [0,1] \to \mathbb{R}^N$ such that $\bigcup_{k\geq 0} V_k \subset f([0,1])$ and the Hölder constant of f satisfies

$$H \lesssim_{N,s} r_0(1 + S_{\mathscr{V}}^{s,+}) \lesssim_{N,s} \operatorname{diam} E + \frac{S_E^{s,+}}{(\operatorname{diam} E)^{s-1}}.$$

Because $\bigcup_{k\geq 0} V_k$ is dense in E, the curve $f([0,1])$ also contains the set E. □

5.3. A refinement of Theorem 5.1

The parameterization in Theorem 5.1 can be made in such a way so that the sequence of maps F_k obtained are *essentially 2-to-1* in the sense of the following proposition.

Proposition 5.7. *Let $\mathscr{V} = (V_k, \rho_k)_{k\geq 0}$ and α_0 satisfy the hypothesis of Theorem 5.1 and let $x_0 \in V_0$. There exists a sequence of piecewise linear maps $F_k : [0,1] \to X$ with the following properties.*

(1) *For all $k \geq 0$, $F_k(0) = x_0 = F_k(1)$.*
(2) *For all $k \geq 0$, there exists $G_k \subset [0,1]$ such that $F_k|G_k$ is 2-1 and $F_k([0,1] \setminus G_k)$ is a finite set.*
(3) *For all $k \geq 0$, $F_k([0,1]) \supset V_k$; for all $x \in V_{k+1}$, $\operatorname{dist}(x, F_k([0,1])) \leq C^* \rho_{k+1} r_0$.*
(4) *For all $k \geq 0$, $\|F_k - F_{k+1}\|_\infty \lesssim_{C^*,\xi_2} \rho_{k+1} r_0$.*
(5) *For all $k \geq 0$, the map F_k is Lipschitz with $\operatorname{Lip}(F_k) \lesssim_{s,C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}}^s)\rho_k^{1-s}$.*

The maps F_k converge uniformly to a $(1/s)$-Hölder map $F : [0,1] \to X$ whose image contains $\bigcup_{k\geq 0} V_k$, the parameterization F starts and ends at x_0 in the sense of (1), and the Hölder constant of F satisfies $H \lesssim_{s,C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}}^s)$.

Proof. Following the algorithm of §3, we construct for each $k \geq 0$, four families $\mathscr{E}_k$, $\mathscr{B}_k$, $\mathscr{F}_k$, $\mathscr{N}_k$ of intervals in $[0,1]$ and a continuous piecewise linear map $f_k : [0,1] \to X$ that satisfy (P1)–(P7). In Step 0, we may assume that $f_0(0) = f_0(1) = x_0$. Thus, $f_k(0) = f_k(1) = x_0$ for all $k \geq 0$. Moreover, for all $x \in V_{k+1}$,

$$\operatorname{dist}(x, F_k([0,1])) \leq \operatorname{dist}(x, V_k) < C^* r_0 \rho_{k+1}$$

by (V4). From the construction, $\|f_k - f_{k+1}\|_\infty \lesssim_{A^*} \rho_{k+1} r_0$. Hence $\|f_k - f_{k+1}\|_\infty \lesssim_{C^*,\xi_2} \rho_{k+1} r_0$. We have shown that the maps f_k satisfy properties (1), (3) and (4).

As for property (2), we already know from (P4) that $f_k|\bigcup \mathscr{E}_k$ is 2-to-1. We proceed to modify f_k on each $I \in \mathscr{B}_k$. From the algorithm in §3, recall that for each $I \in \mathscr{B}_k$, there exists unique $I' \in \mathscr{B}_k$, $I' \neq I$ such that $f_k(I) = f_k(I')$. Enumerate

$$\mathscr{B}_k = \{I_1, I_1', \ldots, I_l, I_l'\},$$

where $f_k(I_j) = f_k(I_j')$. Starting with $I_1 = (a_1, b_1)$, define $\tilde{f}_k|I_1$ so that

(a) $\tilde{f}_k|I_1$ is piecewise linear and continuous, $\tilde{f}_k(a_1) = f_k(a_1)$ and $\tilde{f}_k(b_1) = f_k(b_1)$;

(b) $\mathcal{H}^1(\tilde{f}_k(I_1)) \leq |f_k(a_1) - f_k(b_1)| + \rho_k r_0$;
(c) $\tilde{f}_k(I_1) \cap \bigcup_{I \in \mathscr{E}_k} f_k(I)$ is a finite set.

Let $\psi_1 : I_1' \to I_1$ be the unique orientation-reversing linear map from I_1' onto I_1. Then define $\tilde{f}_k|I_1' = (\tilde{f}_k|I_1) \circ \psi_1$. For induction, assume that we have defined $\tilde{f}_k$ on

$$I_1, I_1', \ldots, I_{r-1}, I_{r-1}'.$$

Define $\tilde{f}_k|I_r$ as with I_1, only this time we require that the set

$$\tilde{f}_k(I_r) \cap \left(\bigcup_{i=1}^{r-1} \tilde{f}_k(I_i) \cup \bigcup_{i=1}^{r-1} \tilde{f}_k(I_i') \cup \bigcup_{I \in \mathscr{E}_k} f_k(I) \right)$$

be finite. Let $\psi_r : I_r' \to I_r$ be the unique orientation-reversing linear map from I_r' onto I_r and define $\tilde{f}_k|I_r' = (\tilde{f}_k|I_r) \circ \psi_r$. Extending $\tilde{f}_k|I = f_k|I$ for all $I \in \mathscr{E}_k \cup \mathscr{N}_k \cup \mathscr{F}_k$, we obtain a sequence $\tilde{f}_k$ of maps that satisfy properties (1)–(4).

The rest of the proof is similar to that of Theorem 5.1 and we only sketch the steps. Define the $\mathcal{M}_s$ for each $I \in \mathscr{I}_k$ and define the collections of intervals $\{\Delta_w\}$ and $\{\Delta_w'\}$. For each w, let $\phi_w : \Delta_w' \to \Delta_w$ be the unique affine homeomorphism from Δ_w' onto Δ_w and let $F_k|\Delta_w' = (\tilde{f}_k|\Delta_w) \circ \phi_w$. Although the maps f_k are different from $\tilde{f}_k$, we have by (b) that $\operatorname{diam} f_k(I) \simeq_{\xi_2} \operatorname{diam} \tilde{f}_k(I)$ for all $k \geq 0$ and all $I \in \mathscr{I}_k$. Thus, Lemma 5.2 and Lemma 5.3 still hold with constants depending at most on s, C^*, ξ_1 and ξ_2. Therefore, the maps F_k satisfy properties (1)–(5). □

5.4. A Carleson condition for an upper Ahlfors s-regular curve

Replacing (5.1) in the main theorem with a Carleson-type condition ensures that the Hölder curve is upper Ahlfors regular. This answers a question posed to us by T. Orponen.

Theorem 5.8. *Assume that $X = l^2(\mathbb{R})$ or $X = \mathbb{R}^N$ for some $N \geq 2$. Let $s \geq 1$, let $\mathscr{V} = ((V_k, \rho_k))_{k \geq 0}$ be a sequence of finite sets V_k in X and numbers $\rho_k > 0$ that satisfy properties (V0)–(V4) defined in §2.2. Let $\Lambda \geq C^*$ and $\Lambda^* := \Lambda/(1 - \xi_2)$. Suppose for all $k \geq 0$ and $v \in V_{k+1}$, we are given a line $\ell_{k,v}$ and $\alpha_{k,v} \geq 0$ such that*

$$\sup_{x \in V_{k+1} \cap B(v, 30\Lambda^* \rho_k r_0)} \operatorname{dist}(x, \ell) \leq \alpha_{k,v} \rho_{k+1}. \tag{$\widetilde{\text{V5}}$}$$

If $\Lambda \gg_{\xi_1, \xi_2} C^*$, $\alpha_0 \in (0, \alpha_1]$, and there exists $M < \infty$ such that for all $j \geq 0$ and $w \in V_j$,

$$S_{\mathscr{V}}^s(j, w) := \sum_{k=j}^{\infty} \sum_{\substack{(v,v') \in \mathsf{Flat}(k) \\ v, v' \in B(w, \Lambda \rho_j r_0)}} \tau_s(k, v, v') \rho_k^s + \sum_{k=j}^{\infty} \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha_0 \\ v \in B(w, \Lambda \rho_j r_0)}} \rho_k^s \leq M \rho_j^s, \tag{5.3}$$

then there exists a $(1/s)$-Hölder map $f : [0,1] \to X$ such that $f([0,1]) \supset \bigcup_{k\geq 0} V_k$ and the curve $f([0,1])$ is upper Ahlfors s-regular with constant depending on at most s, C^*, ξ_1, ξ_2, and M.

Proof. By (V0) and (V4), $\operatorname{excess}\left(\bigcup_{k=1}^{\infty} V_k, V_0\right) \leq \rho_1 r_0 + \rho_2 r_0 + \cdots = \frac{r_0}{1-\xi_2} \leq A^* r_0$, where

$$\operatorname{excess}(A,B) = \sup_{x\in A} \inf_{y\in B} |x-y|$$

whenever A and B are nonempty sets in X. Hence $\bigcup_{k=0}^{\infty} V_k \subset B(x_0, 2A^* r_0)$ by (V1). Thus, there exists a $(1/s)$-Hölder map $f : [0,1] \to X$ such that $\Gamma := f([0,1]) \supset \bigcup_{k=0}^{\infty} V_k$ and the Hölder constant of f satisfies $H_f \lesssim_{s,C^*,\xi_1,\xi_2} r_0(1+M)$ by Theorem 5.1, since $S^s_{\mathscr{V}} = S^s_{\mathscr{V}}(0,w) \leq M$. In particular,

$$\mathcal{H}^s(\Gamma) \leq H_f^s\, \mathcal{H}^1([0,1]) \lesssim_{s,C^*,\xi_1,\xi_2,M} r_0^s.$$

Let $x \in \Gamma$ and let $0 < r \leq \operatorname{diam}\Gamma$. Because $\bigcup_{k=0}^{\infty} V_k \subset B(x_0, 2A^* r_0)$, we have $\operatorname{diam}\Gamma \leq 4A^* r_0$. If $r \geq r_0$, then

$$\mathcal{H}^s(\Gamma \cap B(x,r)) \leq \mathcal{H}^s(\Gamma) \lesssim_{s,C^*,\xi_1,\xi_2,M} r_0^s \lesssim_{s,C^*,\xi_1,\xi_2,M} r^s.$$

Otherwise, $0 < r < r_0$, say $\rho_{j+1} r_0 \leq r < \rho_j r_0$ for some integer $j \geq 0$.

Choose $w \in V_j$ such that $|w - x| = \operatorname{dist}(x, V_j)$. By Lemma 5.2, (V0), and (V4),

$$\operatorname{dist}(x, f_j([0,1])) \leq \frac{30\xi_2}{1-\xi_2} A^* \rho_j r_0.$$

Because $\alpha_0 \leq \alpha_1$, the longest line segment drawn between vertices in V_j has length at most $14A^* \rho_{k-1} r_0$. By (V0), it follows that

$$\operatorname{excess}(f_j([0,1]), V_j) \leq \frac{7}{\xi_1} A^* \rho_j r_0.$$

Thus, $\operatorname{dist}(x, V_j) \lesssim_{\xi_1,\xi_2} A^* \rho_j r_0 \ll_{\xi_1,\xi_2} \Lambda \rho_j r_0$. For all $k \geq 0$, define

$$\widetilde{\rho_k} := \frac{\rho_{j+k}}{\rho_j}.$$

Define $\widetilde{V_0} := V_j \cap B(w, \Lambda \rho_j r_0)$. Then, for each $k \geq 1$, recursively define $\widetilde{V_k}$ to be set of all $x \in V_{j+k} \cap B(w, \Lambda \rho_j r_0)$ such that $\operatorname{dist}(x, \widetilde{V}_{k-1}) \leq \Lambda \rho_{j+k} r_0$. Then $\widetilde{\mathscr{V}} = (\widetilde{V_k}, \widetilde{\rho_k})_{k\geq 0}$ satisfy (V0)–(V4) with respect to $\widetilde{x}_0 = w$ and $\widetilde{r}_0 = \rho_j r_0$, $\widetilde{C^*} = \Lambda$, $\widetilde{\xi_1} = \xi_1$, and $\widetilde{\xi_2} = \xi_2$. For all $k \geq 0$ and $v \in \widetilde{V}_k$, assign $\widetilde{\ell}_{k,v} := \ell_{j+k,v}$ and $\widetilde{\alpha}_{k,v} = \alpha_{j+k,v}$. Then $\widetilde{\mathscr{V}}$ satisfies (V5) with respect to $\widetilde{\ell}_{k,v}$ and $\widetilde{\alpha}_{k,v}$ by ($\widetilde{\text{V5}}$). Moreover, by (5.3),

$$S^s_{\widetilde{\mathscr{V}}} = \sum_{k=0}^{\infty} \sum_{(v,v')\in \widetilde{\mathsf{Flat}(k)}} \widetilde{\tau}(k,v,v') \widetilde{\rho_k}^s + \sum_{k=0}^{\infty} \sum_{\substack{v\in\widetilde{V}_k \\ \widetilde{\alpha_{k,v}} \geq \alpha_0}} \widetilde{\rho_k}^s = \frac{S_{\mathscr{V}}(j,w)}{\rho_j^s} \leq M.$$

Thus, by Theorem 5.1, there is a $(1/s)$-Hölder map g with Hölder constant $H_g \lesssim_{s,\Lambda,\xi_1,\xi_2} \widetilde{r_0}(1+M)$ such that $g([0,1])$ contains $\bigcup_{k\geq 0} \widetilde{V}_k$. Because the algorithm in §3 works locally in the image, $\operatorname{dist}(x, V_j) \lesssim_{\xi_1,\xi_2} A^*\rho_j r_0 \ll_{\xi_1,\xi_2} \Lambda\rho_j r_0$, and $r < \rho_j r_0$, we can guarantee that $g([0,1])$ contains $f([0,1]) \cap B(x,r)$ provided that Λ is sufficiently large. Therefore,

$$\mathcal{H}^s(\Gamma \cap B(x,r)) \leq H_g^s\, \mathcal{H}^1([0,1]) \lesssim_{s,\Lambda,\xi_1,\xi_2,} \widetilde{r_0}(1+M) \lesssim_{s,C^*,\xi_1,\xi_2,M} (\rho_j r_0)^s \lesssim_{s,C^*,\xi_1,\xi_2,M} r^s,$$

where the final inequality holds because $\rho_{j+1} r_0 \leq r$. □

6. Lipschitz parameterization

Using the method above, we obtain the following refinement of the sufficient half of the Analyst's TST in Hilbert space, which is originally due to Jones [32] in the Euclidean case and due to Schul [48] in the infinite-dimensional case.

Proposition 6.1 *(Sufficient half of the Analyst's Traveling Salesman with Nets). Assume that $X = l^2(\mathbb{R})$ or $X = \mathbb{R}^N$ for some $N \geq 2$. Let $\mathscr{V} = (V_k, \rho_k)_{k\geq 0}$ be a sequence of finite sets V_k in X and numbers $\rho_k > 0$ that satisfy properties (V0)–(V5) defined in §2.2. If*

$$S_{\mathscr{V}} := \sum_{k=0}^{\infty} \sum_{v\in V_k} \alpha_{k,v}^2 \rho_k < \infty, \tag{6.1}$$

then for every $x_0 \in V_0$, we can find a sequence of piecewise linear maps $F_k : [0,1] \to X$ with the following properties.

(1) *For all $k \geq 0$, $F_k(0) = x_0 = F_k(1)$.*
(2) *For all $k \geq 0$, there exists $G_k \subset [0,1]$ such that $F_k|G_k$ is 2-1 and $F_k([0,1] \setminus G_k)$ is a finite set.*
(3) *For all $k \geq 0$, $F_k([0,1]) \supset V_k$; for all $x \in V_{k+1}$, $\operatorname{dist}(x, F_k([0,1])) \leq C^* \rho_{k+1} r_0$.*
(4) *For all $k \geq 0$, $\|F_k - F_{k+1}\|_\infty \lesssim_{C^*,\xi_2} \rho_{k+1} r_0$.*
(5) *For all $k \geq 0$, the map F_k is Lipschitz with $\operatorname{Lip}(F_k) \lesssim_{C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}})$.*

The maps F_k converge uniformly to a Lipschitz map $F : [0,1] \to X$ whose image contains $\bigcup_{k\geq 0} V_k$, the parameterization F starts and ends at x_0 in the sense of (1), and the Lipschitz constant of F satisfies $L \lesssim_{C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}})$.

