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Inverse problems for multi-valued quasi variational inequalities and noncoercive variational inequalities with noisy data

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ABSTRACT

We study the inverse problem of identifying a variable parameter in variational and quasi-variational inequalities. We consider a quasi-variational inequality involving a multi-valued monotone map and give a new existence result. We then formulate the inverse problem as an optimization problem and prove its solvability. We also conduct a thorough study of the inverse problem of parameter identification in noncoercive variational inequalities which appear commonly in applied models. We study the inverse problem by posing optimization problems using the output least-squares and the modified output least-squares. Using regularization, penalization, and smoothing, we obtain a single-valued parameter-to-selection map and study its differentiability. We consider optimization problems using the output least-squares and the modified output least-squares for the regularized, penalized and smoothened variational inequality. We give existence results, convergence analysis, and optimality conditions. We provide applications and numerical examples to justify the proposed framework.

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1. Introduction

An inverse problem commonly refers to the process of estimating intrinsic features of a physical model from a measured output of the model. For example, when a partial differential equation is characterizing the model, a measurement of its solution can be used to estimate the force term, the involved material parameters, the boundary conditions and the initial conditions, giving rise to a spectrum of inverse problems. The field of inverse problems has attracted a great deal of attention in the recent years because of the ever-growing list of essential applications in domains such as biomedical sciences, finance, engineering,

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social sciences, and related areas. Although most of the research in inverse problems is in the context of partial differential equations, there are studies which deal with inverse problems in variational inequalities, and much recently in quasi-variational inequalities.

The primary objective of this work is twofold. Firstly, we develop a general framework for the identification of a variable parameter in multi-valued quasi-variational inequalities. This generality is motivated by some recent applications which lead to quasi-variational inequalities with multi-valued maps. Secondly, confining to a variational inequality, we study the impact of data contamination on the identification process. Since variational and quasi-variational inequalities model numerous applied phenomenon, the developed identification process is expected to have wide applicability. For instance, applications of variational and quasi-variational inequalities can be found in elastohydrodynamic [1], energy production management [2], equilibrium problems [3–5], frictional elastostatic contact [6], frictionless quasistatic contact with history-dependent stiffness [7], image processing [8], Nash game equilibrium [9], multiobjective elliptic control [10], reaction-diffusion [11], sandpiles formation [12], shape optimization [13], superconductivity models [14], and numerous others.

To give a brief overview of the related research, we begin by recalling that an inverse problem in variational inequalities appears in the elastohydrodynamic lubrication problem (EHL). The EHL problem results in a variational inequality in which the unknown is the pressure u, and the coefficient a is known. However, due to the significant theoretical and computational obstacles in solving the EHL problem, an efficient two-step procedure is typically designed. In this procedure, the first step comprised of an inverse problem of parameter identification in a variational inequality where the sought parameter is in the primary operator and on the right-hand side of the inequality, see [15]. Inspired by the EHL problem, Hintermüller [16] studied the inverse problem of parameter identification for a variational inequality and besides a rigorous treatment of the analytical aspects, also presented a detailed computational framework. In the same vein, Gonzalez [17] explored the inverse problem of identifying multiple parameters in an elliptic variational inequality and provided an existence result, see also [18]. In earlier work, Hasanov [19] presented useful results for the boundary inverse problem for elliptic variational inequalities. In another contribution [20], the authors gave a detailed numerical treatment of the inverse elasticity problem with Signorini's condition. In [21], the authors focussed on the theoretical aspects of the identification inverse problem in a nonlinear variational inequality. Recently, Kupenko and Manzo [22] investigated the inverse problem of parameter identification for a variational inequality with anisotropic p-Laplacian. Barabasz, Gajda-Zagórska, and Migórski [23] presented a hybrid algorithm for solving inverse problems in elasticity. Migórski and Ochal [24] studied an inverse problem of parameter identification for a non-linear parabolic boundary hemi-variational inequality. We also note that Hoffmann and Sprekels [25] were among the first ones to study parameter identification in variational inequalities. However, in [25], in contrast to the most papers on inverse problems where an optimization framework is a preferred choice, the authors developed an iterative scheme that is based on the construction of regularized time-dependent problems containing the original problem as the asymptotic steady state. We note that in [26–28], the authors studied identification in simplified quasi-variational inequalities and coercive variational inequalities. Recently inverse problems have been also studied in [29, 30] for variational-hemivariational inequalities modelling frictional contact problems in solid mechanics.

The main contributions of this work are as follows:

(1) By relaxing the coercivity condition, we give a new existence result for a quasi-variational inequality with a multi-valued monotone map. The existence theory for multi-valued quasi-variational inequalities is quite challenging and is still in a developmental stage. Among other technical difficulties, solving a quasi-variational inequality requires solving a variational inequality and a fixed point problem simultaneously.
(2) We formulate the inverse problem of parameter identification in a quasi-variational inequality as an equivalent optimization problem. We develop an abstract regularization framework for the inverse problem which is suitable for identifying discontinuous coefficients.
(3) Restricting to the case of noncoercive variational inequalities (see [31]), we thoroughly investigate the impact of data perturbation on the identification process. We study the inverse problem by posing optimization problems using the output least-squares and the modified output least-squares. Using regularization, penalization, and smoothing, we obtain a single-valued parameter-to-selection map and explore its differentiability. We then consider optimization problems using the output least-squares and the modified output least-squares for the regularized, penalized and smoothened variational inequality. We give existence results, convergence analysis, and optimality conditions. We present applications and some preliminary numerical examples to justify the proposed framework.

2. Problem formulation

Assume that B is a real Banach space and A is a nonempty, closed, and convex subset of B. Here the space B is the parameter space whereas the set A imposes feasibility restriction on the sought parameters. Assume that V is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and let V^* be the dual of V. We will pose the variational and quasi-variational inequalities in the space V. Assume that Z is another real Hilbert space such that V is continuously imbedded in Z. We will take the measured data in the space Z. Assume that C is a nonempty, closed, and convex subset of V and assume that $K : C \rightrightarrows C$ is a set-valued map such that for every $u \in C$, the set $K(u)$ is a closed, and convex subset of C. Assume that $F : V \rightrightarrows V^*$ is a set-valued map and $m \in V^*$. We define a trilinear form $T : B \times V \times V \to \mathbb{R}$ with $T(a, u, v)$ symmetric in u, v, and assume the following continuity and positivity conditions:

$$T(a, u, v) \leq \beta \|a\|_B \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V,\ a \in B,\ \beta > 0, \tag{1a}$$

$$T(a, u, u) \geq 0, \quad \text{for all } u \in V,\ a \in A. \tag{1b}$$

We consider the quasi-variational inequality where as usual $\langle T(\cdot, \cdot), \cdot \rangle = T(\cdot, \cdot, \cdot)$ for notational simplicity: Given $a \in A$, find $u \in K(u)$ such that for some $w \in F(u)$, we have

$$\langle T(a, u) + w - m, v - u \rangle \geq 0, \quad \text{for every } v \in K(u). \tag{2}$$

Given $a \in A$, quasi-variational inequality (2) of finding $u = u(a)$ constitutes the direct problem in this study. Our focus, however, is on the inverse problem of identifying the variable coefficient a from a measurement z of a solution u of the quasi-variational inequality.

Quasi variational inequality (2) is quite general and conveniently subsumes many variational and quasi-variational inequalities appearing in several applications of interest as special cases. In particular, the involvement of the multi-valued map F serves at least two purposes. Firstly, Kano, Kenmochi, and Murase [32] showed recently that an elasto-plasto model leads to a multi-valued quasi variational inequality. The general results of this paper can be applied to study the inverse problem of identifying material parameters in a simplified elasto-plastic model. Secondly, our results can be applied to study inverse problems for quasi hemi-variational inequalities where the multi-valued map F corresponds to the sum of a monotone map and a generalized derivative. We emphasize that the constraint set $K(u)$ in (2) depends on the unknown u. This makes the study of quasi variational inequalities quite challenging and is one of the reasons that a majority of theoretical and numerical techniques which are readily available for variational inequalities have not yet been extended to quasi variational inequalities.

Note that in the absence of F and m, (2) reduces to the following quasi variational inequality: find $u \in K(u)$ such that

$$\langle T(a, u), v - u \rangle \geq 0, \quad \text{for every } v \in K(u). \tag{3}$$

Quasi variational inequality (3) is one of the most commonly studied problems of this class and is convenient for many applications such as implicit obstacle problem, dam problems, and others. Bensoussan and Lions [33] introduced (3) in connection with an impulse control problem. A general treatment of (3) was given by Mosco [34].

If additionally $K(u) = C$, for every $u \in C$, then (3) recovers the following variational inequality: find $u \in C$ such that

$$\langle T(a,u), v-u\rangle \geq 0, \quad \text{for every } v \in C.$$

The above variational inequality has been extensively studied in the literature and has found numerous applications (see [35–37]). The inverse problem of parameter identification in simpler variational and quasi variational inequalities has also been studied by many authors, see [16, 26] and the references therein.

3. Essential tools

We now gather the essential background material which includes notions of set-valued maps and existence results for variational inequalities. In the following, we specify the strong convergence and the weak convergence by $\to$ and $\rightharpoonup$, respectively. We begin with the following two definitions:

Definition 3.1: Given a real reflexive Banach space X with X^* as its dual, let $A : X \rightrightarrows X^*$ be a set-valued map. The domain and the graph of the map A are denoted by $D(A) := \{u \in X | \, A(u) \neq \emptyset\}$, and $G(A) := \{(u,w) | \, u \in D(A), \ w \in A(u)\}$. The map A is called *monotone*, if $\langle u-v, x-y\rangle \geq 0$, for all $(x,u),(y,v) \in G(A)$. Furthermore, A is called *maximal monotone*, if the graph of the monotone map A is not included in the graph of any other monotone map with the same domain. The map A is called *strongly-weakly demiclosed*, if for any sequence $\{(x_n, w_n)\}$ with $w_n \in A(x_n)$, and $x_n \to x$ and $w_n \rightharpoonup w$, we have $w \in A(x)$.

Definition 3.2: The map $K : C \rightrightarrows C$ is called *M-continuous*, if the following conditions hold:

(M1) For any sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x$, and for each $y \in K(x)$, there exists $\{y_n\}$ such that $y_n \in K(x_n)$ and $y_n \to y$.
(M2) For $y_n \in K(x_n)$ with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, we have $y \in K(x)$.

The following result (see [38, Lemma 1.5.14]) will play a crucial role:

Lemma 3.3: *Let X be a reflexive Banach space with X^* as its dual. Let $A : X \rightrightarrows X^*$ be a monotone map with $\bar{x} \in \operatorname{int}(D(A))$. Then there is a constant $r = r(\bar{x}) > 0$ such that for any $(x,w) \in G(A)$ and $c := \sup\{\|w'\| | \, \|x' - \bar{x}\| \leq r, \text{ and } w' \in A(x')\} < \infty$, we have*

$$\langle w, x - \bar{x}\rangle \geq r\|w\| - (\|x - \bar{x}\| + r)c.$$

The following fixed point theorem by Kluge [39] will play an important role:

Theorem 3.4: *Let X be a reflexive Banach space and let $C \subset X$ be nonempty, convex, and closed. Assume that $\Psi : C \rightrightarrows C$ is a set-valued map such that for every $u \in C$, the set $\Psi(u)$ is nonempty, closed, and convex, and the graph of Ψ is sequentially weakly closed. Suppose that the set $\Psi(C)$ is bounded. Then the map Ψ has at least one fixed point in C.*

The following interesting result can be found in Alber et al. [40].

Lemma 3.5: *Let X be a uniformly convex Banach space with X^* as its dual, let $F : X \rightrightarrows X^*$ be strongly-weakly demiclosed, convex-valued and monotone, let $C \subset \operatorname{int}(\operatorname{dom}(F))$ be closed, and convex, and let $f \in X^*$. Then $x \in C$ solves the variational inequality of finding $x \in C$ so that for some $w \in F(x)$, we have*

$$\langle w - f, z - x \rangle \geq 0, \quad \text{for every } z \in C, \tag{4}$$

if and only if, it solves the following Minty variational inequality: find $x \in C$ such that

$$\langle w - f, z - x \rangle \geq 0 \quad \text{for every } z \in C, \text{ for every } w \in F(z). \tag{5}$$

Furthermore, if J is the normalized duality map, then there exists a unique $x_\epsilon \in C$ such that for some $w_\epsilon \in F(x_\epsilon)$, we have

$$\langle w_\epsilon + \epsilon J(x_\epsilon), y - x_\epsilon \rangle \geq 0, \quad \text{for every } y \in C, \quad \epsilon > 0. \tag{6}$$

We conclude this section by the following lemma taken from Browder [41, Lemma 1].

Lemma 3.6: *Let X be a Banach space with X^* as its dual and let $\{x_n\} \subset X$. Assume that there is a sequence $\{s_n\} \subset \mathbb{R}_+$ with $s_n \downarrow 0$ such that for any $h \in X^*$ there is a constant C_h such that $\langle h, x_n \rangle \leq s_n \|x_n\| + C_h$, for every n. Then the sequence $\{x_n\}$ is bounded.*

4. Identification in multi-valued quasi variational inequalities

Inspired by the efficient use of optimization-based identification process in linear PDEs, and linear variational inequalities, we shall now resort to an optimization formulation for the inverse problem of parameter estimation in multi-valued quasi-variational inequalities. The ill-posed nature of the inverse problems is well-known, and optimization formulations are quite flexible in incorporating a regularization which is essential for a stable identification process. To develop a general regularization framework, which is aimed to identify discontinuous or rapidly-varying parameters (see [42]), we make the following assumptions:

(1) The Banach space B is continuously embedded in a Banach space L. There is another Banach space $\widehat{B}$ that is compactly embedded in L. The set A is a subset of $B \cap \widehat{B}$, closed and bounded in B and closed in L.

