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Symplectic embeddings of four-dimensional polydisks into balls

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We obtain new obstructions to symplectic embeddings of the four-dimensional polydisk $P(a,1)$ into the ball $B(c)$ for $2 \le a \le (\sqrt{7}-1)/(\sqrt{7}-2) \approx 2.549$, extending work done by Hind and Lisi and by Hutchings. Schlenk's folding construction permits us to conclude our bound on c is optimal. Our proof makes use of the combinatorial criterion necessary for one "convex toric domain" to symplectically embed into another introduced by Hutchings (2016). We also observe that the computational complexity of this criterion can be reduced from $O(2^n)$ to $O(n^2)$.

53D05, 53D42

1 Introduction

1.1 New obstructions to embeddings of four-dimensional polydisks

We investigate the question of when one convex toric symplectic four-manifold can be symplectically embedded into another. In particular, we obtain new sharp obstructions to symplectic embeddings of the four-dimensional polydisk $P(a,1)$ into the ball $B(c)$. In addition, we prove that the computational complexity in Hutchings [10] of obstructing symplectic embeddings of convex toric four-manifolds can be reduced.

Four-dimensional toric manifolds are defined as follows.

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Definition 1.1 Let Ω be a domain in the first quadrant of $\mathbb{R}^2$. We then associate to Ω a subset X_Ω of $\mathbb{C}^2$ defined by

$$X_\Omega = \big\{(z_1, z_2) \in \mathbb{C}^2 \mid (\pi|z_1|^2, \pi|z_2|^2) \in \Omega\big\}.$$

X_Ω is a symplectic manifold with symplectic form given by the restriction of the standard form on $\mathbb{C}^2$, namely

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

We call X_Ω the *toric domain* associated to Ω. Suppose that Ω is of the form

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le A, 0 \le y \le f(x)\big\},$$

where $f\colon [0, A] \to \mathbb{R}_{\ge 0}$ is a nonincreasing function. If f is concave, then we say that X_Ω is a *convex toric domain*. If f is convex, then we say that X_Ω is a *concave toric domain*.

Example 1.2 Let Ω be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(a,0)$, and $(0,b)$ for any $a, b > 0$. Then X_Ω is the 4–dimensional ellipsoid

$$E(a,b) = \left\{(z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \le 1\right\}.$$

When $a = b$, the manifold X_Ω is the 4–dimensional ball $B(a) = E(a,a)$. The ellipsoid $E(a,b)$ is both a concave and a convex toric domain, since Ω is the region lying beneath the line $f(x) = (-b/a)x + b$ in the first quadrant of $\mathbb{R}^2$.

Example 1.3 Let Ω be the rectangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(a,0)$, $(0,b)$, and (a,b) for any $a, b > 0$. Then X_Ω is the polydisk

$$P(a,b) = \big\{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \le a, \pi|z_2|^2 \le b\big\}.$$

The polydisk $P(a,b)$ is a convex toric domain, since Ω is the region lying beneath the constant function $f(x) = b$ on the interval $[0,a]$.

In dimension 4, progress has been made on understanding questions concerning symplectic embeddings. In [7], Hutchings associates to any symplectic four-manifold (X, ω) with (contact) boundary a sequence of real numbers

$$0 = c_0(X) \le c_1(X) \le c_2(X) \le \cdots$$

such that if X symplectically embeds into X', then

$$c_k(X) \le c_k(X') \quad \text{for all } k.$$

The c_k are called *ECH capacities* (here ECH stands for "embedded contact homology", which Hutchings uses to define the capacities). Work by Choi, Cristofaro-Gardiner, Frenkel, Hutchings, and Ramos [2] computed the ECH capacities of all concave toric domains, yielding sharp obstructions to certain symplectic embeddings of concave toric domains. Cristofaro-Gardiner [3] showed that ECH capacities give sharp obstructions to symplectic embeddings of any concave toric domain into any convex toric domain. His result generalizes the results of McDuff [11; 12] and Frenkel and Müller [4].

Obstructions via ECH capacities are suboptimal in the case of symplectic embeddings of a convex toric domain into a concave toric domain. For instance, the ECH capacities of polydisks and balls (which Hutchings explicitly computes in [7]) imply that there is no symplectic embedding of $P(2,1)$ into $B(c)$ for $c<2$. However, a result due to Hind and Lisi [5] indicates that $P(2,1)$ does not symplectically embed into $B(c)$ for any $c<3$.

For this reason, Hutchings studied embedded contact homology in a more refined way than is used to define the ECH capacities. As a result, he was able to give a new combinatorial criterion [10, Theorem 1.19] for obstructing symplectic embeddings which we will hereafter term *the Hutchings criterion*. The Hutchings criterion is a somewhat complicated combinatorial condition; we will defer a full description of it to the next section. Hutchings used this criterion to demonstrate several new bounds on embeddings of polydisks into balls, ellipsoids, and polydisks.

Our first result is the following extension of results by Hutchings [10, Theorem 1.4] and Hind and Lisi [5] on symplectic embeddings of polydisks into balls.

Theorem 1.4 *Let*

$$2 \le a \le \frac{\sqrt{7}-1}{\sqrt{7}-2} = 2.54858\ldots.$$

If $P(a,1)$ symplectically embeds into $B(c)$ then

$$c \ge 2+\frac{a}{2}.$$

Remark 1.5 The bound on c in this theorem is optimal: in [13, Proposition 4.3.9], Schlenk uses "symplectic folding" to construct a symplectic embedding $P(a,1)\hookrightarrow B(c)$ whenever $a>2$ and $c>2+a/2$.

Remark 1.6 Hutchings proved the statement of Theorem 1.4 for $2\le a\le 2.4$ using the Hutchings criterion and conjectured that its full statement could also be proven using the Hutchings criterion [9]. Our proof thus answers this conjecture in the affirmative.

The proof of Theorem 1.4 can be found in Section 3. In the appendix we discuss how extending these results for larger values of a is unlikely to be achieved via the Hutchings criterion or its improvement [10, Conjecture A.3] established by Choi [1]. For $a > 4$, it is known that there are symplectic embeddings of $P(a,1)$ into $B(c)$ for some values with $c < 2 + a/2$; see [13, Figure 7.2].

Our other result is Corollary 1.20, which pertains to the technical details of the Hutchings criterion. We give a combinatorial simplification of the Hutchings criterion for obstructing symplectic embeddings, reducing the computational complexity of verifying the existence of obstructions from $O(2^n)$ to $O(n^2)$. We state the result in Section 1.3 after reviewing the necessary background.

1.2 Review of convex generators

We begin by defining the principal combinatorial objects involved in stating the Hutchings criterion. Our exposition closely follows [10, Section 1.3].

Definition 1.7 A *convex integral path* Λ is a path in $\mathbb{R}^2$ such that:

- The endpoints of Λ are $(0, y(\Lambda))$ and $(x(\Lambda), 0)$ for some nonnegative integers $x(\Lambda)$ and $y(\Lambda)$.
- The path Λ is the graph of a piecewise linear concave function $f\colon [0, x(\Lambda)] \to [0, y(\Lambda)]$ with $f'(0) \le 0$, possibly together with a vertical line segment at the right.
- The vertices of Λ (ie the points at which its slope changes) are lattice points.

Definition 1.8 A *convex generator* is a convex integral path Λ such that:

- Each edge of Λ (ie each line segment between two vertices) is labeled e or h.
- Horizontal and vertical edges can only be labeled e.

Because we will work with convex generators frequently, we require a compact notation for them. For any nonnegative, coprime integers a and b and any positive integer m, we will denote by $e^m_{a,b}$ an edge of a convex generator that is labeled e and has displacement vector $(ma, -mb)$. Similarly, $h_{a,b}$ denotes an edge labeled h that has displacement vector $(a, -b)$, while $e^{m-1}_{a,b} h_{a,b}$ denotes an edge labeled h that has displacement vector $(ma, -mb)$. Since a convex generator is uniquely specified by the set of its edges, this notation provides an equivalence between a convex generator and a commutative formal product of symbols $e_{a,b}$ and $h_{a,b}$, where no two distinct factors $h_{a,b}$ and $h_{c,d}$ have $a = c$ and $b = d$ and where there are no factors of $h_{1,0}$ or $h_{0,1}$.

As explained in [10, Section 6], the boundary of any convex toric domain can be perturbed so that for its induced contact form and up to large action, the ECH generators correspond to these convex generators. Before continuing to draw parallels with ECH, we first describe a few useful aspects of convex generators.

