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An Extension of the Kaliszewski Cone to Non-polyhedral Pointed Cones in Infinite-Dimensional Spaces

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Abstract
In this paper, we propose an extension of the family of constructible dilating cones given by Kaliszewski (Quantitative Pareto analysis by cone separation technique, Kluwer Academic Publishers, Boston, 1994) from polyhedral pointed cones in finite-dimensional spaces to a general family of closed, convex, and pointed cones in infinite-dimensional spaces, which in particular covers all separable Banach spaces. We provide an explicit construction of the new family of dilating cones, focusing on sequence spaces and spaces of integrable functions equipped with their natural ordering cones. Finally, using the new dilating cones, we develop a conical regularization scheme for linearly constrained least-squares optimization problems. We present a numerical example to illustrate the efficacy of the proposed framework.

Keywords Constrained convex optimization · Dilating cones · Infinite-dimensional analysis · Perturbation theory · Proper efficiency

Mathematics Subject Classification 90C20 · 90C31 · 90C46

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1 Introduction

Existence and computation of optimal points of optimization problems by means of separation theorems are vital to a wide variety of applied models. However, in general, the separation theorems demand that the ordering cone of the underlying space has a nonempty interior, a requirement that does not hold in many commonly occurring infinite-dimensional models, for example, in optimization problems constrained by partial differential equations (PDE) (see [1] and the references therein). However, most of these problems are formulated either in sequence spaces or Lebesgue function spaces where their ordering cones have empty interior (see for example [2–4]). Numerous optimization formulations of economic models are also studied in these spaces (see [5–7] and references therein).

The conical regularization, introduced in Khan and Sama [1], is a general framework to handle the lack of interior elements in the ordering cones or equivalently the lack of a Slater-type constraint qualification. This method constructs a family of regularized optimization problems by replacing the ordering cone by a family of dilating cones [4,8–11].

At the same time, we note that the dilating cones are also popular objects for computing proper efficient solutions of a vector optimization problem, and in this case, this is independent of solidness of the original cone. We recall that the first notion of proper efficiency was proposed by Kuhn and Tucker [12] who noted that a subset of the efficient set of a multiobjective optimization problem might be *improper*, in the sense that their elements cannot be computed in a satisfactory manner by scalarization. In a subsequent development, several notions of proper efficiency for multiobjective/vector optimization problems have been proposed, see [8,13–18]).

There are different ways of defining dilating cones, for example, by dilatations of the base of the ordering cone, see [16,19]. In this context, Kaliszewski [20] proposed a family of dilating cones by polyhedral cones. This construction is especially interesting because the dilating cones can be expressed in terms of matrices that make them quite tractable numerically (see [17]). The primary objective of this paper is to construct a family of dilating cones for a general closed, convex, and pointed cone in a Banach space and apply it to study an infinite-dimensional quadratic programming problem.

We organize the contents of this paper into seven sections. In Sect. 2, we formulate the general setting and recall the notions that will be used in the subsequent development. In Sect. 3, we define a new family of dilating cones extending the idea given in [20, Section 3.3] and study its relation with the most important families of dilating cones available in the literature. In Sect. 4, we give the explicit representation of the new dilating cones for sequence and function spaces. In Sect. 5, using the new family of dilating cones, we extend the conical regularization scheme established in Khan and Sama [1] for a linearly constrained least-squares optimization problem. In Sect. 6, we present a numerical example illustrating the theory. Section 7 contains some concluding remarks.

2 Preliminaries

We assume that Y is a Banach space, $\|\cdot\|_Y$ denotes its norm, and the closed unit ball is denoted by $B_Y = \{y \in Y : \|y\|_Y \leq 1\}$. We denote by Y^* the dual space of Y and by $\|\cdot\|_{Y^*}$ its norm. Given a nonempty set $F \subset Y$, we denote by $\operatorname{int} F$, $\operatorname{cl} F$ and $\operatorname{cone} F$, the topological interior, the closure and the cone generated by F, respectively. Moreover, F is called solid, if $\operatorname{int} F \neq \emptyset$. Let $D \subset Y$ be a closed, convex, and pointed cone, $D \cap (-D) = \{0\}$, inducing a partial ordering $\leq_D$ on Y; i.e., $y \leq_D z$ iff $z - y \in D$. Moreover, the positive and the strict positive polar cones of D are denoted, respectively, by D^* and $D^\sharp$, that is,

$$D^* = \left\{\mu^* \in Y^* : \mu^*(d) \geq 0 \,\forall\, d \in D\right\},\; D^\sharp = \left\{\mu^* \in Y^* : \mu^*(d) > 0 \,\forall\, d \in D\backslash\{0\}\right\}.$$

For any $\lambda^* \in Y^*$, by $H_{\lambda^*} := \{y \in Y : \lambda^*(y) \geq 0\}$ we denote the associated positive halfspace. A nonempty and convex set $\Theta \subset D$ is called a base of D, if each nonzero element $x \in D$ has a unique representation of the form $x = \lambda\theta$, with $\lambda > 0$ and $\theta \in \Theta$. Throughout this paper, we assume that D has a base Θ, which is equivalent to $D^\sharp \neq 0$. Without any loss of generality, we assume $\Theta = \{y \in D : \beta^*(y) = 1\}$,where $\beta^* \in D^\sharp$ and $\|\beta^*\|_{Y^*} = 1$ (see for instance [21, Theorem 2.1.15]).

The primary objective of this paper is to introduce a new family of dilating cones for D, see [19,22]. We say that $\{D_\varepsilon\}_{0<\varepsilon<1} \subset Y$ is a family of dilating cones for D, if each D_ε is a closed, convex, and pointed cone with nonempty interior such that:

(D1) $D_{\varepsilon'} \subset D_\varepsilon$, for all $0 < \varepsilon' \leq \varepsilon < 1$;
(D2) $D\backslash\{0\} \subset \operatorname{int} D_\varepsilon$ for every $0 < \varepsilon < 1$;
(D3) $D = \bigcap_{0<\varepsilon<1} D_\varepsilon$.

We now recall two important families of dilating cones:

1. Borwein and Zhuang [16] showed that the family $\{D^H_\varepsilon\}_{0<\varepsilon<\delta}$ given by

$$D^H_\varepsilon := \operatorname{cl}\operatorname{cone}(\Theta + \varepsilon B_Y), \tag{1}$$

   where $\delta := \inf\{\|\theta\|_Y : \theta \in \Theta\} \geq 1$, is a family of dilating cones associated with D. Each cone D^H_ε is called a Henig dilating cone. This family has been frequently used to deal with proper efficient solutions in the sense of Henig (see [16]), and also in perturbation methods in infinite-dimensional quadratic programming problems (see, for instance, [1]). A similar family of dilating cones was also given by Sterna-Karwat in [19, Proposition 6.1].
2. Let $Y = \mathbb{R}^n$ be equipped with a norm $\|\cdot\|_{\mathbb{R}^n}$ and let D be polyhedral, that is,

$$D = \left\{\mathbf{y} \in \mathbb{R}^n : \mathbf{a}_i^{\mathrm{T}}\mathbf{y} \geq 0, i = 1, \ldots, l\right\}, \tag{2}$$

   where $\mathbf{a}_i \in \mathbb{R}^n, i = 1, \ldots, l, l \geq n$, and $\mathbf{a}_i^{\mathrm{T}}$ denotes the transpose of $\mathbf{a}_i$. We suppose that the matrix $\mathbf{A}$ whose rows are $\mathbf{a}_i^{\mathrm{T}}, i = 1, \ldots, l$, has full rank, i.e., $\operatorname{rank}(\mathbf{A}) = n$. Then, the cone D is convex, closed and pointed. For the sake of simplicity, we also assume that $\|\mathbf{a}_i\|_{\mathbb{R}^n} = 1$. In this framework, Kaliszewski [20] introduces a family of dilating polyhedral cones associated with D. More precisely, the Kaliszewski dilating cones are defined as

$$K_\varepsilon := \left\{\mathbf{y} \in \mathbb{R}^n : \mathbf{0} \leq_{\mathbb{R}^l_+} \left[\mathbf{I} + \varepsilon \mathbf{1} \cdot \mathbf{1}^{\mathrm{T}}\right] \mathbf{A}\mathbf{y}\right\}, \tag{3}$$

where $\varepsilon > 0$, $\mathbf{I} \in \mathbb{R}^{l \times l}$ is the identity matrix, and $\mathbf{1}^{\mathrm{T}} = [1....1] \in \mathbb{R}^l$ is the unit constant vector (see [20, Section 3.3]).