Proof. Let $\alpha_0 = \alpha_1$ (see (4.2)), which depends only on C^*, ξ_1, and ξ_2. If $(v, v') \in \mathsf{Flat}(k)$, then $\tau_1(k, v, v') \leq 3\alpha_{k,v}^2$ by Lemma 2.7. Thus, by definition of $S^1_{\mathscr{V}}$ (see Theorem 5.1),

$$S^1_{\mathscr{V}} = \sum_{k=0}^{\infty} \sum_{(v,v')\in \mathsf{Flat}(k)} \tau_1(k,v,v')\rho_k + \sum_{k=0}^{\infty} \sum_{\substack{v\in V_k \\ \alpha_{k,v}\geq \alpha_0}} \rho_k$$

$$\leq 6 \sum_{k=0}^{\infty} \sum_{\substack{v\in V_k \\ \alpha_{k,v}< \alpha_0}} \alpha^2_{k,v}\rho_k + \frac{1}{\alpha_0^2} \sum_{k=0}^{\infty} \sum_{\substack{v\in V_k \\ \alpha_{k,v}\geq \alpha_0}} \alpha^2_{k,v}\rho_k \leq \frac{1}{\alpha_0^2} S_{\mathscr{V}}.$$

The conclusion now follows from Proposition 5.7. □

Part II. Applications and further results

In §7, we give an application of the Hölder Traveling Salesman theorem to the geometry of measures. In particular, we obtain sufficient conditions for a pointwise doubling measure in $\mathbb{R}^N$ to be carried by $(1/s)$-Hölder curves, $s > 1$. This extends the work [8,9] by the first author and Schul, which characterizes 1-rectifiable Radon measures in $\mathbb{R}^N$ in terms of geometric square functions. In §8, we use the method of Part I to obtain a Ważewski type theorem for flat continua, which we described above in §1.3. Finally, in §9, we present examples of Hölder curves and of sets that are not contained in any Hölder curve to highlight the rich geometry of sets in $\mathbb{R}^N$ and illustrate the strengths and limitations of our principal results.

7. Fractional rectifiability of measures

One goal of geometric measure theory is to understand the structure of a measure in $\mathbb{R}^N$ through its interaction with families of lower dimensional sets. For an extended introduction, see the survey [6] by the first author. In this section, we use the Hölder Traveling Salesman theorem to establish criteria for *fractional rectifiability* of pointwise doubling measures in terms of L^p Jones beta numbers. In particular, we extend part of the recent work of the first author and Schul [9] on measures carried by rectifiable curves to measures carried by Hölder curves (see Theorem 7.5). The study of fractional (that is, non-integer dimensional) rectifiability of measures was first proposed by Martín and Mattila [37,38] and examined further by the first and third author [10].

7.1. Generalized rectifiability

Let $\mathcal{A}$ be a nonempty family of Borel sets in $\mathbb{R}^N$ and let μ be a Borel measure on $\mathbb{R}^N$. We say that μ is *carried by* $\mathcal{A}$ if there exists a sequence $(A_i)_{i=1}^{\infty}$ of sets in $\mathcal{A}$ such that $\mu(\mathbb{R}^N \setminus \bigcup_i A_i) = 0$. At the other extreme, we say that μ is *singular to* $\mathcal{A}$ if $\mu(A) = 0$ for all $A \in \mathcal{A}$. If μ is σ-finite, then μ can be uniquely written as the sum of a Borel measure $\mu_{\mathcal{A}}$ carried by $\mathcal{A}$ and a Borel measure $\mu^{\perp}_{\mathcal{A}}$ singular to $\mathcal{A}$ (e.g. see the appendix of [10]). These definitions encode several commonly used notions of rectifiability of measures (see [6]).

Let $1 \leq m \leq N-1$. Let $\mathcal{A}$ denote the family of Lipschitz images of $[0,1]^m$ in $\mathbb{R}^N$. We say that a Borel measure μ is *m-rectifiable* if μ is carried by $\mathcal{A}$; we say that μ is *purely m-unrectifiable* if μ is singular to $\mathcal{A}$. A Borel set $E \subset \mathbb{R}^n$ with $0 < \mathcal{H}^m(E) < \infty$ is called m-rectifiable or purely m-unrectifiable if $\mathcal{H}^m \llcorner E$, the m-dimensional Hausdorff measure restricted to E, has that property. The classes of 1-rectifiable sets and purely 1-unrectifiable sets are also called *Besicovitch regular* sets and *Besicovitch irregular* sets, respectively, in reference to the pioneering investigations by Besicovitch [11,12] into the geometry of 1-sets in the plane.

Example 7.1. Let $\Gamma_1, \Gamma_2, \dots$ be a sequence of rectifiable curves in $\mathbb{R}^N$ and let $a_1, a_2, \dots$ be a sequence of positive weights. Then the measure $\mu = \sum_i a_i \mathcal{H}^1 \llcorner \Gamma_i$ is 1-rectifiable. Note that if the closure of $\bigcup_i \Gamma_i$ is $\mathbb{R}^N$ and the weights are chosen so that $\sum_i a_i \mathcal{H}^1(\Gamma_i) = 1$, we get a 1-rectifiable Borel probability measure μ whose support is $\mathbb{R}^N$.

Example 7.2. Let $C \subset \mathbb{R}$ be the middle halves Cantor set (formed by replacing $[0,1]$ with $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and iterating). Then $E = C \times C \subset \mathbb{R}^2$ is a Cantor set of Hausdorff dimension one, $0 < \mathcal{H}^1(E) < \infty$, E is Ahlfors 1-regular in the sense that

$$\mathcal{H}^1(E \cap B(x,r)) \simeq r \quad \text{for all } x \in E \text{ and } 0 < r \leq \operatorname{diam} E,$$

and E is Besicovitch irregular (e.g. see [37]). In particular, the set E is compact and measure-theoretically one-dimensional, but E is not contained in any rectifiable curve.

7.2. L^p Jones beta numbers and rectifiability

Let μ be a Radon measure on $\mathbb{R}^N$, that is, a locally finite Borel regular measure, let $1 \leq m \leq N-1$, let $p > 0$, let $x \in \mathbb{R}^N$, let $r > 0$, and let L be an m-dimensional affine subspace of $\mathbb{R}^N$. We define

$$\beta_p^{(m)}(\mu, x, r, L) := \left(\int_{B(x,r)} \left(\frac{\operatorname{dist}(z, L)}{r} \right)^p \frac{d\mu(z)}{\mu(B(x,r))} \right)^{1/p}, \tag{7.1}$$

$$\beta_p^{(m)}(\mu, x, r) := \inf_L \beta_p^{(m)}(\mu, x, r, L), \tag{7.2}$$

where the infimum is taken over all m-planes L in $\mathbb{R}^N$. The quantity $\beta_p^{(m)}(\mu, x, r)$ is called the *m-dimensional L^p Jones beta number of μ in $B(x,r)$*. The L^p Jones beta numbers were introduced by David and Semmes [22,23] to study quantitative rectifiability of Ahlfors regular sets and boundedness of singular integral operators. The normalization of the measure in (7.1) that we have chosen (i.e. dividing by $\mu(B(x,r))$) ensures that $\beta_p^{(m)}(\mu, x, r) \in [0,1]$ and $\beta_p^{(m)}$ is invariant under dilations $T_\lambda(z) = \lambda z$ in the sense that

$$\beta_p^{(m)}(\mu, x, r) = \beta_p^{(m)}(T_\lambda[\mu], \lambda x, \lambda r), \quad T_\lambda[\mu](E) = \mu(\lambda^{-1} E) \tag{7.3}$$

for all μ, $x \in \mathbb{R}^N$, $r > 0$, and $\lambda > 0$. By monotonicity of the integral,

$$s\mu(B(y,s))^{1/p}\beta_p^{(m)}(\mu,y,s) \leq r\mu(B(x,r))^{1/p}\beta_p^{(m)}(\mu,x,r) \quad \text{when } B(y,s) \subset B(x,r). \tag{7.4}$$

In a pair of papers, Tolsa [51] and Azzam and Tolsa [5] characterize m-rectifiable Radon measures μ on $\mathbb{R}^N$ with $\mu \ll \mathcal{H}^m$ in terms of L^2 Jones beta numbers. The restriction $\mu \ll \mathcal{H}^m$ is equivalent to the upper density bound $\limsup_{r\downarrow 0} r^{-m}\mu(B(x,r)) < \infty$ μ-a.e. (e.g. see [39, Chapter 6]) and implies that the Hausdorff dimension of the measure is at least m (see [41]). The proof that (7.6) implies the measure μ is m-rectifiable uses an intricate stopping time argument in conjunction with David and Toro's *Reifenberg algorithm for sets with holes* [24] to construct bi-Lipschitz images of $\mathbb{R}^m$ inside $\mathbb{R}^N$ that carry μ. For related developments, see [25,26].

Theorem 7.3 *(see [51], [5]). Let μ be a Radon measure on $\mathbb{R}^N$. Assume that*

$$0 < \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^N. \tag{7.5}$$

Then μ is m-rectifiable if and only if

$$\int_0^1 \beta_2^{(m)}(\mu,x,r)^2 \, \frac{\mu(B(x,r))}{r^m} \, \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^N. \tag{7.6}$$

In [9], the first author and Schul characterize 1-rectifiable Radon measures μ on $\mathbb{R}^N$ in terms of L^p Jones beta numbers and the lower density $\liminf_{r\downarrow 0} r^{-1}\mu(B(x,r))$. In contrast with Theorem 7.3, the main theorem in [9] does not require an *a priori* relationship between the null sets of μ and $\mathcal{H}^1$, nor a bound on the Hausdorff dimension of μ. To lighten the notation, we present Badger and Schul's theorem for pointwise doubling measures and refer the reader to [9, Theorem A] for the full result. Although the classes of measures satisfying (7.5) and (7.7) have no direct relationship with each other, *a posteriori* an m-rectifiable measure satisfying (7.5) also satisfies (7.7). The proof that (7.8) implies the measure μ is 1-rectifiable uses a technical extension of the sufficient half of the Analyst's Traveling Salesman theorem. See [9, Proposition 3.6].

Theorem 7.4 *(see [9, Theorem E]). Let μ be a Radon measure on $\mathbb{R}^n$ and let $p \geq 1$. Assume that μ is pointwise doubling in the sense that*

$$\limsup_{r\downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^N. \tag{7.7}$$

Then μ is 1-rectifiable if and only if

$$\int_0^1 \beta_p^{(1)}(\mu, x, r)^2 \frac{r}{\mu(B(x,r))} \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^N. \tag{7.8}$$

7.3. Sufficient conditions for fractional rectifiability

The following theorem is an application of the Hölder Traveling Salesman theorem and generalizes the "sufficient half" of Theorem 7.4 (also see [10, Theorem A]). The exponents p and q in the Hölder case ($s > 1$) are less restrictive than in the Lipschitz case ($s = 1$).

Theorem 7.5. *Let μ be a Radon measure on $\mathbb{R}^N$, let $s > 1$, and let $p, q > 0$. Then*

$$\mu \llcorner \left\{ x \in \mathbb{R}^n : \int_0^1 \beta_p^{(1)}(\mu, x, r)^q \frac{r^s}{\mu(B(x,r))} \frac{dr}{r} < \infty \text{ and } \limsup_{r \downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \right\}$$

is carried by $(1/s)$-Hölder curves.

At the core of the proof of Theorem 7.5 is the following lemma.

Lemma 7.6. *Let μ be a Radon measure in $\mathbb{R}^N$, and let $s > 1$ and $p, q > 0$ be fixed. Given $x_0 \in \mathbb{R}^N$ and parameters $M > 0$, $\theta > 0$, and $P > 0$, let A denote the set of points $x \in B(x_0, 1/2)$ such that*

$$\int_0^1 \beta_p^{(1)}(\mu, x, r)^q \frac{r^s}{\mu(B(x,r))} \frac{dr}{r} \leq M, \tag{7.9}$$

$$\mu(B(x,2r)) \leq P\mu(B(x,r)) \text{ for all } r \in (0,1], \tag{7.10}$$

and let A' denote the set of points in A such that

$$\mu(A \cap B(x,r)) \geq \theta\mu(B(x,r)) \text{ for all } r \in (0,1]. \tag{7.11}$$

Then A' is contained in a $(1/s)$-Hölder curve $\Gamma = f([0,1])$ with Hölder constant depending on at most N, s, p, q, M, P, θ, and $\mu(A)$.

Proof. Let $\{A'_k\}_{k \geq 0}$ be a nested sequence of 2^{-k}-nets in A', so that the sets $V_k \equiv A'_k$ and scales $\rho_k = 2^{-k}$ satisfy conditions (V0)–(V4) of §2 with parameters $r_0 = 1$, $C^* = 2$, $\xi_1 = \xi_2 = 1/2$. Note that

$$A^* = \frac{C^*}{1 - \xi_2} = 4 \quad \text{and} \quad 30A^* = 120.$$

By (7.9),

$$\begin{aligned} M\mu(A) &\geq \int_A \int_0^1 \beta_p^{(1)}(\mu, x, r)^q \frac{r^s}{\mu(B(x,r))} \frac{dr}{r} d\mu(x) \\ &= \sum_{k=9}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} (512r)^s \int_A \frac{\beta_p^{(1)}(\mu, x, 512r)^q}{\mu(B(x, 512r))} d\mu(x) \frac{dr}{r} \end{aligned} \tag{7.12}$$

where in the second line we used the change of variables $r \mapsto 512r$ (note $512 = 2^9$) and Tonelli's theorem. Now, the open balls $\{B(y, 2^{-(k+1)}) : y \in A'_k\}$ are pairwise disjoint, because the points in A'_k are separated by distance at least 2^{-k}. Thus,

$$M\mu(A) \geq \sum_{k=9}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} r^s \sum_{y \in A'_k} \underbrace{\int_{A \cap B(y, 2^{-(k+1)})} \frac{\beta_p^{(1)}(\mu, x, 512r)^q}{\mu(B(x, 512r))} d\mu(x)}_{I(k,y,r)} \frac{dr}{r}. \tag{7.13}$$

Next, we bound $I(k, y, r)$ from below. Fix $k \geq 9$, $y \in A'_k$, and $r \in [2^{-(k+1)}, 2^{-k}]$. Suppose that $x \in A \cap B(y, 2^{-(k+1)})$. Then

$$\mu(B(x, 512r)) \leq P^2 \mu(B(x, 128r)) \leq P^2 \mu(B(y, 129r)) \leq P^2 \mu(B(y, 255 \cdot 2^{-k})) \tag{7.14}$$

by (7.10). Since $B(y, 255 \cdot 2^{-k}) \subset B(x, 256 \cdot 2^{-k}) \subset B(x, 512r)$, it follows that

$$\begin{aligned} \beta_p^{(1)}(y, 255 \cdot 2^{-k}) &\leq \left(\frac{512r}{255 \cdot 2^{-k}} \right) \left(\frac{\mu(B(x, 512r))}{\mu(B(y, 255 \cdot 2^{-k}))} \right)^{1/p} \beta_p^{(1)}(\mu, x, 512r) \\ &\leq 3P^{2/p} \beta_p^{(1)}(\mu, x, 512r) \end{aligned} \tag{7.15}$$

by (7.4). Hence

$$I(k, y, r) \geq 3^{-q} P^{-2-2q/p} \frac{\beta_p^{(1)}(\mu, y, 255 \cdot 2^{-k})^q}{\mu(B(y, 255 \cdot 2^{-k}))} \int_{A \cap B(y, 2^{-(k+1)})} d\mu(x). \tag{7.16}$$

Invoking doubling again, $\mu(B(y, 255 \cdot 2^{-k})) \leq P^9 \mu(B(y, 2^{-(k+1)}))$. Thus, by (7.11),

$$\frac{1}{\mu(B(y, 255 \cdot 2^{-k}))} \int_{A \cap B(y, 2^{-(k+1)})} d\mu(x) \geq P^{-9} \frac{\mu(A \cap B(y, 2^{-(k+1)}))}{\mu(B(y, 2^{-(k+1)}))} \geq P^{-9} \theta. \tag{7.17}$$