(2) For any $\{b_n\} \subset B$ with $b_n \to 0$ in L, any bounded $\{u_n\} \subset V$, and fixed $v \in V$, we have

$$T(b_n, u_n, v) \to 0. \tag{7}$$

(3) $R : \widehat{B} \to \mathbb{R}$ is positive, convex, and lower-semicontinuous with respect to $\|\cdot\|_L$ and satisfies

$$R(a) \geq \tau_1 \|a\|_{\widehat{B}} - \tau_2, \quad \text{for every } a \in A, \text{ for some } \tau_1 > 0,\ \tau_2 > 0. \tag{8}$$

We consider the following output least-squares (OLS) based optimization problem:

$$\min_{a\in A} J_\kappa(a) := \|u(a) - z\|^2 + \kappa R(a). \tag{9}$$

Here $\kappa > 0$ is a regularization parameter, $R(\cdot)$ is the regularizer defined above, $z \in Z$ is the data, and $u = u(a)$ is a solution of quasi variational inequality (2) for the parameter $a \in A$. That is, for $a \in A$, $u = u(a)$ is such that $u \in K(u)$ and for some $w \in F(u)$, we have

$$\langle T(a,u) + w - m, v - u\rangle \geq 0, \quad \text{for every } v \in K(u). \tag{10}$$

The following existence result for (10) shows that (9) is well-defined. In the following, Γ is the set of all $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\sigma(r) \to 0$ as $r \to \infty$.

Theorem 4.1: *Assume that $F : V \rightrightarrows V^*$ is strongly-weakly demiclosed, convex-valued and monotone with $C \subset int(D(F))$, $K : C \rightrightarrows C$ is M-continuous, and $T : B \times V \times V \to \mathbb{R}$ satisfies* (1). *Assume that one of the following two conditions hold*:

(1) *For every $s \in V^*$, there exists $u_s \in \cap_{v\in C} K(v)$ such that for every $z \in D(F)$ with $\|z\|$ sufficiently large, and every $w \in F(z)$ and some $\sigma \in \Gamma$, we have*

$$\langle T(a,z) + w - s, z - u_s\rangle \geq -\sigma(\|z\|)\|z\|. \tag{11}$$

(2) *There exists a bounded set G such that for each $v \in C$, we have $K(v) \cap G \neq \emptyset$ and*

$$\inf_{w\in F(x)} \frac{\langle T(a,x) + w, x - g\rangle}{\|x\|} \to \infty \text{ as } \|x\| \to \infty, \text{ uniformly in } g \in G. \tag{12}$$

Then, for each $a \in A$, the quasi variational inequality (10) *has a solution.*

Proof: Let $a \in A$ be an arbitrary but fixed element. For the given $a \in A$, we define the variational selection $S_a : C \rightrightarrows C$ which assigns to each $v \in C$, the set of all solutions of the variational inequality of finding $u \in K(v)$ such that for some $w \in F(u)$, we have

$$\langle T(a,u) + w - m, z - u \rangle \geq 0, \quad \text{for every } z \in K(v). \tag{13}$$

We will now show that S_a satisfies the conditions of Theorem 3.4 imposed on the map Ψ.

(I) *For any $v \in C$, the set $S_a(v)$ is nonempty.* We will verify the claim under (11), and analogous arguments can be used for (12). Let $\{\varepsilon_n\}$ be a sequence of real positive parameters such that $\varepsilon_n \to 0$ as $n \to \infty$. Clearly, all the conditions of Lemma 3.5 are met, and by (6), for each $n \in \mathbb{N}$, there is $u_n := u_{\varepsilon_n} \in K(v)$ such that for some $w_n \in F(u_n)$, we have

$$\langle T(a,u_n) + w_n + \epsilon_n u_n - m, z - u_n \rangle \geq 0, \quad \text{for every } z \in K(v). \tag{14}$$

We claim that $\{u_n\}$ is bounded. Indeed, if this is not true, then there is a subsequence $\{u_n\}$ such that $\|u_n\| \to \infty$ as $n \to \infty$. Using the above inequality, for every $z \in K(v)$, we have

$$\langle T(a,u_n) + w_n - m, u_n - z \rangle \leq -\epsilon_n \|u_n\| \left[\|u_n\| - \|z\| \right].$$

Now let $s \in V^*$ be arbitrary and take $z = u_s$ given by the coercivity. Taking $z = u_s$ in the above inequality, and by the coercivity, we get

$$\begin{aligned} -\sigma(\|u_n\|)\|u_n\| &\leq \langle T(a,u_n) + w_n - s, u_n - u_s \rangle \\ &\leq -\langle s - m, u_n - u_s \rangle - \epsilon_n \|u_n\| \left[\|u_n\| - \|u_s\| \right] \\ &\leq -\langle s - m, u_n - u_s \rangle, \end{aligned}$$

as $\epsilon_n \|u_n\| [\|u_n\| - \|u_s\|]$ is positive for $\|u_n\|$ sufficiently large. Consequently,

$$\langle s - m, u_n \rangle \leq \sigma(\|u_n\|)\|u_n\| + \langle s - m, u_s \rangle,$$

and hence Lemma 3.6 with $h = s - m$, $s_n := \sigma(\|u_n\|)$ and $C_h := \langle s - m, u_s \rangle$, confirms that $\{u_n\}$ is bounded. Due to the reflexivity of V, we can extract a subsequence $\{u_n\}$ converging weakly to some u. The Minty formulation (see (15) below) of (14) reads

$$\langle T(a,z) + w_z + \epsilon_n z - m, z - u_n \rangle \geq 0,$$
$$\text{for every } w_z \in F(z), \text{ for every } z \in K(v),$$

which when passed to the limit $n \to \infty$, yields

$$\langle T(a,z) + w_z - m, z - u \rangle \geq 0, \quad \text{for every } w_z \in F(z), \text{ for every } z \in K(v),$$

and by using the Minty formulation once again, we obtain (13).

(II) *For every $v \in C$,$S_a(v)$ is closed and convex.* Clearly, $u \in K(v)$ satisfies (13), if and only if, it solves the following Minty variational inequality: for every $z \in K(v)$ and for every $w_z \in F(z)$, we have

$$\langle T(a,z) + w_z - m, z - u \rangle \geq 0, \tag{15}$$

which at once gives that $S_a(v)$ is closed and convex.

(III) *The set $S_a(C)$ is bounded.* This follows from (11) or (12) in a similar way as in part (I).

(IV) *The graph of the variational selection S_a is sequentially weakly closed.* Let $\{(v_n, y_n)\} \subset G(S_a)$ be such that $y_n \rightharpoonup y$ and $v_n \rightharpoonup v$. We claim that $(v, y) \in G(S_a)$. The set C being convex and closed is also weakly closed, and hence $v \in C$. From the containment $(v_n, y_n) \in G(S_a)$, we deduce that $y_n \in K(v_n)$ and that there exists $w_n \in F(y_n)$ such that

$$\langle T(a, y_n) + w_n - m, z - y_n \rangle \geq 0, \quad \text{for every } z \in K(v_n). \tag{16}$$

We shall first show that $\{w_n\}$ is bounded. From $y_n \in K(v_n)$, by using M-continuity of K, we get $y \in K(v)$. Moreover, there is $\{z_n\}$ converging to y and such that $z_n \in K(v_n)$. We set $z = z_n$ in (16), to get

$$\langle T(a, y_n) + w_n - m, z_n - y_n \rangle \geq 0.$$

By applying Lemma 3.3 to $A(\cdot) := F(\cdot) - m$, there are constants $c > 0$ and $r > 0$ such that

$$\begin{aligned} r\|w_n - m\| &\leq \langle w_n - m, y_n - y \rangle + c(r + \|y_n - y\|) \\ &= \langle T(a, y_n) + w_n - m, y_n - z_n \rangle + \langle T(a, y_n), z_n - y_n \rangle \\ &\quad + \langle w_n - m, z_n - y \rangle + c(r + \|y_n - y\|) \\ &\leq \beta \|a\| \|z_n\| \|y_n - z_n\| \\ &\quad + \|w_n - m\| \|z_n - y\| + c(r + \|y_n - y\|), \end{aligned}$$

and consequently,

$$[r - \|z_n - y\|] \|w_n - m\| \leq c(r + \|y_n - y\|) + \beta \|a\| \|z_n\| \|y_n - z_n\|.$$

Since $z_n \to y$, the term $\|z_n - y\|$ can be made arbitrarily small, and since the right-hand side of the above inequality remains bounded, we prove the boundedness of $\{\|w_n - m\|\}$. The key observations here is that $y_n \rightharpoonup y$ means that the term $\|y_n - y\|$ remains bounded.

For a fixed $z \in K(v)$, there is $z_n \in K(v_n)$ with $z_n \to z$, and for any $w_z \in F(z)$, we get

$$\begin{aligned}\langle w_z, y_n - z\rangle &\leq \langle w_z, y_n - z\rangle + \langle T(a, y_n) + w_n - m, z_n - y_n\rangle \\ &\leq \langle w_n, z_n - z\rangle + \langle w_n - w_z, z - y_n\rangle + \langle T(a, z_n) - m, z_n - y_n\rangle \\ &\leq \langle w_n, z_n - z\rangle + \langle T(a, z_n) - m, z_n - y_n\rangle,\end{aligned}$$

by using the monotonicity of F. By passing the above inequality to limit for $n \to \infty$, we get

$$\langle w_z, y - z\rangle \leq \langle T(a, z) - m, z - y\rangle,$$

and by using the Minty formulation, we deduce that $(y, v) \in G(S_a)$.

Summarizing, we have shown that for an arbitrary $a \in A$, the variational selection $S_a : C \rightrightarrows C$ is a set-valued map such that for any $v \in C$, the set $S_a(v)$ is nonempty, closed, and convex and the graph of S_a is sequentially weakly closed. Moreover, $S_a(C)$ is bounded. This means that all assumptions of Theorem 3.4 are fulfilled for the set-valued map $\Psi = S_a : C \rightrightarrows C$. From Theorem 3.4, we get that S_a has at least one fixed point in C, and this yields that the quasi variational inequality (2) has at least one solution. ∎

Remark 4.2: The above result, besides involving the trilinear form, extends the existence results given in [43], from maximal monotone maps to strongly-weakly demiclosed, convex-valued and monotone maps. This extension is particularly useful for an application of this result to quasi-hemi-variational inequalities (see [44, 45]. We acknowledge that (11) was first introduced in [46] in the context of operator equations.

We have the following existence theorem for the inverse problem:

Theorem 4.3: *Assume that the hypothesis of Theorem 4.1 hold. Then for $\kappa > 0$, regularized output least-squares problem (9) has a solution.*

Proof: By its definition, the objective functional J_κ is bounded from below for every feasible $a \in A$, and hence there exists a minimizing sequence $\{a_n\} \subset A$ such that

$$\lim_{n\to\infty} J_\kappa(a_n) = \inf\{J_\kappa(b) |\ b \in A\}. \tag{17}$$

Let u_n be the solution of quasi variational inequality (10) that corresponds to the parameter a_n. Therefore, $u_n \in K(u_n)$ and for some $w_n \in F(u_n)$, we have

$$\langle T(a_n, u_n) + w_n - m, v - u_n\rangle \geq 0, \quad \text{for every } v \in K(u_n). \tag{18}$$

Due to (17), $J_\kappa(a_n)$ is bounded, and hence by using the inequality $\kappa R(a_n) \leq J_\kappa(a_n)$ and condition (8), we deduce that the sequence $\{a_n\}$ is bounded in the

Banach space $\widehat{B}$. By using the compact embedding of $\widehat{B}$ into L, there is a subsequence which converges strongly in L, to some element of A. Keeping the same notation for subsequences as well, let $\{a_n\}$ be the subsequence that converges in L to $\bar{a} \in A$.

We pick $\{u_n\}$ to be a sequence of solutions of (10) that corresponds to the subsequence of parameters $\{a_n\}$ converging in L to $\bar{a} \in A$. We will first show that $\{u_n\}$ remains bounded by using (12) (but similar arguments will work for (11)). For the sake of argument, we suppose that $\{u_n\}$ is unbounded and that there exists a subsequence such that $\|u_n\| \to \infty$ for $n \to \infty$. To use the coercivity condition, we choose an arbitrary $g_n \in K(u_n) \cap G$. Since G is taken to be bounded, the sequence $\{g_n\}$ is bounded. By taking $v = g_n$ in (10), we obtain

$$\langle T(a_n, u_n) + w_n - m, g_n - u_n \rangle \geq 0,$$

which can be rearranged as follows

$$\langle w_n, u_n - g_n \rangle \leq \langle T(a_n, u_n) - m, g_n - u_n \rangle \leq \|T(a_n, u_n)\| \|g_n\| + \|m\| \, \|u_n - g_n\|$$

where we used the fact that $\langle T(a_n, u_n), u_n \rangle \geq 0$. Consequently,

$$\frac{\langle w_n, u_n - g_n \rangle}{\|u_n\|} \leq \beta \|a_n\| \|g_n\| + \|m\| \left(1 + \frac{\|g_n\|}{\|u_n\|} \right),$$

which implies that

$$\lim_{n \to \infty} \frac{\langle w_n, u_n - g_n \rangle}{\|u_n\|} < \infty,$$

and hence contradicting the coercivity condition. This ensures the boundedness of $\{u_n\}$.

By the reflexivity of V, there is a subsequence, again denoted by $\{u_n\}$ which converges weakly to some $\bar{u} \in V$. Since C is closed and convex, it is also weakly closed, and hence $\bar{u} \in C$. Moreover, due to $u_n \in K(u_n)$, we have $\bar{u} \in K(\bar{u})$. Let $\{z_n\}$ be a sequence that converges strongly to $\bar{u}$ and satisfies $z_n \in K(u_n)$. Since

$$\langle T(a_n, u_n) + w_n - m, z - u_n \rangle \geq 0, \quad \text{for every } z \in K(u_n),$$

by setting $z = z_n$, we obtain

$$\langle T(a_n, u_n) + w_n - m, z_n - u_n \rangle \geq 0.$$

We shall now prove the boundedness of $\{w_n\}$. Applying Lemma 3.3 for F, there are constants $c > 0$ and $r > 0$ such that

$$\begin{aligned} r\|w_n\| &\leq \langle w_n, u_n - \bar{u} \rangle + c(r + \|u_n - \bar{u}\|) \\ &= \langle w_n, u_n - z_n \rangle + \langle w_n, z_n - \bar{u} \rangle + c(r + \|u_n - \bar{u}\|) \\ &\leq \langle T(a_n, z_n) - m, z_n - u_n \rangle + \langle w_n, z_n - \bar{u} \rangle + c(r + \|u_n - \bar{u}\|) \\ &\leq \|w_n\| \|z_n - \bar{u}\| + c(r + \|u_n - \bar{u}\|) + \beta \|a_n\| \|z_n\| \|z_n - u_n\|, \end{aligned}$$

and consequently,

$$[r - \|z_n - \bar{u}\|]\|w_n - m\| \leq c(r + \|u_n - \bar{u}\|) + \beta\|a_n\|\|z_n\|\|z_n - u_n\|.$$

Because of $z_n \to \bar{u}$, the term $\|z_n - \bar{u}\|$ can be made arbitrarily small, and because the right-hand side of the above inequality remains bounded, we deduce that $\{\|w_n - m\|\}$ is bounded.