The main advantage of the Kaliszewski's cone K_ε is that it is amenable for numerical computations. However, this construction is limited to polyhedral cones in finite dimensions. In this work, our main objective is to extend the family of Kaliszewski dilating cones to a general cone D in an infinite-dimensional setting and show its applicability to the conical regularization framework. With this aim, first we formulate the notion of an enumerable halfspace representation which will play an important role.

Definition 2.1 Let I be an arbitrary nonempty index set and let $\{\lambda_i^*\}_{i\in I} \subset D^*$. The collection $\{H_{\lambda_i^*}\}_{i\in I}$ is said to be a representation of D if $D = \bigcap_{i\in I} H_{\lambda_i^*}$.

In the following, we assume that there is an enumerable representation of D given by $\{H_{\lambda_i^*}\}_{i\in I} \subset Y$, $I \subset \mathbb{N}$. This assumption is quite general, covering in particular all separable Banach spaces (see, for example, [23, Corollary 7.49]). The closed convex sets admitting an enumerable halfspace representation are usually called constructible sets in the literature (see, for instance, [24]). Without loss of generality, we also assume unit norm $\left\|\lambda_i^*\right\|_{Y^*} = 1$ for every $i \in I$. Summarizing, throughout this paper we assume that D is a based, closed convex (pointed) cone admitting an enumerable representation.

3 An Extension of Kaliszewski Cone to Non-polyhedral Cones in Infinite-Dimensional Spaces

We aim at constructing a family of dilating cones for D and studying its relationship with Henig dilating cones family $\{D_\varepsilon^H\}_{0<\varepsilon<1}$ and Kaliszewski dilating cones $\{K_\varepsilon\}_{0<\varepsilon<1}$.

Let $\varepsilon \geq 0$. For each $i \in I$, set

$$\lambda_{i,\varepsilon}^* := \lambda_i^* + \varepsilon\beta^*. \tag{4}$$

Since $\lambda_i^* \in D^*$, $i \in I$, and $\beta^* \in D^\sharp$, clearly $\lambda_{i,\varepsilon}^* \in D^\sharp$, for all $\varepsilon > 0$. We define

$$D_\varepsilon^K := \bigcap_{i\in I} H_{\lambda_{i,\varepsilon}^*} = \bigcap_{i\in I} \left\{y \in Y : \lambda_{i,\varepsilon}^*(y) \geq 0\right\}. \tag{5}$$

Evidently, D_ε^K is closed and convex, and $D_0^K = D$. Moreover, $\beta^* \in D_\varepsilon^{K\,\sharp}$. Indeed, if this is not the case, then there is $0 \neq d_\varepsilon \in D_\varepsilon^K$ such that $0 \geq \beta^*(d_\varepsilon)$. By definition of D_ε^K, $\lambda_i^*(d_\varepsilon) \geq -\varepsilon\beta^*(d_\varepsilon) \geq 0$, for every $i \in I$. Therefore, $d_\varepsilon \in \bigcap_{i\in I} H_{\lambda_i^*} = D$, implying $\beta^*(d_\varepsilon) > 0$ which contradicts $0 \geq \beta^*(d_\varepsilon)$. Thus, $\beta^* \in D_\varepsilon^{K\sharp}$ and it is clear that

$$\Theta_\varepsilon^K := \left\{d_\varepsilon \in D_\varepsilon^K : \beta^*(d_\varepsilon) = 1\right\}$$

is a base of D_ε^K ($\Theta_0^K = \Theta$). Thus, in particular, we deduce that D_ε^K is also pointed.

Note that the condition $\varepsilon < 1$ when we refer to families $\{K_\varepsilon\}_{0<\varepsilon<1}$ and $\{D_\varepsilon^K\}_{0<\varepsilon<1}$ is not necessary. We just impose it to unify notations. The following result shows that D_ε^K is indeed a family of dilating cones:

Theorem 3.1 *It follows that $\{D_\varepsilon^K\}_{0<\varepsilon<1}$ is a family of dilating cones for D.*

Proof For $\varepsilon \in]0, 1[$, we have observed that D_ε^K is a closed, convex, and pointed cone. It is easily seen that (D1) holds. To prove (D2), we show that $\Theta \subset \operatorname{int} D_\varepsilon^K$ which at once implies that $D\backslash\{0\} \subset \operatorname{int} D_\varepsilon^K$. Let $\theta \in \Theta$ be a fixed element of the basis, $b \in B_Y$ be a fixed element of the closed unit ball and let $\alpha := \varepsilon/(1+\varepsilon)$. We have for each $i \in I$ that $\lambda_{i,\varepsilon}^*(\theta + \alpha b) = \lambda_{i,\varepsilon}^*(\theta) + \alpha\lambda_{i,\varepsilon}^*(b) \geq \varepsilon + \alpha\left[\lambda_i^*(b) + \varepsilon\beta^*(b)\right]$, thus

$$\lambda_{i,\varepsilon}^*(\theta + \alpha b) \geq \varepsilon - \alpha\left[\left\|\lambda_i^*\right\|_{Y^*} + \varepsilon\left\|\beta^*\right\|_{Y^*}\right] = \varepsilon - \alpha[1+\varepsilon] = 0. \tag{6}$$

Since θ and b are arbitrary, this proves

$$\Theta + \alpha B_Y \subset D_\varepsilon^K, \tag{7}$$

so $\Theta \subset \operatorname{int} D_\varepsilon^K$. Finally, clearly $D \subset \bigcap_{0<\varepsilon<1} D_\varepsilon^K$. For the converse, let $w \in \bigcap_{0<\varepsilon<1} D_\varepsilon^K$. Then, by definition we have

$$\lambda_i^*(w) \geq -\varepsilon\beta^*(w) \ \text{ for every } i \in I,\ 0 < \varepsilon < 1.$$

Thus, taking the limit when ε tends to zero in the inequality above we have $\lambda_i^*(w) \geq 0$ for every $i \in I$. Hence, $w \in D$ and (D3) holds. □

For a polyhedral cone in a finite-dimensional space, the family $\{D_\varepsilon^K\}_{0<\varepsilon<1}$ coincides with the Kaliszewski one. Indeed, for $Y = \mathbb{R}^n$, assume that D is given by expression (2), that is $D = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{a}_i^{\mathrm{T}} y \geq 0, i = 1, \ldots, l\}$. We now show that the cone D_ε^K is indeed an extension of Kaliszewski cone K_ε given by expression (3). In fact, we will show that both cones coincide up to a constant factor. In this sense, a natural representation of D is given by $\left\{H_{\mathbf{a}_i^{\mathrm{T}}}\right\}_{i=1,\ldots,l}$, where we identify $\lambda_i^* \equiv \mathbf{a}_i^{\mathrm{T}}$. Note that $\lambda_i^* \in D^*$, $i = 1, 2, \ldots, l$. On the other hand, we can write (3) as follows:

$$K_\varepsilon = \left\{\mathbf{y} \in \mathbb{R}^n : 0 \leq \left[\mathbf{a}_i^{\mathrm{T}} + \varepsilon\left\|\mathbf{a}\right\|_{Y^*}\frac{\mathbf{a}^{\mathrm{T}}}{\|\mathbf{a}\|_{Y^*}}\right]\mathbf{y} \text{ for every } i = 1, \ldots, l\right\},$$

where we denote $\mathbf{a} = \sum_{j=1}^{l} \mathbf{a}_j$. Since A has full rank, it is easy to see that element $\mathbf{a}^{\mathrm{T}}$ defines a strictly positive functional $\beta^* \in D^\natural$, i.e., $\beta^* := \frac{\mathbf{a}^{\mathrm{T}}}{\|\mathbf{a}\|_{\mathbb{R}^n}}$. Then, defining $\gamma = \varepsilon \,\|\mathbf{a}\|_{\mathbb{R}^n}$, it follows that

$$\lambda^*_{i,\gamma} = \lambda^*_i + \gamma\beta^* = \mathbf{a}_i^{\mathrm{T}} + \varepsilon\,\|\mathbf{a}\|_{\mathbb{R}^n}\frac{\mathbf{a}^{\mathrm{T}}}{\|\mathbf{a}\|_{\mathbb{R}^n}}.$$