Definition 1.9 Let Λ_1 and Λ_2 be convex generators. We then say that Λ_1 and Λ_2 *have no elliptic orbit in common* if, when we write out Λ_1 and Λ_2 as formal products, no factor of $e_{a,b}$ appears in both Λ_1 and Λ_2. Likewise, we say that Λ_1 and Λ_2 *have no hyperbolic orbit in common* if, when we write out Λ_1 and Λ_2 as formal products, no factor of $h_{a,b}$ appears in both Λ_1 and Λ_2.

If Λ_1 and Λ_2 are convex generators with no hyperbolic orbit in common, then we define the *product* $\Lambda_1 \cdot \Lambda_2$ to be the convex generator obtained by concatenating the formal product expressions of Λ_1 and Λ_2. This product operation is associative whenever it is defined.

There are several combinatorial quantities associated to a convex generator that will be of interest to us.

Definition 1.10 Let Λ be any convex generator. Then:

(1) The quantity $L(\Lambda)$ is the number of *lattice points interior to and on the boundary of* the region bounded by Λ and the x– and y–axes.
(2) The quantity $m(\Lambda)$ is the *total multiplicity* of all the edges of Λ, ie the total exponent of all factors of $e_{a,b}$ and $h_{a,b}$ in the formal product for Λ. Note that $m(\Lambda)$ is equal to one less than the number of lattice points on the path Λ.
(3) The quantity $h(\Lambda)$ is the number of edges of Λ labeled h.

Remarkably, one can actually express the ECH index in terms of the above combinatorial data associated to convex generators.

Definition 1.11 If Λ is a convex generator, we define the *ECH index* of Λ to be

$$I(\Lambda) = 2(L(\Lambda) - 1) - h(\Lambda).$$

Definition 1.12 Let Λ be a convex generator and let X_Ω be a convex toric domain. We define the *symplectic action* of Λ with respect to X_Ω by

$$A_\Omega(\Lambda) = A_{X_\Omega}(\Lambda) = \sum_{\nu \in \mathrm{Edges}(\Lambda)} \vec{\nu} \times p_{\Omega,\nu}.$$

Here, for any edge ν of Λ, we write $\vec{\nu}$ to denote the displacement vector of ν and $p_{\Omega,\nu}$ to denote any point on the line ℓ parallel to $\vec{\nu}$ and tangent to $\partial\Omega$. Tangency means that ℓ touches $\partial\Omega$ and that Ω lies entirely in one closed half-plane bounded by ℓ. Finally, $\vec{\nu}\times p_{\Omega,\nu}$ denotes the determinant of the matrix whose columns are given by the two vectors.

Next, we compute the symplectic action of any convex generator with respect to our favorite toric domains.

Example 1.13

- If $X_\Omega = P(a,b)$ is a polydisk, then for any convex generator Λ,
$$A_{P(a,b)}(\Lambda) = bx(\Lambda) + ay(\Lambda).$$
- If $X_\Omega = E(a,b)$ is an ellipsoid, then for any convex generator Λ, we have $A_{E(a,b)}(\Lambda) = c$, where the line $bx + ay = c$ is tangent to Λ at some point.

We have yet another definition, which is essential for computing ECH capacities combinatorially.

Definition 1.14 Let X_Ω be a convex toric domain. We say that a convex generator Λ with $I(\Lambda) = 2k$ for some integer k is *minimal* for X_Ω if

- all edges of Λ are labeled e, and
- for any other convex generator Λ' with all edges labeled e such that $I(\Lambda') = 2k$, we have
$$A_\Omega(\Lambda) < A_\Omega(\Lambda').$$

The symplectic action of minimal generators is related to ECH capacities as follows.

Remark 1.15 By [10, Proposition 5.6], if $I(\Lambda) = 2k$ and Λ is minimal for X_Ω, then $A_\Omega(\Lambda) = c_k(X_\Omega)$.

Our final definition will be key to understanding when one convex toric domain can be symplectically embedded into another convex toric domain.

Definition 1.16 Let X_Ω and $X_{\Omega'}$ be convex toric domains and let Λ and Λ' be convex generators. We write $\Lambda \leq_{X_\Omega, X_{\Omega'}} \Lambda'$ or $\Lambda \leq_{\Omega,\Omega'} \Lambda'$ if

(1) $I(\Lambda) = I(\Lambda')$,

(2) $A_\Omega(\Lambda) \leq A_{\Omega'}(\Lambda')$, and

(3) $x(\Lambda) + y(\Lambda) - \frac{1}{2}h(\Lambda) \geq x(\Lambda') + y(\Lambda') + m(\Lambda') - 1$.

In particular, if X_Ω symplectically embeds into X'_Ω, then the resulting cobordism between their (perturbed) boundaries implies that $\Lambda \le_{X_\Omega, X_{\Omega'}} \Lambda'$ is a necessary condition for the existence of an embedded irreducible holomorphic curve with ECH index zero between the ECH generators corresponding to Λ and Λ'. The inequality (3) is what ultimately allowed Hutchings to go "beyond" ECH capacities in his criterion. It emerges from the fact that every holomorphic curve must have nonnegative genus [10, Proposition 3.2].

We now have all the ingredients needed to state the Hutchings criterion and our modification.

1.3 A modification of the Hutchings criterion

The statement of the criterion we use to obstruct symplectic embeddings will be very similar to the one given by Hutchings in [10, Theorem 1.19]. Our modification reduces the amount of computation required to check the criterion, making the following result a formal consequence of Hutchings' original criterion.

Theorem 1.17 (modified Hutchings criterion) *Let X_Ω and $X_{\Omega'}$ be convex toric domains and Λ' be a minimal generator for $X_{\Omega'}$. Suppose that X_Ω symplectically embeds into $X_{\Omega'}$. Then there exists a convex generator Λ, a nonnegative integer n, and factorizations $\Lambda' = \Lambda'_1 \cdots \Lambda'_n$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$ such that:*

(i) $\Lambda_i \le_{\Omega,\Omega'} \Lambda'_i$ *for all i.*

(ii) *For all $i \neq j$, if $\Lambda'_i \neq \Lambda'_j$ or $\Lambda_i \neq \Lambda_j$, then Λ_i and Λ_j have no elliptic orbit in common.*

(iii) $I(\Lambda_i \cdot \Lambda_j) = I(\Lambda'_i \cdot \Lambda'_j)$ *for all $i \neq j$.*

Remark 1.18 The difference between Theorem 1.17 and the original Hutchings criterion [10, Theorem 1.19] is in the third item, where Hutchings' formulation reads:

(iii)$'$ *If S is any subset of $\{1, \dots, n\}$, then $I\big(\prod_{i\in S} \Lambda_i\big) = I\big(\prod_{i\in S} \Lambda'_i\big)$.*

We do not lose any information by replacing (iii) with (iii)$'$ in the Hutchings criterion because of the following proposition and corollary. Definitions of the terms appearing in the proposition and also the proof can be found in Sections 2.1–2.2.

Proposition 1.19 *Let $Z_1, \dots, Z_n$ be relative homology classes, and assume that $CZ^I_\tau(Z_1 + \cdots + Z_n) = CZ^I_\tau(Z_1) + \cdots + CZ^I_\tau(Z_n)$. Then*

$$I(Z_1 + \cdots + Z_n) = \sum_{i<j} I(Z_i + Z_j) - (n-2) \sum_{i=1}^{n} I(Z_i).$$

Moreover, the assumption of Proposition 1.19 is satisfied for the special contact form arising on the boundary of convex toric domains, by the discussion in Step 4 of the proof of [10, Lemma 5.4]. We thus obtain the following corollary since $I(\Lambda)$ is by definition I of any relative homology class between Λ and the empty set.

Corollary 1.20 *Let $\{\Lambda_i'\}_{i=1}^n$ and $\{\Lambda_i\}_{i=1}^n$ be two sets of convex generators such that the Λ_i' have no hyperbolic orbit in common and the Λ_i have no hyperbolic orbit in common. Suppose that for any $1 \le i \le n$,*

$$I(\Lambda_i) = I(\Lambda_i'),$$

and moreover that for any $i \ne j$,

$$I(\Lambda_i \cdot \Lambda_j) = I(\Lambda_i' \cdot \Lambda_j').$$

Then, for any subset $S \subseteq \{1, 2, \ldots, n\}$,

$$I\Big(\prod_{i \in S} \Lambda_i\Big) = I\Big(\prod_{i \in S} \Lambda_i'\Big).$$

We note that while Theorem 1.17 is technically weaker than the original Hutchings criterion, Corollary 1.20 demonstrates that it is actually equivalent to the original Hutchings criterion, [10, Theorem 1.19]. Thus if we want to check whether some Λ obstructs a certain symplectic embedding, it is enough to check whether the conditions in Theorem 1.17 can be satisfied.