Now let $z \in K(x)$ be arbitrary. Then for some $\{z_n\}$ with $z_n \in K(u_n)$ with $z_n \to z$, and for any $w_z \in F(z)$, by the definition of u_n, we have

$$\begin{aligned}\langle w_z, u_n - z\rangle &\leq \langle w_z, u_n - z\rangle + \langle T(a_n, u_n) + w_n - m, z_n - u_n\rangle \\ &= \langle w_n, z_n - z\rangle + \langle w_n - w_z, z - u_n\rangle + \langle T(a_n, u_n) - m, z_n - u_n\rangle \\ &\leq \langle w_n, z_n - z\rangle + \langle T(a_n, z_n) - m, z_n - u_n\rangle\end{aligned}$$

by the monotonicity of F. By passing the above inequality to limit for $n \to \infty$, we obtain

$$\langle T(a, z) + w_z - m, \bar{u} - z\rangle \leq 0.$$

Therefore, we have shown that for every $z \in K(\bar{u})$ and for every $w_z \in F(z)$, we have

$$\langle T(a, z) + w_z - m, z - \bar{u}\rangle \geq 0,$$

and by the Minty formulation, $\bar{u}$ is a solution that corresponds to $\bar{a}$, that is, $\bar{u} = u(\bar{a})$.

Finally, we have

$$\begin{aligned}J_\kappa(\bar{a}) &= \|u(\bar{a}) - z\|^2 + \kappa R(\bar{a}) \\ &\leq \liminf_{n\to\infty} \|u_n - z\|^2 + \kappa \liminf_{n\to\infty} R(a_n) \\ &\leq \liminf_{n\to\infty} J_\kappa(a_n) = \inf\{J_\kappa(a) : a \in A\},\end{aligned}$$

ensuring that $\bar{a}$ is a solution. This completes the proof. ■

Remark 4.4: Since for each coefficient a, the quasi-variational inequality has a multiple solutions, the OLS functional J could be defined as a function of two variables, namely a and a solution u. However, for simplicity in presentation, we don't use that notation.

5. Identification in variational inequalities under data contamination

We shall now concentrate on understanding the impact on the inverse problem of any contamination in the variational inequality data. Getting inspiration from the treatment of ill-posed variational problems, we employ elliptic regularization (see [47]) for the underlying variational inequality. To make full use of the regularization theory, we further discard the coercivity assumption and work under the assumption that the variational inequality is solvable. We continue to work under the general setting concerning the involved spaces and the continuous and positive trilinear form T satisfying (7) and (8).

We will focus on the variational inequality: Given $a \in A$, find $u = u(a) \in K$ such that

$$T(a, u, v - u) \geq \langle m, v - u \rangle, \quad \text{for every } v \in K. \tag{19}$$

Additional conditions are necessary to ensure that (19) is solvable. For example, if K is bounded or some recession conditions hold (see [48, 49]), then (19) is solvable. However, such conditions don't guarantee that the solution is unique. Therefore, to focus on the general case, we assume that for each $a \in A$, the set of all solutions, which we denote by $\mathcal{U}(a)$, is nonempty. The following lemma provides additional information:

Lemma 5.1: *For any $a \in A$, the set of all solutions $\mathcal{U}(a)$ of (19) is closed and convex.*

Proof: Follows by the definition of the set-valued parameter-to-solution map. ∎

Remark 5.2: We note that in some applications, in (19), instead of a, we have $\ell(a)$ where $\ell : A \mapsto A$ is a sufficiently smooth map. For simplicity in presentation, in the forthcoming developments, we don't specify such a map. However all our results can be appropriately modified to include that generality.

We now consider the following two optimization problems:

Find $a \in A$ by solving the output least-squares (OLS) optimization problem

$$\min_{a \in A} J(a) := \frac{1}{2}\|u(a) - z\|_Z^2. \tag{20}$$

Find $a \in A$ by solving the modified output least-squares (MOLS) optimization problem

$$\min_{a \in A} \widetilde{J}(a) := \frac{1}{2} T(a, u(a) - z, u(a) - z). \tag{21}$$

In the above optimization problems, $u(a) \in \mathcal{U}(a)$ and $z \in Z$ is the data. As seen earlier, the OLS functional (20) minimizes the gap between the computed and the observed solution in the norm of the observation space Z, whereas the MOLS functional (21) minimizes the energy associated to the trilinear form. Evidently, (21) requires that $z \in V$. The MOLS objective has been used extensively in the inverse problem of identifying variable parameters in variational equations, see [42, 50–53].

We will now approximate (20) and (21) by a sequence of solutions of optimization problems where the entire data set is contaminated in the following sense.

Let $\{\epsilon_n\}$, $\{\tau_n\}$, $\{\kappa_n\}$, $\{\delta_n\}$, and $\{\nu_n\}$ be sequences of positive reals. Let $\ell \in V^*$ be a given element. For each $n \in \mathbb{N}$, let $m_{\nu_n} \in V^*$ and $\ell_{\delta_n} \in V^*$ be given elements, and let $z_{\delta_n} \in Z$ be the contaminated data such that the following inequalities hold:

$$\|z_{\delta_n} - z\|_Z \leq \delta_n, \tag{22a}$$

$$\|m_{\nu_n} - m\|_{V^*} \leq \nu_n, \tag{22b}$$

$$\|\ell_{\delta_n} - \ell\|_{V^*} \leq \delta_n. \tag{22c}$$

Furthermore, for each $n \in \mathbb{N}$, let $T_{\tau_n} : B \times V \times V \to \mathbb{R}$ be a trilinear form such that

$$T_{\tau_n}(a, u, u) \geq 0, \quad \text{for all } u \in V,\ a \in A. \tag{23a}$$

$$\left|T_{\tau_n}(a, u, v) - T(a, u, v)\right| \leq \tau_n \|a\|_B \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V,\ a \in B. \tag{23b}$$

Moreover, as $n \to \infty$, the sequences $\{\epsilon_n\}$, $\{\tau_n\}$, $\{\kappa_n\}$, $\{\delta_n\}$, and $\{\nu_n\}$ satisfy

$$\left\{\tau_n, \epsilon_n, \kappa_n, \nu_n, \delta_n, \frac{\tau_n}{\epsilon_n}, \frac{\delta_n}{\epsilon_n}, \frac{\nu_n}{\epsilon_n}\right\} \to 0. \tag{24}$$

Finally, let $S : V \times V \to \mathbb{R}$ be a symmetric bilinear such that there are constants $\alpha_0 > 0$ and $\beta_0 > 0$ satisfying the following continuity and coercivity conditions

$$S(u, v) \leq \beta_0 \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V, \tag{25a}$$

$$S(u, u) \geq \alpha_0 \|u\|_V^2, \quad \text{for all } u \in V. \tag{25b}$$

For each $n \in \mathbb{N}$, we now consider the following regularized variational inequality: Given $a \in A$, find $u_{\varsigma_n}(a) \in V$ such that for every $v \in K$, we have

$$\begin{aligned} &T_{\tau_n}(a, u_{\varsigma_n}(a), v - u_{\varsigma_n}(a)) + \epsilon_n S(u_{\varsigma_n}(a), v - u_{\varsigma_n}(a)) \\ &\quad \geq \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v - u_{\varsigma_n}(a)\rangle, \end{aligned} \tag{26}$$

where $\epsilon_n > 0$ is the regularization parameter and for simplicity, we set $\varsigma_n := (\epsilon_n, \tau_n, \nu_n, \delta_n)$.

Evidently, for a fixed $n \in \mathbb{N}$, and for any $a \in A$, (26) has a unique solution $u_{\varsigma_n}(a)$. Hence the regularized parameter-to-solution map $a \to u_{\varsigma_n}(a)$ is well-defined and single-valued.

We will now approximate (20) and (21) by the following families of their regularized analogues: For $n \in \mathbb{N}$, find $a_{\varsigma_n} \in A$ by solving

$$\min_{a \in A} J_{\kappa_n}(a) := \frac{1}{2}\|u_{\varsigma_n}(a) - z_{\delta_n}\|_Z^2 + \kappa_n R(a_{\varsigma_n}), \tag{27}$$

$$\min_{a \in A} \widetilde{J}_{\kappa_n}(a) := \frac{1}{2} T(a, u_{\varsigma_n}(a) - z_{\delta_n}, u_{\varsigma_n}(a) - z_{\delta_n}) + \kappa_n R(a_{\varsigma_n}), \tag{28}$$

where $u_{\varsigma_n}(a)$ is the unique solution of (26), $\kappa_n > 0$, and R is the regularizer defined above.

We begin with the following existence result:

Theorem 5.3: *For every $n \in \mathbb{N}$, optimization problems (27) and (28) are solvable.*

Proof: The proof follows by standard arguments used earlier in this work. For a fixed $n \in \mathbb{N}$, and $a \in A$, the functional $J_{\kappa_n}(a)$ is bounded from below which ensures the existence of a minimizing sequence $\{a_m\}$ in A such that $\lim_{m\to\infty} J_{\kappa_n}(a_m) = \inf\{J_{\kappa_n}(a),\ a \in A\}$. Therefore, the sequence $\{a_m\}$ is bounded in $\|\cdot\|_{\widehat{B}}$ as well. Due to the compact embedding of $\widehat{B}$ into L, $\{a_m\}$ has a subsequence which converges strongly in $\|\cdot\|_L$. Preserving the same notation for subsequences as well, let $\{a_m\}$ be the subsequence converging to some $\bar{a} \in A$. We can show, by using the coercivity of $T + \epsilon_n S$, that the sequence of solutions $\{u^m_{\varsigma_n}\}$ of (26) remains bounded and converges strongly to $u_{\varsigma_n}(\bar{a})$. The optimality of $\bar{a}$ then follows by continuity properties of the norm and of the regularizer R. The proof for (28) is similar. ■

The following result shows that (27) approximates (20) and (28) approximates (21):

Theorem 5.4: *Assume that the following two conditions hold:*

(1) *The set A is bounded in $\widehat{B}$ and for each $a \in A$, the solution set $\mathcal{U}(a)$ is nonempty, and the image set $\mathcal{U}(A)$ is bounded.*
(2) *For any $a \in A$, either $\mathcal{U}(a)$ is a singleton, or $Z = V$, $\ell_{\delta_n}(v) = \langle z_{\delta_n}, v\rangle$, $\ell(v) = \langle z, v\rangle$ and $S(u, v) = \langle u, v\rangle$.*

Then, for every $n \in \mathbb{N}$, optimization problem (27) has a solution a_{ς_n}. Furthermore, there is a subsequence $\{a_{\varsigma_n}\} \subset A$ of solutions of (27) converging to a solution of (20) in $\|\cdot\|_L$. The same conclusions hold for (28) and (21) provided that additionally the following condition holds: For every $a \in A$, and any $u, w \in V$,

$$\|u - z\|_V \le \|w - z\|_V \Rightarrow T(a, u - z, u - z) \le T(a, w - z, w - z). \tag{29}$$

Proof: By arguments used in Theorem 5.3, it can be shown that (20) and (21) have solutions. Moreover, by Theorem 5.3, for each $n \in \mathbb{N}$, (27) has a solution a_{ς_n}. For simplicity, we set $a_n := a_{\varsigma_n}$. Since A is bounded in $\widehat{B}$, the sequence of solutions $\{a_n\}$ is bounded in $\widehat{B}$. Since $\widehat{B}$ is compactly embedding into L, $\{a_n\}$ has a strongly convergent subsequence in $\|\cdot\|_L$. Let $\{a_n\}$ be such a subsequence converging strongly to $\bar{a} \in A$ in $\|\cdot\|_L$. Let $\{u_n\}$, where $u_n := u_{\varsigma_n}(a_n)$, be the sequence of the solutions of (26). That is, we have

$$\begin{aligned} &T_{\tau_n}(a_n, u_n, v - u_n) + \epsilon_n S(u_n, v - u_n) \\ &\quad \ge \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v - u_n\rangle, \quad \text{for every } v \in K. \end{aligned}$$

We claim that $\{u_n\}$ is bounded. By assumption, for every $a \in A$, the solution set $\mathcal{U}(a)$ is nonempty. Let $\tilde{u}_n \in \mathcal{U}(a_n)$ be chosen arbitrarily. Since $\mathcal{U}(A)$ is bounded by assumption, the sequence $\{\tilde{u}_n\}$ is bounded. Moreover, we have

$$T(a_n, \tilde{u}_n, v - \tilde{u}_n) \geq \langle m, v - \tilde{u}_n \rangle, \quad \text{for every } v \in K.$$

We combine the above two variational inequalities, by setting $v = \tilde{u}_n$ in the first one and $v = u_n$ in the second one, and obtain

$$\begin{aligned} &T_{\tau_n}(a_n, \tilde{u}_n, \tilde{u}_n - u_n) - T(a_n, \tilde{u}_n, \tilde{u}_n - u_n) + \epsilon_n S(u_n, \tilde{u}_n - u_n) + \langle m, \tilde{u}_n - u_n \rangle \\ &\quad - \langle m_{\nu_n}, \tilde{u}_n - u_n \rangle - \epsilon_n \langle \ell_{\delta_n}, \tilde{u}_n - u_n \rangle - T_{\tau_n}(a_n, \tilde{u}_n - u_n, \tilde{u}_n - u_n) \geq 0, \end{aligned}$$

and from the fact that $T_{\tau_n}(a_n, \tilde{u}_n - u_n, \tilde{u}_n - u_n) \geq 0$, deduce

$$\begin{aligned} \epsilon_n S(u_n, u_n) &\leq \epsilon_n S(u_n, \tilde{u}_n \rangle + T_{\tau_n}(a_n, \tilde{u}_n, \tilde{u}_n - u_n) \\ &\quad - T(a_n, \tilde{u}_n, \tilde{u}_n - u_n) - \epsilon_n \langle \ell_{\delta_n}, \tilde{u}_n - u_n \rangle. \\ &\quad + \langle m, \tilde{u}_n - u_n \rangle - \langle m_{\nu_n}, \tilde{u}_n - u_n \rangle \\ &\leq \epsilon_n \beta_0 \|u_n\|_V \|\tilde{u}_n\|_V + \tau_n \|a_n\|_B \|\tilde{u}_n\|_V \|\tilde{u}_n - u_n\|_V \\ &\quad + \epsilon_n \|\ell_{\delta_n}\|_{V^*} \|\tilde{u}_n - u_n\|_V + \nu_n \|\tilde{u}_n - u_n\|_V, \end{aligned}$$

which implies

$$\begin{aligned} \|u_n\|_V \leq \frac{\beta_0}{\alpha_0} \|\tilde{u}_n\|_V &+ \left[\frac{\tau_n}{\alpha_0 \epsilon_n} \|a_n\|_B \|\tilde{u}_n\|_V + \frac{\nu_n}{\alpha_0 \epsilon_n} + \frac{\delta_n + \|\ell\|_{V^*}}{\alpha_0} \right] \\ &\times \left[\frac{\|\tilde{u}_n\|_V}{\|u_n\|_V} + 1 \right], \end{aligned}$$

confirming that the sequence $\{u_n\}$ is bounded.