From this, applying definition (5), $K_\varepsilon = D^K_\gamma$. Consequently, $K_\varepsilon = D^K_{\varepsilon\|\mathbf{a}\|_{\mathbb{R}^n}}$.

Remark 3.1 Let $\varepsilon > 0$. In general, $D^K_\varepsilon \neq D^H_\varepsilon$ (see Example 3.2), but taking into account statements (1), (6) and (7), we deduce that for all $\eta \le \alpha := \frac{\varepsilon}{1+\varepsilon}$ we have $\Theta + \eta B_Y \subset \Theta + \alpha B_Y \subset D^K_\varepsilon$, which implies

$$D^H_\eta = \operatorname{cl}\operatorname{cone}(\Theta + \eta B_Y) \subset \operatorname{cl}\operatorname{cone} D^K_\varepsilon = D^K_\varepsilon,\ \ \forall \eta \le \alpha.$$

For the next theorem, we recall that given two nonempty sets $M, N \subset Y$, the Hausdorff distance between M and N is defined as

$$\mathrm{d}_H(M, N) := \inf\{\mu \ge 0 : M \subset N + \mu B_Y, N \subset M + \mu B_Y\}.$$

It is well known that the Hausdorff distance defines a metric on the space of all nonempty closed and bounded sets.

Theorem 3.2 *Assume* $0 < d_H(\Theta, \Theta^K_\varepsilon) < \delta := \inf\{\|\theta\|_Y : \theta \in \Theta\}$ *for some* $\varepsilon > 0$. *One of the following conditions holds*

(a) Θ *is weakly compact,*
(b) Y *is reflexive.*

Then,

$$D^K_\varepsilon \subset D^H_{d_H(\Theta,\Theta^K_\varepsilon)}. \tag{8}$$

Proof First of all, note that since $\mathrm{d}_H(\Theta, \Theta^K_\varepsilon) < \delta$, cone $D^H_{\mathrm{d}_H(\Theta,\Theta^K_\varepsilon)}$ is well defined [see (1)]. It follows that $\mathrm{d}_H(\Theta, \Theta^K_\varepsilon) = \inf\{\mu \ge 0 : \Theta^K_\varepsilon \subset \Theta + \mu B_Y\}$.

Therefore, $\Theta^K_\varepsilon \subset \bigcap_{\mu > \mathrm{d}_H(\Theta,\Theta^K_\varepsilon)} (\Theta + \mu B_Y)$. We are going to prove that

$$\bigcap_{\mu > \mathrm{d}_H(\Theta,\Theta^K_\varepsilon)} (\Theta + \mu B_Y) = \Theta + \mathrm{d}_H(\Theta, \Theta^K_\varepsilon) B_Y.$$

Indeed, inclusion "$\supset$" is clear. For the converse, let $d \in \bigcap_{\mu > \mathrm{d}_H(\Theta,\Theta^K_\varepsilon)} (\Theta + \mu B_Y)$. Thus, we may assume there exist $\{\theta_n\} \subset \Theta$, $\{b_n\} \subset B_Y$ such that

$$d = \lim_{n\to\infty} \left(\theta_n + \left(\frac{1}{n} + \mathrm{d}_H(\Theta, \Theta^K_\varepsilon)\right) b_n\right).$$

Suppose that (a) holds. Since Θ is weakly compact, we can suppose without loss of generality that $\theta_n \overset{w}{\to} \bar{\theta} \in \Theta$, where $\overset{w}{\to}$ denotes the convergence with respect to the weak topology. Thus, since B_Y is convex, $b_n \overset{w}{\to} \frac{d-\bar{\theta}}{\mathrm{d}_H(\Theta,\Theta_\varepsilon^K)} := \bar{b} \in B_Y$. Hence, $d = \bar{\theta} + \mathrm{d}_H(\Theta, \Theta_\varepsilon^K)\bar{b} \in \Theta + \mathrm{d}_H(\Theta, \Theta_\varepsilon^K)B_Y$ and "$\subset$" is also proved.

Thus, $\Theta_\varepsilon^K \subset \Theta + \mathrm{d}_H(\Theta, \Theta_\varepsilon^K)B_Y$ and then

$$D_\varepsilon^K = \mathrm{cl\,cone}\,\Theta_\varepsilon^K \subset \mathrm{cl\,cone}\left(\Theta + \mathrm{d}_H\left(\Theta, \Theta_\varepsilon^K\right)B_Y\right) = D^H_{\mathrm{d}_H(\Theta,\Theta_\varepsilon^K)},$$

as we wanted to prove. Finally, if (b) is satisfied instead of (a), then in this case B_Y is weakly compact and the result follows by reasoning in analogous way as before. □

Remark 3.2 Let us note that assumption $\mathrm{d}_H(\Theta, \Theta_\varepsilon^K) < \delta$ in Theorem 3.2 imposes in particular that $\mathrm{d}_H(\Theta, \Theta_\varepsilon^K)$ is finite. By the properties of the Hausdorff distance, if Θ_ε^K is bounded, then $\mathrm{d}_H(\Theta, \Theta_\varepsilon^K) < \infty$. In the finite-dimensional setting, it is known that Θ_ε^K is bounded, for all $\varepsilon > 0$. Also, it is very known that the following statements are equivalent (see, for instance [25]):

(i) Θ_ε^K is bounded,
(ii) $\beta^* \in \mathrm{int}\left(D_\varepsilon^{K\sharp}\right)$,
(iii) There exists $\alpha \in]0, 1]$ such that $D_\varepsilon^K \subset \{y \in Y : \beta^*(y) \geq \alpha\|y\|\}$.

On the other hand, if Y is reflexive and $\mathrm{int}\left(D_\varepsilon^{K\sharp}\right) \neq \emptyset$, then by [25, Theorem 3.6], we have that $\mathrm{int}\left(D_\varepsilon^{K\sharp}\right) = D_\varepsilon^{K\sharp}$, so in particular (ii) is satisfied, and then Θ_ε^K is bounded. Moreover, under the same conditions we have that Θ_ε^K is a closed convex and bounded set in a reflexive space; thus it is weakly compact.

In the following result, we prove that if D is solid and Θ is bounded, then Θ_ε^K is bounded, for all $\varepsilon > 0$.