Remark 1.21 Checking that (iii)$'$ is satisfied requires comparing two indices of convex generators in $O(2^n)$ different scenarios. Checking that (iii) is satisfied requires comparing two indices in $O(n^2)$ different scenarios. This vast reduction in complexity is beneficial in many circumstances.

Outline of paper Properties of the ECH index of two convex generators, including the proof of Proposition 1.19, are given in Section 2. The proof of the main embedding result, Theorem 1.4, is given in Section 3. The appendix contains a brief discussion on the difficulties in extending Theorem 1.4 via the Hutchings criterion.

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2 Index calculations

In this section we prove Proposition 1.19. A formula for the index of the product of two convex generators is also given via purely combinatorial methods in Section 2.3.

2.1 Preliminary definitions

Let Y be a closed 3–dimensional manifold with a nondegenerate contact form λ. Let $\xi = \ker(\lambda)$ denote the associated contact structure, and let R denote the Reeb vector field determined by λ. A *Reeb orbit* is a map $\gamma\colon \mathbb{R}/T\mathbb{Z} \to Y$, for some $T > 0$, such that $\gamma'(t) = R(\gamma(t))$. Let $\varphi_t\colon Y \to Y$ denote the time-t Reeb flow. The derivative of φ_t at $\gamma(0)$ restricts to a map

$$d\varphi_t\colon (\xi_{\gamma(0)}, d\lambda) \to (\xi_{\gamma(t)}, d\lambda).$$

The *linearized return map* is the map

$$P_\gamma := d\varphi_T\colon (\xi_{\gamma(0)}, d\lambda) \to (\xi_{\gamma(0)}, d\lambda). \tag{1}$$

We say that γ is *elliptic* if the eigenvalues of P_γ are on the unit circle, *positive hyperbolic* if the eigenvalues of P_γ are positive, and *negative hyperbolic* if the eigenvalues of P_γ are negative.

An *orbit set* is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$, where the α_i are distinct embedded Reeb orbits and the m_i are positive integers. We call m_i the *multiplicity* of α_i in α. The homology class of the orbit set α is defined by

$$[\alpha] = \sum_i m_i[\alpha_i] \in H_1(Y).$$

The orbit set α is *admissible* if $m_i = 1$ whenever α_i is positive or negative hyperbolic.

Let τ be a trivialization of ξ over γ, namely an isomorphism of symplectic vector bundles

$$\tau\colon \gamma^*\xi \xrightarrow{\simeq} (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^2.$$

With respect to this trivialization, the linearized flow $(d\varphi_t)_{t\in[0,T]}$ induces an arc of symplectic matrices $P\colon [0,T]\to \mathrm{Sp}(2)$ defined by

$$P_t = \tau(t)\circ d\varphi_t \circ \tau(0)^{-1}.$$

To each arc of symplectic matrices $\{P_t\}_{t\in[0,T]}$ with $P_0 = 1$ and P_T nondegenerate, there is an associated Conley–Zehnder index $CZ(\{P_t\}_{t\in[0,T]})\in\mathbb{Z}$. We define the *Conley–Zehnder index* of γ with respect to τ by

$$CZ_\tau(\gamma) = CZ(\{P_t\}_{t\in[0,T]}).$$

This depends only on the homotopy class of the trivialization τ.

2.2 The ECH index

Let $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$ be Reeb orbit sets in the same homology class, $\sum_i [\alpha_i] = \sum_j [\beta_j] = \Gamma \in H_1(M)$. Let $H_2(Y,\alpha,\beta)$ denote the set of 2–chains Z in Y with $\partial Z = \sum_i m_i\alpha_i - \sum_j n_j\beta_j$, modulo boundaries of 3–chains. The set $H_2(Y,\alpha,\beta)$ is an affine space over $H_2(Y)$.

Given $Z\in H_2(Y,\alpha,\beta)$, we define the *ECH index* to be

$$I(\alpha,\beta,Z) = c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k) - \sum_j \sum_{k=1}^{n_j} CZ_\tau(\beta_j^k),$$

where Q_τ is the relative intersection pairing defined in [8, Section 3.3] and $c_\tau(Z)$ is the relative first Chern class [8, Section 3.2] of ξ over Z with respect to τ. The relative intersection pairing is an analogue of the intersection number $[C]\cdot[C]$ for closed curves C. As a shorthand, we define

$$CZ_\tau^I(\alpha) = \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k).$$

The ECH index does not depend on the choice of trivialization τ.

We note that the Chern class term is linear in the homology class and the relative intersection term is quadratic. The "total Conley–Zehnder" index term CZ_τ^I typically behaves in a complicated way with respect to addition of homology classes. However, we can conclude for the special contact form arising on the boundary of convex toric domains that the total Conley–Zehnder index term is linear by the discussion in Step 4 of the proof of [10, Lemma 5.4]. The addition operation on homology classes to which we refer is spelled out in [6, Lemma 3.10]. Thus, it is reasonable that one only needs to consider ECH indices of one- and two-term products.

Next we restate and prove Proposition 1.19.

Proposition 2.1 *Let* $Z_1, \dots, Z_n$ *be relative homology classes, and assume that* $CZ^I_\tau(Z_1 + \dots + Z_n) = CZ^I_\tau(Z_1) + \dots + CZ^I_\tau(Z_n)$. *Then*

$$I(Z_1 + \dots + Z_n) = \sum_{i<j} I(Z_i + Z_j) - (n-2)\sum_{i=1}^{n} I(Z_i).$$

Proof Let L_τ denote the sum $c_\tau + CZ^I_\tau$, which is linear under our assumptions. Then

$$I\Big(\sum_{i=1}^{n} Z_i\Big) = L_\tau\Big(\sum_{i=1}^{n} Z_i\Big) + Q_\tau\Big(\sum_{i=1}^{n} Z_i\Big) \tag{2}$$

$$= \sum_{i=1}^{n}[L_\tau(Z_i) + Q_\tau(Z_i)] + 2\sum_{i=1}^{n} Q_\tau(Z_i, Z_j) \tag{3}$$

$$= \sum_{i=1}^{n} I(Z_i + Z_j) - (n-2)\sum_{i=1}^{n} I(Z_i). \tag{4}$$

The second line (3) holds here because of the linearity of L_τ, the quadratic property of Q_τ by [6, Equation 3.11], and the fact that $Q_\tau(Z, \cdot)$ is linear in $\cdot$ by definition. The third line (4) holds because the $2Q_\tau(Z_i, Z_j)$ terms coming from the terms in the first sum each appear exactly once, while the terms in the first sum that only depend on Z_i all appear $n-1$ times. □

2.3 The index of the product of two convex generators

While we have already proven Corollary 1.20, we include the following purely combinatorial description of the index of the product of two convex generators. We expect this to be useful to the future study of obstructing symplectic embeddings of other convex toric domains into concave toric domains. Before giving the general formula of the index of the product of two convex generators, we first provide an example to elucidate the combinatorial intuition.

Given a convex generator Λ, we define $\mathbb{A}(\Lambda)$ to be the *area of* P_Λ. Similarly, if ν is an edge of Λ we define $\mathbb{A}_\nu(\Lambda)$ to be the *area of the portion of* P_Λ *lying underneath* ν. We will also need some additional notation as follows. For any convex generator Λ and any edge ν of Λ, we write ν_x and ν_y for the x– and y–coordinates of the displacement vector of ν. We also define the *slope* of ν to be

$$\mu(\nu) = \frac{\nu_y}{\nu_x}.$$

Example 2.2 Let $\Lambda = e_{1,0}^3 e_{2,1} e_{1,3}$, and let $\Gamma = e_{2,1} e_{0,1}^2$. Using (7) along with the additivity of b and h, we have

$$I(\Lambda\cdot\Gamma) = 2\mathbb{A}(\Lambda\cdot\Gamma) + b(\Lambda\cdot\Gamma) - h(\Lambda\cdot\Gamma)$$
$$= 2\mathbb{A}(\Lambda\cdot\Gamma) + b(\Lambda) + b(\Gamma) - h(\Lambda) - h(\Gamma). \tag{5}$$

We can compute $\mathbb{A}(\Lambda\cdot\Gamma)$ by summing the area under each of the edges of $\Lambda\cdot\Gamma = e_{1,0}^3 e_{2,1}^2 e_{1,3} e_{0,1}^2$.