Now let $\{u_n\}$ be a subsequence converging weakly to some $\bar{u} \in K$. We will prove that $\bar{u} \in \mathcal{U}(\bar{a})$. Since a_n is a minimizer of (27), for every $v \in K$, we have

$$T_{\tau_n}(a_n, u_n, v - u_n) + \epsilon_n S(u_n, v - u_n) \geq \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v - u_n \rangle,$$

which due to the positivity of $T + \epsilon_n S$, further implies that for every $v \in K$, we have

$$\begin{aligned} T_{\tau_n}(a_n, v, v - u_n) + \epsilon_n S(v, v - u_n) &\geq T_{\tau_n}(a_n, u_n, v - u_n) + \epsilon_n S(u_n, v - u_n) \\ &\geq \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v - u_n \rangle, \end{aligned}$$

and consequently,

$$T_{\tau_n}(a_n, v, v - u_n) + \epsilon_n S(v, v - u_n) \geq \langle m_{\nu_n} \epsilon_n + \ell_{\delta_n}, v - u_n \rangle,$$

which, by the rearrangement

$$\begin{aligned} T_{\tau_n}(a_n, v, v - u_n) &= T(a_n, v, v - u_n) + T_{\tau_n}(a_n, v, v - u_n) - T(a_n, v, v - u_n) \\ &= T(a_n - \bar{a}, v, v - u_n) + T(\bar{a}, v, v - \bar{u}) + T(\bar{a}, v, \bar{u} - u_n) \\ &\quad + T_{\tau_n}(a_n, v, v - u_n) - T(a_n, v, v - u_n), \end{aligned}$$

leads to the following inequality

$$T(a_n - \bar{a}, v, v - u_n) + T(\bar{a}, v, v - \bar{u}) + T(\bar{a}, v, \bar{u} - u_n) + T_{\tau_n}(a_n, v, v - u_n) \\ - T(a_n, v, v - u_n) + \epsilon_n S(u_n, v - u_n) \geq \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v - u_n \rangle,$$

which due to the imposed conditions when passed to the limit $n \to \infty$, implies that

$$T(\bar{a}, v, v - \bar{u}) \geq \langle m, v - \bar{u} \rangle, \quad \text{for every } v \in K.$$

We now insert $\bar{u} + t(v - \bar{u}) \in K$ with $t \in (0, 1)$ in place of v in the above inequality to get

$$T(\bar{a}, \bar{u}, v - \bar{u}) + tT(\bar{a}, v - \bar{u}, v - \bar{u}) \geq \langle m, v - \bar{u} \rangle,$$

and pass the above inequality to limit $t \to 0$ obtaining

$$T(\bar{a}, \bar{u}, v - \bar{u}) \geq \langle m, v - \bar{u} \rangle, \quad \text{for every } v \in K,$$

which proves that $\bar{u} \in \mathcal{U}(\bar{a})$.

For a fixed $n \in \mathbb{N}$, the optimality of $a_n \in A$ for (27) means that for each $a \in A$, we have

$$J_{\kappa_n}(a_n) := \frac{1}{2}\|u_{\varsigma_n}(a_n) - z_{\delta_n}\|_Z^2 + \kappa_n R(a_n) \leq \frac{1}{2}\|u_{\varsigma_n}(a) - z_{\delta_n}\|_Z^2 + \kappa_n R(a), \quad (30)$$

where $u_{\varsigma_n}(a)$ is the solution of regularized optimization problem (26) for parameter $a \in A$.

Let $(\hat{a}, \hat{u})$ be a solution of (20). Then, (30) confirms that

$$J_{\kappa_n}(a_n) := \frac{1}{2}\|u_{\varsigma_n}(a_n) - z_{\delta_n}\|_Z^2 + \kappa_n R(a_n) \leq \frac{1}{2}\|u_{\varsigma_n}(\hat{a}) - z_{\delta_n}\|_Z^2 + \kappa_n R(\hat{a}),$$

where $u_{\varsigma_n}(\hat{a})$ is the solution of regularized optimization problem (26) for parameter $\hat{a} \in A$.

We first study the behaviour of $u_{\varsigma_n}(\hat{a})$. By the definition of $u_{\varsigma_n}(\hat{a})$, for any $v \in K$, we have

$$T_{\tau_n}(\hat{a}, u_{\varsigma_n}(\hat{a}), v - u_{\varsigma_n}(\hat{a})) + \epsilon_n S(u_{\varsigma_n}(\hat{a}), v - u_{\varsigma_n}(\hat{a})) \\ \geq \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v - u_{\varsigma_n}(\hat{a}) \rangle. \quad (31)$$

By arguments similar to those used at the beginning of this proof, we can show that the sequence $\{u_{\varsigma_n}(\hat{a})\}$ is bounded, and there is a subsequence $\{u_{\varsigma_n}(\hat{a})\}$ converging weakly to some $\bar{u} \in \mathcal{U}(\hat{a})$.

Let us now consider the following variational inequality of finding $\tilde{u} \in \mathcal{U}(\hat{a})$ such that

$$S(\tilde{u}, w - \tilde{u}) \geq \langle \ell, w - \tilde{u} \rangle, \quad \text{for every } w \in \mathcal{U}(\hat{a}), \tag{32}$$

which, due to the coercivity of $S(\cdot, \cdot)$, has a unique solution $\tilde{u}$, and since $\tilde{u} \in \mathcal{U}(\hat{a})$, we have

$$T(\hat{a}, \tilde{u}, v - \tilde{u}) \geq \langle m, v - \tilde{u} \rangle, \quad \text{for every } v \in K. \tag{33}$$

By combining (31) and (33), we obtain

$$\begin{aligned} & T(\hat{a}, u_{\varsigma_n}(\hat{a}) - \tilde{u}, \tilde{u} - u_{\varsigma_n}(\hat{a})) \\ & \quad + T_{\tau_n}(\hat{a}, u_{\varsigma_n}(\hat{a}), \tilde{u} - u_{\varsigma_n}(\hat{a})) - T(\hat{a}, u_{\varsigma_n}(\hat{a}), \tilde{u} - u_{\varsigma_n}(\hat{a})) \\ & \quad + \epsilon_n S(u_{\varsigma_n}(\hat{a}), \tilde{u} - u_{\varsigma_n}(\hat{a})) \geq \langle m_{\nu_n} - m + \epsilon_n \ell_{\delta_n}, \tilde{u} - u_{\varsigma_n}(\hat{a}) \rangle \end{aligned}$$

and by using the positivity of T, we get

$$\begin{aligned} & \left[\frac{\tau_n}{\epsilon_n} \|\hat{a}\|_B \|u_{\varsigma_n}(\hat{a})\|_V + \frac{\nu_n}{\epsilon_n} + \delta_n \right] \|\tilde{u} - u_{\varsigma_n}(\hat{a})\|_V - \ell(\tilde{u} - u_{\varsigma_n}(\hat{a})) \\ & \quad \geq S(u_{\varsigma_n}(\hat{a}), u_{\varsigma_n}(\hat{a}) - \tilde{u}) \\ & \quad \geq S(\tilde{u}, u_{\varsigma_n}(\hat{a}) - \tilde{u}) \end{aligned} \tag{34}$$

which, when passed to the limit $n \to \infty$, yields

$$S(\tilde{u}, \tilde{u} - \bar{u}) \geq \ell(\tilde{u} - \bar{u}). \tag{35}$$

We use (32), (35), and the positivity of the bilinear form, to obtain $0 \geq S(\bar{u} - \tilde{u}, \bar{u} - \tilde{u}) \geq \alpha_0 \|\bar{u} - \tilde{u}\|_V^2$ and hence $\bar{u} = \tilde{u}$. Since $\bar{u}$ is uniquely defined, the whole sequence $u_{\varsigma_n}(\hat{a})$ converges weakly to $\bar{u}$. The convergence is in fact strong because of (34). Indeed, using (34), we can prove that

$$\limsup_{n \to \infty} \|u_{\varsigma_n}(\hat{a}) - \tilde{u}\|_V^2 \leq 0$$

and hence the strong convergence of $\{u_{\varsigma_n}(\hat{a})\}$ to $\bar{u} = \bar{u}(\hat{a})$ follows.

Let us now assume that $\mathcal{U}(a)$ is a singleton for each $a \in A$. Then, by using the weak lower-semicontinuity of the norm and the regularizer R, and (30) (with $a = \hat{a}$), we have

$$\begin{aligned} \frac{1}{2}\|\bar{u} - z\|_Z^2 & \leq \liminf_{n \to \infty} \frac{1}{2} \|u_{\varsigma_n}(a_n) - z_{\delta_n}\|_Z^2 \\ & \leq \liminf_{n \to \infty} \frac{1}{2} \left\{ \|u_{\varsigma_n}(a_n) - z_{\delta_n}\|_Z^2 + \kappa_n R(a_n) \right\} \\ & \leq \liminf_{n \to \infty} \frac{1}{2} \left\{ \|u_{\varsigma_n}(\hat{a}) - z_{\delta_n}\|_Z^2 + \kappa_n R(\hat{a}) \right\} \\ & \leq \limsup_{n \to \infty} \frac{1}{2} \|u_{\varsigma_n}(\hat{a}) - z\|_Z^2 = \frac{1}{2}\|\bar{u}(\hat{a}) - z\|_Z^2, \end{aligned}$$

and since $\mathcal{U}(\cdot)$ is a singleton, we have $\bar{u}(\hat{a}) = \hat{u}$ which proves that $\|\bar{u} - z\|_Z^2 \leq \|\bar{u}(\hat{a}) - z\|_Z^2$, and hence $(\bar{a}, \bar{u}) \in \operatorname{graph}(\mathcal{U})$, where $\bar{u} = u(\bar{a})$, is a solution of (27).

For the case when $\mathcal{U}(\cdot)$ is set-valued map, for any $v \in V$, we take $\ell_{\delta_n}(v) = \langle z_{\delta_n}, v\rangle$ and $S(u, v) = \langle u, v\rangle$. Then, it follows from (32) that for an arbitrary $\breve{u} \in \mathcal{U}(\hat{a})$, we have $\langle \bar{u} - z, \breve{u} - \bar{u}\rangle \geq 0$ which implies that $\|\bar{u} - z\|_V \leq \|\breve{u} - z\|_V$, and hence $\bar{u} = u(\bar{a})$ is the element closest to z among all the elements $\breve{u} \in \mathcal{U}(\hat{a})$.

Using this observation, we have

$$\begin{aligned}\|u(\bar{a}) - z\|_V^2 &\leq \liminf_{n\to\infty} \left\{\|u_{\varsigma_n}(\hat{a}) - z_{\delta_n}\|_V^2 + \kappa_n R(\hat{a})\right\},\\ &\leq \limsup_{n\to\infty} \|u_{\varsigma_n}(\hat{a}) - z\|_V^2\\ &= \|\bar{u}(\hat{a}) - z\|_Z^2 \leq \|\breve{u}(\hat{a}) - z\|_V^2,\end{aligned}$$

where $\breve{u}(\hat{a}) \in \mathcal{U}(\hat{a})$ is arbitrary. In other words, the above inequality confirms the existence of an element $(\bar{a}, u(\bar{a})) \in \operatorname{graph}(\mathcal{U})$ such that for every $(a, u) \in \operatorname{graph}(\mathcal{U})$, we have

$$\|u(\bar{a}) - z\|_V^2 \leq \|u - z\|_V^2$$

and hence $\bar{a} \in A$ is a minimizer of (20). The proof is complete. ■

Remark 5.5: Because it can happen that $\|\bar{u} - z\| \geq \|\hat{u} - z\|$, the optimality of $(\bar{a}, \bar{u}) \in \operatorname{graph}(\mathcal{U})$ is shown by establishing that $\|\bar{u} - z\| \leq \|\hat{u} - z\|$ as by assumption, we have $\|\hat{u} - z\| \leq \|u - z\|$ for all $(a, u) \in \operatorname{graph}(\mathcal{U})$ with $a \in A$. Evidently, if $\mathcal{U}(a)$ is singleton for each $a \in A$, then the supplied arguments remain valid for any S and ℓ.

Remark 5.6: Note that in Theorem 4.3, the boundedness of the minimizing sequence was proved by the aid of the regularizer R. On the other hand, the lack of any regularizer in (20) or (21), prompted the additional condition that A is bounded in $\widehat{B}$. Indeed, analogs of all of our results can be proved for a fix regularization parameter κ in the original and the regularized problems but without the assumption that A is bounded in $\widehat{B}$.

Our next step is to replace variational inequality (26), which is a constraint for the optimization problems (27) and (28) by a variational equation. For this, we recall that a penalty map for K is a bounded, hemi-continuous, monotone map $P : V \to V^*$ such that

$$K = \{v \in V |\ P(v) = 0\}. \tag{36}$$

An example is $P = (I - P_K)$, where I is the identity and P_K is the projection onto K.