Proposition 3.1 *Suppose that D is solid and let $\varepsilon > 0$. If Θ is bounded, then Θ_ε^K is bounded.*

Proof Suppose that Θ is bounded and fix $\bar{d} \in \mathrm{int}\,D$. Then, there exists $\eta > 0$ such that $\bar{d} + \eta B_Y \subset D$. Consequently, $\lambda_i^*(\bar{d} + \eta b) \geq 0$, for all $b \in B_Y$, and for all $i \in I$, which implies that

$$\lambda_i^*(\bar{d}) \geq \eta \sup_{b \in B_Y} |\lambda_i^*(b)| = \eta\|\lambda_i^*\|_{Y^*} = \eta, \ \forall i \in I. \tag{9}$$

Now, let any $\theta_\varepsilon \in \Theta_\varepsilon^K$. By definition of D_ε^K and (9) we have

$$\lambda_i^*\left(\theta_\varepsilon + \frac{\varepsilon}{\eta}\bar{d}\right) = \lambda_i^*(\theta_\varepsilon) + \frac{\varepsilon}{\eta}\lambda_i^*(\bar{d}) \geq -\varepsilon\beta^*(\theta_\varepsilon) + \frac{\varepsilon}{\eta}\eta = -\varepsilon + \varepsilon = 0,$$

so $\theta_\varepsilon + \frac{\varepsilon}{\eta}\bar{d} \in D$, i.e., $\frac{\theta_\varepsilon + \frac{\varepsilon}{\eta}\bar{d}}{\beta^*\left(\theta_\varepsilon + \frac{\varepsilon}{\eta}\bar{d}\right)} = \frac{\theta_\varepsilon + \frac{\varepsilon}{\eta}\bar{d}}{1 + \frac{\varepsilon}{\eta}\beta^*(\bar{d})} \in \Theta$. Since θ_ε is arbitrary, we conclude that $\Theta_\varepsilon^K \subset \left(1 + \frac{\varepsilon}{\eta}\beta^*(\bar{d})\right)\Theta - \frac{\varepsilon}{\eta}\bar{d}$, so Θ_ε^K is bounded, as we wanted to prove. □

Example 3.1 Regarding Remark 3.2 and [21, Example 1.1.3], if D is the Bishop–Phelps cone [26], defined for β^* and $0 < \alpha < 1$ as

$$C(\beta^*, \alpha) := \{y \in Y : \beta^*(y) \geq \alpha\|y\|\},$$

then it is solid and Θ is bounded. Thus, by Proposition 3.1 we know that Θ_ε^K is bounded, for all $\varepsilon > 0$. From this fact we know that $\mathrm{d}_H(\Theta, \Theta_\varepsilon^K) < \infty$, so by Theorem 3.2, inclusion (8) is verified when Y is reflexive and $\mathrm{d}_H(\Theta, \Theta_\varepsilon^K) < \delta$ for small $\varepsilon > 0$.

Remark 3.3 An important application of dilating cones is in connection with cone separation theorems. More precisely, given a closed cone $C \subset Y$ such that $C \cap D = \{0\}$, we seek for a family $\{D_\varepsilon\}_{0<\varepsilon<1}$ of dilating cones for D such that

$$C \cap D_\varepsilon = \{0\}, \quad \text{for all } \varepsilon > 0. \tag{10}$$

The existence of this family is relevant, for instance, in proper efficiency in vector optimization problems (see, for example [16,17,19]).

Following this line, in [19, Proposition 6.1] Sterna-Karwat proved that if $C \subset Y$ is a weakly closed cone and D is a convex cone with a weakly compact base such that $C \cap D = \{0\}$, then $\{D^H_{\frac{1}{n}}\}_{n \geq n_0}$ is a family of dilating cones for D satisfying (10), for $n_0 \in \mathbb{N}$, sufficiently large. Thus, under the hypotheses of Theorem 3.2, assuming $\mathrm{d}_H(\Theta, \Theta_\varepsilon^K) \to 0$ when $\varepsilon \to 0$, we have that for each $n \geq n_0$ there exists $\varepsilon_n > 0$ such that $\mathrm{d}_H(\Theta, \Theta_{\varepsilon_n}^K) \leq \frac{1}{n}$ and we conclude $C \cap D_{\varepsilon_n}^K = \emptyset$ for every $n \geq n_0$.

In the following example, we compute the involved cones for a particular case of cone D to illustrate their construction.

Example 3.2 Let $Y = \mathbb{R}^2$ be equipped with the Euclidean norm and the ordering $D = \mathbb{R}_+ \times \{0\}$. By identifying elements of the dual space with vectors, we consider the strictly positive functional $\beta^* \equiv (1, 0)$ and the representation $\{H_{\lambda_i^*}\}_{i \in \{1,2,3\}}$ given by $\lambda_1^* \equiv (0, 1)$, $\lambda_2^* \equiv (0, -1)$, $\lambda_3^* \equiv (1, 0)$. Then, by a direct computation, we have

$$\begin{aligned} D_\varepsilon^H &= \left\{(x, y) \in \mathbb{R}^2 : \varepsilon\left(1 - \varepsilon^2\right)^{\frac{-1}{2}} x \geq y \geq -\varepsilon\left(1 - \varepsilon^2\right)^{\frac{-1}{2}} x,\ x \geq 0\right\}, \\ D_\varepsilon^K &= \left\{(x, y) \in \mathbb{R}^2 : \varepsilon x \geq y \geq -\varepsilon x,\ x \geq 0\right\}. \end{aligned}$$

On the other hand, we have $\Theta = \{(1, 0)\}$, while $\Theta_\varepsilon^K = \{(1, x) : -\varepsilon \leq x \leq \varepsilon\}$. The Hausdorff distance can be computed straightforward, $d_H(\Theta, \Theta_\varepsilon^K) = \varepsilon$. Then, by Remark 3.1 and Theorem 3.2 we know that $D^H_{\frac{\varepsilon}{1+\varepsilon}} \subset D_\varepsilon^K \subset D_\varepsilon^H$.

4 Particular Cases

In this section, we give an explicit representation of the extended Kaliszewski cone for two fundamental classes of infinite-dimensional ordered vector spaces.

4.1 $Y = \ell^p, D = \ell^p_+, p \in [1, \infty]$

In this part, we deal with the Banach sequence spaces $Y = \ell^p$, $p \in [1, \infty]$, endowed with the natural ordering given by the cone $D = \ell^p_+$, where

$$\ell^p_+ := \left\{ \mathbf{x} = \{x_i\}_{i \in \mathbb{N}} \in \ell^p : x_i \geq 0 \text{ for every } i \in \mathbb{N} \right\}.$$

We distinguish two cases depending on exponent p.

4.1.1 Case $p \in [1, \infty[$

In this case, $Y = \ell^p$ is the Banach space of all sequences $\mathbf{x} = \{x_i\}_{i \in \mathbb{N}}$ equipped with the norm $\|\mathbf{x}\|_p := \|\mathbf{x}\|_{\ell^p} = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} < \infty$. It is known that $D = \ell^p_+$ is a closed, convex pointed cone with empty interior. A natural representation of D is given by the dual functionals $\{\lambda_i^*\}_{i \in \mathbb{N}} = \{\mathbf{e}_i^*\}_{i \in \mathbb{N}} \subset (\ell^p)^*$, such that for $i \in \mathbb{N}$, $\mathbf{e}_i^* \in (\ell^p)^*$ is defined by $\mathbf{e}_i^*(\mathbf{x}) = x_i$ for every $\mathbf{x} \in \ell^p$. Clearly $\left\|\mathbf{e}_i^*\right\|_{(\ell^p)^*} = 1$, for every $i \in \mathbb{N}$. On the other hand, we take a fixed sequence $\beta = \{\beta_i\}_{i \in \mathbb{N}} \in \ell^1$ with strictly positive components $\beta_i > 0$. Let us note that the associated linear functional β^* defined as

$$\beta^*(\mathbf{x}) = \sum_{i=1}^{\infty} \beta_i x_i, \ \ \forall \mathbf{x} \in \ell^p \tag{11}$$