Therefore,

$$I(k, y, r) \geq 3^{-q} P^{-11-2q/p} \theta \, \beta_p^{(1)}(\mu, y, 255 \cdot 2^{-k})^q. \tag{7.18}$$

Combining (7.13) and (7.18), we obtain

$$3^q P^{11+2q/p}\theta^{-1} M\mu(A) \geq \sum_{k=9}^{\infty} \left(\int_{2^{-(k+1)}}^{2^{-k}} r^s \frac{dr}{r} \right) \sum_{y \in A'_k} \beta_p^{(1)}(\mu, y, 255 \cdot 2^{-k})^q. \tag{7.19}$$

In particular, we conclude that

$$\sum_{k=9}^{\infty} \sum_{y \in A'_k} \beta_p^{(1)}(\mu, y, 255 \cdot 2^{-k})^q \, 2^{-ks} \leq \frac{s}{1-2^{-s}} 3^q P^{11+2q/p}\theta^{-1} M\mu(A) < \infty. \tag{7.20}$$

For each $k \geq 9$ and $y \in A'_k$, let $\ell_{k,v}$ be any line such that

$$\beta_p^{(1)}(\mu, y, 255 \cdot 2^{-k}, \ell_{k,v}) \leq 2\beta_p^{(1)}(\mu, y, 255 \cdot 2^{-k}). \tag{7.21}$$

We will now bound the distance of points in $A'_{k+1} \cap B(y, 120 \cdot 2^{-k})$ to $\ell_{k,v}$. Fix any point $z \in A'_{k+1} \cap B(y, 120 \cdot 2^{-k})$ and let $t2^{-k+1} = \operatorname{dist}(z, \ell_{k,v})$. Then

$$\begin{aligned} \beta_p(\mu, y, 255 \cdot 2^{-k}, \ell_{k,v})^q &\geq \left(\frac{\frac{1}{2} t 2^{-(k+1)}}{255 \cdot 2^{-k}} \right)^q \left(\frac{\mu(B(z, \frac{1}{2} t 2^{-(k+1)}))}{\mu(B(y, 255 \cdot 2^{-k}))} \right)^{q/p} \\ &\geq \left(\frac{t}{1020} \right)^q P^{-(q/p)\log_2(1920/t)} \geq \left(\frac{t}{1920} \right)^{q+(q/p)\log_2(P)}, \end{aligned} \tag{7.22}$$

where in the second line we used doubling of μ at z. It follows that

$$\alpha_{k,v} := 2^{k+1} \sup_{z \in A'_{k+1} \cap B(y, 120 \cdot 2^{-k})} \operatorname{dist}(z, \ell_{k,v}) \leq C(p, q, P) \beta_p(\mu, y, 255 \cdot 2^{-k}, \ell_{k,v})^\eta, \tag{7.23}$$

where $\eta[q + (q/p)\log_2(P)] = q$. Therefore, all together,

$$\sum_{k=9}^{\infty} \sum_{y \in A'_k} \alpha_{k,v}^{q+(q/p)\log_2(P)} 2^{-ks} \leq C(s, p, q, M, P, \theta, \mu(A)) < \infty. \tag{7.24}$$

Finally, by Corollary 5.5, the set A' is contained in the Hausdorff limit of A'_k and this is contained in a $(1/s)$-Hölder curve $\Gamma = f([0,1])$ with Hölder constant depending on at most s, p, q, M, P, θ, and $\mu(A)$. □

Theorem 7.5 follows from countably many applications of Lemma 7.6 and a standard density theorem for Radon measures in $\mathbb{R}^N$. See the proof of [10, Theorem 6.7], where a similar argument is employed. We leave the details to the reader.

8. Hölder parameterization of flat continua

The goal of this section is to prove Proposition 1.3, which we now restate.

Proposition 8.1. *There exists a constant $\beta_1 \in (0,1)$ such that if $s > 1$ and $E \subset \mathbb{R}^N$ is compact, connected, $\mathcal{H}^s(E) < \infty$, E is lower Ahlfors s-regular with constant c, and*

$$\beta_E\left(\overline{B}(x,r)\right) \leq \beta_1 \quad \text{for all } x \in E \text{ and } 0 < r \leq \operatorname{diam} E, \tag{8.1}$$

then $E = f([0,1])$ for some injective $(1/s)$-Hölder continuous map $f : [0,1] \to \mathbb{R}^N$ with Hölder constant $H \lesssim_s c^{-1}\mathcal{H}^s(E)(\operatorname{diam} E)^{1-s}$.

The proposition is trivial if E is a singleton (in which case (8.1) is vacuous). Thus, we may assume that $E \subset \mathbb{R}^N$ is a *continuum*; that is, E is compact, connected, and contains at least two points. Furthermore, as the hypothesis and the conclusion are scale-invariant, we may assume without loss of generality that $\operatorname{diam} E = 1$. To complete the proof of the proposition, we mimic the proof of Theorem 5.1, but with a few modifications. In §8.1, we perform a simplified version of the algorithm in §3. Then, in §8.2, we establish an upper bound on $\mathcal{M}_s([0,1])$ in terms of $c^{-1}\mathcal{H}^s(E)$, which fills the role that Proposition 4.11 played for Theorem 5.1. Equipped with this mass bound, the proof of Proposition 8.1 essentially follows by repeating the proof of Theorem 5.1 *mutatis mutandis*.

At the core of the proof of Proposition 8.1 is the following property, which is satisfied by continua that are sufficiently flat at all locations and scales. We defer a proof of Lemma 8.2 to §8.4. An adventurous reader may wish to supply their own proof.

Lemma 8.2. *Suppose that $E \subset \mathbb{R}^N$ is a continuum satisfying*

$$\beta_E\left(\overline{B}(x,r)\right) \leq 2^{-11} \qquad \text{for all } x \in E \text{ and } 0 < r \leq \operatorname{diam} E.$$

Then for all distinct $x, y \in E$ and for all $z \in [x,y]$, there exists $z' \in E$ such that $\pi_{\ell_{x,y}}(z') = z$ and $|z - z'| \leq 2^{-4}|x-y|$, where $\ell_{x,y}$ denotes the line containing x and y.

8.1. Traveling Salesman algorithm for flat continua

Fix a constant $\beta_1 > 0$ (small) to be specified later. Let $E \subset \mathbb{R}^N$ be a continuum satisfying the hypothesis of Proposition 8.1. Without loss of generality, we assume that $\operatorname{diam} E = 1$. Pick x_0 and y_0 such that $|x_0 - y_0| = 1$, set $r_0 = 1$, $C^* = 2$, and $\xi_1 = \xi_2 = 1/2$. Then

$$A^* = \frac{C^*}{1-\xi_2} = 4, \quad 14A^* = 56, \quad 30A^* = 120.$$

Furthermore, the scales $\rho_k = 2^{-k}$ satisfy (V0).

Set $V_0 = \{x_0, y_0\}$, $\alpha_{0,x_0} = 4\beta_E\left(\overline{B}(x_0, 1)\right)$ and $\alpha_{0,y_0} = 4\beta_E\left(\overline{B}(y_0, 1)\right)$, and let ℓ_{0,x_0} and ℓ_{0,y_0} be best fitting lines corresponding to β_E on the closed balls $\overline{B}(x_0, 1)$ and $\overline{B}(y_0, 1)$, respectively.

Suppose that for some $k \geq 0$ and for all $0 \leq j \leq k$, we have defined sets V_j, numbers $\alpha_{j,v} \geq 0$ for all $v \in V_j$, and lines $\ell_{j,v}$ for all $v \in V_j$ satisfying (V5). Choose V_{k+1} to be any maximal $2^{-(k+1)}$-separated subset of E such that $V_{k+1} \supset V_k$. For each $v \in V_{k+1}$, set

$$\alpha_{k+1,v} := \frac{2r_{k+1}}{2^{-(k+2)}}\beta_E\left(\overline{B}(v, r_{k+1})\right) \leq 480\beta_E\left(\overline{B}(v, r_{k+1})\right),$$

where $r_{k+1} := \min\{120 \cdot 2^{-(k+1)}, 1\}$, and let $\ell_{k+1,v}$ be a best fitting line corresponding to $\beta_E\left(\overline{B}(v, r_k)\right)$. The reader may check that the sequence of sets $(V_k)_{k=0}^{\infty}$ satisfy properties (V1)–(V5) in 2.2.

We now specify $\alpha_0 = 512\beta_1 \leq 1/16$. This ensures that $\alpha_{k,v} < \alpha_0$ for all $k \geq 0$ and $v \in V_k$. Moreover, β_1 is sufficiently small that we may invoke Lemma 8.2 for E. With α_0 fixed, carry out a modified version of the algorithm in §3, in which (P4), (P6), and (P7) are replaced by:

(P4′) $f_k|\bigcup \mathscr{E}_k$ is one-to-one.

(P6′) For each $I \in \mathscr{N}_k \cup \mathscr{F}_k$, the image $f_k(I) \in V_k$. For every $v \in V_k$, there exists a unique interval $I \in \mathscr{N}_k \cup \mathscr{F}_k$ such that $v = f_k(I)$.

(P7′) If $\mathscr{I}_k = \mathscr{E}_k \cup \mathscr{B}_k \cup \mathscr{N}_k \cup \mathscr{F}_k$ is enumerated according to the natural order in $[0, 1]$, say $\mathscr{I}_k = \{I_1, \ldots, I_{2l+1}\}$, then the intervals alternate between elements of $\mathscr{N}_k \cup \mathscr{F}_k$ and $\mathscr{E}_k$. (Thus, the family $\mathscr{B}_k = \emptyset$.) Moreover, $\operatorname{card} \mathscr{N}_k = 2$ and $I_1, I_{2l+1} \in \mathscr{N}_k$. The vertices $f_k(I_1)$ and $f_k(I_{2l+1})$ are 1-sided terminal in V_k, while each other vertex $f_k(I_{2j+1})$ is non-terminal in V_k for all $1 \leq j \leq l - 1$.

We now sketch some steps in the modified algorithm.

8.1.1. Step 0

Partition $[0, 1] = [0, 1/3] \cup (1/3, 2/3) \cup [2/3, 1]$ and assign

$$\mathscr{E}_0 = \{(1/3, 2/3)\}, \quad \mathscr{B}_0 = \emptyset, \quad \mathscr{N}_0 = \{[0, 1/3], [2/3, 1]\}, \quad \mathscr{F}_0 = \emptyset.$$

Also set $f_0([0, 1/3]) = x_0$ and $f_0([2/3, 1]) = y_0$, and define $f_0|(1/3, 2/3)$ to be the restriction of the affine map which interpolates between x_0 and y_0. Verifying properties (P1), (P2), (P3), (P4′), (P5), (P6′), and (P7′) is straightforward. We omit the details.

8.1.2. Induction step

Suppose that $\mathscr{E}_k$, $\mathscr{B}_k$, $\mathscr{N}_k$, $\mathscr{F}_k$, and f_k have been defined and satisfy properties (P1), (P2), (P3), (P4′), (P5), (P6′), and (P7′).

By (P7′) and the induction assumption, the procedure in §3.3 is moot, because there are no bridge intervals.

Follow the procedure in §3.4 for $I \in \mathscr{E}_k$ as written, except assign all closed subintervals $\mathscr{N}_{k+1}(I) \cup \mathscr{F}_{k+1}(I)$ generated in I to $\mathscr{F}_{k+1}(I)$ instead of $\mathscr{N}_{k+1}(I)$. Also set $\mathscr{N}_{k+1}(I) = \emptyset$. Below, we check that $\mathscr{B}_{k+1}(I) = \emptyset$; see Lemma 8.3.

Because $\alpha_{k,v} < \alpha_0$ for all $v \in V_k$, the procedure in §3.5 is moot.

Follow the procedure in §3.6 as written.

Replace the procedure in §3.7 as follows.

By property (P7′), only the intervals in $\mathscr{I}_k$ containing 0 and 1 belong to $\mathscr{N}_k$ and f_k maps each of them onto a 1-sided terminal vertex in V_k. Let I be the interval in $\mathscr{N}_k$ containing 0 and let $v = f_k(I)$. Let $(v, v') \in \mathsf{Flat}(k)$ be the unique flat pair with first coordinate v. Choose an orientation for $\ell_{k,v}$ so that $[v, v']$ lies on the right side of v. Enumerate the points in $V_{k+1} \cap B(f_k(I), C^*\rho_{k+1}r_0)$ on the left side of v (including v) as $v_l, v_{l-1}, \dots, v_1 = v$, starting at the leftmost vertex and working right. The construction splits into three cases.

Case 1a. If $l = 1$, then no new points appeared to the left of v and we set $\mathscr{N}_{k+1}(I) = I$ and $f_{k+1}|I = f_k|I$. Set $\mathscr{E}_{k+1}(I) = \mathscr{B}_{k+1}(I) = \mathscr{F}_{k+1}(I) = \emptyset$.

Case 1b. If $l = 2$, then one new point appeared to the left of v, the new point is 1-sided terminal in V_{k+1} and v is non-terminal in V_{k+1}. Subdivide $I = [0, a] = [0, \frac{1}{3}a] \cup (\frac{1}{3}a, \frac{2}{3}a) \cup [\frac{2}{3}a, a]$, set $\mathscr{N}_{k+1}(I) = [0, \frac{1}{3}a]$, $\mathscr{E}_{k+1}(I) = (\frac{1}{3}a, \frac{2}{3}a)$, $\mathscr{F}_{k+1}(I) = [\frac{2}{3}a, a]$, and $\mathscr{B}_{k+1} = \emptyset$. Define $f_{k+1}|I$ by assigning $f_{k+1}([0, \frac{1}{3}a]) = v_2$, $f_{k+1}|(\frac{1}{3}a, \frac{2}{3}a)$ to be the restriction of the affine map that interpolates between v_2 and v_1, and $f_{k+1}([\frac{2}{3}a, a]) = v_1$.

Case 1c. If $l \geq 3$, then subdivide I into $2l - 1$ intervals, alternating between closed and open intervals. Assign the first interval to $\mathscr{N}_{k+1}(I)$ and the subsequent closed intervals to $\mathscr{F}_{k+1}(I)$. Assign the open intervals in $\mathscr{E}_{k+1}(I)$ and set $\mathscr{B}_{k+1}(I) = \emptyset$. The map $f_{k+1}|I$ is the piecewise linear map starting at v_l, connecting v_l to v_{l-1}, …, connecting v_2 to v_1, which is constant on the intervals in $\mathscr{N}_{k+1}(I) \cup \mathscr{F}_{k+1}(I)$ and constant speed on the intervals in $\mathscr{E}_{k+1}(I)$.

Carry out a similar construction for the interval J in $\mathscr{N}_k$ containing 1, but modified so that $\mathscr{N}_{k+1}(J)$ contains only one interval and that interval contains 1.

Because $\alpha_{k,v} < \alpha_0$ for all $v \in V_k$, the procedure in §3.8 is not used.

Lemma 8.3. *For all $k \geq 0$, $\mathscr{B}_k = \emptyset$. Moreover, for all $I \in \mathscr{E}_k$,* $\operatorname{diam} f_k(I) < 3 \cdot 2^{-k}$.

Proof. We need to check that, for flat continua, the procedure in §3.4 does not generate bridge intervals. Given $I \in \mathscr{E}_k$, let v and v' denote the endpoints of $f_k(I)$. Choose an orientation of $\ell_{k,v}$ so that v' lies to the right of v. Enumerate $V_{k+1}(v, v') = \{v_1, \dots, v_n\}$, where $v_1 = v$, $v_n = v'$, and v_{i+1} is the first point to the right of v_i for all $1 \leq i \leq n - 1$. Suppose for contradiction that $\mathscr{B}_{k+1}(I) \neq \emptyset$. Then

$$|v_{j+1} - v_j| \geq 14A^*\rho_{k+1} = 56 \cdot 2^{-(k+1)} \quad \text{for some } 1 \leq j \leq n - 1.$$

Let $x = (v_j + v_{j+1})/2$ denote the midpoint between v_j and v_{j+1}. By Lemma 8.2, there exists $y \in E$ such that $|y - x| \leq (1/16)|v_{j+1} - v_j|$. Thus,

$$\operatorname{dist}(y, V_{k+1}) \geq \operatorname{dist}(x, V_{k+1}) - |y - x| \geq \frac{7}{16}|v_j - v_{j-1}| > 24 \cdot 2^{-(k+1)}.$$

This contradicts our assumption that V_{k+1} is a maximal $2^{-(k+1)}$-separated set for E. Therefore, $\mathscr{B}_{k+1}(I) = \emptyset$ for all $I \in \mathscr{E}_k$. The only other instances in the algorithm where bridge intervals could be generated are in the procedures in §§3.5 and 3.8. However, since $\alpha_{k,v} < \alpha_0$ for all $k \geq 0$ and $v \in V_k$, these procedures were never used.