For $n \in \mathbb{N}$, a given parameter $\iota_n > 0$ and the penalty map P, consider the penalized-regularized variational equation: Find $u_{\varsigma_n}(a) \in V$ such that for every

$v \in V$, we have

$$T_{\tau_n}(a, u_{\varsigma n}(a), v) + \epsilon_n S(u_{\varsigma n}(a), v) + \frac{1}{\iota_n}\left\langle P(u_{\varsigma n}(a)), v\right\rangle = \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v\rangle. \quad (37)$$

where for notational simplicity, we set $\varsigma_n := (\epsilon_n, \tau_n, \nu_n, \delta_n, \iota_n)$.

We have the following existence result for regularized penalized equations (37).

Theorem 5.7: *For each $n \in \mathbb{N}$, and for any $a \in A$, (37) has a unique solution $u_{\varsigma n}(a)$.*

Proof: The proof follows from the ellipticity of $T + \epsilon_n S$ and the monotonicity of P. ■

We will now formulate analogues of (27) and (28) where the underlying regularized variational inequality has been replaced by the regularized-penalized variational equation: For $n \in \mathbb{N}$, find $a_{\varsigma_n} \in A$ by solving the following optimization problem:

$$\min_{a\in A} J_{\kappa_n}(a) := \frac{1}{2}\|u_{\varsigma n}(a) - z_{\delta_n}\|_Z^2 + \kappa_n R(u_{\varsigma_n}), \quad (38)$$

$$\min_{a\in A} \widetilde{J}_{\kappa_n}(a) := \frac{1}{2} T_{\tau_n}(a, u_{\varsigma n}(a) - z_{\delta_n}, u_{\varsigma n}(a) - z_{\delta_n}) + \kappa_n R(u_{\varsigma_n}), \quad (39)$$

where $u_{\varsigma_n}(a)$ is the unique solution of (37), $\kappa_n > 0$, and R is the regularizer given above.

We give the following existence and convergence result:

Theorem 5.8: *Assume that conditions* (1) *and* (2) *of Theorem 5.4 hold. For each $n \in \mathbb{N}$, optimization problem* (38) *has a solution $\bar{a}_{\varsigma_n}$. There is a sequence $\{(\bar{a}_{\varsigma_n}, \bar{u}_{\varsigma_n})\}$, where $\bar{u}_{\varsigma_n} = \bar{u}_{\varsigma n}(\bar{a}_{\varsigma_n})$ is a solution of* (37) *corresponding to $\bar{a}_{\varsigma_n}$, such that for $n \to \infty$, we have $\bar{a}_{\varsigma_n} \to \tilde{a}$ in L, $\bar{u}_{\varsigma_n} \rightharpoonup \tilde{u}$ in V, where $\tilde{a}$ is a solution of* (20) *and $\tilde{u} = u(\tilde{a})$ is a solution of* (19). *The same relationships hold for* (39) *and* (20) *under the condition* (29).

Proof: For a fixed $n \in \mathbb{N}$, the existence of a solution $\bar{a}_{\varsigma_n}$ of (38) follows by arguments analogous to those used in the proof of Theorem 5.3. For the convergence, we note that the sequence $\{\bar{a}_{\varsigma_n}\} \subset A$ is bounded in $\|\cdot\|_{\widehat{B}}$. Therefore, due to the compact embedding of $\widehat{B}$ into L, there is a subsequence that converges strongly in $\|\cdot\|_L$ to some $\tilde{a} \in A$. Let $\bar{u}_{\varsigma_n}$ be the sequence of solutions of (37) corresponding to $\bar{a}_{\varsigma_n}$. Clearly, the sequence $\{\bar{u}_{\varsigma_n}\}$ is bounded as well. Consequently, there is a subsequence $\{\bar{u}_{\varsigma_n}\}$ which converges weakly to some $\tilde{u} \in V$. We will show that

$\tilde{u} \in K$. By (37), for every $v \in V$, we have

$$\left\langle P(\bar{u}_{\varsigma_n}(a)), v \right\rangle = \iota_n \left[\langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v \rangle - T_{\tau_n}(\bar{a}_{\varsigma_n}, u_{\varsigma_n}(a), v) - \epsilon_n S(u_{\varsigma_n}(a), v) \right], \tag{40}$$

proving $\|P(\bar{u}_{\varsigma_n}(a))\|_V \to 0$ as $\iota_n \to 0$. By the monotonicity of P, for every $v \in V$, we have

$$0 \le \lim_{n\to\infty} \langle P(v) - P(\bar{u}_{\varsigma_n}), v - \bar{u}_{\varsigma_n} \rangle = \langle P(v), v - \tilde{u} \rangle. \tag{41}$$

For an arbitrary $z \in V$, we set $v = \tilde{u} + tz$ where $t > 0$, and obtain $\langle P(\tilde{u} + tz), z \rangle \ge 0$, which due to the hemicontinuity of P, when passed to limit $t \to 0$ gives $\langle P(\tilde{u}), z \rangle \ge 0$, confirming $P(\tilde{u}) = 0$ or equivalently $\tilde{u} \in K$.

By rearranging (37), for every $v \in K$, we have

$$T_{\tau_n}(\bar{a}_{\varsigma_n}, u_{\varsigma_n}(a), v - \bar{u}_{\varsigma_n}) + \epsilon_n S(u_{\varsigma_n}(a), v - \bar{u}_{\varsigma_n}) - \frac{1}{\iota_n} \left\langle P(v) - P(\bar{u}_{\varsigma_n}), v - \bar{u}_{\varsigma_n} \right\rangle$$
$$+ \frac{1}{\iota_n} \left\langle P(v), v - \bar{u}_{\varsigma_n} \right\rangle = \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v - \bar{u}_{\varsigma_n} \rangle,$$

and the monotonicity of P and the fact that $P(v) = 0$, for any $v \in K$, imply that for every $v \in K$, we have

$$T_{\tau_n}(\bar{a}_{\varsigma_n}, u_{\varsigma_n}(a), v - \bar{u}_{\varsigma_n}) + \epsilon_n S(u_{\varsigma_n}(a), v - \bar{u}_{\varsigma_n}) \ge \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v - \bar{u}_{\varsigma_n} \rangle, \tag{42}$$

which when passed to the limit $n \to \infty$, confirms that

$$T(\tilde{a}, v, v - \tilde{u}) \ge \langle m, v - \tilde{u} \rangle, \quad \text{for every } v \in K.$$

Setting $v = \tilde{u} + t(v - \tilde{u})$, with $t \in (0, 1]$, we obtain $T(\tilde{a}, \tilde{u}, v - \tilde{u}) + tT(\tilde{a}, v - \tilde{u}, v - \tilde{u}) \ge \langle m, v - \tilde{u} \rangle$, and by taking $t \to 0$, we get

$$T(\tilde{a}, \tilde{u}, v - \tilde{u}) \ge \langle m, v - \tilde{u} \rangle, \quad \text{for every } v \in K, \tag{43}$$

verifying that $\tilde{u} = u(\tilde{a})$.

We know that $\{\bar{u}_{\varsigma_n}\}$ converges weakly to $\tilde{u}$. However, the fact that $\{\bar{u}_{\varsigma_n}\}$ converges strongly to $\tilde{u}$ can be proved by following the arguments used in the proof of Theorem 5.3.

Now let $a_0 \in A$ be a solution of (20) and let u_0 be the corresponding solution of (19). For a_0, let $\bar{u}_{\varsigma_n}(a_0)$ be the unique solution of the penalized equation: Find $u \in V$ such that

$$T_{\tau_n}(a_0, u, v) + \epsilon_n S(u, v) + \frac{1}{\iota_n} \langle P(u), v \rangle = \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v \rangle, \quad v \in V$$

Then, in view of the above discussion, firstly, $\bar{u}_{\varsigma_n}(a_0) \to u_0$ as $n \to \infty$, and secondly, $(a_0, \bar{u}_{\varsigma_n}(a_0))$ is a feasible point for optimization problem (38). The proof of the optimality is then quite similar to the one given in Theorem 20. ∎

The key advantage of replacing the variational inequality by the penalized equation is that for the latter, the parameter-to-solution map is smooth, provided that the penalty map enjoys smoothness. For this, we replace P by its smooth approximation. For simplicity, we take $P(u) = (I - P_K)(u)$, and for parameters $\varrho_n \geq 0$, define a family of its smooth approximation $P_{\varrho_n} : V \to V$, satisfying the following conditions (partly motivated by [15]):

(PC1) For any ϱ_n, P_{ϱ_n} is monotone, $\mathrm{Null}(P_{\varrho_n}) = K$, and for $v \in V$, $P_{\varrho_n}(v) \to P(v)$ as $\varrho_n \to 0$.

(PC2) For any ϱ_n, the derivative P'_{ϱ_n} of P_{ϱ_n} exists at every point and satisfies the following:

$$\left\langle P'_{\varrho_n}(u)v, v\right\rangle \geq 0, \quad \text{for every } u, v \in V, \tag{44}$$

$$\left\langle P'_{\varrho_n}(u)v, P_K(u)\right\rangle = 0, \quad \text{for every } u, v \in V. \tag{45}$$

For $n \in \mathbb{N}$, the given penalty parameter $\iota_n > 0$, and the corresponding smooth approximation of the penalty map P_{ϱ_n}, we now consider the smooth penalized-regularized variational equation: Find $u_{\varsigma_n} \in V$ such that for every $v \in V$, we have

$$T_{\tau_n}(a, u_{\varsigma_n}(a), v) + \epsilon_n S(u_{\varsigma_n}(a), v) + \frac{1}{\iota_n}\left\langle P_{\varrho_n}(u_{\varsigma_n}(a)), v\right\rangle = \langle m_{\nu_n} + \epsilon_n \ell_{\delta_n}, v\rangle, \tag{46}$$

where we continue to use the short-hand notation $\varsigma_n := (\epsilon_n, \tau_n, \nu_n, \delta_n, \iota_n, \varrho_n)$.

The following ensures the smoothness of the parameter-to-solution map:

Theorem 5.9: *For a fixed $n \in \mathbb{N}$, the map $a \to u_{\varsigma_n}(a)$ is differentiable at any point a in the interior of A (assumed to be nonempty). For any direction $\delta a \in B$, the derivative $\delta u_{\varsigma_n} = Du_{\varsigma_n}(a)(\delta a)$ is the unique solution of the following variational equation*

$$\begin{aligned} &T_{\tau_n}(a, \delta u_{\varsigma_n}, v) + \epsilon_n S(\delta u_{\varsigma_n}, v) + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma_n})\delta u_{\varsigma_n}, v\right\rangle \\ &\quad = -T_{\tau_n}(\delta a, u_{\varsigma_n}, v), \quad \forall\, v \in V. \end{aligned} \tag{47}$$

Proof: For a fixed $n \in \mathbb{N}$, the differentiability follows from the implicit function theorem applied to the map $G : A \times V \to V$ given by

$$(a, u_{\varsigma_n}(a)) \to T_{\tau_n}(a, u_{\varsigma_n}(a), \cdot) + \epsilon_n S(u_{\varsigma_n}(a), \cdot) + \frac{1}{\iota_n}\left\langle P_{\varrho_n}(u_{\varsigma_n}(a)), \cdot\right\rangle$$

where $T_{\tau_n}(a, u_{\varsigma_n}(a), \cdot)$, $S(u_{\varsigma_n}(a), \cdot)$, and $\langle P_{\varrho_n}(u_{\varsigma_n}(a)), \cdot\rangle$ are the dual element given by the Riesz representation theorem. The derivative $D_u G(a, u_{\varsigma_n}(a)) : A \times$

$V \to V$ is given by

$$D_u G(a, u_{\varsigma n})(\delta u) = T_{\tau_n}(a, \delta u, \cdot) + \epsilon_n S(\delta u, \cdot) + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u)(\delta u_{\varsigma n})\cdot\right\rangle.$$

For any $w \in V$, the following equation

$$T_{\tau_n}(a, \delta u, \cdot) + \epsilon_n S(\delta u, \cdot) + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma n})(\delta u), \cdot\right\rangle = \langle w, \cdot\rangle.$$

is uniquely solvable. Therefore $D_u G(a, \cdot)(u_{\varsigma n}) : V \to V$ is surjective and the differentiability follows from the implicit function theorem. Finally, from (37), for every $v \in V$, we have

$$\begin{aligned}
&T_{\tau_n}(a, \delta u_{\varsigma n}, v) + T_{\tau_n}(\delta a, u_{\varsigma n}, v) + \epsilon_n S(\delta u_{\varsigma n}, v)\\
&+ \frac{1}{\varepsilon}\left\langle P'_{\varrho_n}(u_{\varsigma n})\delta u_{\varsigma n}, v\right\rangle = 0, \quad \forall\, v \in V,
\end{aligned}$$

and (47) follows. The proof is complete. ■

Remark 5.10: As customary, we will redefine the role of A by assuming that there is a slightly larger open set on which the parameter-to-solution map is defined and differentiable. This simplification will allow to use optimality conditions on the closed set A.

We now consider analogues of (38) and (39) where the constraint regularized variational equation has been replaced by the smooth regularized-penalized variational equation: For $n \in \mathbb{N}$, find $a_{\varsigma n} \in A$ by solving the following optimization problems:

$$\min_{a \in A} J_n(a) := \frac{1}{2}\|u_{\varsigma n}(a) - z_{\delta_n}\|_Z^2 + \kappa_n R(u_{\varsigma n}), \tag{48}$$

$$\min_{a \in A} \widetilde{J}_n(a) := \frac{1}{2} T_{\tau_n}(a, u_{\varsigma n}(a) - z_{\delta_n}, u_{\varsigma n}(a) - z_{\delta_n}) + \kappa_n R(u_{\varsigma n}), \tag{49}$$

where $u_{\varsigma n}(a)$ is the unique solution of (46), $\kappa_n > 0$, and R is the regularizer defined above.