For any edge ν of Λ, the region underneath ν in $\Lambda\cdot\Gamma$ will be essentially the same shape as the region under ν in Λ, except that ν may be higher up (ie its endpoints may have larger y–coordinates) in the product $\Lambda\cdot\Gamma$. To see this, notice that the y–coordinate of the lower-right endpoint of ν in Λ is

$$y_\Lambda = \sum_{\substack{\sigma\in\mathrm{Edges}\,(\Lambda)\\ \mu(\sigma)<\mu(\nu)}} \sigma_y,$$

while the y–coordinate of the lower-right endpoint of ν in $\Lambda\cdot\Gamma$ is

$$y_{\Lambda\cdot\Gamma} = \sum_{\substack{\sigma\in\mathrm{Edges}\,(\Lambda\cdot\Gamma)\\ \mu(\sigma)<\mu(\nu)}} \sigma_y.$$

Thus, every edge σ of Γ that is steeper than ν will contribute a term of σ_y to $y_{\Lambda\cdot\Gamma}$ which is not in y_Λ, so that the edge ν in $\Lambda\cdot\Gamma$ will be translated upwards by σ_y relative to the position of ν in Λ. This translation is equivalent to taking the region beneath ν in Λ and adding a rectangle to the bottom of it. So $A_{\Lambda\cdot\Gamma}(\nu)$ will be equal to $A_\Lambda(\nu)$ plus the area of several rectangles added beneath ν. Thinking of area in this way allows us to break up the area under each edge in $\Lambda\cdot\Gamma$ into individual contributions from different edges, as shown in Figure 1.

One important feature of this figure is how we split up the area under the edge $e_{2,1}^2$ in $\Lambda\cdot\Gamma$. Because both Λ and Γ have an edge of slope $-\frac{1}{2}$, we treat these as separate and compute areas underneath them individually, even though they combine to form one edge in $\Lambda\cdot\Gamma$. This is important because whichever copy of $e_{2,1}$ is on the left (in the figure we've shown it as the one from Λ, but it would not have affected the answer if we'd put the one from Γ on the left instead) has one rectangle underneath it contributed by the other copy of $e_{2,1}$.

We can now compute $\mathbb{A}(\Lambda\cdot\Gamma)$ by summing up the area contributions of each region of $\Lambda\cdot\Gamma$ shown in Figure 1. Let R be the sum of the areas of all the rectangles added

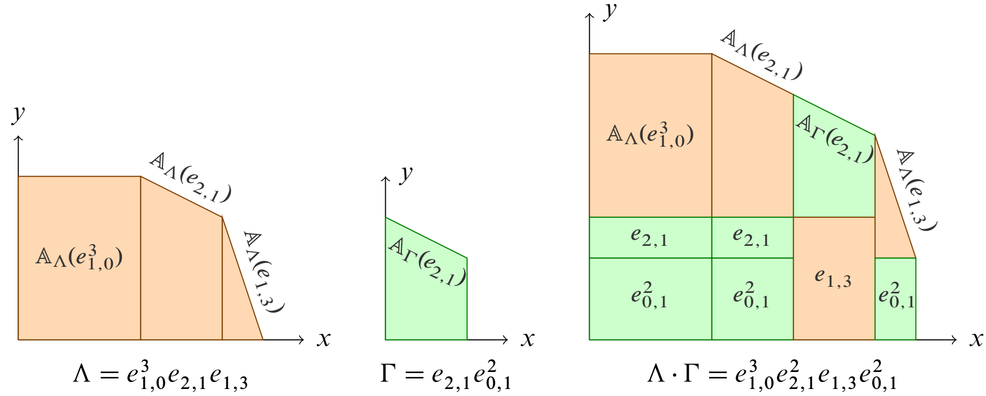

Figure 1: The graph on the right shows $\Lambda\cdot\Gamma$ broken up into pieces of area from Λ and Γ along with rectangles added by taking the product. Rectangles that were added by taking the product are labeled with the edge that necessitated that rectangle. The graphs on the left and center show Λ and Γ for comparison.

by taking the product as described above (that is, all the rectangles underneath $\Lambda\cdot\Gamma$ in the figure except the one labeled $\mathbb{A}_\Lambda(e^3_{1,0})$). Then

$$\begin{aligned}\mathbb{A}(\Lambda\cdot\Gamma) &= \mathbb{A}_\Lambda(e^3_{1,0})+\mathbb{A}_\Lambda(e_{2,1})+\mathbb{A}_\Gamma(e_{2,1})+\mathbb{A}_\Lambda(e_{1,3})+R\\ &= \mathbb{A}(\Lambda)+\mathbb{A}(\Gamma)+R.\end{aligned}$$

Plugging back into (5) and applying (7) then gives

$$\begin{aligned}I(\Lambda\cdot\Gamma) &= 2(\mathbb{A}(\Lambda)+\mathbb{A}(\Gamma)+R)+b(\Lambda)+b(\Gamma)-h(\Lambda)-h(\Gamma)\\ &= (2\mathbb{A}(\Lambda)+b(\Lambda)-h(\Lambda))+(2\mathbb{A}(\Gamma)+b(\Gamma)-h(\Gamma))+2R\\ &= I(\Lambda)+I(\Gamma)+2R.\end{aligned}\tag{6}$$

Equation (6) is precisely the sort of expression we want for the index of the product of two convex generators. By generalizing the above arguments as follows, we obtain a formula for the product of two arbitrary generators with no hyperbolic orbit in common, with an explicit expression for R.

Proposition 2.3 *Let Λ and Γ be any two convex generators that have no hyperbolic orbit in common. Then*

$$I(\Lambda\cdot\Gamma)=I(\Lambda)+I(\Gamma)+2\sum_{\nu\in\mathrm{Edges}(\Lambda)}\ \sum_{\substack{\sigma\in\mathrm{Edges}(\Gamma)\\ \mu(\sigma)\le\mu(\nu)}}\nu_x\sigma_y+2\sum_{\nu\in\mathrm{Edges}(\Gamma)}\ \sum_{\substack{\sigma\in\mathrm{Edges}(\Lambda)\\ \mu(\sigma)<\mu(\nu)}}\nu_x\sigma_y.$$

3 On symplectic embeddings of a polydisk into a ball

Our main goal in this section is to prove Theorem 1.4, that for $2 \le a \le (\sqrt{7}-1)/(\sqrt{7}-2)$, if $P(a,1)$ symplectically embeds into $B(c)$ then $c \ge 2 + a/2$. Before proceeding we need some preliminary results. In Section 3.1 we provide some notation and prove a useful formula for the index of a convex generator via Pick's theorem. In Section 3.2 we prove a necessary result regarding the nature of repeated factors in the Hutchings criterion.

With these results in hand, the plan of attack will be to assume that the statement of Theorem 1.4 is false and apply the modified Hutchings criterion, Theorem 1.17, to the generator $\Lambda' = e_{1,1}^d$ for a suitable choice of d. By [10, Lemma 2.1] this is a minimal generator for $B(c)$. This gives us an integer n, a convex generator Λ, and factorizations $\Lambda' = \Lambda'_1 \cdots \Lambda'_n$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$. To obtain a contradiction, we show that no choice of the Λ'_i and Λ_i is possible. We do so in three steps:

(1) We prove that for sufficiently large m, there is no convex generator Λ such that $\Lambda \le_{P(a,1),B(c)} e_{1,1}^m$. If we choose d to be very large, this will imply that we cannot have $n = 1$. This step is the content of Proposition 3.7, which is proved in Section 3.3.

(2) We use Proposition 3.4 to show that there cannot exist any $i \ne j$ such that $\Lambda'_i = \Lambda'_j$ and $\Lambda_i = \Lambda_j$. In conjunction with Step 1, this will imply that the set of all possible values of n is bounded. This step is the content of Proposition 3.8, which is proved in Section 3.4.