Similarly, suppose to get a contradiction that there exists $I \in \mathscr{E}_k$ such that $\operatorname{diam} f_k(I) \geq 3 \cdot 2^{-k}$. Let v and v' denote the endpoints of $f_k(I)$, and let $x = (v+v')/2$ denote their midpoint. By Lemma 8.2, there exists $y \in E$ such that $|y - x| < (1/16)|v - v'|$. Then

$$\operatorname{dist}(y, V_k) \geq \operatorname{dist}(x, V_k) - |x - y| \geq \frac{7}{16}|v - v'| \geq \frac{21}{16}2^{-k}.$$

This contradicts our assumption that V_{k+1} is a maximal $2^{-(k+1)}$-separated set for E. □

Verifying properties (P1), (P2), (P3), (P4′), (P5), (P6′), and (P7′) for $\mathscr{E}_{k+1}$, $\mathscr{B}_{k+1}$, $\mathscr{N}_{k+1}$, $\mathscr{F}_{k+1}$, and f_{k+1} is again routine. We leave the details to the reader.

8.2. Mass estimate

Let $\mathcal{M}_s([0,1])$ be defined as in §4.

Lemma 8.4. $\mathcal{M}_s([0,1]) \leq 48^s c^{-1} \mathcal{H}^s(E)$.

Proof. Fix a finite tree T over $[0,1]$ of depth m (see §4) and suppose that

$$\partial T = \{(k_1', J_1), (k_1, I_1), \ldots, (k_l', J_l), (k_l, I_l), (k_{l+1}', J_{l+1})\},$$

enumerated according to the orientation of $[0,1]$ so that

$$\{I_1, \ldots, I_l\} \subseteq \bigcup_{k \geq 0} \mathscr{E}_k \qquad \text{and} \qquad \{J_1, \ldots, J_{l+1}\} \subseteq \bigcup_{k \geq 0} (\mathscr{N}_k \cup \mathscr{F}_k).$$

The first interval $J_1 \in \mathscr{N}_{k_1'}$ and the last interval $J_{l+1} \in \mathscr{N}_{k_{l+1}'}$, since they contain 0 and 1, respectively. The remaining intervals $J_i \in \mathscr{F}_{k_i'}$, because they do not contain 0 or 1.

For each $1 \leq i \leq l$, let x_i denote the midpoint of $f_{k_i}(I_i)$.

Claim 8.5. *One one hand, the set $E \cap B(x_i, (3/16)2^{-k_i})$ is nonempty for all $1 \leq i \leq l$. On the other hand, the family $\left\{E \cap B(x_i, (1/4)2^{-k_i}) : 1 \leq i \leq l\right\}$ is pairwise disjoint.*

Proof. Given $1 \leq i \leq l$, let v_i and v_i' denote the endpoints of $f_{k_i}(I_i)$. By Lemma 8.3,

$$|v_i - v_i'| < 3 \cdot 2^{-k_i}.$$

Hence there exists $z_i \in E \cap B(x_i, (1/16)|v_i - v_i'|) \subset E \cap B(x_i, (3/16) \cdot 2^{-k_i})$ by Lemma 8.2.

Suppose in order to reach a contradiction that there exists

$$z \in E \cap B(x_i, (1/4)2^{-k_i}) \cap B(x_j, (1/4)2^{-k_j})$$

for some $i \neq j$ with $0 \leq k_i \leq k_j \leq m$.

Case 1. Suppose that $k_j \geq k_i + 3$. Then

$$|v_j - x_i| \leq |v_j - x_j| + |x_j - z| + |z - x_i| < \frac{3}{2} \cdot 2^{-k_j} + \frac{1}{4} 2^{-k_j} + \frac{1}{4} \cdot 2^{-k_i} < \frac{1}{2} \cdot 2^{-k_i}$$

and similarly for v_j'. Because $\{v_j, v_j'\} \subset B(x_i, \frac{1}{2}2^{-k_i})$, we conclude that v_j and v_j' lie between v_i and v_i' with respect to the linear ordering of $V_i \cap B(v_i, 120 \cdot 2^{-k_i})$. It follows that I_j is contained in I_i, but $I_j \neq I_i$. This contradicts the assertion that $I_i \in \partial T$.

Case 2. Suppose that $k_j \leq k_i + 2$. Then

$$|v_i - v_j| \leq |v_i - x_i| + |x_i - z| + |z - x_j| + |x_j - v_j| < \frac{3}{2} 2^{-k_i} + \frac{1}{4} \cdot 2^{-k_i} + \frac{1}{4} \cdot 2^{-k_j} + \frac{3}{2} 2^{-k_j} < 8 \cdot 2^{-k_j},$$

$$|v_i' - v_j| \leq |v_i' - v_i| + |v_i - v_j| < 3 \cdot 2^{-k_i} + 8 \cdot 2^{-k_j} \leq 20 \cdot 2^{-k_j}.$$

In particular, v_i, v_i', v_j, v_j' belong to $V_j \cap B(v_j, 120 \cdot 2^{-k_j})$, which is linearly ordered by Lemma 2.2, where v_j and v_{j+1} are consecutive points. Assume that v_i and v_i' both lie on the left or the right side of $[v_j, v_j']$, say without loss of generality that the appear from left to right as v_j, v_j', v_i, v_i'. Then

$$|x_i - v_j'| \geq \frac{1}{2} |v_i' - v_i| \geq \frac{1}{2} \cdot 2^{-k_i} > \frac{1}{4} \cdot 2^{-k_i}.$$

It follows that $B(v_j, \frac{1}{4} \cdot 2^{-k_j}) \cap B(v_i, \frac{1}{4} \cdot 2^{-k_i})$ is empty, which contradicts our assumption. Thus, one of v_i or v_i' lies on the left side of $[v_j, v_j']$ and the other lies on the right side. Then $I_j \subset I_i$. If $k_j \geq k_i + 1$, then we reach the same contradiction as in Case 1. If $k_j = k_i$, then it follows that $I_j = I_i$, which contradicts our assumption that $i \neq j$. □

We now continue with the proof of Lemma 8.4. By Claim 8.5, we can find balls $B(z_i, (1/16)2^{-k_i})$ centered in E for all $1 \leq i \leq l$, which are pairwise disjoint. Moreover, because E is lower Ahlfors regular, we have $c[(1/16)2^{-k_i}]^s \leq \mathcal{H}^s(E \cap B(z_i, (1/16)2^{-k_i}))$ for all $1 \leq i \leq l$. Therefore, by Lemma 8.3 and additivity of measures on disjoint sets,

$$\sum_{(k,I) \in \partial T} (\operatorname{diam} f_k(I))^s = \sum_{i=1}^{l} (\operatorname{diam} f_{k_i}(I_i))^s \leq \sum_{i=1}^{l} (3 \cdot 2^{-k_i})^s \leq 48^s c^{-1} \mathcal{H}^s(E).$$

Because T was an arbitrary finite tree over $[0,1]$, we obtain the corresponding inequality for the total mass $\mathcal{M}_s([0,1])$. □

Corollary 8.6. *If $k \geq 0$, $I \in \mathscr{I}_k$, and a is an endpoint of I, then*

$$\mathcal{M}_s(k, I) \leq 48^s c^{-1}\, \mathcal{H}^s(E \cap B(f_k(a), 6 \cdot 2^{-k})).$$

Proof. Let T be a finite tree over (k, I). By Lemma 8.3, the image of any edge interval in ∂T is contained in a ball centered at $f_k(a)$ of radius at most

$$3 \cdot 2^{-k} + 3 \cdot 2^{-(k+1)} + \cdots = 6 \cdot 2^{-k}.$$

The conclusion follows by repeating the proof of Lemma 8.4. □

8.3. Proof of Proposition 8.1

With Lemma 8.4 in hand, follow the proof of Theorem 5.1 in §5.1, *mutatis mutandis*. Construct families of intervals $\mathscr{E}'_k$, $\mathscr{N}'_k$, and $\mathscr{F}'_k$ as in §5.1. We are free to specify the following additional constraints.

- If $I \in \mathscr{E}_k$, say $I \in \mathscr{E}_k(I_0)$ for some $I_0 \in \mathscr{E}_{k-1} \cup \mathscr{N}_{k-1}$, then the corresponding interval $I' \in \mathscr{E}'_k$ satisfies

$$\operatorname{diam} I' = \frac{\mathcal{M}_s(k, I)}{\mathcal{M}_s([0,1])} + \frac{1}{\operatorname{card}(\mathscr{E}_k(I_0))} \left(\operatorname{diam} I_0 - \sum_{J \in \mathscr{I}_k(I_0)} \frac{\mathcal{M}_s(k, J)}{\mathcal{M}_s([0,1])} \right). \tag{8.2}$$

- If $I \in \mathscr{F}_k$, then the corresponding interval $I' \in \mathscr{F}'_k$ satisfies $\operatorname{diam} I' = 0$. That is, I' is a singleton.
- If $I \in \mathscr{N}_k$, then the corresponding interval $I' \in \mathscr{N}'_k$ satisfies $\operatorname{diam} I' = \dfrac{\mathcal{M}_s(k, I)}{\mathcal{M}_s([0,1])}$.

Lemma 8.7. *For any $\epsilon > 0$, there exists $k_0 \geq 0$ such that $\operatorname{diam} I \leq \epsilon$ for all $k \geq k_0$ and $I \in \mathscr{I}'_k$.*

Proof. Fix $\epsilon > 0$. Note that if $J \in \mathscr{E}_l$, then $\operatorname{card}(\mathscr{E}_{l+1}(I)) \geq 2$ by Lemma 8.2, because $\operatorname{diam} f_l(J) \geq 2^{-l}$ and V_{l+1} is a maximal $2^{-(l+1)}$-separated set in E.

Suppose that $I \in \mathscr{I}_k$. Set $J_k = I$. Inductively, given $J_l \in \mathscr{I}_l$, let J_{l-1} denote the unique interval in $\mathscr{I}_{l-1}$ with $J_l \subset J_{l-1}$. That is,

$$I = J_k \subset J_{k-1} \subset \cdots \subset J_1 \subset J_0 = [0,1].$$

For each $i \in \{0, \ldots, k\}$, let $J'_i \in \mathscr{I}'_i$ be the interval associated to J_i. We claim that for each $i \in \{0, \ldots, k\}$

$$\operatorname{diam} J_i' \le \sum_{l=0}^{i} 2^{l-i} \frac{\mathcal{M}_s(l, J_l)}{\mathcal{M}_s([0,1])}. \tag{8.3}$$

We prove (8.3) by induction. For $i = 0$ the claim is clear. Assume that (8.3) is true for $0 \le i < k$. If $J_{i+1} \in \mathscr{F}_{i+1}(J_i)$, then $\operatorname{diam} J_{i+1}' = 0$ and (8.3) is clear. If $J_{i+1} \in \mathscr{F}_{i+1}(J_i)$, then $\operatorname{diam} J_{i+1}' = \mathcal{M}_s(i+1, J_{i+1})/\mathcal{M}_s([0,1])$ and (8.3) is again clear. If $J_{i+1} \in \mathscr{E}_{i+1}(J_i)$, then by (8.2) and the induction hypothesis,

$$\operatorname{diam} J_{i+1}' \le \frac{\mathcal{M}_s(i+1, J_{i+1})}{\mathcal{M}_s([0,1])} + \frac{1}{2} \operatorname{diam} J_i' \le \sum_{l=0}^{i+1} 2^{l-(i+1)} \frac{\mathcal{M}_s(l, J_l)}{\mathcal{M}_s([0,1])}.$$

This establishes (8.3).

Choose an integer l_0 sufficiently large so that $2^{-l_0} l_0 \le \epsilon/2$. If $k \ge l_0$, then by (8.3) and the fact that $\mathcal{M}_s([0,1]) \ge (\operatorname{diam} E)^s = 1$,

$$\operatorname{diam} I' \le \sum_{l=0}^{l_0-1} 2^{l-k} \frac{\mathcal{M}_s(l, J_l)}{\mathcal{M}_s([0,1])} + \sum_{l=l_0}^{k} 2^{l-k} \frac{\mathcal{M}_s(l, J_l)}{\mathcal{M}_s([0,1])} \le \frac{\epsilon}{2} + 2\,\mathcal{M}_s(l_0, J_{l_0}).$$

Thus, by Corollary 8.6,

$$\operatorname{diam} I' \le \frac{\epsilon}{2} + 2\,\frac{48^s}{c} \sup_{x \in E} \mathcal{H}^s(E \cap B(x, 6 \cdot 2^{-l_0}))$$

whenever $k \ge l_0$. Now, because E is compact, $\mathcal{H}^s(E) < \infty$, and $\mathcal{H}^s$ has no atoms,

$$\lim_{n \to \infty} \sup_{x \in E} \mathcal{H}^s(E \cap B(x, 6 \cdot 2^{-n})) = 0.$$

Hence, by choosing l_0 even larger if necessary, we can ensure that

$$\sup_{x \in E} 2\,\frac{48^s}{c} \mathcal{H}^s(E \cap B(x, 6 \cdot 2^{-l_0})) < \frac{\epsilon}{2}.$$

Therefore, $\operatorname{diam} I' < \epsilon$ for all $k \ge l_0$ provided that l_0 is sufficiently large depending on ϵ and E. □

Following the proof of Theorem 5.1 in §5.1, we obtain a sequence of maps $F_k : [0,1] \to \mathbb{R}^N$ and a $(1/s)$-Hölder continuous map $F : [0,1] \to \mathbb{R}^N$ satisfying the following properties.

(1) For each $k \ge 0$, there exist (possibly degenerate) closed intervals $[0, a_k]$ and $[b_k, 1]$ such that $F_k|[0, a_k]$ and $F|[0, b_k]$ are constant maps and $F(0), F(1) \in V_k$.
(2) For each $k \ge 0$, the map $F_k|(a_k, b_k)$ is an injective piecewise linear map connecting $F_k(0)$ to $F_k(1)$ along line segments between points in V_k of length at most $3 \cdot 2^{-k}$.

(3) If $x \in V_k \setminus \{F_k(0), F_k(1)\}$ for some $k \geq 0$, then $F_k^{-1}(x) = F_j^{-1}(x)$ for all $k \geq j$ (because intervals in $\mathscr{F}_j$ are frozen).
(4) The maps F_k converge uniformly to F and $F([0,1])$ contains $\bigcup_{k=0}^{\infty} V_k$.
(5) The Hölder constant of F satisfies

$$H \leq \frac{1}{\xi_1}\left(\mathcal{M}_s([0,1])r_0^{1-s} + 60A^* r_0 \frac{\xi_2}{1-\xi_2}\right) \lesssim_s c^{-1}\mathcal{H}^s(E)(\operatorname{diam} E)^{1-s}.$$

It remains to show that $F([0,1]) = E$ and F is injective.

On one hand, since each V_k is a maximal 2^{-k}-separated set in E,

$$F([0,1]) \supset \overline{\textstyle\bigcup_{k=0}^{\infty} V_k} = E$$

by (3). On the other hand, if $x \in F([0,1])$, say $x = F(t)$, then

$$\operatorname{dist}(x, E) \leq \liminf_{k\to\infty} \operatorname{dist}(F_k(t), E) = 0$$

by (2). Thus, $F([0,1]) = E$.

To check injectivity, we first establish a lemma. We say that an interval I *separates* two numbers $x < y$ if $x < z < y$ for all $z \in I$.

Lemma 8.8. *Let $0 \leq x < y \leq 1$. If k_0 is the least integer $k \geq 0$ such that there exists an interval in $\mathscr{E}_k'$ separating x and y, then $|F_k(x) - F_k(y)| \gtrsim 2^{-k_0}$ for all $k \geq k_0$.*

Proof. Fix $0 \leq x < y \leq 1$, let k_0 be as in the statement of the lemma and let $k \geq k_0$. Let $I_0 \in \mathscr{E}_{k_0}$ be such that I_0 separates x and y. The proof is divided into two cases.