belongs to $\left(\ell^p_+\right)^\natural$. We consider the normalized element $\beta_p^* = \left\{\beta_i^p\right\}_{i \in \mathbb{N}} = \|\beta^*\|^{-1}_{(\ell^p)^*} \beta^*$. By the usual identification of $(\ell^p)^*$ with ℓ^q, where q is the conjugate exponent satisfying $p^{-1} + q^{-1} = 1$ for $p \in]1, \infty[$ and $q = \infty$ for $p = 1$, we have $\|\beta^*\|_{(\ell^p)^*} = \|\beta\|_q$. Consequently $\beta_i^p = \|\beta\|_q^{-1} \beta_i$ for every $i \in \mathbb{N}$. The corresponding functionals $\left\{\lambda^*_{i,\varepsilon}\right\}_{i \in \mathbb{N}}$ given by (4) verify $\lambda^*_{i,\varepsilon}(\mathbf{x}) = \left(\mathbf{e}_i^* + \varepsilon \beta_p^*\right)(\mathbf{x}) = x_i + \varepsilon \beta_p^*(\mathbf{x}) = x_i + \varepsilon \|\beta\|_q^{-1} \sum_{i=1}^{\infty} \beta_i x_i$, and the corresponding dilating cone D_ε^K, $\varepsilon > 0$ is defined by

$$D_\varepsilon^K = \left\{ \mathbf{x} \in \ell^p : x_i \geq -\varepsilon \|\beta\|_q^{-1} \sum_{i=1}^{\infty} \beta_i x_i \text{ for every } i \in \mathbb{N} \right\}.$$

4.1.2 Case $p = \infty$

Here $Y = \ell^\infty$, which is the Banach space of all bounded sequences $\mathbf{x} = \{x_i\}_{i \in \mathbb{N}}$ equipped with the norm $\|\mathbf{x}\|_\infty := \|\mathbf{x}\|_{\ell^\infty} = \max\{|x_i| : i \in \mathbb{N}\} < \infty$, and $D = \ell^\infty_+$. It is known that D is in this case a solid cone. Furthermore, although ℓ^∞ is not separable, the dual canonical basis $\{\lambda_i^*\}_{i \in \mathbb{N}} = \{\mathbf{e}_i^*\}_{i \in \mathbb{N}} \subset (\ell^p)^*$ determines also a unit norm halfspace representation of ℓ^∞_+. The dual element β^* given by expression (11) is also a strictly positive functional $\beta^* \in \left(\ell^\infty_+\right)^\natural$, and we can compute its norm directly, $\|\beta^*\|_{(\ell^\infty)^*} = \|\beta\|_1$. Therefore, $\beta^*_\infty = \left\{\beta_i^\infty\right\}_{i \in \mathbb{N}} = \|\beta\|_1^{-1} \beta^*$ is such that $\|\beta^*\|_{(\ell^\infty)^*} = 1$, and the corresponding dilating cone D_ε^K, $\varepsilon > 0$ is defined by

$$D_\varepsilon^K = \left\{ \mathbf{x} \in \ell^\infty : x_i \geq -\varepsilon \, \|\beta\|_1^{-1} \sum_{i=1}^{\infty} \beta_i x_i \text{ for every } i \in \mathbb{N} \right\}.$$

4.2 $Y = L^p(\Omega), D = L_+^p(\Omega), p \in [1, \infty]$

Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be a bounded convex domain. $L^p(\Omega)$ is the space of integrable functions endowed with the ordering given by the cone $D = L_+^p(\Omega)$, where

$$L_+^p(\Omega) = \left\{ f \in L^p(\Omega) : f(x) \geq 0 \text{ a.e in } \Omega \right\}.$$

As before, we distinguish two cases depending on exponent p.

4.2.1 Case $p \in [1, \infty[$

In this case, $Y = L^p(\Omega)$ is endowed with the norm $\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(s)|^p \, \mathrm{d}s\right)^{\frac{1}{p}}$. It is known that $D = L_+^p(\Omega)$ is a closed, convex pointed cone with empty interior. In the following, we define a representation of the cone D following the idea given in Jadamba et al. [9], see also [11]. For this aim, we consider a family of convex partitions $\{\Delta^\delta\}_{\delta>0}$ of Ω, where δ is a real parameter, see [9,27]. Each partition $\left\{\Delta^\delta\right\}$ consists of a finite number of closed and convex sets $\left\{\Delta_i^\delta\right\} \subset \Omega$ $(i = 1, \ldots, T(\delta))$ such that $\sum_{\Delta_i^\delta \in \Delta^\delta} \left|\Delta_i^\delta\right| = |\Omega|$, where $T(\delta) = \#\Delta^\delta$ is the cardinality of $\{\Delta^\delta\}$, and $|\Omega|$ is the Lebesgue measure of Ω. We assume that diameter of the family tends to zero, that is,

$$\operatorname{diam}(\Delta^\delta) = \max_{i=1,\ldots,T(\delta)} \operatorname{diam}(\Delta_i^\delta) \underset{\delta \to 0}{\to} 0.$$

Without any loss of generality, in the sequel we set $\delta \equiv \operatorname{diam}(\Delta^\delta)$.

Let $q = \frac{p}{p-1}$ be the conjugate exponent of p, where we assume $q = \infty$ for $p = 1$. Now for each $\delta > 0$, $i \in \{1, \ldots, T(\delta)\}$, we define $\lambda_{p,i}^{\delta*} \in L^p(\Omega)^*$ by

$$\lambda_{p,i}^{\delta*}(f) = \left|\Delta_i^\delta\right|^{-\frac{p-1}{p}} \int_{\Delta_i^\delta} f(s) \mathrm{d}s, \text{ for every } f \in L^p(\Omega).$$

For the particular case $p = 1$, we have $\left|\Delta_i^\delta\right|^{-\frac{p-1}{p}} = \left|\Delta_i^\delta\right|^{-0} = 1$, and therefore

$$\lambda_{1,i}^{\delta*}(f) = \int_{\Delta_i^\delta} f(s) \mathrm{d}s, \text{ for every } f \in L^1(\Omega).$$

By Hölder's inequality,

$$\left|\lambda_{p,i}^{\delta*}(f)\right| \leq \left|\Delta_i^\delta\right|^{-\frac{p-1}{p}} \int_{\Delta_i^\delta} |f(s)| \, \mathrm{d}s = \left|\Delta_i^\delta\right|^{-\frac{p-1}{p}} \|f\|_{L^p(\Omega)} \left\|\chi_{\Delta_i^\delta}\right\|_{L^q(\Omega)} = \|f\|_{L^p(\Omega)}.$$

Therefore, $\left|\lambda^{\delta *}_{p,i}(f)\right| \leq \|f\|_{L^p(\Omega)}$, thus $\left\|\lambda^{\delta *}_{p,i}\right\|_{L^p(\Omega)^*} \leq 1$. In fact $\left\|\lambda^{\delta *}_{p,i}\right\|_{L^p(\Omega)^*} = 1$, since $\left|\lambda^{\delta *}_{p,i}(\chi_{\Delta_i^\delta})\right| = \left\|\chi_{\Delta_i^\delta}\right\|_{L^p(\Omega)}$. The family $\{H_{\lambda^{\delta *}_{p,i}}\}_{i=1,\dots,T(\delta);\delta>0}$ is a unit norm representation of $L^p_+(\Omega)$. Note that we can always consider an enumerable representation by taking a sequence $\{\delta_n\} \downarrow 0$.