(3) Using Steps 1 and 2, we show that there is a maximum possible index of the product $\prod_{i=1}^n \Lambda'_i$ which does not depend on d. On the other hand, this product must be equal to $\Lambda' = e_{1,1}^d$. Because of Step 1, we will be able to pick d to be arbitrarily large, which will make the index of Λ' arbitrarily large, resulting in a contradiction. This step is contained in the proof of Theorem 1.4, which is given in Section 3.5.

In the appendix we discuss the difficulties in using the Hutchings criterion to extend Theorem 1.4.

3.1 A helpful lemma via Pick's theorem

We first fix some notation and then prove a useful formula for the index of a convex generator. For any convex generator Λ, let P_Λ be the region bounded by Λ and the x– and y–axes. Recall that we define $\mathbb{A}(\Lambda)$ to be the *area of* P_Λ.

Definition 3.1 For any convex generator Λ, we define

$$b(\Lambda) = x(\Lambda) + y(\Lambda) + m(\Lambda).$$

Recall that the formal product 1 is the path Λ with no edges which starts and ends at $(0,0)$. Note that $b(\Lambda)$ computes the lattice points on the boundary of any $\Lambda \neq 1$ if and only if Λ does not lie entirely on one axis.

Remark 3.2 The operator b is additive under products of convex generators. In other words, for any convex generators Λ and Γ, we have

$$\begin{aligned} b(\Lambda \cdot \Gamma) &= x(\Lambda \cdot \Gamma) + y(\Lambda \cdot \Gamma) + m(\Lambda \cdot \Gamma) \\ &= x(\Lambda) + x(\Gamma) + y(\Lambda) + y(\Gamma) + m(\Lambda) + m(\Gamma) \\ &= b(\Lambda) + b(\Gamma). \end{aligned}$$

Using the above notation, we can now prove a useful formula for the index of a convex generator.

Lemma 3.3 *Let Λ be any convex generator. Then*

$$I(\Lambda) = 2\mathbb{A}(\Lambda) + b(\Lambda) - h(\Lambda). \tag{7}$$

Proof First, suppose that Λ lies entirely on one axis. If $\Lambda = e_{1,0}^x$ for some $x \geq 0$, we have

$$I(\Lambda) = 2x = 2 \cdot 0 + 2x - 0 = 2\mathbb{A}(\Lambda) + b(\Lambda) - h(\Lambda).$$

The case where $\Lambda = e_{0,1}^y$ for some $y \geq 0$ is analogous.

Next, suppose that Λ does not lie entirely on one axis. Since P_Λ is the region bounded by Λ and the x– and y–axes, Pick's theorem yields

$$\mathbb{A}(\Lambda) = i(P_\Lambda) + \tfrac{1}{2} b(P_\Lambda) - 1,$$

where $i(P_\Lambda)$ is the number of lattice points in the interior of P_Λ and $b(P_\Lambda)$ is the number of lattice points on the boundary of P_Λ. Rearranging and noting that $L(\Lambda) = i(P_\Lambda) + b(P_\Lambda)$, we obtain

$$L(\Lambda) = i(P_\Lambda) + b(P_\Lambda) = \mathbb{A}(\Lambda) + \tfrac{1}{2} b(P_\Lambda) + 1 = \mathbb{A}(\Lambda) + \tfrac{1}{2} b(\Lambda) + 1,$$

where the last equality follows from the fact that Λ does not lie entirely on one axis. We can then use this expression for $L(\Lambda)$ to compute $I(\Lambda)$:

$$I(\Lambda) = 2(L(\Lambda) - 1) - h(\Lambda) = 2\mathbb{A}(\Lambda) + b(\Lambda) - h(\Lambda). \tag{8}$$

□

3.2 Repeated factors in the Hutchings criterion

We will prove the following proposition.

Proposition 3.4 *Let Λ and Λ' be nontrivial convex generators with no edges labeled h. Suppose that Λ and Λ' satisfy* (1) *and* (3) *of Definition 1.16 and that $I(\Lambda \cdot \Lambda) = I(\Lambda' \cdot \Lambda')$. Then*

$$8\mathbb{A}(\Lambda') \leq (b(\Lambda') - 1)^2.$$

Moreover, Λ must be of the form $e_{x,y}$, where $x, y \in \mathbb{Z}_{>0}$ are coprime and satisfy

$$xy = 2\mathbb{A}(\Lambda') \quad \text{and} \quad x + y = b(\Lambda') - 1.$$

Equivalently, x and y must be nonnegative coprime integers such that

$$\{x, y\} = \left\{ \frac{b(\Lambda')-1 \pm \sqrt{(b(\Lambda')-1)^2 - 8\mathbb{A}(\Lambda')}}{2} \right\}. \tag{9}$$

Proof We will make repeated use of (7) of Lemma 3.3. Using (7) along with the additivity of b, we get

$$I(\Lambda \cdot \Lambda) = 2\mathbb{A}(\Lambda \cdot \Lambda) + b(\Lambda \cdot \Lambda) - h(\Lambda \cdot \Lambda) = 2\mathbb{A}(\Lambda \cdot \Lambda) + 2b(\Lambda). \tag{10}$$

Recall that P_Λ denotes the region bounded by Λ and the x– and y–axes. Then the region bounded by $\Lambda \cdot \Lambda$ and the x– and y–axes is P_Λ dilated by a factor of 2, which has 4 times the area of P_Λ; ie

$$\mathbb{A}(\Lambda \cdot \Lambda) = 4\mathbb{A}(\Lambda). \tag{11}$$

Substituting (11) into (10) and using (7) again yields

$$I(\Lambda \cdot \Lambda) = 8\mathbb{A}(\Lambda) + 2b(\Lambda) = 4\mathbb{A}(\Lambda) + 2(2\mathbb{A}(\Lambda) + b(\Lambda)) = 4\mathbb{A}(\Lambda) + 2I(\Lambda).$$

Likewise for Λ' we obtain

$$I(\Lambda' \cdot \Lambda') = 4\mathbb{A}(\Lambda') + 2I(\Lambda').$$

Because we assumed $I(\Lambda \cdot \Lambda) = I(\Lambda' \cdot \Lambda')$, we have

$$4\mathbb{A}(\Lambda) + 2I(\Lambda) = 4\mathbb{A}(\Lambda') + 2I(\Lambda').$$

Since $I(\Lambda) = I(\Lambda')$, we have

$$\mathbb{A}(\Lambda) = \mathbb{A}(\Lambda'). \tag{12}$$

Now, because of (7), we have

$$I(\Lambda) = 2\mathbb{A}(\Lambda) + b(\Lambda) = I(\Lambda') = 2\mathbb{A}(\Lambda') + b(\Lambda').$$

Combining this equation with (12) gives

$$b(\Lambda) = x(\Lambda) + y(\Lambda) + m(\Lambda) = b(\Lambda'). \tag{13}$$

On the other hand, the fact that $\Lambda \leq_{\Omega,\Omega'} \Lambda'$ implies that

$$x(\Lambda) + y(\Lambda) \geq b(\Lambda') - 1. \tag{14}$$

Since $m(\Lambda) > 0$, the only way that (13) and (14) can simultaneously hold is if (14) is an equality and we have $m(\Lambda) = 1$. So Λ must have the form $e_{x,y}$, where $\gcd(x, y) = 1$. This allows us to compute properties of Λ explicitly, so that (14) becomes

$$x(\Lambda) + y(\Lambda) = x + y = b(\Lambda') - 1, \tag{15}$$

and (12) becomes

$$\mathbb{A}(\Lambda) = \frac{xy}{2} = \mathbb{A}(\Lambda')$$

or equivalently

$$xy = 2\mathbb{A}(\Lambda'). \tag{16}$$

Using (15) and (16) to solve for x and y yields (9). Finally, we note that since x and y are real, the square roots in (9) must be real. □

Remark 3.5 There are a few interesting interactions between the conditions of Theorem 1.17 and Proposition 3.4. For instance, Proposition 3.4 allows us to rewrite (ii) of Theorem 1.17 as:

(ii) *For all $i \neq j$, if Λ_i and Λ_j have any elliptic orbit $e_{x,y}$ in common, then $\Lambda_i = \Lambda_j = e_{x,y}$.*

In addition, by arguing as in the proof of Theorem 1.4, one can sometimes use Theorem 1.17(i) along with Proposition 3.4 to prove that the set of possible values of $I(\Lambda')$ is bounded. This type of argument appears in Section 3.5.

3.3 Elimination of sufficiently large convex generators

We first prove some useful inequalities on the x and y endpoints of certain convex generators.