Case 1. Assume that $k \leq k_0 + 3$. By Remark 3.5 and minimality of k_0, there exist at most 13 intervals $I \in \mathscr{E}_{k_0}'$ separating x from y. Therefore, there exist consecutive intervals $J_1, \ldots, J_l \in \mathscr{I}_k$ such that $x, y \in \bigcup_{i=1}^{l} J_i$, $I_0 \subset \bigcup_{i=1}^{l} J_i$ and $l \leq 15$. Let a be an endpoint of I_0. Since $\operatorname{diam} F_k(J_i) \leq 3 \cdot 2^{-k_0}$ for all $i \in \{1, \ldots, l\}$, the points $F_k(x)$ and $F_k(y)$ are in $B := B(F_k(a), 45 \cdot 2^{-k_0})$. Let ℓ be a best fitting line for B and let π be the orthogonal projection on ℓ. Since $\beta_E(B) \leq 2^{-11}$, the points of $F_k(\bigcup_{i=1}^{l} J_i) \cap B$ are linearly ordered according to their projection on ℓ. In particular, $|z - w| \leq 2|\pi(z) - \pi(w)|$ for all $z, w \in F_k(\bigcup_{i=1}^{l} J_i) \cap B$. Thus,

$$|F_k(x) - F_k(y)| \geq |\pi_\ell(F_k(x)) - \pi_\ell(F_k(y))| \geq \operatorname{diam} \pi_\ell(F_k(I_0)) \geq \frac{1}{2} \operatorname{diam} F_k(I_0) \geq \frac{1}{2} 2^{-k_0}.$$

Case 2. Assume that $k \geq k_0 + 4$. For each integer $i \geq 0$, let $P_i(x, y)$ be the endpoints of intervals in $\mathscr{E}_{k_0+4+i}'$ lying between x and y. Let x_i (resp. y_i) be the leftmost (resp. rightmost) element of $P_i(x, y)$. For all $i \geq 0$,

$$x \leq x_{i+1} \leq x_i < y_i \leq y_{i+1} \leq y,$$

and I_0 separates x_0 from y_0. As in Case 1, if ℓ is a best fitting line for $F_{k_0}(a)$, then

$$|F_{k_0+3}(x_0)-F_{k_0+3}(y_0)| \geq |\pi_\ell(F_{k_0+3}(x_0))-\pi_\ell(F_{k_0+3}(y_0))| \geq \operatorname{diam} \pi_\ell(F_{k_0}(I_0)) \geq \frac{9}{10}2^{-k_0}.$$

Now, each x_{i+1} (resp. y_{i+1}) is contained in the closure of an interval in $\mathscr{E}'_{k+4+i}$ which has x_i (resp. y_i) as an endpoint. This fact along with Lemma 8.3 yields

$$\begin{aligned} |F_{k_0+4+i}(x_i) - F_{k_0+5+i}(x_{i+1})| &\leq 3 \cdot 2^{-k_0-4-i}, \\ |F_{k_0+4+i}(y_i) - F_{k_0+5+i}(y_{i+1})| &\leq 3 \cdot 2^{-k_0-4-i}. \end{aligned}$$

Therefore, by the triangle inequality,

$$\begin{aligned} |F_k(x) - F_k(y)| &\geq |F_{k_0+4}(x_0) - F_{k_0+4}(y_0)| - \sum_{i=0}^{\infty} |F_{k_0+4+i}(x_i) - F_{k_0+5+i}(x_{i+1})| \\ &\qquad - \sum_{i=0}^{\infty} |F_{k_0+4+i}(y_i) - F_{k_0+5+i}(y_{i+1})| \\ &\geq \frac{9}{10}2^{-k_0} - 6 \cdot 2^{-k_0-4} - 6 \cdot 2^{-k_0-4}. \end{aligned}$$

Hence $|F_k(x) - F_k(y)| \geq (3/20)2^{-k_0}$ and the proof is complete. □

Suppose that $x, y \in [0,1]$ with $x < y$. By Lemma 8.7, there exist intervals $I \in \mathscr{E}'_k$ that separate x and y provided that k is sufficiently large. If k_0 is the least such integer, then $|F(x) - F(y)| \gtrsim 2^{-k_0} > 0$ by Lemma 8.8. This shows that F is injective and completes the proof of Proposition 8.1.

8.4. Proof of Lemma 8.2

We first give an auxiliary estimate.

Lemma 8.9. *Let $E \subset \mathbb{R}^N$, $x \in E$, and $r > 0$. If $y \in E \cap \overline{B}(x,r)$, $|y-x| \geq 64\beta_E(\overline{B}(x,r))r$, and $\ell_{x,y}$ is the line passing through x and y, then*

$$\operatorname{dist}(z, \ell_{x,y}) \leq 4\beta_E(\overline{B}(x,r)) \left(1 + \frac{1.1r}{|y-x|}\right) r \quad \text{for all } z \in E \cap B(x,r).$$

Proof. Let $z \in E \cap \overline{B}(x,r)$, $z \neq x$. Let ℓ be a best fitting line for E in $\overline{B}(x,r)$. Then $\operatorname{dist}(x,\ell)$, $\operatorname{dist}(y,\ell)$, and $\operatorname{dist}(z,\ell)$ are bounded above by $\beta_E(\overline{B}(x,r))2r$. Let $\ell_x = \ell - x$. Then $x \in \ell_x$ and $\operatorname{dist}(y,\ell_x)$ and $\operatorname{dist}(z,\ell_x)$ are bounded above by $2\beta_E(\overline{B}(x,r))2r$. If $y \in \ell_x$, then we have $\operatorname{dist}(z,\ell_{x,y}) = \operatorname{dist}(z,\ell_x) \leq 2\beta_E(\overline{B}(x,r))2r$ and we are done.

To continue, suppose that $y \notin \ell_x$ and let $y' = \pi_{\ell_x}(y)$ and let $z' = \pi_{\ell_x}(z)$. Define

$$w = x + \frac{|z' - x|}{|y' - x|}(y - x) \in \ell_{x,y}.$$

Since $z' \in \ell_x$ between x and y', we have that $z' = x + |z' - x||y' - x|^{-1}(y' - x)$. Therefore,

$$\operatorname{dist}(z', \ell_{x,y}) \leq |z' - w| = |y' - y|\frac{|z' - x|}{|y' - x|} \leq |y' - y|\frac{r}{|y' - x|}.$$

Thus, by the triangle inequality,

$$\operatorname{dist}(z, \ell_{x,y}) \leq |z' - z| + |y' - y|\frac{r}{|y' - x|} \leq 2\beta_E(\overline{B}(x, r))\left(1 + \frac{r}{|y' - x|}\right)2r.$$

Since $\operatorname{dist}(y, \ell_x) \leq 2\beta_E(\overline{B}(x, r))2r \leq (1/16)|x - y|$, we have

$$1.1|y' - x| \geq (1 + 3(1/16)^2)|y' - x| \geq |y - x|$$

by Lemma 2.1, applied with $V = \{x, y\}$. Therefore,

$$\operatorname{dist}(z, \ell_{x,y}) \leq 2\beta_E(\overline{B}(x, r))\left(1 + \frac{1.1r}{|y - x|}\right)2r. \quad \square$$

We now give a proof of the key lemma.

Proof of Lemma 8.2. Without loss of generality, we may assume that $\operatorname{diam} E = 1$. Fix $x, y \in E$ and let $n \geq 0$ be the unique integer such that

$$2^{-(n+1)} < |x - y| \leq 2^{-n}.$$

Case 1. Suppose that $n \in \{0, 1\}$. Let ℓ be the line containing x and y. Since $\beta_E(\overline{B}(x, 1)) \leq 2^{-11}$ and since $|x - y| > 2^{-2}$, by Lemma 8.9 we have that

$$\begin{aligned}\sup_{w \in E} \operatorname{dist}(w, \ell) = \sup_{w \in E \cap \overline{B}(x,1)} \operatorname{dist}(w, \ell) &\leq \left(1 + \frac{1.1}{|x - y|}\right)4\beta_E(\overline{B}(x, 1)) \\ &\leq \frac{21.6}{2^{11}} \leq \frac{1}{2^6} < \frac{|x - y|}{2^4}.\end{aligned}$$

Therefore, E is contained inside the tube $T := \overline{B}(x, 1) \cap B(\ell, 2^{-4}|x - y|)$. Let D be a closed $(N - 1)$-ball centered at z, perpendicular to ℓ and of radius $2^{-4}|x - y|$. In other words, D is the set of all points in $\overline{B}(z, 2^{-6})$ whose projection on ℓ is z. Then D cuts T into two pieces, one containing x and another containing y. By connectedness of E, we must have $D \cap E \neq \emptyset$.

Case 2. Suppose that $n \geq 2$. The procedure here is roughly the same as that in Case 1, with the difference that the tube T is replaced by a more complicated set. By connectedness of E, for each $k \in \{1, \dots, n-1\}$, there exists a point $y_k \in B(x, 2^{-k}) \cap E$. For each $k \in \{1, \dots, n-1\}$ let ℓ_k be the line containing x and y_k. Let also ℓ_n be the line containing x and y.

Working as in Case 1, we can show that for each $k \in \{1, \dots, n\}$,

$$E \cap \overline{B}(x, 2^{-(k-1)}) \subset T_k := \overline{B}(x, 2^{-(k-1)}) \cap B(\ell_k, 2^{-5} 2^{-(k-1)}).$$

Since $\operatorname{diam} E = 1$, we also have $E \subset T_1$. For each $k \in \{1, \dots, n-1\}$, let $T_{k,1}$, $T_{k,2}$ be the two components of $T_k \setminus \overline{B}(x, 2^{-k})$. Set

$$T = T_{1,1} \cup \cdots \cup T_{n-1,1} \cup T_n \cup T_{n-1,2} \cup \cdots \cup T_{1,2}.$$

The sets $T_{1,1}, \dots, T_{n-1,1}, T_n, T_{n-1,2}, \dots, T_{1,2}$ intersect at most in pairs. In particular,

(1) if $i \in \{1, 2\}$, then $T_{1,i} \cap T_{m,j} = \emptyset$ unless $m \in \{1, 2\}$ and $j = i$;
(2) if $k \in \{2, \dots, n-2\}$ (if any) and $i \in \{1, 2\}$, then $T_{k,i} \cap T_{m,j} = \emptyset$ unless $m \in \{k-1, k, k+1\}$ and $j = i$;
(3) if $i \in \{1, 2\}$, then $T_{n-1,i} \cap T_{m,j} = \emptyset$ unless $m \in \{n-2, n-1\}$ and $j = i$;
(4) $T_n \cap T_{m,j} = \emptyset$ unless $m = n-1$.

As with Case 1, if D is an $(N-1)$-ball centered at z, perpendicular to ℓ_n and of radius $2^{-4} 2^{-n}$, then D cuts T_n into two pieces, one containing x and another containing y. Consequently, D cuts T into two pieces, one containing x and another containing y. By connectedness of E and the fact that $E \subset T$, we must have $D \cap E \neq \emptyset$. □

9. Examples

In this section, we give examples of Hölder curves and of sets that are not contained in Hölder curves to illuminate Theorem 1.1, Proposition 1.3, and Theorem 5.1.

9.1. Hölder curves that are non-flat in all scales

First up, we show that condition (1.2) in Theorem 1.1 is not necessary for a bounded set to be contained in a $(1/s)$-Hölder curve when $s > 1$. In contrast, when $s = 1$, condition (1.1) in the Analyst's Traveling Salesman theorem is necessary and sufficient for a bounded set to be contained in a rectifiable curve.

Let $N \geq 2$ and $1 \leq m \leq N-1$ be integers. Given a nonempty set $E \subset \mathbb{R}^N$ and an N-cube $Q \subset \mathbb{R}^N$ with $E \cap Q \neq \emptyset$, define the m-dimensional beta number

$$\beta_E^{(m)}(Q) := \inf_P \sup_{x \in E \cap Q} \frac{\operatorname{dist}(x, P)}{\operatorname{diam} Q}$$

where the infimum is taken over all m-planes P in $\mathbb{R}^N$. If $E \cap Q = \emptyset$, set $\beta_E^{(m)}(Q) = 0$. Note that $\beta_E^{(1)}(Q) = \beta_E(Q)$ as defined in §1 and that $\beta_E^{(m)}(Q) \leq \beta_E^{(n)}(Q)$ whenever $m \geq n$.

Proposition 9.1. *For any $N \geq 2$ and any $s \in (1, N]$, there exists a $(1/s)$-Hölder curve $E \subset \mathbb{R}^N$ such that*

$$\sum_{\substack{Q \in \Delta(\mathbb{R}^N) \\ \beta_E^{(N-1)}(3Q) \geq (6\sqrt{N})^{-1}}} (\operatorname{diam} Q)^s = \infty. \tag{9.1}$$

The construction splits into three cases. Before proceeding, we introduce some notation. Given a cube $Q \subset \mathbb{R}^N$, denote by $\Delta(Q)$ the set of dyadic cubes in $\Delta(\mathbb{R}^N)$ that are contained in Q. Moreover, given positive integers $m \leq N$, there exists a polynomial $P_{N,m}$ of degree m with the following property: If $n \in \mathbb{N}$ and $\{Q_1, \dots, Q_{N^n}\}$ is a partition of $[0,1]^N$ into N-cubes of side-length $1/n$, then

$$\operatorname{card}\{Q_i : Q_i \text{ intersects the } m\text{-skeleton of } \partial[0,1]^N\} = P_{N,m}(n).$$

Recall that if $I_1, \dots, I_N$ are nondegenerate compact intervals, and $Q = I_1 \times \cdots \times I_N$ is an N-cube, then the m-skeleton of Q is the union of sets $I_1' \times \cdots \times I_N'$ where $I_j' = I_j$ for m indices j and $I_j' = \partial I_j$ for the remaining $N - m$ indices j. Finally, we note that if K is the set of vertices of a cube Q in $\mathbb{R}^N$ and P is an $(N-1)$-plane, then

$$\operatorname{dist}(x, P) \geq (2\sqrt{N})^{-1} \operatorname{diam} K \qquad \text{for all } x \in K. \tag{9.2}$$

Case 1: $s = N$. We simply take $E = [0,1]^N$. It is well known that there exists a $(1/N)$-Hölder parametrization $f : [0,1] \to E$. On the other hand, by (9.2),

$$\sum_{\substack{Q \in \Delta(\mathbb{R}^N) \\ \beta_E^{(N-1)}(3Q) \geq (2\sqrt{N})^{-1}}} (\operatorname{diam} Q)^N \geq \sum_{k=0}^{\infty} \sum_{\substack{Q \in \Delta([0,1]^N) \\ \operatorname{diam} Q = \sqrt{N} 2^{-k}}} (\operatorname{diam} Q)^N = \sum_{k=0}^{\infty} 2^{Nk} (\sqrt{N} 2^{-k})^N = \infty.$$

Case 2: $s \in (1, N) \setminus \mathbb{N}$. Let m be the integer part of s. Since the degree of $P_{N,m}$ is strictly less than s and strictly larger than $s - 1$, we can fix $n \in \mathbb{N}$ such that

$$n^s - (n-2)^s < P_{N,m}(n) < n^s - 1. \tag{9.3}$$

By (9.3) and the Intermediate Value Theorem, there exists $\lambda \in (\frac{1}{n}, 1 - \frac{2}{n})$ such that

$$P_{N,m}(n) n^{-s} + \lambda^s = 1. \tag{9.4}$$

Partition $[0,1]^N$ into N-cubes of side-lengths $1/n$ and let $\{Q_i\}_{i=1}^l$ ($l = P_{N,m}(n)$) be those cubes that intersect the m-skeleton of $[0,1]^N$. Let also $Q_0 = [1/n, 1/n + \lambda]^N$. For

each $i = 0, \dots, l$, let ϕ_i be a similarity of $\mathbb{R}^N$ such that $\phi_i([0,1]^N) = Q_i$. Finally, define $E \subset \mathbb{R}^N$,

$$E := \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k \in \{0,\dots,l\}^k} \phi_{i_1} \circ \cdots \circ \phi_{i_k}([0,1]^N).$$

Since the maps $\{\phi_0, \dots, \phi_l\}$ satisfy the open set condition, E is Ahlfors regular [31]. By (9.4) the Hausdorff dimension of E is equal to s, so E is s-regular.