In the following optimality condition for (48), $P'_{\varrho_n}(u_{\varsigma n})^*$ is the adjoint of $P'_{\varrho_n}(u_{\varsigma n})$:

Theorem 5.11: *Assume that conditions* (1) *and* (2) *of Theorem* 5.4 *hold. For each* $n \in \mathbb{N}$, *optimization problem* (48) *has a minimizer* $a_{\varsigma n} \in A$. *Moreover, for any such minimizer* $a_{\varsigma n}$ *of* (48), *there exists* $p_{\varsigma n} \in V$ *such that*

$$T_{\tau_n}(a, p_{\varsigma n}, v) + \epsilon_n S(p_{\varsigma n}, v) + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma n})^* p_{\varsigma n}, v\right\rangle$$
$$= \langle u_{\varsigma n} - z_{\delta_n}, v\rangle_Z, \quad \forall\, v \in V, \tag{50}$$

$$T_{\tau_n}(a_{\varsigma n} - a, u_{\varsigma n}, p_{\varsigma n}) - \kappa_n\left(R(a_{\varsigma n}) - R(a)\right) \geq 0, \quad \forall\, a \in A. \tag{51}$$

Proof: For a fixed $n \in \mathbb{N}$, the existence of a solution $a_{\varsigma_n} \in A$ of (48) follows by the standard arguments. Moreover, such a solution a_{ς_n} satisfies the following variational inequality as a necessary optimality condition

$$DJ_n(a_{\varsigma_n})(a - a_{\varsigma_n}) \geq \kappa_n \left(R(a_{\varsigma_n}) - R(a)\right), \quad \text{for every } a \in A, \tag{52}$$

where $J_n(a) := \frac{1}{2}\|u_{\varsigma_n}(a) - z\|_Z^2$, and hence for any $a \in A$, we have

$$DJ_n(a_{\varsigma_n})(a - a_{\varsigma_n}) = \left\langle Du_{\varsigma_n}\left(a_{\varsigma_n}\right)(a - a_{\varsigma_n}), u_{\varsigma_n} - z\right\rangle_Z.$$

Let us now consider the adjoint variational equation: Find $p_{\varsigma_n} \in V$ such that

$$\begin{aligned} T_{\tau_n}(a_{\varsigma_n}, p_{\varsigma_n}, v) + \epsilon_n S(p_{\varsigma_n}, v) + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma_n})p_{\varsigma_n}, v\right\rangle \\ = \langle u_{\varsigma_n} - z_{\delta_n}, v\rangle_Z, \quad \forall\, v \in V. \end{aligned} \tag{53}$$

Let p_{ς_n} be the unique solution of (53). Setting $\delta u_{\varsigma_n} := Du_{\varsigma_n}(a_{\varsigma_n})(a - a_{\varsigma_n})$, we have

$$\begin{aligned} \langle u_{\varsigma_n} - z_{\delta_n}, \delta u_{\varsigma_n}\rangle_Z &= T_{\tau_n}(a_{\varsigma_n}, p_{\varsigma_n}, \delta u_{\varsigma_n}) + \epsilon_n S(p_{\varsigma_n}, \delta u_{\varsigma_n}) \\ &\quad + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma_n})^* p_{\varsigma_n}, \delta u_{\varsigma_n}\right\rangle \\ &= T_{\tau_n}(a_{\varsigma_n}, \delta u_{\varsigma_n}, p_{\varsigma_n}) + \epsilon_n S(\delta u_{\varsigma_n}, p_{\varsigma_n}) \\ &\quad + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma_n})\delta u_{\varsigma_n}, p_{\varsigma_n}\right\rangle \\ &= T_{\tau_n}(a_{\varsigma_n} - a, u_{\varsigma_n}, p_{\varsigma_n}), \end{aligned}$$

where we used derivative formula (47). Consequently,

$$DJ_n(a_{\varsigma_n})(a - a_{\varsigma_n}) = T_{\tau_n}(a_{\varsigma_n} - a, u_{\varsigma_n}, p_{\varsigma_n}),$$

and (51) follows by substituting this expression in (52). The proof is complete. ■

Under additional ellipticity condition on T, we now give the following optimality conditions for the original problem (20):

Theorem 5.12: *Assume that conditions* (1) *and* (2) *of Theorem* 5.4 *hold. Assume that the adjoint solutions* $\{p_{\varsigma_n}\}$ *remain bounded. Then, there exists a minimizer* $\tilde{a}$ *of* (20) *and* $\tilde{u} \in V, \tilde{p} \in V, \tilde{\lambda} \in V^*$ *with* $\tilde{\lambda}(\tilde{u}) = 0$ *and*

$$T(\tilde{a}, \tilde{u}, v - \tilde{u}) \geq \langle m, v - \tilde{u}\rangle, \quad \textit{for every } v \in K, \tag{54a}$$

$$T(\tilde{a}, \tilde{p}, v) + \tilde{\lambda}(v) = \langle \tilde{u} - z, v\rangle_Z, \quad \textit{for every } v \in V, \tag{54b}$$

$$T(\tilde{a} - a, \tilde{u}, \tilde{p}) \geq 0, \quad \textit{for every } a \in A. \tag{54c}$$

Proof: For $n \in \mathbb{N}$, let $\{a_{\varsigma_n}\}$ be a sequence of solutions of (48), let u_{ς_n} be the corresponding solutions of the penalized equation (46), and let p_{ς_n} be the solutions of the adjoint equation (53). By arguments similar to those used in Theorem 5.8, we can show that there is a subsequence that converges in $\|\cdot\|_L$ to some $\tilde{a} \in A$. We point out that u_{ς_n} solve the smooth penalized equations and therefore the arguments need to be slightly modified. The crucial observation is that (40) holds for P_{ϱ_n}, however, due to the assumption $P_{\varrho_n}(u) \to P(u)$ for $n \to \infty$, the conclusion of (41) holds and the rest. Other arguments go through without any change. Moreover, the proof of the strong convergence of u_{ς_n} to $\tilde{u} = \tilde{u}(\tilde{a})$ holds without any change and hence (54a) holds.

By assumption, the adjoint solutions are bounded, and hence there is a weakly convergent subsequence $\{p_{\varsigma_n}\} \subset V$ such that $p_{\varsigma_n} \rightharpoonup \tilde{p} \in V$. Define two functionals $\lambda_{\varsigma_n}, \tilde{\lambda} : V \to \mathbb{R}$ by

$$\lambda_{\varsigma_n}(v) := \frac{1}{\iota_n}\left\langle P'_{\varsigma_n}(u_{\varsigma_n})^* p_{\varsigma_n}, v\right\rangle = \left\langle u_{\varsigma_n} - z_{\delta_n}, v\right\rangle - T_{\tau_n}(a_{\varsigma_n}, p_{\varsigma_n}, v) - \epsilon_n S(p_{\varsigma_n}, v), \tag{55}$$

$$\tilde{\lambda}(v) := \left\langle \tilde{u} - z, v\right\rangle - T(\tilde{a}, \tilde{p}, v). \tag{56}$$

Clearly, $\lambda_{\varsigma_n}, \tilde{\lambda} \in V^*$, and the convergence $a_{\varsigma_n} \to \tilde{a}$ in $\|\cdot\|_L$ and $u_{\varsigma_n} \to \tilde{u}$ in $\|\cdot\|_V$ yields

$$\begin{aligned}\lambda_{\varsigma_n}(v) = \left\langle u_{\varsigma_n} - z_{\delta_n}, v\right\rangle - T_{\tau_n}(a_{\varsigma_n}, p_{\varsigma_n}, v)\\ - \epsilon_n S(p_{\varsigma_n}, v) \to \left\langle \tilde{u} - z, v\right\rangle - T(\tilde{a}, \tilde{p}, v) = \tilde{\lambda}(v)\end{aligned}$$

as $n \to \infty$. Since this convergence is true for each $v \in V$, we deduce that the sequence $\{\lambda_{\varsigma_n}\}$ converges weakly to $\tilde{\lambda}$. We take $v = P_K(u_{\varsigma_n})$ in (55) and obtain

$$\lambda_{\varsigma_n}(P_K(\bar{u}_{\varsigma_n})) = \frac{1}{\varsigma_n}\left\langle P'_{\varsigma_n}(u_{\varsigma_n})^* \bar{p}_{\varsigma_n}, P_K(u_{\varsigma_n})\right\rangle = 0.$$

By the continuity of the projection map, we get $0 = \lambda_{\varsigma_n}(P_K(u_{\varsigma_n})) \to \tilde{\lambda}(\tilde{u})$ for $n \to \infty$, and consequently, $\tilde{\lambda}(\tilde{u}) = 0$. For (54c), it suffices to take limits in (51). In fact, since $a_{\varsigma_n} \to \tilde{a}$, $u_{\varsigma_n} \to \tilde{u}$, and $p_{\varsigma_n} \rightharpoonup \tilde{p}$, by using the properties of T, and by taking the limit $n \to \infty$ in

$$T_{\tau_n}(a_{\varsigma_n} - a, u_{\varsigma_n}, p_{\varsigma_n}) \geq \kappa_n \left(R(a_{\varsigma_n}) - R(a)\right)$$

we get $T(\tilde{a} - a, \tilde{u}, \tilde{p}) \geq 0$, which completes the proof. ■

We now give an optimality condition for the MOLS based optimization problem (49):

Theorem 5.13: *Assume that conditions* (1) *and* (2) *of Theorem* 5.4 *hold. For each* $n \in \mathbb{N}$, *optimization problem* (49) *has a minimizer* $a_{\varsigma_n} \in A$. *Moreover, for any such* *solution a_{ς_n} of (49), there is $q_{\varsigma_n} \in V$ such that for each $a \in A$ and each $v \in V$, we have*

$$T_{\tau_n}(a_{\varsigma_n}, q_{\varsigma_n}, v) + \epsilon_n S(q_{\varsigma_n}, v) + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma_n})^* q_{\varsigma_n}, v\right\rangle$$
$$= T_{\tau_n}(a_{\varsigma_n}, u_{\varsigma_n} - z_{\delta_n}, v\rangle, \tag{57}$$
$$\frac{1}{2} T_{\tau_n}(a - a_{\varsigma_n}, u_{\varsigma_n} - z_{\delta_n}, u_{\varsigma_n} - z_{\delta_n})$$
$$+ T(a - a_{\varsigma_n}, u_{\varsigma_n}, q_{\varsigma_n}) \geq \kappa_n \left(R(a_{\varsigma_n}) - R(a)\right). \tag{58}$$

Proof: We will follow the scheme of Theorem 5.11. For a fixed, $n \in \mathbb{N}$, let $a_{\varsigma_n} \in A$ be a solution of (49), and u_{ς_n} be the corresponding solution of the penalized regularized variational equation. Then

$$D\widetilde{J}_n(a_{\varsigma_n})(a - a_{\varsigma_n}) \geq \kappa_n \left(R(a_{\varsigma_n}) - R(a)\right), \quad \text{for every } a \in A,$$

where $\widetilde{J}_n(a) := \frac{1}{2} T_{\tau_n}(a_{\varsigma_n}, u_{\varsigma_n} - z_{\delta_n}, u_{\varsigma_n} - z_{\delta_n})$.

By using the notation $\delta u_{\varsigma_n} := D u_{\varsigma_n}(a_{\varsigma_n})(a - a_{\varsigma_n})$, we have

$$D\widetilde{J}_n(a_{\varsigma_n})(a - a_{\varsigma_n}) = \frac{1}{2} T_{\tau_n}(a - a_{\varsigma_n}, u_{\varsigma_n} - z_{\delta_n}, u_{\varsigma_n} - z_{\delta_n})$$
$$+ T_{\tau_n}(a_{\varsigma_n}, \delta u_{\varsigma_n}, u_{\varsigma_n} - z_{\delta_n}). \tag{59}$$

We now consider the adjoint equation of finding $q_{\varsigma_n} \in V$ such that

$$T_{\tau_n}(a_{\varsigma_n}, q_{\varsigma_n}, v) + \epsilon_n S(q_{\varsigma_n}, v) + \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma_n})^* q_{\varsigma_n}, v\right\rangle$$
$$= T_{\tau_n}(a_{\varsigma_n}, z_{\delta_n} - u_{\varsigma_n}, v), \quad \forall\, v \in V. \tag{60}$$

Clearly, (59) is uniquely solvable, and let q_{ς_n} be its unique solution. Then,

$$T_{\tau_n}(a_{\varsigma_n}, u_{\varsigma_n} - z_{\delta_n}, \delta u_{\varsigma_n})$$
$$= -T_{\tau_n}(a_{\varsigma_n}, q_{\varsigma_n}, \delta u_{\varsigma_n}) - \epsilon_n S(q_{\varsigma_n}, \delta u_{\varsigma_n}) - \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma_n})^* q_{\varsigma_n}, \delta u_{\varsigma_n}\right\rangle$$
$$= -T_{\tau_n}(a_{\varsigma_n}, \delta u_{\varsigma_n}, q_{\varsigma_n}) - \epsilon_n S(\delta_a u_{\varsigma_n}, q_{\varsigma_n}) - \frac{1}{\iota_n}\left\langle P'_{\varrho_n}(u_{\varsigma_n}) \delta u_{\varsigma_n}, q_{\varsigma_n}\right\rangle$$
$$= T_{\tau_n}(a - a_{\varsigma_n}, u_{\varsigma_n}, q_{\varsigma_n})$$

by using the derivative characterization Theorem 5.9. Using this in (59), yields

$$D\widetilde{J}_n(a_{\varsigma_n})(a - a_{\varsigma_n}) = \tfrac{1}{2} T_{\tau_n}(a - a_{\varsigma_n}, u_{\varsigma_n} - z_{\delta_n}, u_{\varsigma_n} - z_{\delta_n})$$
$$+ T_{\tau_n}(a - a_{\varsigma_n}, u_{\varsigma_n}, q_{\varsigma_n}),$$

which at once gives the desired inequality. ■

Under additional ellipticity condition on T, finally, we have the following optimality conditions for the original MOLS:

Theorem 5.14: *Assume that conditions* (1) *and* (2) *of Theorem* 5.4 *hold. Assume that the adjoint solutions* $\{p_{\varsigma_n}\}$ *remain bounded. There exists a solution* $\bar{a}$ *of* (21) *and* $\bar{u} \in V, \bar{q} \in V, \bar{\lambda} \in V^*$ *with* $\bar{\lambda}(\bar{u}) = 0$ *and*

$$T(\bar{a}, \bar{q}, v) + \bar{\lambda}(v) = T(\bar{a}, z - \bar{u}, v) \quad \text{for each } v \in V, \tag{61a}$$

$$T(\bar{a}, \bar{u}, v - \bar{u}) \geq \langle m, v - \bar{u}\rangle, \quad \text{for each } v \in K, \tag{61b}$$

$$\frac{1}{2}T(a - \bar{a}, \bar{u} - z, \bar{u} - z) \geq T(\bar{a} - a, \bar{u}, \bar{q}), \quad \text{for each } a \in A. \tag{61c}$$

Proof: The proof follows from the arguments used in the proof of Theorem 54. ■

Remark 5.15: The adjoint equations (53) and and (59) are entirely comparable with the regularized-penalized equation (37). We note that although the adjoint solutions are uniquely defined (due to the elliptic regularization by S), their boundedness would require additional information. For example, milder coercivity, or the existence of $\tilde{p}$ or $\bar{q}$ satisfying the corresponding equations would suffice. This approach is akin to the elliptic regularization of the variational inequalities where we proved that the regularized solutions are bounded provided that the original variational inequality is solvable.