Lemma 3.6 *Let $a > 1$ and $c < 2 + a/2$, and suppose d and Λ are such that $\Lambda \le_{P(a,1),B(c)} e_{1,1}^{d}$. Then*

$$x(\Lambda) < \Big(2+\frac{a}{2}\Big)d - ay(\Lambda) \tag{17}$$

and

$$y(\Lambda) < \frac{d(a-2)+2}{2(a-1)}. \tag{18}$$

Proof By Example 1.13, we have

$$A_{P(a,1)}(\Lambda) = x(\Lambda) + ay(\Lambda).$$

Our assumptions tell us

$$x(\Lambda) + ay(\Lambda) = A_{P(a,1)}(\Lambda) \le A_{B(c)}(e_{1,1}^{d}) = cd < \Big(2+\frac{a}{2}\Big)d \tag{19}$$

and

$$x(\Lambda) + y(\Lambda) \ge x(e_{1,1}^{d}) + y(e_{1,1}^{d}) + m(e_{1,1}^{d}) - 1 = 3d - 1. \tag{20}$$

We solve for $x(\Lambda)$ in (19), obtaining

$$x(\Lambda) < \Big(2+\frac{a}{2}\Big)d - ay(\Lambda).$$

Combining (19) and (20) gives

$$3d - 1 + (a-1)y(\Lambda) \le x(\Lambda) + ay(\Lambda) < \Big(2+\frac{a}{2}\Big)d.$$

Solving for $y(\Lambda)$ shows

$$y(\Lambda) < \frac{d\big(\frac{a}{2}-1\big)+1}{a-1} = \frac{d(a-2)+2}{2(a-1)}.$$ □

We now use the above lemma to eliminate sufficiently large convex generators from consideration in the proof of Theorem 1.4.

Proposition 3.7 *Let*

$$2 \le a < \frac{\sqrt{7}-1}{\sqrt{7}-2},$$

and suppose that $c < 2 + a/2$. Then there exists some $d_a \ge 1$ such that for any $d > d_a$ and any convex generator Λ, we have $\Lambda \not\le_{P(a,1),B(c)} e_{1,1}^{d}$.

Proof Fix some d, and suppose there exists $\Lambda \leq e^d_{1,1}$. Let $x = x(\Lambda)$ and $y = y(\Lambda)$. Because Λ is convex, it lies inside the rectangle $[0, x] \times [0, y]$. Thus, the maximum possible value of $L(\Lambda)$ occurs when Λ contains all the lattice points inside this rectangle, and the largest $I(\Lambda)$ could be is when Λ contains all these lattice points and has no edges labeled h. Noting also that $I(\Lambda) = I(e^d_{1,1}) = d(d+3)$, we see that

$$2((x+1)(y+1)-1) = 2(x+1)(y+1)-2 \geq I(\Lambda) = d(d+3)$$

or equivalently

$$0 \geq d(d+3)+2-2(x+1)(y+1).$$

The substitution of (17) into this equation yields

$$0 > 2ay^2 - y((4+a)d+2-2a) + d(d+3) - (4+a)d. \tag{21}$$

We now wish to substitute (18) into (21), while still maintaining a valid inequality. This is permissible provided the right-hand side of (21) is nonincreasing with respect to increasing y. Notice that the derivative of the right-hand side of (21) with respect to y is

$$4ay-(4+a)d-2+2a.$$

Substituting (18) into this expression gives us

$$4ay-(4+a)d-2+2a < \frac{d(a^2-7a+4)+2(a^2+1)}{a-1}. \tag{22}$$

Now, a^2-7a+4 has roots $(7 \pm \sqrt{33})/2 \approx 0.628,\ 6.372$. Since a is between these two roots, we have $a^2-7a+4 < 0$. So the expression in (22) will be negative for all d above some sufficiently large value d_1. In this case, we can substitute (18) into the right-hand side of (21) and multiply by $2(a-1)^2$ to obtain

$$0 > (-3a^2+10a-6)d^2 - 2(2a^2+a-1)d + 4(a^2-a+1). \tag{23}$$

The coefficient of d^2 in (23) is negative for sufficiently large a, and its roots are $(5 \pm \sqrt{7})/3 \approx 0.7848,\ 2.5486$. Note that $(5+\sqrt{7})/3 = (\sqrt{7}-1)/(\sqrt{7}-2)$. Because our value of a is between these two roots, we can conclude that the coefficient of d^2 is positive. Thus, if d is larger than some sufficiently large value d_2, the right-hand side of (23) will be positive, a contradiction.

We have shown that if $d > d_1$ and $d > d_2$, then the existence of Λ results in a contradiction. Since d_1 and d_2 depend only on a by construction, setting $d_a = \max\{d_1, d_2\}$ now yields the desired statement. □

3.4 Elimination of repeated factors of convex generators

Proposition 3.8 *Let $2 \le a \le 3$, $c < 2 + a/2$, and $d \ge 1$. Then, for any convex generator Λ, at least one of the following holds:*

(i) $\Lambda \not\leq_{P(a,1),B(c)} e_{1,1}^{d}$.

(ii) $I(\Lambda \cdot \Lambda) \neq I(e_{1,1}^{2d})$.

Proof To obtain a contradiction, suppose there exists a Λ such that $\Lambda \le_{P(a,1),B(c)} e_{1,1}^{d}$ and $I(\Lambda \cdot \Lambda) = I(e_{1,1}^{2d})$. We can then apply Proposition 3.4 with $\Lambda' = e_{1,1}^{d}$. Noting that $A(\Lambda') = d^2/2$ and $b(\Lambda') = 3d$, we get $\Lambda = e_{x,y}$, where

$$x = \frac{3d-1 \pm \sqrt{5d^2-6d+1}}{2} \tag{24}$$

and

$$y = \frac{d^2}{x}. \tag{25}$$

On the other hand, $\Lambda \le_{P(a,1),B(c)} e_{1,1}^{d}$ implies that

$$x + ay = x(\Lambda) + ay(\Lambda) = A_{P(a,1)}(\Lambda) \le A_{B(c)}(e_{1,1}^{d}) = cd < \Big(2 + \frac{a}{2}\Big)d.$$

Substituting in our expression (25) for y and multiplying by x gives

$$x^2 + ad^2 < \Big(2 + \frac{a}{2}\Big)xd.$$

We then substitute in our expression (24) for x and multiply by 4 to get

$$(3d-1)^2 \pm (6d-2)\sqrt{5d^2-6d+1} + 5d^2 - 6d + 1 + 4ad^2 < (4+a)d\big(3d - 1 \pm \sqrt{5d^2-6d+1}\big)$$

or equivalently

$$(2+a)d^2 + (a-8)d + 2 \pm (2d - 2 - ad)\sqrt{5d^2-6d+1} < 0.$$

The left-hand side of this equation can be factored:

$$\big(-d - 1 \pm \sqrt{5d^2-6d+1}\big)\big((3-a)d - 1 \pm \sqrt{1-6d+5d^2}\big) < 0. \tag{26}$$

The zeros of the left factor (if they exist) occur when

$$(-d-1)^2 = 5d^2 - 6d + 1,$$

ie when $d = 0$ or $d = 2$. Likewise, the zeros of the right factor (if they exist) occur when

$$((3-a)d-1)^2 = 5d^2 - 6d + 1$$

or equivalently when

$$d((-a^2+6a-4)d-2a) = 0.$$

This equation holds when $d = 0$ and when $d = 2a/(-a^2+6a-4)$. Note that for all $2 \le a \le 3$, we have $1 \le 2a/(-a^2+6a-4) < 2$.

Suppose the sign of the square roots in (26) is positive. Then, both factors in (26) go to ∞ as $d \to \infty$, and both possible zeros of both factors are actually zeros of these factors. If $d \ge 2$, then d is at least as large as all of the zeros of the left-hand side of (26), which means that the left-hand side of (26) is nonnegative, a contradiction. The only remaining option is $d = 1$. *In this case, the left-hand side of* (26) *is again nonnegative, a contradiction.*

Next, suppose the sign of the square roots in (26) is negative. Then, both factors of the left-hand side of (26) go to $-\infty$ as $d \to \infty$, and none of the possible zeros of the left-hand side of (26) is an actual zero. This implies that the left-hand side of (26) is always positive, a contradiction. □

3.5 Proof of Theorem 1.4

Throughout this proof, the symbol $\le$ between two convex generators abbreviates the symbol $\le_{P(a,1),B(c)}$.