Lemma 9.2. *The set E is connected.*

Proof. Set $\mathcal{W} = \bigcup_{k \geq 0} \{0, \dots, l\}^k$ with the convention that $\{0, \dots, l\}^0$ is the empty word $\emptyset$ and $\phi_\emptyset$ is the identity map of $\mathbb{R}^N$. For each $w \in \mathcal{W}$, let $\mathcal{K}_w$ denote the 1-skeleton of $\phi_w([0,1]^N)$. The proof is based now on two observations. First, by the choice of cubes $Q_1, \dots, Q_l$, it follows that $\mathcal{K}_w \subset E$ for all $w \in \mathcal{W}$. Second, $\mathcal{K}_w \cap \mathcal{K}_{wi} \neq \emptyset$ for all $w \in \mathcal{W}$ and $i \in \{0, \dots, l\}$.

Now fix $x \in E$. There exists a sequence of words $(w_n)_{n \geq 0}$ in $\mathcal{W}$ such that w_0 is the empty word, $w_{n+1} = w_n i_n$ with $i_n \in \{0, \dots, l\}$, and $x \in \bigcap_{n \geq 0} \phi_{w_n}(x)$. The set $\bigcup_{n \geq 0} \mathcal{K}_{w_n}$ is a path that joins x with the origin. Hence E is connected. □

By Lemma 9.2, the fact that the Hausdorff dimension of E is s, and Theorem 4.12 in [45], there exists a $(1/s)$-Hölder map $f : [0,1] \to \mathbb{R}^N$ such that $f([0,1]) = E$. It remains to show (9.1). We first prove a lemma.

Lemma 9.3. *If $Q \in \Delta([0,1]^N)$ is a dyadic cube that intersects E, then there exists a dyadic cube $Q' \subset 3Q$ such that $\operatorname{diam} Q' \geq (3n)^{-1} \operatorname{diam} Q$ and $\beta_E^{(N-1)}(3Q') \geq (6\sqrt{N})^{-1}$.*

Proof. Fix $x \in Q \cap E$ and let $i_1, i_2, \dots$ be a sequence of numbers in $\{0, \dots, l\}$ such that

$$x \in \bigcap_{k=1}^{\infty} \phi_{i_1} \circ \cdots \circ \phi_{i_k}([0,1]^N).$$

Let k_0 be the smallest positive integer such that $\phi_{i_1} \circ \cdots \circ \phi_{i_{k_0}}([0,1]^N) \subset 3Q$ and define K to be the set of vertices of $\phi_{i_1} \circ \cdots \circ \phi_{i_{k_0}}([0,1]^N)$. Since each ϕ_i has a scaling factor at least $1/n$, by minimality of k_0 we have that $\operatorname{diam} K \geq (1/n) \operatorname{diam} Q$.

Let Q' be a dyadic cube in $\Delta(3Q)$ (possibly $Q' = Q$) of minimal diameter such that $K \subset 3Q'$. We claim that

$$\frac{1}{3} \operatorname{diam} K \leq \operatorname{diam} Q' \leq \operatorname{diam} K. \tag{9.5}$$

The lower inequality is clear. If $\operatorname{diam} K < \operatorname{diam} Q$, then, since K has edges parallel to the axes, K is contained in $3Q_0$ for some dyadic cube $Q_0 \subset 3Q$ with $\operatorname{diam} Q_0 = \frac{1}{2} \operatorname{diam} Q'$, which is a contradiction. That establishes the upper inequality of (9.5).

By (9.2), (9.5), and the fact that $K \subset E$,

$$\beta_E^{(N-1)}(3Q') \geq \beta_K^{(N-1)}(3Q') \geq \frac{(2\sqrt{N})^{-1} \operatorname{diam} K}{\operatorname{diam} 3Q} = \frac{(6\sqrt{N})^{-1} \operatorname{diam} K}{\operatorname{diam} Q} \geq (6\sqrt{N})^{-1}.$$

This proves the lemma. □

By Ahlfors s-regularity of E, there exists a constant $C > 1$ such that

$$\operatorname{card}\{Q \in \Delta([0,1]^N) : \operatorname{diam} Q = \sqrt{N}2^{-k} \text{ and } Q \cap E \neq \emptyset\} \geq C^{-1}2^{sk}.$$

Fix a positive integer k_0 such that $2^{k_0} > 3n$. For $k \in \mathbb{N}$, set

$$\mathcal{Q}_k = \{Q \in \Delta([0,1]^N) : \operatorname{diam} Q \in [\sqrt{N}2^{-k}, \sqrt{N}2^{-k-k_0}] \text{ and } \beta_E^{(N-1)}(3Q) \geq (6\sqrt{N})^{-1}\}.$$

By Lemma 9.3,

$$\operatorname{card} \mathcal{Q}_k \geq 3^{-N} \operatorname{card}\{Q \in \Delta([0,1]^N) : \operatorname{diam} Q = \sqrt{N}2^{-k} \text{ and } Q \cap E \neq \emptyset\} \geq C^{-1}3^{-N}2^{sk}.$$

Therefore,

$$\begin{aligned}\sum_{\substack{Q \in \Delta(\mathbb{R}^N) \\ \beta_E^{(N-1)}(3Q) \geq (6\sqrt{N})^{-1}}} (\operatorname{diam} Q)^N &\geq \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{Q}_{kk_0}} (\operatorname{diam} Q)^N \\ &\geq \sum_{k=0}^{\infty} C^{-1}3^{-N}2^{skk_0}(\sqrt{N})^s 2^{-s(k+1)k_0} = \infty.\end{aligned}$$

Case 3: $s \in \{2, \ldots, N-1\}$. Fix $n \in \mathbb{N}$ large enough so that $P_{N,s-1}(n) < n^s$. Partition $[0,1]^N$ into N-cubes with disjoint interiors and side-lengths $1/n$ and let $\{Q_1, \ldots, Q_l\}$ $(l = n^s)$ be a collection of such cubes so that the set $\bigcup_{k=1}^{l} Q_i$ is connected and contains the $(s-1)$-skeleton of $[0,1]^N$. The rest of the construction is similar to Case 2 and is left to the reader.

9.2. Ahlfors regular curves without Hölder parametrizations

Next, for all $s > 1$, we construct Ahlfors s-regular curves that are not contained in any $(1/s)$-Hölder curve. The basic strategy is take a disconnected set, which is not contained in a Hölder curve, and then extend the set to transform it into an s-regular curve. We call the curves that we construct "Cantor ladders".

Proposition 9.4. *Let $N \in \mathbb{N}$ with $N \geq 2$, let $s \in (1, N)$, and let $m \in \mathbb{N}$ with $m \leq s$. There exists an Ahlfors s-regular curve $E \subset \mathbb{R}^N$, which is not contained in a (m/s)-Hölder image of $[0,1]^m$.*

We treat the cases $s \in \mathbb{N}$ and $s \notin \mathbb{N}$ separately. Given $m \in \mathbb{N}$, let $\mathcal{W}_m$ be the set of finite words formed by the letters $\{1, \dots, m\}$ including the empty word $\emptyset$. We denote by $|w|$ the number of letters a word has with the convention $|\emptyset| = 0$.

Case 1. Suppose that $s \in \{2, 3, \dots, N-1\}$. Let $D_\emptyset = [0,1]^2$. Given a square $D_w \subset \mathbb{R}^2$ for some $w \in \mathcal{W}_4$, let D_{w1}, D_{w2}, D_{w3}, D_{w4} be the four corner squares in D_w with $\operatorname{diam} D_{wi} = (1/4) \operatorname{diam} D_w$. Let $\mathcal{C}_1$ be the Cantor set in $\mathbb{R}^2$ defined by

$$\mathcal{C}_1 = \bigcap_{k=0}^{\infty} \bigcup_{\substack{w \in \mathcal{W}_4 \\ |w|=k}} D_w.$$

For each $i = 1, \dots, 2^{|w|}$, define $D_{w,i} = D_w \times \{(2i-1)2^{-|w|-1}\}$,

$$K_2 = (\mathcal{C}_1 \times [0,1]) \cup \bigcup_{w \in \mathcal{W}} \bigcup_{i=0}^{2^{|w|}-1} D_{w,i} \qquad \text{and} \qquad E = K_2 \times [0,1]^{s-2} \times \{0\}^{N-s-1}.$$

Here and for the rest of §9.2, we use the convention $A \times \{0\}^0 = A$.

Case 2. Suppose that $s \in (1, N) \setminus \mathbb{N}$. Let $p = s - \lfloor s \rfloor$ be the fractional part of s. Let $I_\emptyset = [0,1]$. Given an interval $I_w = [a_w, b_w]$ for some $w \in \mathcal{W}_2$, let

$$I_{w1} = [a_w, a_w + 2^{-p}(b_w - a_w)] \qquad \text{and} \qquad I_{w2} = [b_w - 2^{-p}(b_w - a_w), b_w].$$

Let $\mathcal{C}_p$ denote the Cantor set in $\mathbb{R}$ defined by

$$\mathcal{C}_p = \bigcap_{k=0}^{\infty} \bigcup_{\substack{w \in \mathcal{W}_2 \\ |w|=k}} I_w.$$

Let S be the bi-Lipschitz embedded image of $([0,1], |\cdot|^{\frac{1}{p+1}})$ into $\mathbb{R}^2$. For each $w \in \mathcal{W}_2$, let S_w be a rescaled copy of S whose endpoints are the right endpoint of I_{w1} and the left endpoint of I_{w2}. For each $w \in \mathcal{W}_2$ and $i = 1, \dots, 2^{|w|} - 1$, define

$$S_{w,i} = S_w + (0, (2i-1)2^{-|w|-1})$$

and define

$$K_{p+1} = (\mathcal{C}_p \times [0,1]) \cup \bigcup_{w \in \mathcal{W}} \bigcup_{i=0}^{2^{|w|}} S_{w,i} \qquad \text{and} \qquad E = K_{p+1} \times [0,1]^{s-p-1} \times \{0\}^{N+p-s-1}.$$

Verification of the desired properties of E is the same for the two cases, so we only treat Case 1. By Theorem 2.1 in [38], there exists no (m/s)-Hölder map $f : [0,1]^m \to \mathbb{R}^N$ whose image contains $\mathcal{C}_1 \times [0,1]^{s-1} \times \{0\}^{N-s-1}$. We show that E is a curve in §9.2.1 and we prove s-regularity of E in §9.2.2.

9.2.1. E is a curve

By the Hahn-Mazurkiewicz theorem [30, Theorem 3.30], to show that E is a curve it is enough to show that E is compact, connected, and locally connected.

For compactness, it is easy to see that $K_2 \subset [0,1]^3$, hence $E \subset [0,1]^N$. Moreover, as $|w| \to \infty$, the squares $D_{w,i}$ accumulate on $\mathcal{C}\times[0,1]$. Therefore, K_2 is closed. Consequently, E is compact.

To settle both connectedness and local connectedness, we prove that there exists $C > 1$ such that for all pairs of points $x, y \in E$ there exists a path joining x with y of diameter at most $C|x-y|^{1/2}$. Clearly, it suffices to show the claim for K_2 instead of E. Fix $x, y \in K_2$ and let w_0 be the word in $\mathcal{W}_4$ of maximum word-length such that the projections of x and y on $\mathbb{R}^2 \times \{0\}$ are contained in D_{w_0}. This means that $|x-y| \geq \frac{1}{2}4^{-|w_0|}$. Choose $i_0 \in \mathbb{N}$ such that $\operatorname{dist}(x, D_{w_0,i_0}) \leq 2 \cdot 2^{-|w_0|}$. If x_0 and y_0 are the projections of x and y onto D_{w_0,i_0}, respectively, then

$$\max\{|x_0 - x|, |y_0 - y|\} \lesssim 2^{-|w_0|} + 4^{-|w_0|} + |x-y| \simeq 2^{-|w_0|}.$$

There exist sequences $(w_n)_{n\in\mathbb{N}}, (u_n)_{n\in\mathbb{N}}$ of words in $\mathcal{W}_4$ and sequences $(i_n)_{n\in\mathbb{N}}, (j_n)_{n\in\mathbb{N}}$ of positive integers such that

(1) $|w_n| = |u_n| = |w_0| + n$;
(2) the orthogonal projection of x (resp. y) on $\mathbb{R}^2$ is contained in D_{w_n} (resp. D_{u_n});
(3) there exists $x_n \in D_{w_n,i_n}$ such that

$$\max\{|x - x_n|, |y - y_n|\} \leq 4^{-|w_0|-n}\sqrt{2} + 2^{-|w_0|-n}.$$

Properties (1) and (2) imply that $D_{w_0} \supsetneq D_{w_1} \supsetneq D_{w_2} \supsetneq \cdots$ and $D_{w_0} \supsetneq D_{u_1} \supsetneq D_{u_2} \supsetneq \cdots$, while property (3) implies that the Hausdorff distances

$$\operatorname{dist}_H(D_{w_n,i_n}, D_{w_{n+1},i_{n+1}}) \lesssim 2^{-|w_0|-n} \quad \text{and} \quad \operatorname{dist}_H(D_{u_n,j_n}, D_{u_{n+1},j_{n+1}}) \lesssim 2^{-|w_0|-n}.$$

Let $\gamma_0 \subset K_2$ be the line segment joining x_0 with y_0. For each $n \geq 0$, let $z_n \in D_{w_n,i_n}$ be a corner point and let z_n' be its projection on $D_{w_{n+1},i_{n+1}}$. Also, let $p_n \in D_{u_n,j_n}$ be a corner point and let p_n' be its projection on $D_{u_{n+1},j_{n+1}}$. Consider the curve

$$\gamma = \gamma_0 \cup \bigcup_{n\in\mathbb{N}} ([x_n, z_n] \cup [z_n, z_n'] \cup [z_n', x_{n+1}]) \cup \bigcup_{n\in\mathbb{N}} ([y_n, p_n] \cup [p_n, p_n'] \cup [p_n', y_{n+1}]),$$

which is a subset of K_2 and joins x with y. Then

$$\begin{aligned}\operatorname{diam}\gamma &\lesssim \operatorname{diam}\gamma_0 + \sum_{n\geq 0}\operatorname{diam}\gamma_n + \sum_{n\geq 0}\operatorname{diam}\sigma_n \\ &\leq |x_0 - y_0| + \sum_{n\geq 0}(|x_n - z_n| + |z_n - z_n'| + |z_n' - x_{n+1}|)\end{aligned}$$

$$+\sum_{n\geq 0}(|y_n-p_n|+|p_n-p_n'|+|p_n'-y_{n+1}|)$$
$$\lesssim 4^{-|w_0|}+\sum_{n\geq 0}(2^{-|w_0|-n}+4^{-|w_0|-n}+2^{-|w_0|-n})\lesssim 2^{-|w_0|}\simeq |x-y|^{1/2}.$$

9.2.2. E is s-regular

We show s-regularity for E. Because the product of regular compact spaces of dimension s_1 and s_2 is (s_1+s_2)-regular, to show that E is s-regular, it suffices to show that K_2 is 2-regular. Fix $x\in K_2$ and $r\in(0,\operatorname{diam} K_2)$.

We first show that

$$\mathcal{H}^2(B(x,r)\cap K_2)\gtrsim r^2. \tag{9.6}$$

If $x\in\mathcal{C}_1\times[0,1]$, then (9.6) follows from the 2-regularity of $\mathcal{C}_1\times[-1,1]$. If $x\in D_{w,i}$ and $r\leq 10\operatorname{diam} D_w$, then (9.6) follows from the 2-regularity of $D_{w,i}$. If $x\in D_{w,i}$ and $r\geq 10\operatorname{diam} D_w$, then there exists $z\in(\mathcal{C}_1\times[0,1])\cap B(x,r)$ such that $B(z,r/2)\subset B(x,r)$ and (9.6) follows from the 2-regularity of $B(z,r/2)\cap K_2$.

For the upper regularity of K_2, instead of working with balls $B(x,r)$, it is more convenient to use cubes

$$Q(x,r)=x+[-r/2,r/2]^3\qquad x\in K_2,\quad r>0.$$

Without loss of generality, we may assume that $r=4^{-k_0}$ for some $k_0\in\mathbb{N}$. For each $k\geq 0$, let

$$\mathcal{D}_k(x,r)=\{D_{w,i}:Q(x,r)\cap D_{w,i}\neq\emptyset \text{ and } |w|=k\}.$$

Then by the 2-regularity of $\mathcal{C}_1\times[-1,1]$, it suffices to show that

$$\sum_{k\geq 0}\sum_{D_{w,i}\in\mathcal{D}_k(x,r)}\mathcal{H}^2(Q(x,r)\cap D_{w,i})\lesssim r^2.$$

The following lemma will let us estimate the above sum. In the sequel, we denote by $m_0\geq 0$ the smallest integer for which there exists $D_{w,i}\in\mathcal{D}(x,r)$ with $|w|=m_0$.