6. Identification in a simplified elastic-plastic torsion model

We consider an elastic-plastic torsion problem for visco-elastic material studied by Kano, Kenmochi, and Murase [32]: find $u \in H_0^1(\Omega)$ and $w \in \beta(u)$ such that $|\nabla u| \leq k_0(u)$, a.e on Ω with

$$\int_\Omega a\nabla u \cdot \nabla(v - u) + \int_\Omega w(v - u) \geq \int_\Omega f(v - w),$$

for all $v \in H_0^1(\Omega)$ with $|\nabla v| \leq k_c(v)$ a.e. on Ω. Here Ω is suitable domain in $\mathbb{R}^2$, $f \in L^2(\Omega)$, a is the materail material and $\beta(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with linear growth at $\pm\infty$. Contrary to [32], a does not explicitly depend on u or its gradient.

In the above model, our objective is to identify the parameter a from a measurement of its u. To reformulate the above problem in the framework supplied above, we set $V = H_0^1(\Omega)$ and $B = L^\infty(\Omega)$. Let κ^* be a constant, and let κ_c be a Lipschitz continuous real function on $\mathbb{R}$ such that $0 < \kappa_c(r) \leq \kappa^*$, for every $r \in \mathbb{R}$. Define a convex set C and $K : C \rightrightarrows C$ by

$$C = \{w \in V |\ |\nabla w| \leq \kappa^*, \text{ a.e. on } \Omega\},$$

$$K(v) = \{w \in V |\ |\nabla w| \leq \kappa_c(v), \text{ a.e. on } \Omega\},$$

which satisfy conditions (M1) and (M2) of Theorem 4.3 (see [32]).

We focus on identifying a in the quasi-variational inequality: Find $u \in K(u)$ such that

$$T(a, u, v - u) + \langle Fu, v - u \rangle \geq \langle m, v - u \rangle, \quad \text{for every } v \in K(u),$$

where $T(a, u, v) = \int_\Omega a \nabla u \nabla v$ and $F = \beta$. It can be shown that the trilinear form $T(a, u, v) = \int_\Omega a \nabla u \nabla v$ is indeed, elliptic, continuous, and satisfies (7).

For the regularizer, we recall that the total variation of $f \in L^1(\Omega)$ is given by

$$\mathrm{TV}(f) = \sup \left\{ \int_\Omega f(\nabla \cdot g) : g \in \left(C^1(\Omega)\right)^N, \ |g(x)| \leq 1 \text{ for all } x \in \Omega \right\}$$

where $|\cdot|$ represents the Euclidean norm. Clearly, if $f \in W^{1,1}(\Omega)$, then $\mathrm{TV}(f) = \int_\Omega |\nabla f|$.

If $f \in L^1(\Omega)$ satisfies $\mathrm{TV}(f) < \infty$, then f is said to have bounded variation, and the Banach space $\mathrm{BV}(\Omega)$ is defined by $\mathrm{BV}(\Omega) = \{f \in L^1(\Omega) : \mathrm{TV}(f) < \infty\}$ being endowed with the norm $\|f\|_{\mathrm{BV}(\Omega)} = \|f\|_{L^1(\Omega)} + \mathrm{TV}(f)$. The functional $\mathrm{TV}(\cdot)$ is a seminorm on $\mathrm{BV}(\Omega)$ and is often called the BV-seminorm.

We set $L = L^1(\Omega)$, $\widehat{B} = \mathrm{BV}(\Omega)$, and $R(a) = TV(a)$, and define two sets

$$A_1 = \{a \in L^\infty(\Omega) |\ c_1 \leq a(x) \leq c_2, \text{ a.e. in } \Omega, \},$$
$$A_2 = \{a \in L^\infty(\Omega) |\ c_1 \leq a(x) \leq c_2, \text{ a.e. in } \Omega, \ \mathrm{TV}(a) \leq c_3\},$$

where c_1, c_2 and c_3 are positive constants. Clearly, both sets are compact in L, whereas A_2 is bounded in $\|\cdot\|_{\widehat{B}}$. It is known that $L^\infty(\Omega)$ is continuously embedded in $L^1(\Omega)$, $\mathrm{BV}(\Omega)$ is compactly embedded in $L^1(\Omega)$, and $TV(\cdot)$ is convex and lower-semicontinuous in $L^1(\Omega)$-norm. Summarizing, all the imposed conditions are satisfied and consequently the developed framework ensures the identification of a discontinuous parameter in the above model.

7. Numerical results

We now report the outcome of two preliminary experiments to show the numerical feasibility of the identification in variational inequalities. The first experiment shows, for the coercive case, a slightly better reconstruction by the MOLS objective. In the second experiment, we look at the impact of data contamination. In both of the experiments, we use piecewise linear finite element discretization for numerically solving the discrete analogs of the optimality systems for the OLS and the MOLS objectives. We solve the discrete optimality systems by a Damped Gauss-Newton iteration with an Armijo rule line search. For the numerical experiments, we choose $\Omega = [0, 1] \times [0, 1]$.

7.1. A coercive example

In this experiment, we identify a in the following variational inequality of finding $u \in K := \{u \in H_0^1(\Omega)|\ u(x) \geq 0,\ \text{a.e. } \Omega\} \subset V := H_0^1(\Omega)$ such that

$$\int_\Omega (a + 0.01)^2 \nabla u \nabla (v - u) \geq \int_\Omega m(a)(v - u), \quad \text{for every } v \in K, \tag{62}$$

where $\Omega \subset \mathbb{R}^2$ is a suitable domain. We choose $B = L^\infty(\Omega)$ and for $a_0 > 0, a_1 > 0$, define

$$A := \{a \in L^\infty(\Omega)|\ 0 < a_0 \leq a \leq a_1 \text{ a.e. in } \Omega\}.$$

The exact solution in this setting is given by: $\bar{a}(x_1, x_2) = 1 + 0.25(\sin^2(2\pi x_1) + \cos(2\pi x_2))$. Note that (62) is slightly more general than (19) where instead of a we have $\ell(a) = (a + 0.01)^2$ and the functional $m(a) = a - \pi\cos(2\pi x_1) + 0.5\pi \sin(2\pi x_2)$ depends on the coefficient. Therefore, our results needs to be slightly modified to take into account these generalities.

Tables 1 and 2 show that the MOLS functional yields slightly better reconstruction than the OLS functional. The reconstruction error in the OLS is about 8% in the discrete L^2-norm whereas for the MOLS it is between 3% and 5%, see Figures 1–4.

7.2. A non-coercive example

We now consider the inverse problem of identifying a variable parameter a in the variational inequality of finding $u \in K \subset V := H_0^1(\Omega)$ such that

$$\int_\Omega a \nabla u \nabla (v - u) \geq \int_\Omega f(v - u)\, dx, \quad \text{for every } v \in K, \tag{63}$$

Table 1. Example 7.1: Reconstruction error for the OLS.

h	$\frac{\|a^h - I_h\bar{a}_0\|_{L^2(\Omega)}}{\|I_h\bar{a}_0\|_{L^2(\Omega)}}$	$\frac{\|u^h - u_0^h\|_{L^2(\Omega)}}{\|u_0^h\|_{L^2(\Omega)}}$	$\frac{\|a^h - I_h\bar{a}_0\|_{L^\infty(\Omega)}}{\|I_h\bar{a}_0\|_{L^\infty(\Omega)}}$	$\frac{\|u^h - u_0^h\|_{L^\infty(\Omega)}}{\|u_0^h\|_{L^\infty(\Omega)}}$
0.0707107	0.087	0.110	0.154	0.105
0.0565685	0.085	0.087	0.154	0.085
0.0471405	0.086	0.061	0.191	0.061
0.0404061	0.084	0.052	0.198	0.054
0.0353553	0.081	0.059	0.164	0.071

Table 2. Example 7.1: Reconstruction error for the MOLS.

h	$\frac{\|a^h - I_h\bar{a}_0\|_{L^2(\Omega)}}{\|I_h\bar{a}_0\|_{L^2(\Omega)}}$	$\frac{\|u^h - u_0^h\|_{L^2(\Omega)}}{\|u_0^h\|_{L^2(\Omega)}}$	$\frac{\|a^h - I_h\bar{a}_0\|_{L^\infty(\Omega)}}{\|I_h\bar{a}_0\|_{L^\infty(\Omega)}}$	$\frac{\|u^h - u_0^h\|_{L^\infty(\Omega)}}{\|u_0^h\|_{L^\infty(\Omega)}}$
0.0707107	0.031	0.016	0.069	0.017
0.0565685	0.030	0.015	0.070	0.015
0.0471405	0.030	0.015	0.080	0.020
0.0404061	0.032	0.016	0.096	0.027
0.0353553	0.051	0.023	0.145	0.049

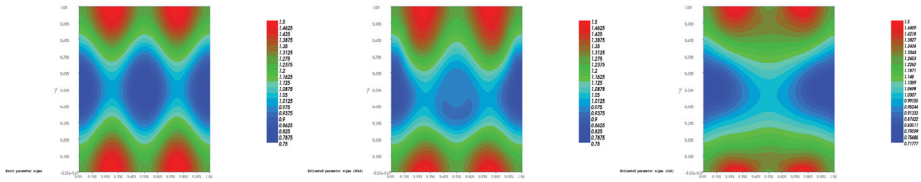

Figure 1. Reconstruction for $h = 0.0353553$. The left figure shows the exact coefficient, the middle figure shows the coefficient by the MOLS approach, and the right figure shows the coefficient by the OLS approach.

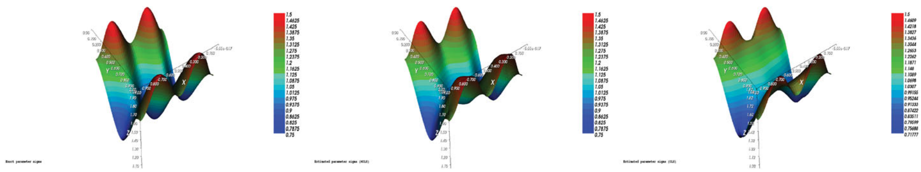

Figure 2. Reconstruction for $h = 0.0353553$. The left figure shows the exact coefficient, the middle figure shows the coefficient by the MOLS approach, and the right figure shows the coefficient by the OLS approach.

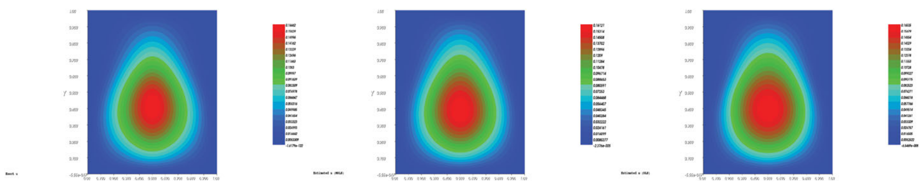

Figure 3. Reconstruction for $h = 0.0353553$. The left figure shows the exact solution, the middle figure shows the solution by the MOLS approach, and the right figure shows the solution by the OLS approach.

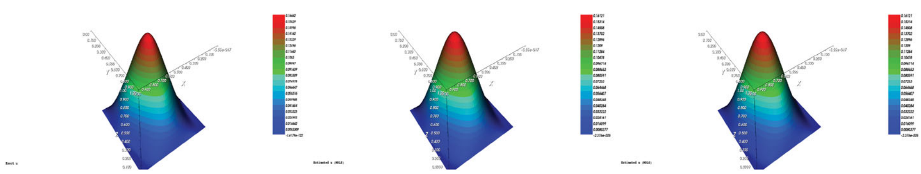

Figure 4. Reconstruction for $h = 0.0353553$. The left figure shows the exact solution, the middle figure shows the solution by the MOLS approach, and the right figure shows the solution by the OLS approach.

where $\Omega \subset \mathbb{R}^2$ is a suitable domain and $K = \{u \in H_0^1(\Omega) |\ u(x) \geq 0,\ \text{a.e. } \Omega\}$. We choose $B = L_\infty(\Omega)$, and for a given positive constant a_1, define

$$A := \{a \in L^\infty(\Omega) |\ 0 \leq a \leq a_1 \text{a.e. in } \Omega\}.$$

The exact data to be $\bar{u}(x_1, x_2) = x_1(1 - x_2)x_2(1 - x_2)$ and $\bar{a}(x_1, x_2) = x_1 x_2$. Since a vanishes at 0, the ellipticity fails for the associated trilinear form. We solve the optimality conditions for (27). In numerical computations, we consider H^1 norm as regularizer and data z is obtained by directly interpolating $\bar{u}$.

We take $h = 0.0707107$ and a keep a regularization parameter κ. Numerical experiments show that reconstruction is quite indifferent for $\varepsilon_n \in [0, 0.0001]$, see Table 3 for $\varepsilon_n = 0.0001$.

To study the influence of the noise, we consider the contaminated data $z_{\delta_n} = z + \delta_n \eta(t)$ with $\eta(t)$ uniformly distributed in $[0, 1]$. We consider two noise levels, $\delta_n \in \{0.0001, 0.001\}$. The reconstruction rate is stable for these noise levels, see Table 3 and Figures 5 and 6. For bigger noise levels, the proposed algorithm becomes unstable and the reconstruction rate is quite poor. We can check this fact for the particular case $\delta_n = 0.01$ in Figure 7.

Table 3. Reconstruction Error for the OLS for $\epsilon = 0.0001$, $h = 0.0707107$ for different noise levels.