Proof of Theorem 1.4 Suppose by way of contradiction that $c < 2 + a/2$ and that $P(a,1)$ symplectically embeds into $B(c)$. By Proposition 3.7, there exists some d_a such that for any $d > d_a$, there is no convex generator Λ satisfying $\Lambda \le e_{1,1}^d$. For any $d \in \mathbb{Z}_{>0}$, define

$$N_d = \#\{\Lambda \mid \Lambda \le e_{1,1}^d\},$$

and let

$$N = \sum_{d=1}^{d_a} dN_d.$$

Note that for any d, there are a finite number of convex generators with index equal to $I(e_{1,1}^d)$, which implies that the N_d and N are finite.

Now, fix any integer $D > N$. The generator $\Lambda' = e_{1,1}^D$ is minimal for $B(c)$ by [10, Lemma 2.1]. So we can apply Theorem 1.17 to obtain a convex generator Λ, an integer n, and factorizations $\Lambda' = \Lambda'_1 \cdots \Lambda'_n$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$ satisfying the three numbered conditions of Theorem 1.17.

Suppose that there exists some $i \neq j$ such that $\Lambda'_i = \Lambda'_j$ and $\Lambda_i = \Lambda_j$. Then let $\Gamma = \Lambda_i = \Lambda_j$, and write $\Lambda'_i = \Lambda'_j = e^d_{1,1}$ for some d. Condition (i) of Theorem 1.17 implies that $\Gamma \leq e^d_{1,1}$, and condition (iii) of Theorem 1.17 implies

$$I(\Gamma \cdot \Gamma) = I(\Lambda'_i \cdot \Lambda'_j) = I(e^{2d}_{1,1}).$$

However, the values of Γ and d then contradict the statement of Proposition 3.8. So, for all $i \neq j$, we must have either $\Lambda'_i \neq \Lambda'_j$ or $\Lambda_i \neq \Lambda_j$.

We claim that with this constraint, it is impossible to have $I(\Lambda') = I\big(\prod_{i=1}^n \Lambda'_i\big)$:

As before, by Proposition 3.7, there exists some d_a such that for any $d > d_a$, there is no convex generator Λ satisfying $\Lambda \leq e^d_{1,1}$. Thus for all $d > d_a$,

$$N_d = \#\{\Lambda \mid \Lambda \leq e^d_{1,1}\} = 0.$$

Assuming $d \leq d_a$, the maximum possible value of $I\big(\prod_{i=1}^n \Lambda'_i\big)$ must then occur when there is precisely one choice of i such that $\Lambda'_i = e^d_{1,1}$ and $\Lambda_i = \eta$, for any $\eta \leq e^d_{1,1}$.

When $\Lambda'_i = e^d_{1,1}$ and $\Lambda_i = \eta$, we obtain

$$I\Big(\prod_{i=1}^n \Lambda'_i\Big) = I\Big(\prod_{d=1}^{d_a} \prod_{i=1}^{N_d} e^d_{1,1}\Big) = I(e_{1,1}^{\sum_{d=1}^{d_a} dN_d}) = I(e^N_{1,1}) = N(N+3).$$

If we again fix any integer $D > N$ then the generator $\Lambda' = e^D_{1,1}$ is minimal for $B(c)$ by [10, Lemma 2.1]. Thus

$$I(\Lambda') = I(e^D_{1,1}) = D(D+3) > N(N+3).$$

Any other choice of the Λ_i appearing in the factorization of Λ must be a subset of the above choice of the Λ_i. As a result, $I\big(\prod_{i=1}^n \Lambda'_i\big)$ will be even smaller. Thus, there are no possible choices for the Λ_i such that $I\big(\prod_{i=1}^n \Lambda'_i\big) = I(\Lambda')$, contradicting the fact that $I\big(\prod_{I=1}^n \Lambda'\big) = I(\Lambda')$.

To obtain the statement that if $P\big((\sqrt{7}-1)/(\sqrt{7}-2), 1\big)$ symplectically embeds into $B(c)$ then $c \geq 2 + a/2$, we appeal to the following limiting argument. Let $a_0 = (\sqrt{7}-1)/(\sqrt{7}-2)$. We have just proven, for all $a < a_0$, if $P(a,1)$ symplectically embeds into $B(c)$ then $c \geq 2 + a/2$. Thus if $P(a_0, 1)$ symplectically embeds into $B(c)$ then $c \geq 2 + a_0/2$. □

Appendix: Difficulties extending Theorem 1.4 via the Hutchings criterion

Theorem 1.4 implies that symplectic folding yields optimal embeddings of $P(a,1)$ into $B(c)$ whenever

$$2 \leq a \leq \frac{\sqrt{7}-1}{\sqrt{7}-2} = 2.54858\dots.$$

For $a > (\sqrt{7}-1)/(\sqrt{7}-2)$, our method of proving Theorem 1.4 breaks down. More specifically, the proof of Proposition 3.7 relies on the fact that $a < (\sqrt{7}-1)/(\sqrt{7}-2)$ in order to conclude that the coefficient of d^2 in (23) is positive, yielding a contradiction for sufficiently large d. When a is larger than this value, the conclusions of the proposition will no longer hold, so we will no longer be able to consider convex generators $e_{1,1}^d$ for arbitrarily large d in the proof of Theorem 1.4.

It is natural to ask whether this upper bound on a can be extended by applying the Hutchings criterion and using different methods of proof than those used in Theorem 1.4. As this discussion concerns approaches which don't work, the more technical proofs are omitted from this version of this paper, but they can be found in Appendix A of the most recent arXiv version.

Since $e_{1,1}^d$ is a minimal generator for $B(c)$ for all $d \geq 1$, we might try applying the Hutchings criterion to $e_{1,1}^d$ for some specific, not necessarily large choice of d, allowing us to avoid the use of Proposition 3.7. We would then argue as follows. For some fixed $a > (\sqrt{7}-1)/(\sqrt{7}-2)$, suppose we have some $c < 2 + a/2$ such that $P(a,1)$ symplectically embeds into $B(c)$. We can apply the modified Hutchings criterion, Theorem 1.17, to $\Lambda' = e_{1,1}^d$ to obtain an integer n, a convex generator Λ, and factorizations $\Lambda' = \Lambda_1' \cdots \Lambda_n'$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$ satisfying Theorem 1.17(i)–(iii).

To obstruct the symplectic embedding we assumed to exist, we must show that no possible choice of the Λ_i and Λ_i' exists. In particular, we must show that there exists no convex generator Γ such that $\Gamma \leq_{P(a,1),B(c)} e_{1,1}^d$: otherwise, we will not be able to obstruct the possibility that $n = 1$, $\Lambda_1' = \Lambda' = e_{1,1}^d$, and $\Lambda_1 = \Lambda = \Gamma$.

However, we can actually prove that for any $a > (\sqrt{7}-1)/(\sqrt{7}-2)$ and any $d \geq 1$, there is some $c < 2 + a/2$ and some convex generator Γ such that $\Gamma \leq_{P(a,1),B(c)} e_{1,1}^d$ for every $d \geq 1$. This implies that it is impossible to improve on the results of Theorem 1.4 by applying the Hutchings criterion to convex generators of the form $e_{1,1}^d$. The proof of this fact relies on the following construction of a convex generator satisfying certain constraints.

Lemma A.1 *Let $d \geq 9$. Then there exists some convex generator $\Lambda = e_{1,0}^F e_{m,1} e_{0,1}^V$ such that*

$$0 \leq F \leq \tfrac{1}{2}(3d - 1 + \sqrt{7d^2 - 3}), \tag{27}$$

$$V = \tfrac{1}{2}(3d - 2 - \sqrt{7d^2 - 6d + 4F}), \tag{28}$$

$$m = \tfrac{1}{2}(3d - 2 + \sqrt{7d^2 - 6d + 4F}) - F. \tag{29}$$

The proof of this lemma follows by direct, albeit brute force, calculations. This lemma then shows that applying the Hutchings criterion to $e_{1,1}^d$ for any $d \geq 1$ cannot improve upon Theorem 1.4.

Proposition A.2 *Let*

$$a \geq \frac{\sqrt{7}-1}{\sqrt{7}-2} = 2.54858\ldots.$$

For any $d \geq 1$, there exists some $\epsilon > 0$ and some convex generator Λ such that $\Lambda \leq_{P(a,1),B(c)} e_{1,1}^d$, where $c = 2 + a/2 - \epsilon$.