Lemma 9.5. *Let $m_0\geq 0$ be the smallest integer for which $\mathcal{D}_{m_0}(x,r)\neq\emptyset$.*

(1) *If $k>m_0$ and $\mathcal{D}_k(x,r)\neq\emptyset$, then $k\geq 2k_0$.*
(2) *If Q' is the projection of $Q(x,r)$ on $\mathbb{R}^2\times\{0\}$, then for all $k\geq 0$,*

$$\operatorname{card}\{D_w:D_w\cap Q'\neq\emptyset \text{ and } |w|=k\}\leq 1+4^k4^{-k_0}.$$

(3) *For each* $w \in \mathcal{W}_4$,

$$\operatorname{card}\{i : D_{w,i} \cap Q(x,r) \neq \emptyset\} \leq 1 + 2^{|w|+1} 4^{-k_0}.$$

(4) *For each* $k \geq 0$, $\operatorname{card} \mathcal{D}_k(x,r) \leq (1 + 4^k 4^{-k_0})(1 + 2^{k+1} 4^{-k_0})$.
(5) *We have*

$$\sum_{D_{w,i} \in \mathcal{D}_{m_0}(x,r)} \mathcal{H}^2(D_{w,i} \cap Q(x,r)) \lesssim r^2.$$

Proof. For (1), recall that if $|w| > m_0$, then the vertical distance between $D_{w,i}$ and D_{w_0,i_0} is at least $2^{-|w|}$. Since $r = 4^{-k_0}$, the cube $Q(x,r)$ can not intersect any $D_{w,i}$, unless $4^{-k_0} \geq 2^{-|w|}$. Thus, $|w| \geq 2k_0$.

For (2), we first note that if $k \leq k_0$, then Q' can intersect at most one square D_w with $|w| = k$. We now use induction to show that for all $k \geq k_0$,

$$\operatorname{card}\{D_w : D_w \cap Q' \neq \emptyset \text{ and } |w| = k\} \leq 4^k 4^{-k_0}.$$

For $k = k_0$, it is true. Suppose that the claim is also true for some $k \geq k_0$. Then Q' intersects D_w with $|w| = k+1$ if and only if there exists w' with $|w'| = k$ such that $Q \cap D_{w'} \neq \emptyset$ and $D_w \subset D_{w'}$. Since each square of generation k contains 4 squares of generation $k+1$,

$$\begin{aligned}\operatorname{card}\{D_w : D_w \cap Q' \neq \emptyset \text{ and } |w| = k+1\} &\leq 4 \operatorname{card}\{D_w : D_w \cap Q' \neq \emptyset \text{ and } |w| = k\} \\ &\leq 4^{k+1} 4^{-k_0}.\end{aligned}$$

For (3), fix $w \in \mathcal{W}_4$. Recall that the vertical height of $Q(x,r)$ is $2r = 2 \cdot 4^{-k_0}$ and that the vertical distance between $D_{w,i}$ and $D_{w,j}$ with $i \neq j$ is at least $2^{-|w|}$. Therefore,

$$\operatorname{card}\{i : D_{w,i} \cap Q(x,r) \neq \emptyset\} \geq 1 + (2r)/2^{-|w|} = 1 + 2^{|w|+1} 4^{-k_0}.$$

Claim (4) is immediate from (2) and (3).

It remains to show (5). On one hand, if $m_0 > k_0$, then by (4), $\operatorname{card}(\mathcal{D}_{m_0}(x,r)) = 1$. Hence (5) follows from the 2-regularity of squares $D_{w,i}$. On the other hand, if $m_0 \leq k_0$, then by (4),

$$\sum_{D_{w,i} \in \mathcal{D}_{m_0}(x,r)} \mathcal{H}^2(D_{w,i} \cap Q(x,r)) \leq \operatorname{card}(\mathcal{D}_{m_0}(x,r))(4^{-m_0})^2 \lesssim 2^{-m_0} 4^{-2k_0} \leq r^2. \quad \square$$

By Lemma 9.5, we have

$$\sum_{D_{w,i}\in\mathcal{D}(x,r)} \mathcal{H}^2(B(x,r)\cap D_{w,i}) \leq \sum_{D_{w,i}\in\mathcal{D}_{m_0}(x,r)} \mathcal{H}^2(D_{w,i}) + \sum_{k=2k_0}^{\infty} \sum_{D_{w,i}\in\mathcal{D}_k(x,r)} \mathcal{H}^2(D_{w,i})$$
$$\lesssim r^2 + \sum_{k=2k_0}^{\infty} 2^k 4^{-k_0} 4^k 4^{-k_0} 4^{-2k}.$$

Finally,

$$\sum_{k\geq 2k_0}^{\infty} 2^k 4^{-k_0} 4^k 4^{-k_0} 4^{-2k} = 4^{-2k_0} \sum_{k\geq k_0} 2^{-k} \lesssim 4^{-3k_0} \lesssim r^2.$$

Therefore, K_2 is 2-regular.

9.3. A compact countable set that is not contained in any Hölder cube

Proposition 9.6. *For each $N \in \mathbb{N}$, $N \geq 2$, there exists a compact and countable set $E \subset \mathbb{R}^N$ with one accumulation point such that for any $m \in \{1, \ldots, N-1\}$ and any $s \in [1, N/m)$, the set E is not contained in a $(1/s)$-Hölder image of $[0,1]^m$.*

Corollary 9.7. *For each $N \in \mathbb{N}$, $N \geq 2$, there exists a compact and countable set $E \subset \mathbb{R}^N$ with one accumulation point such that E is not contained in a rectifiable curve.*

For each integer $k \geq 0$, define $\mathcal{G}_k^0$ to be the union of all vertices of all dyadic cubes in $\mathbb{R}^N$ that are contained in $[0,1]^N$ and have side length 2^{-k}. By a simple combinatorial argument, $\operatorname{card}(\mathcal{G}_0^k) = (2^k+1)^N$ for all $k \geq 0$.

Let ϕ_0 be the identity map, and for each $k \geq 1$, define a map $\phi_k : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\phi_k(x) = (k+1)^{-2} x + \left(0, \ldots, 0, 2\sum_{i=1}^{k} i^{-2}\right).$$

Set $A := \sum_{i=1}^{\infty} i^{-2} = \pi^2/6$, and define the set

$$E := \{(0, \ldots, 0, 2A)\} \cup \bigcup_{k=0}^{\infty} \phi_k(\mathcal{G}_k^0).$$

The set E is clearly countable. If $(x_1, \ldots, x_N) \in E$, then $|x_i| \leq 1$ for all $i = 1, \ldots, N-1$ while $|x_N| \leq 2A$. Therefore, E is bounded. Moreover, the only accumulation point of E is the point $(0, \ldots, 0, 2A)$ which is contained in E. Thus, E is closed.

Next, we claim that

$$|x - y| \geq 2^{-k}(k+1)^{-2} \qquad \text{for all } x \in \phi_k(\mathcal{G}_k^0) \text{ and all } y \in E \setminus \{x\}. \tag{9.7}$$

Indeed, if $x, y \in \mathcal{G}_k^0$, then inequality (9.7) is clear. Otherwise, $\operatorname{dist}(\mathcal{G}_k^0, E \setminus \mathcal{G}_k^0) \geq (k+1)^{-2}$, and thus, (9.7) holds again.

Suppose in order to get a contradiction that there exists a $(1/s)$-Hölder continuous map $f : [0,1]^m \to \mathbb{R}^N$ such that $E \subset f([0,1]^m)$. Let H be the Hölder constant of f. For each $k \geq 0$ and $x \in \mathcal{G}_k^0$, fix a point $w_{k,x}$ such that $f(w_{k,x}) = x$ and set

$$B_{k,x} = B(w_{k,x}, \tfrac{1}{2} H^{-s} 2^{-ks} (k+1)^{-2s}).$$

Inequality (9.7) implies that the balls $B_{k,x}$ are mutually disjoint. Moreover, it is easy to see that each $B_{k,x}$ is contained in $[-1,2]^m$. Therefore,

$$\begin{aligned} 1 \gtrsim_m \mathcal{H}^m([-1,2]^m) &\geq \sum_{k=0}^{\infty} \sum_{x \in \mathcal{G}_k^0} \mathcal{H}^m(B_{k,x}) \\ &\gtrsim_{H,s} \sum_{k=0}^{\infty} (2^k+1)^N \frac{2^{-skm}}{(k+1)^{2sm}} \simeq_N \sum_{k=0}^{\infty} \frac{2^{k(N-ms)}}{(k+1)^{2s}}. \end{aligned}$$

Since $N > ms$, the sum on the right hand side diverges and we reach a contradiction.

9.4. Flat curves with finite $\mathcal{H}^s$ measure and no $(1/s)$-Hölder parametrizations

The following example shows that the assumption of lower s-regularity can not be dropped from Proposition 1.3.

Proposition 9.8. *For any $\beta_0 \in (0,1)$, there exists $s_0 \in (1,2)$ with the following property. For any $s \in (1, s_0)$ there exists a curve $E \subset \mathbb{R}^2$ such that*

(1) $\mathcal{H}^s(E) < \infty$ *and*
(2) $\beta_E(Q) < \beta_0$ *for all* $Q \in \Delta(\mathbb{R}^N)$,

but E is not contained in any $(1/s)$-Hölder image of $[0,1]$.

Before proceeding, we recall a well-known construction method for snowflakes in $\mathbb{R}^2$. Let $\mathbf{p} = (p_0, p_1, \dots)$ be sequence of numbers in $[1/4, 1/2)$. Let Γ_0 be the segment $[0,1] \times \{0\}$, oriented from $(0,0)$ to $(1,0)$. Assume that we have constructed an oriented polygonal arc Γ_k with 4^k edges. Define Γ_{k+1} to be the polygonal arc constructed by replacing each edge e of Γ_k by a rescaled and rotated copy of the oriented polygonal arc in Fig. 4 with $p = p_k$, so that the new oriented arc lies to the left of e. A snowflake arc $\mathcal{S}_{\mathbf{p}}$ is obtained by taking the limit of Γ_k, just as in the construction of the usual von Koch snowflake.

Remark 9.9. For any $\epsilon > 0$, there exists $p^* > 1/4$ (small) such that if a snowflake is built with parameters $1/4 \leq p_k \leq p^*$ for all $k \geq 0$, then $\beta_{\Gamma_k}(B(x,r)) \leq \epsilon r$ for all $k \geq 0$, $x \in \Gamma_k$, and $r > 0$.

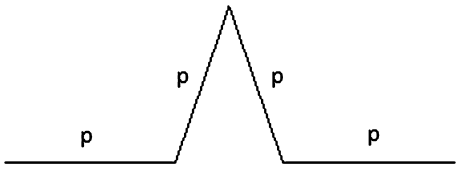

Fig. 4. Generator for a snowflake curve.

Fix $\beta_0 \in (0,1)$. By the preceding remark, there exists $p^* \in (1/4, 1/2)$ such that $\beta_{\mathcal{S}_{p_0}}(Q) < \beta_0$ for all $Q \in \Delta(\mathbb{R}^N)$. Set $\mathbf{p} = (p^*, p^*, \dots)$, set $s_0 = -\log 4/\log p_0$, and fix $s \in [1, s_0)$. It is well-known that there exists a $(1/s_0)$-bi-Hölder homeomorphism $\Phi : [0,1] \to \mathcal{S}_{\mathbf{p}}$; e.g., see [18,46].

We now construct a self-similar Cantor set in $[0,1]$ in the following way. Let $I_\emptyset = [0,1]$. Assuming we have constructed $I_w = [a_w, b_w]$ for some $w \in \{1,2\}^n$, let

$$I_{w1} = [a_w, a_w + (b_w - a_w)2^{-s_0/s}] \quad \text{and} \quad I_{w2} = [b_w - (b_w - a_w)2^{-s_0/s}, b_w].$$

Define $E' = \bigcap_{n=0}^{\infty} \bigcup_{w \in \{1,2\}^n} I_w$. For each component J of $[0,1] \setminus E'$, let γ_J be the line segment joining the endpoints of $\Phi(J)$. Then define

$$E = \Phi(E') \cup \bigcup_J \gamma_J,$$

where the union is taken over all components J of $[0,1] \setminus E'$. Since E' is s/s_0-regular and Φ is $(1/s_0)$-bi-Hölder,

$$\mathcal{H}^s(E) = \mathcal{H}^s(\Phi(E')) + \sum_J \mathcal{H}^s(\gamma_J) \leq C\mathcal{H}^{s/s_0}(E') < \infty.$$

Since $\Phi(E') \subset \mathcal{S}_{\mathbf{p}}$ and γ_J are line segments, we have $\beta_E(Q) < \beta_0$ for all $Q \in \Delta(\mathbb{R}^N)$. Finally, by Theorem 2.1 in [38], there does not exist a $(1/s)$-Hölder map $f : [0,1] \to \mathbb{R}^2$ whose image contains $\Phi(E')$ (and consequently E).

9.5. Sharpness of exponent 1 in Theorem 5.1

To wrap up, we show that Theorem 5.1 does not hold if numbers $\tau_s(k, v, v')$ are replaced by $\tau_s(k, v, v')^p$ with $p > 1$. When $s = 1$, this follows from the necessary half of the Analyst's Traveling Salesman theorem. Thus, we may focus on the case $s > 1$.

Proposition 9.10. *Let $p > 1$, let $s > 1$ be sufficiently close to 1, and let $\alpha_0 > 0$ be sufficiently close to 0. There exists a sequence of finite sets $\{(V_k, \rho_k)\}_{\geq 0}$ of numbers and finite sets in $\mathbb{R}^2$ satisfying (V0)–(V5) such that*

$$\sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha_0}} \rho_k^s r_0^s + \sum_{(v,v') \in \mathsf{Flat}(k)} \tau_s(k, v, v')^p \rho_k^s r_0^s < \infty \tag{9.8}$$

but there does not exist a $(1/s)$*-Hölder map* $f : [0,1] \to \mathbb{R}^2$ *such that* $\bigcup_{k\geq 0} V_k \subset f([0,1])$.

Let $s > 1$ and $n_0 \in \mathbb{N}$ be constants to be specified below. Fix a number

$$0 < q < \min\{1/s, (p-1)/s\}.$$

For each $n \in \mathbb{N}$, let

$$t_k = \sqrt{\frac{1}{4^{1/s}}\left(1 + \frac{1}{k+n_0}\right)^{2q} - \frac{1}{4}}.$$

Construct a sequence of polygonal arcs Γ_k as in §9.4 with parameters

$$p_k = 1/4 + t_k^2.$$

We may assume that numbers p_k are in $[1/4, 1/2)$ by taking n_0 to be sufficiently large. For each $k \geq 0$, we define a finite set $V_k \subset \Gamma_k$ as follows. Define $V_0 := \{v_{0,1}, v_{0,2}\}$, where $v_{0,1} = (0,0)$ and $v_{0,2} = (1,0)$. Suppose that for some $k \geq 0$ we have defined a set

$$V_k = \{v_{k,1}, \dots, v_{k,N_k}\}, \qquad N_k = 2^k + 1,$$

where points $v_{k,i}$ are enumerated according to the orientation of Γ_k. For each $i = 1, \dots, 2^k + 1$, set $v_{k+1,2i-1} = v_{k,i}$, and assign $v_{k+1,2i}$ to the point of Γ_{k+1} that lies between $v_{k+1,2i-1}$ and $v_{k+1,2i+1}$ and is equal distance to $v_{k+1,2i-1}$ and $v_{k+1,2i+1}$ (the peak of the triangle in Fig. 4). Define the quantities

$$r_0 = 1, \quad C^* = 2, \quad \xi_1 = 2^{-1/s}, \quad \xi_2 = \frac{1 + 2^{-1/s}}{2}, \quad \rho_0 = 1, \quad \rho_k = 2^{-k/s}\frac{(k+1+n_0)^q}{(2+n_0)^q}. \tag{9.9}$$

For each $k \geq 0$ and $v \in V_k$, define

$$\alpha_{k,v} := \inf_{\ell} \sup_{x \in V_{k+1} \cap B(v, 30A^*\rho_k r_0)} \frac{\operatorname{dist}(x,\ell)}{\rho_{k+1} r_0},$$

where the infimum is taken over all lines ℓ in $\mathbb{R}^2$ and A^* is as in §2.2. Let $\ell_{k,v}$ be a line ℓ, which realizes the number $\alpha_{k,v}$.