Noise δ_n	$\frac{\|a^h - I_h\bar{a}\|_{L^2(\Omega)}}{\|I_h\bar{a}\|_{L^2(\Omega)}}$	$\frac{\|u^h - \bar{u}^h\|_{L^2(\Omega)}}{\|\bar{u}^h\|_{L^2(\Omega)}}$	$\frac{\|a^h - I_h\bar{a}\|_{L^\infty(\Omega)}}{\|I_h\bar{a}\|_{L^\infty(\Omega)}}$	$\frac{\|u^h - \bar{u}^h\|_{L^\infty(\Omega)}}{\|\bar{u}^h\|_{L^\infty(\Omega)}}$
0	0.013	0.009	0.019	0.024
1e−04	0.014	0.010	0.021	0.026
1e−03	0.027	0.008	0.033	0.037

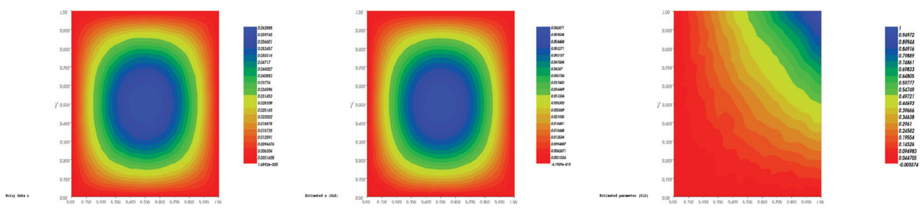

Figure 5. $h = 0.0707107$, $\kappa = \epsilon = 0.0001$. Reconstruction with noise level $\delta_n = 0.001$.

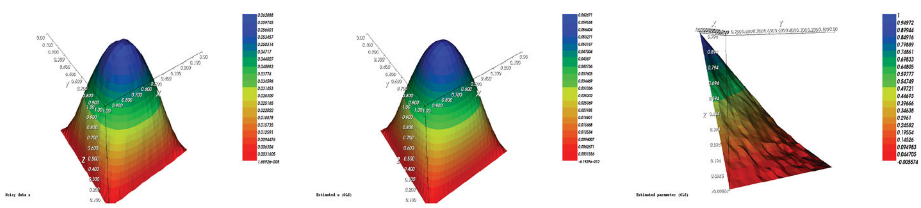

Figure 6. $h = 0.0707107$, $\epsilon = 0.0001$. Reconstruction with noise level $\delta_n = 0.001$.

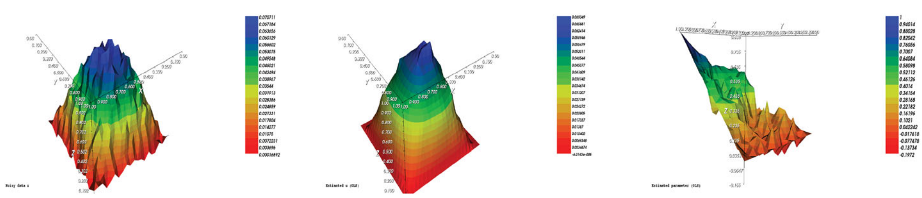

Figure 7. $h = 0.0707107$. Reconstruction for noise level $\delta_n = 0.01$.

8. Concluding remarks

We presented identification results for multi-valued quasi-variational inequalities defined on general constraint sets depending on the unknown solution. On the other hand, for noncoercive variational inequalities, we combined elliptic regularization with a penalization and smoothing which permitted us to derive new optimality conditions. A comparison between the two approaches sheds some light onto to difficulties associated with an adequate treatment of quasi-variational inequalities. For the general constraints, as considered in this work, the parameter-to-solution map is typically multi-valued. Therefore, its smoothness needs to be studied by tools from variational and set-valued optimization. On the other hand, although non-coercive variational inequalities also have a set-valued parameter-to-solution map, the elliptic regularization results in a single-valued parameter-to-regularized solution map. Furthermore, a penalization, coupled with smoothing, permits to study its differentiability and eventually leads to optimality conditions. We emphasize that it is expected that for smaller regularization parameters, the numerical solution will show some instability because, for the unregularized problem, the parameter-to-solution map is in general nonsmooth. Moreover, the developed techniques are not readily available for quasi-variational inequalities, the elliptic regularization does not render a single-valued parameter to solution map, and a general penalization approach is not available. A possible way is to restrict to the so-called general solution and apply the developed tools through that. Some of these possibilities will be addressed in future research.

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References

[1] Hu B. A quasi-variational inequality arising in elastohydrodynamics. SIAM J Math Anal. 1990;21(1):18–36.
[2] Boulaaras S, Bencheikh le Hocine MA, Haiour M. The finite element approximation in a system of parabolic quasi-variational inequalities related to management of energy production with mixed boundary condition. Comput Math Model. 2014;25(4):530–543.
[3] Barbagallo A. Regularity results for evolutionary nonlinear variational and quasi-variational inequalities with applications to dynamic equilibrium problems. J Global Optim. 2008;40(1–3):29–39.
[4] Daniele P. A remark on a dynamic model of a quasi-variational inequality. Rend Circ Mat Palermo. 1997;2(Suppl. 48):91–100. Equilibrium problems with side constraints. Lagrangean theory and duality, II (Scilla, 1996).
[5] Maugeri A. Variational and quasi-variational inequalities in network flow models. Recent developments in theory and algorithms. In: Variational inequalities and network equilibrium problems (Erice, 1994). New York: Plenum; 1995. p. 195–211.
[6] Motreanu D, Sofonea M. Quasivariational inequalities and applications in frictional contact problems with normal compliance. Adv Math Sci Appl. 2000;10(1):103–118.
[7] Sofonea M, Matei A. History-dependent quasi-variational inequalities arising in contact mechanics. European J Appl Math. 2011;22(5):471–491.
[8] Lenzen F, Becker F, Lellmann J, et al. A class of quasi-variational inequalities for adaptive image denoising and decomposition. Comput Optim Appl. 2013;54(2):371–398.
[9] Ton BA. Time-dependent quasi-variational inequalities and the Nash equilibrium. Nonlinear Anal Ser A Theory Methods. 2000;41(7–8):1057–1081.
[10] Borzì A, Kanzow C. Formulation and numerical solution of Nash equilibrium multiobjective elliptic control problems. SIAM J Control Optim. 2013;51(1):718–744.
[11] Naito K. A system of quasivariational inequalities and its application to reaction diffusion equations. Publ Res Inst Math Sci. 1985;21(4):677–698.
[12] Barrett JW, Prigozhin L. A quasi-variational inequality problem arising in the modeling of growing sandpiles. ESAIM Math Model Numer Anal. 2013;47(4):1133–1165.
[13] Abouchabaka J, Aboulaïch R, Nachaoui A, et al. Quasi-variational inequality and shape optimization for solution of a free boundary problem. COMPEL. 1999;18(2):143–164.
[14] Barrett JW, Prigozhin L. A quasi-variational inequality problem in superconductivity. Math Models Methods Appl Sci. 2010;20(5):679–706.
[15] Bayada G, El Alaoui Talibi M. An application of the control by coefficients in a variational inequality for hydrodynamic lubrication. Nonlinear Anal Real World Appl. 2000;1(2):315–328.
[16] Hintermüller M. Inverse coefficient problems for variational inequalities: optimality conditions and numerical realization. M2AN Math Model Numer Anal. 2001;35(1):129–152.
[17] González GA. Theoretical framework of an identification problem for an elliptic variational inequality with bilateral restrictions. J Comput Appl Math. 2006;197(1):245–252.
[18] Adly S, Bergounioux M, Ait Mansour M. Optimal control of a quasi-variational obstacle problem. J Global Optim. 2010;47(3):421–435.
[19] Hasanov A. Inverse coefficient problems for elliptic variational inequalities with a nonlinear monotone operator. Inverse Probl. 1998;14(5):1151–1169.
[20] Zheng C, Cheng X, Liang K. Numerical analysis of inverse elasticity problem with Signorini's condition. Commun Comput Phys. 2016;20(4):1045–1070.
[21] Yang R, Ou YH. Inverse coefficient problems for nonlinear elliptic variational inequalities. Acta Math Appl Sin Engl Ser. 2011;27(1):85–92.

[22] Kupenko OP, Manzo R. Approximation of an optimal control problem in coefficient for variational inequality with anisotropic p-Laplacian. Nonlinear Differ Equ Appl. 2016;23(3):Art. 35, 18.
[23] Barabasz B, Gajda-Zagórska E, Migórski S, et al. A hybrid algorithm for solving inverse problems in elasticity. Int J Appl Math Comput Sci. 2014;24(4):865–886.
[24] Migórski S, Ochal A. An inverse coefficient problem for a parabolic hemivariational inequality. Appl Anal. 2010;89(2):243–256.
[25] Hoffmann KH, Sprekels J. On the identification of parameters in general variational inequalities by asymptotic regularization. SIAM J Math Anal. 1986;17(5):1198–1217.
[26] Gwinner J, Jadamba B, Khan AA, et al. Identification in variational and quasi-variational inequalities. J Convex Anal. 2018;25(2):545–569.
[27] Khan AA, Motreanu D. Inverse problems for quasi-variational inequalities. J Global Optim. 2018;70:401–411.
[28] Migorski S, Khan AA, Zheng S. Inverse problems for nonlinear quasi-variational inequalities with an application to implicit obstacle problems of p-Laplacian type. Inverse Probl. 2019;35(3):035004.
[29] Migórski S, Zeng B. Convergence of solutions to inverse problems for a class of variational-hemivariational inequalities. Discrete Contin Dyn Syst B. 2018; 22 pages. DOI:10.3934/dcdsb.2018172
[30] Zeng B, Migorski S. Variational-hemivariational inverse problems for unilateral frictional contact. Appl. Anal. 2018; 21 pages. DOI:10.1080/00036-811.2018.1491037
[31] Shi P, Shillor M. Noncoercive variational inequalities with application to friction problems. Proc Roy Soc Edinburgh Sect A. 1991;117(3–4):275–293.
[32] Kano R, Kenmochi N, Murase Y. Existence theorems for elliptic quasi-variational inequalities in Banach spaces. In: Recent advances in nonlinear analysis. Hackensack (NJ): World Sci. Publ.; 2008. p. 149–169.
[33] Bensoussan A, Lions J-L. Nouvelle formulation de problèmes de contrôle impulsionnel et applications. C R Acad Sci Paris Sér A-B. 1973;276:A1189–A1192.
[34] Mosco U. Implicit variational problems and quasi variational inequalities. In: Nonlinear operators and the calculus of variations (Summer School, Univ. Libre Bruxelles, Brussels, 1975). Berlin: Springer; 1976. p. 83–156. (Lecture notes in math.; vol. 543).
[35] Kovtunenko VA, Leugering G. A shape-topological control of variational inequalities. Eurasian Math J. 2016;7(3):41–52.
[36] Maugeri A, Raciti F. On existence theorems for monotone and nonmonotone variational inequalities. J Convex Anal. 2009;16(3–4):899–911.
[37] Myśliński A. Structural optimization of variational inequalities using piecewise constant level set method. In: System modeling and optimization. Heidelberg: Springer; 2013. p. 407–416. (vol. 391 of IFIP adv. inf. commun. technol.).
[38] Alber Y, Ryazantseva I. Nonlinear ill-posed problems of monotone type. Dordrecht: Springer; 2006.
[39] Kluge R. On some parameter determination problems and quasi-variational inequalities. In: Theory of nonlinear operators (Proc. Fifth Internat. Summer School, Central Inst. Math. Mech. Acad. Sci. GDR, Berlin, 1977). vol. 6 of Abh. Akad. Wiss. DDR, Abt. Math. Naturwiss. Tech., 1978. Berlin: Akademie-Verlag; 1978. p. 129–139.
[40] Alber Y, Butnariu D, Ryazantseva I. Regularization of monotone variational inequalities with Mosco approximations of the constraint sets. Set-Valued Anal. 2005;13(3):265–290.
[41] Browder FE. On a principle of H. Brézis and its applications. J Funct Anal. 1977;25(4):356–365.
[42] Gockenbach MS, Khan AA. An abstract framework for elliptic inverse problems. I. An output least-squares approach. Math Mech Solids. 2007;12(3):259–276.

[43] Khan AA, Tammer C, Zalinescu C. Regularization of quasi-variational inequalities. Optimization. 2015;64(8):1703–1724.
[44] Liu Z. Generalized quasi-variational hemi-variational inequalities. Appl Math Lett. 2004;17(6):741–745.
[45] Liu Z, Zeng B. Optimal control of generalized quasi-variational hemivariational inequalities and its applications. Appl Math Optim. 2015;72(2):305–323.
[46] Guan Z, Kartsatos AG, Skrypnik IV. Ranges of densely defined generalized pseudomonotone perturbations of maximal monotone operators. J Differ Equ. 2003;188(1):332–351.
[47] Han W. A regularization procedure for a simplified friction problem. Math Comput Model. 1991;15(8):65–70.
[48] Adly S, Goeleven D, Théra M. Recession mappings and noncoercive variational inequalities. Nonlinear Anal. 1996;26(9):1573–1603.
[49] Tomarelli F. Noncoercive variational inequalities for pseudomonotone operators. Rend Sem Mat Fis Milano. 1991;61:141–183. (1994).
[50] Clason C, Khan AA, Sama M, et al. Contingent derivatives and regularization for noncoercive inverse problems. Optimization. 2018. DOI:10.1080/02331934.2018.1442448
[51] Gibali A, Jadamba B, Khan AA, et al. Gradient and extragradient methods for the elasticity imaging inverse problem using an equation error formulation: a comparative numerical study. In: Nonlinear analysis and optimization. Providence (RI): Amer. Math. Soc.; 2016. p. 65–89. (vol. 659 of Contemp. math.).
[52] Jadamba B, Khan AA, Oberai A, et al. First-order and second-order adjoint methods for parameter identification problems with an application to the elasticity imaging inverse problem. Inverse Probl Sci Eng. 2017;25(12):1768–1787.
[53] Jadamba B, Khan AA, Rus G, et al. A new convex inversion framework for parameter identification in saddle point problems with an application to the elasticity imaging inverse problem of predicting tumor location. SIAM J Appl Math. 2014;74(5):1486–1510.