Proof First, note that when $d = 1$, we have $e_{1,0}^2 \leq_{P(a,1),B(c)} e_{1,1}$ for any $c \geq 2$, and when $d = 2$, we have $e_{1,0}^5 \leq_{P(a,1),B(c)} e_{1,1}^2$ for any $c \geq 2.5$. Since 2 and 2.5 are less than $2 + a/2$ for any possible value of a, the desired statement follows for $d = 1, 2$. Moreover, if $3 \leq d \leq 8$, we can define $\Lambda = e_{1,0}^F e_{m,1}$, where

$$F = \tfrac{1}{2}(d^2 - 3d + 2) \quad \text{and} \quad m = \tfrac{1}{2}(-d^2 + 9d - 6).$$

F and m are positive integers for all $3 \leq d \leq 8$. In addition we have

$$\begin{aligned} x(\Lambda) + y(\Lambda) &= F + m + 1 \\ &= \tfrac{1}{2}(6d - 4) + 1 \\ &= 3d - 1 \\ &= x(e_{1,1}^d) + y(e_{1,1})^d + m(e_{1,1}^d) - 1, \end{aligned} \tag{30}$$

and (8) yields

$$\begin{aligned} I(\Lambda) &= 2F + m + 2F + m + 2 \\ &= 2d^2 - 6d + 4 - d^2 + 9d - 6 + 2 \\ &= d^2 + 3d \\ &= I(e_{1,1}^d). \end{aligned} \tag{31}$$

Finally,

$$A_{P(a,1)}(\Lambda) = x(\Lambda) + ay(\Lambda) = 3d - 2 + a,$$

so that $A_{P(a,1)}(\Lambda) < (2+a/2)d$ whenever

$$a > \frac{2(d-2)}{d-2} = 2.$$

Because $a > 2$ by assumption, we must have $A_{P(a,1)}(\Lambda) < (2+a/2)d$. Then, for any $0 < \epsilon \leq (2+a/2) - A_{P(a,1)}(\Lambda)/d$, we obtain

$$A_{P(a,1)}(\Lambda) \leq (2+a/2-\epsilon)d = A_{B(2+a/2-\epsilon)}(e^d_{1,1}).$$

This equation, together with (30) and (31), implies that $\Lambda \leq_{P(a,1),B(c)} e^d_{1,1}$ for $c = 2+a/2-\epsilon$, as desired.

We are left with the case where $d \geq 9$. Here, we can apply Lemma A.1 to construct some convex generator $\Lambda = e^F_{1,0}e_{m,1}e^V_{0,1}$ satisfying (27), (28), and (29). We will prove that $\Lambda \leq_{P(a,1),B(c)} e^d_{1,1}$ for some c of the desired form. First, notice that

$$\begin{aligned} x(\Lambda)+y(\Lambda) &= (F+m)+(V+1) \\ &= \tfrac{1}{2}(3d-2-\sqrt{J}) + \tfrac{1}{2}(3d-2+\sqrt{J}) + 1 \\ &= 3d-1 \\ &= x(e^d_{1,1}) + y(e^d_{1,1}) + m(e^d_{1,1}) - 1, \end{aligned} \tag{32}$$

where

$$J = 7d^2 - 6d + 4F.$$

Moreover, using (8) and substituting in (32) gives

$$\begin{aligned} I(\Lambda) &= 2\mathbb{A}(\Lambda) + x(\Lambda) + y(\Lambda) + m(\Lambda) \\ &= 2F(V+1) + m(2V+1) + 3d - 1 + F + V + 1 \\ &= 2F(V+1) + 2Vm + 3d - 1 + F + V + m + 1 \\ &= 2V(F+m) + 2F + 3d - 1 + x(\Lambda) + y(\Lambda). \end{aligned}$$

Substituting in (32) again and using the definitions of m and V produces

$$\begin{aligned} I(\Lambda) &= \tfrac{1}{2}(3d-2-\sqrt{J})(3d-2+\sqrt{J}) + 2F + 6d - 2 \\ &= 2 - 3d + d^2 - 2F + 2F + 6d - 2 \\ &= d^2 + 3d = I(e^d_{1,1}). \end{aligned} \tag{33}$$

In light of (33) and (32), we see that $\Lambda \leq_{P(a,1),B(c)} e^d_{1,1}$ if and only if $A_{P(a,1)}(\Lambda) \leq A_{B(c)}(e^d_{1,1})$. We will show

$$A_{P(a,1)}(\Lambda) < \left(2+\frac{a}{2}\right)d. \tag{34}$$

Then, for any $0 < \epsilon \le (2 + a/2) - A_{P(a,1)}(\Lambda)/d$, we have

$$A_{P(a,1)}(\Lambda) \le \Big(2 + \frac{a}{2} - \epsilon\Big)d = A_{B(2+a/2-\epsilon)}(e^d_{1,1}),$$

so that $\Lambda \le_{P(a,1),B(c)} e^d_{1,1}$, where $c = 2 + a/2 - \epsilon$.

To prove (34), we first use (32) and (28) to compute $A_{P(a,1)}(\Lambda)$:

$$\begin{aligned} A_{P(a,1)}(\Lambda) &= x(\Lambda) + ay(\Lambda) \\ &= 3d - 1 + (a-1)y(\Lambda) \\ &= 3d - 1 + (a-1)\big(\tfrac{1}{2}(3d - 2 - \sqrt{J}) + 1\big) \end{aligned}$$

Using this calculation, (34) is equivalent to

$$(a-1)(3d - \sqrt{J}) < (a-2)d + 2.$$

Rearranging produces

$$\sqrt{J} - d - 2 < a(\sqrt{J} - 2d).$$

Since $\sqrt{J} - 2d \ge \sqrt{7d^2 - 6d} - 2d > 0$ for all $d > 2$, the above inequality becomes

$$\frac{\sqrt{J} - d - 2}{\sqrt{J} - 2d} < a. \tag{35}$$

The left-hand side of (35) is increasing for all F and all $d > 2$, and its limit as $d \to \infty$ is

$$\frac{\sqrt{7}-1}{\sqrt{7}-2} = 2.54858\ldots.$$

Since a is at least this limit value by assumption and $d \ge 9$, we conclude that (35) is true, hence so is (34). □

Now that we know we cannot use any generator of the form $e^d_{1,1}$ to improve upon the results of Theorem 1.4, we might ask if we can apply the Hutchings criterion to any other generator for the ball.

First, we investigate other possibilities for minimal generators. These must *uniquely* minimize the symplectic action among all convex generators of equal index. The following lemma shows that in every index grading other than those of the $e^d_{1,1}$, the action with respect to any ball is nonuniquely minimized, so that the $e^d_{1,1}$ are the only minimal generators for $B(c)$. The proof of this lemma is by direct construction.

Lemma A.3 *Let $c > 0$, and let k be a positive integer such that $2k \ne I(e^d_{1,1})$ for all $d \ge 1$. Then there exist two distinct convex generators which minimize the symplectic action with respect to $B(c)$ among convex generators with index $2k$.*

As a result, we cannot apply Theorem 1.17 to any convex generators other than the $e^d_{1,1}$ in order to understand symplectic embeddings into the ball. Combined with Proposition A.2, this implies that in fact, Theorem 1.17 cannot be used to extend the upper bound on a in the statement of Theorem 1.4.

The improvement of the Hutchings criterion [10, Conjecture A.3], proven in Choi [1], allows the statement of Theorem 1.17 to be weakened so that one need only assume that all edges of Λ' are labeled e (as opposed to the requirement that Λ' be minimal). As a result, one could conceivably improve upon Theorem 1.4 using a nonminimal generator.

For instance, we could try to apply the Hutchings criterion to the convex generators constructed in Lemma A.3, which nonuniquely minimize the symplectic action in their index grading. However, preliminary evidence suggests that these generators (as well as all others of equal index and symplectic action) will do no better than the $e^d_{1,1}$.

Moreover, [10, Conjecture A.3] would also allow one to use a generator that does not minimize the symplectic action at all. This choice would likely weaken the action inequality in the definition of $\leq$ between convex generators for most relevant cases. Thus the Hutchings criterion should on the whole yield weaker combinatorial conditions for nonminimal generators than it does for minimal ones. In short, some possibility remains to extend the statement of Theorem 1.4 to larger values of a using the Hutchings criterion, but it will require methods beyond the scope of this paper.

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