Now, we define a strictly positive functional $\beta^*_p \in L^p_+(\Omega)^\natural$ of unit norm by

$$\beta^*_p(f) = |\Omega|^{-\frac{p-1}{p}} \int_\Omega f(s)\mathrm{d}s, \text{ for every } f \in L^p(\Omega).$$

Therefore, for each $\varepsilon, \delta > 0$, $i \in \{1, \dots, T(\delta)\}$, the functionals $\lambda^{\delta *}_{p,\varepsilon,i}$ are given by

$$\lambda^{\delta *}_{p,\varepsilon,i}(f) = \lambda^{\delta *}_{p,i}(f) + \varepsilon\beta^*_p(f) = \left|\Delta_i^\delta\right|^{-\frac{p-1}{p}} \int_{\Delta_i^\delta} f(s)\mathrm{d}s + \varepsilon\,|\Omega|^{-\frac{p-1}{p}} \int_\Omega f(s)\mathrm{d}s,$$

for every $f \in L^p(\Omega)$. Thus, extended Kaliszewski cone D^K_ε, $\varepsilon > 0$, is given by

$$D^K_\varepsilon = \bigcap_{\delta>0} \left\{ f \in L^p(\Omega) : \left|\Delta_i^\delta\right|^{-\frac{p-1}{p}} \int_{\Delta_i^\delta} f(s)\mathrm{d}s + \varepsilon\,|\Omega|^{-\frac{p-1}{p}} \int_\Omega f(s)\mathrm{d}s \geq 0 \right.$$
$$\left. \forall i = 1, \dots, T(\delta) \right\}$$

for $p \in [1, \infty)$. In particular, for the case $p = 1$,

$$D^K_\varepsilon = \bigcap_{\delta>0} \left\{ f \in L^1(\Omega) : \int_{\Delta_i^\delta} f(s)\mathrm{d}s + \varepsilon \int_\Omega f(s)\mathrm{d}s \geq 0\, \forall i = 1, \dots, T(\delta) \right\}.$$

4.2.2 Case $p = \infty$

In this case, we consider that $Y = L^\infty(\Omega)$ is the space of essentially bounded functions on Ω endowed with the norm $\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}$, and we consider the associated cone $D = L^\infty_+(\Omega)$, which has nonempty interior. We can repeat the previous process in order to get a unit norm representation for this case. In this sense, for each $\delta > 0$, $i \in \{1, \dots, T(\delta)\}$, we define $\lambda^{\delta *}_{\infty,i} \in L^\infty(\Omega)^*$ by

$$\lambda^{\delta *}_{\infty,i}(f) = \left|\Delta_i^\delta\right|^{-1} \int_{\Delta_i^\delta} f(s)\mathrm{d}s, \text{ for every } f \in L^\infty(\Omega).$$

Clearly, $\left\|\lambda^{\delta *}_{\infty,i}\right\|_{L^\infty(\Omega)^*} = 1$, and the family of halfspaces $\left\{H_{\lambda^{\delta *}_{\infty,i}}\right\}_{i=1,\dots,T(\delta);\delta>0}$ is a unit norm enumerable representation of the cone $L^\infty_+(\Omega)$. We also define the strictly positive functional $\beta^*_\infty \in D^\natural$ of unit norm by

$$\beta^*_\infty(f) = |\Omega|^{-1} \int_\Omega f(s)\mathrm{d}s, \text{ for every } f \in L^\infty(\Omega).$$

Taking into account all these facts, in this case extended Kaliszewski cone is given by $D_\varepsilon^K = \bigcap_{\delta>0} \left\{ f \in L^\infty(\Omega) : \left|\Delta_i^\delta\right|^{-1} \int_{\Delta_i^\delta} f(s)\mathrm{d}s + \varepsilon\,|\Omega|^{-1} \int_\Omega f(s)\mathrm{d}s \geq 0 \text{ for every } i \in T(\delta) \right\}$.

5 Conical Regularization Based on Extended Kaliszewski Dilating Cones

5.1 Statement of the Problem

Let X and V be Hilbert spaces equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_V$. Following the general assumptions made in Sect. 2, we recall Y is a Banach normed space, and D is a based, closed pointed convex cone which admits a countable representation of halfspaces. As before, by Θ we denote its base generated by a functional $\beta^* \in D^\natural$. Under these conditions, cone D_ε^K can be constructed for each $\varepsilon > 0$. We consider the following linearly constrained least-squares problem:

$$
\begin{aligned}
(P): \quad & \text{Minimize } J(x) := \frac{1}{2}\|Sx - v_d\|_V^2 + \frac{\kappa}{2}\|x - x_d\|_X^2 \\
& \text{subject to } Gx \leq_D w, \; x \in X.
\end{aligned}
$$

Here $S : X \to V$, $G : X \to Y$ are bounded linear operators, $\kappa > 0$ is a given parameter, $v_d \in V$, $x_d \in X$ and $w \in Y$ are given elements. Evidently, (P) has unique solution $x_0 \in X$.

For each $\varepsilon > 0$, let D_ε^K be the corresponding extended Kaliszewski cone given by (5). Following [1], the associated conically regularized problems are then defined by replacing the cone D in problem (P) by D_ε^K :

$$
\begin{aligned}
(P_\varepsilon): \; & \text{Minimize } J(x) := \frac{1}{2}\|Sx - v_d\|_V^2 + \frac{\kappa}{2}\|x - x_d\|_X^2 \\
& \text{subject to } \lambda_i^*(Gx) + \varepsilon\beta^*(Gx) \leq \lambda_i^*(w) + \varepsilon\beta^*(w) \text{ for every } i \in I, x \in X.
\end{aligned}
$$

We will make the following mild assumption.

$$
Gx_0 - w \in -D\backslash\{0\}. \tag{12}
$$

Condition (12) provides a non-trivial feasible point such that the regularized optimality system is solvable (see [1]) and not equivalent to an equality constrained problem. We will denote by x_ε the unique solution to (P_ε), and we also use the notation $q_\varepsilon := Gx_\varepsilon - w$ for $\varepsilon \geq 0$. In the following result, we apply the general conical regularization scheme given in [1, Theorems 3.1, 3.2, 3.3]. These results were established for a Hilbert space Y but, since proofs only use properties of normed spaces, they can be easily extended for a general Banach space Y.

Theorem 5.1 *The following statements hold:*

(i) *For each* $\varepsilon > 0$, *there exists unique solution* x_ε *to problem* (P_ε).
(ii) $\{x_\varepsilon\} \to x_0$ *for* $\varepsilon \downarrow 0$.
(iii) *For each* $\varepsilon \in]0, 1[$, *there exists a Lagrange multiplier* $\mu_\varepsilon^* \in D_\varepsilon^{K^*}$ *such that*

$$DJ(x_\varepsilon) + \mu_\varepsilon^* \circ G = 0 \text{ in } X^*, \tag{13a}$$

$$\mu_\varepsilon^*(Gx_\varepsilon - w) = 0, \tag{13b}$$

$$\lambda_i^*(Gx_\varepsilon - w) + \varepsilon\beta^*(Gx_\varepsilon - w) \le 0 \quad \text{for every } i \in I. \tag{13c}$$

In the following, by μ_ε^* we denote any multiplier verifying KKT conditions (13). Without loss of generality, we assume $\mu_\varepsilon^* \neq 0$. Otherwise, by Theorem 5.1 we can take limit in KKT condition (13a) to get $DJ(x_\varepsilon) = 0 \to DJ(x_0) = 0$, and this implies the trivial case $x_\varepsilon = x_0$. In the next theorem, we are going to establish a priori estimate, for which we need first the following technical result.

Lemma 5.1 *For any multiplier* $\mu_\varepsilon^* \in D_\varepsilon^{K^*}$ *for* (P_ε), *we have* $\left\{\mu_\varepsilon^*(q_0)\right\} \to 0$ *for* $\varepsilon \to 0$.