Lemma 9.11. *There exist choices of* s *and* n_0 *so that the following properties hold.*

(1) *For all* $k \geq 0$ *and* $i \in \{1, \dots N_k\}$, *we have* $|v_{k,i} - v_{k,i+1}| = \rho_k$.
(2) *The sequence* $\{(V_k, \rho_k)\}_{k\geq 0}$ *satisfies (V0)–(V5) with the parameters given in (9.9).*
(3) *For all* $k \geq 0$ *and* $v \in V_k$, *we have* $\alpha_{k,v} \leq \alpha_0$, *where* α_0 *is as in Definition 2.4.*

(4) *For all* $k \geq 0$ *and* $i \in \{1, \dots, N_k\}$,

$$\mathsf{Flat}(k) = \{(v_{k,i}, v_{k,i+1}) : i = 1, \dots, 2^k\} \quad \textit{and}$$
$$V_{k+1,i}(v_{k,i}, v_{k,i+1}) = \{v_{k,i}, v_{k+1,2i}, v_{k,i}\}.$$

Proof. For (1), we work by induction. The claim is true for $k = 0$ by the choice of points $v_{0,1}$ and $v_{0,2}$. Assume the claim is true for some $k \geq 0$. By the Pythagorean theorem,

$$\begin{aligned} |v_{k+1,2i-1} - v_{k+1,2i}| = |v_{k,i} - v_{k+1,2i-1}| &= (4^{-1} + t_1^2)^{1/2} |v_{k,i} - v_{k,i+1}| \\ &= (4^{-1} + t_1^2)^{1/2} \rho_k = \rho_{k+1}. \end{aligned}$$

In similar fashion, one can compute $|v_{k+1,2i} - v_{k+1,2i+1}|$ and the proof of (1) is complete.

Claim (3) is immediate from Remark 9.9 by taking s sufficiently close to 1 and n_0 sufficiently large.

For (V0), we have

$$\frac{\rho_{k+1}}{\rho_k} = 2^{-1/s} \left(\frac{k + 2 + n_0}{k + 1 + n_0} \right)^q.$$

Clearly, $\rho_{k+1} > \xi_1 \rho_k$. On the other hand, since $2^{-1/s} < \xi_2 < 1$, if n_0 is sufficiently large, then $\rho_{k+1} \leq \xi_2 \rho_k$. Properties (V1), (V2), and (V5) are immediately satisfied by our construction. For (V4), fix a point $v_{k+1,2i} \in V_{k+1} \setminus V_k$. By (1), we have $|v_{k+1,2i} - v_{k+1,2i+1}| = \rho_{k+1}$ and (V4) is satisfied.

For (V3), claim (3), and claim (4), we apply induction on k. For $k = 0$ (V3) is immediate by the choice of parameters. For claim (3), we note that $\alpha_{0,v} = 0$ for all $v \in V_0$, since V_0 contains only 2 points. For the same reason, claim (4) is satisfied when $k = 0$.

To show (V3), we note by (3) that the closest point of V_{k+1} to $v_{k+1,2i}$ are the points $v_{k,i}$ and $v_{k,i+1}$. Therefore,

$$\min_{v \in V_{k+1} \setminus \{v_{k+1,2i}\}} |v - v_{k+1,2i}| = |v_{k+1,2i} - v_{k,i}| = \rho_{k+1}.$$

Similarly, by (3), the closest point of V_{k+1} to $v_{k+1,2i+1} = v_{k,i+1}$ are the points $v_{k+1,2i}$ and $v_{k+1,2(i+1)}$ (or only one point of these two if $i = 0$ or $i = 2^k$}) and the above inequality also applies.

Finally, to show (4), we apply (3) and the arguments in the proof of (V3). Namely, if $v_{k,i} \in V_k$ with $k \in \{2, 2^k - 1\}$, then $\alpha_{v_{k,i}} < \alpha_0$ and $v_{k,i}$ lies between points $v_{k,i-1}$ and $v_{k,i+1}$. Therefore,

$$\mathsf{Flat}(k) = \{(v_{k,i}, v_{k,i+1}) : i = 1, \dots, 2^k\}.$$

Furthermore, the only point of V_{k+1} lying between $v_{k,i}$ and $v_{k,i+1}$ is $v_{k+1,2i}$. Thus,

$$V_{k+1,i}(v_{k,i}, v_{k,i+1}) = \{v_{k,i}, v_{k+1,2i}, v_{k,i}\}. \quad \square$$

We now show that there does not exist a $(1/s)$-Hölder map $f : [0,1] \to \mathbb{R}^2$ whose image contains $\bigcup_{k\geq 0} V_k$. Contrary to the claim, assume that such a map f exists and let H be its Hölder constant. For each $v_{k,i} \in V_k$, fix $w_{k,i} \in [0,1]$ such that $f(w_{k,i}) = v_{k,i}$. Then

$$|w_{k,i} - w_{k,j}| \gtrsim_{H,s} |v_{k,i} - v_{k,j}|^s \gtrsim_{n_0,q,s} 2^{-k}(k+1)^{sq}.$$

Therefore,

$$1 \geq 2^k \min_{i=1,\dots,2^k+1} |w_{k,i} - w_{k,j}| \gtrsim_{H,s,n_0,q} (k+1)^{sq},$$

which diverges as $k \to \infty$ and we reach a contradiction.

It remains to check (9.8). By Lemma 9.11, it suffices to show that

$$\sum_{k=0}^{\infty} \sum_{i=1}^{2^k} \tau_s(k, v_{k,i}, v_{k,i+1})^p |v_{k,i} - v_{k,i+1}|^s < \infty.$$

By the Mean Value Theorem,

$$\begin{aligned}
\tau_s(k, v_{k,i}, v_{k,i+1}) &= \frac{|v_{k,i} - v_{k+1,2i}|^s + |v_{k+1,2i} - v_{k,i+1}|^s - |v_{k,i} - v_{k,i+1}|^s}{|v_{k,i} - v_{k,i+1}|^s} \\
&= 2\frac{(k+2+n_0)^{sq} - (k+1+n_0)^{sq}}{(k+2+n_0)^{sq}} \\
&\lesssim_{n_0,s,q} \frac{1}{k+1}.
\end{aligned}$$

Finally, since $sq - p < -1$,

$$\begin{aligned}
\sum_{k=0}^{\infty} \sum_{i=1}^{2^k} \tau_s(k, v_{k,i}, v_{k,i+1})^p |v_{k,i} - v_{k,i+1}|^s &\lesssim_{n_0,s,q} \sum_{k=0}^{\infty} \frac{2^k}{(k+1)^p} (2^{-k/s}(k+1)^q)^s \\
&= \sum_{k=0}^{\infty} (k+1)^{sq-p} < \infty.
\end{aligned}$$

Appendix A. Tours on connected, finite simple graphs

A *finite simple graph* $G = (V, E)$ in a Banach space X is a finite set of points $V \subset X$ (called *vertices* of G) along with a set $E \subseteq \{\{v, v'\} : \text{distinct } v, v' \in V\}$ (called *edges* of G). We may identify edges $\{v, v'\}$ in the graph with the (unoriented) line segments $[v, v']$ in X. A graph is *connected* if every pair of vertices in the graph can be joined by

a sequence of edges in the graph. The *valence* of a vertex v in G is the number of edges in G that contain v.

Proposition A.1. *Let G be a connected, finite simple graph in X. Assume that every vertex in G has valence at most 2. For any vertex v_0 in G and any nondegenerate compact interval Δ, there exists a collection $\mathcal{I}$ of open intervals, whose closures are mutually disjoint and contained in the interior of Δ, and there exists a continuous map $g : \Delta \to G$ with the following properties.*

(1) *The endpoints of Δ are mapped onto v_0.*
(2) *For every vertex v of G, there exists at least one component J of $\Delta \setminus \bigcup \mathcal{I}$ such that $g(J) = v$. Conversely, for every component J of $\Delta \setminus \bigcup \mathcal{I}$, there exists a vertex v of G such that $g(J) = v$.*
(3) *Each interval in $\mathcal{I}$ is mapped linearly onto some edge e of G. Conversely, for each edge e of G, there exist exactly two intervals $I \in \mathcal{I}$ such that $g(I) = e$.*
(4) *For any vertex v in G, there exists a component J of $g^{-1}(v) \cap \Delta \setminus \bigcup \mathcal{I}$ such that for any edge e containing v as an endpoint, there exists $I \in \mathcal{I}$ such that $\overline{I} \cap J \neq \emptyset$ and $g(I) = e$.*

Proof. If we only desired properties (1)–(3), then we could prove the proposition without any restriction on the valence of the vertices by simple induction on the number of edges. The restriction on the valence of the vertices ensures the graph has one of two simple forms that make it easy to describe maps g satisfying properties (1)–(4). Thus, let G be a connected, finite simple graph in X, and assume that every vertex in G has valence at most 2. The conclusion being trivial otherwise, we may assume that G contains at least two vertices. There are two possibilities. In each case, we will construct the family $\mathcal{I}$ and the map g, but leave verification of properties (1) through (4) to the reader.

Case 1: Suppose that every vertex of G has valence 2 (i.e. G is a "cycle"). Then we can find an enumeration $\{u^1, \dots, u^k\}$ of the vertices of G so that the edges of G are precisely $\{[u^i, u^{i+1}] : i = 1, \dots, k\}$, where we set $u^{k+1} = u^1$. Without loss of generality, assume that $u^1 = v_0$. Let $\mathcal{I} = \{I_1, \dots, I_{2k}\}$ be open intervals, enumerated according to the orientation of Δ, whose closures are mutually disjoint and contained in the interior of Δ. Then there exists a continuous, surjective map $g : \Delta \to G$ such that

(1) g is linear on each I_i and constant on each component of $\Delta \setminus \bigcup_{i=1}^{2k} I_i$;
(2) for each $i \in \{1, \dots, k\}$, g maps I_i linearly onto $[u^i, u^{i+1}]$ and maps the left endpoint of I_i onto u^i;
(3) for each $i \in \{k+1, \dots, 2k\}$, g maps I_i linearly onto $[u^{i-k}, u^{i-k+1}]$ and maps the left endpoint of I_i onto u^{i-k}.

That is, g winds twice around the graph, starting and ending at $v_0 = u^1$.

Case 2: Suppose that least one vertex of G has valence 1 (i.e. G is an "arc"). Then we can find an enumeration $\{u^1, \dots, u^k\}$ of the vertices of G so that the edges of G are precisely $\{[u^i, u^{i+1}] : i = 1, \dots, k-1\}$. In this case, u^1 and u^k have valence 1 and all other vertices have valence 2. Assume that $v_0 = u^l$ for some $1 \leq l \leq k$. Let $\mathcal{I} = \{I_1, \dots, I_{2(k-1)}\}$ be open intervals, enumerated according to the orientation of Δ, whose closures are mutually disjoint and contained in the interior of Δ. Then there exists a continuous, surjective map $g : \Delta \to G$ such that

(1) g is linear on each I_i and constant on each component of $\Delta \setminus \bigcup_{i=1}^{2(k-1)} I_i$;
(2) for each $1 \leq i \leq l-1$ (if any), g maps I_i linearly onto $[u^{l-i}, u^{l-i+1}]$ and maps the left endpoint of I_i onto u^{l-i+1};
(3) for each $i \in \{l, \dots, l+k-2\}$, g maps I_i linearly onto $[u^{i-l+1}, u^{i-l+2}]$ and maps the left endpoint of I_i onto u^{i-l+1};
(4) for each $l+k-1 \leq i \leq 2(k-1)$ (if any), g maps I_i linearly onto $[u^{2k-2+l-i}, u^{2k-1+l-i}]$ and maps the left endpoint of I_i onto $u^{2k-1+l-i}$.

That is, g walks along the graph from $v_0 = u^l$ towards u^1, walks from u^1 to u^k, and walks from u^k back to u^l. □

Appendix B. From Lipschitz to Hölder parameterizations

The following method of obtaining Hölder parameterizations from Lipschitz ones is well known, see e.g. [49, Lemma VII.2.8]. We include Lemma B.1 and its proof here to have a clear statement about the dependence of the Hölder constant of the map f.

Lemma B.1. *Let $s > 1$, $M > 0$, $0 < \xi_1 \leq \xi_2 < 1$, $\alpha > 0$, $\beta > 0$, and $j_0 \in \mathbb{Z}$. Let $(X, |\cdot|)$ be a Banach space. Suppose that ρ_j $(j \geq j_0)$ is a sequence of scales and $f_j : [0, M] \to X$ $(j \geq j_0)$ is a sequence of Lipschitz maps satisfying*

(1) *$\rho_{j_0} = 1$ and $\xi_1 \rho_j \leq \rho_{j+1} \leq \xi_2 \rho_j$ for all $j \geq j_0$,*
(2) *$|f_j(x) - f_j(y)| \leq A_j |x - y|$ for all $j \geq j_0$, where $A_j \leq \alpha \rho_j^{1-s}$, and*
(3) *$|f_j(x) - f_{j+1}(x)| \leq B_j$ for all $j \geq j_0$, where $B_j \leq \beta \rho_j$.*

Then f_j converges uniformly to a map $f : [0, M] \to X$ such that

$$|f(x) - f(y)| \leq H |x - y|^{1/s} \quad \text{for all } x, y \in [0, M],$$

where H is a finite constant depending only on $\max(M, 1/M)$, ξ_1, ξ_2, α, and β; see (B.7).

Proof. Define $f : [0, M] \to X$ pointwise by $f(x) = f_{j_0}(x) + \sum_{k=j_0}^{\infty} (f_{k+1}(x) - f_k(x))$. Then f exists and is the uniform limit of the maps f_j by (3), because $\sum_{k=j_0}^{\infty} B_k < \infty$. In fact, for all $j \geq j_0$ and $x \in [0, M]$,

$$|f(x) - f_j(x)| \leq \sum_{k=j}^{\infty} |f_{j+1}(x) - f_j(x)| \leq \sum_{k=j}^{\infty} \beta \rho_k \leq \frac{\beta}{1-\xi_2} \rho_j. \tag{B.1}$$

Suppose that $x, y \in [0, M]$ with $x \neq y$. Then there is a unique integer $j \geq j_0$ such that

$$M\rho_{j+1}^s < |x - y| \leq M\rho_j^s. \tag{B.2}$$

By the triangle inequality, (2), and (B.1),

$$\begin{aligned} |f(x) - f(y)| &\leq |f_j(x) - f_j(y)| + |f(x) - f_j(x)| + |f(y) - f_j(y)| \\ &\leq \alpha \rho_j^{1-s} |x - y| + \frac{2\beta}{1-\xi_2} \rho_j. \end{aligned} \tag{B.3}$$

By the first inequality in (B.2) and (1), we have

$$\rho_j < \frac{1}{M^{1/s} \xi_1} |x - y|^{1/s}. \tag{B.4}$$

Hence, by the second inequality in (B.2), we have

$$\rho_j^{1-s} |x - y| \leq M\rho_j < \frac{M}{M^{1/s} \xi_1} |x - y|^{1/s}. \tag{B.5}$$

Combining (B.3)–(B.5) yields $|f(x) - f(y)| \leq h|x - y|^{1/s}$ for all $x, y \in [0, M]$, where

$$h = \frac{1}{M^{1/s} \xi_1} \left(\alpha M + \frac{2\beta}{1-\xi_2} \right). \tag{B.6}$$

If $M \geq 1$, then $1/M^{1/s} \leq 1$, while if $M \leq 1$, then $1/M^{1/s} \leq 1/M$. Thus, it follows that $|f(x) - f(y)| \leq H|x - y|^{1/s}$ for all $x, y \in [0, M]$, where

$$H = \frac{\max(1, 1/M)}{\xi_1} \left(\alpha M + \frac{2\beta}{1-\xi_2} \right) = \frac{\alpha}{\xi_1} \max(1, M) + \frac{2\beta}{\xi_1 (1-\xi_2)} \max(1, 1/M) \tag{B.7}$$

depends only on $\max(M, 1/M)$, ξ_1, ξ_2, α, and β. □

Remark B.2. Lemma B.1 is often used with geometric scales $\rho_j = \rho^j$ (i.e. $\xi_1 = \xi_2 = \rho$). However, separating the parameters ξ_1 and ξ_2 provides additional flexibility that can make constructing examples easier; see e.g. §9.5.

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