Proof By the Taylor expansion of J at $x = x_\varepsilon$, we have

$$J(x_0) - J(x_\varepsilon) - DJ(x_\varepsilon)(x_0 - x_\varepsilon) = \frac{\kappa}{2}\|x_0 - x_\varepsilon\|_X^2 + \frac{1}{2}\|Sx_0 - Sx_\varepsilon\|_V^2 .$$

Applying KKT conditions (13a) and (13b), we have

$$DJ(x_\varepsilon)(x_0 - x_\varepsilon) = -(\mu_\varepsilon^* \circ G)(x_0 - x_\varepsilon) = -\mu_\varepsilon^*(Gx_0 - Gx_\varepsilon) = -\mu_\varepsilon^*(Gx_0 - w)$$
$$= -\mu_\varepsilon^*(q_0),$$

and consequently

$$\frac{\kappa}{2}\|x_0 - x_\varepsilon\|_X^2 + \frac{1}{2}\|Sx_0 - Sx_\varepsilon\|_V^2 = J(x_0) - J(x_\varepsilon) + \mu_\varepsilon^*(q_0). \tag{14}$$

Due to $\{x_\varepsilon\} \to x_0$, we have $J(x_\varepsilon) \to J(x_0)$, and therefore it follows from (14) that $\left\{\mu_\varepsilon^*(q_0)\right\} \to 0$ for $\varepsilon \to 0$. The proof is complete. □

We have the following error estimate:

Theorem 5.2 *For* $\varepsilon > 0$, *set* $\delta_\varepsilon := \alpha\left\|\mu_\varepsilon^*\right\|_{Y^*}$, *where* $\alpha := (1+\varepsilon)^{-1}\varepsilon$. *Then* $\{\delta_\varepsilon\} \to 0$ *and the following estimate holds for sufficiently small* ε :

$$\frac{\kappa}{2}\|x_0 - x_\varepsilon\|_X^2 + \frac{1}{2}\|Sx_0 - Sx_\varepsilon\|_V^2 \le J(x_0) - J(x_\varepsilon) - \delta_\varepsilon\left|\beta^*(q_0)\right| . \tag{15}$$

Proof By definition $q_0 \in -D$, furthermore $q_0 \neq 0$ by assumption (12). Therefore, the element $-q_0\beta^*(-q_0)^{-1}$ is well defined and by definition $-q_0\beta^*(-q_0)^{-1} \in \Theta$.

By Remark 3.1, $\mu_\varepsilon^* \in D_\alpha^{H*}$, therefore by applying [1, Theorem 4.1] which holds for a general Banach space, we have

$$\mu_\varepsilon^* \left(-q_0\beta^*(-q_0)^{-1}\right) \geq (1+\varepsilon)^{-1}\varepsilon \left\|\mu_\varepsilon^*\right\|_{Y^*} = \delta_\varepsilon.$$

By Lemma 5.1, $\left\{\mu_\varepsilon^*(q_0)\right\} \to 0$, and hence it follows from the previous inequality that $\{\delta_\varepsilon\} \to 0$ for $\varepsilon \to 0$. Furthermore, we also have $\mu_\varepsilon^*(q_0) \leq -\delta_\varepsilon \left|\beta^*(q_0)\right|$. Combining this expression with (14), we finally get (15). □

We will now introduce the conical regularization scheme for the two classes of infinite-dimensional problems introduced previously.

5.2 Conical Regularization Scheme for $Y = \ell^p$, $D = \ell_+^p$, $p \in [1, \infty]$

Let any fixed $p \in [1, \infty]$, and let $Y = \ell^p$, $D = \ell_+^p$. Denoting $Gx = \{\mathbf{Gx}_i\}_{i\in\mathbb{N}} \in \ell^p$, by (Q^p) we denote the following instance of problem (P)

$$\begin{aligned}(Q^p) : \text{Minimize } J(x) &:= \frac{1}{2}\left\|Sx - v_d\right\|_V^2 + \frac{\kappa}{2}\left\|x - x_d\right\|_U^2 \\ \text{subject to } &\mathbf{Gx}_i \leq w_i \text{ for every } i \in I, \quad x \in X.\end{aligned}$$

Following Sect. 4.1, for each $\varepsilon > 0$, the corresponding regularized problem is given by

$$\begin{aligned}(Q_\varepsilon^p) : \text{Minimize } \; &J(x) := \frac{1}{2}\left\|Sx - v_d\right\|_V^2 + \frac{\kappa}{2}\left\|x - x_d\right\|_U^2 \\ \text{s.t. } &\mathbf{Gx}_i + \varepsilon\left\|\beta\right\|_q^{-1}\sum_{i=1}^{\infty}\beta_i\mathbf{Gx}_i \leq w_i + \varepsilon\left\|\beta\right\|_q^{-1}\sum_{i=1}^{\infty}\beta_i w_i \text{ for every } i \in I, \quad x \in X,\end{aligned}$$

for the case $p \in [1, \infty[$, and

$$\begin{aligned}(Q_\varepsilon^\infty) : \text{Minimize } \; &J(x) := \frac{1}{2}\left\|Sx - v_d\right\|_V^2 + \frac{\kappa}{2}\left\|x - x_d\right\|_U^2 \\ \text{s.t. } &\mathbf{Gx}_i + \varepsilon\left\|\beta\right\|_1^{-1}\sum_{i=1}^{\infty}\beta_i\mathbf{Gx}_i \leq w_i + \varepsilon\left\|\beta\right\|_1^{-1}\sum_{i=1}^{\infty}\beta_i w_i \text{ for every } i \in I, \quad x \in X,\end{aligned}$$

for the case $p = \infty$.

5.3 Conical Regularization Scheme for $Y = L^p(\Omega)$, $D = L_+^p(\Omega)$, $p \in [1, \infty]$

We consider $Y = L^p(\Omega)$, $D = L_+^p(\Omega)$, and we denote $y_x = Gx \in L^p(\Omega)$. In this case, problem (P) takes the form

$$\begin{aligned}(\bar{Q}^p) : \text{Minimize } J(x) &:= \frac{1}{2}\|Sx - v_d\|_V^2 + \frac{\kappa}{2}\|x - x_d\|_U^2 \\ \text{subject to } & y_x(s) \leq w(s) \text{ a.e. in } \Omega, \quad x \in X.\end{aligned}$$

For each $\varepsilon > 0$, following Sect. 4.2, we define the corresponding regularized problem for three separate cases depending on exponent p. If $p = 1$, the regularized problem takes the form

$$\begin{aligned}(\bar{Q}_\varepsilon^1) : \text{Minimize } & J(x) := \frac{1}{2}\|Sx - v_d\|_V^2 + \frac{\kappa}{2}\|x - x_d\|_U^2 \\ \text{s.t. } & \int_{\Delta_i^\delta}(y_x(s) - w(s))\mathrm{d}s + \varepsilon \int_\Omega (y_x(s) - w(s))\mathrm{d}s \leq 0, \\ & \text{for every } i = 1, \ldots, T(\delta),\ \delta > 0,\ x \in X.\end{aligned}$$

For $p \in]1, \infty[$ we have

$$\begin{aligned}(\bar{Q}_\varepsilon^p) : \text{Minimize } & J(x) := \frac{1}{2}\|Sx - v_d\|_V^2 + \frac{\kappa}{2}\|x - x_d\|_U^2 \\ \text{s.t. } & \left|\Delta_i^\delta\right|^{-\frac{p-1}{p}} \int_{\Delta_i^\delta}(y_x(s) - w(s))\mathrm{d}s + \varepsilon\,|\Omega|^{-\frac{p-1}{p}} \int_\Omega (y_x(s) - w(s))\mathrm{d}s \leq 0, \\ & \text{for every } i = 1, \ldots, T(\delta),\ \delta > 0,\ x \in X,\end{aligned}$$

while for $p = \infty$ the corresponding regularized problem is given by

$$\begin{aligned}(\bar{Q}_\varepsilon^\infty) : \text{Minimize } & J(x) := \frac{1}{2}\|Sx - v_d\|_V^2 + \frac{\kappa}{2}\|x - x_d\|_U^2 \\ \text{s.t. } & \left|\Delta_i^\delta\right|^{-1} \int_{\Delta_i^\delta}(y_x(s) - w(s))\mathrm{d}s + \varepsilon\,|\Omega|^{-1} \int_\Omega (y_x(s) - w(s))\mathrm{d}s \leq 0, \\ & \text{for every } i = 1, \ldots, T(\delta),\ \delta > 0,\ x \in X.\end{aligned}$$

6 A Numerical Example

We now present a numerical example to illustrate the theoretical framework. We consider a nonregular ℓ^p-constrained optimization problem, which is a quadratic modification of a linear example given in [28]. We consider fixed exponent $1 \leq p < \infty$, and the quadratic problem

$$(Q) : \quad \text{Minimize } J(x) = \frac{1}{2}(x-1)^2 \text{ subject to } Gx \leq_{\ell_+^p} \mathbf{w}, \quad x \in \mathbb{R},$$

where $G : \mathbb{R} \to \ell^p$ is the bounded linear operator defined by $Gx = x\mathbf{a} = \left\{xi^{-2}\right\}_{i\in\mathbb{N}}$, with $\mathbf{a} = \left\{i^{-2}\right\}_{i\in\mathbb{N}}$, and $\mathbf{w}\in\ell^p$ is the sequence defined by $\mathbf{w} = \left\{i^{-3}\right\}_{i\in\mathbb{N}}$.

This problem is of type (Q^p) by taking $X = \mathbb{R}$, $Y = \ell^p$, $D = \ell_+^p$, $\kappa = 1$, $S = v_d = 0$, $x_d = 1$. The feasible set F_p of (Q) can be explicitly computed as follows:

$$\begin{aligned} F_p &= \left\{x \in \mathbb{R}: Gx - \mathbf{w} \in -\ell_+^p\right\} \\ &= \left\{x \in \mathbb{R}: xi^{-2} \leq i^{-3} \text{ for every } i = 1, \ldots, \infty\right\} =]-\infty, 0]. \end{aligned}$$

Clearly, $x_0 = 0$ is the unique solution to (Q).

Problem (Q) has no Lagrange multiplier. Otherwise, there would exist

$$\mu_0^* = \{\mu_i^0\}_{i \in \mathbb{N}} \in (\ell_+^p)^* \equiv \ell_+^q,$$

q being the conjugate exponent of p, such that

$$DJ(x_0) + \mu_0^* \circ G = 0 \Leftrightarrow -1 + \mu_0^* \circ G = 0 \Leftrightarrow \mu_0^* \circ G = 1, \tag{16}$$

$$\mu_0^*(Gx_0 - \mathbf{w}) = 0 \Leftrightarrow \mu_0^*(\mathbf{w}) = 0 \Leftrightarrow \sum_{i=1}^{\infty} \mu_i^0 i^{-3} = 0. \tag{17}$$

From (16), since $\mu_i^0 i^{-3} \geq 0$, we necessarily have $\mu_i^0 = 0$, for every $i \in \mathbb{N}$. Therefore $\mu_0^* = 0$, which contradicts (17), since in this case we get the absurd $0 = 1$. Therefore, problem (Q) is not regular.

Now, for each $\varepsilon > 0$, the regularized problem is given by

$$\begin{aligned} (Q_\varepsilon): \text{Minimize } & J(x) = \frac{1}{2}(x-1)^2 \\ & \text{subject to } e_i^*(Gx - \mathbf{w}) + \varepsilon\beta_p^*(Gx - \mathbf{w}) \leq 0, \quad \text{for every } i \in \mathbb{N},\ x \in \mathbb{R}. \end{aligned}$$

For the computations, we will take $\beta^* = \sum_{i=1}^{\infty} \frac{1}{i^2} e_i^*$. By definition, the feasible set is given by

$$\begin{aligned} F_{p,\varepsilon} &= \{x \in \mathbb{R} : e_i^*(Gx) + \varepsilon\beta_p^*(Gx) \leq e_i^*(\mathbf{w}) + \varepsilon\beta_p^*(\mathbf{w}) \text{ for every } i = 1, \ldots, \infty\} \\ &= \left\{x \in \mathbb{R} : x \leq \frac{\varepsilon\beta_p^*(\mathbf{w}) + \frac{1}{i^3}}{\varepsilon\beta_p^*(\mathbf{a}) + \frac{1}{i^2}} \text{ for every } i = 1, \ldots, \infty\right\}. \end{aligned}$$

Thus, $x_{p,\varepsilon} = \inf_{i=1,\ldots,\infty} \frac{\varepsilon\beta_p^*(\mathbf{w}) + \frac{1}{i^3}}{\varepsilon\beta_p^*(\mathbf{a}) + \frac{1}{i^2}}$. And we have that $\lim_{\varepsilon \to 0} x_{p,\varepsilon} = 0$, indeed,

$$\begin{aligned} \lim_{\varepsilon \to 0} x_{p,\varepsilon} &= \lim_{\varepsilon \to 0} \inf_{i=1,\ldots,\infty} \frac{\varepsilon\beta_p^*(\mathbf{w}) + \frac{1}{i^3}}{\varepsilon\beta_p^*(\mathbf{a}) + \frac{1}{i^2}} = \inf_{i=1,\ldots,\infty} \left[\lim_{\varepsilon \to 0} \frac{\varepsilon\beta_p^*(\mathbf{w}) + \frac{1}{i^3}}{\varepsilon\beta_p^*(\mathbf{a}) + \frac{1}{i^2}}\right] \\ &= \inf_{i=1,\ldots,\infty} \frac{1}{i} = 0. \end{aligned}$$

In Tables 1, 2 and 3, we present numerical results.

We consider three exponents $p \in \{1, 2, 100\}$ and a finite sequence of parameter ε for each case. We have solved numerically these problems by using GeoGebra [29]. Numerical results show that $\lim_{\varepsilon \to 0} x_{p,\varepsilon} = 0$ holds, and the conical regularization

Table 1 Problem (Q^1)

ε	$E(\varepsilon) = x_{1,\varepsilon}$
1.e−01	6.246e−01
1.e−02	3.535e−01
1.e−03	1.786e−01
1.e−04	8.605e−02
1.e−05	4.062e−02
1.e−06	1.900e−02
1.e−07	8.851e−03
1.e−08	4.115e−03

Table 2 Problem (Q^2)

ε	$E(\varepsilon) = x_{2,\varepsilon}$
1.e−01	6.195e−01
1.e−02	3.496e−01
1.e−03	1.764e−01
1.e−04	8.495e−02
1.e−05	4.010e−02
1.e−06	1.875e−02
1.e−07	8.736e−03
1.e−08	4.061e−03

Table 3 Problem (Q^{100})

ε	$E(\varepsilon) = x_{100,\varepsilon}$
1.e−01	5.630e−01
1.e−02	3.084e−01
1.e−03	1.537e−01
1.e−04	7.365e−02
1.e−05	3.468e−02
1.e−06	1.620e−02
1.e−07	7.544e−03
1.e−08	3.506e−03

scheme is effective for this case. There are no significant differences between different exponents. Experimentally, we notice an order convergence rate of $\mathcal{O}(h^k)$ with $k = 0.33$ in all the three cases. In fact, we observed that the same convergence rate for larger exponents p as well.

7 Conclusions

By assuming that D is a closed, convex, and pointed cone that admits an enumerable halfspace representation, we have constructed a family of dilating cones for D, also defined in terms of an enumerable intersection of halfspaces. This construction extends the one due to Kaliszewski given for polyhedral cones in the finite-dimensional space. We studied the relation of the proposed family of dilating cones with the most commonly used families of dilating cones. We used the new dilating cones to develop a conical regularization scheme for a linearly constrained least-squares problem focusing mainly on the case the constraint space is either ℓ^p or L^p. This new tool allows extending the results of classical monograph in vector optimization [20] from polyhedral finite-dimensional to more general infinite-dimensional cones. One of our future goals is to explore this idea, especially in given computable optimality conditions for efficient solutions of vector optimization problems.

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